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Espaces de modules de surfaces plates et leur forme volume

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ESPACES DE MODULES DE SURFACES PLATES ET LEUR FORME VOLUME

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Résumé

Dans cette thèse, nous nous intéressons aux trois types de surfaces plates à singularités coniques suivants :

- surfaces de translation à bord géodésique,

- surfaces avec forêt effaçante, et

- surfaces plates homéomorphes à la sphère $\mathbb{S}^2$.

Nous étudions les espaces de modules de ces surfaces et relions leurs propriétés aux propriétés de l'espace de modules des surfaces de translation.

Les résultats principaux de cette thèse sont les suivants : nous montrons tout d'abord que les espaces de modules en question sont tous des orbifolds. Plus précisement, ces espaces sont des quotients des variétés plates affines complexes par des groupes agissant proprement discontinument. Dans un deuxième temps, nous construisons de manière uniforme une forme volume sur chacun de ces espaces. Notons que les surfaces de translation (fermées) sont un cas particulier des surfaces de translation à bord géodésiques. Dans ce cas, notre forme volume est égale, à une constante multiplicative près, à la forme volume habituelle définie par l'application de périodes.

Dans [Th], Thurston étudie l'espace de modules des surfaces plates polyèdrales, il montre que cet espace est muni d'une structure métrique hyperbolique complexe. Nous montrerons que la forme volume induite par la métrique hyperbolique complexe coïncide, à une constante multiplicative près, avec notre forme volume.

Pour les surfaces de translation à bord géodésique dont le bord est non-vide, ainsi que les surfaces avec forêt effaçante, nous définissons des fonctions d'énergie sur leur espace de modules qui tiennent compte de l'aire de la surface, et de la longueur du bord, ou des arbres. Nous montrons que les volumes de ces espaces renormalisés par cette énergie sont finis. Nous retrouvons, comme cas particuliers, le fait que l'espace de modules des surfaces de translation, et l'espace de modules des structures métriques plates sur la sphère sont de volume fini.

Abstract

In this thesis, we are interested in three types of flat surfaces :

- translation surfaces with geodesic boundary,
- flat surfaces with erasing forest, and
- spherical flat surfaces.

We study the moduli spaces of those surfaces, and relate their properties to those of moduli spaces of (closed) translation surfaces.

The main results of this thesis are the followings : first, we prove that the moduli spaces under consideration are orbifolds. More precisely, they are quotients of flat complex affine manifolds by some groups acting properly discontinuously. Next, we define a volume form on each of those moduli spaces by similar method. Note that (closed) translation surfaces are a particular case of translation surfaces with geodesic boundary. In this case, up to a multiplication constant, our volume form equals the usual one, which is defined by the period mapping.

In [Th], Thurston studies the moduli space of flat surfaces isometric to polyhedra, he shows that this moduli space can be equipped with a complex hyperbolic metric structure. We prove that the volume form induced by the complex hyperbolic metric and our volume form coincide, up to a multiplication constant.

For translation surfaces with geodesic boundary, and flat surfaces with erasing forest, we define some energy functions, which involve the area of the surface, and the length of its boundary, or the total length of the trees in the forest, on their moduli spaces respectively. We prove that the volumes of our moduli spaces normalized by these energy functions are finite. We deduce from this result the fact that the volumes of the moduli space of translation surfaces, and the volume of the moduli space of flat metric structures on the sphere are finite.

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Chapitre 1

Introduction

1.1 Surface plate à singularités coniques

Soit Σ une surface compacte, fermée, orientée, c'est-à-dire une variété de dimension 2, compacte, sans bord. On dit que Σ est une *surface plate à singularités coniques* lorsqu'elle est munie d'une structure métrique Euclidienne en dehors d'un sous-ensemble fini *Sing* telle que, pour tout x appartenant à *Sing*, un voisinage de x est modelé sur un cône. Les premiers exemples de telles surfaces sont des polyèdres avec la métrique induite par la métrique Euclidienne de $\mathbb{R}^3$. Pour ces surfaces, les seuls points singuliers sont les sommets, les points à l'intérieur d'une face sont évidemment réguliers, ainsi que les points à l'intérieur d'une arête car ceux-ci ont un voisinage isométrique à l'union de deux demi-disques plongés dans $\mathbb{R}^2$. Dans le cas des polyèdres, tout sommet admet un voisinage isométrique à un cône dont l'angle au sommet est strictement plus petit que 2π. Les surfaces plates en géneral ne vérifient pas cette propriété.

Les tores plats, *i.e.* quotients de $\mathbb{R}^2$ par des réseaux $\mathbb{Z}u \oplus \mathbb{Z}v$, avec $u, v \in \mathbb{R}^2$ indépendants, sont d'autres exemples de surfaces plates. On construit également des surfaces plates dont le genre est plus grand que 1 (avec forcément des singularités), par exemple par revêtement ramifié des tores plats.

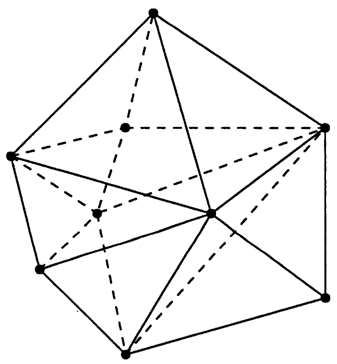
Pour les surfaces à bord, nous introduisons la notion de *surface plate à singularités coniques et à bord géodésique*, pour simplifier, que nous appelons surfaces plates à bord géodésique pour simplifier. Une surface plate à bord géodésique est une surface dont l'intérieur est munie d'une structure surface plate à singularités coniques (comme ci-dessus), et dont le bord est une union finie de segments géodésiques. Les exemples les plus simples de telles surfaces sont des polygones munis de la métrique induite par celle de $\mathbb{R}^2$. Comme dans le cas des surfaces fermées, on peut avoir des surfaces plates à bord géodésique de tout genre.

Il existe un lien important entre l'étude des surfaces plates et la théorie de surface de Riemann : si Σ est une surface plate, alors la structure surface plate induit une structure conforme sur $\Sigma \setminus \{Sing\}$ qui s'étend uniquement en une structure conforme de Σ, et on a ainsi une surface de Riemann avec des points marqués qui sont les points singuliers de Σ. Inversement, étant donnée une surface de Riemann Σ avec des points marqués, un théorème de Troyanov assure qu'il existe dans la classe conforme de Σ une structure surface plate à singularités coniques dont les points singuliers sont les points marqués, avec les angles coniques fixés, de plus, une telle structure est unique à homothétie près (voir [Tr1]).

Les espaces de modules des surfaces plates ayant des singularités coniques fixées sont l'objet de nombreuses recherches, un bref aperçu des résultats concernant ce sujet est présenté dans les paragraphes qui suivent.

1.2 Métrique polyèdrale sur la sphère

Dans son article [Th], Thurston s'intéresse aux espaces de modules des surfaces plates isométriques aux polyèdres. Soit x un point singulier sur une surface plate, dont le voisinage est isométrique à un cône d'angle θ. On appelle le nombre $2\pi - \theta$ la *courbure* en x. Pour toute surface plate isométrique à un polyèdre, tous les points singuliers sont de courbure positive. Par le théorème de Gauss-Bonnet, la somme de courbures de tous les points singuliers d'une surface plate polyèdre doit être égale à 4π.



Soient $\kappa_1, \dots, \kappa_n$, $(n \geqslant 3)$, n nombres réels appartenant à l'intervalle $(0, 2\pi)$, et vérifiant :

$$\kappa_1 + \dots + \kappa_n = 4\pi.$$

On note $C(\kappa_1, \dots, \kappa_n)$ l'espace de modules des surfaces plates homéomorphes à $\mathbb{S}^2$, ayant n points singuliers de courbures $(\kappa_1, \dots, \kappa_n)$ à homothétie près. Cet espace n'est pas complet en général : si $\kappa_i + \kappa_j < 2\pi$, alors la distance entre les points singuliers de courbures κ_i et κ_j peut être réduite à zéro de façon que l'aire de la surface limite reste finie. On peut donc compléter $C(\kappa_1, \dots, \kappa_n)$ par les espaces $C(\tilde{\kappa}_{I_1}, \dots, \tilde{\kappa}_{I_k})$, où $(I_1, \dots, I_k)$ est une partition de l'ensemble $\{1, \dots, n\}$, et

$$\tilde{\kappa}_{I_j} = \sum_{i \in I_j} \kappa_i < 2\pi.$$

Pour ces espaces de modules, Thurston obtient le résultat suivant :

Théorème (Thurston) *Soient* $(\kappa_1, \ldots, \kappa_n)$, $(n \geqslant 3)$, *n nombres réels dans l'intervalle* $(0, 2\pi)$ *dont la somme est* 4π. *Alors, l'espace de modules* $C(\kappa_1, \ldots, \kappa_n)$ *est une variété hyperbolique complexe de dimension* $n-3$, *dont la complétion est une variété hyperbolique complexe à cônes de volume fini. La complétion de* $C(\kappa_1, \ldots, \kappa_n)$ *est un orbifold si et seulement si pour tout couple* (κ_i, κ_j) *tel que* $i \neq j$ *et* $s = \kappa_i + \kappa_j < 2\pi$, *on a :*

i) Soit $(2\pi - s)$ *divise* 2π,

ii) Soit $\kappa_i = \kappa_j$ *et* $\pi - \kappa_i$ *divise* 2π.

Pour construire les cartes locales, Thurston utilise des triangulations par segments géodésiques des surfaces dans $C(\kappa_1, \ldots, \kappa_n)$, en associant aux $n-2$ arêtes particulières $n-2$ nombres complexes obtenus par une application developpante. Par cette construction, le voisinage d'un point dans $C(\kappa_1, \ldots, \kappa_n)$ est identifié au quotient d'un ouvert dans $\mathbb{C}^{n-2}$ par l'action de $\mathbb{C}^*$.

Dans ces coordonnées, l'aire d'une surface dans $C(\kappa_1, \ldots, \kappa_n)$ est donnée par une forme Hermitienne $\mathbf{H}$ de signature $(1, n-3)$. Plus précisement, si S est la surface dans $C(\kappa_1, \ldots, \kappa_n)$ représentée par un vecteur $Z \in \mathbb{C}^{n-2}$, alors l'aire de S est donnée par ${}^t\overline{Z} \cdot \mathbf{H} \cdot Z$. La métrique hyperbolique complexe de $C(\kappa_1, \ldots, \kappa_n)$ est la métrique qui est induite localement par la forme Hermitienne $\mathbf{H}$ sur le quotient.

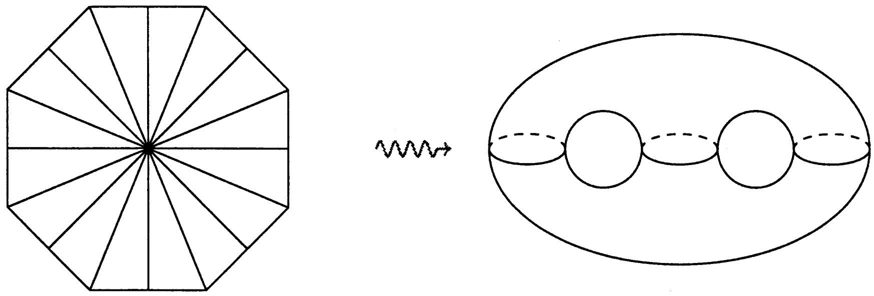
1.3 Surface de translation

Soient Σ une surface plate à singularités coniques, et γ une courbe fermée contenue dans $\mathrm{int}(\Sigma) \setminus \{singularités\}$. Soit p un point de γ, on note $Hol_p(\gamma)$ l'holonomie de γ considérée comme un lacet avec point de base p. En général, $Hol_p(\gamma)$ est un élément de $SO(2) \ltimes \mathbb{R}^2$, le groupe d'isométries de $\mathbb{E}^2$ ($\mathbb{R}^2$ muni de la métrique Euclidienne) préservant l'orietation.

Si Σ est une surface telle que pour toute courbe fermée γ dans $\mathrm{int}(\Sigma) \setminus \{singularités\}$, l'holonomie de γ est une translation (dans ce cas le point de base n'a pas d'importance), alors on dit que Σ est une *surface de translation.* Une caractéristique des surfaces de translation est qu'un rayon géodésique ne s'intersecte jamais lui-même transversalement, autrement-dit, soit le rayon est une géodésique fermée, soit il rencontre un point singulier, soit il se prolonge infiniment. Par conséquent, étant donnée une direction $\theta \in [0, 2\pi)$, on peut définir un feuilletage sur une surface de translation en géodésiques dans cette direction.

Si x est un point singulier d'une surface de translation Σ, l'angle du cône en x doit être un multiple entier de 2π. Notons que cette propriété est nécessaire mais pas suffisante pour caractériser les surfaces de translation.

Il est clair que les tores plats sont des surfaces de translations mais ils ne sont pas les seuls. Pour construire un exemple de surface de translation qui n'est pas un tore, considérons un octogone dont les côtés opposés sont parallèles et de même longueur. En recollant les côtes opposés de cet octagone, on obtient une surface compacte, sans bord, de genre 2. Comme les identifications sont des isométries de $\mathbb{E}^2$, cette nouvelle surface hérite de l'octagone au départ une structure métrique plate à singularités côniques. Remarquons que les huit sommets de l'octagone s'identifient en un seul point de la surface, qui est l'unique point singulier dont l'angle conique est 6π. Puisque les côtés opposés de l'octagone sont parallèles, leur identification est réalisée par une translation de $\mathbb{R}^2$, par conséquent, l'holonomie de toute courbe fermée ne passant pas par le point singulier de la surface est une translation, on peut donc conclure que la surface obtenue est bien une surface de translation.



En parallèle avec des surfaces de translation, on a aussi la notion de surface de demi-translation. Une *surface de demi-translation* est une surface plate telle que l'holonomie de toute courbe fermée est un élément du group $\{\pm \mathrm{Id}\} \ltimes \mathbb{R}^2$. Comme le cas des surfaces de translation, un segment géodésique sur une surface de demi-translation n s'intersecte jamais lui-même transversalement. Il s'ensuit qu'étant donnée une direction $\theta \in [0;\pi)$, on peut définir un feuilletage d'une telle surface en géodésiques parallèles à cette direction. Une condition nécessaire mais pas suffisante pour avoir une surface de demi-translation est que l'angle du cône en tout point singulier doit être un multiple entier de π. Un exemple de surface de demi-translation est la sphère $\mathbb{S}^2$ munie d'une métrique plate avec 4 points singuliers dont les angles coniques sont tous égaux à π.

Dans la suite de ce paragraphe, nous allons rappeler quelques propriétés importantes de l'espace de modules des surfaces de translation.

1.3.1 Espace de modules

Notons d'abord que l'on a l'identification suivante :

$$\left\{\begin{array}{l}\text{Surface de translation d'aire finie avec}\\ \text{un feuilletage en droites parallèles}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{1-forme holomorphe sur une}\\ \text{surface de Riemann}\end{array}\right\}.$$

Fixons les entiers $g \geqslant 2$, et $k_1, \dots, k_n,\ k_i \geqslant 1,\ i = 1, \dots, n$, tels que

$$k_1 + \cdots + k_n = 2g - 2 \tag{1.1}$$

On note $\mathcal{H}(k_1, \dots, k_n)$ l'ensemble des couples (M, ω) à isomorphisme près, où M est une surface de Riemann compacte, sans bord de genre g, et ω est une 1-forme holomorphe définie sur M dont les zéros sont d'ordre $k_1, \dots, k_n$. Deux couples (M, ω) et (M', ω') sont isomorphes s'il existe un isomorphisme de surfaces de Riemann $h : M \longrightarrow M'$ tel que $h^*\omega' = \omega$.

Par le théorème de Riemann-Roch, pour qu'une telle 1-forme existe, les entiers $g, k_1, \dots, k_n$ doivent vérifier (1.1). On appelle $\mathcal{H}(k_1, \dots, k_n)$ une *strate* de l'espace de modules des 1-formes holomorphes. En utilisant l'identification ci-dessus, on peut considérer $\mathcal{H}(k_1, \dots, k_n)$ comme l'espace de modules des surfaces de translations ayant n singularités d'angles $(k_1 + 1)2\pi, \dots, (k_n + 1)2\pi$, avec un feuilletage en droites parallèles spécifié.

Il est bien connu que $\mathcal{H}(k_1, \dots, k_n)$ est un orbifold complexe algébrique, et que

$$\dim_{\mathbb{C}} \mathcal{H}(k_1, \dots, k_n) = 2g + n - 1.$$

1.3.2 Forme volume

Soit (M, ω) un point dans $\mathcal{H}(k_1, \dots, k_n)$, on note $p_1, \dots, p_n$ les n zéros de ω. Soient $\gamma_1, \dots, \gamma_{2g+n-1}$ une famille de courbes sur M qui représente une base dans $H_1(M, \{p_1, \dots, p_n\}; \mathbb{Z})$ telle que $\{\gamma_1, \dots, \gamma_{2g}\}$ forment une base symplectique standard de $H_1(M, \mathbb{Z})$, et γ_{2g+i} est un arc joignant p_1 à p_{i+1}.

Considérons l'application suivante dite *application de périodes* :

$$\begin{array}{rccl}\Phi : & \mathcal{U} & \longrightarrow & \mathbb{C}^{2g+n-1} \simeq \mathbb{R}^{2(2g+n-1)}\\ & (M, \omega) & \longmapsto & (\int_{\gamma_1} \omega, \dots, \int_{\gamma_{2g+n-1}} \omega).\end{array}$$

où $\mathcal{U}$ est un voisinage de (M, ω) dans $\mathcal{H}(k_1, \dots, k_n)$.

Cette application est une carte locale de $\mathcal{H}(k_1, \dots, k_n)$. Soit $\phi \in \mathbb{C}^{2g+n-1}$ l'image de (M, ω) par Φ, alors l'aire de M est donnée dans cette carte locale par la formule suivante :

$$\mathbf{Aire}_\omega(M) = \frac{\imath}{2} \int_M \omega \wedge \overline{\omega} = \frac{1}{2} \sum_{i=1}^{g} (\phi_i \overline{\phi}_{g+i} - \overline{\phi}_i \phi_{g+i}).$$

Soit $\lambda_{2(2g+n-1)}$ la mesure de Lebesgue de $\mathbb{C}^{2g+n-1}$. Considérons la forme volume $\mu_0 = \Phi^* \lambda_{2(2g+n-1)}$ définie au voisinage de (M, ω). Comme les bases de $H_1(M, \{p_1, \dots, p_n\}; \mathbb{Z}) \simeq \mathbb{Z}^{2g+n-1}$ sont liées par des matrices dans $SL(2g+n-1,\mathbb{Z})$, la forme volume μ_0 ne dépend pas du choix de la famille $\{\gamma_1,\dots,\gamma_{2g+n-1}\}$, et est donc bien définie sur $\mathcal{H}(k_1,\dots,k_n)$.

Considérons maitenant le sous-ensemble $\mathcal{H}_1(k_1,\dots,k_n)$ de $\mathcal{H}(k_1,\dots,k_n)$ qui contient tous les couples (M,ω) tels que

$$\int_M \omega\wedge\overline{\omega}=1.$$

Dans une carte locale défine par l'application de périodes Φ, l'ensemble $\mathcal{H}_1(k_1,\dots,k_n)\cap\mathcal{U}$ est envoyé sur un ouvert dans

$$\mathbf{Q}_1=\{\phi\in\mathbb{C}^{2g+n-1}\mid\frac{1}{2}\sum_{i=1}^{g}(\phi_i\overline{\phi}_{g+i}-\overline{\phi}_i\phi_{g+i})=1\}.$$

La mesure de Lebesgue $\lambda_{2(2g+n-1)}$ induit naturellement une forme volume $\lambda^1_{2(2g+n-1)}$ sur $\mathbf{Q}_1$. Soit $\mu_0^1=\Phi^*\lambda^1_{2(2g+n-1)}$, on en déduit que μ_0^1 est une forme volume bien définie sur $\mathcal{H}_1(k_1,\dots,k_n)$.

Le théorème suivant a été démontré par H.Masur, et W.A.Veech

Théorème (H.Masur, W.A. Veech) *Le volume de chaque strate $\mathcal{H}_1(k_1,\dots,k_n)$ est fini :*

$$\mathbf{Vol}(\mathcal{H}_1(k_1,\dots,k_n))=\int_{\mathcal{H}_1(k_1,\dots,k_n)}d\mu_0^1<\infty.$$

Dans un article récent [EO], A. Eskin et A. Okounkov donnent une méthode pour calculer le volume des strates $\mathcal{H}_1(k_1,\dots,k_n)$.

1.3.3 Action de $SL_2(\mathbb{R})$

Soient Σ une surface de translation. Etant donné un élément A du groupe $SL_2(\mathbb{R})$, on peut construire une autre surface de translation, notée par $A\cdot\Sigma$, de manière suivante : soit $\{\varphi_i, i\in\mathcal{I}\}$ un atlas définissant la structure surface de translation de Σ, on note $\{\tilde{\varphi}_i, i\in\mathcal{I}\}$ un autre atlas dont les cartes $\tilde{\varphi}_i$ sont définies par :

$$\tilde{\varphi}_i=A\circ\varphi.$$

Comme les changements de cartes $\varphi_j\circ\varphi_i^{-1}$ sont des translations de $\mathbb{R}^2$ (si leur domaine de définition est non-vide), les changements de cartes $\tilde{\varphi}_j\circ\tilde{\varphi}_i=A\circ(\varphi_j\circ\varphi_i^{-1})\circ A^{-1}$ sont aussi des translations de $\mathbb{R}^2$. Les cartes $\{\tilde{\varphi}_i,\ i\in\mathcal{I}\}$ définissent donc une structure surface de translation sur Σ, on note cette nouvelle surface $A\cdot\Sigma$. On peut vérifier sans difficulté que $A\cdot\Sigma$ a le même nombre de points singuliers avec les mêmes angles que Σ.

On obtient ainsi une action de $SL_2(\mathbb{R})$ sur l'espace de modules des surfaces de translation. Cette action de $SL_2(\mathbb{R})$ peut être réalisée plus concrètement : si Σ est une surface de translation obtenue par le recollement des polygones $P_1, \dots, P_j$ dans $\mathbb{R}^2$, alors $A \cdot \Sigma$ est la surface obtenue par le même recollement appliqué aux polygones $A(P_1), \dots, A(P_j)$.

Pour mieux comprendre cette action de $SL_2(\mathbb{R})$, soient (M, ω) un couple dans $\mathcal{H}(k_1, \dots, k_n)$, et $(\gamma_1, \dots, \gamma_{2g+n-1})$ une base de $H_1(M, \{p_1, \dots, p_n\}; \mathbb{Z})$, où $\{p_1, \dots, p_n\}$ est l'ensemble des zéros de ω. On note Σ la surface de translation définie par (M, ω), et suppose que $\gamma_i,\ i = 1, \dots, 2g+n-1$, est une union des segments géodésiques à extrémités dans $\{p_1, \dots, p_n\}$, un tel segment géodésique est appelé un *lien selle* de Σ.

Par définition, on a un homéomorphisme φ de Σ dans $A \cdot \Sigma$ qui envoie l'ensemble des points singuliers de Σ sur l'ensemble des points singuliers de $A \cdot \Sigma$.

En identifiant $\mathbb{C}$ à $\mathbb{R}^2$, pour tout $z \in \mathbb{C}$, on note $A(z)$ l'image du vecteur $z \in \mathbb{R}^2$ par A. Soit s un lien selle de Σ, alors $\varphi(s)$ est aussi un lien selle de $A \cdot \Sigma$. Supposons que $A \cdot \Sigma$ est définie par un couple (M', ω') dans $\mathcal{H}(k_1, \dots, k_n)$, on a alors :

$$\int_{\varphi(s)} \omega' = A\left(\int_s \omega\right).$$

Par conséquent, si $\Phi((M, \omega)) = (\phi_1, \dots, \phi_{2g+n-1})$ dans la carte locale associée à $\{\gamma_1, \dots, \gamma_{2g+n-1}\}$ (par l'application de périodes), alors $\Phi((M', \omega')) = (A(\phi_1), \dots, A(\phi_{2g+n-1}))$ dans la carte locale associée à $\{\varphi(\gamma_1), \dots, \varphi(\gamma_{2g+n-2})\}$. On en déduit que dans ces cartes locales, l'action de A est donnée par la matrice :

$$\tilde{A} = \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A \end{pmatrix}.$$

Comme $\det(\tilde{A}) = 1$, $\tilde{A}$ préserve donc la mesure de Lebesgue de $\mathbb{C}^{2g+n-1} = \mathbb{R}^{2(2g+n-1)}$, il s'ensuit que la forme volume μ_0 est invariante par l'action de A.

On peut remarquer sans difficulté que, pour tout $A \in SL_2(\mathbb{R})$, on a $\mathbf{Aire}(\Sigma) = \mathbf{Aire}(A \cdot \Sigma)$, ce qui signifie que A préserve l'ensemble $\mathcal{H}_1(k_1, \dots, k_n)$. Comme A préserve la forme volume μ_0, il en résulte que A préserve aussi la forme volume μ_0^1 de $\mathcal{H}_1(k_1, \dots, k_n)$.

De la même façon que le groupe $SL_2(\mathbb{R})$, on peut également considérer l'action du sous-groupe à un paramètre $\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R}\}$ sur $\mathcal{H}(k_1, \ldots, k_n)$. L'action de ce sous-groupe définit naturellement un flot sur l'espace de modules $\mathcal{H}(k_1, \ldots, k_n)$, qui est appelé *le flot géodésique de Teichmüller*.

Concernant les actions de $SL_2(\mathbb{R})$ et de $\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R}\}$, on a le théorème suivant :

Théorème (H.Masur, W.A.Veech) *Les actions de $SL_2(\mathbb{R})$ et de $\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, t \in \mathbb{R}\}$ sont ergodiques par rapport à la forme volume μ_0^1 sur chaque composante connexe de $\mathcal{H}_1(k_1, \ldots, k_n)$.*

Notons $\mathcal{H}_g$ l'union de toutes les strates $\mathcal{H}(k_1, \ldots, k_n)$ telles que $k_1 + \cdots + k_n = 2g - 2$. On a une projection naturelle de $\mathcal{H}_g$ sur $\mathcal{M}_g$ l'espace de modules des surfaces de Riemann compactes, fermées, de genre g. L'orbite d'un couple $(M, \omega) \in \mathcal{H}(k_1, \ldots, k_n) \subset \mathcal{H}_g$ par $SL_2(\mathbb{R})$ induit le diagramme commutative suivant

$$\begin{array}{ccc} SL_2(\mathbb{R}) & \longrightarrow & \mathcal{H}_g \\ \downarrow & & \downarrow \\ \mathbb{H}^2 \simeq SL_2(\mathbb{R})/SO(2) & \xrightarrow{f} & \mathcal{M}_g \end{array}$$

où f est un immersion isométrique pour la métrique de Teichmüller de $\mathcal{M}_g$. L'image de $\mathbb{H}^2$ par cette application est la projection d'un *disque de Teichmüller* dans l'espace de Teichmüller $\mathcal{T}_g$.

1.4 Motivation

En géométrie symplectique, il est d'usage d'étudier les déformations d'une variété symplectique par une famille continue de paramètres, en particulier lorsqu'elle est obtenue par réduction symplectique. Ici, nous nous proposons d'étudier des déformations de l'espace de modules des surfaces de translation dans le cadre des surfaces plates. Nous allons considérer des surfaces plates dont les angles aux points singuliers sont fixés, sur lesquelles il existe une union disjointe d'arbres dont le complémentaire est une surface de translation. Lorsque ces arbres se rétrécissent en points isolés, on obtient une surface de translation usuelle. Nous appelons des arbres ayant cette propriété les arbres effaçants, et leur union une forêt effaçante.

On peut remarquer aussitôt que les surfaces plates polyèdrales vérifient l'hypothèse précédente car le complémentaire de n'importe quel arbre sur la sphère est topologiquement un disque. Ceci nous permet de retrouver des résultats déjà connus, notamment par Thurston, pour les surfaces plates polyèrales.

La première question que nous allons étudier est la structure, et la dimension de ces espaces. Nous voudrons ensuite savoir s'il existe des formes volumes sur ces espaces, et établir le lien entre ces formes volumes et la forme volume de l'espace de modules des surfaces de translation. De plus, comme dans les cas des surfaces plates polyèdrales et surfaces de translation, nous souhaitons montrer que les espaces de modules en question sont de volume fini, et éventuellement, calculer leur volume.

Les résultats obtenus dans cette thèse nous donnent des réponses à ces questions. Plus précisement, nous construisons une structure plate affine complexe pour ces espaces de modules. Nous définissons en suite une forme de volume sur ces espaces qui, dans les cas de surfaces de translation, et de surfaces plates polyèdrales, est égale aux formes volumes habituelles à une constante multiplicative près. Nous montrons que l'intégrale des fonctions d'énergie, qui sont définies à partir de l'aire de la surface, et de la longueur des branches, par rapport à cette forme volume est finie. Notons que ce résultat nous permet de donner une nouvelle preuve du fait que le volume de chaque strate de l'espace de modules des surfaces de translation est fini.

Dernière remarque, la méthode que nous allons développer pour étudier les surfaces avec arbres effaçants s'adapte naturellement dans le cas des surfaces de translation avec bord, lequel inclut les polygones de $\mathbb{R}^2$, et sera le premier cadre naturel de nos travaux.

1.5 Présentation des résultats

1.5.1 Surface de translation à bord géodésique

Les premiers résultats de cette thèse concernent l'espace de modules des surfaces de translation à bord géodésique. Plus précisement, on va s'intéresser aux surfaces plates à singularités coniques dont le bord est une union finie de segments géodésiques satisfaisant la condition suivante : l'holonomie de toute courbe fermée contenue dans l'intérieur de la surface, et ne passant pas par des points singuliers est une translation de $\mathbb{R}^2$.

Fixons les données suivantes :

- Les entiers g, n, m, et $s_1, \dots, s_m,\ s_j \geqslant 1$;
- Les nombres réels $\alpha_1, \dots, \alpha_n$, avec $\alpha_i \in 2\pi\mathbb{N}$, et $\beta_1, \dots, \beta_m$, avec $\beta_j \in 2\pi\mathbb{Z}$, tels que :

$$(\alpha_1 + \cdots + \alpha_n) + (\beta_1 + \cdots + \beta_m) = 2\pi(2g + m + n - 2) \tag{1.2}$$

On note $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, où $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$, et $\bar{\beta} = ((s_1, \beta_1), \ldots, (s_m, \beta_m))$, l'ensemble des couples (Σ, ξ), où Σ est une surface de translation à bord géodésique vérifiant les conditions suivantes :

- Σ a n points singuliers à l'intérieur numérotés de 1 à n tels que l'angle du cône au i-ème point est α_i,

- $\partial\Sigma$ a m composantes connexes numérotées de 1 à m telles que la i-ème composante est l'union de s_j segments géodésiques, et la somme des angles aux extrémités de ces segments vaut $\beta_j + s_j\pi$,

et ξ est un champ de vecteur parallèle normalisé (la longueur de tout vecteur de ce champ est 1) sur Σ.

Remarque : Par le théorème de Gauss-Bonnet, pour que $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ soit non-vides, les angles $\alpha_1, \ldots, \alpha_n$, et $\beta_1, \ldots, \beta_m$ doivent vérifier (1.2).

Avec ces données, nous avons :

Théorème 1.5.1 *$\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ est le quotient d'une variété plate affine complexe de dimension :*

$$\begin{cases} 2g + n - 1, & si\ m = 0; \\ \sum_{j=1}^{m} s_j + 2g + m + n - 2, & si\ m > 0. \end{cases}$$

par l'action d'un groupe agissant proprement discontinument.

Ce théorème résulte du Théorème 2.2.7 et de la Proposition 2.2.8. Les cartes locales de $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ sont construites à partir des triangulations géodésiques des surfaces dans $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$.

Comme dans le cas des surfaces de translation sans bord, il existe une action du groupe $SL_2(\mathbb{R})$ sur $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, et nous avons (cf. Théorème 2.2.9 et Proposition 2.6.2) :

Théorème 1.5.2 *Il existe une forme volume μ_{Tr} sur $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ invariante par l'action du groupe $SL_2(\mathbb{R})$.*

Au cas où $m = 0$, $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ s'identifie à l'espace de module $\mathcal{H}(k_1, \ldots, k_n)$, avec $\alpha_i = (k_i + 1)2\pi$, rappelons que nous avons la forme volume μ_0 sur $\mathcal{H}(k_1, \ldots, k_n)$ qui est définie par l'application de périodes. Nous avons (cf. Proposition 2.2.10) :

Proposition 1.5.3 *Il existe sur chaque composante connexe de $\mathcal{H}(k_1, \ldots, k_n)$ une constante λ telle que $\mu_{\mathrm{Tr}} = \lambda\mu_0$.*

1.5.2 Surface plate avec forêt effaçante

Soit Σ une surface plate compacte, sans bord, une *forêt effaçante* sur Σ est une union disjointe d'arbres $\hat{A} = A_1 \sqcup \cdots \sqcup A_m$ telle que :

- Tout point singulier de Σ est un sommet d'un arbre dans $\hat{A}$.
- Pour toute courbe fermée γ sur Σ, si $\gamma \cap \hat{A} = \varnothing$, alors l'holonomie de γ est une translation.

Si toutes les arêtes d'un arbre sur Σ sont des segments géodésiques, alors on dit que cet arbre est géodésique. Une forêt est dite *géodésique* si tous ses arbres sont géodésiques.

Fixons m arbres topologiques $\mathcal{A}_1, \dots, \mathcal{A}_m$. Nous autorisons le cas limite où certains arbres peuvent être des points isolés. Notons $k_j, j = 1, \dots, m$, le nombre de sommets de $\mathcal{A}_j$, et posons $k_0 = 0$. Choisissons une numérotation des sommets de $\mathcal{A}_1, \dots, \mathcal{A}_m$ telle que les sommets de $\mathcal{A}_j, j = 1, \dots, m$, sont numérotés par $\{k_0 + \cdots + k_{j-1} + 1, \dots, k_0 + \cdots + k_j\}$. Notons $\hat{\mathcal{A}}$ la famille $\{\mathcal{A}_1, \dots, \mathcal{A}_m\}$, et posons

$$n = \sum_{j=1}^{m} k_j.$$

Soient g un entier, et $\alpha_1, \dots, \alpha_n$, n nombres réels positifs tels que

$$\begin{array}{rcl} \alpha_1 + \cdots + \alpha_n & = & (2g + n - 2)2\pi, \text{ et} \\ \alpha_{k_0+\cdots+k_{j-1}+1} + \cdots + \alpha_{k_0+\cdots+k_j} & \in & 2\pi\mathbb{N}. \end{array}$$

Notons $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$, où $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, l'espace de modules des triplets $(\Sigma, \hat{A}, \xi)$, où

- Σ est une surface plate compacte, sans bord,
- $\hat{A} = A_1 \sqcup \cdots \sqcup A_m$ est une forêt effaçante géodésique sur Σ telle que A_j est isomorphe à $\mathcal{A}_j$ (deux arbres sont isomorphes s'il existe une application de l'un à l'autre qui définit une bijetion entre deux ensembles de sommets, et une bijection entre deux ensembles d'arêtes), et
- ξ est un champ de vecteur parallèle défini sur $\Sigma \setminus \hat{A}$ dont tous les vecteurs sont de norme 1.

Nous supposons en plus que l'isomorphisme entre A_j et $\mathcal{A}_j$ envoie le i-ème sommet de $\mathcal{A}_j$ sur un point dont l'angle du cône associé est α_i.

Remarque Par définition, tout point singulier de Σ est un sommet d'un arbre de la forêt $\hat{A}$, mais on peut avoir des sommets qui ne sont pas des points singuliers de Σ (l'angle du cône en ces points est 2π).

Il s'avère que la méthode utilisée pour étudier l'espace de modules des surfaces de translation à bord géodésique peut s'appliquer dans cette situation, et nous obtenons (cf. Théorème 3.1.10, et Corollaire 3.1.8) :

Théorème 1.5.4 $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ *est le quotient d'une sous variété plate affine complexe de l'espace des surfaces de translations à bord géodésique (avec des données appropriées) de dimension*

$$\begin{cases} 2g+n-1, & \textit{si } \alpha_i \in 2\pi\mathbb{N},\ \forall i=1,\dots,n, \\ 2g+n-2, & \textit{sinon.} \end{cases}$$

par l'action d'un groupe agissant proprement discontinument, préservant une forme volume.

Notons que l'on n'a pas d'action de $SL_2(\mathbb{R})$ sur $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ dans le cas général.

1.5.3 Surface plate sphérique

Par *surface plate sphérique*, on entend une surface plate homéomorphe à la sphère $\mathbb{S}^2$. Soit Σ une surface plate sphérique, il n'est pas difficile de montrer qu'il existe un arbre géodésique sur Σ dont les sommets sont les points singuliers. Un tel arbre est automatiquement effaçant car son complémentaire dans Σ est un disque. Cette observation nous amène à considérer les surfaces plates sphériques comme un cas parliculier des surfaces plates avec arbres effaçants.

Fixons n réels positifs $\alpha_1, \dots, \alpha_n$, tels que

$$\alpha_1 + \dots + \alpha_n = 2\pi(n-2).$$

Notons $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$, où $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$, l'espace de modules des surfaces plates homéomorphes à la sphère ayant n singularités d'angles $\alpha_1, \dots, \alpha_n$, et $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ l'ensemble $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^* \times \mathbb{S}^1$. Nous avons (cf. Théorème 4.1.1) :

Théorème 1.5.5 $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ *est le quotient d'une variété plate affine complexe de dimension* $n-2$ *par l'action d'un groupe agissant properment discontinument, et préservant une forme volume* μ_{Tr}.

