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RELATIVE EXACTNESS MODULO A POLYNOMIAL MAP AND ALGEBRAIC $(\mathbb{C}^p, +)$-ACTIONS

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ABSTRACT. — Let $F = (f_1, \dots, f_q)$ be a polynomial dominating map from $\mathbb{C}^n$ to $\mathbb{C}^q$. We study the quotient $\mathcal{T}^1(F)$ of polynomial 1-forms that are exact along the generic fibres of F, by 1-forms of type $\mathrm{d}R + \sum a_i \mathrm{d}f_i$, where $R, a_1, \dots, a_q$ are polynomials. We prove that $\mathcal{T}^1(F)$ is always a torsion $\mathbb{C}[t_1, \dots, t_q]$-module. Then we determine under which conditions on F we have $\mathcal{T}^1(F) = 0$. As an application, we study the behaviour of a class of algebraic $(\mathbb{C}^p, +)$-actions on $\mathbb{C}^n$, and determine in particular when these actions are trivial.

RÉSUMÉ (*Exactitude relative modulo une application polynomiale et actions algébriques de* $(\mathbb{C}^p, +)$)

Soit $F = (f_1, \dots, f_q)$ une application polynomiale dominante de $\mathbb{C}^n$ dans $\mathbb{C}^q$. Nous étudions le quotient $\mathcal{T}^1(F)$ des 1-formes polynomiales qui sont exactes le long des fibres génériques de F, par les 1-formes du type $\mathrm{d}R + \sum a_i \mathrm{d}f_i$, où $R, a_1, \dots, a_q$ sont des polynômes. Nous montrons que $\mathcal{T}^1(F)$ est toujours un $\mathbb{C}[t_1, \dots, t_q]$-module de torsion. Nous déterminons ensuite sous quelles conditions sur F ce module est réduit à zéro. En application, nous étudions le comportement d'une classe d'actions algébriques de $(\mathbb{C}^p, +)$ sur $\mathbb{C}^n$, et nous déterminons en particulier quand ces actions sont triviales.

1. Introduction

Let $F = (f_1, \dots, f_q)$ be a dominating polynomial map from $\mathbb{C}^n$ to $\mathbb{C}^q$ with $n > q$. Let $\Omega^k(\mathbb{C}^n)$ be the space of polynomial differential k-forms on $\mathbb{C}^n$.

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For simplicity, we denote by $\mathbb{C}[F]$ the algebra generated by $f_1, \dots, f_q$, and by $\mathbb{C}(F)$ its fraction field. Our purpose in this paper is to compare two notions of relative exactness modulo F for polynomial 1-forms, and to deduce some consequences on some algebraic groups actions.

The first notion is the *topological relative exactness.* A polynomial 1-form ω is topologically relatively exact (in short: TR-exact) if ω is exact along the generic fibres of F. More precisely this means there exists a Zariski open set U in $\mathbb{C}^q$ such that, for any y in U, the fibre $F^{-1}(y)$ is non-critical and non-empty, and ω has null integral along any loop γ contained in $F^{-1}(y)$.

The second notion is the *algebraic relative exactness.* A polynomial 1-form is algebraically relatively exact (in short: AR-exact) if it is a coboundary of the De Rham relative complex of F (see [13]). Recall this complex is given by the spaces of relative forms

$$\Omega^k_F = \Omega^k(\mathbb{C}^n)/\sum \mathrm{d}f_i \wedge \Omega^{k-1}(\mathbb{C}^n)$$

and the morphisms $\mathrm{d}_F : \Omega^k_F \to \Omega^{k+1}_F$ induced by the exterior derivative.

Definition 1.1. — The module of relative exactness of F is the quotient $\mathcal{T}^1(F)$ of TR-exact 1-forms by AR-exact 1-forms. This is a $\mathbb{C}[F]$-module under the multiplication rule $(P(F), \omega) \mapsto P(F)\omega$.

For holomorphic germs, Malgrange implicitly compared these notions of relative exactness in [13]. He proved that the first relative cohomology group of the germ F is zero if the singular set of F has codimension ≥ 3; in this case, $\mathcal{T}^1(F)$ is reduced to zero. In [2], Berthier and Cerveau studied the relative exactness of holomorphic foliations, and introduced a similar quotient. For polynomials in two variables, Gavrilov [9] proved that $\mathcal{T}^1(f) = 0$ if every fibre of f is connected and reduced. Concerning polynomial maps, we first prove the following result.

Proposition 1.2. — *If F is a dominating map, then $\mathcal{T}^1(F)$ is a torsion $\mathbb{C}[F]$-module.*

In other words, every TR-exact 1-form ω can be written as

$$P(F)\omega = \mathrm{d}R + a_1\mathrm{d}f_1 + \cdots + a_q\mathrm{d}f_q$$

where $R, a_1, \dots, a_q$ are all polynomials. In [3], the author in collaboration with Alexandru Dimca studied in a comprehensive way the torsion of this module for any polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$. We are going to extend these results in any dimension and determine when $\mathcal{T}^1(F)$ is zero.

Let $F : X \to Y$ be a morphism of algebraic varieties, where Y is equidimensionnal and X may be reducible. A property $\mathcal{P}$ on the fibres of F is *k-generic* if the set of points y in Y whose fibre $F^{-1}(y)$ does not satisfy $\mathcal{P}$ has codimension $> k$ in Y. A *blowing-down* is an irreducible hypersurface V in $\mathbb{C}^n$ such that $F(V)$ has codimension ≥ 2 in $\mathbb{C}^q$. If no such hypersurface exists, we say that F has no blowing-downs. Finally F is non-singular in codimension 1 if its singular set has codimension ≥ 2. It is easy to prove that a non-singular map in codimension 1 has no blowing-downs.

Definition 1.3. — The map F is primitive if its fibres are 0-generically connected and 1-generically non-empty.

Then we show that a polynomial map F is primitive if and only if every polynomial R locally constant along the generic fibres of F can be written as $R = S(F)$, where S is a polynomial. So this definition extends the notion of primitive polynomial (*cf.* [8]).

Definition 1.4. — The map F is quasi-fibered if F is non-singular in codimension 1, its fibres are 1-generically connected and 2-generically non-empty. The map F is weakly quasi-fibered if F has no blowing-downs, its fibres are 1-generically connected and 2-generically non-empty.

Theorem 1.5. — *Let F be a primitive mapping. If F is a quasi-fibered mapping, then $\mathcal{T}^1(F) = 0$. If F is weakly quasi-fibered, then every TR-exact 1-form ω splits as $\omega = \mathrm{d}R + \omega_0$, where R is a polynomial and $\omega_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q = 0$.*

We apply these results to the study of algebraic $(\mathbb{C}^p,+)$-actions on $\mathbb{C}^n$. Such an action is a regular map $\varphi : \mathbb{C}^p \times \mathbb{C}^n \to \mathbb{C}^n$ such that

$$\varphi\big(u, \varphi(v,x)\big) = \varphi(u+v, x)$$

for all u, v, x. Geometrically speaking, φ is obtained by integrating a system $\mathcal{D} = \{\partial_1, \ldots, \partial_p\}$ of derivations on $\mathbb{C}[x_1, \ldots, x_n]$ that are pairwise commuting and locally nilpotent (see [11]), that is :

$$\forall f \in \mathbb{C}[x_1, \ldots, x_n],\ \exists\, k \in \mathbb{N}, \quad \partial_i^k(f) = 0.$$

The ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^\varphi$ is the set of polynomials P such that

$$P \circ \varphi = P.$$

Finally φ is *free at the point* x if the orbit of x has dimension p, and *free* if it is free at any point of $\mathbb{C}^n$. The set of points where φ is not free is an algebraic set denoted $\mathcal{NL}(\varphi)$.

Definition 1.6 (condition (H)). — An algebraic $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$ satisfies condition (H) if its ring of invariants is isomorphic to a polynomial ring in $n-p$ variables.

Under this condition, φ is provided with a *quotient map* F (see [16]) defined as follows: If $f_1, \ldots, f_{n-p}$ denote a set of generators of $\mathbb{C}[x_1, \ldots, x_n]^\varphi$, then

$$F : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-p}, \quad x \longmapsto \big(f_1(x), \ldots, f_{n-p}(x)\big).$$

The generic fibres of F are orbits of the action, but this map need not define a topological quotient: For instance, it does not separate all the orbits. The action φ is *trivial* if it is conjugate by a polynomial automorphism of $\mathbb{C}^n$ to

$$\varphi_0(t_1,\dots,t_p;x_1,\dots,x_n)=(x_1+t_1,\dots,x_p+t_p,x_{p+1},\dots,x_n).$$

We are going to search under which conditions the actions satisfying (H) are trivial. According to a result of Rentschler [18], every fix-point free algebraic $(\mathbb{C},+)$-action on $\mathbb{C}^2$ is trivial. We know [15] that (H) is always satisfied for $(\mathbb{C},+)$-actions on $\mathbb{C}^3$, but we still do not know if fixed-point free $(\mathbb{C},+)$-actions on $\mathbb{C}^3$ are trivial (see [11]). In dimension ≥ 4, the works [11], [21] of Nagata and Winkelmann prove that (H) need not be satisfied. For $(\mathbb{C},+)$-actions satisfying this condition, Deveney and Finston [6] proved that φ is trivial if its quotient map defines a locally trivial $(\mathbb{C},+)$-fibre bundle on its image.

We are going to see how this last result extends via relative exactness. Let φ be a $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$ satisfying (H), and consider the following operators:

$$\begin{aligned}[\mathcal{D}]:(R_1,\dots,R_p)&\longmapsto \det\big(\big(\partial_i(R_j)\big)\big),\\ J:(R_1,\dots,R_p)&\longmapsto \det(\mathrm{d}R_1,\dots,\mathrm{d}R_p,\mathrm{d}f_1,\dots,\mathrm{d}f_{n-p}).\end{aligned}$$

We say that $[\mathcal{D}]$ (resp. J) vanishes at the point x if, for any polynomials $R_1,\dots,R_p$, we have

$$[\mathcal{D}](R_1,\dots,R_p)(x)=0\quad\big(\text{resp. }J(R_1,\dots,R_p)(x)=0\big).$$

The zeros of $[\mathcal{D}]$ correspond to the points of $\mathcal{NL}(\varphi)$, and the zeros of J are the singular points of F. We generalise Daigle's [4] Jacobian Formula for $(\mathbb{C},+)$-actions.

