CAHIERS DU BURO

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Contributions to the Theory of Tournaments Part IV A Comparaison of Tournaments Through Probabilistic Completions

Cahiers du Bureau universitaire de recherche opérationnelle. Série Recherche, tome 32 (1979), p. 21-34

<http://www.numdam.org/item?id=BURO_1979__32__21_0>

CONTRIBUTIONS TO THE THEORY OF TOURNAMENTS

PART IV (1)

A Comparison of Tournaments Through Probabilistic Completions

T. V. NARAYANA and F. AGYEPONG

1 – INTRODUCTION

Hartigan [3] has introduced the concept of probabilistic completion of a knock-out tournament (simply tournament hereafter) and studied in detail the classical case of a tournament with $n = 2^t$ players. He has indicated how these ideas could be extended to more general tournaments : given the result of a tournament as a rooted tree, he states a recursive formula for calculating the probabilistic completion of the tournament. Narayana and Zidek [5] have studied a class of random tournaments with a single outlier from a different point of view, namely, the probability distribution of the number of rounds played by each player in a tournament. We study Hartigan's "deterministic" model from a similar point of view in this paper and obtain the probability distribution of the number of rounds played by each player. In the classical case at least, Hartigan's results on estimating the expected strengths of players (and their variances) can be derived very simply by this alternative approach.

A novel interpretation yielded by this approach, together with a very simple result of Hartigan, permits us to obtain estimated strengths explicity for the "king of the mountain" tournament or T_4 introduced by Narayana and Zidek. This tournament is valid for an arbitrary number n of players ; also, since all rooted trees can be expressed in terms of T_4, we are able to compare various tournaments or more generally "tournament patterns". Numerical comparisons can be made, at least for small values of n, between T_4 and the classical case or its immediate generalizations which are valid for all n. The conclusions we reach in the deterministic case are generally similar to the comparisons of tournaments made in the random model – making due allowances for the plethora of criteria available in the deterministic case.

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(1) Les parties I et II font l'objet des références [5] et [6]. La partie III est parue dans *Proceedings V Iranian Math. Conf.* (Chiraz 1974).

2. TERMINOLOGY AND PRELIMINARY RESULTS

Let us consider a tournament with n players, no two of whom have the same strength. The stronger player always wins in any encounter, so that the only element of randomness arises in the manner in which the players are matched. We rank the n players in decreasing order of strength ; player n is the strongest, player $(n - 1)$ loses to n but beats the remaining players and so on. When there is no possible confusion, we shall use the integer i $(i = 1, \ldots, n)$ to denote both the player and his strength. It is convenient to refer to a general knock-out tournament ([3], p. 503) with n such players as a "deterministic" model tournament to distinguish it from the "random" model tournament introduced in [5]. The random model consists of $(n - 1)$ players between whom encounters are decided at random, while there is a single (outlier) player who beats all others with probability p. Indeed a more general model which includes both the above cases is being studied by Narayana.

A few important tournaments or tournament patterns defined in [5] are studied in the deterministic model, namely, T_1 , T_2 , T_4 and the classical case with $n = 2^t$ players. We denote for brevity the classical case by C. We first consider C and state two preliminary remarks.

Remark 1. The number of possible ways of matching 2^t players in C is $\dfrac{2^t\,!}{2^{2^t - 1}}$.

Remark 2. If $s\,(i\,,k)$ = probability that player i survives round k ,

then $$s\,(i\,,k) = \binom{i-1}{2^k - 1} \Big/ \binom{2^t - 1}{2^k - 1} = \binom{2^t - 2^k}{2^t - i} \Big/ \binom{2^t - 1}{2^t - i} . \qquad (1)$$

The last equality in (1) follows trivially ; the remaining proofs are omitted as more general results can be established cf. Capell and Narayana [1]. Indeed Narayana obtained the above equation as a very special case of the general model mentioned above.