Comme dans les cas des surface de translation avec bord, ou celui des surfaces avec forêt effaçante, la forme volume μ_{Tr} dans 1.5.5 est définie à l'aide des triangulations géodésiques des surfaces dans $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$. Notons que, à la différence des surfaces avec forêt effaçante en général, ici nous n'avons pas besoin de spécifier un arbre effaçant particulier sur la surface.

Notons $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ l'ensemble des surfaces d'aire 1 dans $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$. Dans le cas où tous les angles α_i sont plus petits que 2π, le travail de Thurston donne une forme volume μ_{Hyp} sur $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ qui provient de la métrique hyperbolique complexe. La forme volume μ_{Tr} de $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ induit aussi une forme volume sur $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$, notons celle-ci $\hat{\mu}^1_{\mathrm{Tr}}$. Nous allons montrer que $\hat{\mu}^1_{\mathrm{Tr}} = \lambda\mu_{\mathrm{Hyp}}$, où λ est une constante dépendant de $(\alpha_1, \dots, \alpha_n)$ (cf. Proposition 4.4.1). Une conséquence directe de ce fait est

Proposition 1.5.6 *Si $\alpha_i < 2\pi$, pour tout $i \in \{1, \dots, n\}$, alors*

$$\hat{\mu}^1_{\mathrm{Tr}}(\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*) < +\infty.$$

1.5.4 Intégration des fonctions d'énergie

Revenons au cas des surfaces de translation à bord géodésique. Rappelons que $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ est l'espace de modules des couples (Σ, ξ), où Σ est une surface de translation à bord géodésique, et ξ est un champ de vecteur parallèle constant sur Σ. Nous définissons une fonction d'énergie $\mathcal{F}$ sur $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ par :

$$\mathcal{F}((\Sigma, \xi)) = \exp(-\mathbf{Aire}(\Sigma) - \ell^2(\partial\Sigma)),$$

où $\ell(\partial\Sigma)$ est la longueur du bord de Σ.

Pour les surfaces avec forêt effaçante, nous avons une fonction d'énergie similaire :

$$\begin{array}{rccl} \mathcal{F}^{\mathrm{et}}: & \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}) & \longrightarrow & \mathbb{R} \\ & (\Sigma, \hat{\mathcal{A}}, \xi) & \longmapsto & \exp(-\mathbf{Aire}(\Sigma) - \ell^2(\hat{\mathcal{A}})) \end{array}$$

où $\ell(\hat{\mathcal{A}})$ est la somme de longueur totale des arbres de la forêt $\hat{\mathcal{A}}$. Rappelons que nous avons défini une forme volume μ_{Tr} sur $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, ainsi que sur $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$. Nous avons alors (cf. Théorème 5.1.1) :

Théorème 1.5.7 *a) Si le bord des surfaces dans $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ est non-vide alors :*

$$\int_{\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})} \mathcal{F} d\mu_{\mathrm{Tr}} < +\infty,$$

b) Si les arbres dans la famille $\hat{\mathcal{A}}$ ne sont pas tous des points isolés, alors

$$\int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})} \mathcal{F}^{\mathrm{et}} d\mu_{\mathrm{Tr}} < +\infty.$$

En utilisant ce résultat, nous obtenons une nouvelle preuve du fait que le volume de toute strate $\mathcal{H}_1(k_1, \dots, k_n)$ par rapport à la forme volume μ^1_0 est fini (cf. Proposition 5.5.1).

Pour les espaces de modules des surfaces plates sphériques, inspirés du résultat de Thurston, en utilisant le Théorème 1.5.7, nous obtenons un résultat plus général (cf. Théorème 5.1.2)

Théorème 1.5.8 *L'intégrale de la fonction $(\Sigma, e^{\imath\theta}) \longmapsto \exp(-\mathbf{Aire}(\Sigma))$ par rapport à la forme volume μ_{Tr} sur $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ est finie.*

$$\int_{\mathcal{M}(\mathbb{S}^2,\bar{\alpha})} \exp(-\mathbf{Aire}) d\mu_{\mathrm{Tr}} < \infty.$$

Par conséquent, le volume de $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ *est fini.*

Remark: Veech [V2] a trouvé ce résultat pour une forme volume qui est définie différemment.

1.6 Sommaire

La suite de cette thèse est organisée comme suit :

- **Chapitre 2 :** dans ce chapitre, nous traiterons le cas des surfaces de translation à bord géodésique. Nous montrerons d'abord que, pour toute surface de translation à bord géodésique, il existe toujours une triangulation par segments géodésiques dont l'ensemble des sommets contient l'ensemble des points singuliers. Nous montrons ensuite qu'une telle triangulation permet de définir des coordonnées locales d'une variété plate affine complexe $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. Par définition, $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ est le quotient de $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ par l'action d'un groupe $\Gamma(S,\mathcal{V})$, nous montrerons que l'action de $\Gamma(S,\mathcal{V})$ est proprement discontinue.

  Sur les cartes locales de $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$, qui sont définies par des triangulations géodésiques, une forme volume peut être définie de façon naturelle. Nous montrons que cette forme volume ne dépend pas du choix de la triangulation. Cela résulte du fait que, pour une surface de translation ou de demi-translation, avec ou sans bord, étant données deux triangulations géodésiques dont les ensembles de sommets coincident et contiennent l'ensemble des points singuliers, alors on peut transformer l'une à l'autre par une suite de changements élémentaires (cf. Théorème 2.6.2). Nous obtenons ainsi une forme volume μ_{Tr} bien définie sur $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. Comme l'action de $\Gamma(S,\mathcal{V})$ préserve cette forme volume, celle-ci induit une forme volume sur $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$.

  Comme les surfaces de translation fermées sont un cas particulier des surfaces de translation à bord géodésique, la forme volume μ_{Tr} est bien définie sur chacune des strate $\mathcal{H}(k_1,\dots,k_n)$. Nous montrerons, enfin, que sur chacune des composantes connexes de $\mathcal{H}(k_1,\dots,k_n)$, la forme volume μ_{Tr} est égale à $\lambda\mu_0$, où λ est une constante non-nulle, et μ_0 est la forme volume définie par l'application de périodes.

- **Chapitre 3 :** ce chapitre concerne les surfaces plates avec arbres effaçants. Avec le même schéma que Chapitre 2, nous montrons que $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ est le quotient d'une variété plate affine complexe

$\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$, qui est une sous variété de $\mathcal{T}_{\text{T}}(\bar{\alpha}'; \bar{\beta}')$, avec des donées $\bar{\alpha}', \bar{\beta}'$ appropriées, par l'action d'un groupe $\Gamma(S_g, \hat{\mathcal{A}})$ agissant proprement discontinument. Ensuite, nous prouvons l'existence d'une forme volume μ_{Tr} sur $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ qui est invariante par l'action de $\Gamma(S_g, \hat{\mathcal{A}})$, cette forme volume induit donc une forme volume sur $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$.

- **Chapitre 4 :** dans ce chapitre nous nous concentrerons sur les surfaces plates sphériques. Remarquons d'abord qu'il existe, sur toute surface plate sphérique, un arbre géodésique connectant tous les points singuliers, et un tel arbre est automatiquement effaçant car son complémentaire est un disque. Cette observation nous permet de considérer les surfaces plates sphériques comme un cas particulier des surfaces plates avec forêt effaçante. Ainsi, nous démontrons aisément que $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ est un orbifold complexe de dimension $n-2$.

  La preuve de l'existence d'une forme volume μ_{Tr}, analogue à celles définies dans les deux chapitres précédents, est un peu plus délicate, car nous ne choisissons pas auparavant un arbre effaçant. Néanmoins, nous pouvons prouver que deux triangulations géodésiques d'une surface plate sphérique dont l'ensemble des sommets coincide avec l'ensemble des points singuliers peuvent être transformées l'une à l'autre par des changements élémentaires (cf. Théorème 4.3.2). Cela nous permet de définir μ_{Tr} sur $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$.

  Nous terminerons ce chapitre par la comparaison entre la forme volume $\hat{\mu}^1_{\text{Tr}}$, induite par μ_{Tr}, et la forme volume μ_{Hyp}, qui provient de la métrique hyperbolique complexe définie par Thurston, sur $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$, dans le cas où tous les angles coniques sont inférieurs à 2π.

- **Chapitre 5 :** dans ce chapitre, nous montrons que les intégrales des fonctions $\mathcal{F}$ et $\mathcal{F}^{\text{et}}$, définies sur $\mathcal{M}_{\text{T}}(\bar{\alpha}; \bar{\beta})$ et $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ respectivement, par rapport à la forme volume μ_{Tr} sont finies. Nous prouvons ensuite le fait que le volume des strates $\mathcal{H}_1(k_1, \dots, k_n)$ est fini comme une conséquence de ce résultat. Finalement, nous prouvons que le volume de $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ par rapport à la forme volume $\hat{\mu}^1_{\text{Tr}}$, qui est induite par μ_{Tr}, est fini. Notons que pour le cas particulier où tous les angles coniques sont inférieurs à 2π, ce résultat a été déjà connu par le travail de Thurston, et le même résultat a été trouvé par Veech dans [V2] pour une autre forme volume.

Pour des raisons pratiques, le reste de cette thèse sera rédigé en anglais. L'auteur s'en excuse pour des inconvénients éventuellement causés au lecteur par ce choix, et le remercie pour sa compréhension.

Chapitre 2

Translation surfaces with boundary

2.1 Introduction

Translation surfaces are flat surfaces with conical singularities verifying the following condition : the holonomy of every closed curve, which does not contain any singularity, is an Euclidean translation. On a translation surface, one can define a *parallel vector field* on the complement of the singularities. There exists a system of local charts defining the flat metric structure such that, on each chart, this vector field is mapped to a vertical vector field on a domain of $\mathbb{R}^2$. Any pair (Σ, ξ), where Σ is a closed translation surface, and ξ is a parallel vector field on Σ, can be identified to a pair (M, ω), where M is a closed Riemann surface, and a holomorphic 1-form on M. The zeros of ω are the singularities of metric structure on Σ, zeros of order k, $k = 0, 1, 2, \dots$, correspond to singularities of angles $2\pi(k+1)$.

Let g be the genus of Σ, and $k_1, \dots, k_n$ be the orders of the zeros of ω. By the Riemann-Roch Theorem, one has

$$k_1 + \cdots + k_n = 2g - 2.$$

Fix $k_1, \dots, k_n$ and let $\mathcal{H}(k_1, \dots, k_n)$ denote the moduli space of pairs (M, ω), where M is closed, and the holomorphic 1-form ω has exactly n zeros with orders $k_1, \dots, k_n$. The space $\mathcal{H}(k_1, \dots, k_n)$ is also called a *stratum* of the moduli space of translation surfaces of genus g, where g can be computed by the above equation. It is well known that $\mathcal{H}(k_1, \dots, k_n)$ is a complex orbifold of dimension $2g+n-1$.

Let (M, ω) be a pair in $\mathcal{H}(k_1, \dots, k_n)$. The zeros of ω are denoted by $x_1, \dots, x_n$, and their orders by is k_i respectively. Let $\{\gamma_1, \dots, \gamma_{2g+n-1}\}$ be a set of curves on M which is a generating family of the group $H_1(M, \{x_1, \dots, x_n\}; \mathbb{Z})$. For any element (M', ω') close to (M, ω) in $\mathcal{H}(k_1, \dots, k_n)$, we denote $\{\gamma'_1, \dots, \gamma'_{2g+n-1}\}$ the corresponding curves on M'. We can then define a map Φ from a neighborhood of (M, ω) into $\mathbb{C}^{2g+n-1}$, which sends a pair (M', ω') to the vector $(\int_{\gamma'_1} \omega', \dots, \int_{\gamma'_{2g+n-1}} \omega')$. The map Φ is called the *period mapping*.

Let $\lambda_{2(2g+n-1)}$ denote the Lebesgue measure of $\mathbb{C}^{2g+n-1} \simeq \mathbb{R}^{2(2g+n-1)}$. Since two generating families of $H_1(M, \{x_1, \dots, x_n\}; \mathbb{Z})$ are related by an element of the group $SL(2g+n-1, \mathbb{Z})$, the volume form $\Phi^*\lambda_{2(2g+n-1)}$ is well defined on $\mathcal{H}(k_1, \dots, k_n)$. We denote this volume form μ_0.

Let $\mathcal{H}_1(k_1, \dots, k_n)$ denote the subspace of $\mathcal{H}(k_1, \dots, k_n)$ consisting of pairs (M, ω) such that $\int_M |\omega|^2 = 1$. An element of $\mathcal{H}_1(k_1, \dots, k_n)$ corresponds to a translation surface of area 1. The volume form μ_0 induces a volume form μ_1 on $\mathcal{H}_1(k_1, \dots, k_n)$. It is proved by Masur [M] and Veech [V1] that the volume of $\mathcal{H}_1(k_1, \dots, k_n)$ is finite. In [EO], Eskin and Okounkov compute the volume of several samples of $\mathcal{H}_1(k_1, \dots, k_n)$. They actually give a method to compute the volume of every stratum $\mathcal{H}_1(k_1, \dots, k_n)$, and give numerical results for some of them.

In this chapter, we are interested in translation surfaces with boundary such that every boundary component is a finite union of geodesic segments. Let Σ be such a translation surface. A point x in Σ is *regular* if either :

- x is a point in the interior of Σ, and x has a neighborhood isometric to a disk $\{z \in \mathbb{C} : |z| < \epsilon\}$ with ϵ small, or

- x is a point in the boundary of Σ, and x has a neighborhood isometric to a half disk $\{z \in \mathbb{C} : |z| < \epsilon,\ \mathrm{Im} z \geqslant 0\}$.

Similarly to closed translation surfaces, on any translation surface with geodesic boundary, we can define parallel vector fields on the complement of the singularities and the boundary. Let C be a boundary component of Σ, and ξ be a parallel vector field on Σ. Let $c : \mathbb{S}^1 \longrightarrow \Sigma$ be a simple, closed C^1 curve freely homotopic to C. Assume that for every t in $\mathbb{S}^1$, the tangent vector $v(t) = \dot{c}(t) \neq 0$. Let $\Theta : \mathbb{S}^1 \longrightarrow \mathbb{R}$ denote the function which maps t to the angle between $v(t)$ and the vertical vector $\xi(c(t))$. We define the *cone angle* of C to be the number $\int_{\mathbb{S}^1} d\Theta$. Observe that the cone angle of a boundary component of any translation surface belongs to the set $\{2k\pi,\ k \in \mathbb{Z}\}$, and it does not depend on the choices of c and ξ.

Let g, n, m be three positive integers. Fix n numbers $\alpha_1, \dots, \alpha_n$ with $\alpha_i \in 2\pi\mathbb{N}$, and m pairs of numbers $(\beta_1, s_1), \dots, (\beta_m, s_m)$, with β_j in $2\pi\mathbb{Z}$, and s_j in $\mathbb{N}$. We consider the moduli space of translation surfaces Σ of genus g having n singularities in the interior, and m boundary components denoted by $C_1, \dots, C_m$ such that :

- the n singularities in the interior of Σ have cone angles $\alpha_1, \dots, \alpha_n$.

- the cone angle associated to the component C_j is β_j, $j = 1, \dots, m$.

- there exists a subset Q_j of C_j containing exactly s_j points such that $C_j \setminus Q_j$ is a union of open geodesic segments.

Let $\bar{\alpha}$ denote the sequence $\{\alpha_1 \dots, \alpha_n\}$, and $\bar{\beta}$ denote the sequence $\{(\beta_1, s_1), \dots, (\beta_m, s_m)\}$. Let $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ denote the moduli space of surfaces described above. The main results of this chapter is that $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ is a complex affine orbifold, and moreover, we can specify a volume form μ_{Tr} on $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$. When $m = 0$, $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ can be identified to the space $\mathcal{H}(k_1, \dots, k_n)$, with $\alpha_i = 2\pi(k_i + 1)$, $i = 1, \dots, n$. In this case, for each connected component of $\mathcal{H}(k_1, \dots, k_n)$, there exists a constant λ such that $\mu_{\mathrm{Tr}} = \lambda \mu_0$.

2.2 Definitions and main results

We start with some basic definitions :

2.2.1 Flat surface and translation surface

Definition 2.2.1 (Flat Surface with Conical Singularities and Geodesic Boundary) *Let Σ be a compact, connected surface, possibly with boundary. Let $\{p_1, p_2, \dots, p_{n_1}\}$ be a finite subset of the interior of Σ, and $\{q_1, q_2, \dots, q_{n_2}\}$ be a finite subset of the boundary of Σ. We say that Σ is a* flat surface with geodesic boundary, *having* conical singularities *at $p_1, \dots, p_{n_1}$, and* corners *at $q_1, \dots, q_{n_2}$, if $\Sigma \setminus \{p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2}\}$ is equipped with an Euclidean metric structure verifying the following conditions :*

(i) For each $i \in \{1, \dots, n_1\}$, there exists $\theta_i > 0$ such that p_i has a neighborhood isometric to a small disk around the origin in $\mathbb{R}^2$, which is equipped with the metric $g_{\theta_i}(r, \theta) = dr^2 + (\frac{\theta_i}{2\pi})^2 r^2 d\theta^2$ in the polar coordinates. The number θ_i is called the cone angle *at p_i.*

(ii) For each $j \in \{1, \dots, n_2\}$, there exists $\eta_j > 0$ such that q_j has a neighborhood isometric to small upper half disk around the origin in $\mathbb{R}^2$, which is equipped with the metric $g_{\eta_j}(r, \theta) = dr^2 + (\frac{\eta_j}{\pi})^2 r^2 d\theta^2$ in the polar coordinates. The number η_j is called the corner angle *at q_j.*

(iii) $\partial\Sigma \setminus \{q_1 \dots, q_{n_2}\}$ is a finite set of open geodesic segments.

In the sequel, 'a flat surface' is a flat surface with conical singularities whose boundary, if not empty, is geodesic.

Let $\Sigma; (p_1, \dots, p_{n_1}); (q_1, \dots, q_{n_2})$ be as in Definition 2.2.1. Let $\theta_1, \dots, \theta_{n_1}$ be the cone angles at $p_1, \dots, p_{n_1}$ respectively, and $\eta_1, \dots, \eta_{n_2}$ be the corner angles at $q_1, \dots, q_{n_2}$ respectively. Let $\chi(\Sigma)$ denote the Euler characteristic of Σ. We have the following formula

$$\sum_{i=1}^{n_1} \theta_i + \sum_{j=1}^{n_2} \eta_j = 2\pi(n_1 + \frac{n_2}{2} - \chi(\Sigma)). \tag{2.1}$$

This is a consequence of the Gauss-Bonnet Formula (see [Tr1]).

Definition 2.2.2 (Translation Surface) *A translation surface Σ is a flat surface verifying the following condition : if c is a closed curve in the interior of Σ which does not contain any singular point, then the holonomy of c is a translation of the Euclidean plane $\mathbb{R}^2$.*

Note that the cone angle at any singular point in the interior of a translation surface must be an integral multiple of 2π. The corner angle at a singular point on the boundary of a translation surface may not belong to the set $\pi\mathbb{Z}$, but the sum of all corner angles at the singular points on each boundary component must be an integral multiple of π.

We define as usual the length of a piece-wise C^1 curve, and denote $\mathbf{d}$ the induced distance on a flat surface. Note that for any pair of points (x, y) of a flat surface, there always exists a curve piece-wise geodesic joining x and y whose length is $\mathbf{d}(x, y)$.

Definition 2.2.3 (Normalized Parallel Vector Field) *Let Σ be a translation surface. A* parallel vector field *on Σ is a vector field defined in the interior of Σ except at singular points, which is nowhere zero, and in local charts of the Euclidean metric structure, all the lines determined by the vectors of this field are parallel. A parallel vector field is said to be* normalized *if the norm of all of its vectors is one.*

Remark: : A parallel vector field exists if and only if Σ is a translation surface.

From now on, by 'translation surface' (with or without boundary), we will mean a 'translation surface with a distinguished parallel vector field on it'.

Let Σ be a translation surface, and ξ be a parallel vector field on Σ. Assume that the boundary of Σ is not empty, and let C be a component of $\partial\Sigma$. We assume in addition that C is oriented coherently with the orientation of Σ.

Definition 2.2.4 (Cone Angle associated to a Boundary Component) *Let $c : \mathbb{S}^1 \longrightarrow \Sigma$ be a C^1, simple, closed curve which is contained in the interior of Σ, and freely homotopic to $\overline{C}$, where $\overline{C}$ is the curve C with opposite orientation. Assume that c does not contain any singular point of Σ. For every $t \in \mathbb{S}^1$, let $\Theta(t)$ denote the angle between the vector $v(t) = c'(t)$, and the vector $\xi(c(t))$. The* cone angle associated to the component *C is defined to be the number*

$$\int_{\mathbb{S}^1} d\Theta(t).$$

Remark:

a. The cone angle associated to any component of $\partial\Sigma$ belongs to the set $2\pi\mathbb{Z}$.

b. This cone angle does not depend on the choices of the curve c and the field ξ.

c. If C contains s corners with corners angles $\eta_1, \dots, \eta_s$, then the cone angle associated to C equals $\sum_{j=1}^{s} \eta_j - s\pi$.

Now, fix three non-negative integers g, n, m such that $2g + n + m - 2 > 0$. Let $\alpha_1, \dots, \alpha_n$ be n real numbers in $2\pi\mathbb{N}$, and $\beta_1, \dots, \beta_m$ be m numbers in $2\pi\mathbb{Z}$ such that

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{m} \beta_j = 2\pi(2g + n + m - 2). \tag{2.2}$$

Let $s_1, \dots, s_m$ be m positive integers. In this chapter, we will fix a compact connected translation surface S of genus g, whose boundary has m components denoted by $C_1, \dots, C_m$ verifying the following hypothesis :

- There are n points $p_1, \dots, p_n$ in the interior of S such that the cone angle at p_i is α_i, $i = 1, \dots, n$.
- The cone angles associated to the C_j is β_j, $j = 1, \dots, m$.
- For $j = 1, \dots, m$, there exists a subset Q_j of C_j consisting of s_j points such that $C_j \setminus Q_j$ is a union of open geodesic segments.

Let $\mathcal{P}$ denote the set $\{p_1, \dots, p_n\}$, and $\mathcal{V}$ denote $\mathcal{P} \cup (Q_1 \cup \dots \cup Q_m)$. Let $\hat{S}$ denote the double of S, and let $\hat{\mathcal{V}}$ denote the finite subset of $\hat{S}$ arising from $\mathcal{V}$. The flat metric structure of S induces a flat metric structures on $\hat{S}$ whose all the singularities are contained in the set $\hat{\mathcal{V}}$. Note that we have Riemann surface structure on $\hat{S} \setminus \hat{\mathcal{V}}$ which is induced by the metric structure.

Given a homeomorphism f of S, we denote $\hat{f}$ the homeomorphism of $\hat{S}$ arising from f. We call $\hat{f}$ the double of f.

First, we have :

Definition 2.2.5 (Mapping Class Group) *We denote* $\mathrm{Homeo}^+(S, \mathcal{V})$ *the group of orientation preserving homeomorphisms of S which fix every point in the set* $\mathcal{V}$. *Let* $\mathrm{Homeo}_0^+(S, \mathcal{V})$ *denote the normal subgroup of* $\mathrm{Homeo}^+(S, \mathcal{V})$ *consisting of all homeomorphisms f such that double $\hat{f}$ of f is isotopic to* $\mathrm{Id}_{\hat{S}}$ *by an isotopy fixing all the points in* $\hat{\mathcal{V}}$. *The* mapping class group *of S preserving* $\mathcal{V}$ *is defined to be the quotient group* $\mathrm{Homeo}^+(S, \mathcal{V})/\mathrm{Homeo}_0^+(S, \mathcal{V})$, *which will be denoted by* $\Gamma(S, \mathcal{V})$.

Remark:

a. Let f be a homeomorphism of S which fixes all the points in $\mathcal{V}$. If f can be connected to the identity of S by an isotopy fixing all the points in $\mathcal{V}$, then clearly f is an element in $\mathrm{Homeo}_0^+(S, \mathcal{V})$.

b. Consider S as an embedded surface in $\hat{S}$. The boundary of S becomes then a union of simple curves $c_1, \dots, c_k$ joining points in $\hat{\mathcal{V}}$. By Lemma A.0.1, given a homeomorphism f of S, if $\hat{f}$ is a homeomorphism isotopic to the identity of $\hat{S}$ by an isotopy fixing all the points in $\hat{\mathcal{V}}$, then there exists an isotopy from $\hat{f}$ to $\mathrm{Id}_{\hat{S}}$ which preserves every curve in the family $\{c_1, \dots, c_k\}$. As a consequence, we see that $\mathrm{Homeo}_0^+(S, \mathcal{V})$ is the set of all homeomorphisms of S which are isotopic to Id_S by an isotopy fixing all the points in $\mathcal{V}$.

Let $\bar{\alpha}$ and $\bar{\beta}$ denote the sets $\{\alpha_1, \dots, \alpha_n\}$ and $\{(\beta_1, s_1), \dots, (\beta_m, s_m)\}$ respectively.

Now, if $\phi : S \longrightarrow \Sigma$ is a homeomorphism of flat surfaces, we denote $\hat{\phi}$ the induced homeomorphism from $\hat{S}$ onto $\hat{\Sigma}$.

We denote $\widetilde{\mathcal{T}}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$ the set of pairs (Σ, ϕ), where Σ is a translation surface of genus g whose boundary has m components, and $\phi : S \longrightarrow \Sigma$ is a homeomorphism verifying the following conditions :

1. For $i = 1, \dots, n$, $\phi(p_i)$ is a point in the interior of Σ with cone angle α_i.

2. For $j = 1, \dots, m$, $\phi(C_j)$ is a component of $\partial\Sigma$ with associated cone angle β_j.

3. For $j = 1, \dots, m$, $\phi(C_j \setminus Q_j)$ is a union of open geodesic segments in a component of $\partial\Sigma$.

We define an equivalence relation on $\widetilde{\mathcal{T}}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$ as follows : two pairs (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are equivalent if and only if there exists an isometry $h : \Sigma_1 \longrightarrow \Sigma_2$ such that the homeomorphism $\phi_2^{-1} \circ h \circ \phi_1 : S \longrightarrow S$ is an element of $\mathrm{Homeo}_0^+(S, \mathcal{V})$. The equivalence class of a pair (Σ, ϕ) will be denoted by $[(\Sigma, \phi)]$.

Let $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$ denote the space of equivalence classes of this relation. Obviously, the group $\Gamma(S, \mathcal{V})$ acts on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$. The quotient space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*/\Gamma(S, \mathcal{V})$ is denoted by $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$.

Definition 2.2.6 (Teichmüller space of translation surfaces) *The* Teichmüller space of translation surfaces with parallel vector field *is the set of all pairs* $([(\Sigma, \phi)], \xi)$*, where* $[(\Sigma, \phi)]$ *is an element of* $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$*, and* ξ *is a normalized parallel vector field on* Σ*. We denote this space* $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$*.*

The moduli space of translation surfaces with parallel vector field *is the quotient space* $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})/\Gamma(S,\mathcal{V})$, *it is denoted by* $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$.

Note that in the case $g = n = 0$, and $m = 1$, the space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ is just the moduli space of Euclidian metric structures with geodesic boundary on a closed disk.

Remark: The group $\mathbb{S}^1$, identified to the rotations of the Euclidean plane, acts naturally on the space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$: if R_θ is the rotation of angle θ, and $([(\Sigma,\phi)],\xi)$ is an element in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$, then $R_\theta \cdot ([(\Sigma,\phi)],\xi) = ([(\Sigma,\phi)], R_\theta\cdot\xi)$, where $R_\theta\cdot\xi$ is the parallel vector field defined as follows : at every point where ξ is defined, $R_\theta\cdot\xi$ is the vector obtained by rotating ξ an angle θ. This action of $\mathbb{S}^1$ endows $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ with a principal $\mathbb{S}^1$-bundle structure over $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})^*$.

2.2.2 Main results

Recall that a flat complex affine manifold is a C^∞ manifold which admits an atlas whose transition maps are complex linear transformations. With g, $\bar{\alpha}$, and $\bar{\beta}$ as above, we can now state the main results of this chapter

Theorem 2.2.7 ($\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ is a Flat Complex Affine Manifold) *The space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ is a flat complex affine manifold of dimension :*

- $2g+n-1$ *if* $m=0$.
- $\sum_{j=1}^{m} s_j + 2g+m+n-2$ *if* $m>0$.

Regarding the moduli space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$, we have

Proposition 2.2.8 *The action of the mapping class group $\Gamma(S,\mathcal{V})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ is properly discontinuous.*

and

Theorem 2.2.9 (Existence of volume form on $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$) *There exists on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ a volume form which is invariant by the action of $\Gamma(S,\mathcal{V})$.*

By Theorem 2.2.8, and Theorem 2.2.9, we have a well defined volume form on $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. Let μ_{Tr} denote the volume form in Theorem 2.2.9. This volume form is defined by using the local charts of the complex affine structure of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$.

When $m=0$, *i.e.* when the surfaces under consideration are closed, set

$$k_i = \frac{\alpha_i}{2\pi} - 1,\ i = 1,\dots,n.$$

We can then identify the moduli space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ to $\mathcal{H}(k_1,\dots,k_n)$. Recall that $\mathcal{H}(k_1,\dots,k_n)$ is the moduli space of pairs (M,ω) where M is a closed Riemann surface of genus g, and ω is a holomorphic 1-form on M which has n zeros with orders $k_1,\dots,k_n$. Let μ_0 denote the volume form on $\mathcal{H}(k_1,\dots,k_n)$ which is defined by using the period mapping. The following proposition gives the relation between μ_0 and μ_{Tr}.

Proposition 2.2.10 *On each connected component of $\mathcal{H}(k_1,\dots,k_n)$, there exists a constant λ such that $\mu_{\mathrm{Tr}} = \lambda\mu_0$.*

Remark that, similarly to the case of closed translation surfaces, we have an action of $SL(2,\mathbb{R})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ which is defined in a natural way. This action commutes with the action of the group $\Gamma(\tilde{g},\tilde{n})$, and hence it descends onto an action of $SL(2,\mathbb{R})$ on the moduli space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. We have

Proposition 2.2.11 *The volume form μ_{Tr} is invariant by the action of the action of $SL(2,\mathbb{R})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$, and hence on $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$.*

The chapter is organized as follows, in Section 2.3, and Section 2.4, we prove Theorem 2.2.7. Proposition 2.2.8 is proved in Section 2.5. Section 2.6 is devoted to the proof of the fact that any two *admissible triangulations* of a translation surface can be transformed one into the other by elementary moves. The construction of the volume form μ_{Tr} is given in Section 2.7. The comparison Proposition 2.2.10 is proved in Section 2.8. Finally, in Section 2.9, we show that the volume form μ_{Tr} is invariant by the action of $SL(2,\mathbb{R})$.

2.3 Admissible triangulation

2.3.1 Introduction

Let $([(\Sigma,\phi)],\xi)$ be an element in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. Following the method of Thurston in [Th], we construct local charts of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ about $([(\Sigma,\phi)],\xi)$ by using geodesic triangulations of Σ. In view of this construction, we first define :

Definition 2.3.1 (Admissible triangulation) *An* admissible triangulation *of* $[(\Sigma, \phi)]$ *is a triangulation* T *of* Σ *such that :*

- *The set of vertices of* T *is the set* $V = \phi(\mathcal{V})$.
- *Every edge of* T *is a geodesic segment.*

By assumption, the surface Σ has n singular points $x_1, \ldots, x_n$ in its interior with cone angles $\alpha_1, \ldots, \alpha_n$ respectively. Let $Y_1, \ldots, Y_m$ denote the components of the boundary of Σ so that the cone angle associated to Y_j is β_j. There exist s_j distinct points $y_{1j}, \ldots, y_{s_j j}$ on Y_j which divide Y_j into s_j geodesic segments. We consider the set $V = \{x_1, \ldots, x_n; y_{11}, \ldots, y_{s_m m}\}$ as the set of singular points of Σ even though some of them may be regular.

The main results of this section are the following two propositions :

Proposition 2.3.2 (Existence of admissible triangulations) *There exists a triangulation* T *of* Σ *with the following properties :*

(i) The set of vertices of T *is* V.

(ii) Every edge of T *is a geodesic segment.*

Remark: Given an admissible triangulation T of Σ, one can find $2g + m + n - 1$ edges of T such that the complement of the union these edges and the boundary $\partial\Sigma$ is a topological open disk. This set of edges will be called a *family of primitive edges* of T.

By Proposition 2.3.2, we know that admissible triangulations exist on any translation surface in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^*$. For the proof of Theorem 2.2.7, we also need the following

Proposition 2.3.3 (Uniqueness of admissible triangulations up to isotopy) *Let* T_1 *and* T_2 *be two admissible triangulations of* $[(\Sigma, \phi)]$. *Let* $\hat{\Sigma}$ *be the double of* Σ *which is equipped with the induced flat metric. Let* $\hat{V}$ *be the finite subset of* $\hat{\Sigma}$ *which is induced from* $V = \phi(\mathcal{V})$.

As usual, for any homeomorphism φ *of* Σ, *let* $\hat{\varphi}$ *be the homeomorphism of* $\hat{\Sigma}$ *that lifts* φ. *Suppose that there exists an homeomorphism* $\varphi : \Sigma \longrightarrow \Sigma$ *such that :*

- $\hat{\varphi}$ *is isotopic to the identity of* $\hat{\Sigma}$ *by an isotopy fixing the set* $\hat{V}$ *;*

- $\varphi(\mathrm{T}_1) = \mathrm{T}_2$,

then $\mathrm{T}_1 = \mathrm{T}_2$.

Remark: Geodesic triangulations of flat surfaces whose vertex set is the set of singularities have already appeared in [KMS]. The fact that (closed) translation surfaces always admit such triangulations (Proposition 2.3.2) is well known, since every translation surface can be constructed by gluing some rectangles (zippered rectangles). For flat surface in general, possibly with boundary, this fact is also already known (see [BS] for further information), we give a proof of this fact here below only for the sake of completeness.

2.3.2 Proof of Proposition 2.3.2

Proposition 2.3.2 is a consequence of the following lemmas :

Lemma 2.3.4 *If* $(m,n) \neq (0,1)$*, then there exist* $m+n-1$ *geodesic segments with endpoints in* V *such that if we cut the surface* Σ *along those segments, then we will obtain a translation surface whose boundary has only one component, and the new surface contains no singularities in the interior.*

Proof: Consider the following algorithm :

- If $m = 0$ and $n > 1$, then choose a path c of minimal length joining two distinct points in $V = \{x_1, \dots, x_n\}$. The path c contains an arc c_0 which joins two distinct points of V, and contains no others points of V in its interior. Cut open Σ along the arc c_0, we obtain a new translation surface with boundary. Let Σ' denote the new surface, and V' denote the finite subset of Σ which arises from the set V. The boundary of the new surface has one component, and V' contains $n-2$ points in the interior of Σ'.

- If $\partial\Sigma \neq \varnothing$ and $n > 0$, then choose a path c of minimal length from a point in $V_1 = \{x_1, \dots, x_n\} = V \cap \mathrm{int}(\Sigma)$ to a point in $V_2 = \{y_{11}, \dots, y_{s_1 1}; \dots; y_{1m}, \dots, y_{s_m m}\} = V \cap \partial\Sigma$. The path c contains an arc c_0 joining a point in V_1 to a point in V_2 which stays in the interior of Σ except the endpoint in V_2. Since c is of minimal length, it does not have self-intersection, and the same is true for c_0. Cut open the surface Σ along c_0, we get a new translation surface with boundary. Let Σ' denote the new surface, and let V' denote the finite subset of Σ' which arises from the set V of Σ. Note that the boundary of Σ' has also m components as Σ, but V' contains at most $n-1$ points in the interior of Σ'.

- If $\partial\Sigma$ contains more than one component, and $n = 0$, then choose a path c of minimal length joining two points of V which are contained in two different components of $\partial\Sigma$. Remark that c does not have self-intersection. The path c contains an arc c_0 joining two points of V which is contained in the interior of Σ, except the endpoints. Cut open the surface Σ along the arc c_0, we obtain a new translation surface with boundary. Let Σ' denote the new surface, by construction, the boundary of Σ' has $m-1$ components. Let V' denote the finite subset of Σ which arises from the subset V of Σ.

The algorithm above can be applied again to the pair (Σ', V'), and we can continue until we get a translation surface whose boundary has only one component, with no singular points in the interior. This proves lemma. □

By Lemma 2.3.4, we can restrict the proof of the proposition to the cases : $(m, n) = (0, 1)$ and $(m, n) = (1, 0)$. Next, we show the following

Lemma 2.3.5 *Assume that $(m, n) = (0, 1)$ or $(m, n) = (1, 0)$, then there exist $2g$ geodesic segments on Σ with endpoints in V such that if we cut Σ along those segments, then we obtain a disk.*

Proof: We will only prove this lemma for the case $(m, n) = (1, 0)$, the other case can be showed by similar arguments. We proceed by induction :

- If $g = 0$, then Σ is already a disk, we have nothing to prove.

- If $g > 0$, take a point y in the set V, and consider a non-separating closed curve γ whose base-point is y which is not homotopic to $\partial\Sigma$. Let γ_0 be the closed curve with minimal length in the homotopy class (with fixed endpoints) of γ. The curve γ_0 is a union of geodesic segments whose endpoints are contained in V. Since γ_0 is not homotopic to $\partial\Sigma$, it follows that γ_0 contains an geodesic arc a joining two points in V which is not contained in $\partial\Sigma$. Note that the two endpoints of a may coincide. Since Σ is a translation surface, the arc a cannot have self-intersection. Hence, we can cut Σ along the arc a to obtain a surface of genus $g-1$ whose boundary contains two components.