Proposition 1.7. — *Let φ be an algebraic $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$ satisfying condition (H). Then there exists an invariant polynomial E such that*

$$[\mathcal{D}]=E\times J.$$

From a geometric viewpoint, this means that $\mathcal{NL}(\varphi)$ is the union of an invariant hypersurface and of the singular set of F. In particular E is constant if $\operatorname{codim}\mathcal{NL}(\varphi)\geq 2$.

Theorem 1.8. — *Let φ be an algebraic $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$ satisfying condition (H). If E is constant and F is quasi-fibered, then φ is trivial.*

Therefore the assumption "quasi-fibered" correspond to some regularity in the way that F fibres the orbits. In particular the action is trivial if F defines a topological quotient, *i.e.* if F is smooth surjective and separates the orbits.

Corollary 1.9. — *Let φ be an algebraic $(\mathbb{C},+)$-action on $\mathbb{C}^n$ satisfying condition (H). If F is quasi-fibered, there exists a polynomial P such that φ is conjugate to the action*

$$\varphi'(t;x_1,\dots,x_n)=\big(x_1+tP(x_2,\dots,x_n),x_2,\dots,x_n\big).$$

Corollary 1.10. — *Every algebraic $(\mathbb{C}^{n-1},+)$-action φ on $\mathbb{C}^n$ such that $\operatorname{codim}\mathcal{NL}(\varphi)\geq 2$ is trivial. In particular φ is free.*

We end up with counter-examples illustrating the necessity of the conditions of Theorem 1.8 and its corollaries.

2. Proof of Proposition 1.2

In this section, we establish the first proposition announced in the introduction in two steps. First we describe a TR-exact 1-form ω on every generic fibre of F. Second we "glue" all these descriptions by using the uncountability of complex numbers. To that purpose, we use the following definitions.

For any ideal I, we denote by

$$I\Omega^1(\mathbb{C}^n)$$

the space of polynomial 1-forms with coefficients in I. We introduce the equivalence relation:

$$\omega\simeq\omega'\ [I]\iff \omega-\omega'\in \mathrm{d}\Omega^0(\mathbb{C}^n)+\sum\Omega^0(\mathbb{C}^n)\mathrm{d}f_i+I\Omega^1(\mathbb{C}^n).$$

This equivalence is compatible with the structure of $\mathbb{C}[F]$-module given by the natural multiplication, since $\mathrm{d}\Omega^0(\mathbb{C}^n)+\sum\Omega^0(\mathbb{C}^n)\mathrm{d}f_i$ and $I\Omega^1(\mathbb{C}^n)$ are both $\mathbb{C}[F]$-modules.

Lemma 2.1. — *Let $F^{-1}(y)$ be a non-empty non-critical fibre of F, where $y=(y_1,\dots,y_q)$. A polynomial 1-form ω is exact on $F^{-1}(y)$ if and only if there exists a polynomial R and some polynomial 1-forms $\eta_1,\dots,\eta_q$ such that*

$$\omega=\mathrm{d}R+\sum_i(f_i-y_i)\eta_i.$$

Proof. — Since ω is exact on $F^{-1}(y)$, it has an holomorphic integral R on this fibre. Since $F^{-1}(y)$ is a smooth affine variety, R is a regular map by Grothendieck's Theorem (see [7, p. 182]). In other words, R is the restriction to $F^{-1}(y)$ of a polynomial, which will also be denoted by R. The $(q+1)$-form $(\omega-\mathrm{d}R)\wedge\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q$ vanishes on $F^{-1}(y)$. Since $F^{-1}(y)$ is non-critical, $(f_1-y_1),\dots,(f_q-y_q)$ define a local system of parametres at any point of $F^{-1}(y)$. So the ideal $((f_1-y_1),\dots,(f_q-y_q))$ is reduced and we get:

$$(\omega-\mathrm{d}R)\wedge\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q\equiv 0\quad [f_1-y_1,\dots,f_q-y_q].$$

The q-form $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q$ never vanishes on $F^{-1}(y)$. By de Rham Lemma (see [19]), there exist some polynomials α_i and some polynomial 1-forms η_i such that

$$\omega - \mathrm{d}R = \sum_{i=1}^{q} \alpha_i \,\mathrm{d}f_i + \sum_{i=1}^{q} (f_i - y_i)\eta_i$$

which can be rewritten as

$$\omega = \mathrm{d}\Big(R + \sum_{i=1}^{q} \alpha_i (f_i - y_i)\Big) + \sum_{i=1}^{q} (f_i - y_i)(\eta_i - \mathrm{d}\alpha_i). \qquad \square$$

Proof of Proposition 1.2. — Let ω be a TR-exact 1-form. Let us show there exists a non-zero polynomial P such that $P(F)\omega \simeq 0$ $[(0)]$. By Lemma 2.1, there exists a non-empty Zariski open set U in $\mathbb{C}^q$ such that, for any $y = (y_1, \ldots, y_q)$ in U

$$\omega \simeq 0 \quad [f_1 - y_1, \ldots, f_q - y_q].$$

We proceed to an elimination of $f_1 - y_1, \ldots, f_q - y_q$. For any point $y = (y_{i+1}, \ldots, y_q)$ in $\mathbb{C}^{q-i}$, we denote by $I_i(y)$ the following ideal:

$$I_i(y) = (f_{i+1} - y_{i+1}, \ldots, f_q - y_q).$$

By convention, $\mathbb{C}^0$ is the space reduced to a point, and $I_q(y) = (0)$. Let us show by induction on $i \le q$ the following property:

There exists a non-empty Zariski open set U_i in $\mathbb{C}^{q-i}$ such that, for any point y in U_i, there exists a non-zero polynomial P in $\mathbb{C}[t_1, \ldots, t_i]$ for which

$$P(f_1, \ldots, f_i)\omega \simeq 0 \quad \big[I_i(y)\big].$$

This property is true for $i = 0$. Assume it holds to the order $i < q$, and let U_i be such a Zariski open set. We may assume that U_i is a principal open set, *i.e.* $U_i = \{f(y) \neq 0\}$. Write

$$f = \sum_{k \le s} f_k(t_{i+2}, \ldots, t_q) t_{i+1}^k,$$

and set $U_{i+1} = \{f_s(y') \neq 0\}$. Let $y' = (y_{i+2}, \ldots, y_q)$ be a point in U_{i+1}. For any z such that $f(z, y') \neq 0$, the point $y = (z, y')$ belongs to U_i. By induction, there exist a non-zero polynomial P^z and a polynomial 1-form η^z such that:

$$P^z(f_1, \ldots, f_i)\omega \simeq (f_{i+1} - z)\eta^z \quad \big[I_{i+1}(y')\big].$$

For any such z, fix a 1-form η^z satisfying this equivalence. The system $\{\eta^z\}$ thus obtained is an uncountable subset of $\Omega^1(\mathbb{C}^n)$. Since $\Omega^1(\mathbb{C}^n)$ has countable dimension, these forms cannot be linearly independent. There exist some distinct values $z_1, \ldots, z_m$ and some non-zero constants $(\beta_1, \ldots, \beta_m)$ such that:

$$\beta_1 \eta^{z_1} + \cdots + \beta_m \eta^{z_m} = 0.$$

Since the equivalence relation is compatible with the structure of $\mathbb{C}[F]$-module, we get with the previous relations:

$$\Big(\sum_{j=1}^{m}\beta_j P^{z_j}(f_1,\dots,f_i)\prod_{k\neq j}(f_{i+1}-z_k)\Big)\omega \simeq 0 \quad \big[I_{i+1}(y')\big].$$

None of the β_j (resp. P^{z_j}) is zero by construction. Thus the polynomial

$$\widetilde{P}=\sum_{j=1}^{m}\beta_j P^{z_j}(t_1,\dots,t_i)\prod_{k\neq j}(t_{i+1}-z_k)$$

is non-zero, and satisfies the relation $\widetilde{P}(f_1,\dots,f_{i+1})\omega \equiv 0$ $[I_{i+1}(y')]$. Since we can perform this process for any point y' in U_{i+1}, the induction is proved. □

3. A factorisation lemma

In this section, we prove an extension of the first Bertini's Theorem and Stein's Factorisation Theorem (see [20, p. 139], and [10, p. 280]) to the case of reducible varieties. This result is certainly well-known but I could not find a proper reference for it. So I prefer to give a proof of it, based on Zariski's Main Theorem.

Lemma 3.1. — *Let $F : X \to Y$ be a dominating morphism of complex affine varieties, where X is equidimensional and Y is irreducible. Let R be a regular map on X. Assume that:*

- *the fibres of F are generically connected;*
- *the restriction of F to any irreducible component of X is dominating;*
- *the map $G=(F,R)$ is everywhere singular on X.*

Then R coincides on a dense open set of X with $\alpha(F)$, where α is a rational map on Y. In this case, R is said to factor through F.

Proof. — Since the map $G : X \to Y\times\mathbb{C}$ is everywhere singular, G cannot be dominating. So there exists an element P of $\mathbb{C}[Y][t]$ such that $P(F,R)=0$ on X. Note that P has degree >0 with respect to t, because F is a dominating map. Under the previous assumptions, there exists a Zariski open set U in Y such that:

- for any irreducible component X' of X, U is contained in $F(X')$;
- for any point y in U, $F^{-1}(y)$ is connected;
- for any point y in U, the polynomial $P(y,t)$ is non-zero.

Let y be a point in U. Since $P(y,R)=0$ on $F^{-1}(y)$, R is locally constant on $F^{-1}(y)$. Since R is regular and $F^{-1}(y)$ is connected, R is constant on $F^{-1}(y)$. So we can define the correspondence $\alpha : U \to \mathbb{C}$ that maps any point y of U to the unique value that takes R on $F^{-1}(y)$. Consider its graph:

$$Z=\big\{(y,\alpha(y)),\ y\in U\big\}.$$

If X' is an irreducible component of X, then Z coincides with $\overline{G(X'\cap F^{-1}(U))}$. So Z is constructible for the Zariski topology, and $\overline{Z}$ is irreducible. Therefore $\overline{Z}$ defines in $Y\times\mathbb{C}$ a rational correspondence from Y to $\mathbb{C}$ in the sense of Zariski (see [17, p. 29–51]). By Zariski's Main Theorem, α coincides with a rational map on Y. Let U' be an open set contained in U where α is regular. Then $F^{-1}(U')$ is a dense open subset of X. Moreover R and $\alpha(F)$ coincide on $F^{-1}(U')$ by construction. □

4. Blowing-downs and primitive mappings

In this section, we give some properties of blowing-downs and primitive mappings. For this class of maps, we will establish a *division lemma* (see Section 5) that is the key-point for the proof of Theorem 1.4. Let F be a polynomial dominating map from $\mathbb{C}^n$ to $\mathbb{C}^q$, and let $S(F)$ be its set of singular points. We introduce the following sets:

$$B(F) = \big\{y\in\mathbb{C}^q,\ F^{-1}(y) \textit{ is non-empty and not connected}\big\},$$

$$E(F) = \textit{union of blowing-downs of } F,$$

$$I(F) = \big\{y\in\mathbb{C}^q,\ F^{-1}(y) \textit{ is empty}\big\}.$$

Let H be the GCD of all q-minors of $\mathrm{d}F$, and set:

$$\omega_F = \frac{\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q}{H}.$$

Note that for all polynomials P and R, we have

$$P(F)\mathrm{d}R\wedge\omega_F = \mathrm{d}\big(P(F)R\big)\wedge\omega_F.$$

Since the sets $B(F), E(F), I(F)$ are all constructible for the Zariski topology, it makes sense to consider their codimensions. Recall that F is primitive if its fibres are 0-generically connected and 1-generically non-empty, *i.e.* $\operatorname{codim} B(F)\geq 1$ and $\operatorname{codim} I(F)\geq 2$.