3. ESTIMATED STRENGTH OF THE PLAYERS IN C AND T_4

Before obtaining the estimated strengths of the players in C, we obtain the mean and variance of the round in which player $i < 2^t$ is defeated. Clearly from

(1), $r(i,k) = s(i,k) - s(i,k+1)$ is the probability that player $i < 2^t$ plays exactly $k+1$ rounds, $k = 0, 1, \ldots, t-1$. Thus :

$$E(X) = \sum_{k=0}^{t-1} (k+1)\, r(i,k) = \sum_{k=0}^{t-1} (k+1)\,[s(i,k) - s(i,k+1)] = \sum_{k=0}^{t-1} s(i,k), \quad (2)$$

and similarly,

$$E((X+1)X) = \sum_{k=0}^{t-1} (k+1)\,k\,(s(i,k) - s(i,k+1)) = 2 \,.\, \sum_{k=1}^{t-1} k\, s(i,k) \,. \quad (3)$$

From (2) and (3), the expected value M_i and variance V_i of the number of rounds played by player i can be easily obtained. Closed formulae for (2), (3) are available at least when $i = (2^t - 1)$ and $(2^t - 2)$, and a table of M_i, V_i is given when $t = 2, 3, 4$.

Tables of M_i, V_i for $t = 2, 3, 4$

	$t = 2$	
i	M_i	V_i
4	2	0
3	1.666	0.222
2	1.333	0.222
1	1	0

	$t = 3$	
i	M_i	V_i
8	3	0
7	2.428	0.431
6	2	0.471
5	1.686	0.444
4	1.457	0.305
3	1.286	0.204
2	1.143	0.122
1	1	0

	$t = 4$	
i	M_i	V_i
16	4	0
15	3.267	0.862
14	2.762	0.981
13	2.407	0.887
12	2.147	0.761
11	1.950	0.65
10	1.790	0.557
9	1.658	0.476
8	1.544	0.403
7	1.444	0.335
6	1.355	0.273
5	1.275	0.217
4	1.202	0.166
3	1.133	0.116
2	1.067	0.062
1	1	0

It appears very easy to prove that :

$$M_1 < M_2 < \ldots < M_{n-1} < M_n, \tag{4}$$

where $n = 2^t$; on the other hand, letting i stand for the smallest integer such that $M_i \geqslant 2$, the tables seem to indicate for all n, $n - i + 1 = \left\{ \dfrac{n-1}{3} \right\}$. We recognise a conjecture very similar to the one outlier case [5, equation (35)].

We now show how to obtain simple explicit expressions for the estimated strengths (and their variances) in C. Although the results were stated to us first by Moon [4], we have obtained them independently by a completely different argument. Let us consider a fixed integer k, $1 \leqslant k \leqslant t$, and let us suppose that we are given that player i survives round k. From (1), this implies that $i \geqslant 2^k$, since $s(i, k) > 0$; we now want to estimate the strength of the player X beaten by i in round k. Surely, the strength of X, which is a random variable, is an integer ν satisfying $2^{k-1} \leqslant \nu < i$; and using (1) again,

$$p^*(\nu, k) = \frac{s(\nu, k-1)}{\sum\limits_{\nu=2^{k-1}}^{i} s(\nu, k-1)} \tag{5}$$

denotes the (conditional) probability that i beat ν in round k. For $k = 1$, $s(\nu, 0)$ is undefined, but from our definition of $s(i, k)$, $s(\nu, 0) = 1$. Thus

$$E(X) = \sum_{\nu=2^{k-1}}^{i-1} \nu\, p^*(\nu, k),$$

so that when $k = 1$,

$$E(X) = i/2. \tag{6}$$

Further, for $k > 1$, using (1) and (5),

$$E(X) = \frac{\sum\limits_{\nu=2^{k-1}}^{i-1} \nu \binom{\nu-1}{2^{k-1}-1}}{\sum\limits_{\nu=2^{k-1}}^{i-1} \binom{\nu-1}{2^{k-1}-1}} = \frac{2^{k-1}\binom{i}{2^{k-1}+1}}{\binom{i-1}{2^{k-1}}} = i\,\frac{2^{k-1}}{2^{k-1}+1}.$$