  Let Σ' denote the new surface. By construction, Σ' is also a translation surface with geodesic boundary. Let C_1', C_2' denote the two components of $\partial\Sigma'$. Let V' denote the finite subset of $\partial\Sigma'$ which arises from the set V. Consider a path c of minimal length from a point in $V' \cap C_1'$ to another point in $V' \cap C_2'$. This path contains an arc c_0 with one endpoint in $V' \cap C_1'$, and the other endpoint in $V' \cap C_2'$. The arc c_0 has no self-intersections because c is of minimal length. Hence, we can cut Σ' along c_0 to obtain a translation surface of genus $g-1$ whose boundary contains only one component. Like Σ and Σ', the new surface has no singular points in its interior. This allows us to conclude by induction. □

Lemma 2.3.4 and Lemma 2.3.5 imply :

Lemma 2.3.6 *There exist $2g+m+n-1$ geodesic segments on Σ with endpoints in V such that if we cut open Σ along those segments, we will have a flat surface homeomorphic to a disk, which has no singular points in the interior.*

To complete the proof of 2.3.2 we need the following :

Lemma 2.3.7 *Let S be a flat surface with geodesic boundary, homeomorphic to a closed disk. Suppose that S has no singular points in the interior. Let V be a finite subset of ∂S such that $\partial S \setminus V$ is a union of open geodesic segments. Then there exists a triangulation of S by geodesic segments whose set of vertices is V.*

Proof: Let $a_1, \dots, a_r$ denote the points in V following an orientation. Let $\overline{a_i a_{i+1}}$ denote the geodesic segment contained in ∂S whose endpoints are a_i and a_{i+1}, for $i = 1, \dots, r$, with the convention $a_{r+1} = a_1$. We know, by the Gauss-Bonnet Theorem, that the sum of all the angles at $a_1, \dots, a_r$ is $(r-2)\pi$. We prove the lemma by induction.

- For the case $r = 3$, we have a triangle, and there is nothing to prove.

- If $r > 3$, it suffices to prove that there exists a geodesic segment which is contained in the interior of S joining two singular points in ∂S.

  Suppose that all the angles at the corners $a_1, \dots, a_r$ are less than π. Consider the path s of minimal length joining a_1 and a_3. Since $r \geqslant 4$, a_1 and a_3 are not adjacent. Because the angle at every singular point is less than π, $s \cap \partial S = \{a_1, a_3\}$, which means that s is a geodesic segment contained inside S, and we are done.

  Now, suppose that there exists a singular point whose angle is greater than or equal to π. Without loss of generality, we can assume that this point is a_1. For every $i = 2, \dots, r$, consider a path s_i of minimal length from a_1 to a_i. The path s_i is a union of geodesic segments. If one of its segment is contained in the interior of S then we are done. If not, s_i is either

$$c_i^1 = \bigcup_{j=1}^{i-1} \overline{a_j a_{j+1}},$$

  or

$$c_i^2 = \bigcup_{j=i}^{r} \overline{a_j a_{j+1}}.$$

Since we have

$$\mathbf{leng}(c_i^1) + \mathbf{leng}(c_i^2) = \sum_{j=1}^{r} \mathbf{leng}(\overline{a_j a_{j+1}})$$

which is independent of i, there exists $k \in \{2, \dots, r\}$ such that $s_i = c_i^1$, for every $i = 2, \dots, k$, and $s_i = c_i^2$, for every $i = k+1, \dots, r$. Now, if c_k^1 is a path of minimal length from a_1 to a_k, then all the angles at $a_2, \dots, a_{k-1}$ are greater than or equal to π. Similarly, if c_{k+1}^2 is a path of minimal length from a_1 to a_{k+1}, then the angles at $a_{k+2}, \dots, a_r$ are all greater than or equal to π. As a consequence, among the angles at $a_1, \dots, a_r$, there are at least $r-2$ angles greater than or equal to π, but this is impossible according to the Gauss-Bonnet Theorem. Therefore, there must be a geodesic segment which is contained inside S, and the lemma is then proved. □

Proposition 2.3.2 follows immediately from Lemma 2.3.7 and Lemma 2.3.6 above. □

2.3.3 Proof of Proposition 2.3.3

Proposition 2.3.3 follows from the following lemma :

Lemma 2.3.8 *Let Σ be a flat surface without boundary. Let $V = \{x_1, \dots, x_n\}$ be a finite subset of Σ such that $\Sigma \setminus V$ contains only regular points, and suppose that $\chi(\Sigma \setminus V) < 0$. Let γ and γ' be two simple geodesic arcs of Σ having the same endpoints in V (the two endpoints may coincide). Assume that γ and γ' are homotopic with fixed endpoints relative to V, then we have $\gamma \equiv \gamma'$.*

Proof: We first observe that there exist no Euclidean structures on a closed disk such that its boundary is the union of two geodesic segments. This is just a consequence of the Gauss-Bonnet Theorem.

Since $\chi(\Sigma \setminus V) < 0$, the universal covering of $\Sigma \setminus V$ is the open disk $\Delta = \{z \in \mathbb{C} : \ |z| < 1\}$. The flat metric structure on $\Sigma \setminus V$ give rise to a flat metric structure on Δ (which is not complete). Now, let $\tilde{\gamma}$ be a lift of γ in Δ whose endpoints are contained in the boundary of Δ . By lifting the homotopy from γ to γ', we get a lift $\tilde{\gamma}'$ of γ' which has the same endpoints as $\tilde{\gamma}$. Note that by assumption, $\tilde{\gamma}$ and $\tilde{\gamma}'$ are two geodesic in Δ.

The two curves $\tilde{\gamma}$ and $\tilde{\gamma}'$ may have intersections, but in any case, we can find (at least) an open disk D which is bounded by two arcs, one is a subsegment of $\tilde{\gamma}$, the other is a subsegment of $\tilde{\gamma}'$. Consequently, the open disk D is isometric to the interior of an Euclidian disk which is bounded by two geodesic segments. Since such a disk cannot exist, the lemma follows. □

Back to the proof of 2.3.3. Let $\hat{\mathrm{T}}_1$ and $\hat{\mathrm{T}}_2$ denote the triangulations of $\hat{\Sigma}$ which are induced by T_1 and T_2 respectively. By assumption, we have $\hat{\mathrm{T}}_2 = \hat{\varphi}(\hat{\mathrm{T}}_1)$, where $\hat{\varphi}$ is a homeomorphism of $\hat{\Sigma}$ which is isotopic to the identity by an isotopy fixing the common vertex set of $\hat{\mathrm{T}}_1$ and $\hat{\mathrm{T}}_2$ which is $\hat{V}$.

Since every edge of $\hat{\mathrm{T}}_1$ and $\hat{\mathrm{T}}_2$ is a simple geodesic segment, Lemma 2.3.8 implies immediately that $\hat{\mathrm{T}}_1 = \hat{\mathrm{T}}_2$. Therefore we have $\mathrm{T}_1 = \mathrm{T}_2$, and Proposition 2.3.3 follows. □

2.4 Flat complex affine structure on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$

In this section, we give the proof of Theorem 2.2.7. Recall that we have a fixed a translation surface S, whose set of singular points in the interior are denoted by $p_1,\dots,p_n$, and boundary components of S are denoted by $C_1,\dots,C_m$. The cone angle at p_i is $\alpha_i,\ i=1,\dots,n$, and the cone angle associated to C_j is $\beta_j,\ j=1,\dots,m$. For each $j\in\{1,\dots,m\}$, Q_j is a finite subset of C_j such that $C_j\setminus Q_j$ is a union of s_j open geodesic segments. The points in Q_j are denoted by $\{q_{1j},\dots,q_{s_jj}\}$. Let $\mathcal{V}$ denote the set $\{p_1,\dots,p_n\}\cup_{j=1}^m Q_j$.

Let $\mathcal{TR}(S)$ denote the set of all equivalence classes of triangulations (not necessarily geodesic) of S whose vertex set is $\mathcal{V}$, where two triangulations are equivalent if they are isotopic relative to $\mathcal{V}$. Let $\mathcal{T}$ be an element of $\mathcal{TR}(S)$. We denote $\mathcal{U}_\mathcal{T}$ the subset of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ consisting of pairs $([(\Sigma,\phi)],\xi)$ such that there exists a homeomorphism ϕ' in the same equivalence class as ϕ, *i.e.* $\phi^{-1}\circ\phi'\in\mathrm{Homeo}_0^+(S,\mathcal{V})$, which maps $\mathcal{T}$ onto an admissible triangulation of Σ.

Proposition 2.3.2 implies that the family $\{\mathcal{U}_\mathcal{T}\ :\ \mathcal{T}\in\mathcal{TR}(S)\}$ covers the space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. We will define coordinate charts on $\mathcal{U}_\mathcal{T}$ for each $\mathcal{T}$ in $\mathcal{TR}(S)$.

2.4.1 Definition of the local charts $\Psi_\mathcal{T}$

Given an equivalence class of triangulations $\mathcal{T}$ in $\mathcal{TR}(S)$, let $([(\Sigma,\phi)],\xi)$ be a point in $\mathcal{U}_\mathcal{T}$. By definition, we can assume that $\mathrm{T}=\phi(\mathcal{T})$ is an admissible triangulation of Σ. By Proposition 2.3.3, we know that T is unique.

Let N_1 be the number of edges of T, and N_2 be the number of triangles of T. By computing the Euler characteristic of Σ, we see that :

$$N_1 = 3(2g+n+m-2)+2\sum_{j=1}^m s_j \text{ and } N_2 = 2(2g+n+m-2)+\sum_{j=1}^m s_j.$$

We construct a map from $\mathcal{U}_{\mathcal{T}}$ to $\mathbb{C}^{N_1}$ as follows :

Choose an orientation for every edge of T. For each triangle Δ in T, there exists an isometric embedding of this triangle into $\mathbb{R}^2$ such that the vector field ξ is mapped to the constant vertical vector field $(0,1)$, defined on the image of Δ. By this embedding, each oriented side of the triangle Δ is mapped into a vector in $\mathbb{R}^2 \simeq \mathbb{C}$. As a consequence, we can associate to every oriented edge e of T a complex number $z(e)$. Note that, even though each edge e in the interior of Σ belongs to two distinct triangles, the complex number $z(e)$ is well defined because the vector field ξ is parallel and normalized. The procedure above defines a map from $\mathcal{U}_{\mathcal{T}}$ into $\mathbb{C}^{N_1}$. Let $\Psi_{\mathcal{T}}$ denote this map.

We get immediately the following important observations :

Lemma 2.4.1 *i) Let e_i, e_j, e_k be three edges of* T *which bound a triangle. Then we have*

$$\pm z(e_i) \pm z(e_j) \pm z(e_k) = 0, \tag{2.3}$$

where the signs are determined by the orientation of e_i, e_j and e_k.

ii) If $e_1, \dots, e_k$ are the k edges of T *which bound an open disk in Σ, then we have*

$$\pm z(e_1) \pm \cdots \pm z(e_k) = 0, \tag{2.4}$$

where, again, the signs are determined by the orientations of the edges.

Proof: Assertion $i)$ is straight forward. Assertion $ii)$ follows from $i)$. Namely, let D denote the disk bounded by $e_1, \dots, e_k$. The disk D is divided into triangles by the triangulation T. By $i)$, three sides of a triangle verify (2.3). Note that every edge of T inside D belongs to two distinct triangles. If for each triangle, we choose the orientation of its boundary coherently with the orientation of the surface, and write the corresponding equation according to this orientation, then, by taking the sum over all the triangles inside D, we get (2.4). □

Let $\mathrm{S}_{\mathcal{T}}$ denote the linear equation system consisting of N_2 equations of type 2.3 corresponding to the triangles of $\mathcal{T}$. From what we have seen, the vector $\Psi_{\mathcal{T}}([(\Sigma,\phi)],\xi)$ is a solution of the system $\mathrm{S}_{\mathcal{T}}$.

Let $\mathrm{V}_{\mathcal{T}}$ denote the subspace of $\mathbb{C}^{N_1}$ consisting of solutions of the system $\mathrm{S}_{\mathcal{T}}$. We have

Lemma 2.4.2 *$\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is an open subset of $\mathrm{V}_{\mathcal{T}}$.*

Proof: The fact that $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is contained in $\mathrm{V}_{\mathcal{T}}$ is a direct consequence of Lemma 2.4.1.

Now, let Z be the image of $([(\Sigma, \phi)], \xi)$ by $\Psi_{\mathcal{T}}$, and let $Z' = (z'_1, \dots, z'_{N_1})$ be a vector in a neighborhood of Z in $\mathrm{V}_{\mathcal{T}}$. Using the triangulation T of Σ, we construct a flat surface from Z' as follows :

. Construct an Euclidean triangle from z'_i, z'_j, z'_k if z'_i, z'_j, z'_k verify an equation of type (2.3).

. Identify two sides of two distinct triangles if they correspond to the same complex number z'_i.

Clearly by this construction we obtain a translation surface Σ' homeomorphic to Σ. The surface Σ' has n singular points of cone angles $\alpha_1, \dots, \alpha_n$ in the interior, and the boundary of Σ' has m components with associated cone angles $\beta_1, \dots, \beta_j$.

Moreover, we also get a triangulation T' of Σ' by geodesic segments. Each triangle in T' corresponds to a triangle in $\mathbb{R}^2$ specified by three complex numbers which are coordinates of Z', hence we get a normalized parallel vector field ξ' on Σ' which is defined by the constant vertical vector field $(0, 1)$ on the Euclidean plan $\mathbb{R}^2$.

Define an orientation preserving homeomorphism

$$f : \Sigma \longrightarrow \Sigma'$$

as follows : f maps each edge of T onto the corresponding edge of T' (*i.e.* the edge of T that corresponds to the same coordinate), and the restriction f on each triangle of T is a linear transformation of $\mathbb{R}^2$. Let ϕ' denote the map

$$\phi' = f \circ \phi : S \longrightarrow \Sigma'.$$

It follows that the pair $([(\Sigma', \phi')], \xi')$ represents a point of $\mathcal{U}_{\mathcal{T}}$ close to $([(\Sigma, \phi)], \xi)$. By construction, it is clear that $Z' = \Psi_{\mathcal{T}}([(\Sigma', \phi')], \xi')$. Hence, we deduce that $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is an open set of $\mathrm{V}_{\mathcal{T}}$. □

2.4.2 Injectivity of $\Psi_{\mathcal{T}}$

Lemma 2.4.3 *The map $\Psi_{\mathcal{T}}$ is injective.*

Proof: Let $([(\Sigma_1, \phi_1)], \xi_1)$ and $([(\Sigma_2, \phi_2)], \xi_2)$ be two points in $\mathcal{U}_{\mathcal{T}}$ such that $\Psi_{\mathcal{T}}([(\Sigma_1, \phi_1)], \xi_1) = \Psi_{\mathcal{T}}([(\Sigma_2, \phi_2)], \xi_2)$. By definition, we can assume that $\mathrm{T}_1 = \phi_1(\mathcal{T})$ and $\mathrm{T}_2 = \phi_2(\mathcal{T})$ are admissible triangulations of Σ_1 and Σ_2 respectively. By Proposition 2.3.3, we know that T_1 and T_2 are unique.

Now, the hypothesis $\Psi_{\mathcal{T}}([(\Sigma_1, \phi_1)], \xi_1) = \Psi_{\mathcal{T}}([(\Sigma_2, \phi_2)], \xi_2)$ implies that there exists an isometry

$$h : \Sigma_1 \longrightarrow \Sigma_2,$$

which maps each triangle of T_1 onto a triangle of T_2, and also ξ_1 onto ξ_2. It follows that the homeomorphism

$$\phi_2^{-1} \circ h \circ \phi_1 : S \longrightarrow S$$

fixes all the points in $\mathcal{V}$, and preserves each triangles of $\mathcal{T}$. We deduce that the map $\phi_2^{-1} \circ h \circ \phi_1$ is isotopic to the identity of S by an isotopy fixing all the points in $\mathcal{V}$. Therefore, by definition, we have $([(\Sigma_1, \phi_1)], \xi_1) = ([(\Sigma_2, \phi_2)], \xi_2)$. □

2.4.3 Computation of dimension of $\mathrm{V}_{\mathcal{T}}$

Lemma 2.4.4 $\dim_{\mathbb{C}} \mathrm{V}_{\mathcal{T}} = \begin{cases} 2g+n-1, & \textit{if } m=0; \\ 2g+n+m-2+\sum_{j=1}^{m} s_j, & \textit{otherwise.} \end{cases}$

Proof: Recall that $\mathrm{V}_{\mathcal{T}}$ is the subspace of $\mathbb{C}^{N_1}$ consisting of solutions of the system $\mathrm{S}_{\mathcal{T}}$. Since the system $\mathrm{S}_{\mathcal{T}}$ contains N_2 equations, we have

$$\dim \mathrm{V}_{\mathcal{T}} \geqslant N_1 - N_2 = \sum_{j=1}^{m} s_j + 2g + m + n - 2. \tag{2.5}$$

Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{U}_{\mathcal{T}}$, and T be the admissible triangulation of Σ which is the image of $\mathcal{T}$ by ϕ.

Let $a_1, a_2, \dots, a_{s_1+\dots+s_m}$ denote the edges of T which are contained in the boundary of Σ. Choose a family of primitive edges in T which will be denoted by $b_1, \dots, b_{2g+m+n-1}$. Recall that for any oriented edge e of T, $z(e)$ is the complex number associated to e in the construction of $\Psi_{\mathcal{T}}$.

By definition, we have $\mathrm{int}(\Sigma) \setminus \cup_{j=1}^{2g} b_j$ is an open disk. Using Lemma 2.4.1 *ii*), we deduce that if e is any edge of T which does not belong to the set $\{a_1, \dots, a_{s_1+\dots+s_m}, b_1, \dots, b_{2g+m+n-1}\}$, then $z(e)$ can be written as a linear combination of $z(a_1), \dots, z(a_{s_1+\dots+s_m}), z(b_1), \dots, z(b_{2g+m+n-1})$, whose coefficients are determined by the triangulation T. Note that the coefficients of these linear functions belong the set $\{-1, 0, 1\}$. We deduce

$$\dim \mathrm{V}_{\mathcal{T}} \leqslant \sum_{j=1}^{m} s_j + 2g + m + n - 1. \tag{2.6}$$

Suppose that the edges $a_1, \dots, a_{s_1+\dots+s_m}$ are oriented coherently with the orientation of the surface Σ. Apply (2.4) to the disk $\mathbf{D} = \mathrm{int}(\Sigma) \setminus \cup_{j=1}^{2g+m+n-1} b_j$, we get

$$z(a_1) + \cdots + z(a_{s_1+\cdots+s_m}) = 0. \tag{2.7}$$

The numbers $z(b_j)$, $j = 1, \ldots, 2g+m+n-1$, do not appear in the equation (2.7) because each of the edges b_j belongs to two different triangles.

Here, we have two issues :

- Case 1 : $m = 0$, that is the surface Σ is closed. In this case, the equation (2.7) is void. However, this also means that the sum of all equations in the system $\mathrm{S}_{\mathcal{T}}$, with appropriate choices of signs, is the trivial equation $0 = 0$. This implies $\mathrm{rank}(\mathrm{S}_{\mathcal{T}}) \leqslant N_2 - 1$. Hence

$$\dim \mathrm{V}_{\mathcal{T}} \geqslant N_1 - (N_2 - 1) = 2g + n - 1. \tag{2.8}$$

  From (2.6) and (2.8), we conclude that $\dim \mathrm{V}_{\mathcal{T}} = 2g + n - 1$.

- Case 2 : $m > 0$, that is the boundary of Σ is not empty. The equation (2.7) implies that the vector $(z(a_1), \ldots, z(a_{s_1+\cdots+s_m}), z(b_1), \ldots, z(b_{2g+m+n-1}))$ belongs to a hyperplane of $\mathbb{C}^{(s_1+\cdots+s_m)+2g+m+n-1}$. Therefore we have

$$\dim \mathrm{V}_{\mathcal{T}} \leqslant \sum_{j=1}^{m} s_j + 2g + m + n - 2. \tag{2.9}$$

  From (2.5) and (2.9), we conclude that $\dim \mathrm{V}_{\mathcal{T}} = \sum_{j=1}^{m} s_j + 2g + m + n - 2$.

□

2.4.4 Coordinate change

Let $\mathcal{T}_1, \mathcal{T}_2$ be two equivalence classes of triangulations in $\mathcal{TR}(S)$. Suppose that $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2} \neq \varnothing$, and let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2} \neq \varnothing$. Let $\mathrm{T}_1, \mathrm{T}_2$ be the admissible triangulations of Σ corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. As usual, we denote $\Psi_{\mathcal{T}_1}, \Psi_{\mathcal{T}_2}$ the local charts on $\mathcal{U}_{\mathcal{T}_1}$ and $\mathcal{U}_{\mathcal{T}_2}$ respectively. We have :

Lemma 2.4.5 *There exists an invertible complex linear map*

$$\mathbf{L} : \mathbb{C}^{N_1} \longrightarrow \mathbb{C}^{N_1}$$

such that $\Psi_{\mathcal{T}_2}([(\Sigma', \phi')], \xi') = \mathbf{L} \circ \Psi_{\mathcal{T}_1}([(\Sigma', \phi')], \xi')$, *for every* $([(\Sigma', \phi')], \xi')$ *in a neighborhood of* $([(\Sigma, \phi)], \xi)$.

Proof: Let e be an edge of T_2. Let Δ_i, $i \in I$, denote the triangles in T_1 such that $\Delta_i \cap \mathrm{int}(e) \neq \varnothing$, $\forall i \in I$.

Using the developing map, we can construct a polygon $\mathbf{P}$ in $\mathbb{R}^2$ by gluing isometric copies of Δ_i's ($i \in I$), such that e corresponds to a diagonal $\tilde{e}$ inside $\mathbf{P}$. The polygon $\mathbf{P}$ may contain several copies of a single Δ_i. By this construction, we get a map :

$$\varphi : \mathbf{P} \longrightarrow \Sigma,$$

which is locally isometric, such that $\varphi(\tilde{e}) = e$.

Since the map φ sends geodesic segments in the boundary of $\mathbf{P}$ onto edges of T_1, it follows that the complex numbers associated to the edge e can be written as linear function of the complex numbers associated to the edges corresponding to geodesic segments in the boundary of $\mathbf{P}$. Note that the coefficients of these linear functions are unchanged if we replace $([(\Sigma, \phi)], \xi)$ by another pair $([(\Sigma', \phi')], \xi')$ nearby in $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$, and this argument is reciprocal between T_1 and T_2. We deduce that the coordinate change between $\Psi_{\mathcal{T}_1}$ and $\Psi_{\mathcal{T}_2}$, in a neighborhood of $([(\Sigma, \phi)], \xi)$, is a complex linear transformation of $\mathbb{C}^{N_1}$ which sends $V_{\mathcal{T}_1}$ onto $V_{\mathcal{T}_2}$. The lemma is then proved. □

The proof of Theorem 2.2.7 is now complete. □

2.4.5 Remark

Let $\mathcal{T}$ be an equivalence class in $\mathcal{TR}(S)$. Let $\mathcal{U}_{\mathcal{T}}, \Psi_{\mathcal{T}}, V_{\mathcal{T}}$ be as in the proof of 2.2.7. We already know that $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is an open set in $V_{\mathcal{T}}$, but more can be said about $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$.

Consider $\mathcal{T}$ as a particular triangulations of S. Choose a numbering for the set of edges of $\mathcal{T}$, and an orientation for each edge.

To each triangle Δ_α in $\mathcal{T}, \alpha = 1, \dots, N_2$, we can associate a Hermitian form $\mathbf{H}_\alpha$ of $\mathbb{C}^{N_1}$ as follows : if the sides of Δ_α are denoted by e_i, e_j, e_k, then $\mathbf{H}_\alpha(Z, W) = \frac{i}{4}(z_i \overline{w}_j - z_j \overline{w}_i)$, where $Z = (z_1, \dots, z_{N_1})$, and $W = (w_1, \dots, w_{N_1})$ are vectors in $\mathbb{C}^{N_1}$.

The Hermitian form $\mathbf{H}_\alpha$ verifies the following property : if $Z = \Psi_{\mathcal{T}}([(\Sigma, \phi)], \xi)$, then $|\mathbf{H}_\alpha(Z, Z)|$ is equal to the area of the triangle $\phi(\Delta_\alpha)$ in Σ. By interchanging z_i and z_j if necessary, we can assume that $\mathbf{H}_\alpha(Z, Z) > 0$ for every $\alpha = 1, \dots, N_2$.

Now, let Z be a vector in $V_{\mathcal{T}}$, let $\Sigma(Z)$ denote the surface obtained by the method described in the inverse construction of $\Psi_{\mathcal{T}}$. The necessary and sufficient condition for $\Sigma(Z)$ to be a translation surface homeomorphic to S is that

$$\mathrm{H}_\alpha(Z, Z) > 0, \quad \text{for every } \alpha = 1, \ldots, N_2.$$

Therefore, $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is the set $\{Z \in \mathrm{V}_{\mathcal{T}} \mid \mathrm{H}_\alpha(Z, Z) > 0, \ \forall \alpha = 1, \ldots, N_2\}$.

2.5 Properness of the action of Mapping Class Group

In this paragraph, we prove Proposition 2.2.8. First, we recall some basic dfinitions of the Teichmüller Theory.

2.5.1 Elements of Teichmüller Theory

We refer to [Ga] for a more detailed presentation of this important theory.

Quasiconformal mappings

Let D be a domain of the complex plane $\mathbb{C}$, and $f : D \longrightarrow \mathbb{C}$ a function defined on D. Assume that the function f is written as $f(x, y) = u(x, y) + \imath v(x, y)$. We say that f is absolutely continuous on lines, and abbreviate by ACL, if for every rectangle R in D with sides parallel to the x-axis and y-axis, both $u(x, y)$ and $v(x, y)$ are absolutely continuous on almost every horizontal line and almost every vertical line in R. The functions u and v will then have partial derivatives u_x, u_y, v_x, v_y almost everywhere in D. In general, the partial derivatives u_x, u_y, v_x, v_y are only distributions since they are not defined everyw-here.

The complex derivatives of f are defined by

$$f_z = \frac{1}{2}(f_x - \imath f_y) \text{ and } f_{\bar{z}} = \frac{1}{2}(f_x + \imath f_y).$$

Definition 2.5.1 (Analytic definition of Quasiconformal Mapping) *Let f be a homeomorphism from a domain $D \subset \mathbb{C}$ to another domain $D' \subset \mathbb{C}$. The map f is K-*quasiconformal *$(K > 1)$ if*

(i) f is ACL in D, and

(ii) $|f_{\bar{z}}| \leqslant k|f_z|$ almost everywhere, where $k = \frac{K-1}{K+1} < 1$.

The minimal possible value of K for which (ii) holds is called the dilatation *of f.*

The quasiconformal mappings verify the following property, if f_1 is K_1-quasiconformal and f_2 is K_2-quasiconformal, then $f_2 \circ f_1$ is $K_1 K_2$-quasiconformal.

The Teichmüller space $\mathcal{T}(\tilde{g}, \tilde{n})$

Let $\tilde{S}$ be a Riemann surface of genus $\tilde{g}$ without boundary, and $\{\tilde{p}_1, \dots, \tilde{p}_{\tilde{n}}\}$ be $\tilde{n}$ points of $\tilde{S}$. Let $\widetilde{\mathcal{T}}(\tilde{g}, \tilde{n})$ denote the set of all pairs (X, f), where X is a Riemann surface, and $f : \tilde{S} \longrightarrow X$ is a quasi-conformal homeomorphism. We can define an equivalence relation on $\widetilde{\mathcal{T}}(\tilde{g}, \tilde{n})$ as follows : (X, f) and (X', f') are equivalent if and only if there exists a *conformal* homeomorphism $h : X \longrightarrow X'$, such that the quasi-conformal map $f'^{-1} \circ h \circ f : \tilde{S} \longrightarrow \tilde{S}$ is isotopic to the identity by an isotopy fixing the points $\tilde{p}_1, \dots, \tilde{p}_{\tilde{n}}$. By definition, the *Teichmüller space* $\mathcal{T}(\tilde{g}, \tilde{n})$ is the space of equivalence classes of this equivalence relation. The equivalence class of a pair (X, f) is denoted by $[(X, f)]$.

Teichmüller metric

Let (X_1, f_1) and (X_2, f_2) be two pairs in $\widetilde{\mathcal{T}}(\tilde{g}, \tilde{n})$. The *Teichmüller distance* between $[(X_1, f_1)]$ and $[(X_2, f_2)]$ is defined by

$$\mathbf{d}_{\text{Teich}}([(X_1, f_1)], [(X_2, f_2)]) = \frac{1}{2} \inf\{\log K(f_2 \circ f \circ f_1^{-1})\},$$

where the infimum is taken over all quasi-conformal homeomorphisms f of $\tilde{S}$ which can be deformed into $\mathrm{Id}_{\tilde{S}}$ by an isotopy fixing every point in the set $\{\tilde{p}_1, \dots, \tilde{p}_{\tilde{n}}\}$, and $K(f_2 \circ f \circ f_1^{-1})$ is the dilation of $f_2 \circ f \circ f_1^{-1} : X_1 \longrightarrow X_2$. The Teichmüller distance between two equivalence classes in $\mathcal{T}(\tilde{g}, \tilde{n})$ does not depend on the representatives to be used in this definition.

Action of Modular Group $\Gamma(\tilde{g}, \tilde{n})$ on $\mathcal{T}(\tilde{g}, \tilde{n})$

The mapping class group $\Gamma(\tilde{g}, \tilde{n})$ the group of all quasi-conformal homeomorphisms of $\tilde{S}$ which is identity on the set $\{\tilde{p}_1, \dots \tilde{p}_{\tilde{n}}\}$, modulo the connected component of identity (of $\tilde{S}$).

The mapping class group $\Gamma(\tilde{g}, \tilde{n})$ acts on $\mathcal{T}(\tilde{g}, \tilde{n})$ as follows. Let $[h]$ be an element of $\Gamma(\tilde{g}, \tilde{n})$ which is represented by a quasiconformal map $h : S \longrightarrow S$. Let $[(X, f)]$ be an equivalence class in $\mathcal{T}(\tilde{g}, \tilde{n})$. We have :

$$[h] \cdot [(X, f)] = [(X, f \circ h)].$$

It is well known that the action of $\Gamma(\tilde{g}, \tilde{n})$ on $\mathcal{T}(\tilde{g}, \tilde{n})$ is properly discontinuous with respect to the topology induced by the Teichmüller metric.

2.5.2 Embedding of the group $\Gamma(S, \mathcal{V})$

Let $\tilde{g} = g+m-1$, $\tilde{n} = 2n+\sum_{j=1}^{m} s_j$. By definition, the double $\hat{S}$ of S is a closed surface of genus $\tilde{g}$, and the subset $\hat{\mathcal{V}}$ of $\hat{\Sigma}$ contains $\tilde{n}$ points. If φ is a homeomorphism of S, we denote $\hat{\varphi}$ the homeomorphism of $\hat{S}$ that lifts φ. We have

Lemma 2.5.2 *The homomorphism $\varphi \longmapsto \hat{\varphi}$ induces an embedding of the group $\Gamma(S, \mathcal{V})$ into the group $\Gamma(\tilde{g}, \tilde{n})$.*

Proof: Since any homeomorphism is isotopic to a diffeomorphism, and a diffeomorphism is quasi-conformal, given an homeomorphism $\hat{\varphi}$ of $\hat{S}$, there always exists a quasi-conformal homeomorphism $\hat{\varphi}'$ which is isotopic to $\hat{\varphi}$. As a consequence, we can define map from $\Gamma(S, \mathcal{V})$ into $\Gamma(\tilde{g}, \tilde{n})$ by associating to the equivalence class of φ in $\Gamma(S, \mathcal{V})$ the equivalence class of the quasi-conformal $\hat{\varphi}'$ in $\Gamma(\tilde{g}, \tilde{n})$. This map is clearly a homomorphism.

If $\hat{\varphi}'$ is isotopic to $\mathrm{Id}_{\hat{S}}$, then so is $\hat{\varphi}$. By definition of $\Gamma(S, \mathcal{V})$, this implies that φ is in the equivalence class of Id_S. We deduce that the homomorphism defined above is injective, and the lemma follows. $\square$

2.5.3 A Mapping from $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ to $\mathcal{T}(\tilde{g}, \tilde{n})$

There is a natural map F from $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ into $\mathcal{T}(\tilde{g}, \tilde{n})$, which we will call the *forgetting map*.

Given a point $([(\Sigma, \phi)], \xi)$ in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, let $\hat{\Sigma}$ be the double of Σ which is equipped with the induced flat metric, and $\hat{\phi}$ be the homeomorphism from $\hat{S}$ onto $\hat{\Sigma}$ that lifts ϕ. Note the flat metric structure on $\hat{\Sigma}$ induces a conformal structure on the open dense set $\hat{\Sigma} \setminus \hat{\phi}(\hat{\mathcal{V}})$ of $\hat{\Sigma}$, and since $\hat{\phi}(\hat{\mathcal{V}})$ is finite, this conformal structure can be extended uniquely into a conformal structure on $\hat{\Sigma}$. Let $\hat{\phi}'$ be any quasi-conformal map from $\hat{S}$ onto $\hat{\Sigma}$ which is isotopic to $\hat{\phi}$ by an isotopy which is constant on the set $\hat{\mathcal{V}}$ of $\hat{S}$.

The map F is defined as follows : the image by F of the pair $([(\Sigma, \phi)], \xi)$ in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ is the equivalence class of the pair $(\hat{\Sigma}, \hat{\phi}')$ in $\mathcal{T}(\tilde{g}, \tilde{n})$, where $\hat{\Sigma}$ is now considered as a Riemann surface.

Proposition 2.5.3 *The map* F *is continuous.*

Proof: Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, and $\{([(\Sigma_k, \phi_k)], \xi_k),\ k \in \mathbb{N}\}$ be a sequence in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ converging to $([(\Sigma, \phi)], \xi)$. We can suppose that the map $\hat{\phi} : \hat{S} \longrightarrow \hat{\Sigma}$ that lifts ϕ is quasi-conformal so that we can write $\mathrm{F}([(\Sigma,\phi)],\xi)=[(\hat{\Sigma},\hat{\phi})]$.

Let T be an admissible triangulation of Σ, and $\mathcal{T}$ be the equivalence class of $\phi^{-1}(\mathrm{T})$ in $\mathcal{TR}(S)$. By definition, $([(\Sigma,\phi)],\xi)$ is a point in $\mathcal{U}_{\mathcal{T}}$. Without loss of generality, we can assume that the sequence $\{([(\Sigma_k,\phi_k)],\xi_k),\ k\in\mathbb{N}\}$ is also contained in $\mathcal{U}_{\mathcal{T}}$.

As we have seen in the proof of Theorem 2.2.7, there exists a local chart $\Psi_{\mathcal{T}}$ of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ which is defined on $\mathcal{U}_{\mathcal{T}}$. Put $Z=\Psi_{\mathcal{T}}([(\Sigma,\phi)],\xi)$, and $Z_k=\Psi_{\mathcal{T}}([(\Sigma_k,\phi_k)],\xi_k)$. By assumption we have $Z_k\overset{k\to\infty}{\longrightarrow}Z$ in $\mathbb{C}^{N_1}$.

Recall that, by the definition of $\Psi_{\mathcal{T}}$, for every point $([(\Sigma',\phi')],\xi')$ in $\mathcal{U}_{\mathcal{T}}$, we can write $\phi'=f\circ\phi$, where $f:\Sigma\longrightarrow\Sigma'$ is a homeomorphism such that

- $f(\mathrm{T})$ is an admissible triangulation of Σ' denoted by T'.
- f sends an edge of T onto an edge of T', and the restriction of f' into the a triangle of T is a linear transformation of $\mathbb{R}^2$.

Therefore, for every $k\in\mathbb{N}$, we can assume that $\phi_k=f_k\circ\phi$, where $f_k:\Sigma\longrightarrow\Sigma_k$ is a homeomorphism with the same properties as f above.

Let $\hat{\mathrm{T}}$ be the geodesic triangulation of $\hat{\Sigma}$ which is induced by T, and let $\hat{f}_k$ be the homeomorphism from $\hat{\Sigma}$ onto $\hat{\Sigma}_k$ that lifts f_k. It follows immediately that $\hat{f}_k$ maps $\hat{\mathrm{T}}$ onto a geodesic triangulation of $\hat{\Sigma}_k$, and we can assume that $\hat{\phi}_k=\hat{f}_k\circ\hat{\phi}$.

Since $\hat{f}_k$ is clearly quasi-conformal, and by assumption, $\hat{\phi}$ is also quasi-conformal, it follows that $\hat{\phi}_k$ is also quasi-conformal. Therefore, we can write

$$\mathrm{F}([(\Sigma_k,\phi)_k],\xi_k)=[(\hat{\Sigma}_k,\hat{\phi}_k)],\ \forall k.$$

All we need to prove is that

$$\mathbf{d}_{\mathrm{Teich}}([(\hat{\Sigma},\hat{\phi})],[(\hat{\Sigma}_k,\hat{\phi}_k)])\overset{k\to\infty}{\longrightarrow}0.$$

It is clear that, as Z_k tends to Z, the restriction of $\hat{f}_k$ on each triangle of $\hat{\mathrm{T}}$ tends to identity, which implies that

$$\lim_{k\to\infty}K(\hat{f}_k)=1,$$

where $K(\hat{f}_k)$ is the dilatation of $\hat{f}_k$. By the definition of $\mathbf{d}_{\mathrm{Teich}}$, it follows that

$$\lim_{k\to\infty} \mathbf{d}_{\mathrm{Teich}}([(\hat{\Sigma},\hat{\phi})],[(\hat{\Sigma}_k,\hat{\phi}_k)])=0,$$

and the proposition follows. □

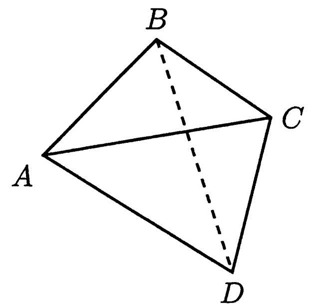
2.5.4 Proof of Proposition 2.2.8

By definition, the map F is obviously $\Gamma(S,\mathcal{V})$-equivariant. By Lemma 2.5.2, we know that $\Gamma(S,\mathcal{V})$ is a subgroup of $\Gamma(\tilde{g},\tilde{n})$. It is well known that the action of $\Gamma(\tilde{g},\tilde{n})$ is properly discontinuous on $\mathcal{T}(\tilde{g},\tilde{n})$. Since F is continuous, and $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ and $\mathcal{T}(\tilde{g},\tilde{n})$ are clearly locally compact, we deduce that the action of $\Gamma(S,\mathcal{V})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ is properly discontinuous. □

2.6 Changes of triangulations

Let $[(\Sigma,\phi)]$ be an element of the space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})^*$, we have seen that an admissible geodesic triangulation of Σ (cf. Definition 2.3.1) allows us to construct a local chart for $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$. In this section, we are interested in relations between geodesic triangulations of Σ. More precisely, we want to answer the question : How to go from an admissible triangulation to another one. This will play a crucial role in our construction of the volume form on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$.