Proposition 4.1. — *A polynomial map $F:\mathbb{C}^n\to\mathbb{C}^q$ is primitive if and only if any polynomial R such that $\mathrm{d}R\wedge\omega_F=0$ belongs to $\mathbb{C}[F]$.*

Proof. — Assume that F is primitive. Let R be a polynomial such that $\mathrm{d}R\wedge\omega_F=0$. Then the map $G=(F,R)$ is everywhere singular. Since the generic fibres of F are connected, R factors through F by the factorisation lemma. Let us set

$$R = b(F)/a(F),$$

where a,b are relatively prime. Let us show by absurd that a is constant. Assume not, and let a' be an irreducible factor of a. For any point y in $V(a')-I(F)$, there exists a point x such that $F(x)=y$, which implies that $a(y)R(x)=b(y)=0$. So b vanishes on $V(a')-I(F)$. Since $I(F)$ has codimension ≥ 2 in $\mathbb{C}^n$, $V(a')-I(F)$ is dense in $V(a')$ and b vanishes on $V(a')$. By Hilbert's Nullstellensatz, a' divides b, contradicting the fact that a and b are relatively prime. Thus a is constant and R belongs to $\mathbb{C}[F]$.

Assume now that any polynomial R such that $\mathrm{d}R\wedge\omega_F=0$ belongs to $\mathbb{C}[F]$. The q-form ω_F is obviously non-zero, and the polynomials f_i are algebraically independent. So F is a dominating map.

Let us prove first that $\operatorname{codim}(B(F))\geq 1$. By Bertini First Theorem (see [20, p. 139]), it suffices to show that $\mathbb{C}(F)$ is algebraically closed in $\mathbb{C}(x_1,\dots,x_n)$. Let R be a rational fraction that is algebraic over $\mathbb{C}(F)$. Let

$$P(z,t_1,\dots,t_q)=\sum_{k\leq s}a_k(t_1,\dots,t_q)z^k$$

be a nonzero polynomial such that $P(R,f_1,\dots,f_q)=0$. We choose P of minimal degree with respect to z. Since $P(R,f_1,\dots,f_q)=0$, the denominator of R divides $a_s(F)$. By derivation and wedge product, we get:

$$\frac{\partial P}{\partial z}(R,f_1,\dots,f_q)\mathrm{d}R\wedge\omega_F=0.$$

Since P has minimal degree, $\mathrm{d}R\wedge\omega_F=0$ and $\mathrm{d}(a_s(F)R)\wedge\omega_F=0$. As $a_s(F)R$ is a polynomial, it belongs to $\mathbb{C}[F]$ and R lies in $\mathbb{C}(F)$.

Let us show by absurd that $\operatorname{codim}(I(F))\geq 2$. Assume not, and let $C=V(f)$ be a codimension 1 irreducible component of $\overline{I(F)}$, where f is reduced. Since the intersection $V(f)\cap F(\mathbb{C}^n)$ has codimension ≥ 2, there exists a polynomial P vanishing on $V(f)\cap F(\mathbb{C}^n)$ and not divisible by f. The function $P(F)$ vanishes on $V(f(F))$. By Hilbert's Nullstellensatz, there exists an integer n such that $P^n(F)$ is divisible by $f(F)$. The function P^n/f is rational non-polynomial, and $R=P^n(F)/f(F)$ belongs to $\mathbb{C}[x_1,\dots,x_n]$. Since R satisfies the equation $\mathrm{d}R\wedge\omega_F=0$, R belongs to $\mathbb{C}[F]$, hence a contradiction. $\square$

For $q=1$, a mapping F is primitive if and only if its generic fibres are connected. Indeed any non-constant polynomial map from $\mathbb{C}^n$ to $\mathbb{C}$ has to be surjective. In this way, the definition of primitive mapping extends the notion of primitive polynomial (see [8]).

Example 1. — The polynomial $F(x,y)=x^2$ is not primitive because its generic fibres are not connected. Note that $\mathrm{d}x\wedge\mathrm{d}(x^2)=0$, but x does not belong to $\mathbb{C}[x^2]$.

Example 2. — Consider the mapping $F:\mathbb{C}^3\to\mathbb{C}^2$, $(x,y,z)\mapsto(x,xy)$. The function y satisfies the relation $\mathrm{d}y\wedge\mathrm{d}x\wedge\mathrm{d}(xy)=0$ but does not belong to $\mathbb{C}[x,xy]$. So F is not a primitive mapping although its generic fibres are connected. The obstruction lies in the fact that $\overline{I(F)}=\{(y_1,y_2),\ y_1=0\}$, so $\operatorname{codim}(I(F))=1$.

Example 3. — Consider the mapping $F:\mathbb{C}^3\to\mathbb{C}^2$, $(x,y,z)\mapsto(xy,zy)$. It is easy to see that F is onto and that its generic fibres are isomorphic to $\mathbb{C}^*$. So F is a primitive mapping.

Recall that a blowing-down is an hypersurface of $\mathbb{C}^n$ that is mapped by F to a set of codimension ≥ 2. For instance, the plane $\{y=0\}$ in $\mathbb{C}^3$ is a blowing-down of the map $F(x,y,z)=(xy,zy)$.

Proposition 4.2. — *Any blowing-down of F is contained in $S(F)$.*

Proof. — Let V be a blowing-down of F, and let W denote the Zariski closure of $F(V)$. Then W is irreducible and there exists a dense open set W' of W, consisting only of smooth points of W and containing $F(V)$. So $V'=F^{-1}(W')\cap V$ is a dense open set of V. For any smooth point x in V', the differential of the restriction of F to V has rank $\leq \dim W' \leq q-2$. The differential $dF(x)$ maps the hyperplane T_xV to a space of dimension $\leq q-2$. So $\mathrm{d}F(x)$ maps $\mathbb{C}^n$ to a space of dimension $\leq q-1$, and F is singular at x. Since any smooth point of V' is a singularity of F and $S(F)$ is closed, we have the inclusion $V\subset S(F)$. □

5. The Division Lemma

In this section, we are going to establish the essential tool for the proof of Theorem 1.4. Let ω be a TR-exact 1-form ω. By Proposition 1.2, there exists a non-zero polynomial P in $\mathbb{C}[t_1,\dots,t_q]$, and some polynomials $R,a_1,\dots,a_q$ in $\mathbb{C}[x_1,\dots,x_n]$ such that:

$$P(F)\omega=\mathrm{d}R+a_1\mathrm{d}f_1+\cdots+a_q\mathrm{d}f_q.$$

By using the wedge product with ω_F, we get :

$$\mathrm{d}R\wedge\omega_F=P(F)\omega\wedge\omega_F\equiv 0\quad\big[P(F)\big].$$

Assume there exist some polynomials $S,b_1,\dots,b_q$ such that $\omega=\mathrm{d}S+\sum_i b_i\mathrm{d}f_i$. By an obvious computation, we get

$$\omega\wedge\omega_F=\mathrm{d}S\wedge\omega_F\quad\text{and}\quad \mathrm{d}(R-P(F)S)\wedge\omega_F=0.$$

Since F is primitive, there exists a polynomial A such that $R=A(F)+P(F)S$.

More generally, let R be a polynomial satisfying the equation

$$\mathrm{d}R\wedge\omega_F\equiv 0\quad\big[P(F)\big].$$

R is said to be *$\mathcal{E}$-divisible* by $P(F)$ if there exist some polynomials A and S such that

$$R=A(F)+P(F)S.$$

In this section we are going to determine under which conditions a polynomial R satisfying this equation is $\mathcal{E}$-divisible by $P(F)$.

Division Lemma. — *Let F be a primitive mapping from $\mathbb{C}^n$ to $\mathbb{C}^q$. Let P be an element of $\mathbb{C}[t_1,\dots,t_q]$, and R a polynomial in $\mathbb{C}[x_1,\dots,x_n]$ satisfying the equation $\mathrm{d}R\wedge\omega_F\equiv 0\ [P(F)]$. Assume that:*

- *$V(P)\cap B(F)$ has codimension ≥ 2 in $\mathbb{C}^q$;*
- *$V(P(F))\cap E(F)$ has codimension ≥ 2 in $\mathbb{C}^n$;*
- *$V(P)\cap I(F)$ has codimension ≥ 3 in $\mathbb{C}^q$.*

Then R is $\mathcal{E}$-divisible by $P(F)$.

5.1. The Weak Division Lemma. — In this subsection, we are going to establish a weak version of the division lemma. A polynomial R is said to be *weakly $\mathcal{E}$-divisible* by $P(F)$ if there exists a polynomial B coprime to P such that $B(F)R$ is $\mathcal{E}$-divisible by $P(F)$.

Weak Division Lemma. — *Let F be a primitive mapping from $\mathbb{C}^n$ to $\mathbb{C}^q$. Let P be an irreducible polynomial of $\mathbb{C}[t_1,\dots,t_q]$. Let R be a polynomial in $\mathbb{C}[x_1,\dots,x_n]$ satisfying the equation $\mathrm{d}R\wedge\omega_F\equiv 0\ [P(F)]$. Assume that:*

- *$V(P)\cap B(F)$ has codimension ≥ 2 in $\mathbb{C}^q$;*
- *$V(P(F))\cap E(F)$ has codimension ≥ 2 in $\mathbb{C}^n$.*

Then R is weakly $\mathcal{E}$-divisible by $P(F)$.

The proof splits in two steps. Consider a polynomial R satisfying the equation $\mathrm{d}R\wedge\omega_F\equiv 0\ [P(F)]$.