The last two equations express the important fact that for all k, $1 \leqslant k \leqslant t$,

$$E(X)/i = 2^{k-1}/(2^{k-1}+1). \tag{8}$$

In other words, the ratio of the expected strength of the player beaten by any i in any round to the strength of i himself *is independent of* i and depends only on the round where i beats X. As the strongest player 2^t survives all rounds, the expected strength of the player A_ℓ beaten by him in round ℓ ($\ell = 1, \ldots, t$) is

$$\frac{2^{\ell-1}}{2^{\ell-1}+1}\,2^t. \tag{9}$$

Following Hartigan [3] and Moon [4] we may call these players A_ℓ of type $\{\ell\}$; players of type $\{k, \ell\}$ ($1 \leqslant k < \ell$) are the players beaten by A_ℓ in round k, $1 \leqslant k < \ell$. Using (8), once again, the estimated strength of a player of type $\{k, \ell\}$ is :

$$2^t \times \frac{2^{k-1}}{2^{k-1}+1} \times \frac{2^{\ell-1}}{2^{\ell-1}+1} \qquad \text{for} \qquad 1 \leqslant k < \ell \leqslant t.$$

Generally, a player of type $\{a, b, \ldots, \ell\}$ ($1 \leqslant a < b < \ldots < \ell \leqslant t$) has expected strength :

$$\frac{2^{a-1}}{2^{a-1}+1} \times \frac{2^{b-1}}{2^{b-1}+1} \times \ldots \times \frac{2^{\ell-1}}{2^{\ell-1}+1}\,2^t, \tag{10}$$

a result first announced by Moon [4].

One advantage of the simpler derivation we present here is that the same method is applicable, with very little change, for higher moments. Indeed, let $X^{[j]} = X(X+1)\ldots(X+j-1)$. Then $E(X^{[j]})$ can be obtained by a repetition of the argument given in (8), and we can easily obtain variances and higher moments of the strengths of players of type $\{a, b, \ldots, \ell\}$. Given that i beats X in round k, we have

$$E(X^{[j]}\,|\,i \to X) = \frac{2^{k-1}}{2^{k-1}+j}\,i^{[j]}\ ;$$

so that for a player of type $\{a, b, \ldots, \ell\}$, we have,

$$E(X^{[j]}) = \frac{2^{a-1}}{2^{a-1}+j}\,\frac{2^{b-1}}{2^{b-1}+j}\cdots\frac{2^{\ell-1}}{2^{\ell-1}+j}\,2^t(2^t+1)\ldots(2^t+j-1).$$

4 – ESTIMATED STRENGTHS OF THE PLAYERS IN T_4 AND THE MAIN THEOREM

Let us consider a deterministic tournament with n players, whose strengths – as well as whose names – are $n, n-1, \ldots, 1$. If we play a knock-out tournament using the pattern T_4, it is easily seen that the result of such a tournament can be represented by a special kind of rooted tree. These trees consist of a "main" chain" or "trunk" and all "branches" from the trunk are chains of length 1. Using the case $n = 4$ for an illustration, we have the four trees of Fig. 1, Section 5 as possible outcomes of T_4. Clearly, given any tree with $(n - 1)$ players, we can associate with it two trees playing T_4 with n players – namely, the two trees obtained by assuming that the last player beats or is beaten by the winner of the tournament with $(n - 1)$ players. It is easy to establish a one-one correspondence between the 2^{n-2} compositions of $(n - 1)$ and the rooted trees which represent the possible outcomes of tournament T_4 with n players. Letting $(a_1, \ldots, a_k)$ where $a_1 + \ldots + a_k = n - 1$ represent such a tree, the probability with which such a tree arises, is shown by induction to be :

$$p(a_1, \ldots, a_k) = \frac{2}{n(n-a_1)(n-a_1-a_2)\ldots(n-a_1-\ldots-a_k)}. \tag{11}$$

We assume, as usual, that all permutations of the players are equally likely in deriving (11). We summarise these results formally in the following lemma.