Let us start with the simplest example : let $ABCD$ be a convex quadrilateral in $\mathbb{R}^2$. There are only two ways to triangulate $ABCD$: one by adding the diagonal AC, and the other by adding the diagonal BD.



This example suggests

Definition 2.6.1 (Elementary Move and Connected Triangulations) *Let Σ be a flat surface with geodesic boundary. Let T be a triangulation of Σ by geodesic segments whose set of vertices contains the set of singularities of Σ. An* elementary move *of T is a transformation as follows : take two adjacent triangles of T which form a convex quadrilateral, replace the common side of the two triangles by the other diagonal of the quadrilateral (if these two triangles have more than one common sides, just take one of them). After such a move, we obtain evidently a another geodesic triangulation of Σ with the same set of vertices as T.*

Let $\mathrm{T}_1, \mathrm{T}_2$ be two geodesic triangulations of Σ whose sets of vertices coincide. We say that T_1 and T_2 are connected *if there exists a sequence of elementary moves which transform T_1 into T_2.*

In this section, we prove the following theorem

Theorem 2.6.2 *Let Σ be a flat surface with geodesic boundary. Let $p_1, \dots, p_n$ denote the singularities of Σ. Suppose that Σ satisfies the following condition*

$$(Q') \qquad \text{for every closed curve } c \subset \mathrm{int}(\Sigma \setminus \{p_1, \dots, p_n\}), \text{ we have } \mathbf{orth}(c) \in \{\pm \mathrm{Id}\},$$

where $\mathbf{orth}(c)$ is the orthogonal part of the holonomy of c. Let $\mathrm{T}_1, \mathrm{T}_2$ be two geodesic triangulations of Σ such that the set of vertices of T_i is $\{p_1, \dots, p_n\}$, $i = 1, 2$, then T_1 and T_2 are connected.

Remark: The changes of triangulations by elementary moves, which are also called *flips*, are already studied in the context of flat surfaces (not necessarily translation surfaces). In this general situation, Theorem 2.6.2 is already known, it results from the fact that any geodesic triangulation whose vertex set contains all the singularities can be transformed by flips into a special one, called Delaunay triangulation, which is unique up to some flips (see [BS] for further detail). However, we would like to introduce another proof of this fact in the case of translation surfaces. The proof we present here is based on an observation on polygons, and uses some basic properties of translation and semi-translation surfaces.

We start by proving the following fact about Euclidean polygons :

Lemma 2.6.3 *Let P be a polygon in $\mathbb{R}^2 \simeq \mathbb{E}^2$. Let T be a triangulations of P whose edges are diagonals. Let d be a diagonal of P which is contained inside P, but not an edge of T. Then there exists a sequence of elementary moves which transform T into a triangulation containing d.*

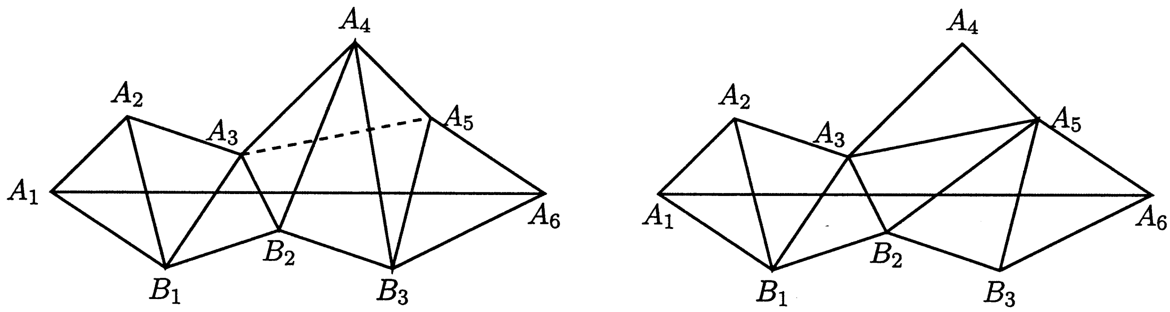
Remark: In this situation, we only consider triangulations whose edges are diagonals of P, by 'diagonal of P' we mean a geodesic segment contained inside P whose endpoints are vertices of P.

Proof: Since the diagonal d is not contained in T, it intersects some edges of T. Let m be the number of intersection points of d and the diagonals in T. Note that we only count intersection points which are not vertices of the polygon P. These m intersection points divide d into $m+1$ sub-segments, each sub-segment is contained in a triangle of T. The union of these $m+1$ triangles is a polygon P_1 which contains d as a diagonal. The number of sides of P_1 is $m+3$. Obviously, we get a triangulation T_1 of P_1 which is induced by T. Note that d intersects all the diagonals in T_1. It suffices to show that there exists a sequence of elementary moves in P_1 that transform T_1 into a triangulation containing d. We prove this by induction.

. If $m = 1$, then P_1 is a quadrilateral, and an elementary move suffices to transform T_1 into a triangulation containing d.

. For $m > 1$, let $a_1, \dots, a_m$ denote the set of edges of T_1. By construction we have $d \cap a_i \neq \varnothing$ for every $i = 1, \dots, m$. We will show that there exist elementary moves which transform T_1 into another triangulation T_2 of P_1 such that d intersects at most $m-1$ diagonals in T_2.

  Equip the plane $\mathbb{R}^2$ with the Cartesian coordinates such that d is a horizontal segment contained in the Ox axis. Let $x : \mathbb{R}^2 \longrightarrow \mathbb{R}$, and $y : \mathbb{R}^2 \longrightarrow \mathbb{R}$ denote the two coordinate functions. Let $A_1, \dots, A_r$, and $B_1, \dots, B_s$ denote the vertices of P_1 such that $y(A_1) = y(A_r) = 0, x(A_1) < x(A_r),\ y(A_i) > 0$, for $i = 2, \dots, r-1$, and $y(B_j) < 0$, for $j = 1, \dots, s$. The points $A_1, \dots, A_r$ are ordered in the clockwise sense, and the points $B_1, \dots, B_s$ are ordered in the counter-clockwise sense. Note that, since $m > 1$, we can always assume that $r \geqslant 4$.



  There exists i_0, $2 \leqslant i_0 < r$, such that $y(A_{i_0}) \geqslant y(A_i), \forall i \in \{1, \dots, r\}$, and $y(A_{i_0}) > y(A_i)$ if $i < i_0$. By assumption, we see that the segment $\overline{A_{i_0-1}A_{i_0+1}}$ is a diagonal of P_1. Since $r \geqslant 4$, we have $\overline{A_{i_0-1}A_{i_0+1}} \neq \overline{A_1A_r}$. Clearly, the segment $\overline{A_{i_0-1}A_{i_0+1}}$ does not intersect $d = \overline{A_1A_r}$ since both $y(A_{i_0-1})$ and $y(A_{i_0+1})$ must be positive or zero, and at least one of them is strictly positive. Moreover, the number of intersection points of $\overline{A_{i_0-1}A_{i_0+1}}$ with the diagonals in T_1 is strictly less than m. By induction assumption, there exists a sequence of elementary moves which transform T_1 into a new triangulation T_2 of P_1 which contains $\overline{A_{i_0-1}A_{i_0+1}}$.

Now, the triangulation T_2 contains m diagonals, one of them is $\overline{A_{i_0-1}A_{i_0+1}}$. We have seen that $\overline{A_{i_0-1}A_{i_0+1}}$ does not intersect d. It follows that d intersects at most $m-1$ diagonals in T_2, and hence we are done.

□

Corollary 2.6.4 *Let* P *be a polygon in the Euclidean plane* $\mathbb{E}^2$. *Let* T_1 *and* T_2 *be two triangulations of* P *by diagonals. Then there exists a sequence of elementary moves which transform* T_1 *into* T_2.

Proof: Let n be the number of sides of P. We show this corollary by induction.

- If $n=4$ there are two possibilities :
  - P is not convex. In this case, P has only one triangulation, hence $\mathrm{T}_1=\mathrm{T}_2$.
  - P is convex. In this case, if $\mathrm{T}_1 \neq \mathrm{T}_2$, then T_2 is obtained from T_1 by an elementary move.
- For $n>4$, if the triangulations T_1 and T_2 have a common edge, then we are done since this common edge divides P into two polygons whose numbers of sides are strictly less than n. We are left with the case where T_1 and T_2 have no common edges. In this case, choose an arbitrary edge d of T_2, by Lemma 2.6.3, there exists a sequence of elementary moves which transform T_1 into a new triangulation T'_1 which contains d. The corollary is then proved. □

2.6.1 Proof of Theorem 2.6.2

Let g be the genus of Σ, and p be the number of components of its boundary. Observe that every geodesic triangulation of Σ whose set of vertices is $\{p_1,\dots,p_n\}$ must contain all the geodesic segments on the boundary of Σ.

Let n_1 be the number of singular points on the boundary of Σ, and n_2 be the number of singular points in the interior of Σ. By the computation of Euler characteristic of Σ, we see that the triangulations T_1 and T_2 have the same number N_e of edges. We have

$$N_e = 3(\frac{2}{3}n_1+n_2+2g+p-2).$$

Let $k, 0 \leqslant k \leqslant N_e$, be the number of common edges of T_1 and T_2. Since the boundary of Σ contains n_1 edges, we have $k \geqslant n_1$. If $k=N_e$, then $\mathrm{T}_1=\mathrm{T}_2$. Assume that $n_1 \leqslant k < N_e$, we will proceed by induction.

Given a geodesic triangulation T on Σ, let e be a geodesic segment joining two vertices of T. If e is not contained in T, then, using a developing map, one can construct an Euclidian polygon P_e in $\mathbb{R}^2$ which is composed by isometric copies of the triangles in T which are crossed by e. Note that a triangle Δ in T may have several copies inside P, the number of those copies is equal to the number of connected components of the set $\mathrm{int}(e) \cap \mathrm{int}(\Delta)$. By construction, there exists a map

$$\varphi_e : \mathrm{P}_e \longrightarrow \Sigma,$$

which is locally isometric, and there exits a diagonal $\tilde{e}$ of P such that $\varphi_e(\tilde{e}) = e$. Remark that $\varphi_e^{-1}(\mathrm{T})$ is a triangulation of P by diagonals. We will call P_e the *developing polygon of e with respect to* T.

First, let us prove the following technical lemma

Lemma 2.6.5 *Let* P *be a polygon in* $\mathbb{R}^2$ *whose vertices are denoted by* $A_1, A_2, A_3, B_1, \dots, B_l$. *Let* $x : \mathbb{R}^2 \longrightarrow \mathbb{R}$, *and* $y : \mathbb{R}^2 \longrightarrow \mathbb{R}$ *denote the two coordinate functions of* $\mathbb{R}^2$. *Assume that the vertices of* P *verify the following conditions :*

+ (A_1, A_2, A_3) *are ordered in the clock-wise sense ;*

+ $y(A_i) \geqslant 0,\ i = 1, 2, 3,\ y(A_1) < y(A_2)$, *and* $y(A_2) \geqslant y(A_3)$.

+ $y(B_j) < 0,\ j = 1, \dots, l$;

+ $B_1, \dots, B_l$ *are ordered in the counter-clockwise sense.*

+ *For all* $j \in \{1, \dots, l\}$, *the segment* $\overline{A_2B_j}$ *is a diagonal of* P.

Let T *denote the triangulation of* P *by the diagonals* $\overline{A_2B_1}, \dots, \overline{A_2B_l}$. *Let* $\{s_0, \dots, s_k\}$ *be a family of disjoint horizontal segments in* P *whose endpoints are contained the boundary of* P, *where* s_0 *is a segment lying on the horizontal axis* $y = 0$. *Let* r *be the number of intersection points of the edges of* T *with the set* $\cup_{i=0}^k s_i$. *Then there exists a sequence of elementary moves which transform* T *into a new triangulation* T′ *whose edges intersect the set* $\cup_{i=0}^k s_i$ *at at most* $r - 1$ *points.*

Proof: Consider the following algorithm :

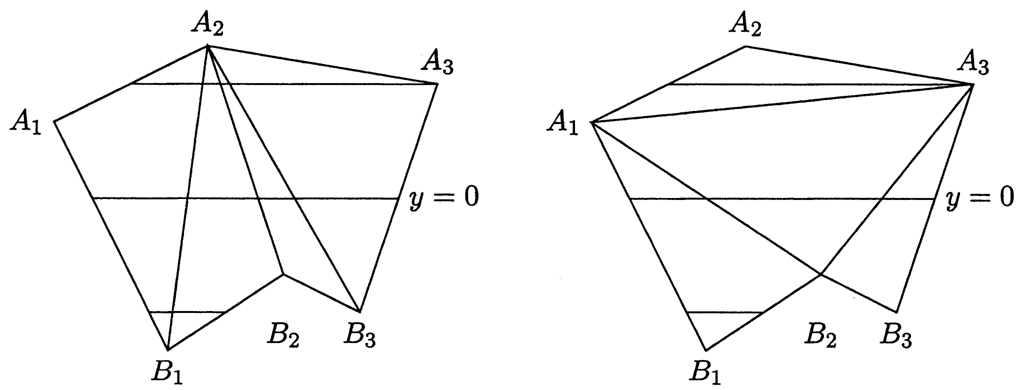
Let j_0 be the smallest index such that $y(B_{j_0}) = \min\{y(B_j) : j = 1, \dots, l\}$, that is $y(B_j) > y(B_{j_0})$ for all $j < j_0$, and $y(B_j) \geqslant y(B_{j_0}),\ \forall j = 1, \dots, l$.

1. If P is a quadrilateral, that is $l = 1$, then P must be convex. Apply an elementary move inside P and stop the algorithm.

2. If $1 < j_0 < l$, then consider the quadrilateral $A_2B_{j_0-1}B_{j_0}B_{j_0+1}$. By the choice of j_0, this quadrilateral is convex. Hence, we can apply an elementary move inside it, and the algorithm stops.

3. If $j_0 = 1$ and $l \geqslant 2$, then consider the quadrilateral $A_2A_1B_1B_2$. Observe that this quadrilateral is convex. Apply an elementary move inside it. By this move, we get a new triangulation of P which contains the triangle $\Delta A_1B_1B_2$. Cut off this triangle from P. Replace P by the remaining sub-polygon and restart the algorithm.

4. If $j_0 = l > 1$, then consider the quadrilateral $A_2A_3B_lB_{l-1}$. Since this quadrilateral is convex, we can apply an elementary move inside it, then cut off the triangle $\Delta A_3B_lB_{l-1}$. Replace P by the remaining sub-polygon and restart the algorithm.



Observe that, at each step of the algorithm above, the number of intersection points of the set $\cup_{i=0}^{k} s_i$ with the edges of the new triangulation cannot exceed the number of intersection points with those of the ancien one. Indeed, suppose that we are in the case $1 < j_0 < l$, by the choice of j_0, we have $y(B_{j_0}) \leqslant \min\{y(B_{j_0-1}), y(B_{j_0+1})\}$, and $y(A_2) \geqslant \max\{y(B_{j_0-1}), y(B_{j_0+1})\}$, consequently, if a horizontal segment s_i intersects $\overline{B_{j_0-1}B_{j_0+1}}$, then it must intersect $\overline{A_2B_{j_0}}$. Therefore, the number of intersection points does not increase. The same argument works for the other cases.

Moreover, at the final step of the algorithm, *i.e.* case 1. or 2., we replace a diagonal intersecting the segment s_0 by another one which does not intersect s_0. Hence, by this algorithm, we get a new triangulation T′ of P whose edges have strictly less intersection points with the set $\cup_{i=0}^{k} s_i$ than those of T_4. □

Let $a_1, \ldots, a_{N_e}$, and $b_1, \ldots, b_{N_e}$ denote the edges of T_1 and T_2 respectively. We can assume that $a_i = b_i$, for $i = 1, \ldots, k$. All we need to prove is the following

Proposition 2.6.6 *There exists a sequence of elementary moves which transform* T_1 *into a new triangulation containing* $b_1, \ldots, b_k$*, and* b_{k+1}.

Proof: Since b_{k+1} is not an edge of T_1, it must intersect some edges of T_1. Let P be the developing polygon of b_{k+1} with respect to T_1. Let $\varphi : P \longrightarrow \Sigma$ be the associated immersion. Let T_3 be the triangulation of P by diagonals which is induced by T_1, (*i.e.* $T_3 = \varphi^{-1}(T_1)$). By definition, each diagonal in T_3 is mapped by φ onto an edge of T_1 which intersects b_{k+1}. Finally, let d be the diagonal of P such that $\varphi(d) = b_{k+1}$. Observe that d intersects all the diagonals which are edges of T_3.

Let m be the number of intersection points of b_{k+1} with the edges of T_1 excluding the two endpoints of b_{k+1}. Note that b_{k+1} may intersect an edge of T_1 more than once. By construction, the polygon P is triangulated by m diagonals, hence it has $m + 3$ sides.

We prove the proposition by induction.

- If $m = 1$, then P is a quadrilateral. The quadrilateral P must be convex because its two diagonals intersect. If P is mapped by φ to a single triangle of T_1, then there is a singular point of Σ with cone angle strictly less than π. But this is impossible since, for every closed curve c in $\mathrm{int}(\Sigma \setminus \{p_1, \ldots, p_n\})$, we have $\mathbf{orth}(c) \in \{\pm \mathrm{Id}\}$. Thus, we conclude that φ maps int(P) isometrically onto a quadrilateral consisting of two triangles in T_1. Clearly, by applying the elementary move inside $\varphi(P)$, we obtains a new triangulation which contains b_{k+1}.

- If $m > 1$, it is enough to show that there exists a sequence of elementary moves which transform T_1 into a new triangulation T_1' containing $b_1 = (a_1), \ldots, b_k = (a_k)$, such that b_{k+1} intersects the edges of T_1' at most $m - 1$ times.

  Equip the plane $\mathbb{R}^2$ with a system of Cartesian coordinates such that d is a horizontal segment lying in the axis Ox. Let $x : \mathbb{R}^2 \longrightarrow \mathbb{R}$, and $y : \mathbb{R}^2 \longrightarrow \mathbb{R}$ denote the two coordinate functions. Let $A_1, \ldots, A_r$ denote the vertices of P such that $y(A_i) > 0$, and $B_1, \ldots, B_s$ denote the vertices of P such that $y(B_j) < 0$. Let A_0 and A_{r+1} denote the left and the right endpoints of d respectively. We set, by convention, $B_0 = A_0$, and $B_{s+1} = A_{r+1}$. Since P has $m + 3$ vertices, we have $r + s + 2 = m + 3$. We can assume that $r \geqslant s$ (if it is not the case, reverse the orientation of Oy). We name the vertices of P such that $A_0, \ldots, A_{r+1}$ are ordered in the clockwise sense, and $B_0, \ldots, B_{s+1}$ are ordered in the counter-clockwise sense.

  Without loss of generality, we can assume that $r \geqslant 2$ because $m > 1$. Let i_0 be the smallest index such that $y(A_{i_0}) = \max\{y(A_i) : i = 1, \ldots, r\}$, that is $y(A_{i_0}) \geqslant y(A_i)\ \forall i = 1, \ldots, r$, and

$y(A_{i_0}) > y(A_i)$ if $i < i_0$. Consider the sub-polygon P_1 of P, which consists of all triangles in T_3 having A_{i_0} as a vertex. The vertices of P_1 are $A_{i_0-1}, A_{i_0}, A_{i_0+1}$ and $B_{j_0}, \dots, B_{j_0+l}$. The polygon P_1 is triangulated by the diagonals $\overline{A_{i_0}B_{j_0}}, \dots, \overline{A_{i_0}B_{j_0+l}}$. Let T_4 denote this triangulation of P_1.

By Lemma 2.6.7 below, we know that φ maps $\mathrm{int}(\mathrm{P}_1)$ bijectively onto an open domain Q_1 in Σ. Therefore, any elementary move inside P_1 induces an elementary move inside Q_1.

Since $b_1, \dots, b_k, b_{k+1}$ are edges of the triangulation T_2, we have $\mathrm{int}(b_i) \cap \mathrm{int}(b_{k+1}) = \varnothing$, $\forall i = 1, \dots, k$. Recall that $b_1, \dots, b_k$ are also edges of the triangulation T_1, from this we deduce that $\mathrm{int}(b_i) \cap \mathrm{Q}_1 = \varnothing$, since if e is an edge of T_1 and $\mathrm{int}(e) \cap \mathrm{Q}_1 \neq \varnothing$, then $\mathrm{int}(e) \cap \mathrm{int}(b_{k+1}) \neq \varnothing$. This implies that an elementary move inside Q_1 does not affect the edges $b_1, \dots, b_k$.

Consider the intersection of P_1 and the inverse image of b_{k+1} by φ. A priori, this set is a family of geodesic segments with endpoints in the boundary of P_1. Clearly, the segment $s_0 = \overline{A_0A_{r+1}} \cap \mathrm{P}_1$ is contained in the set $\mathrm{P}_1 \cap \varphi^{-1}(b_{k+1})$. Since Σ satisfies $(\mathcal{Q}')$, all the segments in this family are parallel, therefore, all of them are parallel to the horizontal axis. Let r be the number of intersection points of the set $\mathrm{P}_1 \cap \varphi^{-1}(b_{k+1})$ and the edges of T_4.

Now, Lemma 2.6.5 shows that there exists a sequence of elementary moves which transform T_4 into a new triangulation whose edges intersect the set $\mathrm{P}_1 \cap \varphi{-}1(b_{k+1})$ at at most $r-1$ points. It follows that there exists a sequence of elementary moves inside the domain Q_1 which transform T_1 into a new triangulation of Σ whose edges have at most $m-1$ intersection points with b_{k+1}. As we have seen, those elementary moves do not affect the edges $b_1, \dots, b_k$. By induction, the proposition is then proved. □

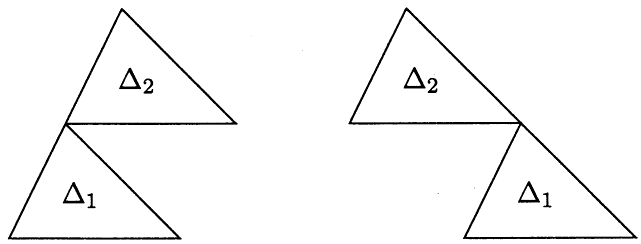
We need the following lemma to complete the proof of 2.6.6

Lemma 2.6.7 *With the same notations as in the proof of 2.6.6, the restriction of φ onto $\mathrm{int}(\mathrm{P}_1)$ is an isometric embedding.*

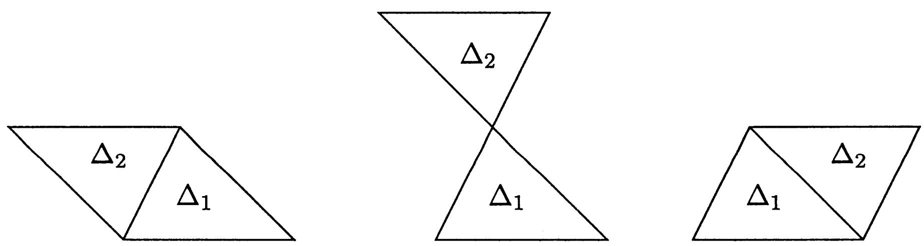
Proof: Since φ maps each triangle of T_3 onto a triangle of T_1, it is enough to show that the images by φ of the triangles of T_3 which are contained in P_1 are all distinct.

Suppose that there exist two triangles Δ_1 and Δ_2 such that $\varphi(\Delta_1) = \varphi(\Delta_2)$. Since φ is locally isometric, and by assumption, the orthogonal part of the holonomy of any closed curve in $\mathrm{int}(\Sigma \setminus \{p_1, \dots, p_n\})$ is either Id or $-$Id, it follows that either $\Delta_2 = \Delta_1 + v$, or $\Delta_2 = -\Delta_1 + v$, where $-\Delta_1$ is the image of Δ_1 by $-$Id, and $v \in \mathbb{R}^2$. Note that, by definition, the triangles Δ_1 and Δ_2 have a common vertex, which is A_{i_0}.

- If $\Delta_2 = \Delta_1 + v$, exclude the case $\Delta_1 \equiv \Delta_2$, we have two possible configurations. In these both cases, we see that the angle of P_1 at the point A_{i_0} is at least π. But, by assumption, this is impossible since we have $y(A_{i_0}) > y(A_{i_0-1})$ and $y(A_{i_0}) \geqslant y(A_{i_0+1})$.



- If $\Delta_2 = -\Delta_1 + v$, we have three possible configurations. In the case where Δ_1 and Δ_2 have only one common vertex, we see that the angle of P_1 at A_{i_0} must be greater than π, which is, as we have seen above, impossible. In the other two cases, Δ_1 and Δ_2 are adjacent. As we have seen, this implies the existence of a singular point of Σ with cone angle strictly less than π. This is again impossible.



The lemma is then proved. $\square$

2.7 Volume form on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$

Our aim in this section is to define the volume form μ_{Tr} on the space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ which is invariant by the action of the group $\Gamma(\tilde{g}, \tilde{n})$. The construction of this volume form relies on the local charts defined in the proof of Theorem 2.2.7.

Recall that, if $\mathrm{L} : \mathrm{E} \longrightarrow \mathrm{F}$ is a linear map between (real) vector spaces which is surjective, then given a volume form μ_{E} on E, and a volume form μ_{F} on F, one can define a volume form μ on $\ker(\mathrm{L})$ as follows : let E_1 be a subspace of E so that $\mathrm{E} = \mathrm{E}_1 \oplus \ker(\mathrm{L})$, the restriction L_1 of L on E_1 is then a linear isomorphism, the volume form μ on $\ker(\mathrm{L})$ is defined to be the one such that :

$$\mu_{\mathrm{E}} = \mu \wedge \mathrm{L}_1^* \mu_{\mathrm{F}}.$$

Remark that μ does not depend on the choice of E_1.

2.7.1 Definition of the volume form μ_{Tr}

Let us start by recalling some basic properties of the local charts $\Psi_{\mathcal{T}}$ which are defined in Section 2.4. Let $\mathcal{T}$ be a triangulation of S representing an equivalence class in $\mathcal{TR}(S)$. Let $\mathcal{U}_{\mathcal{T}}$ be the subset of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ consisting of all pairs $([(\Sigma,\phi)],\xi)$ such that the homeomorphism ϕ maps $\mathcal{T}$ onto an admissible triangulation of Σ. The local chart $\Psi_{\mathcal{T}}$ is defined on $\mathcal{U}_{\mathcal{T}}$ with image in $\mathrm{V}_{\mathcal{T}}$, which is a subspace of $\mathbb{C}^{N_1}$, where N_1 is the number of edges of $\mathcal{T}$. The image of $\mathcal{U}_{\mathcal{T}}$ is an open set of $\mathrm{V}_{\mathcal{T}}$.

Let $a_1,\dots,a_{N_2}$ denote the vectors of $(\mathbb{C}^{N_1})^*$ which correspond to the equations of the system $\mathbf{S}_{\mathcal{T}}$. A vector a_i is said to be *normalized* if each of its coordinates belongs to the set $\{-1,0,1\}$. We have two cases :

- Case 1 : $m > 0$. In this case, we have shown that $\mathrm{rank}(\mathbf{S}_{\mathcal{T}}) = N_2$ (see Lemma 2.4.4). Consider the complex linear map $\mathbf{A}_{\mathcal{T}} : \mathbb{C}^{N_1} \longrightarrow \mathbb{C}^{N_2}$, which is defined in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ by the matrix

$$\mathbf{A}_{\mathcal{T}} = \begin{pmatrix} a_1 \\ \vdots \\ a_{N_2} \end{pmatrix}.$$

  The map $\mathbf{A}_{\mathcal{T}}$ is then surjective, and $\mathrm{V}_{\mathcal{T}} = \ker \mathbf{A}_{\mathcal{T}}$. The map $\mathbf{A}_{\mathcal{T}}$ is said to be *normalized* if each row of its matrix in the canonical basis is normalized.

  Let λ_{2N_1} et λ_{2N_2} denote the Lebesgue measures on $\mathbb{C}^{N_1} \simeq \mathbb{R}^{2N_1}$ and $\mathbb{C}^{N_2} \simeq \mathbb{R}^{2N_2}$ respectively. Since $\mathbf{A}_{\mathcal{T}}$ is surjective, λ_{2N_1} and λ_{2N_2} induce a volume form $\nu_{\mathcal{T}}$ on $\mathrm{V}_{\mathcal{T}}$ via the following exact sequence :

$$0 \longrightarrow \mathrm{V}_{\mathcal{T}} \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathcal{T}}} \mathbb{C}^{N_2} \longrightarrow 0.$$

- Case 2 : $m = 0$. In this case, we have $\mathrm{rank}(\mathbf{S}_{\mathcal{T}}) = N_2 - 1$ (see Lemma 2.4.4), hence $\mathrm{rank}(\mathbf{A}_{\mathcal{T}}) = N_2 - 1$. If the vectors $a_1,\dots,a_{N_2}$ are normalized, and the their signs are chosen suitably, we have $a_1 + \dots + a_{N_2} = 0$. Thus, without loss of generality, we can assume that $\mathrm{Im}\mathbf{A}_{\mathcal{T}} = \mathbf{W}$, where

$\mathbf{W}$ is the complex hyperplane of $\mathbb{C}^{N_2}$ defined by $\mathbf{W} = \{(z_1, \dots, z_{N_2}) \in \mathbb{C}^{N_2} : z_1 + \dots + z_{N_2} = 0\}$.

Let $\lambda'_{2(N_2-1)}$ denote the volume form of $\mathbf{W}$ which is induced by the Lebesgue measure of $\mathbb{C}^{N_2}$. The volume forms λ_{2N_1} and $\lambda'_{2(N_2-1)}$ induce a volume form $\nu_{\mathcal{T}}$ on $\mathrm{V}_{\mathcal{T}}$ via the following exact sequence :

$$0 \longrightarrow \mathrm{V}_{\mathcal{T}} \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathcal{T}}} \mathbf{W} \longrightarrow 0.$$

In both cases, let $\mu_{\mathcal{T}}$ denote the volume form $\Psi^*_{\mathcal{T}}\nu_{\mathcal{T}}$ which is defined on $\mathcal{U}_{\mathcal{T}}$.

2.7.2 Invariance by coordinate changes

To show that the volume forms $\mu_{\mathcal{T}}$, $\mathcal{T} \in \mathcal{TR}(S)$, give a well-defined volume form on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, we need to prove that whenever $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2} \neq \varnothing$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ represent two different equivalence classes in $\mathcal{TR}(S)$, then we have

$$\mu_{\mathcal{T}_1} = \mu_{\mathcal{T}_2} \text{ on } \mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}.$$

Let us begin with

Proposition 2.7.1 *Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$. Let T_1 and T_2 be two admissible triangulations of Σ corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Assume that T_2 is obtained by T_1 by an elementary move, then $\mu_{\mathcal{T}_1} = \mu_{\mathcal{T}_2}$ on $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$.*

Proof: Suppose that the elementary move occurs in a quadrilateral Q which is formed by two triangles Δ_1 and Δ_2 of T_1. Note that the edge of T_1 which is removed by this elementary move is contained in the interior of Σ.

Let $Z = (z_1, \dots, z_{N_1})$ denote the image of $([(\Sigma, \phi)], \xi)$ by $\Psi_{\mathcal{T}_1}$. We can assume that

. z_1 is associated to the common side of Δ_1 and Δ_2.

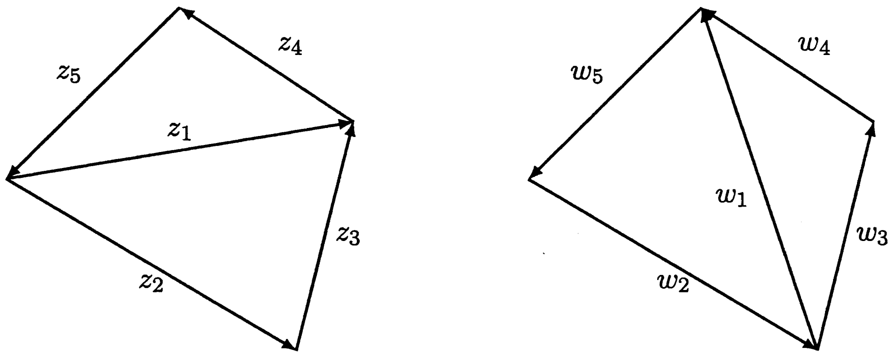
. z_2, z_3 are associated to the other sides of Δ_1 such that $\{-z_1, z_2, z_3\}$ is the oriented boundary of Δ_1.

. z_4, z_5 are associated to the other sides of Δ_2 such that $\{z_1, z_4, z_5\}$ is the oriented boundary of Δ_2.

We have

$$-z_1 + z_2 + z_3 = 0, \tag{2.10}$$

$$z_1 + z_4 + z_5 = 0. \tag{2.11}$$



After the move, the quadrilateral Q is divided into two triangles Δ_1' and Δ_2'. Let $W = (w_1, \ldots, w_{N_1})$ denote the image of $([\Sigma, \phi)], \xi)$ by $\Psi_{\mathcal{T}_2}$. We can assume that

. w_1 is associated to the common edge of Δ_1' and Δ_2'.

. w_i is associated to the oriented edge corresponding to z_i, for every $i = 2, \ldots, N_1$.

We have then

$$- w_1 + w_3 + w_4 = 0, \tag{2.12}$$

$$w_1 + w_2 + w_5 = 0. \tag{2.13}$$

We see that the equations (2.10) and (2.11) are contained in the system $\mathbf{S}_{\mathcal{T}_1}$, and the equations (2.12) and (2.13) are contained in the system $\mathbf{S}_{\mathcal{T}_2}$. The other equations of $\mathbf{S}_{\mathcal{T}_2}$ are the same as those of $\mathbf{S}_{\mathcal{T}_1}$ with z_i replaced by w_i, for $i = 2, \ldots, N_1$. Note that z_1 does not appear in any equation of $\mathbf{S}_{\mathcal{T}_1}$ other than (2.10) and (2.11). Similarly, w_1 does not appear in any equation of $\mathbf{S}_{\mathcal{T}_2}$ other than (2.12) and (2.13).

Let $\mathbf{A}_{\mathcal{T}_1}$ denote the normalized linear map associated to $\mathbf{S}_{\mathcal{T}_1}$. The matrix of $\mathbf{A}_{\mathcal{T}_1}$ in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ is of the form

$$\mathbf{A}_{\mathcal{T}_1} = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & 1 & \cdots & 0 \\ 0 & * & * & * & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & * & * & * & * & \cdots & * \end{pmatrix}.$$

Similarly, let $\mathbf{A}_{\mathcal{T}_2}$ denote the normalized linear map associated to $\mathbf{S}_{\mathcal{T}_2}$ whose matrix in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ is of the form

$$\mathbf{A}_{\mathcal{T}_2} = \begin{pmatrix} -1 & 0 & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 0 \\ 0 & * & * & * & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & * & * & * & * & \cdots & * \end{pmatrix}.$$

From what has been said, the i-th row of the matrix $\mathbf{A}_{\mathcal{T}_2}$ is the same as the i-th row of the matrix $\mathbf{A}_{\mathcal{T}_1}$, for every $i = 3, \ldots, N_2$.

Let $\mathbf{F} : \mathbb{C}^{N_1} \longrightarrow \mathbb{C}^{N_1}$ be the linear map which is defined in the canonical basis of $\mathbb{C}^{N_1}$ by the matrix

$$\mathbf{F} = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Now, observe that $\mathbf{A}_{\mathcal{T}_2} \circ \mathbf{F} = \mathbf{A}_{\mathcal{T}_1}$. As a consequence, the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \mathbf{A}_{\mathcal{T}_1} & \longrightarrow & \mathbb{C}^{N_1} & \xrightarrow{\mathbf{A}_{\mathcal{T}_1}} & \mathbb{C}^{N_2} & \longrightarrow & 0 \\ & & \downarrow \mathbf{H} & & \downarrow \mathbf{F} & & \| \mathrm{Id} & & \\ 0 & \longrightarrow & \ker \mathbf{A}_{\mathcal{T}_2} & \longrightarrow & \mathbb{C}^{N_1} & \xrightarrow{\mathbf{A}_{\mathcal{T}_2}} & \mathbb{C}^{N_2} & \longrightarrow & 0 \end{array}$$

The isomorphism $\mathbf{H} : \ker \mathbf{A}_{\mathcal{T}_1} \longrightarrow \ker \mathbf{A}_{\mathcal{T}_2}$, which is induced by $\mathbf{F}$, is the coordinate change $\Psi_{\mathcal{T}_2} \circ \Psi_{\mathcal{T}_1}^{-1}$.