First we show that its restriction to $V(P(F))$ factors through F. So there exist two polynomials A, B, with B coprime to P, such that $B(F)R-A(F)$ vanishes on $V(P(F))$. If $h_1^{n_1}\cdots h_r^{n_r}$ is the irreducible decomposition of $P(F)$ in $\mathbb{C}[x_1,\dots,x_n]$, then $h_1\cdots h_r$ divides $B(F)R-A(F)$.

Second we prove that every factor h_i divides $B(F)R-A(F)$ with multiplicity $\geq n_i$.

Lemma 5.1. — *Let P be an irreducible polynomial in $\mathbb{C}[t_1,\dots,t_q]$. Let h be an irreducible factor of $P(F)$. Let R be a polynomial satisfying the equation $\mathrm{d}R\wedge\omega_F\equiv 0\ [h]$. Then the map $G: V(h)\to V(P)\times\mathbb{C},\ x\mapsto (F(x),R(x))$ is everywhere singular.*

Proof. — It suffices to show that the collection of 1-forms $\mathrm{d}R, \mathrm{d}h, \mathrm{d}f_1,\dots,\mathrm{d}f_q$ has rank $\leq q$ at any point x of $V(h)$. We are going to check that whenever you choose $q+1$ forms in this collection, their wedge product is divisible by h. Consider the first case, when this wedge product contains all the forms $\mathrm{d}f_1,\dots,\mathrm{d}f_q$. Then it is either equal to $\mathrm{d}R\wedge\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q$ or to $\mathrm{d}h\wedge\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q$. By assumption $\mathrm{d}R\wedge\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q$ is divisible by h. To see that the second one is divisible by h, factor $P(F)=Qh^m$, where Q is coprime to h and $m\geq 1$.

By wedge product, we get:

$$\mathrm{d}P(F)\wedge \mathrm{d}f_1\wedge\cdots\wedge \mathrm{d}f_q = mh^{m-1}Q\,\mathrm{d}h\wedge \mathrm{d}f_1\wedge\cdots\wedge \mathrm{d}f_q \\ +h^m\mathrm{d}Q\wedge \mathrm{d}f_1\wedge\cdots\wedge \mathrm{d}f_q = 0.$$

This yields $Q\,\mathrm{d}h\wedge \mathrm{d}f_1\wedge\cdots\wedge \mathrm{d}f_q\equiv 0[h]$. Since Q is coprime to h, we find:

$$\mathrm{d}h\wedge \mathrm{d}f_1\wedge\cdots\wedge \mathrm{d}f_q\equiv 0\quad [h].$$

Consider now the second case, when $\mathrm{d}R$ and $\mathrm{d}h$ appear in the wedge product. Assume first that $q>1$. Up to a reordering of the forms $\mathrm{d}f_i$, we may assume that this wedge product is equal to $\mathrm{d}R\wedge\mathrm{d}h\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q$. Since $P(F)=Qh^m$ where Q is coprime to h, we get by derivation:

$$\mathrm{d}\big\{P(F)\big\} = \sum_{i=1}^{q}\frac{\partial P}{\partial t_i}(F)\mathrm{d}f_i\equiv 0\quad [h^{m-1}].$$

By wedge product, we find:

$$\begin{aligned}\frac{\partial P}{\partial t_1}(F)H\omega_F &= \frac{\partial P}{\partial t_1}(F)\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_q\\ &= \mathrm{d}\{P(F)\}\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q\equiv 0\quad [h^{m-1}].\end{aligned}$$

By construction, the coefficients of ω_F have no common factors. Thus h^{m-1} divides $\partial P/\partial t_1(F)H$. Then write:

$$\begin{aligned}\frac{\partial P}{\partial t_1}(F)H\mathrm{d}R\wedge\omega_F &= \mathrm{d}R\wedge\mathrm{d}\{P(F)\}\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q\\ &= \mathrm{d}R\wedge\mathrm{d}\{Qh^m\}\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q.\end{aligned}$$

Since $\mathrm{d}R\wedge\omega_F$ is divisible by h, we get

$$\mathrm{d}R\wedge\mathrm{d}\{Qh^m\}\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q\equiv 0\quad [h^m]$$

which leads to

$$mQh^{m-1}\mathrm{d}R\wedge\mathrm{d}h\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q\equiv 0\quad [h^m].$$

Since Q is coprime to h, we deduce:

$$\mathrm{d}R\wedge\mathrm{d}h\wedge\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q\equiv 0\quad [h].$$

If $q=1$, we do the same computation and forget the wedge product with $\mathrm{d}f_2\wedge\cdots\wedge\mathrm{d}f_q$. □

LEMMA 5.2. — *Let P be an irreducible polynomial in $\mathbb{C}[t_1,\dots,t_q]$. Let $h_1^{n_1}\cdots h_r^{n_r}$ be the irreducible decomposition of $P(F)$ in $\mathbb{C}[x_1,\dots,x_n]$. Let R be a polynomial such that $\mathrm{d}R\wedge\omega_F\equiv 0$ $[h_1\cdots h_r]$. Assume that:*

- *$V(P)\cap B(F)$ has codimension ≥ 2 in $\mathbb{C}^q$;*
- *$V(P(F))\cap E(F)$ has codimension ≥ 2 in $\mathbb{C}^n$.*

Then there exist two polynomials A, B, where B is coprime to P, such that $B(F)R-A(F)$ is divisible by $h_1\cdots h_r$.

Proof. — By the previous lemma applied to all the irreducible components of $V(P(F))$, we can see that the map:

$$G : V\big(P(F)\big) \longrightarrow V(P)\times\mathbb{C}, \quad x \longmapsto \big(F(x), R(x)\big)$$

is singular. Since $V(P(F))\cap E(F)$ has codimension ≥ 2, none of the hypersurfaces $V(h_i)$ is a blowing-down. So F maps every $V(h_i)$ densely on $V(P)$. Since $V(P)\cap B(F)$ has codimension ≥ 2, the generic fibres of $F : V(P(F)) \to V(P)$ are connected. By the factorisation lemma, there exists a rational map α on $V(P)$ such that $R = \alpha(F)$ on $V(P(F))$. Write α as A/B, where B is coprime to P. The polynomial $B(F)R - A(F)$ vanishes on $V(P(F))$. By Hilbert's Nullstellensatz, it is divisible by $h_1\cdots h_r$. □

Proof of the Weak Division Lemma. — Let P be an irreducible polynomial in $\mathbb{C}[t_1,\dots,t_q]$. Let $h_1^{n_1}\cdots h_r^{n_r}$ be the irreducible decomposition of $P(F)$ in $\mathbb{C}[x_1,\dots,x_n]$. Let R be a polynomial such that $\mathrm{d}R\wedge\omega_F \equiv 0\ [P(F)]$. Then R satisfies the equation

$$\mathrm{d}R\wedge\omega_F \equiv 0 \quad [h_1\cdots h_r].$$

By the previous lemma, there exist some polynomials A, B, where B is coprime to P, such that $S = B(F)R - A(F)$ is divisible by $h_1\cdots h_r$. Factor S as $S_0h_1^{k_1}\cdots h_r^{k_r}$, where S_0 is coprime to each h_i. Let us show by absurd that $k_i \geq n_i$ for any i.

Assume there exists an index i such that $k_i/n_i < 1$. Let i_0 be an index for which the ratio k_i/n_i is minimal, and let u/v be its irreducible decomposition. By construction, we have $0 < u/v < 1$. The function

$$L = S^v/P(F)^u = S_0^v h_1^{vk_1-un_1}\cdots h_r^{vk_r-un_r}$$

is polynomial, since $u/v \leq k_i/n_i$ implies $vk_i - un_i \geq 0$. Moreover L satisfies the equation $\mathrm{d}L\wedge\omega_F \equiv 0\ [h_1\cdots h_r]$. Indeed if $vk_i - un_i > 0$, then L is divisible by h_i and $L = L_ih_i$. We set $P(F) = P_ih_i^{n_i}$, where P_i is coprime to h_i. By an easy computation, we get:

$$\mathrm{d}P(F)\wedge\omega_F = P_in_ih_i^{n_i-1}\mathrm{d}h_i\wedge\omega_F + h_i^{n_i}\mathrm{d}P_i\wedge\omega_F = 0.$$

Since P_i is coprime to h_i, we deduce $\mathrm{d}h_i\wedge\omega_F \equiv 0\ [h_i]$, and this implies:

$$\mathrm{d}L\wedge\omega_F = L_i\mathrm{d}h_i\wedge\omega_F + h_i\mathrm{d}L_i\wedge\omega_F \equiv 0 \quad [h_i].$$

If $vk_i - un_i = 0$, set $S = S_ih_i^{k_i}$. By derivation and wedge product, we get:

$$S\mathrm{d}L\wedge\omega_F = S_ih_i^{k_i}\mathrm{d}L\wedge\omega_F = vL\mathrm{d}S\wedge\omega_F.$$

By an easy computation, we obtain

$$\mathrm{d}S\wedge\omega_F = B(F)\mathrm{d}R\wedge\omega_F \equiv 0 \quad [h_i^{n_i}]$$

which implies

$$S_i\mathrm{d}L\wedge\omega_F \equiv 0 \quad [h_i^{n_i-k_i}].$$

Since $n_i - k_i > 0$ and S_i is coprime to h_i, we deduce $\mathrm{d}L \wedge \omega_F \equiv 0\ [h_i]$. Thus $\mathrm{d}L \wedge \omega_F$ is divisible by $h_1 \cdots h_r$. By Lemma 5.2, there exist two polynomials A', B', where B' is coprime to P, such that $B'(F)L - A'(F) \equiv 0\ [h_1 \cdots h_r]$.

Let us show by absurd that $vk_i - un_i = 0$ for any i. Assume that h_i divides L. By the previous relation, h_i divides $A'(F)$. Since $V(h_i)$ is not a blowing-down and P is irreducible, A' is divisible by P, which implies:

$$B'(F)L \equiv 0 \quad [h_1 \cdots h_r].$$

Since none of the $V(h_j)$ are blowing-downs and every h_j divides $P(F)$, every h_j is coprime to $B'(F)$. So L is divisible by $h_1 \cdots h_r$, contradicting its construction.

