COURS DE L'INSTITUT FOURIER

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On Some Family of Contractible Hypersurfaces in C^4

Cours de l'institut Fourier, tome S24 (1993), p. 57-75
<http://www.numdam.org/item?id=CIF_1993__S24__57_0>

Journées Singulières et Jacobiennes
GRENOBLE
26-28 mai 1993 (57–75)

On some family of contractible hypersurfaces in $\mathbf{C}^4$

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I. Introduction. In this paper we study a class of contractible smooth hypersurfaces in $\mathbf{C}^4$ which is interesting in connection with the problem of linearizing of a $\mathbf{C}^*$-action on $\mathbf{C}^3$ and the Abhyankar-Sathaye conjecture. Some of these hypersurfaces were found in [D], and the complete list of them was presented in [Ru]. An equivalent but less explicit description of this class is given in [K1]. Every hypersurface from this class admit a $\mathbf{C}^*$-action with one fixed point. Each of them is also diffeomorphic to $\mathbf{R}^6$, but it is unknown whether there is a hypersurface in this class which is isomorphic to $\mathbf{C}^3$. If such hypersurface does exist then the $\mathbf{C}^*$-action is non-linearizable on it [Ru], i.e. we have a counterexample to the linearizing problem. Moreover this hypersurface would be a counterexample to the Abhyankar-Sathaye conjecture [D], [Ru], [K2]. On the other hand P. Russell proved that if there is no $\mathbf{C}^3$ in this class then the linearizing problem may be studied under some additional simplifying assumption [Ru]. To a great extend the linearizing problem with this extra assumption was studied in [KR]. In [K1], [K2], [Ru] it was shown that some of these hypersurfaces (but not all of them) have nonnegative Kodaira logarithmic dimension. In this paper we shall present other algebraic obstacles for many hypersurfaces from this class to be $\mathbf{C}^3$. The technique we are using here is simpler and the results are stronger than in previous papers. But we have to pay for this by long computation. The main result may be described as follows. The $\mathbf{C}^*$-action on a hypersurface X from this class generates a linear representation of $\mathbf{C}^*$ on the tangent space of the fixed point.

This tangent representation is of form $(x,y,z) \to (\lambda^{-a}x, \lambda^{b}y, \lambda^{c}z)$ where a,b,c are natural, $\lambda \in \mathbf{C}^*$, and (x,y,z) is a coordinate system on the tangent space. Put $\alpha_1 = GCD(b,c)$, $\alpha_2 = GCD(a,c)$, $\alpha_3 = GCD(a,b)$.

Theorem A. *There is no dominant morphism from $\mathbf{C}^3$ into X when $\alpha_1 \geq 2$, $\alpha_2 > 3$, $\alpha_3 > 3$.*

In particular none of hypersurfaces in $\mathbf{C}^4$ from [D] is isomorphic to $\mathbf{C}^3$.

2. Preliminaries. In this section we recall the linearizing problem and the construction of the hypersurfaces from [Ru].

Consider a complex algebraic variety X and a $\mathbf{C}^*$-action on it, $(\lambda, p) \to \lambda \cdot p$, where $\lambda \in \mathbf{C}^*$ and $p \in X$. Recall that this action is algebraic if the map $\mathbf{C}^* \times X \to X, (\lambda, p) \to \lambda \cdot p$ is a morphims of complex algebraic varieties. The simplest example of an algebraic $\mathbf{C}^*$-action on $\mathbf{C}^3$ is a linear action on $\mathbf{C}^3$ given by

$$(x,y,z) \to (\lambda^a x, \lambda^b y, \lambda^c z) \tag{2.1}$$

where (x,y,z) is a coordinate system on $\mathbf{C}^3$, $\lambda \in \mathbf{C}^*$, a,b,c are integers. These integers a,b,c are called the weights of the action.

The linearizing conjecture. *Every $\mathbf{C}^*$-action on $\mathbf{C}^3$ is equivalent to a linear one up to a polynomial coordinate substitution.*

A function f on an algebraic manifold X is semi-invariant (or quasi-invariant) of weight $l \in \mathbf{Z}$ relative to a $\mathbf{C}^*$-action G if for every $\lambda \in \mathbf{C}^*$ we have $f \circ G(\lambda) = \lambda^l f$. Note that the linearizing conjecture just claims the existence of a semi-invariant coordinate system. This conjecture is true in all cases except for one, the answer to which is unknown yet (see [Ru], [KR], [B] for details). Following [Ru] we call this case the "hard-case" $\mathbf{C}^*$-action. This "hard-case" $\mathbf{C}^*$-action on a threefold X can be described by the two following conditions.

(i) The $\mathbf{C}^*$-action has only one fixed point o. In this case the action generates the tangent representation on T_oX which is of form (2.1). Its weights are also called the weights of the $\mathbf{C}^*$-action.

(ii) The weights a, b, c of the $\mathbf{C}^*$-action are nonzero and the sign of a is different from the signs of b and c.

In [Ru] P. Russell found an interesting connection between the "hard-case" of the linearizing problem and the class of hypersurfaces in $\mathbf{C}^4$ described below.

Let a', b', c' be pairwise prime natural, let $b' \geq c'$, and let ω be the group of a'-roots of unity. Consider an ω-action on $\mathbf{C}^2$ given by $(\bar{u}, \bar{v}) \to (\lambda^{c'}\bar{u}, \lambda^{b'}\bar{v})$, where $\lambda \in \omega$, and $(\bar{u}, \bar{v})$ is a coordinate system. Consider a polynomial $\bar{f}$ which is semi-invariant relative to ω and is satisfying two assumptions.

Assumption (a). The fiber $L = \{\bar{f} = 0\}$ is isomorphic to a line and L meets the axis $\bar{u} = 0$ normally at the origin and $r - 1$ other points $(r \geq 2)$. Hence, without loss of generality one may suppose that $\bar{f}(\bar{u}, \bar{v}) = \bar{v} +$ high order terms. In particular the weight of $\bar{f}$ is b'. (We treat b' here as an element of $\mathbf{Z}_{a'}$).

Assumption (b). Consider the Laurent polynomial $\bar{F} = \bar{s}^{-b'}\bar{f}(\bar{s}^{c'}\bar{u}, \bar{s}^{b'}\bar{v})$ where $\bar{s}$ is a new variable. The first assumption on $\bar{f}$ implies that $\bar{F}$ can be rewritten as a Laurent polynomial $\bar{F}(\bar{w}, \bar{w}^{-1}, \bar{u}, \bar{v})$ with $\bar{w} = \bar{s}^{a'}$. Assume that the function $\bar{F}$ does not depend on $\bar{w}^{-1}$, i.e. $\bar{F}$ is a polynomial $\bar{F}(\bar{w}, \bar{u}, \bar{v})$ in $\bar{w}, \bar{u}, \bar{v}$.

Note that $\bar{F}$ is semi-invariant of weight b' under the $\mathbf{C}^*$-action $(\bar{w}, \bar{u}, \bar{v}) \to (\lambda^{-a'}\bar{w}, \lambda^{c'}\bar{u}, \lambda^{b'}\bar{v})$. Choose pairwise prime natural $\alpha_1, \alpha_2, \alpha_3$ such that $(\alpha_1, a') = (\alpha_2, b') = (\alpha_3, c') = 1$.