Here, we have two cases :

- Case 1 : $m > 0$. We have $\dim_{\mathbb{C}} \ker \mathbf{A}_{\mathcal{T}_1} = \dim_{\mathbb{C}} \ker \mathbf{A}_{\mathcal{T}_2} = \sum_{j=1}^{m} s_j + 2g + n - 2$. In this case, by definition, the volume forms $\nu_{\mathcal{T}_1}$ and $\nu_{\mathcal{T}_2}$ are induced by the Lebesgue measures λ_{2N_1} and λ_{2N_2} on $\ker \mathbf{A}_{\mathcal{T}_1}$ and $\ker \mathbf{A}_{\mathcal{T}_2}$ respectively. Since $|\det \mathbf{F}| = 1$, we deduce that $\mathbf{H}^* \nu_{\mathcal{T}_2} = \nu_{\mathcal{T}_1}$. Therefore, the forms $\mu_{\mathcal{T}_1}$ and $\mu_{\mathcal{T}_2}$ coincide in a neighborhood of $([(\Sigma, \phi)], \xi)$.

- Case 2 : $m = 0$. We have $\dim_{\mathbb{C}} \ker \mathbf{A}_{\mathcal{T}_1} = \dim_{\mathbb{C}} \ker \mathbf{A}_{\mathcal{T}_2} = 2g + n - 1$, we can assume that $\mathrm{Im}\mathbf{A}_{\mathcal{T}_1} = \mathrm{Im}\mathbf{A}_{\mathcal{T}_2} = \mathbf{W}$, where $\mathbf{W}$ is the complex hyperplane of $\mathbb{C}^{N_2}$ defined above. In this case, the volume forms $\nu_{\mathcal{T}_1}$ and $\nu_{\mathcal{T}_2}$ are induced by λ_{2N_1} and $\lambda'_{2(N_2-1)}$, where $\lambda'_{2(N_2-1)}$ is the volume form on $\mathbf{W}$. Since we also have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \mathbf{A}_{\mathcal{T}_1} & \longrightarrow & \mathbb{C}^{N_1} & \xrightarrow{\mathbf{A}_{\mathcal{T}_1}} & \mathbf{W} & \longrightarrow & 0 \\ & & \downarrow \mathbf{H} & & \downarrow \mathbf{F} & & \| \mathrm{Id} & & \\ 0 & \longrightarrow & \ker \mathbf{A}_{\mathcal{T}_2} & \longrightarrow & \mathbb{C}^{N_1} & \xrightarrow{\mathbf{A}_{\mathcal{T}_2}} & \mathbf{W} & \longrightarrow & 0 \end{array}$$

it follows that $\mathbf{H}^*\nu_{\mathcal{T}_2} = \nu_{\mathcal{T}_1}$. Hence we get the same conclusion. □

Corollary 2.7.2 *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two triangulations of S which represent two different equivalence classes in $\mathcal{TR}(S)$. Assume that $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2} \neq \varnothing$, then $\mu_{\mathcal{T}_1} = \mu_{\mathcal{T}_2}$ on $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$.*

Proof: Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$. Let $\mathrm{T}_1, \mathrm{T}_2$ be the two admissible triangulations of Σ which correspond to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. By Theorem 2.6.2, we know that T_2 can be obtained from T_1 by a sequence of elementary moves. Proposition 2.7.1 tells us that the volume forms corresponding to two admissible triangulations which differ from each other by an elementary move are equal. The corollary is then proved. □

By Corollary 2.7.2, we see that the volume forms $\mu_{\mathcal{T}}$, $\mathcal{T} \in \mathcal{TR}(S)$ give rise to a well defined volume form on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$. From now on, we denote this volume form μ_{Tr}.

2.7.3 Invariance by the action of Mapping Class Group

To complete the proof of Theorem 2.2.9, we need the following :

Proposition 2.7.3 *The volume form μ_{Tr} is invariant by the action of $\Gamma(\tilde{g}, \tilde{n})$.*

Proof: The fact that μ_{Tr} is invariant by the action of the group $\Gamma(\tilde{g}, \tilde{n})$ is quite clear from the definition. Let γ be an element of $\Gamma(\tilde{g}, \tilde{n})$, and suppose that $\gamma([(\Sigma_1, \phi_1)], \xi_1) = ([(\Sigma_2, \phi_2)], \xi)$. By definition there exits then an isometry

$$h : \Sigma_1 \longrightarrow \Sigma_2,$$

such that $\phi_2^{-1} \circ h \circ \phi_1 \in \mathrm{Homeo}^+(S, \mathcal{V})$. The isometry h sends an admissible triangulation of Σ_1 onto an admissible triangulation of Σ_2, from which we deduce that γ preserves the volume form μ_{Tr}. □

The proof of Theorem 2.2.9 is now complete.

2.8 Proof of Proposition 2.2.10

In this paragraph, we will always assume that $m = 0$, because of this additional hypothesis, we replace $\mathcal{T}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ by $\mathcal{T}_{\mathrm{T}}(\bar{\alpha})$, and $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ by $\mathcal{M}_{\mathrm{T}}(\bar{\alpha})$ to simplify the notations.

2.8.1 Flat surface defined by holomorphic 1-form

In this paragraph we suppose that $g \geqslant 2$. Let M be a compact Riemann surface of genus g, without boundary, and ω be a holomorphic 1-form on M. Let $x_1,\dots,x_n$ denote the zeros of ω, and $k_1,\dots,k_n$ denote their orders respectively. It is well known that ω defines a flat metric on M such that the cone angle at x_i is $2\pi(k_i+1), i = 1,\dots,n$. In this situation, we consider $\{x_1,\dots,x_n\}$ as the set of singularities of the flat surface, even though some of these points are actually regular (k_i may be zero). Note that the 1-form ω also determines a singular foliation of M by 'vertical' geodesics. A flat surface defined by a holomorphic 1-form is a translation surface.

Fix a sequence $k_1,\dots,k_n$ of non-negative integers such that $k_1+\dots+k_n = 2g-2$. Let $\mathcal{H}(k_1,\dots,k_n)$ denote the moduli space of holomorphic 1-form having n zeros of orders $k_1,\dots,k_n$. By definition, $\mathcal{H}(k_1,\dots,k_n)$ is the quotient space of the set of all pairs (M,ω) as above by the following equivalence relation : (M_1,ω_1) and (M_2,ω_2) are equivalent if and only if there exists a conformal homeomorphism $f : M_1 \longrightarrow M_2$ such that $f^*\omega_2 = \omega_1$.

It is well known that $\mathcal{H}(k_1,\dots,k_n)$ is a complex algebraic orbifold of dimension $2g+n-1$. Let (M_0,ω_0) be a pair in $\mathcal{H}(k_1,\dots,k_n)$. Let $\{\gamma_1^0,\dots,\gamma_{2g+n-1}^0\}$ denote a basis of the homology group $H_1(M_0,\{x_1^0,\dots,x_n^0\},\mathbb{Z}) \simeq \mathbb{Z}^{2g+n-1}$, where $x_1^0,\dots,x_n^0$ denote the zeros of ω_0. We can consider every pair (M,ω) in a neighborhood of (M_0,ω_0) as a deformation of (Σ_0,ω_0) so that we can specify a basis of $H_1(M,\{x_1,\dots,x_n\},\mathbb{Z})$, where $x_1,\dots,x_n$ denote the zeros of ω, corresponding to $\gamma_1^0,\dots,\gamma_{2g+n-1}^0$. The curves in this basis will be denoted by $\gamma_1,\dots,\gamma_{2g+n-1}$. It follows that the map

$$\Phi : (M,\omega) \longmapsto (\int_{\gamma_1}\omega,\dots,\int_{\gamma_{2g+n-1}}\omega) \in \mathbb{C}^{2g+n-1} \cong \mathbb{R}^{2(2g+n-1)},$$

defines a local coordinate chart of $\mathcal{H}(k_1,\dots,k_n)$ in a neighborhood of (Σ_0,ω_0). This is the *period mapping*. The pull-back by Φ of the Lebesgue measure on $\mathbb{C}^{2g+n-1} \simeq \mathbb{R}^{2(2g+n-1)}$ is a well defined volume form on $\mathcal{H}(k_1,\dots,k_n)$. We denote this volume form μ_0.

Assume in addition that the integers $k_1, \dots, k_n$ are pairwise distinct. In this case, we can identify $\mathcal{H}(k_1, \dots, k_n)$ to the space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha})$, with $\alpha_i = 2\pi(k_i + 1),\ i = 1, \dots, n$. Remark that if $k_1, \dots, k_n$ are not pairwise distinct, then the space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha})$ is a finite covering of $\mathcal{H}(k_1, \dots, k_n)$.

2.8.2 Proof of Proposition 2.2.10

Let (M, ω) be a pair in $\mathcal{H}(k_1, \dots, k_n)$. Let Σ denote the induced translation surface. Let $x_1, \dots, x_n$ denote its singularities so that the cone angle at x_i is $2\pi(k_i + 1)$. The vertical geodesic flow determined by ω induces a normalized parallel vector field on $\Sigma \setminus \{x_1, \dots, x_n\}$. Let ξ denote this vector field. The pair (M, ω) in $\mathcal{H}(k_1, \dots, k_n)$ is then identified to the element $(\Sigma, \{x_1, \dots, x_n\}, \xi)$ in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha})$.

Let T be a geodesic triangulation of Σ whose set of vertices coincides with the set of singularities of Σ, we know such triangulations exist by Proposition 2.3.2. Note that, in this case, any geodesic triangulation whose set of vertices coincides with the set of singularities is admissible.

Recall that a *family of primitive edges* of T is a set of $2g + n - 1$ edges of T such that the complement of the union of those edges is a topological open disk. Remark that such a family always exists because it corresponds to a maximal tree in the dual graph of T. Let $\{b_1, \dots, b_{2g+n-1}\}$ be a family of primitive edges of T. Observe that $\{b_1, \dots, b_{2g+n-1}\}$ is a basis of the group $H_1(\Sigma, \{x_1, \dots, x_n\}, \mathbb{Z})$.

Let $\phi : S \longrightarrow \Sigma$ be a quasi-conformal homeomorphism which maps p_i to $x_i,\ i = 1, \dots, n$. Let $\mathcal{T}$ denote the equivalence class of the triangulation $\phi^{-1}(\mathrm{T})$ in $\mathcal{TR}(S)$. Let $\Psi_{\mathcal{T}}$ be the local chart associated to $\mathcal{T}$. As usual, let $\mathbf{S}_{\mathcal{T}}$ denote the system of linear equations associated to $\mathcal{T}$. Let $\mathrm{V}_{\mathcal{T}}$ be the space of solutions of $\mathbf{S}_{\mathcal{T}}$, and $\mathbf{A}_{\mathcal{T}}$ be the normalized linear map associated to $\mathbf{S}_{\mathcal{T}}$. We can assume that

$$\mathrm{Im}\mathbf{A}_{\mathcal{T}} = \mathbf{W} = \{(z_1, \dots, z_{N_2}) \in \mathbb{C}^{N_2} \mid z_1 + \dots + z_{N_2} = 0\}.$$

Note that here $N_1 = 4(2g + n - 1) - 3$, $N_2 = 3(2g + n - 1) - 2$, and $\dim_{\mathbb{C}} \mathrm{V}_{\mathcal{T}} = 2g + n - 1$. By $\Psi_{\mathcal{T}}$, a neighborhood of $(\Sigma, \{x_1, \dots, x_n\}, \xi)$ in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha})$ is identified to an open set of $\mathrm{V}_{\mathcal{T}}$.

There exists a neighborhood $\mathcal{U}$ of $(\Sigma, \{x_1, \dots, x_n\}, \xi)$ such that, for any point $(\Sigma', \{x'_1, \dots, x'_n\}, \xi')$ in $\mathcal{U}$, there exists a quasi-conformal homeomorphism $f_{\Sigma'} : \Sigma \longrightarrow \Sigma'$ which maps T onto an admissible triangulation T$'$ of Σ'. Let $b'_i,\ i = 1, \dots, 2g + n - 1$, denote the image of b_i by $f_{\Sigma'}$. The segments $\{b'_1, \dots, b'_{2g+n-1}\}$ form a basis of the group $H_1(\Sigma', \{x'_1, \dots, x'_n\}, \mathbb{Z})$. Hence, we can define a local chart of $\mathcal{H}(k_1, \dots, k_n)$ by the following period mapping

$$\begin{array}{cccc} \Phi : & \mathcal{U} & \longrightarrow & \mathbb{C}^{2g+n-1} \\ & (\Sigma', \{x'_1, \dots, x'_n\}, \xi') \simeq (M', \omega') & \longmapsto & (\int_{b'_1} \omega', \dots, \int_{b'_{2g+n-1}} \omega') \end{array}$$

By the construction of $\Psi_{\mathcal{T}}$, we can assume that if $\Psi_{\mathcal{T}}(\Sigma', \{x'_1, \dots, x'_n\}, \xi') = (z_1, \dots, z_{N_1})$, then the complex numbers $z_1, \dots, z_{2g+n-1}$ are associated to the edges $b'_1, \dots, b'_{2g+n-1}$. It follows that the map

$$\Psi_{\mathcal{T}} \circ \Phi^{-1} : \Phi(\mathcal{U}) \subset \mathbb{C}^{2g+n-1} \longrightarrow \mathbb{C}^{N_1}$$

maps $(z_1, \dots, z_{2g+n-1})$ to $(z_1, \dots, z_{2g+n-1}, z_{2g+n}, \dots, z_{N_1})$. We deduce that $\Psi_{\mathcal{T}} \circ \Phi^{-1}$ is an injective linear map. Hence, $\Psi_{\mathcal{T}} \circ \Phi^{-1}$ is a restriction into $\Phi(\mathcal{U})$ of an isomorphism from $\mathbb{C}^{2g+n-1}$ onto $\mathrm{V}_{\mathcal{T}}$.

Let $\lambda_{2(2g+n-1)}$ denote the Lebesgue measure of $\mathbb{C}^{2g+n-1} \simeq \mathbb{R}^{2(2g+n-1)}$. By definition, $\mu_0 = \Phi^*\lambda_{2(2g+n-1)}$.

Let $\lambda'_{2(N_2-1)}$ be the volume form of $\mathbf{W}$ which is induced by the Lebesgue measure of $\mathbb{C}^{N_2}$, and $\nu_{\mathcal{T}}$ be the volume form on $\mathrm{V}_{\mathcal{T}}$ which is induced by λ_{2N_1} and $\lambda'_{2(N_2-1)}$ via the following exact sequence

$$0 \longrightarrow \mathrm{V}_{\mathcal{T}} \longrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathcal{T}}} \mathbf{W} \longrightarrow 0.$$

By definition, the volume form μ_{Tr} on a neighborhood of $(\Sigma, \{x_1, \dots, x_n\}, \xi)$ is $\Psi_{\mathcal{T}}^*\nu_{\mathcal{T}}$. Clearly, on $\mathbb{C}^{2g+n-1}$ we have

$$(\Psi_{\mathcal{T}} \circ \Phi^{-1})^*\nu_{\mathcal{T}} = \lambda\lambda_{2g+n-1},$$

where λ is a non-zero constant. This implies $\mu_{\mathrm{Tr}} = \lambda\mu_0$ on a neighborhood of $(\Sigma, \{x_1, \dots, x_n\}, \xi)$. We deduce that μ_{Tr}/μ_0 is locally constant. Consequently, μ_{Tr}/μ_0 is constant on every connected component of $\mathcal{H}(k_1, \dots, k_n)$. □

2.9 Action of $SL_2(\mathbb{R})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$

There is an action of the group $SL_2(\mathbb{R})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ which is defined as follows : let $([(\Sigma, \phi)], \xi)$ be an element of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Let $\{f_\alpha : U_\alpha \longrightarrow \mathbb{R}^2\}$ be an atlas defining the flat metric structure on Σ, then $\{A \circ f_\alpha\}$ is an atlas of another flat metric structure on Σ. Since all the transition functions are translations of $\mathbb{R}^2$, it follows that $\{A \circ f_\alpha\}$ defines a translation surface structure on Σ. Let $A \cdot \Sigma$ denote the new translation surface. We define the image of $[(\Sigma, \phi)]$ by A to be the equivalence class of the pair $(A \cdot \Sigma, \phi)$, that is, while the flat metric structure on Σ is modified by A, the marking map ϕ stays the unchanged. To define the image of the parallel vector field ξ on $A \cdot \Sigma$, we choose an atlas $\{f_\alpha : U_\alpha \longrightarrow \mathbb{R}^2\}$ of Σ such that, for every α, $f_{\alpha*}\xi$ is the constant vertical vector filed $(0, 1)$ on $f_\alpha(U_\alpha)$. The image of ξ on $A \cdot \Sigma$ is defined to be the pull-back of the vertical vector field $(0, 1)$ on $A \circ f_\alpha(U_\alpha)$. Let $A \cdot ([(\Sigma, \phi)], \xi)$ denote the image of $([(\Sigma, \phi)], \xi)$ by A. It is easy to verify that $A \cdot ([(\Sigma, \phi)], \xi)$ is also an element of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$. We have then defined an action of every $A \in SL_2(\mathbb{R})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$.

Remark: One can check out easily that the action of $SO(2) \subset SL_2(\mathbb{R})$ by this definition is equivalent to the rotations of the normalized parallel vector field on each translation surface.

From the definitions, it follows immediately that the action of $SL_2(\mathbb{R})$ commutes with the action of the mapping class group $\Gamma(\tilde{g}, \tilde{n})$ on $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$. Hence, we also get an action of $SL_2(\mathbb{R})$ on the moduli space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$. Furthermore, we have

Proposition 2.9.1 *The volume form μ_{Tr} is invariant by the action of $SL_2(\mathbb{R})$.*

Proof: Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, and T be an admissible triangulation of Σ. Let $\mathcal{T}$ be the equivalence class of $\phi^{-1}(\mathrm{T})$ in $\mathcal{TR}(S)$. Let $\mathcal{U}_{\mathcal{T}}$ be the associated domain of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, and $\Psi_{\mathcal{T}}$ be the associated local chart.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of the group $SL_2(\mathbb{R})$. By definition, it is clear that the action of A preserve the domain $\mathcal{U}_{\mathcal{T}}$.

By the local chart $\Psi_{\mathcal{T}}$, we identify $\mathcal{U}_{\mathcal{T}}$ to an open set in a subspace $\mathrm{V}_{\mathcal{T}}$ of $\mathbb{C}^{N_1}$. By definition, the induced action of A on $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ verifies

$$A \cdot (z_1, \dots, z_{N_1}) = (A(z_1), \dots, A(z_{N_1})), \ \forall (z_1, \dots, z_{N_1}) \in \Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}}),$$

where the complex numbers $A(z_i)$ is defined as follows : if $z_i = x_i + \imath y_i$, with $x_i, y_i \in \mathbb{R}$, then $A(z_i) = u_i + \imath v_i$, with

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

If we identify $\mathbb{C}^{N_1}$ to $\mathbb{R}^{2N_1}$, the action of A on $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is the restriction of the action of the following matrix :

$$\begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A \end{pmatrix}.$$

Now, recall that the volume form μ_{Tr} is induced by the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbf{X}$, where $\mathbf{X}$ is either $\mathbb{C}^{N_2}$ or $\mathbf{W}$, via the complex linear map $\mathbf{A}_{\mathcal{T}}$. We have the following commutative diagram :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{V}_{\mathcal{T}} & \longrightarrow & \mathbb{C}^{N_1} & \xrightarrow{\mathbf{A}_{\mathcal{T}}} & \mathbf{X} & \longrightarrow & 0 \\ & & \downarrow A & & \downarrow A & & \downarrow A & & \\ 0 & \longrightarrow & \mathrm{V}_{\mathcal{T}} & \longrightarrow & \mathbb{C}^{N_1} & \xrightarrow{\mathbf{A}_{\mathcal{T}}} & \mathbf{X} & \longrightarrow & 0 \end{array}$$

where we have used the same notation A to denote the action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathrm{V}_{\mathcal{T}}, \mathbb{C}^{N_1}$, and $\mathbf{X}$ by applying this matrix to each complex coordinate. Clearly, this action of A preserves the Lebesgue measures on $\mathbb{C}^{N_1}$ and $\mathbf{X}$. Therefore, A preserves the induced volume form on $\mathrm{V}_{\mathcal{T}}$. The proposition is then proved. □

Remark: Proposition 2.2.10 can be deduced from Proposition 2.9.1 as follows : define a function f on $\mathcal{H}(k_1, \dots, k_n)$ by

$$f = \frac{d\mu_{\mathrm{Tr}}}{d\mu_0}.$$

The function f is then continuous. Since both μ_{Tr} and μ_0 are $SL(2,\mathbb{R})$-invariant, so is f. But we know that the action of $SL(2,\mathbb{R})$ is ergodic on each connected component of $\mathcal{H}(k_1, \dots, k_n)$. Hence, f is constant on each connected component of $\mathcal{H}(k_1, \dots, k_n)$.

Chapitre 3

Flat surface with erasing trees

3.1 Definitions and main results

3.1.1 Flat surface with conical singularities and erasing trees

Let Σ be a flat surface. A *tree* in Σ is the image of an embedding from a topological tree into Σ. We consider an isolate point as a special tree, which has only one vertex and no edges. A *forest* in Σ is a finite disjoint union of trees in Σ. A tree in Σ is said to be *geodesic* if each of its edges is a geodesic segment in Σ. A forest is said to be *geodesic* if it is a union of geodesic trees.

Definition 3.1.1 (Erasing tree and erasing forest) *Let Σ be a compact connected flat surface without boundary. Let $p_1, \ldots, p_n$ denote the singular points of Σ. An* erasing tree *(resp.* erasing forest*) in Σ is a tree (resp. forest) whose vertex set contains all the singular points of Σ such that, if c is a closed curve in Σ which does not intersect this tree (resp. forest), then the holonomy of c is a translation of $\mathbb{R}^2$ (the orthogonal part of the holonomy is trivial).*

Given a flat surface with an erasing forest, one can define

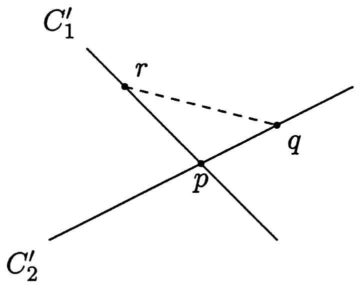
Definition 3.1.2 (Normalized Parallel Vector Field) *Let Σ be a compact, connected flat surface without boundary. Assume that there exists on Σ an erasing forest $\hat{A}$. A* parallel vector field *on the complement of $\hat{A}$ is a vector field which is nowhere zero such that, in local charts of the Euclidean metric structure, all the lines determined by the vectors of this field are parallel. A parallel vector field is said to be* normalized *if all of its vectors are of norm one.*

The next proposition shows that geodesic trees always exist on flat surfaces.

Proposition 3.1.3 (Existence of geodesic trees) *Let Σ be flat surface without boundary. Let $\{p_1, \ldots, p_n\}$ denote the singularities of Σ. Then there exists a geodesic tree whose vertices are $\{p_1, \ldots, p_n\}$.*

Proof: Let C_1 be a path from p_1 to p_2 whose length is minimal. The path C_1 is a finite union of geodesic segments whose endpoints are singular points of Σ. Apart from p_1 and p_2, C_1 can contain other points in $\{p_1, \dots, p_n\}$. Since C_1 is a path of minimal length, it has no self intersections. By renumbering the set of singular points if necessary, we can assume that C_1 is a path joining p_1 and p_r via the points $p_2, \dots, p_{r-1}$. Note that for every point $p \in C_1$, the length of the path from p_1 to p along C_1 is the distance $\mathbf{d}(p_1, p)$ between them.

If $r = n$, then we have obtained a geodesic tree whose vertices are $\{p_1, \dots, p_n\}$. If $r < n$, let C_2 be a path from p_1 to p_{r+1} whose length is minimal. If $C_1 \cap C_2 = \{p_1\}$, then we get a geodesic tree which contains at least $r+1$ singular points as vertices. If this is not the case, we prove that C_2 can not intersect C_1 transversely at a regular point.



Suppose that p is a regular point where C_2 intersects C_1 transversely. Let V be a neighborhood of p such that $S_1 = V \cap C_1$ and $S_2 = V \cap C_2$ are two geodesic segments, and p is the unique common point of S_1 and S_2. Let C_1' be the paths from p_1 to p along C_1 and C_2' be the path from p_1 to p along C_2, we have

$$\mathbf{leng}(C_1') = \mathbf{leng}(C_2') = \mathbf{d}(p_1, p).$$

Let q be a point in $S_2 \setminus C_2'$, and r be a point in $S_1 \cap C_1'$. Let $\overline{pq}$ denote the sub-segment of S_2 whose endpoints are p and q, and $\overline{pr}$ denote the sub-segments of S_1 whose endpoints are p and r. We have

$$\mathbf{d}(p_1, q) = \mathbf{d}(p_1, p) + \mathbf{leng}(\overline{pq}),$$

and

$$\mathbf{d}(p_1, p) = \mathbf{d}(p_1, r) + \mathbf{leng}(\overline{pr}).$$

Since p is a regular point of Σ, if we choose the points q and r close enough to p, the geodesic segment $\overline{qr}$ joining q and r will be contained in the neighborhood V, and we have

$$\mathbf{leng}(\overline{qr}) < \mathbf{leng}(\overline{pr}) + \mathbf{leng}(\overline{pq}).$$

It follows that

$$\mathbf{d}(p_1, q) = \mathbf{d}(p_1, r) + \mathbf{leng}(\overline{pr}) + \mathbf{leng}(\overline{pq}) > \mathbf{d}(p_1, r) + \mathbf{leng}(\overline{qr}).$$

The above inequality is in contradiction with the definition of the distance $\mathbf{d}$. Thus, we conclude that C_2 cannot intersect C_1 transversely at a regular point. This implies that the last intersection point of C_1 and C_2, that is the intersection point of furthest distance from p_1, must be a singular point p_k of Σ. Omit the part of C_2 from p_1 to p_k , we obtain a geodesic tree connecting at least $r+1$ singular points of Σ.

Let C_3 denote the new tree. For any point p of C_3, the length of the unique path from p_1 to p along C_3 is the distance $\mathbf{d}(p_1, p)$. This property allows us to conclude by an induction argument. □

Recall that a *closed translation surface* is a flat surface such that, for any closed curve γ which does not contain any singularity of the metric structure, we have $\mathbf{orth}(\gamma) = \mathrm{Id}$, where $\mathbf{orth}(\gamma)$ is the orthogonal part of the holonomy of γ. A *spherical flat surface* is a flat surface homeomorphic to the sphere $\mathbb{S}^2$. Proposition 3.1.3 implies

Corollary 3.1.4 *i) There exists on any closed translation surface a geodesic erasing tree.*

ii) There exists on any spherical flat surface a geodesic erasing tree.

Proof: The existence of a geodesic tree whose set of vertices is precisely the set of singular points of the flat surface is guaranteed by Proposition 3.1.3. By definition of translation surface, such a tree is obviously erasing, and $i)$ follows. Note that on a (closed) translation surface we have already an erasing forest which is the union of all singular points.

For spherical flat surfaces, by Proposition 3.1.3, there exists on any spherical flat surface a geodesic tree whose set of vertices is precisely the set of singular points. Since the complement of a tree in a sphere is an topological open disk, the holonomy of any closed curve in this complement must be Id. Therefore, we get an erasing tree, and $ii)$ follows. □

3.1.2 Main results

We fix two integers $g \geqslant 0$, $n > 0$, such that $2g + n - 2 > 0$, and positive real numbers $\alpha_1, \dots, \alpha_n$ verifying $\alpha_1 + \cdots + \alpha_n = 2\pi(2g + n - 2)$.

In the sequel of this chapter, S_g will be fixed a compact connected flat surface of genus g, without boundary. Assume that there exists a geodesic erasing forest $\hat{\mathcal{A}} = \sqcup_{i=1}^m \mathcal{A}_i$ on S_g, where each $\mathcal{A}_i$ is a geodesic tree. Let $p_1, \dots, p_n$ denote the vertices of the trees in $\hat{\mathcal{A}}$, and assume that the cone angle at p_i is α_i. Recall that, by definition, all the singular points of S_g are contained in the set $\{p_1, \dots, p_n\}$, but some of the points p_i may be regular. We also assume that at least one of the trees in $\hat{\mathcal{A}}$ is not a point.

Definition 3.1.5 (Mapping class group preserving a forest) *Let* $\mathrm{Homeo}^+(S_g, \hat{\mathcal{A}})$ *denote the group of orientation preserving homeomorphisms of* S_g *which fix the points* $\{p_1, \dots, p_n\}$*, and preserve the set* $\hat{\mathcal{A}}$*. Let* $\mathrm{Homeo}_0^+(S_g, \hat{\mathcal{A}})$ *be the normal subgroup of* $\mathrm{Homeo}^+(S_g, \hat{\mathcal{A}})$ *consisting of all elements which can be connected to* Id_{S_g} *by an isotopy fixing the points* $p_1, \dots, p_n$*.*

The mapping class group *of* S_g *preserving the trees in* $\hat{\mathcal{A}}$ *is the quotient group*

$$\Gamma(S_g, \hat{\mathcal{A}}) = \mathrm{Homeo}^+(S_g, \hat{\mathcal{A}})/\mathrm{Homeo}_0^+(S_g, \hat{\mathcal{A}}).$$

Remark: It follows from Lemma A.0.1 that, if f is a homeomorphism of S_g which is isotopic to identity by an isotopy fixing every point the set $\{p_1, \dots, p_n\}$, then there exists an isotopy $H_t : S_g \times [0;1] \longrightarrow S_g$ from f to Id_{S_g} such that $H_t(\hat{\mathcal{A}}) = \hat{\mathcal{A}},\ \forall t \in [0;1]$.

Without loss of generality, we can assume that there exist the integers $k_0, k_1, \dots, k_m$ such that $k_0 = 0$, $\sum_{j=1}^{m} k_j = n$, and the set of vertices of $\mathcal{A}_j$ is $\{p_{k_0+\dots+k_{j-1}+1}, \dots, p_{k_0+\dots+k_j}\}$ for every $j \in \{1, \dots, m\}$. The angles $\alpha_1, \dots, \alpha_n$ must satisfy the following condition :

$$\alpha_{k_0+\dots+k_{j-1}+1} + \dots + \alpha_{k_0+\dots+k_j} \in 2\pi\mathbb{N},\ \forall j \in \{1, \dots, m\}.$$

Let $\bar{\alpha}$ denote the set $\{\alpha_1, \dots, \alpha_n\}$. Let $\widetilde{\mathcal{T}}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ denote the set of pairs (Σ, ϕ), where Σ is a flat surface of genus g, and $\phi : S_g \longrightarrow \Sigma$ is an orientation preserving homeomorphism which maps $\hat{\mathcal{A}}$ onto a geodesic erasing forest of Σ.

We define an equivalence relation on $\widetilde{\mathcal{T}}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ as follows : two pairs (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are equivalent if there exists an isometry $h : \Sigma_1 \longrightarrow \Sigma_2$ such that the homeomorphism $\phi_2^{-1} \circ h \circ \phi_1$ is an element of $\mathrm{Homeo}_0^+(S_g, \hat{\mathcal{A}})$. The equivalence class of a pair (Σ, ϕ) will be denoted by $[(\Sigma, \phi)]$. Let $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ denote the space of equivalence classes of this relation.

Obviously, the group $\Gamma(S_g, \hat{\mathcal{A}})$ acts on $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$. The quotient space $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*/\Gamma(S_g, \hat{\mathcal{A}})$ is the *moduli space of flat surfaces with marked erasing trees* and denoted by $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$.

We denote $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ the set of equivalence classes $[(\Sigma, \phi)]$ where Σ is a flat surface of area one, and $\mathcal{M}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ the quotient space $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*/\Gamma(S_g, \hat{\mathcal{A}})$.

Definition 3.1.6 (Teichmüller space of flat surfaces with erasing forest) *The* Teichmüller space of flat surfaces with marked erasing forest and parallel vector field *is the set of all pairs* $([(\Sigma, \phi)], \xi)$*, where* $[(\Sigma, \phi)]$ *is an element of* $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$*, and* ξ *is a normalized parallel vector field on* $\Sigma \setminus \phi(\hat{\mathcal{A}})$*. We denote* *this space* $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$.

The moduli space of flat surfaces with marked erasing forest and normalized parallel vector field *is the quotient space* $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})/\Gamma(S_g, \hat{\mathcal{A}})$ *and denoted by* $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$.

Remark:

- The group $\mathbb{S}^1$, identified to the rotations of the Euclidean plane, acts naturally on the space $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$: if R_θ is the rotation of angle θ and $([(\Sigma, \phi)], \xi) \in \mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$, then $R_\theta \cdot ([(\Sigma, \phi)], \xi) = ([(\Sigma, \phi)], R_\theta \cdot \xi)$, where $R_\theta \cdot \xi$ is the parallel vector field defined as follows : at every point where ξ is defined, $R_\theta \cdot \xi$ is the vector obtained by rotating ξ an angle θ. This action of $\mathbb{S}^1$ endows $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ with a principal $\mathbb{S}^1$-bundle structure over $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$.

- The space $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ has also a $\mathbb{C}^*$-bundle structure over $\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$: for each element $[(\Sigma, \phi)] \in \mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, let ξ be a normalized parallel vector field on $\Sigma \setminus \phi(\hat{\mathcal{A}})$, the fiber over $[(\Sigma, \phi)]$ is the set of pairs $(r \cdot [(\Sigma, \phi)], R_\theta \cdot \xi)$, with $r \in \mathbb{R}_+^*, \theta \in \mathbb{S}^1$, where $r \cdot [(\Sigma, \phi)]$ is the multiplication of the metric on Σ by r while ϕ stays unchanged.

We can now state the main results of this chapter.

Proposition 3.1.7 ($\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ is embedded into $\mathcal{T}(g, n)$) *Let* $\mathcal{T}(g, n)$ *denote the Teichmüller space of conformal structures, and* $\Gamma(g, n)$ *denote the usual modular group of the punctured surface* $S_g \backslash \{p_1, \dots, p_n\}$.

a) There exists an injective map $\Theta : \mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^* \longrightarrow \mathcal{T}(g, n)$.

b) There exists also a monomorphism $\sigma : \Gamma(S_g, \hat{\mathcal{A}}) \longrightarrow \Gamma(g, n)$ *with respect to which* Θ *is equivariant*.

The definitions of Θ and σ are quite natural. Namely, since a flat metric structure implies a conformal structure, an equivalence class of $\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ is contained in an equivalence class of $\mathcal{T}(g, n)$, this defines Θ. By definition, a homeomorphism in $\text{Homeo}^+(S_g, \hat{\mathcal{A}})$ fixes the set $\{p_1, \dots, p_n\}$, hence it represents an element in the modular group $\Gamma(g, n)$, this defines σ.

Endow the space $\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ with the topology inherited from $\mathcal{T}(g, n)$, we get then a topology on $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ which is induced by the $\mathbb{C}^*$-bundle structure over $\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$. We have :

Corollary 3.1.8 *The action of the group* $\Gamma(S_g, \hat{\mathcal{A}})$ *on* $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ *is properly discontinuous.*

Proof: Since $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ is a $\mathbb{C}^*$-bundle over $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, and the action of $\Gamma(S_g, \hat{\mathcal{A}})$ preserves this bundle structure, it is enough to show that the action of $\Gamma(S_g, \hat{\mathcal{A}})$ on $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ is properly discontinuous. But this is a direct consequence of Proposition 3.1.7, since we know that the action of $\Gamma(g,n)$ on $\mathcal{T}(g,n)$ is properly discontinuous. □

Now, let us slit open the surface S_g along every tree $\mathcal{A}_j$ in the forest $\hat{\mathcal{A}}$, if $\mathcal{A}_j$ is not a point. The surface obtained, which will be denoted by $S_g^\natural$, is then a translation surface with geodesic boundary. If the tree $\mathcal{A}_j$ has $k_j > 1$ vertices (hence, $k_j - 1$ edges), then the vertices of $\mathcal{A}_j$ give rise to $2(k_j-1)$ points in the boundary component of $S_g^\natural$ corresponding to $\mathcal{A}_j$ whose complement are $2(k_j-1)$ open geodesic segments. Let $\mathcal{V}^\natural$ denote the finite subset of $S_g^\natural$ which arises from the set $\{p_1, \dots, p_n\}$.

Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$, by definition, $\phi(\mathcal{A}_j)$ is a geodesic tree of Σ. Slit open the surface Σ along every tree $\phi(\mathcal{A}_j)$ if $\mathcal{A}_j$ is not a point, and let $\Sigma^\natural$ denote the new surface. Observe that $\Sigma^\natural$ is also a translation surface with geodesic boundary homeomorphic to $S_g^\natural$. The homeomorphism ϕ from S_g onto Σ induces a homeomorphism $\phi^\natural$ from $\mathbb{S}_g^\natural$ onto $\Sigma^\natural$ which maps each geodesic segment on the boundary of $S_g^\natural$ onto a geodesic segment on the boundary of $\Sigma^\natural$. The normalized parallel vector field ξ on Σ induces also a normalized parallel vector field on $\Sigma^\natural$ which will be denoted again by ξ. It follows that we get a point in the Teichmüller space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$, which is represented by the pair $([(\Sigma^\natural, \phi^\natural)], \xi)$, where the data $\bar{\alpha}'$, and $\bar{\beta}'$ are determined by the angles $\bar{\alpha}$ and the forest $\hat{\mathcal{A}}$.

Let Ξ denote the map from $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ into $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$ which associates to a pair $([(\Sigma, \phi)], \xi)$ in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ the pair $([(\Sigma^\natural, \phi^\natural)], \xi)$ constructed as above. First, we have

Proposition 3.1.9 *The map Ξ is well defined.*

Proof: We need to show that if (Σ_1, ϕ_1) and (Σ_2, ϕ_2) represent the same point in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ then $(\Sigma_1^\natural, \phi_1^\natural)$ and $(\Sigma_2^\natural, \phi^\natural)$ represent the same point in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')^*$.