Since $vk_i - un_i = 0$, v divides n_i for any i. As $0 < u/v < 1$, v is strictly greater than 1 and $P(F) = T^v$, where T belongs to $\mathbb{C}[x_1, \ldots, x_n]$. This implies:

$$\mathrm{d}\{P(F)\} \wedge \omega_F = vT^{v-1}\mathrm{d}T \wedge \omega_F = 0.$$

Since F is primitive, T belongs to $\mathbb{C}[F]$ by Proposition 4.1. Therefore P is the v-th power of some polynomial, which contradicts the irreducibility of P. □

5.2. Proof of the Division Lemma. — Let R be a polynomial satisfying the equation $\mathrm{d}R \wedge \omega_F \equiv 0\ [P(F)]$. From an analytic viewpoint, the weak division lemma asserts that R coincides on $V(P(F))$ with $\alpha(F)$, where α is a rational function on $V(P)$. In order to prove the Division Lemma, we are going to show that α is regular if $V(P) \cap I(F)$ has codimension ≥ 3. In other words, we are going to eliminate the "poles" of α.

Recall that an ideal I in a local ring R is $\mathcal{M}$-primary if I contains some power of the maximal ideal $\mathcal{M}$ of R. We denote by $\mathcal{O}_{\mathbb{C}^q,y}$ the ring of germs of regular functions at the point y in $\mathbb{C}^q$. For simplicity, we set:

$$\mathbb{C}[[X]] = \mathbb{C}[[x_1, \ldots, x_n]] \quad \text{and} \quad \mathbb{C}[[T]] = \mathbb{C}[[t_1, \ldots, t_q]].$$

LEMMA 5.3. — *Let $I = (g_1, \ldots, g_n)$ be an $\mathcal{M}$-primary ideal in $\mathbb{C}[[X]]$. If the classes of the formal series $\{e_1, \ldots, e_\mu\}$ form a basis of the vector space $\mathbb{C}[[X]]/I$, then $\{e_1, \ldots, e_\mu\}$ is a basis of the $\mathbb{C}[[g_1, \ldots, g_n]]$-module $\mathbb{C}[[X]]$.*

Proof. — Since $(g_1, \ldots, g_n)$ is $\mathcal{M}$-primary, $\mathbb{C}[[X]]$ is a finitely generated $\mathbb{C}[[g_1, \ldots, g_n]]$-module (see [1]). By Nakayama Lemma (*cf.* [20, p. 283]), $\{e_1, \ldots, e_\mu\}$ forms a minimal set of generators of this module. Let us show by absurd that $e_1, \ldots, e_\mu$ are $\mathbb{C}[[g_1, \ldots, g_n]]$-linearly independent.

Assume there exist some formal series $a_i(y_1, \ldots, y_n)$, not all equal to zero, such that $\sum_k a_k(g_1, \ldots, g_n)e_k = 0$. Up to a linear change of coordinates on $y_1, \ldots, y_n$, which is equivalent to replacing $g_1, \ldots, g_n$ by another set of formal series generating the same ideal, we may assume there exists an index i for which $a_i(y_1, 0, \ldots, 0) \neq 0$. By setting $a_i(x_1, 0, \ldots, 0) = b_i(x_1)$, we find:

$$b_1(g_1)e_1 + \cdots + b_\mu(g_1)e_\mu \equiv 0 \quad [g_2, \ldots, g_n].$$

Let m be the minimum of the orders of all formal series $b_1,\dots,b_\mu$. Then

$$b_i(x_1) = x_1^m c_i(x_1)$$

for any i, and $c_i(0)\neq 0$ for at least one of them. Thus we get:

$$g_1^m\{c_1(g_1)e_1+\cdots+c_\mu(g_1)e_\mu\}\equiv 0 \quad [g_2,\dots,g_n].$$

Since $(g_1,\dots,g_n)$ is $\mathcal{M}$-primary, $g_1,\dots,g_n$ is a regular sequence (*cf.* [20, p. 227]) and g_1 is not a zero-divisor modulo $[g_2,\dots,g_n]$. We deduce:

$$c_1(0)e_1+\cdots+c_\mu(0)e_\mu\equiv 0 \quad [g_1,g_2,\dots,g_n].$$

So $c_1(0)=\cdots=c_\mu(0)=0$, hence contradicting the fact that not all $c_i(0)$ are zero. □

LEMMA 5.4. — *Let y be a point in $\mathbb{C}^q$ such that the fibre $F^{-1}(y)$ is non-empty of dimension $(n-q)$. Let P,B,A be three elements of $\mathbb{C}[t_1,\dots,t_q]$ such that $A(F)$ belongs to the ideal $(P(F),B(F))\mathbb{C}[x_1,\dots,x_n]$. Then A belongs to $(P,B)\mathcal{O}_{\mathbb{C}^q,y}$.*

Proof. — Let x be a point in $F^{-1}(y)$ where the fibre has local dimension $(n-q)$. For simplicity, we may assume $x=0$ and $y=0$. There exists a q-dimensional vector space, defined by some linear equations $\ell_1,\dots,\ell_{n-q}$ and intersecting locally $F^{-1}(0)$ only at 0. By Ruckert's Nullstellensatz (see [1]), the ideal $(f_1,\dots,f_q,\ell_1,\dots,\ell_{n-q})$ is $\mathcal{M}$-primary in the ring $\mathbb{C}[[X]]$. Let $\{e_1,\dots,e_\mu\}$ be a basis of the vector space $\mathbb{C}[[X]]/(f_1,\dots,f_q,\ell_1,\dots,\ell_{n-q})$ such that $e_1=1$. By Lemma 5.3, $\{e_1,\dots,e_\mu\}$ is a basis of the $\mathbb{C}[[f_1,\dots,\ell_{n-q}]]$-module $\mathbb{C}[[X]]$. Let R,S be two polynomials in $\mathbb{C}[x_1,\dots,x_n]$ such that

$$A(F)=P(F)R+Q(F)S.$$

If $R_1(f_1,\dots,\ell_{n-q})$ and $S_1(f_1,\dots,\ell_{n-q})$ denote their first coordinate in the basis $\{e_1,\dots,e_\mu\}$, we get:

$$P(F)R_1(f_1,\dots,\ell_{n-q})+B(F)S_1(f_1,\dots,\ell_{n-q})=A(F).$$

After reduction modulo $\ell_1,\dots,\ell_{n-q}$, this implies:

$$P(F)R_1(F,0)+B(F)S_1(F,0)=A(F).$$

Thus A belongs to the ideal $(P,B)\mathbb{C}[[T]]$. Since $\mathcal{O}_{\mathbb{C}^q,0}$ is a Zariski ring and $\mathbb{C}[[T]]$ is its $\mathcal{M}$-adic completion, we get $(P,B)\mathbb{C}[[T]]\cap\mathcal{O}_{\mathbb{C}^q,0}=(P,B)\mathcal{O}_{\mathbb{C}^q,0}$ (see [14, p. 171–172]). So A belongs to $(P,B)\mathcal{O}_{\mathbb{C}^q,0}$. □

LEMMA 5.5. — *Let P,B,A be three polynomials in $\mathbb{C}[t_1,\dots,t_q]$ such that $A(F)$ belongs to $(P(F),B(F))\mathbb{C}[x_1,\dots,x_n]$. If $V(P(F),B(F))$ has codimension ≥ 2 and $V(P(F))\cap I(F)$ has codimension ≥ 3, then A belongs to $(P,B)\mathbb{C}[t_1,\dots,t_q]$.*

Proof. — This lemma is obvious if $V(P,B)$ is empty. We assume it is not, and consider the varieties $X = V(P(F), B(F))$ and $Y = V(P,B)$. By assumption, $P(F)$ and $B(F)$ are coprime and X is equidimensionnal of codimension 2 in $\mathbb{C}^n$. Moreover P, B are coprime and Y is equidimensionnal of codimension 2 in $\mathbb{C}^q$. As $V(P) \cap I(F)$ has codimension ≥ 3, the restriction

$$F_R : X \longrightarrow Y, \quad x \longmapsto F(x)$$

is a dominating map. We construct a dense open set U in Y such that $F^{-1}(y)$ has dimension $(n-q)$ for any y in U. Let X_i be any irreducible component of X. If $F(X_i)$ has codimension ≥ 3, fix a dense open set U_i in Y that does not meet $F(X_i)$. If $F(X_i)$ has codimension 2, we apply the theorem on the dimension of fibres to $F_R : X_i \to \overline{F(X_i)}$. There exists an open set V_i contained in $F(X_i)$ such that $F^{-1}(y) \cap X_i$ has dimension $(n-q)$ for any y in V_i. If U' is the intersection of all U_i and V' is the union of all V_i, then $U = U' \cap V'$ is a dense open set in Y, and $F^{-1}(y)$ has dimension $(n-q)$ for any y in U.

By Lemma 5.4, A belongs to $(P,B)\mathcal{O}_{\mathbb{C}^q,y}$ for any y in U. This means there exists a polynomial β_y such that $\beta_y(y) \neq 0$ and $\beta_y A$ belongs to $(P,Q)\mathbb{C}[t_1,\dots,t_q]$. The zero set of P, B and the β_y, when y runs through U, has codimension ≥ 3 since it is contained in $Y - U$. The ideal J generated by P, B and the β_y has depth ≥ 3. Since $\mathbb{C}[t_1,\dots,t_q]$ is catenary, J contains a polynomial β such that P, B, β is a regular sequence. By construction $\beta A \equiv 0\ [P,B]$. As β is not a zero divisor modulo (P,B), A belongs to $(P,B)\mathbb{C}[t_1,\dots,t_q]$. □

Proof of the Division Lemma. — Let R be a polynomial satisfying the equation $\mathrm{d}R \wedge \omega_F \equiv 0\ [P(F)]$. Assume that $V(P) \cap B(F)$ has codimension ≥ 2, $V(P(F)) \cap E(F)$ has codimension ≥ 2 and $V(P) \cap I(F)$ has codimension ≥ 3. By the Weak Division Lemma, there exist two polynomials A, B, where B is coprime to P, and a polynomial S such that:

$$B(F)R - A(F) = P(F)S.$$

Let us show by absurd that $X = V(P(F), B(F))$ has codimension ≥ 2. Assume that X contains an hypersurface V. Then F maps V to $Y = V(P,B)$, which codimension is ≥ 2 since P and B are coprime. So V is a blowing-down, and this contradicts the assumption on $V(P(F)) \cap E(F)$.

Since $A(F)$ belongs to $(P(F), B(F))\mathbb{C}[x_1,\dots,x_n]$ and $V(P) \cap I(F)$ has codimension ≥ 3, A belongs to $(P,B)\mathbb{C}[t_1,\dots,t_q]$ by Lemma 5.5. There exist some polynomials P_1, B_1 such that $A = PP_1 + BB_1$. Thus we deduce:

$$B(F)\big\{R - B_1(F)\big\} = P(F)\big\{S - P_1(F)\big\}.$$

Since $X = V(P(F), B(F))$ has codimension 2, $P(F)$ and $B(F)$ are coprime. So $P(F)$ divides $R - B_1(F)$ and the Division Lemma is proved. □

5.3. Proof of Theorem 1.5. — Let F be a primitive mapping that is either quasi-fibered or weakly quasi-fibered. By definition, the following conditions hold:

- $B(F)$ has codimension ≥ 2 in $\mathbb{C}^q$;
- $E(F)$ is empty;
- $I(F)$ has codimension ≥ 3 in $\mathbb{C}^q$.