Lemma 1. The possible outcomes of T_4 with $n \geqslant 2$ players may be represented by rooted trees which correspond to the compositions $(a_1, \ldots, a_k)$ of the integer $(n - 1)$. Further, using T_4 as our playing pattern, the probability of the tree $(a_1, \ldots, a_k)$ is given by (11).

We next note that players in T_4, given any rooted tree $(a_1, \ldots, a_k)$, can be divided into two categories : the $(k + 1)$ players on the main trunk, and the remaining players on the side branches. We shall refer to them, with the exception of the winner, in what follows as players of type $\{a, b, \ldots, \ell\}$ and players who lose immediately respectively. As it is easily seen from our main theorem, this notation does not lead to any confusion, except for some (unimportant) arbitrariness in designating the lowest player on the trunk. We now state the

Main Theorem. In T_4, the estimated strength for a player i of type $\{a, b, \ldots, \ell\}$ is given by

$$E(X) = n \cdot \frac{a}{a+1} \cdot \frac{b}{b+1} \cdot \ldots \cdot \frac{\ell}{\ell+1}\,. \tag{12}$$

If a player i loses immediately to a player of type $\{a, b, \ldots, \ell\}$ his estimated strength is

$$E(X) = \frac{n}{2} \frac{a}{a+1} \cdot \frac{b}{b+1} \cdot \ldots \cdot \frac{\ell}{\ell+1} \cdot \tag{13}$$

Proof. As the proof of this theorem is only a slight generalisation of the proof in the classical case, we indicate it very briefly. Clearly, the estimated strength of the winner is n, and a moment's reflexion enables us to see that the estimated strengths of the $(a_1 - 1)$ players losing immediately to the winner are $n/2$ respectively. (Note also that the player on the main trunk beaten by the winner is of type $\{n - a_1\}$, consistent with the definition of T_4).

Let us suppose we are given that player i entered the tournament before or on round k. Then denoting by $t(i, k)$ the probability that i survives round k, we have

$$t(i, k) = \binom{i-1}{k} \Big/ \binom{n-1}{k}, \tag{14}$$

since we essentially have to choose k players other than i and weaker than him in order for i to survive round k. If X denotes the player beaten by i in round k, X is a random variable with possible values $1, 2, \ldots, i-1$. Consider the two mutually exclusive cases listed below for calculating $E(X)$.

Case 1. X enters round k and not before i.e. X immediately loses to i. Then, clearly,

$$E(X) = \frac{1}{i-1} \{1 + \ldots + (i-1)\} = i/2. \tag{15}$$

Case 2. X survives round $(k-1)$ so that it is i who received byes before round k. In this case

$$E(X) = \sum_{\nu=1}^{i-1} \nu t^*(\nu, k-1), \tag{16}$$

where, as in the classical case,

$$t^*(\nu, k-1) = t(\nu, k-1) \Big/ \sum_{\nu=1}^{i-1} t(\nu, k-1).$$

Noting,

$$t^*(\nu, k-1) = \binom{\nu-1}{k-1} \Big/ \binom{i-1}{k}, \tag{17}$$

we have, from (16) and (17),

$$E(X)/i = k/(k+1). \tag{18}$$

As before, (15), (18) express the important fact that $E(X)/i$ is independent of i in Case 1 and also in Case 2, depending in the latter case only on k, the round in which i beats X. Equations (12), (13) now follow in exactly the same way as in C and the theorem is proved.