Theorem 2.1 [Ru]. *The hypersurface* $\bar{X} = \{(x, y, z, t) \in \mathbf{C}^4 \mid z^{\alpha_2} + \bar{F}(y^{\alpha_1}, t^{\alpha_3}, x) = 0\}$ *is smooth contractible.*

Put $a = a'\alpha_2\alpha_3$, $b = b'\alpha_1\alpha_3$, $c = c'\alpha_1\alpha_2$. Then we have the $\mathbf{C}^*$-action on $\bar{X}$ given by $(x, y, z, t) \to (\lambda^{c\alpha_2}x, \lambda^{-a}y, \lambda^{b}z, \lambda^{c}t)$. The origin is the only fixed point. Since (y, z, t) is a semi-invariant local coordinate system in a neighborhood of the origin, it is again a "hard-case" $\mathbf{C}^*$-action.

Example: $a' = b' = c' = 1$, $\bar{f}(\bar{u},\bar{v}) = \bar{v} + \bar{v}^2 + \bar{u}$. Then $\bar{F}(\bar{w},\bar{u},\bar{v}) = \bar{v} + \bar{w}\bar{v}^2 + \bar{u}$ and $\bar{X} = \{(x,y,z,t) \in \mathbf{C}^4 \mid x + x^2y^{\alpha_1} + z^{\alpha_2} + t^{\alpha_3} = 0\}$.

Note that, by the Abhyankar-Moh-Suzuki theorem [AM], [Su], there exists a polynomial $\bar{g}(\bar{u},\bar{v})$ such that $\mathbf{C}[\bar{u},\bar{v}] = \mathbf{C}[\bar{f},\bar{g}]$. Put $u = \bar{f}(\bar{u},\bar{v})$ and $v = \bar{g}(\bar{u},\bar{v})$. Then $\bar{u} = f(u,v)$ and $\bar{v} = g(u,v)$ where f, g are polynomials. Since $\bar{f}$ is semi-invariant relative ω, the polynomial $\bar{g}$ can be chosen to be also semi-invariant relative ω. In this case u, v, f, g are semi-invariants with weights b', c', c', b' respectively. Put $F(w, w^{-1}, u, v) = s^{-c'} f(s^{b'}u, s^{c'}v)$ where $w = s^{a'}$. Note that F does not depend on w^{-1} automatically since $b' \geq c'$, i.e. F is a polynomial $F(w,u,v)$ in w,u,v. Put $X = \{(x,y,z,t) \in \mathbf{C}^4 \mid t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x) = 0\}$.

Theorem 2.2 [**Ru**]. *The hypersurfaces $\overline{X}$ and X are isomorphic.*

Definition. We say that the weak linearizing conjecture holds if every "hard-case" $\mathbf{C}^*$-action on $\mathbf{C}^3$ is linearizable under the following additional condition: the weights of this action are pairwise prime.

In many case the "weak" linearizing conjecture was investigated in [KR].

Theorem 2.3 [**Ru**]. *If every hypersurface X as above is not isomorphic to $\mathbf{C}^3$, then the linearizing conjecture can be reduced to the "weak" linearizing conjecture.*

3. Main Idea. Let an affine algebraic hypersurface X in $\mathbf{C}^4$ is given by a polynomial equation $P(x,y,z,t) = 0$ in a coordinate system (x,y,z,t). Suppose that there exists a dominant morphism $\varphi : \mathbf{C}^3 \to X$ and that (ζ,η,θ) is coordinate system on $\mathbf{C}^3$. Then x,y,z,t may be treated as polynomials in ζ,η,θ. Consider the Jacobi matrix of the polynomials x,y,z with respect to ζ,η,θ and denote by $J(t)$ its determinant. We shall use the notation $J(z), J(y)$, and $J(x)$ in the similar meaning. Consider the partial derivatives P_x, P_y, P_z, P_t of P with respect to x,y,z,t.

Lemma 3.1. *Let P_x, P_y, P_z, P_t have no common zeros on X. Then $J(x), J(y), J(z), J(t)$ are divisible by $P_x \circ \varphi$, $P_y \circ \varphi$, $P_z \circ \varphi$, $P_t \circ \varphi$ respectively.*

Proof. Since $P \circ \varphi \equiv 0$ the derivatives of $P \circ \varphi$ with respect to ζ,η,θ must be identically zero. In other words $J \cdot P' \equiv 0$ where J is the 3×4-matrix of partial derivatives of x, y, z, t with respect to ζ, η, θ, and P' is the 4-vector function with components $P_x \circ \varphi$, $P_y \circ \varphi$, $P_z \circ \varphi$, $P_t \circ \varphi$. Application of Kramer's rule to this linear system implies $J(x) \cdot P_y \circ \varphi = \pm J(y) \cdot P_x \circ \varphi$, $J(z) \cdot P_y \circ \varphi = \pm J(y) \cdot P_z \circ \varphi$, and $J(t) \cdot P_y \circ \varphi = \pm J(y) \cdot P_t \circ \varphi$. Therefore $J(y)$ is divisible by $P_y \circ \varphi$. The other statements can be obtained in a similar manner.

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Denote the degrees of x, y, z, t, $P_x \circ \varphi$, $P_y \circ \varphi$, $P_z \circ \varphi$, and $P_t \circ \varphi$ (as polynomials in ζ, η, θ) by $d_x, d_y, d_z, d_t, D_x, D_y, D_z, D_t$ respectively. Since $degJ(x) \geq d_y + d_z + d_t - 3$ and the similar inequalities hold for $degJ(y)$, $degJ(z)$, $degJ(t)$, we have

Corollary 3.2. *If P_x, P_y, P_z, P_t have no common zeros on X, then*

$$\begin{aligned} d_y + d_z + d_t &\geq D_x + 3 \\ d_x + d_z + d_t &\geq D_y + 3 \\ d_x + d_y + d_t &\geq D_z + 3 \\ d_x + d_y + d_z &\geq D_t + 3. \end{aligned}$$

In particular,

$$\begin{aligned} 3(d_x + d_y + d_z + d_t) &\geq 12 + D_x + D_y + D_z + D_t \\ 3d_y + 2d_x + 2d_t + 2d_z &\geq 9 + D_x + D_z + D_t. \end{aligned}$$

4. Demonstration of the main idea. It is shown here that the algebraic varieties given by

(a) $x + x^{d-1}y + y^{d-a}z^a + t^d = 0$ where $d - 1 > a > 1$ and a is relatively prime with d and $d - 1$ (so $a > 2$) or by

(b) $x + x^2y^a + z^b + t^c = 0$ where $a \geq 2, b > 3, c > 3$

are not the images of $\mathbf{C}^3$ under polynomial mappings.

It is worth mentioning that the list (a) presents the contractible hypersurfaces in $\mathbf{C}^4$ described in [D]. In case (a)

$D_x = (d-2)d_x + d_y$

$D_z = (d-a)d_y + (a-1)d_z$

$D_t = (d-1)d_t$

from which Corollary 3.2 implies

$2d_x + 3d_y + 2d_z + 2d_t \geq (d-2)d_x + (d-a+1)d_y + (a-1)d_z + (d-1)d_t + 9.$

There are no solutions in natural d_x, d_y, d_z, d_t with our restrictions on a and d and, thus, φ cannot exist.

In case (b)

$D_y = 2d_x + (a-1)d_y$

$D_t = (c-1)d_z$

$D_z = (b-1)d_z$

$D_x = d_x + ad_y$

from which Corollary 3.2 implies

$3(d_x + d_y + d_z + d_t) \geq 3d_x + (2a-1)d_y + (b-1)d_z + (c-1)d_t + 12.$

There are no solutions in natural d_x, d_y, d_z, d_t with our restrictions on a, b and c, and, thus, φ cannot exist.