Let ω be a TR-exact 1-form. By Proposition 1.2, there exists a non-zero polynomial P, and some polynomials $R, a_1, \ldots, a_q$ such that:

$$P(F)\omega = \mathrm{d}R + a_1\mathrm{d}f_1 + \cdots + a_q\mathrm{d}f_q.$$

By wedge product with ω_F, we can see that R satisfies the equation $\mathrm{d}R\wedge\omega_F \equiv 0$ $[P(F)]$. According to the above conditions, $V(P)\cap B(F)$ has codimension ≥ 2 in $\mathbb{C}^q$, $V(P(F))\cap E(F)$ is empty and $V(P)\cap I(F)$ has codimension ≥ 3 in $\mathbb{C}^q$. By the Division Lemma, there exist some polynomials A and S such that $R = A(F) + P(F)S$. Therefore a simple calculation yields:

$$P(F)\omega = P(F)\mathrm{d}S + \sum_{k=1}^{q}\Big(a_k + S\frac{\partial P}{\partial t_k}(F) + \frac{\partial A}{\partial t_k}(F)\Big)\mathrm{d}f_k.$$

Let c_k denote the coefficient of $\mathrm{d}f_k$ in this sum. Then $\sum_k c_k\mathrm{d}f_k$ is divisible by $P(F)$. If ω_0 is that quotient, we can see

$$\omega_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q = 0$$

which implies the second part of the theorem. If now F is quasi-fibered, then it is non-singular in codimension 1. By De Rham Lemma (see [19]), ω_0 can be written as $\sum_k d_k\mathrm{d}f_k$, where all d_k are polynomials. Therefore ω is AR-exact. ☐

6. Recalls on $(\mathbb{C}^p, +)$-actions

An algebraic $(\mathbb{C}^p,+)$-action φ on an affine variety X consists of a regular map $\varphi : \mathbb{C}^p \times X \to X$ such that:

$$\forall (u,v) \in \mathbb{C}^p \times \mathbb{C}^p,\ \forall x \in X, \quad \varphi\big(u, \varphi(v,x)\big) = \varphi(u+v, x).$$

We denote by $\mathbb{C}[X]^\varphi$ its ring of invariants, *i.e.* the space of regular functions f such that $f \circ \varphi = f$. The action φ can be defined as the composition of p pairwise commuting algebraic $(\mathbb{C},+)$-actions φ_i. These latter are the restriction of φ to the i-th coordinate of $\mathbb{C}^p$. To each φ_i corresponds the derivation $\partial_i = \varphi_i^*(\mathrm{d}/\mathrm{d}t_i)_{t_i=0}$, which enjoys the remarkable property of being locally nilpotent (see the introduction). Moreover these derivations commute pairwise. Conversely if $\{\partial_1, \ldots, \partial_p\}$ is a system of locally nilpotent pairwise commuting derivations, the exponential map

$$\exp(t_1\partial_1 + \cdots + t_p\partial_p)(f) = \sum_{k\geq 0} \frac{(t_1\partial_1 + \cdots + t_p\partial_p)^k(f)}{k!}$$

defines a morphism of algebras from $\mathbb{C}[X]$ to $\mathbb{C}[X] \otimes \mathbb{C}[t_1, \ldots, t_p]$. This morphism induces a regular map $\varphi : \mathbb{C}^p \times X \to X$ that is an $(\mathbb{C}^p, +)$-action on X. In this case, φ is said to be *generated* by $\{\partial_1, \ldots, \partial_p\}$.

Definition 6.1. — A commutative p-distribution $\mathcal{D}$ is a system of locally nilpotent pairwise commuting derivations $\partial_1, \ldots, \partial_p$. Its ring of invariants $\mathbb{C}[X]^{\mathcal{D}}$ is the intersection of the kernels of the ∂_i on $\mathbb{C}[X]$.

If φ is generated by $\mathcal{D}$, then $\mathbb{C}[X]^{\mathcal{D}}$ is the ring of invariants of φ. Indeed, by definition of φ via the exponential map, a regular function f is invariant by φ if and only if $\partial_i(f) = 0$ for any i. Recall that the action φ is free at x if the stabilizer of x is reduced to zero, or in other words if the orbit of x has dimension p. Let $[\mathcal{D}]$ be the operator defined at the introduction. We introduce its evaluation at x:

$$[\mathcal{D}](x) : (R_1, \ldots, R_p) \longmapsto \det\big(\big(\partial_i(R_j)\big)\big)(x).$$

Lemma 6.2. — *Let φ be an algebraic $(\mathbb{C}^p, +)$-action on X, and let $\mathcal{D}$ be its commutative p-distribution. Then φ is not free at x if and only if $[\mathcal{D}](x)$ is the null map.*

Proof. — Assume first that φ is not free at x. Let $(u_1, \ldots, u_p)$ be a non-zero element of the stabilizer of x. Let φ^u be the $(\mathbb{C}, +)$-action defined by

$$\varphi^u_t(y) = \varphi_{tu_1,\ldots,tu_p}(y).$$

Starting from the relation $\varphi^u_1(x) = x$, we get by an obvious induction that $\varphi^u_m(x) = x$ for any integer $m > 0$. So $\varphi^u_t(x) = x$ for any t in $\mathbb{C}$, and x is a fixed point of φ^u. For any regular function R, we get by derivation

$$\sum u_i\partial_i(R)(x) = 0$$

which implies for any p-uple $(R_1, \ldots, R_p)$:

$$[\mathcal{D}](x)(R_1, \ldots, R_p) = \det\big(\big(\partial_i(R_j)\big)\big)(x) = 0.$$

Assume now that $[\mathcal{D}](x)$ is the null map. Let $(\partial_i)_x$ be the evaluation map of ∂_i at x, *i.e.* the map $R \mapsto \partial_i(R)(x)$. As $\mathbb{C}$-linear forms on $\mathbb{C}[X]$, the $(\partial_i)_x$ are not linearly independent. There exists a non-zero p-uple $(u_1, \ldots, u_p)$ such that $\sum_i u_i(\partial_i)_x = 0$. Since the ∂_i are locally nilpotent and commute pairwise, the derivation $\delta = u_1(\partial_1) + \cdots + u_p(\partial_p)$ is itself locally nilpotent. So δ generates the action φ^u defined by $\varphi^u_t(y) = \varphi_{tu_1,\ldots,tu_p}(y)$. Since $\sum_i u_i(\partial_i)_x = 0$, x is a fixed point of φ^u as can be seen via the exponential map. Therefore the stabilizer of x is not reduced to zero. □

Let $\mathcal{D} = \{\partial_1, \dots, \partial_p\}$ be a commutative p-distribution on $\mathbb{C}[X]$. Since the exponential map defines a morphism of algebras, the map

$$\deg_D : \mathbb{C}[X] \longrightarrow \mathbb{N} \cup \{-\infty\},\ f \longmapsto \deg_{t_1,\dots,t_p}\bigl\{\exp(t_1\partial_1 + \cdots + t_p\partial_p)(f)\bigr\}$$

satisfies all the axioms of a degree function: This is the *degree relative to* $\mathcal{D}$. By construction, the ring of invariants of $\mathcal{D}$ is the set of regular functions of degree ≤ 0. If A is a domain, we denote by $\mathrm{Fr}(A)$ its fraction field. The following lemma is due to Makar-Limanov [12].

Lemma 6.3. — *Let A be a domain of characteristic zero. Let ∂ be a non-zero locally nilpotent derivation on A and let A^∂ be its kernel. Then* $\mathrm{Fr}(A)$ *is isomorphic to* $\mathrm{Fr}(A^\partial)(t)$. *In particular, for any subfield k of* $\mathrm{Fr}(A^\partial)$, *the transcendence degrees satisfy the relation:*

$$\deg\mathrm{tr}_k\bigl\{\mathrm{Fr}(A^\partial)\bigr\} = \deg\mathrm{tr}_k\bigl\{\mathrm{Fr}(A)\bigr\} - 1.$$

Proof. — Since ∂ is non-zero locally nilpotent, there exists an element f of A such that $\partial(f) \neq 0$ and $\partial^2(f) = 0$. So $g = \partial(f)$ is invariant. It is then easy to check by induction on p that every element P of A, of degree p for ∂, can be written in a unique way as $g^pP = a_0 + \cdots + a_pf^p$, where all the a_i are invariant. □

We end these recalls with the *factorial closedness* property, which is essential for rings of invariants (see [4], [5]).

Definition 6.4. — Let B a UFD and let A be a subring of B. A is factorially closed in B if every element P of B which divides a non-zero element Q of A belongs to A.

Lemma 6.5. — *Let X be an affine variety such that $\mathbb{C}[X]$ is a UFD. Let $\mathcal{D}$ be a commutative p-distribution on X. Then $\mathbb{C}[X]^{\mathcal{D}}$ is factorially closed in $\mathbb{C}[X]$.*

Proof. — Let Q be a non-zero element of $\mathbb{C}[X]^{\mathcal{D}}$, and let P divide Q in $\mathbb{C}[X]$. By considering the degree relative to $\mathcal{D}$, we get

$$\deg_{\mathcal{D}}(Q) = \deg_{\mathcal{D}}(P) + \deg_{\mathcal{D}}(Q/P) = 0.$$

This implies $\deg_{\mathcal{D}}(P) = 0$, and P is invariant with respect to $\mathcal{D}$. □

7. Jacobian description of p-distributions

Let φ be an algebraic $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$, satisfying the condition (H). Let $\mathcal{D}$ be its commutative p-distribution, and let F be its quotient map. In this section we are going to prove proposition 1.7. The main idea is to construct a system of rational coordinates for which calculations will be simple. We obtain this system by adding some polynomials s_i to $f_1, \dots, f_{n-p}$. By analogy with $(\mathbb{C},+)$-actions, we denote them as "rational slices" (see [4], [6]). With these coordinates, we show there exists an invariant fraction E such that $[\mathcal{D}] = E \times J$, and there only remains to show that E is a polynomial.

DEFINITION 7.1. — Let $\mathcal{D}$ be a commutative p-distribution on $\mathbb{C}[x_1, \dots, x_n]$. A diagonal system of rational slices is a collection $\{s_1, \dots, s_p\}$ of polynomials such that the matrix $(\partial_i(s_j))$ is diagonal and all its diagonal coefficients are non-zero invariant with respect to $\mathcal{D}$.