Our discussion of the main theorem is contained in the following comments.

a) We could have obtained C as a special case of the main theorem. As Case 1 does not arise in C, C is conceptually simpler than T_4 – in fact, $s(i, k)$ as defined by (1) is of the same form as $t(i, k)$ and the main theorem, suitably simplified, applies verbatim to C as well. Indeed, using the same arguments as in T_4, recursion formulae for estimated strengths can be easily calculated for any rooted tree – a result first stated by Hartigan.

b) Our method of proof makes it evident that if the original strengths of the players were $A + B, 2A + B, \ldots, nA + B$, with $A \neq 0$, the estimated strengths of the players undergo the same linear transformation. In particular, it is valid to define any such linear transformation of (12), (13) with $A \neq 0$, as the estimated "generalized strengths" of the players. In many problems it is natural to require that the sum of these generalized strengths add up to $n(n+1)/2$ with A, B integers. Clearly, the choice $B = 0, A = 1$ gives the strengths of the players as defined by Moon [4] and in this paper ; the only other choice $B = n + 1, A = -1$ leads to estimating final ranks of the players as in Hartigan [3]. Hence our approach not only simplifies the original proofs of both Moon and Hartigan, but actually shows their essential equivalence : the sum of the estimated strength and the estimated rank of any player is constant. Finally, the use of generalized strengths simplifies the calculation of expected strengths of the players in two independent knock-out tournaments of m and n players – i.e. where there are no comparisons between the two sets : this is indeed the essence of Hartigan's method [3, Section 5].

c) Finally, we state explicit formulae for calculating higher moments for players in T_4, as the proofs are analogous to C.

If i loses immediately to n, then

$$E(i^{[j]}) = n^{[j]}/(j+1). \tag{19}$$

If i is of type $\{a, b, \ldots, \ell\}$, then

$$E(i^{[j]}) = n^{[j]} \cdot \frac{a}{a+j} \cdot \frac{b}{b+j} \cdot \ldots \cdot \frac{\ell}{\ell+j} \tag{20}$$

If i loses immediately to somebody of type $\{a, b, \ldots, \ell\}$, then

$$E(i^{[j]}) = \frac{n^{[j]}}{j+1} \cdot \frac{a}{a+j} \cdot \ldots \cdot \frac{\ell}{\ell+j} \tag{21}$$

It is easy to verify for T_4 the following formula, which is valid for any rooted tree and any random knock-out tournament with n players :

$$\Sigma\, E(i^{[j]}) = \frac{n^{[j+1]}}{(j+1)} = j\,!\binom{n+j}{j+1}. \tag{22}$$

The Σ in equation (22) is over all n players in the tournament and the proof of (22) is similar to that of the comments in Section 5.

5 – A COMPARISON OF TOURNAMENTS

We will consider in this section a comparison of tournaments for small values of n, using for illustrative purposes T_4 with 4 players (Fig. 1) and some tournaments with 6 players (Fig. 2). The four rooted trees which are the outcomes of T_4 in this case, their compositions and their probabilities as well as the estimated strengths of the players and their variances are given below. We note that the total variance for each tree (e.g. for [2, 1] we obtain $10/9 = 0 + 2/9 + 2/3 + 2/9$) is indicated in the last row and we remark that the tree [2, 1] also represents C with 4 players.

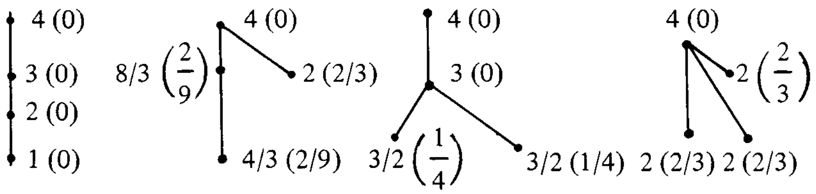
To illustrate one criterion on which comparisons of tournaments may be based, suppose we are given a deterministic model with n players, but we do not know which player has strength i ($i = 1, \ldots, n$). We label the players randomly

FIGURE 1

Rooted trees for T_4 with 4 players.