5. Partial derivatives of P have no common zeros. Let $f(u,v)$, $F(w,u,v) = s^{-c'}f(s^{b'}u, s^{c'}v)$ be the same as in preliminaries (recall that $w = s^{a'}$). Let $P(x,y,z,t) = t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x)$ and $X = \{(x,y,z,t) \in \mathbf{C}^4 \mid P(x,y,z,t) = 0\}$.

Lemma 5.1. *The partial derivatives P_x, P_y, P_z, P_t of P have no common zeros on X.*

Proof. By Theorems 2.1 and 2.2, the threefold X is smooth irreducible. Since P is not a power of another polynomial, by construction, the partial derivatives of P cannot have common zeros on X.

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6. Some corollaries of the Abhyankar-Moh-Suzuki theorem. Let $h(u,v) = \sum a_{ij}u^iv^j$ be a polynomial and let I be the set of indices such that $a_{ij} \neq 0$ iff $(i,j) \in I$. In order to make notation shorter we shall say that h is $\sum_{(i,j)\in I} u^iv^j$ up to nonzero coefficients.

Let f be an irreducible polynomial whose Newton polygon N_f is a right triangle. Let $f_0(u,v)$ be the sum of monomials from f that corresponds to the hypotenuse of N_f.

Definition. We call f_0 the quasi-leading part of f.

Let the zero fiber of an irreducible polynomial f of degree ≥ 2 be isomorphic to $\mathbf{C}$. It follows from the Epimorphism Theorem [AM] that the Newton polygon of f is a right triangle and its quasi-leading part $f_0(u,v)$ is either $(u+v^k)^l$ or $(u^k+v)^l$ up to nonzero coefficients.

Lemma 6.1. *Let f be as above and let $f_0(u,v) = (u+v^k)^l$ where $k \geq 2$. Then there exists a polynomial automorphism A of $\mathbf{C}[u,v]$ such that*

(1) $A = A_m \circ \cdots \circ A_1$ where for every odd index i the automorphism A_i has form $(u,v) \to (u+q_i(v), v)$ (up to nonzero coefficients), for every even i the automorphism A_i has form $(u,v) \to (u, v+q_i(u))$ (up to nonzero coefficients);

(2) $\deg q_i \geq 2$ for all i. Moreover, if $f(0,0)=0$ then $q_i(0)=0$ for each i;

(3) $f(u,v) = A(u)$ if m is odd and $f(u,v) = A(v)$ if m is even.

Proof. Consider the substitution $\hat{u} = u + v^k$ and $\hat{v} = v$. Then the function f on $\mathbf{C}^2$ coincides with a polynomial $\hat{f}(\hat{u},\hat{v})$. One can see that $deg_{\hat{v}}\hat{f} < deg_v f = kl$. By the Epimorphism Theorem, either $\hat{f}_0(\hat{u},\hat{v}) = (\hat{u}+\hat{v}^{\hat{k}})^l$ with $\hat{k} < k$, or $\hat{f}_0(\hat{u},\hat{v}) = (\hat{u}^r + \hat{v})^n$ with $r \geq 2$ and $rn = l$. The rest follows by induction.
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Definition. A function Deg on $\mathbf{C}[x_1,\ldots,x_n]$ is called a weighted degree if the following properties of the usual degree function hold for Deg:

(1) $Deg\, h$ is a nonnegative rational number for every nonzero $h \in \mathbf{C}[x_1,\ldots,x_n]$, $Deg\, h = -\infty$ for $h=0$, and if $Deg\, h = 0$ then h is a nonzero constant;

(2) if $Deg\, h_1 < Deg\, h_2$ then $Deg\,(h_1+h_2) = Deg\, h_2$;

(3) if $Deg\, h_1 = Deg\, h_2$ then $Deg\,(h_1+h_2) \leq Deg\, h_2$:

(4) $Deg\, h_1 h_2 = Deg\, h_1 + Deg\, h_2$.

Lemma 6.2. *Let $f, A_1, \ldots, A_m$, $q_1, \ldots, q_m$ be the same as in Lemma 6.1 and let Deg be a weighted degree. Put $A^i = A_i \circ \cdots \circ A_1$, $f_i = A^i(u)$, $g_i = A^i(v)$ for all $i \leq m$. Then $\frac{\partial}{\partial v} f_i(u,v) = f_i^v(u,v) + r_i^v(u,v)$, $\frac{\partial}{\partial v} g_i(u,v) = g_i^v(u,v) + p_i^v(u,v)$ where*

(1) $f_i^v(u,v) = q_i'(g_{i-1})q_{i-1}'(f_{i-2})\cdots q_1'(v)$ *for every odd* $i \geq 1$ *and* $g_i^v(u,v) = q_{i-1}'(f_{i-2})\cdots q_1'(v)$ *for odd* $i \geq 3$;

(2) $f_i^v(u,v) = q_{i-1}'(g_{i-2})q_{i-2}'(f_{i-3})\cdots q_1'(v)$ *and* $g_i^v = q_i'(f_{i-1})f_i^v(u,v)$ *for even* i;

(3) $Deg\, f_i^v > Deg\, r_i^v$ *and* $Deg\, g_i^v > Deg\, p_i^v$.

Similarly, $\frac{\partial}{\partial u}f_i(u,v) = f_i^u(u,v) + r_i^u(u,v)$, $\frac{\partial}{\partial u}g_i(u,v) = g_i^u(u,v) + p_i^u(u,v)$ *where* $Deg\, f_i^u > Deg\, r_i^u$, $Deg\, g_i^u > Deg\, p_i^u$; *for odd* $i \geq 3$ $g_i^u = q_{i-1}'(f_{i-2})q_{i-2}'(g_{i-3})\cdots q_2'(f_1)$ *and* $f_i^u = q_i'(g_{i-1})g_i^u$; *for even* $i \geq 4$ $f_i^u = q_{i-1}'(g_{i-2})\cdots q_2'(f_1)$ *and for even* $i \geq 2$ $g_i^u = q_i'(f_{i-1})f_i^u$.

Proof. We shall use induction. For $i = 1,2,3$ the statement can be easily checked. Assume it is true for $i-1$. Since $f_i = f_{i-1}$ for even i consider the case of odd i. Then $f_i = f_{i-1} + q_i(g_{i-1})$ and $\frac{\partial}{\partial v}f_i = \frac{\partial}{\partial v}f_{i-1} + q_i'(g_{i-1})\frac{\partial}{\partial v}g_{i-1} = f_{i-1}^v + r_{i-1}^v + q_i'(g_{i-1})(g_{i-1}^v + p_{i-1}^v) = f_i^v + r_i^v$ where $r_i^v = f_{i-1}^v + r_{i-1}^v + q_i'(g_{i-1})p_{i-1}^v$. Note that $Deg\, f_i^v > Deg\, f_{i-1}^v$ since $f_i^v = q_i'(g_{i-1})q_{i-1}'(f_{i-2})f_{i-1}^v$. Hence $Deg\, f_i^v > Deg\, r_i^v$. For $\frac{\partial}{\partial v}g_i$ the statement is true since $g_i = g_{i-1}$, and induction works. Clearly one can repeat the argument for $\frac{\partial}{\partial u}f_i$ and $\frac{\partial}{\partial u}g_i$.
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Corollary 6.3. *Let* k *be the degree of* q_1 *and let* $h_1(u,v)$, $h_2(u,v)$ *be polynomials on* $\mathbf{C}^2$. *Then* $Deg\,\frac{\partial f}{\partial v} = Deg\, q_1'(v)\frac{\partial f}{\partial u} = Deg\,\frac{\partial f}{\partial u} + (k-1)Deg\, v$, *and* $Deg\left(h_1\frac{\partial f}{\partial v} + h_2\frac{\partial f}{\partial u}\right) = Deg\,\frac{\partial f}{\partial u} + Deg\, h_3$, *where* $h_3 = q_1'(v)h_1(u,v) + h_2(u,v)$.