LEMMA 7.2. — *Every commutative p-distribution $\mathcal{D}$ satisfying condition (H) admits a diagonal system of rational slices $\{s_1, \dots, s_p\}$.*

Proof. — Let $\mathcal{D}_k$ be the commutative $(p-1)$-distribution $\{\partial_1, \dots, \partial_{k-1}, \partial_{k+1}, \dots, \partial_p\}$, and let $\mathbb{C}[x_1, \dots, x_n]^{\mathcal{D}_k}$ be its ring of invariants. By induction on Lemma 6.3, we get:

$$\deg \operatorname{tr}_{\mathbb{C}} \operatorname{Fr}\bigl(\mathbb{C}[x_1, \dots, x_n]^{\mathcal{D}_k}\bigr) \geq n - p + 1.$$

Since $\mathbb{C}[x_1, \dots, x_n]^{\mathcal{D}}$ is isomorphic to a polynomial ring in $n - p$ variables, ∂_k cannot be identically zero on $\mathbb{C}[x_1, \dots, x_n]^{\mathcal{D}_k}$. For any k, there exists a polynomial s_k such that $\partial_k(s_k) \neq 0$, $\partial_k^2(s_k) = 0$ and $\partial_i(s_k) = 0$ if $i \neq k$. The collection $\{s_1, \dots, s_p\}$ is a diagonal system of rational slices. □

LEMMA 7.3. — *Let $\mathcal{D}$ be a commutative p-distribution satisfying the condition (H). Let $\{s_1, \dots, s_p\}$ be a diagonal system of rational slices. Then the map $G = (s_1, \dots, s_p, f_1, \dots, f_{n-p})$ is dominating.*

Proof. — Let us show by absurd that G is dominating. Assume that G is not, and let Q be an element of $\mathbb{C}[z_1, \dots, z_p, y_1, \dots, y_{n-p}]$ such that $Q(G) = 0$. We assume Q to have minimal degree with respect to the variables $z_1, \dots, z_p$. By derivation, we get for all i:

$$\frac{\partial Q}{\partial z_i}(G)\partial_i(s_i) = \partial_i\bigl(Q(G)\bigr) = 0.$$

Since $\partial_i(s_i) \neq 0$, this implies $\partial Q/\partial z_i(G) = 0$. By minimality of the degree, we deduce that $\partial Q/\partial z_i = 0$ for all i. So Q belongs to $\mathbb{C}[y_1, \dots, y_{n-p}]$. Therefore the f_i are not algebraically independent, and we obtain:

$$\deg \operatorname{tr}_{\mathbb{C}} \mathbb{C}(F) < n - p.$$

But $\mathbb{C}[F]$ is the ring of invariants of $\mathcal{D}$. By induction with Lemma 6.3, we find that $\deg \operatorname{tr}_{\mathbb{C}} \mathbb{C}(F) \geq n - p$, hence a contradiction. □

LEMMA 7.4. — *Let $\mathcal{D}$ be a commutative p-distribution satisfying (H). Let $\{s_1, \dots, s_p\}$ be a diagonal system of rational slices. Then*

$$\mathbb{C}[x_1, \dots, x_n] \subset \mathbb{C}(f_1, \dots, f_{n-p})[s_1, \dots, s_p].$$

Proof. — Let us show by induction on $r \geq 0$ that every polynomial of degree r with respect to $\mathcal{D}$ belongs to $\mathbb{C}(f_1,\dots,f_{n-p})[s_1,\dots,s_p]$. For $r=0$, this is obvious because every polynomial of degree zero is invariant, and belongs to $\mathbb{C}[f_1,\dots,f_{n-p}]$. Assume the property holds to the order r. Let R be a polynomial of degree $r+1$ with respect to $\mathcal{D}$. By definition, the polynomials $\partial_i(R)$ have all degree $\leq r$. By induction, there exist some elements P_i of $\mathbb{C}(y_1,\dots,y_{n-p})[z_1,\dots,z_p]$ such that $\partial_i(R) = P_i(G)$ for all i. Since $\mathcal{D}$ is commutative, we get for all (i,j):

$$\frac{\partial P_j}{\partial z_i}(G)\partial_i(s_i) = \partial_i \circ \partial_j(R) = \partial_j \circ \partial_i(R) = \frac{\partial P_i}{\partial z_j}(G)\partial_j(s_j).$$

By construction, there exists a non-zero polynomial S_i in $\mathbb{C}[y_1,\dots,y_{n-p}]$ such that $\partial_i(s_i) = S_i(F)$. Since G is dominating, this yields for all (i,j):

$$S_i\frac{\partial P_j}{\partial z_i} = S_j\frac{\partial P_i}{\partial z_j}\cdot$$

The differential 1-form $\omega = \sum P_i/S_i \mathrm{d}z_i$ is polynomial in the variables z_i. By the above equality, ω is closed with respect to z_i. So ω is exact and there exists an element P of $\mathbb{C}(y_1,\dots,y_{n-p})[z_1,\dots,z_p]$ such that $\omega = \mathrm{d}P$. Therefore $\partial_i(R - P\circ G) = 0$ for all i, and the function $R - P\circ G$ is rational and invariant with respect to $\mathcal{D}$. Since the ring of invariants of $\mathcal{D}$ is factorially closed, $R - P\circ G$ belongs to $\mathbb{C}(f_1,\dots,f_{n-p})$. So R belongs to $\mathbb{C}(f_1,\dots,f_{n-p})[s_1,\dots,s_p]$, hence proving the induction. □

Following exactly the same argument, we can prove the equality

$$\mathbb{C}[x_1,\dots,x_n] = \mathbb{C}[f_1,\dots,f_{n-p}][s_1,\dots,s_p]$$

if the matrix $(\partial_i(s_j))$ is the identity. In this case G is an algebraic automorphism. In any case, the previous lemma asserts that G is always a birational automorphism of $\mathbb{C}^n$.

Lemma 7.5. — *Let $\mathcal{D}$ be a commutative p-distribution satisfying (H). Let $\{s_1,\dots,s_p\}$ be a diagonal system of rational slices. Then*

$$\partial_1(s_1)\dots\partial_p(s_p) \times J = J(s_1,\dots,s_p) \times [\mathcal{D}].$$

Proof. — For any p-uple of polynomials $(R_1,\dots,R_p)$, there exist some rational functions P_i such that $R_i = P_i(G)$. On one hand, we get by the chain rule:

$$\begin{aligned} J(R_1,\dots,R_p) &= \det\bigl(\mathrm{d}(P_1,\dots,P_p,y_1,\dots,y_{n-p})\bigr)(G)\det(\mathrm{d}G)\\ &= \det\bigl((\partial P_i/\partial z_j)\bigr)(G)J(s_1,\dots,s_p). \end{aligned}$$

On the other hand, we have the following relation:

$$[\mathcal{D}](R_1,\dots,R_p) = \det\bigl((\partial_i(R_j))\bigr) = \det\Bigl(\Bigl(\sum_k \frac{\partial P_j}{\partial z_k}(G)\partial_i(s_k)\Bigr)\Bigr).$$

Since the matrix $(\partial_i(s_j))$ is diagonal, this yields

$$[\mathcal{D}](R_1,\ldots,R_p)=\det\Big(\Big(\frac{\partial P_i}{\partial z_j}\Big)\Big)(G)\partial_1(s_1)\cdots\partial_p(s_p)$$

which implies $\partial_1(s_1)\cdots\partial_p(s_p)J(R_1,\ldots,R_p)=J(s_1,\ldots,s_p)\times[\mathcal{D}](R_1,\ldots,R_p)$. □

LEMMA 7.6. — *Let $\mathcal{D}$ be a commutative p-distribution satisfying the condition (H). Let $\{s_1,\ldots,s_p\}$ be a diagonal system of rational slices. Then $J(s_1,\ldots,s_p)$ is invariant.*

Proof. — For simplicity, we denote by J' the jacobian of every map from $\mathbb{C}^n$ to $\mathbb{C}^n$. Since $\{s_1,\ldots,s_p\}$ is a diagonal system of rational slices, we get via the exponential map the relation $s_i\circ\varphi=s_i+t_i\partial_i(s_i)$, and this yields:

$$\begin{aligned}J'(s_1\circ\varphi,\ldots,s_p\circ\varphi,f_1\circ\varphi,\ldots,f_{n-p}\circ\varphi)&\\ =J'\big(s_1+t_1\partial_1(s_1),\ldots,s_p+t_p\partial_p(s_p),f_1,\ldots,f_{n-p}\big).&\end{aligned}$$

Since every $\partial_i(s_i)$ belongs to $\mathbb{C}[F]$, we deduce:

$$\begin{aligned}J'(s_1\circ\varphi,\ldots,s_p\circ\varphi,f_1\circ\varphi,\ldots,f_{n-p}\circ\varphi)&=J'(s_1,\ldots,s_p,f_1,\ldots,f_{n-p})\\&=J(s_1,\ldots,s_p).\end{aligned}$$

Moreover we find by the chain rule:

$$J'(s_1\circ\varphi,\ldots,s_p\circ\varphi,f_1\circ\varphi,\ldots,f_{n-p}\circ\varphi)=J'(s_1,\ldots,s_p,f_1,\ldots,f_{n-p})(\varphi)\times J'(\varphi).$$

Since φ is an automorphism of $\mathbb{C}^n$ for any $(t_1,\ldots,t_p)$, the polynomial $J'(\varphi)$ never vanishes. So it is non-zero constant. As $\varphi_{0,\ldots,0}$ is the identity, $J'(\varphi)\equiv 1$ and that implies

$$J(s_1\circ\varphi,\ldots,s_p\circ\varphi,f_1\circ\varphi,\ldots,f_{n-p}\circ\varphi)=J(s_1,\ldots,s_p,f_1,\ldots,f_{n-p})(\varphi)$$

which leads to $J(s_1,\ldots,s_p)(\varphi)=J(s_1,\ldots,s_p)$. Thus $J(s_1,\ldots,s_p)$ is invariant. □

Proof of Proposition 1.7. — Let $\mathcal{D}$ be a commutative p-distribution satisfying the condition (H). By Lemmas 7.5 and 7.6, there exist two non-zero invariant polynomials E_1 and E_2 such that:

$$E_1\times[\mathcal{D}]=E_2\times J.$$

Since $\mathbb{C}[F]$ is factorially closed in $\mathbb{C}[x_1,\ldots,x_n]$, we may assume that E_1 and E_2 have no common factor. Let us show by absurd that E_1 is non-zero constant. Assume that E_1 is not constant. By definition of J, E_1 divides all the coefficients of the $(n-p)$-form $\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_{n-p}$. So the hypersurface $V(E_1)$ is contained in the singular set of F. But that contradicts a result of Daigle [4], that asserts that F is non-singular in codimension 1. □

8. Trivialisation of algebraic $(\mathbb{C}^p,+)$-actions

In this section, we are going to establish Theorem 1.8. The main idea is to refine a diagonal system of rational slices, in order to get the coordinate functions of an algebraic automorphism that conjugates φ to the trivial action.