Composition	[1, 1, 1]	[2, 1]	[1, 2]	[3]
Probability	1/12	1/4	1/6	1/2

Tree



Total Variance	0	10/9	1/2	2

with the numbers $l, \ldots, n$ and assume as usual that the true strengths of the players are equally likely to be any one of the n ! permutations of $(1, 2, \ldots, n)$. The statistician is faced with estimating the true strengths after making a certain number $k \geqslant 0$ of comparisons between them i.e. he chooses a vector $\underline{M} = (M_1, \ldots, M_n)$ and asserts M_i is the estimated strength of player labelled i. The loss to the satistician with such a choice $\underline{M}$ is $L(\underline{M}, \underline{T}) = \sum_{i=1}^{n} (T_i - M_i)^2$ where $\underline{T} = (T_1, \ldots, T_n)$ is the vector of true strengths. After $k > 0$ comparisons have been made, we define $R_k(\underline{M})$, the risk of the statistician as $R_k(\underline{M}) = \frac{1}{n(P)} \sum_P L(\underline{M}, \underline{T})$ where the sum $\sum_P$ is taken over the set P of permutations consistent with the k comparisons made, and $n(P)$ is the number of elements in P. Clearly if $k = 0$, i.e. no comparisons have been made, $n(P) = n$! and the best choice for M is $\underline{M} = \left(\frac{n+1}{2}, \ldots, \frac{n+1}{2}\right)$. For this $\underline{M}$, the minimum risk $R_0(\underline{M}) = \frac{n(n^2-1)}{12}$ is attained as it is easy to prove that this choice minimizes the statistician's risk. In a knock-out tournament, $k = n - 1$ and we place the further restriction that once a player loses, he plays no more. We shall make this assumption throughout this section, although such a restriction is not essential for most of our comments.

Further, if $n = 2^t$, and the playing pattern is C, only one tree can be the outcome of the tournament. Clearly, the minimum risk to the statistician is attained if we choose the vector $\underline{M}$ in accordance with the estimated strengths in C. Indeed, from the fact $\sum_{i=1}^{n} (x_i - K)^2$ is minimised for fixed $x_1, \ldots, x_n$ when $K = \bar{x}$, we notice that for each player the "individual variance" is minimised by choosing the corresponding estimated strength in C *a fortiori* the total variance is minimised by this choice. On the other hand, if $n \neq 2^t$ or some other tournament, (T_4, say), is played, the outcome of the tournament can be any one of a set of rooted trees. Of course, given the rooted tree which is the outcome of a tournament, the main theorem enables the statistician to choose $\underline{M}$ so as to minimise his risk. However, a moment's reflection convinces us that even before the tournament is played (i.e. even if the rooted tree which is the particular outcome of the tournament is unknown) a strategy for minimising the risk exists. Formal proofs of the comments and elementary results which follow would be tedious ; we shall state them without proof.

I. Every random knock-out tournament with n players [5] is equivalent to a probability distribution on the rooted trees with n nodes. As illustrations, see figures 1 and 2. Of course, given an arbitrary probability distribution on the trees, there may be no corresponding tournament. The characterization of these tournament distributions is not attempted here.

II. Let $\underline{P} = (p_1, \ldots, p_\ell)$ be a tournament distribution on the trees $\bar{T}_1, \ldots, \bar{T}_\ell$. Let $\underline{M}_i = (M_1^i, \ldots, M_n^i)$ be the *ordered* estimated strengths of the players in tree T_i $(i = 1, \ldots, \ell)$. Then the minimum risk for this tournament is achieved by choosing the vector $\underline{M} = (M_1, \ldots, M_n)$ where

$$M_t = \sum_{i=1}^{\ell} p_i M_t \quad (t = 1, \ldots, n). \tag{23}$$

Furthermore, the total variance or minimum risk is given for this choice of $\underline{M}$ by

$$\sum_{t=1}^{n} (t^2 - M_t^2) = \frac{n(n+1)(2n+1)}{6} - \sum_{t=1}^{n} M_t^2. \tag{24}$$