By definition, there exists an isometry

$$h : \Sigma_1 \longrightarrow \Sigma_2,$$

such that $\phi_2^{-1} \circ h \circ \phi_1$ is isotopic to Id_{S_g} by an isotopy fixing the points $\{p_1, \dots, p_n\}$. Let $h^\natural$ be the isometry from $\Sigma_1^\natural$ onto $\Sigma_2^\natural$ which is induced by h.

By Lemma A.0.1, we can assume that the isotopy H_t from $\phi_2^{-1} \circ h \circ \phi_1$ to Id_{S_g} preserves the forest $\hat{\mathcal{A}}$, therefore H_t induces an isotopy from ${\phi_2^\natural}^{-1} \circ h^\natural \circ \phi_1^\natural$ to $\mathrm{Id}_{S_g^\natural}$, which is identity on the set $\mathcal{V}^\natural$. By definition, it follows that the pairs $(\Sigma_1^\natural, \phi_1^\natural)$ and $(\Sigma_2^\natural, \phi_2^\natural)$ represent the same point in $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')^*$. $\square$

We have the following

Theorem 3.1.10 *i) The map Ξ is injective, continuous, and the set $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$ is a special complex affine sub-manifold of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$ (meaning that the coordinate changes of $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$, which are induced by those of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$, preserve a volume form) of dimension*

- $2g + n - 1$ *if* $\alpha_i \in 2\pi\mathbb{N}$ *for every* $i \in \{1, \ldots, n\}$.
- $2g + n - 2$ *otherwise.*

ii) There exists a volume form on $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$ whose pull-back by Ξ gives a volume on $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ which is invariant by the action of the group $\Gamma(S_g, \hat{\mathcal{A}})$.

A direct consequence of Theorem 3.1.10 is the following

Corollary 3.1.11 *The space $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ is a flat complex affine manifold of dimension*

- $2g + n - 1$ *if* $\alpha_i \in 2\pi\mathbb{N}$ *for every* $i \in \{1, \ldots, n\}$.
- $2g + n - 2$ *otherwise.*

There exists on $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ a volume form invariant by the action of the group $\Gamma(S_g, \hat{\mathcal{A}})$, which will be denoted by μ_{Tr}.

3.2 The embedding of $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ into $\mathcal{T}(g, n)$

3.2.1 Conformal metrics with conical singularities on a Riemann surface

In this subsection, we follow loosely the definitions in [Tr1]. Let S be a compact Riemann surface, possibly with boundary. A *conformal (singular) metric* g on S is defined by a local expression

$$h = \rho(z)|dz|^2,$$

where z is a local coordinate on S, and ρ is a positive measurable function.

A *(real) divisor* on S is simply a formal sum :

$$\mathbf{div} = \sum_{i=1}^{n_1} s_i p_i + \sum_{j=1}^{n_2} t_j q_j,$$

where $p_i \in \mathrm{int}(S)$ $(i = 1, \ldots, n_1)$, $q_j \in \partial S$ $(j = 1, \ldots, n_2)$, and $s_1, \ldots, s_{n_1}, t_1, \ldots, t_{n_2}$ are real numbers.

We will always suppose that the real numbers $s_1, \ldots, s_{n_1}$ and $t_1, \ldots, t_{n_2}$ satisfy the following condition :

$$s_i > -1;\ i = 1, \ldots, n_1 \text{ and } t_j > -\frac{1}{2};\ j = 1, \ldots, n_2.$$

The set $\{p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}\}$ is called the *support* of $\mathbf{div}$ and denoted by $\mathrm{supp}(\mathbf{div})$. The real number

$$|\mathbf{div}| = \sum_{i=1}^{n_1} s_i + \sum_{j=1}^{n_2} t_j,$$

is called the *degree* of the divisor $\mathbf{div}$.

A conformal metric h on S is said to *represent* the divisor $\mathbf{div}$ if h is a smooth Riemannian metric on $S \setminus \mathrm{supp}(\mathbf{div})$ such that :

$$(*) \quad \begin{cases} \forall i \in \{1, \ldots, n_1\}, & h = e^{2u}|z_i|^{2s_i}|dz_i|^2 \text{ on a neighborhood } U_i \text{ of } p_i, \\ \forall j \in \{1, \ldots, n_2\}, & h = e^{2v}|w_j|^{4t_j}|dw_j|^2 \text{ on a neighborhood } V_j \text{ of } q_j, \end{cases}$$

where z_i (resp. w_j) is a holomorphic coordinate on U_i (resp. V_j) such that $z_i(p_i) = 0$ (resp. $w_j(q_j) = 0$), and $u : U_i \longrightarrow \mathbb{R}$ (resp. $v : V_j \longrightarrow \mathbb{R}$) is a continuous function of class C^2 on $U_i - \{p_i\}$ (resp. on $V_j - \{q_j\}$).

The point p_i is then said to be a *conical singularity* of angle $\theta_i = 2\pi(s_i + 1)$. The point q_j is said to be a *corner* of angle $\eta_j = 2\pi(t_j + \frac{1}{2})$. Observe that $\mathbb{C}$, equipped with the metric $|z|^{2s}|dz|^2$, is isometric to an Euclidean cone of angle $\theta = 2\pi(s + 1)$. Similarly, the upper half plane $U = \{z \in \mathbb{C} : \mathrm{Im} z \geqslant 0\}$, equipped with the metric $|z|^{4t}|dz|^2$, is isometric to an Euclidean corner of angle $\eta = \pi(2t + 1)$.

If h is a conformal metric with conical singularities on S, let K_h denote the curvature of h, this is real function which is defined on $S \setminus \{$ singularities of $h\}$. An Euclidean conformal metric, with conical singularities, representing $\mathbf{div}$ is then a conformal metric h satisfying the following conditions :

- For each p_i, $i = 1, \ldots, n_1$, there exists a conformal coordinate z defined in a neighborhood of p_i such that $h = |z|^{2s_i}|dz|^2$.

- For each $q_j,\ j = 1, \dots, n_2$, there exists a conformal coordinate w defined in a neighborhood of q_j such that $h = |w|^{4t_j}|dw|^2$.

- $K_h \equiv 0$ on $S \setminus \mathrm{supp}(\mathbf{div})$.

Let S be a compact Riemannian surface, possibly with boundary, and $\mathbf{div}$ be a real divisor of S satisfying the condition $(*)$. The *Euler characteristic* of the pair $(S, \mathbf{div})$ is defined to be

$$\chi(S, \mathbf{div}) = \chi(S) + |\mathbf{div}|.$$

We have (see [Tr1])

Theorem 3.2.1 (Gauss-Bonnet formula) *Let h be a conformal metric representing* $\mathbf{div}$, *then*

$$\frac{1}{2\pi}\int\int_S K_h dA_h + \frac{1}{2\pi}\int_{\partial S} k_h dh = \chi(S, \mathbf{div}),$$

where K_h is the curvature, dA_h is the area element and k_h is the geodesic curvature of h.

Corollary 3.2.2 *If h is a conformal flat metric with conical singularities and geodesic boundary, representing* $\mathbf{div}$, *then we have*

$$\sum_{i=1}^{n_1}\theta_i + \sum_{j=1}^{n_2}\eta_j = 2\pi(n_1 + \frac{n_2}{2} - \chi(S)),$$

where θ_i is the cone angle at p_i ($i = 1, \dots, n_1$) and η_j is the corner angle at q_j ($j = 1, \dots, n_2$).

We quote here an important result which is proved in [Tr1] :

Proposition 3.2.3 ([Tr1], Proposition 2) *Let S be a compact Riemannian surface, possibly with boundary, and* $\mathbf{div}$ *a real divisor on S such that $\chi(S, \mathbf{div}) = 0$. Then there exists on S a conformal metric representing* $\mathbf{div}$ *such that $\partial S \setminus \mathrm{supp}(\mathbf{div})$ is geodesic. This metric is unique up to homothety.*

3.2.2 Proof of Proposition 3.1.7

a) Let Σ be a flat surface having n conical singularities homeomorphic to S_g. The flat metric structure on Σ induces a conformal structure on $\Sigma \setminus \{$ singularities $\}$. The map Θ is defined as follows : for every pair (Σ, ϕ) which is a representative of an equivalence class in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, let $\bar{\phi}$ be a quasi-conformal homeomorphism from S_g onto Σ in the same isotopy class relative to $\{p_1, \dots, p_n\}$ of ϕ. Since the isotopy class relative to $\{p_1, \dots, p_n\}$ of ϕ contains diffeomorphisms, such a homeomorphism exists. We define $\Theta([(\Sigma, \phi)])$ to be the equivalence class in $\mathcal{T}(g, n)$ which is represented by the pair $(\Sigma \setminus \{x_1, \dots, x_n\}, \bar{\phi})$, where $x_i = \phi(p_i)$ $i = 1, \dots, n$ and $\Sigma \setminus \{x_1, \dots, x_n\}$ is now considered as a Riemann surface. We need to prove :

Lemma 3.2.4 *The map Θ is well defined.*

Proof: We have to prove that two different representatives (Σ_1, ϕ_1) and (Σ_2, ϕ_2) of an equivalence class in $\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ give the same equivalence class in $\mathcal{T}(g,n)$. Let $\bar{\phi}_1, \bar{\phi}_2$ be the quasi-conformal homeomorphisms in the same isotopy class relative to $\{p_1, \dots, p_n\}$ of ϕ_1 and ϕ_2 respectively.

By the definition of $\mathcal{T}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, there exists an isometry $f : \Sigma_1 \longrightarrow \Sigma_2$ such that $\phi_2^{-1} \circ f \circ \phi_1$ is an element of $\text{Homeo}_0^+(S_g, \hat{\mathcal{A}})$. Since an isometry between two flat surfaces is a conformal homeomorphism between the two Riemann surfaces underlying, and $\bar{\phi}_1, \bar{\phi}_2$ are homotopic to ϕ_1, ϕ_2 relative to $\{p_1, \dots, p_n\}$ respectively, it follows that $\bar{\phi}_2^{-1} \circ f \circ \bar{\phi}_1$ is an element of $\mathcal{QC}_0^+(g,n)$. Hence, the pairs $(\Sigma_1 \setminus \{\phi_1(p_1), \dots, \phi_1(p_n)\}, \bar{\phi}_1)$ and $(\Sigma_2 \setminus \{\phi_2(p_1), \dots, \bar{\phi}_2(p_n)\}, \phi_2)$ belong to the same equivalence class in $\mathcal{T}(g,n)$. □

Next, we have :

Lemma 3.2.5 *The map Θ is injective.*

Proof: Let (Σ_1, ϕ_1) and (Σ_2, ϕ_2) be two pairs in $\widetilde{\mathcal{T}}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ such that $\text{Area}(\Sigma_1) = \text{Area}(\Sigma_2) = 1$. Let $\bar{\phi}_1, \bar{\phi}_2$ be two quasi-conformal homeomorphisms isotopic to ϕ_1, ϕ_2 relative to $\{p_1, \dots, p_n\}$ respectively.

Suppose that $(\Sigma_1 \setminus \{\phi_1(p_1), \dots, \phi_1(p_n)\}, \bar{\phi}_1)$ and $(\Sigma_2 \setminus \{\phi_2(p_1), \dots, \phi_2(p_n)\}, \bar{\phi}_2)$ belong to the same equivalence class in $\mathcal{T}(g,n)$, we have to prove that (Σ_1, ϕ_1) and (Σ_2, ϕ_2) also belong to the same equivalence class in $\mathcal{T}_1^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$.

By the definition of $\mathcal{T}(g,n)$, there exists a conformal homeomorphism $h : \Sigma_1 \longrightarrow \Sigma_2$ such that $\bar{\phi}_2^{-1} \circ h \circ \bar{\phi}_1$ is isotopic to Id_{S_g} by an isotopy fixing every point in the set $\{p_1, \dots, p_n\}$. Now, since $\bar{\phi}_i$ is isotopic to ϕ relative to $\{p_1, \dots, p_n\}$, for $i = 1, 2$, it follows that $\phi_2^{-1} \circ h \circ \phi_1$ is also isotopic to Id_{S_g} by an isotopy fixing every point in the set $\{p_1, \dots, p_n\}$.

First, we prove that h is also an isometry between the two flat surfaces Σ_1 and Σ_2.

Let $(x_1, \dots, x_n)$, and $(y_1, \dots, y_n)$ denote the singularities of Σ_1 and Σ_2 respectively, where $x_i = \phi_1(p_i), y_i = \phi_2(p_i),\ i = 1, \dots, n$. Let f_1 and f_2 denote the two flat metrics on Σ_1 and Σ_2 respectively. Let $\mathbf{div}_1$ denote the divisor $\sum_{j=1}^n s_j x_j$, and $\mathbf{div}_2$ denote the divisor $\sum_{j=1}^n s_j y_j$, where s_j satisfies $\alpha_j = 2\pi(s_j + 1)$. By definition, f_i is a conformal flat metric which represents the divisor $\mathbf{div}_i$ on

$\Sigma_i,\ i = 1, 2.$

Since h is a conformal homeomorphism, it follows that $h^* f_2$ is also a conformal flat metric on Σ_1. Since $h(\mathbf{div}_1) = \mathbf{div}_2$, we deduce that $h^* f_2$ represents $\mathbf{div}_1$ too. Now, from Proposition 3.2.3, there exists $\lambda > 0$ such that $f_1 = \lambda\, h^* f_2$. Since we have assumed that $\mathrm{Area}_{f_1}(\Sigma_1) = \mathrm{Area}_{f_2}(\Sigma_2) = 1$, it follows that $\lambda = 1$. Therefore we have $f_1 = h^* f_2$, in other words, h is an isometry from the flat surface Σ_1 onto the flat surface Σ_2.

All we need to prove now is that $\phi_2^{-1} \circ h \circ \phi_1$ preserves the forest $\hat{\mathcal{A}}$. By definition, $\phi_1(\hat{\mathcal{A}})$ is a union of geodesic trees on Σ_1 whose vertices are $x_1, \dots, x_n$. Since h is an isometry of flat surfaces, $h(\phi_1(\hat{\mathcal{A}}))$ is a union of geodesic trees whose vertices are $y_1, \dots, y_n$. Let a be an edge of a tree in $\hat{\mathcal{A}}$. The set $\phi_1(a)$ is a geodesic segment on Σ_1, hence $h(\phi_1(a))$ is a geodesic segment of Σ_2. By definition, $\phi_2(a)$ is also a geodesic segment of Σ_2.

By assumption, there exists an isotopy relative to $\{p_1, \dots, p_n\}$ from $h \circ \phi_1$ to ϕ_2. Now, from Lemma 2.3.8, we have $h(\phi_1(a)) = \phi_2(a)$. Since this is true for every edges in $\hat{\mathcal{A}}$, we conclude that $h \circ \phi_1(\hat{\mathcal{A}}) = \phi_2(\hat{\mathcal{A}})$, or equivalently, $\phi_2^{-1} \circ h \circ \phi_1(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$. It follows immediately that $\phi_2^{-1} \circ h \circ \phi_1 \in \mathrm{Homeo}_0^+(S_g, \hat{\mathcal{A}})$, in other words, (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are equivalent in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$. □

Part a) of Proposition 3.1.7 is now proved.

b) It is well known that $\Gamma(g, n)$ can be identified to the quotient group $\mathrm{Homeo}^+(g, n)/\mathrm{Homeo}_0^+(g, n)$, where $\mathrm{Homeo}^+(g, n)$ is the group of all preserving orientation homeomorphism of S_g which fix every point in the set $\{p_1, \dots, p_n\}$, and $\mathrm{Homeo}_0^+(g, n)$ is the normal subset of $\mathrm{Homeo}^+(g, n)$ consisting of all elements which are isotopic to Id_{S_g} relative to $\{p_1, \dots, p_n\}$.

By definition, it is clear that $\mathrm{Homeo}^+(S_g, \hat{\mathcal{A}})$ is a subgroup of $\mathrm{Homeo}^+(g, n)$, and

$$\mathrm{Homeo}_0^+(S_g, \hat{\mathcal{A}}) = \mathrm{Homeo}^+(S_g, \hat{\mathcal{A}}) \cap \mathrm{Homeo}_0^+(g, n).$$

It follows that $\Gamma(S_g, \hat{\mathcal{A}})$ is a subgroup of $\Gamma(g, n)$. Let $\sigma : \Gamma(S_g, \hat{\mathcal{A}}) \longrightarrow \Gamma(g, n)$ denote the natural imbedding. The morphism σ is obviously injective. Since the actions of $\Gamma(S_g, \hat{\mathcal{A}})$ and $\Gamma(g, n)$ are defined in the same way, the map Θ is equivariant with respect to σ. □

From now on, we can consider $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ as a subset of the Teichmüller space $\mathcal{T}(g, n)$, and $\Gamma(S_g, \hat{\mathcal{A}})$ as a subgroup of $\Gamma(g, n)$, which preserves $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$.

3.3 Injectivity of the map Ξ

Let $X_1 = ([(\Sigma_1, \phi_1)], \xi_1)$ and $X_2 = ([(\Sigma_2, \phi_2)], \xi_2)$ be two points in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ such that $\Xi(X_1) = \Xi(X_2)$. By definition, $\Xi(X_i)$, $i = 1, 2$, is represented by the pair $([(\Sigma_i^\natural, \phi_i^\natural)], \xi_i)$. The assumption $\Xi(X_1) = \Xi(X_2)$ implies that there exists an isometry $h^\natural$ from $\Sigma_1^\natural$ onto $\Sigma_2^\natural$ such that ${\phi_2^\natural}^{-1} \circ h^\natural \circ \phi_1^\natural$ is an element in $\mathrm{Homeo}_0^+(S_g^\natural, \mathcal{V}^\natural)$.

Clearly, the isometry $h^\natural$ induces an isometry h from Σ_1 to Σ_2, which maps the forest $\phi_1(\hat{\mathcal{A}})$ to the forest $\phi_2(A_2)$. Set $\varphi = \phi_2^{-1} \circ h \circ \phi_1 : S_g \longrightarrow S_g$. Remark that $\varphi(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$, therefore $\varphi \in \mathrm{Homeo}^+(S_g, \hat{\mathcal{A}})$. All we need to prove is the following

Lemma 3.3.1 *φ is isotopic to Id_{S_g} by an isotopy fixing all the points in $\{p_1, \dots, p_n\}$.*

Proof: Since $\varphi^\natural = {\phi_2^\natural}^{-1} \circ h^\natural \circ \phi_1^\natural$ belongs to $\mathrm{Homeo}_0^+(S_g^\natural, \mathcal{V}^\natural)$, there exists an isotopy

$$\mathrm{H}^\natural : S_g^\natural \times [0; 1] \longrightarrow S_g^\natural,$$

such that, $\mathrm{H}_0^\natural = \varphi^\natural$, $\mathrm{H}_1^\natural = \mathrm{Id}_{S_g^\natural}$, and $\mathrm{H}_t(\mathcal{V}^\natural) = \mathcal{V}^\natural$, where $\mathrm{H}_t^\natural = \mathrm{H}^\natural(., t)$, $\forall t \in [0; 1]$.

Let $(a, \bar{a})$ be a pair of geodesic segments in the boundary of $S_g^\natural$ which correspond to the same edge $\tilde{a}$ in the forest $\hat{\mathcal{A}}$. The identifications with $\tilde{a}$ induce a homeomorphism $\rho_{\bar{a}}$ from a onto $\bar{a}$. Let f be a homeomorphism of $S_g^\natural$ which is identity on the set $\mathcal{V}^\natural$. The necessary and sufficient condition for f to define a homeomorphism on S_g is that,

$$\text{for every edge } \tilde{a} \text{ in the forest } \hat{\mathcal{A}}, \text{ we have } \rho_{\bar{a}}^{-1} \circ f_{|\bar{a}} \circ \rho_{\bar{a}} = f_{|a} \qquad (*)$$

Lemma 3.3.1 will follow from the following lemma

Lemma 3.3.2 *Given any homeomorphism f of $S_g^\natural$ which is identity on the set $\mathcal{V}^\natural$, there exists a homeomorphism f' of $S_g^\natural$ such that the homeomorphism $\hat{f} = f' \circ f$ verifies the condition $(*)$.*

Proof: We only prove this lemma in the case $\hat{\mathcal{A}}$ contains only one edge a. The general case can be shown by similar argument.

We identify a thin neighborhood N_a of a in $S_g^\natural$ to a rectangle $R_\epsilon = [0; 1] \times [0; \epsilon]$ in $\mathbb{R}^2$, with ϵ positive, such that a is identified to the segment $[0; 1] \times \{0\}$. The map $(\rho_{\bar{a}}^{-1} \circ f_{|\bar{a}} \circ \rho_{\bar{a}}) \circ f_{|a}^{-1}$ induces a homeomorphism q of the segment $[0; 1]$. We define a homeomorphism Q of R_ϵ as follows

$$Q(s, t) = (s + \frac{\epsilon - t}{\epsilon}(q(s) - s), t), \ \forall (s, t) \in [0; 1] \times [0; \epsilon].$$

Note that $q(0) = 0$, and $q(1) = 1$, therefore Q is identity on the two vertical sides of R_ϵ. By definition, Q is identity on the upper side of R_ϵ, and $Q = q$ on the lower side of R_ϵ.

The homeomorphism Q induces a homeomorphism Q' of N_a. We can extend Q' by identity outside N_a to obtain a homeomorphism f' of $S_g^\natural$. By construction, we have

$$f'_{|a} = (\rho_{\bar{a}}^{-1} \circ f_{|\bar{a}} \circ \rho_{\bar{a}}) \circ f_{|a}^{-1},$$

and

$$f'_{|\bar{a}} = \mathrm{Id}_{\bar{a}}.$$

It follows immediately that $\hat{f} = f' \circ f$ verifies the condition $(*)$ on a. The lemma is then proved. □

Back to the proof of 3.3.1. By Lemma 3.3.2, for each $t \in [0;1]$, we can find a homeomorphism H'_t of $S_g^\natural$ such that $\hat{\mathrm{H}}_t = \mathrm{H}'_t \circ \mathrm{H}_t$ verifies the conditions $(*)$. Clearly, the homeomorphisms H'_t can be chosen continuously as a function of t, therefore, $\hat{\mathrm{H}}_t$ induces an isotopy from φ to Id_{S_g} which is identity on the set $\{p_1, \dots, p_n\}$, and the lemma follows. □

Lemma 3.3.1 allows us to conclude that the map Ξ is injective.

3.4 Image of $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ by Ξ

Let $\mathcal{V}^\natural$ denote the finite subset of $S_g^\natural$ arising from the set $\{p_1, \dots, p_n\}$ of S_g. Let $\mathcal{TR}(S_g^\natural)$ be the set of all triangulations of $S_g^\natural$ whose vertex set is $\mathcal{V}^\natural$ modulo homotopy relative to $\mathcal{V}^\natural$.

Let $\mathcal{T}$ be a triangulation in $\mathcal{TR}(S_g^\natural)$, in Section 2.4, we have already defined a subset $\mathcal{U}_\mathcal{T}$ of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$ corresponding to $\mathcal{T}$, and a local chart $\Psi_\mathcal{T}$ defined on $\mathcal{U}_\mathcal{T}$. Let N_1, N_2 be respectively the number of edges, and the number of triangles of $\mathcal{T}$. Recall that we also have a system of linear equations associated to $\mathcal{T}$, which is denoted by $\mathrm{S}_\mathcal{T}$, consisting of N_2 equations. Let $\mathrm{V}_\mathcal{T}$ be the subspace of $\mathbb{C}^{N_1}$ consisting of solutions of the system $\mathrm{S}_\mathcal{T}$. The image of $\mathcal{U}_\mathcal{T}$ by $\Psi_\mathcal{T}$ is then an open subset of $\mathrm{V}_\mathcal{T}$. Since we have assumed that there exists at least a tree in $\hat{\mathcal{A}}$ which is not a point, the boundary of $S_g^\natural$ is not empty, and hence,

$$\dim_{\mathbb{C}} \mathrm{V}_\mathcal{T} = 2g + 2\sum_{j=1}^{m}(k_j - 1) - 2 = 2g + 2(n - m) - 2.$$

Note that the family $\{\mathcal{U}_\mathcal{T},\ \mathcal{T} \in \mathcal{TR}(S_g^\natural)\}$ is an open cover of the space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$. First, we have

Proposition 3.4.1 *For every triangulation $\mathcal{T}$ in $\mathcal{TR}(S_g^\natural)$, the intersection $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap \mathcal{U}_{\mathcal{T}}$ is mapped by Ψ_{T} onto an open subset of a subspace of $\mathrm{V}_{\mathcal{T}}$ of dimension*

- $2g + n - 1$ *if* $\alpha_i \in 2\pi\mathbb{N}, \ \forall i = 1, \dots, n.$
- $2g + n - 2$ *otherwise.*

For each $\mathcal{T}$ in $\mathcal{TR}(S_g^\natural)$, let $\mathrm{V}_{\mathcal{T}}^*$ denote the subspace of $\mathrm{V}_{\mathcal{T}}$ that contains the image of $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap \mathcal{U}_{\mathcal{T}}$ as an open subset. We have then

Proposition 3.4.2 *If $\mathcal{T}_1$ and $\mathcal{T}_2$ represent two different equivalence classes in $\mathcal{TR}(S_g^\natural)$ such that $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2} \neq \varnothing$, then $\Psi_{\mathcal{T}_2} \circ \Psi_{\mathcal{T}_1}^{-1}$ maps $\mathrm{V}_{\mathcal{T}_1}^*$ onto $\mathrm{V}_{\mathcal{T}_2}^*$.*

From Proposition 3.4.1, and Proposition 3.4.2, we get immediately

Corollary 3.4.3 *$\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$ is a special flat complex affine subspace of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$.*

3.4.1 Proof of Proposition 3.4.1

Let $([(\Sigma, \phi)], \xi)$ be a point $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ whose image by Ξ is a point $([(\Sigma^\natural, \phi^\natural)], \xi)$ in $\mathcal{U}_{\mathcal{T}} \subset \mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$. By definition, the homeomorphism $\phi^\natural$ sends the triangulation $\mathcal{T}$ of $S_g^\natural$ onto an admissible triangulation T of $\Sigma^\natural$. The triangulation T of $\Sigma^\natural$ induces a triangulation of Σ by geodesic segments containing the forest $\hat{A} = \phi(\hat{\mathcal{A}})$, whose vertex set is $\{p_1, \dots, p_n\}$. This triangulation of Σ will be denoted by T^*.

Recall that the map $\Psi_{\mathcal{T}}$ associates to each edge of T a complex numbers, the complex number associated to an edge e of T will be denoted by $z(e)$. We start with

Lemma 3.4.4 *If $(e, \bar{e})$ is a pair of edges in the boundary of $\Sigma^\natural$ which corresponds to an edge of a tree $A_j = \phi(\mathcal{A}_j)$ in $\phi(\hat{\mathcal{A}})$, then we have*

$$z(\bar{e}) = -e^{\imath\theta} z(e) \tag{3.1}$$

where the number θ is determined by the angles $\bar{\alpha}$, and the tree $\mathcal{A}_j$.

Proof: Let $\tilde{e}$ denote the edge of A_j which corresponds to the pair $(e, \bar{e})$. Assume that the edges e and $\bar{e}$ are oriented coherently with the orientation of $\Sigma^\natural$. It follows that the orientations of e and $\bar{e}$ induces inverse orientations of $\tilde{e}$, this justifies the minus sign in (3.1).

Let p be the mid-point of $\tilde{e}$, and let γ be a closed curve on the surface Σ such that $\gamma \cap \hat{A} = \{p\}$, where $\hat{A} = \phi(\hat{\mathcal{A}})$.

Observe that θ is the rotation angle of the holonomy of the curve γ. The angle θ is determined from the tree A_j and the angles $\alpha_1, \dots, \alpha_n$ as follows : since A_j is a tree, $A_j \setminus \tilde{e}$ has two connected components. Take one of these components and add to it the segment $\tilde{e}$, we get then a sub-tree A'_j of A_j.

Suppose that $\{x_{i_0}, x_{i_1}, \dots, x_{i_k}\}$ are the vertices of the tree A'_j, where x_{i_0} and x_{i_1} are the endpoints of $\tilde{e}$. Up to a permutation of indices, the curve γ is homotopic to the curve $l_{i_1} \circ l_{i_2} \circ \cdots \circ l_{i_k} \circ \gamma'$, where l_{i_s}, $s = 1, \dots, k$, is a closed curve homologous to a small loop about x_{i_s}, and γ' is a closed curve in $\Sigma \setminus \hat{A}$. Since the rotation $\mathbf{orth}(l_{i_s})$ is of angle α_{i_s}, and the rotation $\mathbf{orth}(\gamma')$ is trivial by definition of erasing forest, it follows that $\mathbf{orth}(\gamma)$ is the rotation of angle $\alpha_{i_1} + \cdots + \alpha_{i_k}$. Hence

$$\theta = \alpha_{i_1} + \cdots + \alpha_{i_k} \mod 2\pi.$$

□

Since the trees in the forest $\hat{\mathcal{A}}$ have totally $(n-m)$ edges, Lemma 3.4.4 implies that coordinates of the vector $\Psi_{\mathcal{T}}([(\Sigma^{\natural}, \phi^{\natural})], \xi) \in \mathbb{C}^{N_1}$ is must verify $(n-m)$ additional equations of type (3.1). Adding those equations to the system $\mathbf{S}_{\mathcal{T}}$, we get a system $\mathbf{S}^*_{\mathcal{T}}$ which contains $N_2 + (n-m)$ linear equations. Let $\mathrm{V}^*_{\mathcal{T}}$ denote the subspace of $\mathbb{C}^{N_1}$ consisting of solutions of $\mathbf{S}^*_{\mathcal{T}}$. We have then

Lemma 3.4.5 *The image of* $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap \mathcal{U}_{\mathcal{T}}$ *by* $\Psi_{\mathcal{T}}$ *is an open subset of* $\mathrm{V}^*_{\mathcal{T}}$.

Proof: Let $Z = (z_1, \dots, z_{N_1})$ denote the image of $([(\Sigma^{\natural}, \phi^{\natural})], \xi)$ by $\Psi_{\mathcal{T}}$. It suffices to show that $\Psi_{\mathcal{T}}(\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap \mathcal{U}_{\mathcal{T}})$ contains neighborhood of Z in $\mathrm{V}^*_{\mathcal{T}}$.

Let $Z' = (z'_1, \dots, z'_{N_1}) \in \mathbb{C}^{N_1}$ be a vector in a neighborhood of Z which is also a solution of the system $\mathbf{S}^*_{\mathcal{T}}$. Using the triangulation T, we construct a flat surface from Z' as follows :

. Construct an Euclidean triangle from z'_i, z'_j, z'_k if z'_i, z'_j, z'_k verify an equation of type (2.3).

. Identify two sides of two distinct triangles if they correspond to the same complex number z'_i.

. Identify the edges corresponding to z'_i and z'_j if z'_i and z'_j satisfy an equation of type (3.1).

Clearly by this construction we obtain a flat surface Σ' homeomorphic to Σ. The surface Σ' also has n conical singularities, and there is a distinguished geodesic erasing forest $\hat{A}'$ on Σ'. Moreover, we also get a triangulation $\mathrm{T}^{*\prime}$ of Σ' by geodesic segments. Each triangle in $\mathrm{T}^{*\prime}$ corresponds to a triangle in $\mathbb{E}^2$

specified by three complex numbers, hence we get a normalized parallel vector field ξ' on $\Sigma' \setminus \hat{A}'$ which is defined by the constant vertical vector field $(0,1)$ on the Euclidean plan $\mathbb{E}^2$.

Define an orientation preserving homeomorphism

$$f : \Sigma \longrightarrow \Sigma',$$

as follows : f maps each edge of T^* onto the corresponding edge of $\mathrm{T}^{*\prime}$, and the restriction f on each triangle is a linear transformation of $\mathbb{R}^2$. Note that the homeomorphism f is then quasi-conformal with respect to the conformal structures on Σ, and Σ'. Let ϕ' denote the map

$$\phi' = f \circ \phi : S_g \longrightarrow \Sigma'.$$

It follows that the pair $([(\Sigma', \phi')], \xi')$ represents a point of $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ close to $([(\Sigma, \phi)], \xi)$. Clearly, by construction, we have $\Psi_{\mathcal{T}}(\Xi([(\Sigma', \phi')], \xi')) = Z'$, and the lemma follows. $\square$

Now, we need to compute the dimension of $\mathrm{V}^*_{\mathcal{T}}$.

Lemma 3.4.6 *We have*

$$\dim_{\mathbb{C}} \mathrm{V}^*_{\mathcal{T}} = \begin{cases} 2g+n-1, & \text{if } \alpha_i \in 2\pi\mathbb{N},\ \forall i = 1, \dots, n; \\ 2g+n-2, & \text{otherwise.} \end{cases}$$

Proof: Since the system $\mathbf{S}_{\mathcal{T}}$ contains already N_2 equations, the system $\mathbf{S}^*_{\mathcal{T}}$ contains $N_2 + (n-m)$ equations, therefore

$$\dim \mathrm{V}^*_{\mathcal{T}} \geqslant N_1 - (N_2 + (n-m)) = 2g+n-2. \tag{3.2}$$

Consider the surface $\Sigma^\natural$ with the admissible triangulation T. Let $a_1, \bar{a}_1, \dots, a_{n-m}, \bar{a}_{n-m}$ denote the edges of T which are contained in the boundary of $\Sigma^\natural$ so that each pair $(a_i, \bar{a}_i)$ corresponds to an edge of a tree in the forest $\hat{A}$ of Σ.

Choose a family of primitive edges in T, note that such a family must contains $2g+m-1$ edges, let $b_1, \dots, b_{2g+m-1}$ denote the edges in this family. As usual, for any edge e of T, let $z(e)$ be the complex number associated to e by $\Psi_{\mathcal{T}}$.

By definition, we have $\mathrm{int}(\Sigma^\natural) \setminus \cup_{j=1}^{2g+m-1} b_j$ is an open disk. Using Lemma 2.4.1, *ii*), we deduce that if e is any edge of T, then $z(e)$ can be written as a linear combination of

$$(z(a_1), z(\bar{a}_1), \dots, z(a_{n-m}), z(\bar{a}_{n-m}); z(b_1), \dots, z(b_{2g+m-1})),$$

with the coefficients in $\{\pm 1, 0\}$. From Lemma 3.4.4, we know that $z(\bar{a}_i) = -e^{\imath\theta_i} z(a_i)$, where θ_i is determined by $\bar{\alpha}$ and $\hat{\mathcal{A}}$. The complex number $z(e)$ is a linear function of

$$(z(a_1), \dots, z(a_{n-m}), z(b_1), \dots, z(b_{2g+m-1})).$$

We deduce that

$$\dim \mathrm{V}^*_{\mathcal{T}} \leqslant 2g + n - 1. \tag{3.3}$$

Apply Lemma 2.4.1, $ii)$ to the disk $\mathbf{D} = \mathrm{int}(\Sigma^{\natural}) \setminus \cup_{j=1}^{2g+m-1} b_j$, we get

$$\sum_{i=1}^{n-m} (z(a_i) + z(\bar{a}_i)) = 0$$

By Lemma 3.4.4, it follows

$$\sum_{i=1}^{n-m} (1 - e^{\imath\theta_i}) z(a_i) = 0. \tag{3.4}$$

Note that the numbers $z(b_j),\ j = 1, \dots, 2g + m - 1$, do not appear in the equation (3.4) because each of the edges b_j belongs to two distinct triangles. Here, we have two issues :

- Case 1 : there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \notin 2\pi\mathbb{N}$. The equation (3.4) is then non-trivial, which means that the vector $(z(a_1), \dots, z(a_{n-m}), z(b_1), \dots, z(b_{2g+m-1}))$ belongs to a hyperplane of $\mathbb{C}^{2g+n-1}$. Therefore we have

$$\dim \mathrm{V}^*_{\mathcal{T}} \leqslant 2g + n - 2. \tag{3.5}$$

  From (3.2) and (3.5), we conclude that $\dim_{\mathbb{C}} \mathrm{V}^*_{\mathcal{T}} = 2g + n - 2$.

- Case 2 : $\alpha_i \in 2\pi\mathbb{N}$ for every i in $\{1, \dots, n\}$. In this case, the equation (3.4) is trivial. However, this also means that the sum of all equations in the system $\mathbf{S}^*_{\mathcal{T}}$, with appropriate choices of signs, is the trivial equation $0 = 0$. This implies $\mathrm{rank}(\mathbf{S}^*_{\mathcal{T}}) \leqslant N_2 + (n - m) - 1$. Hence

$$\dim \mathrm{V}^*_{\mathcal{T}} \geqslant N_1 - (N_2 + n - m - 1) = 2g + n - 1. \tag{3.6}$$

  From (3.3) and (3.6), we conclude that $\dim \mathrm{V}^*_{\mathcal{T}} = 2g + n - 1$.

The lemma is then proved. □

The proof of Proposition 3.4.1 is now complete. □

3.4.2 Proof of Proposition 3.4.2

Let $([(\Sigma,\phi)],\xi)$ be a point in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ such that $([(\Sigma^\natural,\phi^\natural)],\xi)$ be a point in $\mathcal{U}_{\mathcal{T}_1}\cap\mathcal{U}_{\mathcal{T}_2}$. Let $\mathrm{T}_1,\mathrm{T}_2$ be the admissible triangulations of $\Sigma^\natural$ corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. By Theorem 2.6.2, we know that one can transform T_1 into T_2 by a sequence of elementary moves.

Recall that, by definition, $\mathrm{V}_{\mathcal{T}_i}$ is the solution space of $\mathrm{S}_{\mathcal{T}_i}$, $i=1,2$, and $\mathrm{V}^*_{\mathcal{T}_i}$ is the solution space of $\mathrm{S}^*_{\mathcal{T}_i}$, $i=1,2$, where $\mathrm{S}^*_{\mathcal{T}_i}$ is obtained from $\mathrm{S}_{\mathcal{T}_i}$ by adding $(n-m)$ equations of type (3.1). Hence we can consider $\mathrm{V}^*_{\mathcal{T}_i}$ as the intersection of $\mathrm{V}_{\mathcal{T}_i}$ and the solution space V of those additional equations.