Proof of Theorem 1.8. — Let φ be an algebraic $(\mathbb{C}^p,+)$-action on $\mathbb{C}^n$ satisfying the condition (H). Assume that E is constant and that the quotient map F is quasi-fibered. Let $\{s_1,\dots,s_p\}$ be a diagonal system of rational slices. Such a system exists by Lemma 7.2. By Proposition 1.7, we have for any $(p-1)$-uple $(R_1,\dots,R_{i-1},R_{i+1},\dots,R_p)$:

$$J(R_1,\dots,R_{i-1},s_i,R_{i+1},\dots,R_p)=[\mathcal{D}](R_1,\dots,R_{i-1},s_i,R_{i+1},\dots,R_p)/E.$$

Let P_i be the polynomial of $\mathbb{C}[t_1,\dots,t_{n-p}]$ such that $\partial_i(s_i)=P_i(F)$. Since E is constant and $\partial_k(s_i)=0$ if $k\neq i$, the previous equality yields:

$$J(R_1,\dots,R_{i-1},s_i,R_{i+1},\dots,R_p)\equiv 0 \quad \big[P_i(F)\big].$$

If we replace R_k by all the polynomials $x_1,\dots,x_n$, we can see that the coefficients of the differential form $ds_i\wedge \mathrm{d}f_1\wedge\cdots\wedge \mathrm{d}f_{n-p}$ are all divisible by $P_i(F)$. By Daigle's result [4], F is non-singular in codimension 1. So the coefficients of $\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_{n-p}$ have no common factor. Therefore s_i satisfies the equation:

$$\mathrm{d}s_i\wedge\omega_F\equiv 0 \quad \big[P_i(F)\big].$$

By the Division Lemma, there exist some polynomials A_i,S_i such that:

$$s_i=A_i(F)+P_i(F)S_i.$$

By an easy computation, we obtain that $(\partial_i(S_j))$ is the identity. By the remark following Lemma 7.5, we have the equality

$$\mathbb{C}[x_1,\dots,x_n]=\mathbb{C}[f_1,\dots,f_{n-p}][S_1,\dots,S_p]$$

which implies that $G=(S_1,\dots,S_p,f_1,\dots,f_{n-p})$ is an algebraic automorphism of $\mathbb{C}^n$. Let φ_0 be the trivial action generated by the commutative p-distribution $\{\partial/\partial x_1,\dots,\partial/\partial x_p\}$. By using the exponential map, we find that $G\circ\varphi=\varphi_0\circ G$. So φ is trivial. □

Proof of Corollary 1.9. — Let φ be an algebraic $(\mathbb{C},+)$-action on $\mathbb{C}^n$ satisfying (H), generated by the derivation ∂. Assume that the quotient map is quasi-fibered. Since F is nonsingular in codimension 1, the derivation J is locally nilpotent and generates a $(\mathbb{C},+)$-action φ' such that $\mathcal{NL}(\varphi')$ has codimension ≥ 2. By Theorem 1.8, φ' is trivial. Moreover via the automorphism of trivialisation, ∂ is conjugate to $P(x_2,\dots,x_n)\partial/\partial x_1$, where $E=P(F)$ is the factor of Proposition 1.7. □

Proof of Corollary 1.10. — Let φ be an algebraic $(\mathbb{C}^{n-1},+)$-action on $\mathbb{C}^n$, and assume that $\mathcal{NL}(\varphi)$ has codimension ≥ 2. Then the factor E of Proposition 1.7 is constant. Let us prove that φ is trivial. By Theorem 1.8, we only have to show that φ satisfies the condition (H) and that its quotient map is quasi-fibered.

Let f be a non-constant invariant polynomial of minimal homogeneous degree on $\mathbb{C}[x_1,\dots,x_n]$. Then $f-\lambda$ is irreducible for any λ. Indeed if $f-\lambda$ were reducible, all its irreducible factors would be invariant by factorial closedness. But that contradicts the minimality of the degree of f. Since all the fibres of f are irreducible, they are reduced and connected. So f is quasi-fibered, and there only remains to prove that f generates the ring of invariants of φ.

Let us show by induction on r that any invariant polynomial P of homogeneous degree $\leq r$ belongs to $\mathbb{C}[F]$. This is obvious for $r=0$. Assume this is true to the order r, and let P be an invariant polynomial of degree $\leq r+1$. Let x be a point in $\mathbb{C}^n$ where φ is free, and set $y=f(x)$. Since P is invariant, P is constant on the orbit of x. Since this orbit has dimension $n-1$ and that $f^{-1}(y)$ is irreducible, this orbit is dense in $f^{-1}(y)$. So P is constant on $f^{-1}(y)$. By Hilbert's Nullstellensatz, there exists a polynomial Q such that $P=P(x)+(f-y)Q$. The polynomial Q is invariant by factorial closedness and has degree $\leq r$. By induction, Q belongs to $\mathbb{C}[F]$, and so does P, hence giving the result. □

9. A few examples

We can show that the first assertion in Theorem 1.5 is an equivalence. More precisely, a primitive mapping F is quasi-fibered if and only if $\mathcal{T}^1(F)=0$. We will not prove it here, but we would rather give two examples illustrating the necessity of the conditions given in Theorem 1.8. In both cases, the module of relative exactness is not zero. Consider the locally nilpotent derivation on $\mathbb{C}[x,y,z]$:

$$\partial_1 = x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}\cdot$$

Its ring of invariant is generated by x and $xz+y^2$, and its quotient map is defined by:

$$F_1 : \mathbb{C}^3 \longrightarrow \mathbb{C}^2, \quad (x,y,z) \longmapsto (x, xz+y^2).$$

It is easy to check that F_1 is surjective and that $\overline{B(F_1)} = \{(u,v)\in\mathbb{C}^2,\ u=0\}$. So F_1 is not quasi-fibered because its fibres are not 1-generically connected, and the action generated by ∂_1 is not trivial. Second consider the locally nilpotent derivation on $\mathbb{C}[x,y,u,v]$:

$$\partial_2 = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\cdot$$

The polynomials $u, v, xv-yu$ are invariant and generate the ring of invariants of ∂_2. So the corresponding action φ_2 satisfies the condition (H), and its quotient map is given by:

$$F_2 : \mathbb{C}^4 \longrightarrow \mathbb{C}^3, \quad (x,y,u,v) \longmapsto (u,v,xv-yu).$$

By an easy computation, we get that $B(F_2)$ is empty, $S(F_2) = V(x,y)$ and $I(F_2) = \{(r,0,0), r \in \mathbb{C}^*\}$. So F_2 is not quasi-fibered because its fibres are not 2-generically non-empty, and φ_2 is not trivial.

BIBLIOGRAPHY

[1] Abhyankar (S.S.) – *Local analytic geometry*, Academic Press, New York-London, 1964.

[2] Berthier (M.) & Cerveau (D.) – *Quelques calculs de cohomologie relative*, Ann. scient. Éc. Norm. Sup. 4e série, t. **26** (1993), no. 3, pp. 403–424.

[3] Bonnet (P.) & Dimca (A.) – *Relative differential forms and complex polynomials*, Bull. Sci. Math. (2), t. **124** (2000), no. 7, pp. 557–571.

[4] Daigle (D.) – *On some properties of locally nilpotent derivations*, J. Pure Appl. Algebra, t. **114** (1997), pp. 221–230.

[5] Deveney (J.K.) – *G_a-actions on $\mathbb{C}^3$ and $\mathbb{C}^7$*, Comm. Algebra, t. **22** (1994), no. 15, pp. 6295–6302.

[6] Deveney (J.K.) & Finston (D.R.) – *On locally trivial G_a-actions*, Transformation Groups, t. **2** (1997), no. 2, pp. 137–145.

[7] Dimca (A.) – *Singularities and topology of hypersurfaces*, Springer Verlag, New-York-Berlin, 1992.

[8] Dimca (A.) & Poenaru (L.) – *On the connectivity of complex affine hypersurfaces, II*, Topology, t. **39** (2000), pp. 1035–1043.

[9] Gavrilov (L.) – *Petrov modules and zeros of Abelians integrals*, Bull. Sci. Math. (2), t. **122** (1998), no. 7, pp. 571–584.

[10] Hartshorne (R.) – *Algebraic geometry*, Springer Verlag, New York-Heidelberg, 1977.

[11] Kraft (H.) – *Challenging problems on affine n-space*, in *Sém. Bourbaki 1994/1995*, Astérisque, vol. 237, Société Mathématique de France, 1996, exp. 802.

[12] Makar-Limanov (L.) – *On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$*, Israel J. Math., t. **96**, part B (1996), pp. 419–429.

[13] Malgrange (B.) – *Frobenius avec singularités, 2. Le cas général*, Invent. Math., t. **39** (1977), pp. 67–89.

[14] Matsumura (H.) – *Commutative Algebra*, Benjamin, New York, 1970.

[15] Miyanishi (M.) – *Normal affine subalgebras of a polynomial ring*, in *Algebraic and Topological Theories – to the memory of Dr. Takehito Miyata*, Kinokuniya, Tokyo, 1985, pp. 37–51.

[16] Mumford (D.) – *Geometric invariant theory*, Springer Verlag, Berlin-Heidelberg, 1965.

[17] ———, *Algebraic geometry I: Complex projective varieties*, Springer Verlag, Berlin-Heidelberg-New York, 1976.

[18] Rentschler (R.) – *Opérations du groupe additif sur le plan affine*, C. R. Acad. Sci. Paris Sér. I Math., t. **267** (1968), pp. 384–387.

[19] Saito (K.) – *On a generalisation of De Rham Lemma*, Ann. Inst. Fourier (Grenoble), t. **26** (1976), no. 2, pp. 165–170.

[20] Shafarevich (I.R.) – *Basic algebraic geometry, vol. 1–2*, Springer Verlag, Berlin-Heidelberg-New York, 1994.

[21] Winkelmann (J.) – *On free holomorphic* $\mathbb{C}$*-actions on* $\mathbb{C}^n$ *and homogeneous Stein manifolds*, Math. Ann., t. **286** (1990), pp. 593–612.