Lemma 6.4. *Let* $f_1,\dots,f_m, g_1,\dots,g_m, q_1,\dots,q_m, Deg, h_1,h_2,h_3$ *be the same as in Lemma 6.1 and Corollary 6.3. Suppose that* $Deg\, h_3 \geq Deg\, u$. *Then* $Deg\left(h_1\frac{\partial f_i}{\partial v} + h_2\frac{\partial f_i}{\partial u}\right) \geq Deg\, f_i$ *for* $i \geq 1$ *and* $Deg\left(h_1\frac{\partial g_i}{\partial v} + h_2\frac{\partial g_i}{\partial u}\right) \geq Deg\, g_i$ *for* $i \geq 2$. *Moreover, these inequalities are strict if either* $Deg\, h_3 > Deg\, u$, *or* $Deg\, f_{i-1} \geq Deg\, f_i$ *for some odd* i, *or* $Deg\, g_{i-1} \geq Deg\, g_i$ *for some even* i.

Proof. We use induction. For $i = 1$ the statement follows from condition $Deg\,[q_1'(v)h_1(u,v) + h_2(u,v)] \geq Deg\, u$. Assume that i is odd. Then $g_i = g_{i-1}$ and thus $Deg\left[h_1(u,v)\frac{\partial g_i}{\partial v}(u,v) + h_2(u,v)\frac{\partial g_i}{\partial u}(u,v)\right] \geq Deg\, g_i$. By Lemma $Deg\left[h_1\frac{\partial f_i}{\partial v}\right.$

$+h_2\frac{\partial f_i}{\partial u}\Big] = Deg\, f_i^\circ$ and $Deg\left[h_1\frac{\partial g_i}{\partial v}+h_2\frac{\partial g_i}{\partial u}\right] = Deg\, g_i^\circ$ where $f_i^\circ = q_1'(g_{i-1})q_{i-1}'(f_{i-2})\cdots q_2'(f_1)h_3$ and $g_i^\circ = q_{i-1}'(f_{i-2})\cdots q_2'(f_1)h_3$. In particular $f_i^\circ = q_i'(g_{i-1})g_i^\circ$, $g_i^\circ = q_{i-1}'(f_{i-2})f_{i-1}^\circ$ and $Deg\, f_i^\circ = Deg\, q_i'(g_{i-1}) + Deg\, g_i^\circ > Deg\, f_{i-1}^\circ$ Suppose that $Deg\, f_{i-1} \geq Deg\, f_i$. Then $Deg\, f_i^\circ > Deg\, f_{i-1}^\circ \geq Deg\, f_{i-1} \geq Deg\, f_i$ and we have a strict inequality. Now suppose that $Deg\, f_{i-1} < Deg\, f_i$. Recall that $f_i = f_{i-1} + q_i(g_{i-1})$ for odd i and hence $Deg\, f_i = Deg\, q_i(g_{i-1})$. This implies $Deg\, f_i^\circ = Deg\, g_i^\circ + Deg\, q_i'(g_{i-1}) = Deg\, g_i^\circ + Deg\, f_i - Deg\, g_{i-1}$. Since $g_i^\circ = g_{i-1}^\circ$ we have $Deg\, f_i^\circ \geq Deg\, f_i$ by induction. The case of even i can be treated similarly.

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We shall need the following simple fact which immediately follows from Lemma 6.1 and induction.

Lemma 6.5. *Let $q_1(v) = q(v) + v^{k'} + v^k$ up to nonzero coefficients where $deg\, q < k' < k$. Then $f(u,v)$ contains monomials v^{lk} and $v^{lk-k+k'}$ with nonzero coefficients.*

7. Some estimate of the degrees of polynomials. We shall use later the following

Lemma 7.1. *Let u, v be algebraically independent polynomials on $\mathbf{C}^n$. Suppose that $deg\, u^m = deg\, v^k$ for some $m, k > 0$. Then $deg(u^m - v^k) > (m-1)deg\, u - deg\, v$.*

Proof. Extend the degree function to the field of rational functions on $\mathbf{C}^n$ (without the zero function) by putting $deg\, r = deg\, p - deg\, q$ where $r = \frac{p}{q}$ and p, q are polynomials. Let r_1, r_2 be rational functions and let r' be a partial derivative of r with respect to some coordinate. We may suppose that $r' \neq 0$ when $r \neq const$. The following properties are simple:

(1) $deg\, r_1 r_2 = deg\, r_1 + deg\, r_2$;

(2) $deg\, r' \leq deg\, r - 1$;

(3) $deg(log r)' = deg\, r'$ when $deg\, r = 0$;

(4) $deg\, r \geq -deg\, q$.

Put $h = u^m - v^k$. Then $deg h = deg\, u^m + deg\left(1 - \frac{v^k}{u^m}\right)$ by (1), $deg h > deg\, u^m + deg\left(\frac{v^k}{u^m}\right)'$ by (2). Note that the last term has sense since $\frac{v^k}{u^m} \neq const$. Then $deg h > deg\, u^m + deg\left(\frac{kv'u - mu'v}{uv}\right)$ by (3), $deg h > deg\, u^m - deg\, uv$ by (4), and we are done.

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We also need a generalization of this lemma.

Lemma 7.2. *Let x, y, z be algebraically independent polynomials on $\mathbf{C}^n$ and let m, k, l be naturals. Suppose that $deg z^m y^l = deg\, x^k$. Then $deg(z^m y^l - x^k) > (m-1)deg\, z + (l-1)deg\, y - deg\, x$.*

Proof. Extend the degree function to the field of rational functions on $\mathbf{C}^n$ (without the zero function) in the way it was done in Lemma 7.1. Hence properties (1) – (4) from the proof of the previous lemma hold.

Put $h = z^m y^l - x^k$. Then $deg h = deg\, z^m y^l + deg\left(1 - \frac{x^k}{z^m y^l}\right)$ by (1), $deg h > deg\, z^m y^l + deg\left(\frac{x^k}{z^m y^l}\right)'$ by (2). Note that the last term has sense since $\frac{x^k}{z^m y^l} \neq const$. Then $deg h > deg\, z^m y^l + deg\left(\frac{kx'zy - mxz'y - lxzy'}{xzy}\right)$ by (3), $deg h > deg\, z^m y^l - deg\, xzy$ by (4), and we are done.

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Corollary 7.3. *Let x, y, z be algebraically independent polynomials on $\mathbf{C}^n$, let m, l be naturals. Suppose that q is a quadratic polynomial in one variable and $deg\, z^m y^l = 2deg\, x$. Then $deg(z^m y^l - q(x)) > (m-1)deg\, z + (l-1)deg\, y - deg\, x$.*

Proof. It is enough to note that $q(x) = \beta_1(x + \beta_2)^2 - \beta_3$ for some constants $\beta_1, \beta_2, \beta_3$.

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8. The Quasi-leading Part of f. Let f, a', b', c' be the same as in section 2. Let f_0 be the quasi-leading part of f.

Lemma 8.1. *One may suppose that* $f_0(u,v) = (u+v^k)^l$ *where* $k \geq 2$.