Now, the map $\Psi_{\mathcal{T}_2}\circ\Psi^{-1}_{\mathcal{T}_1}$ can be seen as a restriction of a linear isomorphism $\mathbf{L}$ of $\mathbb{C}^{N_1}$ onto $\mathrm{V}_{\mathcal{T}_1}$. Since elementary moves do not affect the edges on the boundary of $\Sigma^\natural$, the linear isomorphism $\mathbf{L}$ preserves the space V, and the proposition follows. □

3.5 Continuity of Ξ

Let $([(\Sigma,\phi)],\xi)$ be a point $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$, and assume that $([(\Sigma^\natural,\phi^\natural)],\xi)$ is contained in $\mathcal{U}_{\mathcal{T}}$, where $\mathcal{T}$ is a representative of an equivalence class in $\mathcal{TR}(S_g^\natural)$. Let $Z=(z_1,\dots,z_{N_1})\in\mathbb{C}^{N_1}$ be the image of $([\Sigma^\natural,\phi^\natural)],\xi)$ in $\mathbb{C}^{N_1}$ by $\Psi_{\mathcal{T}}$. We have proved that Z is contained in the subspace $\mathrm{V}^*_{\mathcal{T}}$ of $\mathbb{C}^{N_1}$. To show the continuity of Ξ, we prove the following proposition

Proposition 3.5.1 *There exists a neighborhood* U *of* Z *in* $\mathrm{V}^*_{\mathcal{T}}$ *such that* $\Xi^{-1}(\Psi^{-1}_{\mathcal{T}}(\mathrm{U}))$ *is a neighborhood of* $([(\Sigma,\phi)],\xi)$ *in* $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$.

3.5.1 Preliminaries

Let U be a neighborhood of Z in $\mathrm{V}^*_{\mathcal{T}}$ such that for any W in U, the construction given in the proof of Lemma 3.4.5 gives a point $([(\Sigma_W,\phi_W)],\xi_W)$ in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$.

Observe that there exists a Hermitian form $\mathbf{H}$ of $\mathbb{C}^{N_1}$, such that, for any W in U, the area of the surface Σ_W is given by $\overline{W}^t\mathbf{H}W$. We define

$$\mathrm{U}_1=\{W=(w_1,\dots,w_{N_1})\in\mathrm{U}:\overline{W}^t\mathbf{H}W=1,\ w_1\in\mathbb{R}\}.$$

We can assume that $\mathbf{Area}(\Sigma)=1$, and apply a rotation to the field ξ so that Z is a vector in U_1. We can also assume that U_1 is a ball.

Let $\Phi_{\mathcal{T}}$ be the map which associates to any vector W in U_1 the point $([(\Sigma_W, \phi_W)]$ in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ (we forget the field ξ_W). Observe that the image of U_1 by $\Phi_{\mathcal{T}}$ is contained in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$.

To prove Proposition 3.5.1, we will prove the following proposition

Proposition 3.5.2 *$\Phi_{\mathcal{T}}(\mathrm{U}_1)$ is a neighborhood of $[(\Sigma, \phi)]$ in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$.*

3.5.2 Proof of 3.5.2 in the case $\alpha_i \in 2\pi\mathbb{N}, \forall i = 1, \dots, n$

In this case, we have seen that $\dim_{\mathbb{C}} \mathrm{V}_{\mathcal{T}}^* = 2g + n - 1$, hence U_1 is a ball of real dimension $2(2g+n-2)$. We remark that, in this case, $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ is locally homeomorphic to the moduli space of closed translation surfaces having n singularities. It is well known that the later is of complex dimension $2g+n-1$, hence so is $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$. It follows that $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ is of real dimension $2(2g+n-2)$. Since $\dim_{\mathbb{R}} \mathrm{U}_1 = \dim_{\mathbb{R}} \mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, to prove that $\Phi_{\mathcal{T}}(\mathrm{U}_1)$ is a neighborhood of $[(\Sigma, \phi)]$ in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, we only need to verify that $\Phi_{\mathcal{T}}$ is continuous, and injective.

The injectivity of $\Phi_{\mathcal{T}}$ follows from the fact that, for if $[(\Sigma_W, \phi_W)] = \Phi_{\mathcal{T}}(W)$, then there exists a unique normalized parallel vector field ξ_W on Σ_W such that $\Psi_{\mathcal{T}}([(\Sigma_W, \phi_W)], \xi_W) = W$.

For the continuity of $\Phi_{\mathcal{T}}$, recall that we have an embedding from $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ into $\mathcal{T}(g, n)$, and the topology on $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ is induced from the topology of $\mathcal{T}(g, n)$ with Teichmüller metric by this embedding. Therefore, it is enough to show that $\Phi_{\mathcal{T}}$ is a continuous map from U_1 into $\mathcal{T}(g, n)$.

Let $\{W_k\}$ be a sequence of vectors converging to a vector W_∞ in U_1. Let $[(\Sigma_k, \phi_k)]$, $k = 1, 2, \dots$, denote the image of W_k, and $[(\Sigma_\infty, \phi_\infty)]$ be the image of W_∞ by $\Phi_{\mathcal{T}}$. By construction, we can assume that

$$\phi_k = f_k \circ \phi_\infty,$$

where f_k is a homeomorphism from Σ_∞ onto Σ_k, which maps the admissible triangulation $\mathrm{T}_\infty = \phi_\infty(\mathcal{T})$ of Σ_∞ onto an admissible triangulation of Σ_k.

Recall that the restriction of f_k into each triangle of T_∞ is a linear map of $\mathbb{R}^2$, therefore f_k is quasi-conformal. As k tends to ∞, the restriction of f_k on each triangle of T_∞ tends to identity, hence the dilatation $K(f_k)$ tends to 1, it implies immediately that the Teichmüller distance between $[(\Sigma_k, \phi_k)]$ and $[(\Sigma_\infty, \phi_\infty)]$ tends to zero. We deduce that $\Phi_{\mathcal{T}}$ is continuous, and the proposition follows. □

3.5.3 Proof of 3.5.2 in the case there exist i such that $\alpha_i \notin 2\pi\mathbb{N}$

In this case, by Proposition B.0.1, we know that there exist a subset $\widetilde{\mathrm{U}}_1$ of $\mathbb{C}^{N_1}$, and a continuous map $\widetilde{\Phi}_{\mathcal{T}}$ from $\widetilde{U}_1$ into $\mathcal{T}(g,n)$ verifying the following conditions :

- $\widetilde{\mathrm{U}}_1$ is homeomorphic to a ball of real dimension $(6g+2n-6)$.
- $\mathrm{U}_1 = \widetilde{\mathrm{U}}_1 \cap \mathrm{V}^*_{\mathcal{T}}$.
- $\Phi_{\mathcal{T}}$ is the restriction of $\widetilde{\Phi}_{\mathcal{T}}$ into U_1.
- $\widetilde{\Phi}_{\mathcal{T}}(\widetilde{\mathrm{U}}_1)$ is a neighborhood of $[(\Sigma,\phi)]$ in $\mathcal{T}(g,n)$.
- For every $W \in \widetilde{\mathrm{U}}_1$, $\widetilde{\Phi}_{\mathcal{T}}(W)$ is represented by a pair $(\Sigma_W, f_W \circ \phi)$, where Σ_W is a flat surface having n singularities with cone angles $\alpha_1,\dots,\alpha_n$, and f_W is a homeomorphism from Σ onto Σ_W mapping the triangulation T onto a triangulation by geodesic segments of Σ_W, whose vertex set is the set of singular points.

Note that the surface $\widetilde{\Phi}_{\mathcal{T}}(W)$ is defined by constructing triangles from the coordinates of W, and gluing them together using $\mathcal{T}$ as pattern.

It follows that, every point X in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})^*$ close enough to $[(\Sigma,\phi)]$ can be written as $\widetilde{\Phi}_{\mathcal{T}}(W)$, with $W \in \widetilde{\mathrm{U}}_1$. In particular, X can be represented as a pair $(\Sigma_W, f_W \circ \phi)$ with the properties described above. By definition, X is represented by a pair (Σ', ϕ'), where Σ' is also a flat surface having n singularities with cone angles $\alpha_1,\dots,\alpha_n$, and ϕ' is a homeomorphism mapping the erasing forest $\hat{\mathcal{A}}$ onto an erasing forest of Σ'.

We can then identity Σ' to Σ_W, and it follows that $f_W \circ \phi$ is isotopic to ϕ' relative to $\{p_1,\dots,p_n\}$. Since both $f_W \circ \phi$ and ϕ' map $\hat{\mathcal{A}}$ onto a geodesic forest, using Lemma 2.3.8, we conclude that $f_W \circ \phi(\hat{\mathcal{A}}) = \phi'(\hat{\mathcal{A}})$. Now, by the definition of $\widetilde{\Phi}_{\mathcal{T}}$, it implies that the vector W belongs to the space $\mathrm{V}^*_{\mathcal{T}}$. Therefore,

$$W \in \mathrm{V}^*_{\mathcal{T}} \cap \widetilde{\mathrm{U}}_1 = \mathrm{U}_1.$$

The proposition is then proved. □

3.5.4 Proof of Proposition 3.5.1

Proposition 3.5.1 is a direct consequence of Proposition 3.5.2. Set $\mathrm{U} = \mathrm{U}_1 \times \mathbb{C}^*$, with U_1 as in Proposition 3.5.2. The set U can be identified to an open subset of $\mathrm{V}^*_{\mathcal{T}}$.

For each $W \in \mathrm{U}_1$, let $[(\Sigma_W, \phi_W)] \in \mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$ be the image of W by $\Phi_{\mathcal{T}}$. There exists a unique normalized parallel vector field ξ_W on Σ_W such that $\Psi_{\mathcal{T}} \circ \Xi([(\Sigma_W, \phi_W)], \xi_W) = W$. We can then extend the map $\Phi_{\mathcal{T}}$ into a map $\hat{\Phi}_{\mathcal{T}}$ which is defined on U such that

$$\Psi_{\mathcal{T}} \circ \Xi \circ \hat{\Phi}_{\mathcal{T}}(W) = W, \; \forall W \in \mathrm{U}.$$

It follows that $\hat{\Phi}_{\mathcal{T}}(\mathrm{U})$ is contained in $\Xi^{-1}(\Psi_{\mathcal{T}}^{-1}(\mathrm{U}))$. From 3.5.2, we know that $\Phi_{\mathcal{T}}(\mathrm{U}_1)$ is a neighborhood of $[(\Sigma, \phi)]$ in $\mathcal{T}_1^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})^*$, therefore $\hat{\Phi}_{\mathcal{T}}(\mathrm{U})$ is a neighborhood of $([(\Sigma, \phi)], \xi)$ in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$. Proposition 3.5.1 is then proved. □

3.6 Volume form on $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$

In this section, we define a volume form on the sub-manifold $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$ of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$, and prove that the pull-back of this this volume form onto $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ is invariant by the action of the group $\Gamma(S_g, \hat{\mathcal{A}})$. The construction of this volume form is similar to the construction of the volume form μ_{Tr} of $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}'; \bar{\beta}')$.

3.6.1 Definitions

Let $\mathcal{T}$ be a triangulation of $S_g^\natural$, which represents an equivalence class in $\mathcal{TR}(S_g^\natural)$. As usual, let N_1, N_2 denote the number of edges, and the number of triangles in $\mathcal{T}$ respectively. Let $\Psi_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \longrightarrow \mathbb{C}^{N_1}$ be the local chart associated to $\mathcal{T}$. Recall that $\Psi_{\mathcal{T}}(\mathcal{U}_{\mathcal{T}})$ is an open subset of the solution space $\mathrm{V}_{\mathcal{T}}$ of a system $\mathbf{S}_{\mathcal{T}}$, which consists of N_2 equations of type (2.3). We have shown that $\Psi_{\mathcal{T}}(\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap \mathcal{U}_{\mathcal{T}})$ is an open subset of the solution space $\mathrm{V}_{\mathcal{T}}^*$ of a system $\mathbf{S}_{\mathcal{T}}^*$, which consists of $N_2 + (n-m)$ equations. The system $\mathbf{S}_{\mathcal{T}}^*$ is obtained from $\mathbf{S}_{\mathcal{T}}$ by adding $(n-m)$ equations of type (3.1).

Let $a_1, \dots, a_{N_2+(n-m)}$ denote the vectors of $(\mathbb{C}^{N_1})^*$ which correspond to the equations of the system $\mathbf{S}_{\mathcal{T}}^*$. A vector a_i is said to be *normalized* if each of its coordinates is either 0, or a complex number of module 1. We have two cases :

- Case 1 : there exist $i \in \{1, \dots, n\}$ such that $\alpha_i \notin 2\pi\mathbb{N}$. In this case, we have seen that $\dim \mathrm{V}_{\mathcal{T}}^* = 2g + n - 2$, hence $\mathrm{rank}(\mathbf{S}_{\mathcal{T}}^*) = N_2 + (n-m)$. Consider the complex linear map $\mathbf{A}_{\mathcal{T}}^* : \mathbb{C}^{N_1} \longrightarrow \mathbb{C}^{N_2+(n-m)}$, which is defined in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2+(n-m)}$ by the matrix

$$\mathbf{A}_{\mathcal{T}} = \begin{pmatrix} a_1 \\ \vdots \\ a_{N_2+(n-m)} \end{pmatrix}.$$

The map $\mathbf{A}_{\mathcal{T}}$ is then surjective, and $\mathrm{V}_{\mathcal{T}}^{*} = \ker \mathbf{A}_{\mathcal{T}}^{*}$. The map $\mathbf{A}_{\mathcal{T}}$ is said to be *normalized* if each row of its matrix in the canonical basis is normalized.

Let λ_{2N_1} et $\lambda_{2(N_2+(n-m))}$ denote the Lebesgue measures on $\mathbb{C}^{N_1} \simeq \mathbb{R}^{2N_1}$ and $\mathbb{C}^{N_2+(n-m)} \simeq \mathbb{R}^{2(N_2+(n-m))}$ respectively. Since $\mathbf{A}_{\mathcal{T}}$ is surjective, λ_{2N_1} and λ_{2N_2} induce a volume form $\nu_{\mathcal{T}}$ on $\mathrm{V}_{\mathcal{T}}$ via the following exact sequence :

$$0 \longrightarrow \mathrm{V}_{\mathcal{T}}^{*} \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathcal{T}}^{*}} \mathbb{C}^{N_2+(n-m)} \longrightarrow 0.$$

- Case 2 : for every $i \in \{1, \dots, n\}$, $\alpha_i \in 2\pi\mathbb{N}$. In this case, $\mathrm{rank}(\mathbf{S}_{\mathcal{T}}^{*}) = N_2 + (n-m) - 1$, hence $\mathrm{rank}(\mathbf{A}_{\mathcal{T}}^{*}) = N_2 - 1$.

If the vectors $a_1, \dots, a_{N_2+(n-m)}$ are normalized, and if their signs are chosen suitably, we have $a_1 + \dots + a_{N_2} = 0$. Thus, without loss of generality, we can assume that $\mathrm{Im}\mathbf{A}_{\mathcal{T}}^{*} = \mathbf{W}$, where $\mathbf{W}$ is the complex hyperplane of $\mathbb{C}^{N_2+(n-m)}$ defined by

$$\mathbf{W} = \{(z_1, \dots, z_{N_2+(n-m)}) \in \mathbb{C}^{N_2+(n-m)} : z_1 + \dots + z_{N_2+(n-m)} = 0\}.$$

Let $\lambda'_{2(N_2+(n-m)-1)}$ denote the volume form of $\mathbf{W}$ which is induced by the Lebesgue measure of $\mathbb{C}^{N_2+(n-m)}$. The volume forms λ_{2N_1} and $\lambda'_{2(N_2+(n-m)-1)}$ induce a volume form $\nu_{\mathcal{T}}$ on $\mathrm{V}_{\mathcal{T}}^{*}$ via the following exact sequence :

$$0 \longrightarrow \mathrm{V}_{\mathcal{T}}^{*} \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathcal{T}}^{*}} \mathbf{W} \longrightarrow 0.$$

In both cases, let $\mu_{\mathcal{T}}$ denote the volume form $\Psi_{\mathcal{T}}^{*}\nu_{\mathcal{T}}$ which is defined on $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap \mathcal{U}_{\mathcal{T}}$.

3.6.2 Invariance by coordinate changes

Let $\mathcal{T}_1$, and $\mathcal{T}_2$ be two triangulations of $S_g^{\natural}$ which represent two different equivalence classes in $\mathcal{TR}(S_g^{\natural})$. Assume that $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap (\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}) \neq \varnothing$. Then we have

Lemma 3.6.1 *$\mu_{\mathcal{T}_1} = \mu_{\mathcal{T}_2}$ on $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap (\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2})$.*

Proof: Let $([(\Sigma^{\natural}, \phi^{\natural})], \xi)$ be a point in $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})) \cap (\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2})$, and let T_1, T_2 be the admissible triangulations of $\Sigma^{\natural}$ corresponding to $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

By Theorem 2.6.2, we can assume that T_2 is obtained from T_1 by only one elementary move. Since an elementary move does not affect the edges of T_1 which are contained in the boundary of $\Sigma^{\natural}$, the equations of type (3.1) in $\mathbf{S}_{\mathcal{T}_1}$ and in $\mathbf{S}_{\mathcal{T}_2}$ are the same. Therefore, we can using the same arguments as in the proof of Proposition 2.7.1, to show that there exists an isomorphism of $\mathbf{F}$ of $\mathbb{C}^{N_1}$ such that $|\det \mathbf{F}| = 1$, and the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{V}^*_{\mathcal{T}_1} & \longrightarrow & \mathbb{C}^{N_1} & \stackrel{\mathbf{A}^*_{\mathcal{T}_1}}{\longrightarrow} & \mathbf{X} & \longrightarrow & 0 \\ & & \downarrow \mathbf{H} & & \downarrow \mathbf{F} & & \|\mathrm{Id} & & \\ 0 & \longrightarrow & \mathrm{V}^*_{\mathcal{T}_2} & \longrightarrow & \mathbb{C}^{N_1} & \stackrel{\mathbf{A}^*_{\mathcal{T}_2}}{\longrightarrow} & \mathbf{X} & \longrightarrow & 0 \end{array}$$

where $\mathbf{X}$ is either $\mathbb{C}^{N_2+(n-m)}$, or $\mathbf{W}$, and the isomorphism $\mathbf{H} : \mathrm{V}^*_{\mathcal{T}_1} \longrightarrow \mathrm{V}^*_{\mathcal{T}_2}$, which is induced by $\mathbf{F}$, is the coordinate change between $\Psi_{\mathcal{T}_2}$ and $\Psi_{\mathcal{T}_1}$. It follows immediately that

$$\nu_{\mathcal{T}_1} = \mathbf{H}^* \nu_{\mathcal{T}_2},$$

and the lemma follows. □

3.6.3 Invariance by action of $\Gamma(S_g, \hat{\mathcal{A}})$

Lemma 3.6.1 implies that the volume forms $\{\mu_{\mathcal{T}} : \mathcal{T} \in \mathcal{TR}(S_g^\natural)\}$ give a well defined volume form on $\Xi(\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}))$. Let μ_{Tr} denote the pull-back of this volume form onto $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$. To complete the proof of Theorem 3.1.10, we need to show

Lemma 3.6.2 *The volume form μ_{Tr} is in variant by the action of $\Gamma(S_g, \hat{\mathcal{A}})$.*

Proof: The fact that μ_{Tr} is invariant by the action of the group $\Gamma(S_g, \hat{\mathcal{A}})$ is quite clear from the definition of $\Gamma(S_g, \hat{\mathcal{A}})$. Let γ be an element of $\Gamma(S_g, \hat{\mathcal{A}})$, and suppose that $\gamma([(\Sigma_1, \phi_1)], \xi_1) = ([(\Sigma_2, \phi_2)], \xi_2)$. By definition there exist an isometry h from Σ_1 onto Σ_2. Note that, by definition, $\phi_2^{-1} \circ h \circ \phi_1$ preserves the forest $\hat{\mathcal{A}}$.

As usual, let $([(\Sigma_i^\natural, \phi_i^\natural)], \xi_i)$ be the image of $([(\Sigma_i, \phi_i)], \xi_i)$ by Ξ, $i = 1, 2$. The isometry h induces then an isometry from $([(\Sigma_1^\natural, \phi_1^\natural)], \xi_1)$ onto $([(\Sigma_2^\natural, \phi_2^\natural)], \xi_2)$. Consequently, an admissible triangulation of $\Sigma_1^\natural$ is mapped by h onto an admissible triangulation of $\Sigma_2^\natural$. Since any two admissible triangulations of $\Sigma^\natural$ are connected by elementary moves, Lemma 3.6.1 allows us to conclude. □

The proof of Theorem 3.1.10 is now complete. □

3.7 A necessary condition for a tree to be erasing

Assume that the forest $\hat{\mathcal{A}}$ contains only one non-trivial tree $\mathcal{A}$, *i.e.* all other trees in $\hat{\mathcal{A}}$ are points, then from the proof of 3.1.10, we get the following

Corollary 3.7.1 *If there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \notin 2\pi\mathbb{N}$, then the tree $\mathcal{A}$ contains at least three vertices.*

Proof: By assumption, $\mathcal{A}$ contains at least two vertices. Assume that $\mathcal{A}$ has exactly two vertices whose cone angles are α_1, α_2. By assumption, both angles α_1, α_2 do not belong to the set $2\pi\mathbb{N}$ since the cone angle at any isolate point in $\hat{\mathcal{A}}$ must be an integral multiple of 2π.

We know that the tree $\mathcal{A}$ has only one edge, this edge corresponds to a pair of geodesic segments $(a, \bar{a})$ on the boundary of $S_g^\natural$. Let ξ be a normalized parallel vector field on $S_g^\natural$, and $\mathcal{T}$ be an admissible triangulation of $S_g^\natural$. Let $\Psi_\mathcal{T}$ be the local chart of $\mathcal{T}_\mathrm{T}(\bar{\alpha}'; \bar{\beta}')$ associated to $\mathcal{T}$. Note that $\mathcal{U}_\mathcal{T}$ contains the point $([(S_g^\natural, \mathrm{Id})], \xi)$.

Let $z(a)$ and $z(\bar{a})$ be the complex numbers associated to a, and $\bar{a}$ respectively by $\Psi_\mathcal{T}$. From Lemma 3.4.4, and (3.4), we have

$$(1 - e^{\imath\theta})z(a) = 0.$$

where $\theta = \alpha_1 \mod 2\pi$. Since $\alpha_1 \notin 2\pi\mathbb{N}$, we have $e^{\imath\theta} \neq 1$. Hence the equation above implies that $z(a) = 0$, which means that the two vertices of $\mathcal{A}$ coincide, and we get a contradiction. $\square$

Chapitre 4

Spherical flat surface

4.1 Introduction

Spherical flat surfaces are flat surfaces which are homeomorphic to the sphere $\mathbb{S}^2$. By Proposition 3.2.3, we know that each homothety class of spherical flat surface with prescribed cone angles at the singularities corresponds to a unique conformal structure on the sphere $\mathbb{S}^2$ with marked points and vice versa.

Let $p_1, \dots, p_n$ be $n \geqslant 3$ points on the standard sphere $\mathbb{S}^2$. Fix a set of n positive real numbers $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ such that $\alpha_1 + \cdots + \alpha_n = 2\pi(n-2)$. The *Teichmüller space of spherical flat surfaces* having n singularities with cone angles $\alpha_1, \dots, \alpha_n$ is the set of equivalence classes of pairs (Σ, ϕ), where

. Σ is a spherical flat surface having n singularities with cone angles $\alpha_1, \dots, \alpha_n$.

. ϕ is a homeomorphism from $\mathbb{S}^2$ to Σ, which sends $\{p_1, \dots, p_n\}$ onto the set of singularities of Σ such that the cone angle at $\phi(p_i)$ is α_i.

. The equivalence class of (Σ, ϕ) is the set of all pairs (Σ, ϕ'), where ϕ' is a homeomorphism isotopic to ϕ by an isotopy which is constant on the set $\{p_1, \dots, p_n\}$.

We denote this Teichmüller space $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})^*$. The equivalence class of a pair (Σ, ϕ) in $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})^*$ will be denoted by $[(\Sigma, \phi)]$. Let $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ denote the product $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})^* \times \mathbb{S}^1$.

Let $\Gamma(0; n)$ denote the modular group of homeomorphisms of $\mathbb{S}^2$ which is identity on the set $\{p_1, \dots, p_n\}$. Clearly, $\Gamma(0; n)$ acts on $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})^*$, the quotient space $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$ is the *moduli space of spherical flat surfaces* having cone angles $\{\alpha_1, \dots, \alpha_n\}$. Note that in this definition, we do not allow exchanges of singularities having with the same cone angle. We denote $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ the subspace of $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$ consisting of all surface of area 1. By Proposition 3.2.3, the space $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ can be identified to the moduli space

$\mathcal{M}(0;n)$ of configurations of n points on the sphere $\mathbb{S}^2$ up to Möbius transformations.

Extend the action of $\Gamma(0;n)$ onto $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ such that $\Gamma(0;n)$ acts trivially on the $\mathbb{S}^1$ part and let $\mathcal{M}(\mathbb{S}^2,\bar{\alpha})$ denote the quotient $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})/\Gamma(0;n)$. The main result of this chapter is the following

Theorem 4.1.1 *a) $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ is a flat complex affine manifold of dimension $n-2$, on which $\Gamma(0;n)$ acts properly discontinuously.*

b) There exists a volume form on $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ which is invariant by the action of the group $\Gamma(0;n)$.

The volume forms mentioned in Theorem 4.1.1, and Theorem 2.2.9 are defined by the same method.

4.2 Flat complex affine structure on $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$

As a direct consequence of Proposition 3.2.3, we can identify $\mathcal{T}_1(\mathbb{S}^2,\bar{\alpha})^*$ to $\mathcal{T}(0;n)$, and hence, $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ to $\mathcal{T}(0;n)\times\mathbb{C}^*$, we endow $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ with the topology induced by this identification. It is well known that $\dim_{\mathbb{C}}\mathcal{T}(0;n)=n-3$, it follows that $\dim_{\mathbb{C}}\mathcal{T}(\mathbb{S}^2,\bar{\alpha})=n-2$.

4.2.1 Definition of local charts

Let $\mathcal{TR}(\mathbb{S}^2,\{p_1,\dots,p_n\})$ denote the set of triangulations of $\mathbb{S}^2$ whose vertex set is $\{p_1,\dots,p_n\}$ modulo isotopy relative to $\{p_1,\dots,p_n\}$. Given a triangulation $\mathcal{T}$ of $\mathbb{S}^2$ which represents an equivalence class in $\mathcal{TR}(\mathbb{S}^2,\{p_1,\dots,p_n\})$, let $\mathcal{U}_{\mathcal{T}}$ denote the subset of $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ consisting of pairs $([(\Sigma,\phi)],e^{\imath\theta})$, such that $\phi(\mathcal{T})$ is a geodesic triangulation of Σ. By Proposition B.0.1, we know that $\mathcal{U}_{\mathcal{T}}$ is an open set in $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$.

Choose a tree $\mathcal{A}$ in $\mathcal{T}$ whose vertex set is $\{p_1,\dots,p_n\}$, for any $([(\Sigma,\phi)],e^{\imath\theta})$ in $\mathcal{U}_{\mathcal{T}}$, $\phi(\mathcal{A})$ is a geodesic erasing tree of Σ. Therefore, we can identify $\mathcal{U}_{\mathcal{T}}$ to an open subset in $\mathcal{T}^{et}(\mathbb{S}^2,\mathcal{A})$. From Theorem 3.1.10, we get a map

$$\Psi_{\mathcal{T},\mathcal{A}}:\mathcal{U}_{\mathcal{T}}\longrightarrow\mathbb{C}^{4n-7},$$

which is injective, and continuous, such that $\Psi_{\mathcal{T},\mathcal{A}}(\mathcal{U}_{\mathcal{T}})$ is an open subset of the solution space $\mathrm{V}^*_{\mathcal{T},\mathcal{A}}$ of a system of linear equations $\mathrm{S}^*_{\mathcal{T},\mathcal{A}}$. Note that, in this case, the system $\mathrm{S}^*_{\mathcal{T},\mathcal{A}}$ has $3n-5$ equations, and $\mathrm{rank}\mathrm{S}^*_{\mathcal{T},\mathcal{A}}=3n-5$, hence $\dim_{\mathbb{C}}\mathrm{V}^*_{\mathcal{T},\mathcal{A}}=(4n-7)-(3n-5)=n-2$. It follows that $\Psi_{\mathcal{T},\mathcal{A}}$ can be considered as a local chart of $\mathcal{T}(\mathbb{S}^2,\bar{\alpha})$ on $\mathcal{U}_{\mathcal{T}}$. It is worth noticing that $\Psi_{\mathcal{T},\mathcal{A}}$ is only defined up to a rotation.

4.2.2 Coordinate changes

Let $\mathcal{T}_1, \mathcal{T}_2$ be two triangulations of $\mathbb{S}^2$ which represent two different equivalence classes in $\mathcal{TR}(\mathbb{S}^2, \{p_1, \dots, p_n\})$. Let $([(\Sigma, \phi)], \xi)$ be a point in $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$, and let T_1, T_2 be the geodesic triangulations of Σ corresponding to $\mathcal{T}_1$, and $\mathcal{T}_2$ respectively. Choose a tree $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) in $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) which connects all the points in $\{p_1, \dots, p_n\}$, and let $\Psi_{\mathcal{T}_1,\mathcal{A}_1}$ and $\Psi_{\mathcal{T}_2,\mathcal{A}_2}$ be the two local charts of $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ corresponding.

Given an edge e of T_2 which is not contained in T_1, let P_e be the developing polygon of e with respect to T_1 (see 2.6.1). By construction, there exists a map φ_e from P_e into Σ which is locally isometric mapping a diagonal of P_e onto e.

The map φ_e sends geodesic segments in the boundary of P_e onto edges of T_1. It follows that the complex number associated to the edge e by the local chart $\Psi_{\mathcal{T}_2,\mathcal{A}_2}$ can be written as a linear function of complex numbers associated to edges of T_1, which correspond the segments in the boundary of P_e, by the local chart $\Psi_{\mathcal{T}_1,\mathcal{A}_1}$. Since the roles of T_1 and T_2 in this reasoning can be interchanged, we deduce that the coordinate change between $\Psi_{\mathcal{T}_1,\mathcal{A}_1}$ and $\Psi_{\mathcal{T}_2,\mathcal{A}_2}$ can be written as a linear isomorphism of $\mathbb{C}^{4n-7}$ which sends $\mathrm{V}^*_{\mathcal{T}_1,\mathcal{A}_1}$ onto $\mathrm{V}^*_{\mathcal{T}_2,\mathcal{A}_2}$. Therefore we can conclude that $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ is a flat complex affine manifold of dimension $n-2$.

4.2.3 Action of $\Gamma(0; n)$

We know that $\Gamma(0;n)$ acts properly discontinuously on $\mathcal{T}(0;n)$. We have seen that $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ can be identified to $\mathcal{T}(0;n) \times \mathbb{C}^*$. Clearly, the action of $\Gamma(0;n)$ on the $\mathbb{C}^*$ factor of the product $\mathcal{T}(0;n) \times \mathbb{C}^*$ is trivial, therefore the action of $\Gamma(0;n)$ on $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ is properly discontinuous. Part $a)$ of Theorem 4.1.1 is now proved.

4.3 Volume form on $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$

4.3.1 Definition

Set $N_1 = 4n-7$, $N_2 = 3n-5$. Let $\mathcal{T}$ be a triangulation of $\mathbb{S}^2$ which represents an equivalence class in $\mathcal{TR}(\mathbb{S}^2, \{p_1, \dots, p_n\})$. Let $\mathcal{A}$ be a tree contained in $\mathcal{T}$, which connects all the points in $\{p_1, \dots, p_n\}$. Let $\Psi_{\mathcal{T},\mathcal{A}}$ be the local chart associated to $(\mathcal{T}, \mathcal{A})$, which is defined on the set $\mathcal{U}_{\mathcal{T}}$.

Let $\mathbf{S}^*_{\mathcal{T},\mathcal{A}}$ be the system of linear equations associated to $\Psi_{\mathcal{T},\mathcal{A}}$, and let $\mathbf{A}^*_{\mathcal{T},\mathcal{A}}$ be the normalized linear map associated to $\mathbf{S}^*_{\mathcal{T},\mathcal{A}}$. In this case, $\mathbf{A}^*_{\mathcal{T},\mathcal{A}}$ is a linear map from $\mathbb{C}^{N_1}$ onto $\mathbb{C}^{N_2}$, which is given, in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, by a matrix whose rows correspond to the equations in $\mathbf{S}^*_{\mathcal{T},\mathcal{A}}$. Recall that every entry of the matrix of $\mathbf{A}^*_{\mathcal{T},\mathcal{A}}$ (in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$) is either zero, or a complex number of module one.

We define $\nu_{\mathcal{T},\mathcal{A}}$ to be the volume form on $\mathrm{V}^*_{\mathcal{T},\mathcal{A}}$ which is induced by the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ via the following exact sequence

$$0 \longrightarrow \mathrm{V}^*_{\mathcal{T},\mathcal{A}} \longrightarrow \mathbb{C}^{4n-7} \xrightarrow{\mathbf{A}^*_{\mathcal{T},\mathcal{A}}} \mathbb{C}^{3n-5} \longrightarrow 0.$$

Let $\mu_{\mathcal{T},\mathcal{A}}$ denote the pull-back of $\nu_{\mathcal{T},\mathcal{A}}$ on $\mathcal{U}_{\mathcal{T}}$. The following proposition shows that the volume form $\mu_{\mathcal{T},\mathcal{A}}$ does not depend on the choice of $\mathcal{A}$

Proposition 4.3.1 *Let $\mathcal{T}$ be a triangulation representing an equivalence class in $\mathcal{TR}(\mathbb{S}^2, \{p_1, \dots, p_n\})$. Let $\mathcal{A}_1, \mathcal{A}_2$ be two trees contained in $\mathcal{T}$, each of which connects all the points in $\{p_1, \dots, p_n\}$.*

Let $\mathbf{A}^*_{\mathcal{T},\mathcal{A}_1}$ and $\mathbf{A}^*_{\mathcal{T},\mathcal{A}_2}$ denote the linear maps from $\mathbb{C}^{N_1}$ onto $\mathbb{C}^{N_2}$ corresponding to $\mathcal{A}_1$, and $\mathcal{A}_2$ respectively. Let $\nu_{\mathcal{T},\mathcal{A}_i}$, $i = 1,2$ denote the volume form on $\mathrm{V}^*_{\mathcal{T},\mathcal{A}_i}$ which is induced from the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$. Let $\mathbf{H} = \Psi_{\mathcal{T},\mathcal{A}_2} \circ \Psi^{-1}_{\mathcal{T},\mathcal{A}_1}$ be the coordinate change between $\Psi_{\mathcal{T},\mathcal{A}_1}$, and $\Psi_{\mathcal{T},\mathcal{A}_2}$, then we have

$$\mathbf{H}^*\nu_{\mathcal{T},\mathcal{A}_2} = \nu_{\mathcal{T},\mathcal{A}_1}.$$

To show that the volume form $\mu_{\mathcal{T},\mathcal{A}}$ actually does not depend on the choice of $\mathcal{T}$, we prove the following theorem

Theorem 4.3.2 *Let Σ be a spherical flat surface. If T_1 and T_2 are two geodesic triangulations of Σ whose sets of vertices coincide, and contain the set of singularities of Σ, then T_1 and T_2 are connected (i.e. one can be transformed into the other by elementary moves).*

Corollary 4.3.3 *The volume forms $\mu_{\mathcal{T},\mathcal{A}}$ agree on overlap domains of local charts, and give a well defined volume form μ_{Tr} on $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ which is invariant by $\Gamma(0;n)$.*

Proof: From Proposition 4.3.1, we know that the volume form $\mu_{\mathcal{T},\mathcal{A}}$ does not depend on the choice of the tree $\mathcal{A}$, therefore, we can write $\mu_{\mathcal{T}}$ instead of $\mu_{\mathcal{T},\mathcal{A}}$.

Let $\mathcal{T}_1, \mathcal{T}_2$ be two triangulations of $\mathbb{S}^2$ which represent two different equivalence classes in $\mathcal{TR}(\mathbb{S}^2, \{p_1, \dots, p_n\})$ such that $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2} \neq \varnothing$. Let $([(\Sigma, \phi)], e^{\imath\theta})$ be a point in $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$, and let T_1, T_2 be two geodesic triangulations of Σ corresponding to $\mathcal{T}_1, \mathcal{T}_2$ respectively. We have to show that $\mu_{\mathcal{T}_1} = \mu_{\mathcal{T}_2}$ on $\mathcal{U}_{\mathcal{T}_1} \cap \mathcal{U}_{\mathcal{T}_2}$.

By Theorem 4.3.2, we only have to consider the case where T_2 is obtained from T_1 by an elementary move. Remark that, in this case, there exists a tree A connecting all the singular points of Σ which is contained in both T_1 and T_2. Therefore, we can consider a neighborhood of $([(\Sigma,\phi)], e^{\imath\theta})$ as an open subset in $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$, where $\hat{\mathcal{A}} = A$ and $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$. It has been shown in Lemma 3.6.1, that in this situation, we have $\mu_{\mathcal{T}_1} = \mu_{\mathcal{T}_2}$. It follows that the volume forms $\{\mu_{\mathcal{T}} : \mathcal{T} \in \mathcal{TR}(\mathbb{S}^2, \{p_1, \dots, p_n\})\}$ give a well defined volume form on $\mathcal{T}(\mathbb{S}^2, \bar{\alpha})$ which will be denoted by μ_{Tr}.