Illustrations. In C with 4 players, $\ell = 1$. We have already calculated the total variance in Figure 1, by taking into account the individual variances. As a check of (24),

$$\frac{4.5.9}{6} - [4^2 + (8/3)^2 + 2^2 + (4/3)^2] = 10/9 .$$

In T_4 with 4 players, $\ell = 4$. The $\underline{M}^{i}$'s and P are given in Figure 1, eg. $\underline{M}^3 = (4,3,3/2,3/2), p_3 = 1/6$. Thus $\underline{M} = (4, 29/12, 23/12, 20/12)$ and the total variance, using (24), is given by 246/144. The variances of the individual positions are (0, 83/144, 83/144, 80/144), checking out once again the total variance.

III. If we base our definition of efficiency of a tournament on the total variance, a natural definition to use is

$$\mathcal{E}\ (T) = 1 - \frac{\dfrac{n\,(n+1)\,(2n+1)}{6} - \Sigma M_t^2}{\dfrac{n\,(n^2-1)}{12}} \tag{25}$$

A table of efficiences of some tournaments with $4 \leq n \leq 8$ is given after Figure 2 . Other criteria for comparing tournaments might be based on the maximum variance of the individual position ($\mathcal{E}_2$ say) or on the variance of the winner or second best player. Since in all knock-out tournaments the winner is determined with no error, $\mathcal{E}_3$ might be based on the "precision" with which the second best man is estimated. *Illustration.* According to $\mathcal{E}_2$, T_4 with 4 players is better than C with 4 players, the maximum variances being 83/144 and 2/3. We note also that variances in T_4, apart from the winner, are nearly equal as compared with C. Further examples can be found in Figure 2.

IV. When the number of comparisons which can be made between n players is not limited to $(n - 1)$, and we remove the restriction that a player who loses once plays no more, it is natural to try to determine that playing pattern or patterns which assure $\mathcal{E}(T) = 1$ with the minimum k. This problem has been considered by Steinhaus [7] and Ford and Johnson [2]. In principle, our method of computing efficiencies would settle the minimum k for $n = 12$, provided an efficient way of deleting "inadmissible strategies" can be formulated. We are investigating this problem.

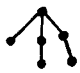
We conclude this section with Figure 2 and Table 2.

The reader interested in numerical details of the calculations of M vectors in Figure 2 and efficiencies in Table 2 is referred to TOURPACK, and APL package on tournaments available at the University of Alberta. We content ourselves with stating here that variances of individual positions are given alongside each M_i in the M vectors in Figure 2 and that 'T_2 Taboo' denotes T_2 with the restriction that winners in round 1 should (as far as possible) not be oposed in round 2, i.e. they play the byes of round 1 in round 2.

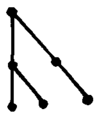
FIGURE 2

Some tournament distributions for 6 players.

Pattern and trees with corresponding P and M vectors

T_1 [2, 2, 1],  ; $P = (1/3, 2/3)$

$$M = [6, 64/15\binom{194}{225}, 56/15\binom{224}{225}, 3\,(2), 32/15\binom{246}{225}, 28/15\binom{196}{225}.$$

T_2 **Taboo** [3, 2], [3, 1, 1],  ,  ; $P = (4/9, 2/9, 2/9, 1/9)$.

$$M = [6, 9/2\binom{9}{20}, 10/3\binom{71}{45}, 16/6\binom{313}{180}, 43/18\binom{2257}{1620}, 35/18\binom{1561}{1620}]$$

T_2 To find P, M note $T_2 \equiv 1/3\,(T_1) + 2/3\,(T_2$ Taboo).

TABLE 2

Efficiencies $\mathcal{E}$ of Some Tournaments

Pattern \ n	4	5	6	7	8
T_1	.778*	.671	.667	.648*	.679*
T_2	.778*	.683	.652	.648*	.679*
T_4	.658	.580	.520	.481	.494
T_2 Taboo	–	–	.650	–	–

* T_1, T_2 are identical as tournaments in these cases, and for $n = 4, 8$ both patterns represent C.

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