Proof. Assume that it is not so, i.e. $f_0(u,v) = (v+u^k)^l$ where $k \geq 1$. Put $\hat{u} = u$ and $\hat{v} = v + u^k$. In these coordinates the function f can be rewritten as a polynomial $\hat{f}(\hat{u},\hat{v})$. Recall that $F = s^{-c'}f(s^{b'}u, s^{c'}v)$ and F can be viewed as a polynomial $F(w,u,v)$ where $w = s^{a'}$. Hence $s^{-c'}f_0(s^{b'}u, s^{c'}v)$ is a polynomial in w,u,v, and this implies that $s^{kb'-c'} = w^n$. Recall that our hypersurface X is given by the equation $P(x,y,z,t) = t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x) = 0$. Put $\hat{F} = \hat{s}^{-c'}\hat{f}(\hat{s}^{b'}\hat{u}, \hat{s}^{c'}\hat{v})$. Again this function is a polynomial $\hat{F}(\hat{w},\hat{u},\hat{v})$ in $\hat{w} = \hat{s}^{a'}, \hat{u}, \hat{v}$. Consider the hypersurface $\hat{X} = \{\hat{t}^{\alpha_3} + \hat{F}(\hat{y}^{\alpha_1}, \hat{z}^{\alpha_2}, \hat{x}) = 0\}$. It is easy to check that $\hat{X}$ is the image of X under the automorphism of C^4 given by $(\hat{x},\hat{y},\hat{z},\hat{t}) = (x + y^{n\alpha_1}z^{k\alpha_2}, y, z, t)$. Hence we can use $\hat{f}$ instead of f. Note that, by construction and by the Abhyankar-Moh-Suzuki theorem, either $\hat{f}_0(\hat{u},\hat{v}) = (\hat{u}+\hat{v}^k)^{\bar{l}}$ up to nonzero coefficients or $\hat{f}_0(\hat{u},\hat{v}) = (\hat{v}+\hat{u}^r)^l$ where $r < k$. The rest follows by induction.

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9. Weighted degrees.

We need the following properties of the weighted degrees which were introduced in section 6.

Lemma 9.1. *Let* $\varphi: C^n \to C^m$ *be a dominant morphism which generates an injective homomorphism* $C[x_1,\ldots,x_m] \to C[y_1,\ldots,y_n]$. *Let Deg be a weighted degree on* $C[x_1,\ldots,x_m]$. *For every* $h \in C[y_1,\ldots,y_n]$ *put* $Deg_1 h = Deg\, h \circ \varphi$. *Then* Deg_1 *is a weighted degree.*

Proof. Properties (2)-(4) and the first part of property (1) from the definition of weighted degrees in section 6 are obvious. Let $Deg_1 h = 0$, then the polynomial $h \circ \varphi$ is constant. Since φ is dominant this implies that h is constant.

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Lemma 9.2. *Let* $\varphi: C[x_1,\ldots,x_n] \to C[y_1,\ldots,y_n]$ *is given by* $(x_1,\ldots,x_n) \to (x_1^m, x_2,\ldots,x_n)$. *Thus we may tread* $C[y_1,\ldots,y_n]$ *as a subalgebra of* $C[x_1,\ldots,x_n]$. *Then every weighted degree Deg on* $C[y_1,\ldots,y_n]$ *can be extended to a weighted* *degree on* $\mathbf{C}[x_1,\ldots,x_n]$.

Proof. Every nonzero polynomial $h \in \mathbf{C}[x_1,\ldots,x_n]$ may be presented uniquely in the form $\sum_{i=0}^{k} x_1^i q_i(y_2,\ldots,y_n)$. Put $Deg\, h = \max\{Deg\, q_i + i Deg\, y_1/m \mid \text{for all } i \text{ with } q_i \neq 0\}$. The properties (1)-(4) of weighted degrees can be easily checked.
○

From now on we fix notation for the rest of the paper. Let $f, u, v, s, w, a', b', c', \alpha_1, \alpha_2$, α_3, F be the same as in section 2. Put $\tilde{u} = s^{b'}u$, $\tilde{v} = s^{c'}v$. Then $F(w,u,v) = s^{-c'}f(\tilde{u},\tilde{v})$. Put $y^{\alpha_1} = w$, $z^{\alpha_2} = u$, $x = v$, $P(x,y,z,t) = t^{\alpha_3} + F(y^{\alpha_1}, z^{\alpha_2}, x)$ and $X = \{(x,y,z,t) \in \mathbf{C}^4 \mid P(x,y,z,t) = 0\}$. Suppose $\varphi : \mathbf{C}^3 \to X$ is a dominant morphism. Then we can treat every regular function on X as a polynomial in ζ, η, θ where (ζ,η,θ) is a coordinate systen on $\mathbf{C}^3$.

We shall need some weighted degrees on polynomial rings. The degree function on $\mathbf{C}[\zeta,\eta,\theta]$ and the morphism φ generate a weighted degree on $\mathbf{C}[x,y,z]$, by Lemma 9.1. This weighted degree generates a weighted degree on $\mathbf{C}[v,w,u]$, by the same lemma, since $v = x, w = y^{\alpha_1}$, and $u = z^{\alpha_2}$. In its turn this weighted degree generates a weighted degree on $\mathbf{C}[v,s,u]$, by Lemma 9.2, since $w = s^{a'}$. The morphism $(v,s,u) \to (\tilde{u},\tilde{v})$ is dominant. Hence the last weighted degree generates a weighted degree on $\mathbf{C}[\tilde{u},\tilde{v}]$. By abusing notation, we denote all these weighted degrees by the same symbol Deg. Thus $deg\, h \circ \varphi(\zeta,\eta,\theta) = Deg\, h(x,y,z)$, $Deg\, h(x,y^{\alpha_1},z^{\alpha_2}) = Deg\, h(v,w,u)$, $Deg\, h(v,s^{a'},u) = Deg\, h(v,w,u)$, and $Deg\, g(\tilde{u},\tilde{v}) = Deg\, g(s^{b'}u, s^{c'}v)$, where h and g are polynomials in three and two variables respectively and the variables in polynomials indicate on which polynomial ring we should consider this Deg function.

By Lemma 8.1, one may suppose that the quasi-leading part of f is $f_0(u,v) = (u+v^k)^l$. Recall that the function f satisfies Lemma 6.1. Let $q_1,\ldots,q_m$ be the same as in Lemma 6.1. The assumption on f_0 implies that $q_1(v) = v^k + q(v)$ where $\deg q < k$. We shall use $u_1 = \tilde{u} + q_1(\tilde{v})$ and $v_1 = \tilde{v}$. Note that the function $f(\tilde{u},\tilde{v})$ can be rewritten as a polynomial $f^1(u_1,v_1)$. As above we introduced the weighted degree on the ring $\mathbf{C}[u_1,v_1]$ such that $Deg\, f(\tilde{u},\tilde{v}) = Deg\, f^1(u_1,v_1)$.

We can treat every regular function on X as a polynomial in ζ, η, θ where (ζ, η, θ) is a coordinate systen on $\mathbf{C}^3$. Denote by P_x, P_y, P_z, P_t the partial derivatives of P. Let $d_x, d_y, d_z, d_t, D_x, D_y, D_z, D_t$ be the degrees of the functions $x, y, z, t, P_x, P_y, P_z, P_t$ as polynomials in ζ, η, θ.

10. Estimates of D_x, D_y, D_z, D_t.

The chain rule implies.

Lemma 10.1. *In the above notation $D_z \geq (\alpha_2 - 1)d_z$, $D_t \geq (\alpha_3 - 1)d_t$, $D_y \geq (\alpha_1 - 1)d_y$.*

The fact that the function F is a polynomial in $w = s^{a'}, u, v$ and $(a', c') = 1$, implies

Lemma 10.2. *The numbers $kc' - b'$ and $(kl-1)c'$ are divisible by a'. Moreover if $f(u,v)$ contains a monomial $v^{kl-k+k'}$ with a nonzero coefficient then $k - k'$ is divisible by a'.*

Lemma 10.3. *Let β be the smallest natural α such that $\alpha a' + c' - b' > 0$. Then the difference of $\frac{\partial \tilde{f}}{\partial \tilde{u}}(\tilde{u}, \tilde{v})$ and a constant is divisible by $s^{\beta a' + c' - b'}$. The difference of $\frac{\partial \tilde{f}}{\partial \tilde{v}}(\tilde{u}, \tilde{v})$ and a constant is divisible by $s^{a'} = y^{\alpha_1}$.*

Proof. Let $\tilde{u}^{n_1}\tilde{v}^{n_2}$ be a monomial in $f(\tilde{u}, \tilde{v})$ such that $n_1 + n_2 > 1$ and $n_1 \geq 1$. Since $f(\tilde{u}, \tilde{v})/s^{c'} = F(w, u, v)$ we have $\tilde{u}^{n_1}\tilde{v}^{n_2} = u^{n_1}v^{n_2}y^{\alpha\alpha_1}s^{c'}$. Hence the corresponding monomial $\tilde{u}^{n_1-1}\tilde{v}^{n_2}$ in $\frac{\partial f}{\partial \tilde{u}}$ is divisible by s to the power of $\alpha a' + c' - b' = (n_1 - 1)b' + c' > 0$. This implies the statement of Lemma for $\frac{\partial f}{\partial \tilde{u}}$. For $\frac{\partial f}{\partial \tilde{v}}$ the proof is similar.
○

In order to obtain an estimate of D_x note that P_x is an element of the ring $\mathbf{C}[x, y, z]$ and thus we may apply the weighted degrees which were introduced in the previous section.

Lemma 10.4. *In the above notation $D_x \geq (k-1)d_x + \alpha_1 d_y$ and $D_x \geq Deg\frac{\partial f}{\partial \tilde{u}}(\tilde{u},\tilde{v}) + (k-1)Deg\tilde{v}$.*

Proof. Note that $P_x \circ \varphi = \left[s^{-c'}\frac{\partial}{\partial x}(s^{c'}P)\right] \circ \varphi = \left[\frac{\partial}{\partial \tilde{v}}f(\tilde{u},\tilde{v})\right] \circ \varphi$. Consider the mapping $\psi = (\tilde{u},\tilde{v}) : Y \to \mathbf{C}^2$. It is easy to check that it is dominant (otherwise the functions x, y, z on X are algebraically dependent). By Corollary 6.3, $Deg\left[\frac{\partial f}{\partial \tilde{v}}(\tilde{u},\tilde{v})\right] = Deg\left[\frac{\partial f}{\partial \tilde{u}}(\tilde{u},\tilde{v})\right] + (k-1)Deg\tilde{v}$. Suppose that $\frac{\partial f}{\partial \tilde{u}} \neq const$. By Lemma 10.3, $Deg\frac{\partial f}{\partial \tilde{u}}(\tilde{u},\tilde{v}) \geq (\beta a' + c' - b')Degs$. Thus $D_x \geq (k-1)d_x + (k-1)c'Deg\,s + (\beta a' - b' + c')Deg\,s = (k-1)d_x + (\beta a' - b' + kc')Deg\,s$. Since the number $\beta a' - b' + kc' > 0$ and divisible by a', $D_x \geq (k-1)d_x + a'Deg\,s = (k-1)d_x + \alpha_1 d_y$. If $\frac{\partial f}{\partial \tilde{u}} = const$ then $f(\tilde{u},\tilde{v}) = \tilde{u} + q_1(\tilde{v})$. In this case the statement of Lemma is clear.
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Corollary 10.5. *In the above notation there is no dominant mapping $\varphi : \mathbf{C}^3 \to X$ if one of the following condition holds:*

(i) $\alpha_1 \geq 2,\ \alpha_2 > 3,\ \alpha_3 > 3,\ k > 3$;

(ii) $\alpha_1 \geq 3,\ \alpha_2 \geq 3,\ \alpha_3 \geq 3,\ k \geq 3$.

Proof. Corollary 3.2 and Lemma 10.1 imply that

$$\begin{aligned}&3(d_x + d_y + d_z + d_t) \geq 12 + D_x + D_y + D_z + D_t \geq \\ &= 12 + (k-1)d_x + (2\alpha_1 - 1)d_y + (\alpha_2 - 1)d_z + (\alpha_3 - 1)d_t\end{aligned}$$

and

$$\begin{aligned}&3d_y + 2(d_x + d_z + d_t) \geq 9 + D_x + D_z + D_t \geq \\ &9 + (k-1)d_x + \alpha_1 d_y + (\alpha_2 - 1)d_z + (\alpha_3 - 1)d_t.\end{aligned}$$

These inequalities have no solutions in natural numbers when the assumption of Corollary holds.
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11. A Better Estimate of D_y. In this section we shall prove some lemmas which help to remove condition $k > 3$ in Corollary 10.5 (i). The first of them is obvious.

Lemma 11.1. *Let $g(u,v)$ be a polynomial such that $g(u,v) = (u+v^k)^l u^{n_1} v^{n_2} + h(u,v)$ and $km_1 + m_2 < kl + kn_1 + n_2$ for every monomial $u^{m_1}v^{m_2}$ from h. Let Deg be the function on $\mathbf{C}[u,v]$ defined in Section 9. If $Deg\, u > kDeg v$, then $Deg\, g(u,v) = Deg\, u^{n_1+l}v^{n_2}$. If $Deg\, u < kDeg\, v$, then $Deg\, g(u,v) = Deg\, u^{n_1}v^{n_2+kl}$. If $Deg\, u = Deg\, v^k$ but $Deg\, u + v^k = Deg\, v^k$, then $Deg\, g(u,v) = Deg\, u^{n_1}v^{n_2+kl} = Deg\, u^{n_1+l}v^{n_2}$.*

Lemma 11.2. *Suppose that either $b' \neq k$ or $c' \neq 1$. Then $D_y \geq kd_x + (\alpha_1 - 1)d_y$. If $b' = k$, $c' = 1$ and $k = 2$ or 3 we still have $D_y \geq 2d_x + (\alpha_1 - 1)d_y$.*

Proof. Recall that $q_1, ..., q_m$ be the same as in Lemma 6.1. If $m = 1$ then the statement of Lemma is obviously true. So suppose that $m > 1$ and, in particular, $\dfrac{\partial f}{\partial \tilde{u}} \neq const.$

Note that $\dfrac{\partial}{\partial y} = \dfrac{\alpha_1}{a'}\dfrac{y^{\alpha_1 - 1}}{s^{a'-1}}\dfrac{\partial}{\partial s} = \dfrac{\alpha_1}{a'}\dfrac{s}{y}\dfrac{\partial}{\partial s}$. Thus it suffices to show that $Deg\left(s^{c'+1}\dfrac{\partial P}{\partial s}\right) \geq kd_x + \alpha_1 d_y + c'Deg\, s$. Note that $Q = s^{c'+1}\dfrac{\partial P}{\partial s} = s\dfrac{\partial}{\partial s}f(\tilde{u},\tilde{v}) - c'f(\tilde{u},\tilde{v})$. Put $Q_1 = s\dfrac{\partial}{\partial s}f(\tilde{u},\tilde{v})$. Then $Q_1 = b'\tilde{u}\dfrac{\partial f}{\partial \tilde{u}}(\tilde{u},\tilde{v}) + c'\tilde{v}\dfrac{\partial f}{\partial \tilde{v}}(\tilde{u},\tilde{v})$. We have to consider several cases:

(1) $Deg\,\tilde{u} > Deg\,\tilde{v}^k$;
(2) $Deg\,\tilde{u} < kDeg\,\tilde{v}$;
(3) $Deg\,\tilde{u} = Deg\,\tilde{v}^k$, $Deg\, s\dfrac{\partial u_1}{\partial s} = Deg[b'\tilde{u} + c'\tilde{v}q_1'(\tilde{v})] = Deg\,\tilde{u}$ where $u_1 = \tilde{u} + q_1(\tilde{v})$;
(4) $Deg\,\tilde{u} = kDeg\,\tilde{v}$, $Deg\, u_1 = Deg\,\tilde{u}$, $Deg\, s\dfrac{\partial u_1}{\partial s} < Deg\,\tilde{u}$;
(5) $Deg\,\tilde{u} = kDeg\,\tilde{v}$, $Deg\, u_1 < Deg\,\tilde{u}$, $Deg\, s\dfrac{\partial u_1}{\partial s} < Deg\,\tilde{u}$.

In the first case the Newton polygon of Q is a right triangle and $DegQ = Deg\,\tilde{u}^l$ by Lemma 11.1. Hence, since $Deg\,\tilde{u} = \alpha_2 Deg\, z + b'Deg\, s > Deg\,\tilde{v} = kd_x + kc'Deg\, s$, we have $DegQ > kld_x + lkc'Deg\, s$. Lemma 10.2 yields that $lkc' = \alpha a' + c'$ for some positive integer α. Thus $DegQ > kld_x + \alpha\alpha_1 d_y + c'Deg\, s$ which implies the desired conclusion for D_y in this case. Case (2) can be treated similarly.

In case (3) consider two situations: $Deg\, u_1 = Deg\, \tilde{u}$ and $Deg\, u_1 < Deg\, \tilde{u}$. When the last inequality holds $DegQ = DegQ_1 = Deg\frac{\partial f}{\partial \tilde{u}}(\tilde{u},\tilde{v})(b'\tilde{u}+c'\tilde{v}q_1'(\tilde{v})) > Deg f(\tilde{u},\tilde{v})$ by Lemma. Thus $DegQ = kDeg\,\tilde{v} + Deg\frac{\partial f}{\partial \tilde{u}}(\tilde{u},\tilde{v}) \geq kd_x + kc'Deg\, s + (\beta a' + c' - b')Deg\, s = kd_x + (\beta a' + kc' - b')Deg\, s + c'Deg\, s$, by Lemma 10.3. Since the number $\beta a' + kc' - b' > 0$ and is divisible by a', we have $DegQ \geq kd_x + \alpha_1 d_y + c'Deg\, s$.

Let $Deg\, u_1 = Deg\, \tilde{u}$. Put $v_1 = v$ and rewrite $f(\tilde{u},\tilde{v})$ as a polynomial $f^1(u_1,v_1)$ in u_1 and v_1. The quasi-leading part of f^1 is $(v_1 + u_1^{k_2})^{l_2}$ up to non-zero coefficients and $k_2 l_2 = l$. Condition $m > 1$ means that $k_2 > 1$. Thus $f^1(u_1,v_1) = \sum_{k_2 n_2 + n_1 \leq l} u_1^{n_1} v_1^{n_2}$ up to coefficients and

$$Q = u_1^{l-1}\left(ls\frac{\partial u_1}{\partial s} - c'u_1\right) + \sum_{\substack{k_2 n_2 + n_1 \leq l \\ n_1 \neq l}} u_1^{n_1 - 1} v_1^{n_2}\left(n_1 s\frac{\partial u_1}{\partial s} - c'u_1 + (n_2 - 1)c'v_1\right).$$

Since $Deg\left(n_1 s\frac{\partial u_1}{\partial s} - c'u_1 + (n_2 - 1)c'v_1\right) \leq Deg\, u_1$, we have $DegQ = deg\, u_1^{l-1} \left(sl\frac{\partial u_1}{\partial s} - c'u_1\right) > Deg\, u_1^{l-1} = k(l-1)Deg\,\tilde{v} = k(l-1)d_x + k(l-1)c'Deg\, s$. If $q_1(\tilde{v}) \neq \tilde{v}^k$ up to nonzero coefficient then $f(\tilde{u},\tilde{v})$ contains monomials $\tilde{v}^{kl}$ and $\tilde{v}^{kl-k+k'}$ with $0 < k' < k$, by Lemma 6.5. Lemma 10.2 implies that $k - k'$ is divisible by a', i.e. $DegQ \geq k(l-1)d_x + (kl - 2k + k')c'Deg\, s + (k-k')Deg\, s \geq k(l-1)d_x + c'Deg\, s + \alpha_1 d_y$ which is the desired inequality. If $q_1(\tilde{v}) = \tilde{v}^k$ then $Deg\left(ls\frac{\partial u_1}{\partial s} - c'u_1\right) \geq \min(b', kc')Deg\, s$ and $DegQ \geq Deg\, u_1^{l-1} + Deg\left(ls\frac{\partial u_1}{\partial s} - c'u_1\right) \geq k(l-1)d_x + \min(klc' + b' - kc', klc')Deg\, s \geq k(l-1)d_x + c'Deg\, s + \min((kl-1)c',\ klc' - c' + b' - kc')Deg\, s \geq k(l-1)d_x + c'Deg\, s + \alpha_1 d_y$ and we are done in case (3).

In case (4)

$$Q_1 = s\frac{\partial f^1}{\partial u_1}\frac{\partial u_1}{\partial s} + s\frac{\partial f^1}{\partial v_1}\frac{\partial v_1}{\partial s} = h(u_1,v_1)\frac{\partial f^1}{\partial u_1} + cv_1\frac{\partial f^1}{\partial v_1}.$$

where $h(u_1,v_1) = b'u_1 + c'v_1 q_1'(v_1) - b'q_1(v_1)$. Since $Deg[q_2'(u_1)h(u_1,v_1) + cv_1] \geq$

$Deg\, q_2'(u_1) \geq Deg\, u_1 \geq Deg\, v_1^k > Deg\, v_1$, we have that $DegQ_1 > Degf^1 = Degf(\tilde{u},\tilde{v})$ and thus

$$DegQ \quad = DegQ_1 = Deg\frac{\partial f^1}{\partial v_1}(u_1,v_1) + Deg[q_2'(u_1)h(u_1,v_1) + c'v_1]$$

by Corollary 6.3. Hence $DegQ_1 \geq Deg\frac{\partial f^1}{\partial v_1}(u_1,v_1) + (k_2-1)kDeg\, v_1$. Recall that $v_1 = \tilde{v} = x\cdot s^{c'}$ and $\frac{\partial f^1}{\partial v_1} = \frac{\partial f}{\partial \tilde{v}} - \frac{\partial f}{\partial \tilde{u}}\cdot q_1'(\tilde{v})$. Consider two subcases: $l_2 > 1$ and $l_2 = 1$.

Suppose that $l_2 > 1$. Then by Lemma 11.1, the monomial $u_1^{k_2(l_2-1)}$ has the greatest Deg among the monomials of $\frac{\partial f^1}{\partial v_1}(u_1,v_1)$, i.e. $Deg\frac{\partial f^1}{\partial v_1} = Deg\, u_1^{k_2(l_2-1)} > Deg\, v_1^k > Degq_1'(\tilde{v})$. According to Lemma 10.3 there exists a constant γ such that $\frac{\partial f}{\partial \tilde{u}} - \gamma$ is divisible by $s^{a'\beta-b'+c'}$. Thus $Deg\frac{\partial f^1}{\partial v_1} = Deg\left(\frac{\partial f^1}{\partial v_1} - \gamma q_1'(v_1)\right) \geq \min(a', a'\beta - b' + c')Deg\, s$ by Lemma 10.2. Therefore $DegQ = DegQ_1 \geq (k_2-1)kd_x + (k_2-1)kc'Deg\, s + \min(a', a'\beta - b' + c')Deg\, s \geq kd_x + \alpha_1 d_y + c'Deg\, s$ and we are done is this subcase.

Now let $l_2 = 1$. This means that $m = 2$ and $k_2 = l$. Suppose that $q_1(\tilde{v}) \neq \tilde{v}^k$. Then for $0 < k' < k$ the polynomial $f(\tilde{u},\tilde{v})$ contains a monomial $\tilde{v}^{k_2k-k+k'}$ with a nonzero coefficient, by Lemma 6.5. Lemma 10.2 implies that $k - k'$ is divisible by a' and thus $a' < k'$. Hence $Deg\, Q > Deg\, q_2'(u_1) = (k_2-1)kDeg\, v_1 = k(k_2-1)d_x + ((k_2-1)k-1)c'Deg\, s + c'Deg\, s \geq k(k_2-1)d_x + \alpha_1 d_y + c'Deg\, s$. When $q_1(\tilde{v}) = \tilde{v}^k$ then $Deg\, h(u_1,v_1) \geq \min(b', kc')Deg\, s$. Using the fact that $Deg\, Q = Deg\, q_2'(u_1)h(u_1,v_1)$ one can repeat the argument from case (3) in order to obtain the desired inequality.

In case (5) note that the conditions $Deg(\tilde{u} + \tilde{v}^k) < Deg\tilde{u}$ and $Deg(b'\tilde{u} + kc'\tilde{v}^k) < Deg\,\tilde{u}$ imply that $b' = kc'$. Since c' and b' are coprime this means that $c' = 1$ and $b' = k$. Consider two subcases (5′) $k = 3$ and (5″) $k = 2$.

(5′). Put $\hat{u} = \hat{u} + \hat{v}^k = s^k(x^k + z^{\alpha_2})$. Since $Deg\,\hat{u}_1 < Deg\,\tilde{u}$, we have $\alpha_2 d_z = kd_x$. By Lemma 7.1 $Deg(z^{\alpha_2} + x^k) > (k-1)d_x - d_z = \left(k - 1 - \frac{k}{\alpha_2}\right)d_x$. Hence $Deg\,\hat{u} > Deg\,\tilde{v}$ for $k = 3$ and $\alpha_2 \geq 3$. Put $\hat{v} = \tilde{v}$. Then $(\hat{u},\hat{v})$ is a semi-invariant (relative to the linear action pf the group of a'-roots of unity) coordinate system on $\mathbf{C}^2$, the weight of $\hat{u}$ is $b' = k$, and the weight of $\hat{v}$ is $c' = 1$. Hence one can represent the function $f(\tilde{u},\tilde{v})$ as a semi-invariant polynomial $\hat{f}(\hat{u},\hat{v})$. Denote the quasi-leading part of $\hat{f}$ by $\hat{f}_0(\hat{u},\hat{v})$. Consider three possible situations:

(i) $\hat{f}_0(\hat{u},\hat{v}) = (\hat{u}+\hat{v}^2)^l$,
(ii) $\hat{f}_0(\hat{u},\hat{v}) = (\hat{u}+\hat{v})^l$, and
(iii) $\hat{f}_0(\hat{u},\hat{v}) = (\hat{u}^{k_2}+\hat{v})^{l_2}$ with $k_2 > 1$ and $k_2 l_2 = l$.
Note that $s\dfrac{\partial \hat{u}}{\partial s} = b'\hat{u} = 3\hat{u}$, $s\dfrac{\partial \hat{v}}{\partial s} = c'\hat{v} = \hat{v}$, and $Q_1 = s\dfrac{\partial \hat{f}}{\partial s}(\hat{u},\hat{v}) = b'\hat{u}\dfrac{\partial \hat{f}}{\partial \hat{u}} + c'\hat{v}\dfrac{\partial \hat{f}}{\partial \hat{v}}$.

Now in cases (ii) and (iii) it suffices to apply Lemma 11.1, since $Deg\,\hat{u}_1 > Deg\,\hat{v}_1$. If (i) holds then we can just repeat the arguments from cases (1)-(4) and we do not need to consider case (5) since $b' \neq 2$. Thus in $(5)'$ $D_y \geq 2d_x + (\alpha_1 - 1)d_y$.

$(5)''$. Consider $u_1 = \tilde{u} + q_1(\tilde{v})$ and $v_1 = \tilde{v}$. Since $a' = 1$, $c' = 1$, and $b' = 2$ we have $u_1 = z^{\alpha_2}y^{2\alpha_1} + q_1(xy^{\alpha_1})$. Recall that $q_1(0) = 0$, i.e. $q_1(v_1) = v_1^2 + v_1$ or $q_1(v_1) = v_1^2$ up to nonzero coefficients. Suppose that $q_1(v_1) = v_1^2 + v_1$. Since $Deg\,u_1 < Deg\,u$ we have $\alpha_2 d_z = 2d_x$. By Corollary 7.3 $Deg\,u_1 > d_x + (\alpha_1 - 1)d_y - d_z = \left(1 - \dfrac{2}{\alpha_2}\right)d_x + (\alpha_1 - 1)d_y > \dfrac{1}{2}(d_x + \alpha_1 d_y) = \dfrac{1}{2}Deg\,v_1$. If $q_1(v_1) = v_1^2$ then $u_1 = y^{2\alpha_1}(z^{\alpha_2} + x^2)$. Application of Lemma 7.1 shows that again $Deg\,u_1 > \left(1 - \dfrac{2}{\alpha_2}\right)dv_1 > \dfrac{1}{2}dv_1$. Represent f as a polynomial $f^1(u_1, v_1)$. Then the quasi-leading part of f^1 is $(u_1^{k_2} + v_1)^{l_2}$ with $k_2 > 1$. We can apply again Lemma 11.1 to show that $D_y \geq 2d_x + (\alpha_1 - 1)d_y$. Lemma is proved.

○

12. Theorem A. *Let $\alpha_1 \geq 2$, $\alpha_2 > 3$, $\alpha_3 > 3$. Then there is no dominant morphism $\varphi : \mathbf{C}^3 \to X$.*

Proof. By Corollary 10.4 it is enough to consider the case when $k = 2$ or 3. Then $D_y > 2d_x + (\alpha_1 - 1)d_y$. Using Lemmas we have $D_x + D_y + D_z + D_t \geq (k+1)d_x + (2\alpha_1 - 1)d_y + (\alpha_2 - 1)d_z + (\alpha_3 - 1)d_t$. Thus the inequality $3(d_x + d_y + d_z + d_t) \geq 12 + D_x + D_y + D_z + D_t$ has no solution in natural numbers. Now Corollary 3.2 implies the desired conclusion.

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