Let γ be an element of $\Gamma(0;n)$, and suppose that $\gamma([(\Sigma_1,\phi_1)], e^{\imath\theta_1}) = ([(\Sigma_2,\gamma_2)], e^{\imath\theta_2})$. We can write $([(\Sigma_i,\phi_i)], e^{\imath\theta_i}) = ([(\overline{\Sigma}_i, \overline{\phi}_i)], z_i)$, $i = 1,2$, with $\mathbf{Area}(\overline{\Sigma}_i) = 1$, and $z_i \in \mathbb{C}^*$.

By definition, we have $z_1 = z_2$, and there exists a conformal homeomorphism h from $\overline{\Sigma}_1$ onto $\overline{\Sigma}_2$ which sends the of singular points of $\overline{\Sigma}_1$ onto the set singular points of $\overline{\Sigma}_2$ respecting the cone angles. From Proposition 3.2.3, we deduce that h is an isometry between $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$.

Since an isometry between two spherical flat surfaces sends geodesic triangulations onto triangulations, the same argument as above shows that μ_{Tr} is invariant by the action of $\Gamma(0;n)$. □

The remainder of this section is devoted to the proofs of Proposition 4.3.1, and Theorem 4.3.2.

4.3.2 Cutting and gluing

Let $\mathcal{T}, \mathcal{A}_1, \mathcal{A}_2$ be as in Proposition 4.3.1. Let $([(\Sigma,\phi)], e^{\imath\theta})$ be a point $\mathcal{U}_{\mathcal{T}}$. Let T denote the geodesic triangulation of Σ corresponding to $\mathcal{T}$, and let A_1, A_2 be the geodesic trees corresponding to $\mathcal{A}_1, \mathcal{A}_2$ respectively.

Let Σ_0^1 and Σ_0^2 denote the flat surface with geodesic boundary obtained by slitting open the surface Σ along the trees A_1 and A_2 respectively. Observe that Σ_i^0, $i = 1,2$, is homeomorphic to a closed disk. Let T_0^1 (resp. T_0^2) denote the geodesic triangulation of Σ_0^1 (resp. Σ_0^2) which is induced by T.

Consider a pair (Σ_0, T_0) where

- Σ_0 is a flat surface homeomorphic to a closed disk, with geodesic boundary, and having no singularities in the interior.

- T_0 is a triangulation of Σ_0 by geodesic segments whose vertex set is contained in the boundary of Σ^0.

- The edges of T_0 on the boundary of Σ_0 are paired up. Two edges in a pair have the same length.

We will call such a pair a *well triangulated flat disk.* Consider the following the following operation :

- Choose a pair of edges $(a, \bar{a})$ of T_0 in the boundary of Σ_0, and an edge b in the interior of Σ_0 so that a and $\bar{a}$ do not belong to the same connected component of $\Sigma_0 \setminus b$.

- Cut Σ_0 along b, then glue two the sub-disks by identifying a to $\bar{a}$.

Clearly, by this operation, we get another pair $(\Sigma_0', \mathrm{T}_0')$ with is also a well triangulated flat disk. We will call this operation the *cutting-gluing operation.*

Observe that, by construction, the pairs $(\Sigma_0^1, \mathrm{T}_0^1)$, and $(\Sigma_0^2, \mathrm{T}_0^2)$ verify the conditions above. We have

Lemma 4.3.4 *The pair $(\Sigma_0^2, \mathrm{T}_0^2)$ can be obtained from $(\Sigma_0^1, \mathrm{T}_0^1)$ by a sequence of cutting-gluing operations.*

Proof: We remark that the trees A_1 and A_2 correspond respectively to two maximal trees A_1^*, A_2^* in the dual graph T^* of the triangulation T. By *maximal tree* we mean a tree whose vertex set contains all the vertices of the dual graph. Any edge of T^* which is not contained in A_i^* is dual to an edge of $A_i,\ i = 1, 2$.

Let e^* be an edge of T^* which is contained in A_2^*, but not in A_1^*. Let v_1^* and v_2^* denote the endpoints of the edge e^*. Since A_1^* is a maximal tree, there exists a path c^* in A_1^* which joins v_1^* to v_2^*. The union of c^* and e^* is then a cycle in the dual graph T^*, it follows that there exists an edge e_1^* in c^*, different from e^*, which is not contained in A_2^*. Replacing e_1^* by e^*, we get a new maximal tree which contains one more common edge with A_2^* than A_1^*.

Thus we can transform A_1^* into A_2^* by a finite sequence of such replacements. Now, we just need to observe that the operation of replacing e_1^* by e^* corresponds to a cutting-gluing operation described above, and the lemma follows. □

4.3.3 Increased exact sequence

Given a well triangulated flat disk (Σ_0, T_0), using a developing map, we can associate to each edge e of T_0 a complex number $z(e)$. The complex numbers associated to the edges of T_0 verify two types of equation

- If e_i, e_j, e_k bound a triangle of T_0, then $\pm z(e_i) \pm z(e_j) \pm z(e_k) = 0$,

- If $(e, \bar{e})$ is a pair of boundary edges of T_0 of the same length, then $z(\bar{e}) = e^{\imath\theta} z(e)$.

Assume that T_0 contains N_1 edges, and choose a numbering of the edges of T_0, we get a linear system $\mathbf{S}_0$ of N_1 variables. Let N_2 be the number of equations of $\mathbf{S}_0$, let $\mathbf{A}_0$ be the matrix associated to $\mathbf{S}_0$, we say that $\mathbf{A}_0$ is *normalized* if every entry of $\mathbf{A}_0$ is zero, or a complex number of module one. Let $a_1, \dots, a_{N_2}$ denote the row vectors of $\mathbf{A}_0$. We also assume that $\mathrm{rank}\mathbf{A}_0 = N_2$.

By definition, $\mathbf{A}_0$ is an element of $\mathbf{M}_{\mathbb{C}}(N_2, N_1)$. Let $Z = (z_1, \dots, z_{N_1})$ be the vector of $\mathbb{C}^{N_1}$ whose coordinates are complex numbers associated to the edges of T_0. Choose an edge e_0 of T_0 which is contained inside Σ_0, and assume that the complex number associated to this edge is z_1. Without loss of generality, we can assume that the first two arrows a_1, a_2 of $\mathbf{A}_0$ verifies

$$a_1 \cdot Z^t = z_1 + z_{i_1} + z_{j_1} \tag{4.1}$$

and

$$a_2 \cdot Z^t = -z_1 + z_{i_2} + z_{j_2} \tag{4.2}$$

We construct a matrix $\hat{\mathbf{A}}_0$ in $\mathbf{M}_{\mathbb{C}}(N_2+1, N_1+1)$ from $\mathbf{A}_0$ and e_0 as follows : let $\hat{a}_1, \dots, \hat{a}_{N_2+1}$ denote the row vectors of $\hat{\mathbf{A}}_0$, then we have

. $\hat{a}_1$ is obtained by from a_1 by adding a zero into the last column.

. $\hat{a}_2$ is obtained from a_2 by replacing -1 by 0 in the first column, and adding a zero into the last column.

. For $j = 3, \dots, N_2$, $\hat{a}_j$ is obtained from a_j by adding a zero into the last column.

. The last row $\hat{a}_{N_2+1}$ is the row vector whose entries in the first, and the last columns are 1, and all other entries are 0.

We will call $\hat{\mathbf{A}}_0$ the *increased normalized matrix* of $\mathbf{A}_0$ associated to the splitting along e_0.

Consider the map

$$\begin{array}{cccc} \mathbf{I}: & \mathbb{C}^{N_1} & \longrightarrow & \mathbb{C}^{N_1+1} \\ & (z_1, \dots, z_{N_1}) & \longmapsto & (z_1, \dots, z_{N_1}, -z_1) \end{array}$$

Observe that, we have

$$\hat{\mathbf{A}}_0 \cdot \mathbf{I} = \begin{pmatrix} \mathbf{A}_0 \\ 0 \end{pmatrix}$$

It follows that $\mathbf{I}$ is a bijection from $\ker \mathbf{A}_0$ onto $\ker \hat{\mathbf{A}}_0$. We will call $\mathbf{I}$ the *embedding associated to* $\hat{\mathbf{A}}_0$.

Let $\hat{\nu}_{\mathrm{T}_0}$ be the volume form on $\ker \mathbf{A}_0$ which is induced from the Lebesgue measures of $\mathbb{C}^{N_1+1}$ and $\mathbb{C}^{N_2+1}$ by the exact sequence

$$0 \longrightarrow \ker \mathbf{A}_0 \xrightarrow{\mathbf{I}} \mathbb{C}^{N_1+1} \xrightarrow{\hat{\mathbf{A}}_0} \mathbb{C}^{N_2+1} \longrightarrow 0.$$

Let ν_{T_0} be the volume form on $\ker \mathbf{A}_0$ which is induced from the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ by the exact sequence

$$0 \longrightarrow \ker \mathbf{A}_0 \hookrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_0} \mathbb{C}^{N_2} \longrightarrow 0.$$

We have the following lemma :

Lemma 4.3.5 $\nu_{\mathrm{T}_0} = c_0 \hat{\nu}_{\mathrm{T}_0}$, *where c_0 is a constant which does not depend on the choice of the edge e_0.*

Proof: Let λ_{2N_1} be the Lebesgue measure of $\mathbb{C}^{N_1}$, and $\hat{\lambda}_{2N_1}$ be the volume form on $\mathbb{C}^{N_1}$ which is induced from the Lebesgue measures of $\mathbb{C}^{N_1+1}$ and $\mathbb{C}$ by the exact sequence

$$0 \longrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{I}} \mathbb{C}^{N_1+1} \xrightarrow{\mathbf{h}} \mathbb{C} \longrightarrow 0,$$

where $\mathbf{h} : (z_1, \dots, z_{N_1+1}) \longmapsto z_1 + z_{N_1+1}$. Set

$$c_0 = \frac{\hat{\lambda}_{2N_1}}{\lambda_{2N_1}}.$$

By definition, the volume form ν_{T_0} is induced from λ_{2N_1} and the Lebesgue measure of $\mathbb{C}^{N_2}$ by the following exact sequence

$$0 \longrightarrow \ker \mathbf{A}_0 \longrightarrow \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_0} \mathbb{C}^{N_2} \longrightarrow 0,$$

Observe that the volume form $\hat{\nu}_{\mathrm{T}_0}$ is defined in the same way with λ_{2N_1} replaced by $\hat{\lambda}_{2N_1}$. Hence the lemma follows. □

4.3.4 Proof of Proposition 4.3.1

By Lemma 4.3.4, it suffices to consider the case where $(\Sigma_0^2, \mathrm{T}_0^2)$ is obtained from $(\Sigma_0^1, \mathrm{T}_0^1)$ by only one cutting-gluing operation. Let e_0 denote the edge along which we cut Σ_0^1, and let (e_1, e_2) denote the pair of edges in the boundary of Σ_0^1 which are identified in this operation. Note that e_0 divides Σ_0^1 into two sub-disks $\mathbf{D}_1$ and $\mathbf{D}_2$, such that e_i is contained in the boundary of $\mathbf{D}_i$, for $i = 1, 2$.

To simplify notations, we identify an oriented edge of T_0 to the complex number which is associated to it. Assume that the edges on the boundary of Σ_0^1 are oriented coherently with the orientation of Σ_0^1.

Let $Z = (z_1, \ldots, z_{N_1})$ be the vector in $\mathbb{C}^{N_1}$ whose coordinates are the complex numbers associated to the edges of T_0^1. Let k the be number of edges of T_0^1 which are contained in the closure of $\mathbf{D}_1$. Without loss of generality, we can assume that $z_1, \ldots, z_k$ are the complex numbers associated to these k edges, with z_1 associated to e_0, and z_k associated to e_1. We also assume that z_{k+1} is the complex number associated to e_2. Since e_1 is identified to e_2, the complex numbers z_k and z_{k+1} must verify the following equation

$$e^{\imath\theta} z_k + z_{k+1} = 0$$

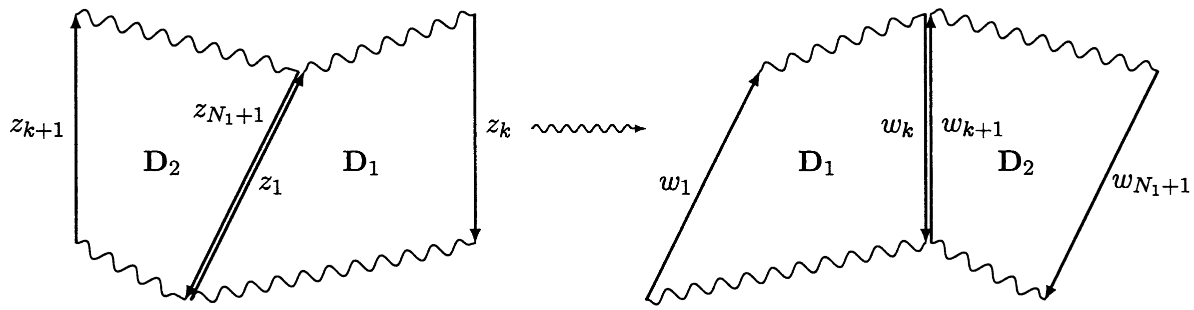
Let $\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1}$, be the increased normalized matrix of $\mathbf{A}^*_{\mathcal{T},\mathcal{A}_1}$ associated to the splitting along the edge e_0. By definition, we can write

$$\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1} = \begin{pmatrix} 1 & * & \ldots & * & 0 \\ 0 & * & \ldots & * & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & * & \ldots & * & 0 \\ 1 & * & \ldots & * & 1 \end{pmatrix}$$

Let $\hat{a}_1, \ldots, \hat{a}_{N_2+1}$ denote the row vectors of the matrix $\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1}$. Note that the vector $\hat{Z} = (z_1, \ldots, z_{N_1}, -z_1)$ belongs to the space $\ker \hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1}$.

Let T_1^1 and T_2^1 denote respectively the triangulations of $\mathbf{D}_1$ and $\mathbf{D}_2$ which are induced by T_0^1. We consider, by convention, that the edge e_0 is split into two edges : e_0^1, which belongs to T_1^1, is oriented in the same orientation as e_0, and e_0^2, which belongs to T_2^1, is oriented in the inverse orientation. By this convention, we can consider the coordinates of $\hat{Z}$ as the complex numbers associated to the edges of T_1^1 and T_2^1, where z_{N_1+1} is associated to e_0^2.

Remark that the cutting-gluing operation consists of rotating the disk $\mathbf{D}_1$ by an angle θ, and gluing $R_\theta(\mathbf{D}_1)$ to $\mathbf{D}_2$ by identifying $R_\theta(e_1)$ to e_2, where R_θ is the rotation of angle θ in $\mathbb{R}^2$.



Let $(w_1, \ldots, w_{N_1}, w_{N_1+1})$ be the complex numbers associated to the edges of $R_\theta(\mathrm{T}_1^1)$ and T_2^1 as follows

. For $i = 1, \ldots, k,$ w_i is associated to $R_\theta(z_i)$.

. For $i = k+1, \ldots, N_1 + 1,$ w_k is associated to z_i.

In other words

. $w_i = e^{\imath\theta} z_i$, for $i = 1, \ldots, k$.

. $w_i = z_i$, for $i = k+1, \ldots, N_1 + 1$.

Let $\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_2}$ be the increased normalized matrix of $\mathbf{A}^*_{\mathcal{T},\mathcal{A}_2}$ associated to the splitting along e'_0, where e'_0 is the edge corresponding to the pair (e_1, e_2). Observe that the vector $\hat{W} = (w_1, \ldots, w_{N_1+1})$ belongs to $\ker \hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_2}$. Let $\hat{b}_1, \ldots, \hat{b}_{N_2}, \hat{b}_{N_2+1}$ denote the row vectors of the matrix of $\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_2}$. We have

- If $\hat{b}_i$ correspond to a triangle, then $\hat{b}_i = \hat{a}_i$.
- If $\hat{b}_i$ correspond to a pair of of boundary edges (e, e'), we have two cases :
  - If e and e' are both contained in the boundary of $R_\theta(\mathbf{D}_1)$, or $\mathbf{D}_2$, then $\hat{b}_i = \hat{a}_i$.
  - If e is contained in $\partial R_\theta(\mathbf{D}_1)$, and e' is contained $\partial \mathbf{D}_2$, suppose that

$$\hat{a}_i \cdot \hat{Z}^t = e^{\imath\theta'} z_i + z_j, \text{ with } i \leqslant k < j$$

then

$$\hat{b}_i \cdot \hat{W}^t = e^{\imath(\theta' - \theta)} w_i + w_j.$$

Now, let $\hat{\mathbf{F}} \in \mathbf{M}_{N_1+1}(\mathbb{C})$ be the following matrix

$$\hat{\mathbf{F}} = \begin{array}{c} \\ \\ \end{array}\begin{pmatrix} e^{\imath\theta} & \dots & \overset{(k)}{0} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & e^{\imath\theta} & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 \end{pmatrix}.$$

We see that $\hat{W}^t = \hat{\mathbf{F}} \cdot \hat{Z}^t$, and clearly, $|\det \hat{\mathbf{F}}| = 1$. From the relations between $\hat{b}_i$ and $\hat{a}_i$, it follows that

$$\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_2} \cdot \hat{\mathbf{F}} = \hat{\mathbf{G}} \cdot \hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1},$$

where $\hat{\mathbf{G}} \in \mathbf{M}_{N_2+1}(\mathbb{C})$ is a diagonal matrix whose diagonal entries are either 1, or $e^{\imath\theta}$. Clearly, we have $|\det \hat{\mathbf{G}}| = 1$.

Let $\mathbf{I}_1, \mathbf{I}_2$ be the linear embeddings of $\mathbb{C}^{N_1}$ into $\mathbb{C}^{\mathrm{N}_1+1}$ associated to $\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1}$, and $\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_2}$ respectively. Note, that in this case, we have

$$\mathbf{I}_1(z_1, \dots, z_{N_1}) = (z_1, \dots, z_{N_1}, -z_1),$$

and

$$\mathbf{I}_2(w_1, \dots, w_{N_1}) = (w_1, \dots, w_{k-1}, w_k, -w_k, w_{k+1}, \dots, w_{N_1}).$$

Now, from the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \mathbf{A}^*_{\mathcal{T},\mathcal{A}_1} & \xrightarrow{\mathbf{I}_1} & \mathbb{C}^{N_1+1} & \xrightarrow{\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_1}} & \mathbb{C}^{N_2+1} & \longrightarrow & 0 \\ & & \downarrow \mathbf{H} & & \downarrow \hat{\mathbf{F}} & & \downarrow \hat{\mathbf{G}} & & \\ 0 & \longrightarrow & \ker \mathbf{A}^*_{\mathcal{T},\mathcal{A}_2} & \xrightarrow{\mathbf{I}_2} & \mathbb{C}^{N_1+1} & \xrightarrow{\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_2}} & \mathbb{C}^{N_2+1} & \longrightarrow & 0 \end{array}$$

where $\mathbf{H}$ is the isomorphism which is induced from $\hat{\mathbf{F}}$ and $\hat{\mathbf{G}}$, we deduce that

$$\mathbf{H}^* \hat{\nu}_{\mathcal{T},\mathcal{A}_2} = \hat{\nu}_{\mathcal{T},\mathcal{A}_1} \tag{4.3}$$

where $\hat{\nu}_{\mathcal{T},\mathcal{A}_i}$, $i = 1, 2$, is the volume form on $\ker \mathbf{A}^*_{\mathcal{T},\mathcal{A}_i}$ which is induced from the Lebesgue measures of $\mathbb{C}^{N_1+1}$ and $\mathbb{C}^{N_2+1}$ via the exact sequence

$$0 \longrightarrow \ker \mathbf{A}^*_{\mathcal{T},\mathcal{A}_i} \xrightarrow{\mathbf{I}_i} \mathbb{C}^{N_1+1} \xrightarrow{\hat{\mathbf{A}}^*_{\mathcal{T},\mathcal{A}_i}} \mathbb{C}^{N_2+1} \longrightarrow 0.$$

Remark that the map $\mathbf{H}$ is the coordinate changes between $\Psi_{\mathcal{T},\mathcal{A}_1}$ and $\Psi_{\mathcal{T},\mathcal{A}_2}$. From Lemma 4.3.5 we know that

$$\frac{\hat{\nu}_{T,\mathcal{A}_1}}{\nu_{T,\mathcal{A}_1}} = \frac{\hat{\nu}_{T,\mathcal{A}_2}}{\nu_{T,\mathcal{A}_2}}.$$

Hence the proposition follows from (4.3). □

4.3.5 Proof of Theorem 4.3.2

Theorem 4.3.2 is of course a consequence of the fact that any geodesic triangulation whose vertex set is the set of singularities can be transformed into a Delaunay triangulation. Here, we give another proof of this fact by using similar ideas to the proof of Theorem 2.6.2.

Let $x_1, \dots, x_n$ denote the vertices of T_1 and T_2. By convention, we consider $\{x_1, \dots, x_n\}$ as the set of singular points of Σ even though some of them may be regular. In what follows, if T is a triangulation of Σ whose vertex set is $\{x_1, \dots, x_n\}$, we will call a tree contained in T which connects all the vertices of T a *maximal tree*.

Let $A_i,\ i = 1, 2$ be a maximal tree of T_i. If $A_1 \equiv A_2$, then the theorem follows from Theorem 2.6.2. Thus, it is enough to prove the following

Proposition 4.3.6 *There exists a sequence of elementary moves which transforms* T_1 *into a triangulation containing* A_2.

We start by the following lemma

Lemma 4.3.7 *If* $c_1, \dots, c_k$ *are geodesic segments with endpoints in* $\{x_1, \dots, x_n\}$ *such that* $\mathrm{int}(c_i) \cap \mathrm{int}(c_j) = \varnothing$ *if* $i \neq j$, *and* $\mathrm{int}(c_i) \cap A_1 = \varnothing, i = 1, \dots, k$, *then there exists a sequence of elementary moves which transforms* T_1 *into a new triangulation containing* A_1, *and all the segments* $c_1, \dots, c_k$.

Proof: This lemma is just a direct consequence of Lemma 2.6.3. Namely, let Σ' denote the flat surface obtained by slitting open the surface Σ along the tree A_1. The surface Σ' is homeomorphic to a closed disk. Let $\mathrm{T}_1^{(0)}$ denote the triangulation of Σ' which is induced by T_1.

Let P_1 be the developing polygon of c_1 with respect to $\mathrm{T}_1^{(0)}$. By definition, the segment c_1 is a diagonal of P_1. By Lemma 2.6.3, there exists a sequence of elementary moves inside P_1 which transforms the triangulation induced by $\mathrm{T}_1^{(0)}$ into a triangulation containing c_1. We get then a new triangulation $\mathrm{T}_1^{(1)}$ of

Σ' which contains c_1.

Let P_2 denote the developing polygon of c_2 with respect to $\mathrm{T}_1^{(1)}$. Since c_1 is an edge of $\mathrm{T}_1^{(1)}$, and, by assumption, $\mathrm{int}(c_1) \cap \mathrm{int}(c_2) = \varnothing$, we have $\mathrm{int}(c_1) \cap \mathrm{int}(\mathrm{P}_2) = \varnothing$. Apply Lemma 2.6.3 to the polygon P_2, we get a new triangulation $\mathrm{T}_1^{(2)}$ of Σ', which contains c_1 and c_2.

Clearly, this procedure can be continued until we get a triangulation $\mathrm{T}_1^{(k)}$ of Σ' which contains all the segments $c_1, \dots, c_k$, and the lemma follows. □

Now, let $a_1, \dots, a_{n-1}$ denote the edges of the tree A_1, and $b_1, \dots, b_{n-1}$ denote the edges of the tree A_2. We will proceed by induction. Suppose that T_1 contains already the k edges $b_1, \dots, b_k$ of A_2. We will show that T_1 can be transformed by a sequence of elementary moves into a new triangulation containing $b_1, \dots, b_k$ and b_{k+1}.

Let m be the number of intersection points of b_{k+1} with the tree A_1 excluding the endpoints of b_{k+1}. If $m = 0$, then Lemma 4.3.7 allows us to get the conclusion. Therefore, if $m \geqslant 1$, all we need to show is the following

Lemma 4.3.8 *The triangulation T_1 can be transformed by elementary moves into a new triangulation T_1' which contains a maximal tree A_1', and the edges $b_1, \dots, b_k$, such that the number of intersecting points of b_{k+1} with A_1', excluding the endpoints of b_{k+1}, is at most $m-1$.*

Proof: We can assume that the endpoints of b_{k+1} are x_1 and x_2. We consider b_{k+1} as a geodesic ray exiting from x_1. Let y_1 denote the first intersection point of b_{k+1} with the tree A_1, which is contained in the interior of an edge $\overline{x_{j_1} x_{j_1+1}}$ of A_1.

Let $\overline{x_1 y_1}$ denote the subsegment of b_{k+1} whose endpoints are x_1 and y_1. Without loss of generality, we can assume that x_{j_1} is contained in the unique path along A_1 from x_1 to x_{j_1+1}.

Cutting open the surface Σ along the tree A_1, we get a flat surface Σ' with geodesic boundary homeomorphic to a close disk. By construction, we have a surjective map :

$$\pi_{A_1} : \Sigma' \longrightarrow \Sigma,$$

verifying the following properties

. $\pi_{A_1}|_{\mathrm{int}(\Sigma')}$ is an isometry,

- $\pi_{A_1}(\partial\Sigma') = A_1$.

- There are $2(n-1)$ geodesic segments in the boundary of Σ' such that the restriction of π_{A_1} into each segment is an isometry.

- For every edge e in A_1, $\pi_{A_1}^{-1}(\mathrm{int}(e))$ is the union of two open segments in the boundary of Σ'.

Let s_1 denote the inverse image of $\overline{x_1 y_1}$ by π_{A_1}, then s_1 is a geodesic segment with endpoints in the boundary of Σ'. Let x'_1 and y'_1 denote the endpoints of s_1 with $\pi_{A_1}(x'_1) = x_1$, and $\pi_{A_1}(y'_1) = y_1$.

Let $x'_1, \dots, x'_{2(n-1)}$ denote the points in $\pi_{A_1}^{-1}(\{x_1, \dots, x_n\})$ following an orientation of $\partial\Sigma'$. By choosing the suitable orientation, we can assume that the point y'_1 is between $x'_{j'_1}$ and $x'_{j'_1+1}$, where $\pi_{A_1}(x'_{j'_1}) = x_{j_1}$, and $\pi_{A_1}(x'_{j'_1+1}) = x_{j_1+1}$.

For every j in $\{1, \dots, 2(n-1)\}$, we denote $\overline{x'_j x'_{j+1}}$ the segment in the boundary of Σ' between x'_j and x'_{j+1}, with the convention $x'_{2n-1} = x'_1$. Note that $\pi_{A_1}(\overline{x'_j x'_{j+1}})$ is an edge of A_1.

Let c_0 be a path in Σ' joining x'_1 and $x'_{j'_1+1}$ with minimal length. First, we prove

Lemma 4.3.9 *We have* $c_0 \cap s_1 = \{x'_1\}$.

Proof: Suppose that $c_0 \cap \mathrm{int}(s_1) \neq \varnothing$, then let y'_2 denote the first intersection point of c_0 with $\mathrm{int}(s_1)$. Let c_1 denote the path from x'_1 to y'_2 along c_0, and let $\overline{x'_1 y'_2}$ denote the subsegment of s_1 with endpoints x'_1 and y'_2.

The path c_1 is a (finite) union of geodesic segments whose endpoints are in the set $\{x'_1, \dots, x'_{2(n-1)}\}$, it follows that c_1 and $\overline{x'_1 y'_2}$ bound a disk D, which is equipped with a flat metric with geodesic boundary. Since the path c_0 is of minimal length, so is the path c_1. It follows that the interior angle between two consecutive segments of c_1 is at least π. Therefore, if the number of segments in c_1 is l, the boundary of D contains then $l+1$ geodesic segments, and the sum of all the interior angles is at least $(l-1)\pi$. But this is impossible by the Gauss-Bonnet Theorem, hence we conclude that $c_0 \cap \mathrm{int}(s_1) = \varnothing$.

The same argument as above shows that y'_1 is not contained in c_0, and the lemma follows. □

Let $\overline{y'_1 x'_{j'_1+1}}$ denote the subsegment of $\overline{x'_{j'_1} x'_{j'_1+1}}$ between $x'_{j'_1+1}$ and y'_1. From Lemma 4.3.9, we see that $s_1 \cup \overline{y'_1 x'_{j'_1+1}} \cup c_0$ is the boundary of a disk D_0 contained in Σ'. We have immediately the following

Lemma 4.3.10 *Let s be a geodesic ray that intersects the interior of* D_0. *If s inters* D_0 *by a point in the path c_0, then s must exit* D_0 *by a point in* $(s_1 \cup \overline{y'_1 x'_{j'_1+1}}) \setminus \{x'_1, x'_{j'_1+1}\}$.

Proof: If s exits D_0 by another point in c_0, then we have a flat disk with geodesic boundary which violates the Gauss-Bonnet Theorem. $\square$

Let $\hat{c}_0$ denote the image of c_0 by π_{A_1}. The path $\hat{c}_0$ is then a finite union of geodesic segments on Σ with endpoints in the set $\{x_1, \dots, x_n\}$. It is clear that $\hat{c}_0$ contains a path $\hat{c}_1$ joining x_1 and x_{j_1+1}. Let us prove the following

Lemma 4.3.11 *The path $\hat{c}_1$ does not contain the segment $\overline{x_{j_1}x_{j_1+1}}$.*

Proof: Suppose, on the contrary, that $\hat{c}_1$ contains $\overline{x_{j_1}x_{j_1+1}}$. This implies that c_0 contains a segment $\overline{x'_{k'}x'_{k'+1}}$, with $k' \neq j'$, such that

$$\pi_{A_1}(\overline{x'_{k'}x'_{k'+1}}) = \pi_{A_1}(\overline{x'_{j'_1}x'_{j'_1+1}}) = \overline{x_{j_1}x_{j_1+1}}.$$

Let y'_2 denote the unique point in $\overline{x'_{k'}x'_{k'+1}}$ such that $\pi_{A_1}(y'_2) = \pi_{A_1}(y'_1) = y_1$. The inverse image of b_{k+1} by π_{A_1} is a sequence of $(m+1)$ geodesic segments of Σ' with endpoints in the boundary of Σ', whose s_1 is the first one.

Let s_2 be the next segment in the sequence. The point y'_2 is one endpoint of s_2, by assumption, y'_2 is an intersection point of the segment s_2 and the disk D_0.Consider the segment s_2 as a geodesic ray exiting from y'_2.

By Lemma 4.3.10, the ray s_2 exits D_0 by a point z'_2 in $(s_1 \cup \overline{y'_1x'_{j'_1+1}}) \setminus \{x'_1, x'_{j'_1+1}\}$. Since the geodesic b_{k+1} is a simple, the point z'_2 can not be contained in s_1. Hence z'_2 must be a point in $\mathrm{int}(\overline{y'_1x'_{j'_1+1}})$.

Now, since the segments $\overline{x'_{j'_1}x'_{j'_1+1}}$ and $\overline{x'_{k'}x'_{k'+1}}$ are identified by π_{A_1}, the point z'_2 is identified to a point y'_3 in $\overline{x'_{k'}x'_{k'+1}}$. Consequently, the argument above can be applied infinitely many times, which implies that the inverse image of b_{k+1} by π_{A_1} contains infinitely many segments, and we have a contradiction to the fact that $\pi^{-1}_{A_1}(b_{k+1})$ contains only $m+1$ segments. $\square$

Since A_1 is a tree, the set $A_1 \setminus \mathrm{int}(\overline{x_{j_1}x_{j_1+1}})$ has two connected components, the one containing x_1 will be denoted by C_1, the other one containing x_{j_1+1} will be denoted by C_2. From Lemma 4.3.11, we know that the path $\hat{c}_1$, which joins x_1 to x_{j_1+1} does not contain $\overline{x_{j_1}x_{j_1+1}}$. Therefore the path $\hat{c}_1$ must contain a segment $\hat{s}$, with endpoints in $\{x_1, \dots, x_n\}$, such that one of the two endpoints is in C_1, and the other is in C_2.

Let s be the inverse image of $\hat{s}$ by π_{A_1}. Evidently, $\hat{s}$ is not an edge of A_1, hence s is a segment contained inside Σ', it follows that $\text{int}(\hat{s}) \cap A_1 = \varnothing$.

Let us prove

Lemma 4.3.12 $\text{int}(\hat{s}) \cap \text{int}(b_i) = \varnothing$, *for every* $i = 1, \ldots, k$.

Proof: Let b'_i, $i = 1, \ldots, k$, denote the inverse image of b_i by π_{A_1}. Since $\text{int}(b_i) \cap A_1 = \varnothing$, b'_i is a geodesic segment contained inside Σ'.

Suppose that $\text{int}(\hat{s}) \cap \text{int}(b_i) \neq \varnothing$, it follows that $\text{int}(b'_i) \cap \text{int}(s) \neq \varnothing$. Let y''_i be the intersection point of $\text{int}(b'_i)$ and $\text{int}(s)$. Recall that s is included in the path c_0. We can then consider the segment b'_i as a ray which inters D_0 by y''_i. By Lemma 4.3.9, we know that b'_i must exit D_0 by a point z''_i which is contained in $s_1 \cup \overline{y'_1 x'_{j'_1+1}}$, but it would imply that either $\text{int}(b_i) \cap b_{k+1} \neq \varnothing$, or $\text{int}(b_i) \cap A_1 \neq \varnothing$, which is impossible by assumption. The lemma is then proved. □

We can now finish the proof of Lemma 4.3.8. Using Lemma 4.3.7, we deduce that there exists a sequence of elementary moves which transforms T_1 into a new triangulation T'_1 containing A_1, the edges $b_1, \ldots, b_k$, and the segment $\hat{s}$. By replacing $\overline{x_{j_1} x_{j_1+1}}$ by $\hat{s}$, we get a new maximal tree A'_1. Let us show that the number of intersection points of b_{k+1} with A'_1, excluding the endpoints of b_{k+1}, is at most $m-1$. We have

$$\begin{aligned}\text{Card}\{\text{int}(b_{k+1}) \cap A'_1\} = \;& \text{Card}\{\text{int}(b_{k+1}) \cap A_1\} - \text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\overline{x_{j_1} x_{j_1+1}})\} + \\ & + \text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\hat{s})\}\end{aligned}$$

Let y be a point in $\text{int}(b_{k+1}) \cap \text{int}(\hat{s})$, and let $y' = \pi_{A_1}^{-1}(y)$. Let b' be the segment in $\pi_{A_1}^{-1}(b_{k+1})$ which contains y'. Note that $y' = b' \cap s$.

By Lemma 4.3.10, and since $\text{int}(b') \cap \text{int}(s_1) = \varnothing$, it follows that b' contains a point z' in $\overline{x'_{j'_1} x'_{j'_1+1}}$. We deduce that there is a one-to-one mapping from $\{\text{int}(b_{k+1}) \cap \text{int}(\hat{s})\}$ into $\{\text{int}(b_{k+1}) \cap \text{int}(\overline{x_{j_1} x_{j_1+1}})\}$. Clearly, the point y_1 does not belong to the image of this map, therefore we have

$$\text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\overline{x_{j_1} x_{j_1+1}})\} \geqslant \text{Card}\{\text{int}(b_{k+1}) \cap \text{int}(\hat{s})\} + 1.$$

It follows immediately that

$$\text{Card}\{\text{int}(b_{k+1}) \cap A'_1\} \leqslant \text{Card}\{\text{int}(b_{k+1}) \cap A_1\} - 1 = m - 1.$$

The proof of Lemma 4.3.8 is now complete. □

From what we have seen, Proposition 4.3.6, and hence Theorem 4.3.2, follow directly from Lemma 4.3.8. □

4.4 Comparison with complex hyperbolic volume form

In this section, we assume that all the angles $\alpha_1, \dots, \alpha_n$ are less than 2π. Put $\kappa_i = 2\pi - \alpha_i,\ i = 1, \dots, n$, we have

$$\kappa_1 + \cdots + \kappa_n = 4\pi.$$

Following Thurston [Th], we denote $C(\kappa_1, \dots, \kappa_n)$ the moduli space of spherical flat surface having n singularities with cone angles $\alpha_1, \dots, \alpha_n$, or equivalently, with curvatures $\kappa_1, \dots, \kappa_n$, up to homothety. In [Th], Thurston proves that $C(\kappa_1, \dots, \kappa_n)$ admits a complex hyperbolic metric structure with finite volume, and the metric closure of $C(\kappa_1, \dots, \kappa_n)$ has cone manifold structure.

The complex hyperbolic metric provides a volume form μ_{Hyp} on $C(\kappa_1, \dots, \kappa_n)$. On the other hand, the volume form μ_{Tr} gives another volume form on $C(\kappa_1, \dots, \kappa_n)$ denoted by $\hat{\mu}^1_{\mathrm{Tr}}$. The volume form $\hat{\mu}^1_{\mathrm{Tr}}$ is defined as follows :

- First, we identify $C(\kappa_1, \dots, \kappa_n)$ to the subset $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ of all surfaces of area 1 in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$. Let $f : \mathcal{M}(\mathbb{S}^2, \bar{\alpha}) \longrightarrow \mathbb{R}$ be the function which associates to a pair (Σ, θ) in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha}) = \mathcal{M}(\mathbb{S}^2, \bar{\alpha})^* \times \mathbb{S}^1$ the area of Σ. The space $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ can be considered as the quotient of the locus $f^{-1}(1)$ by the action of $\mathbb{S}^1$.

- By Theorem 4.1.1, we know that $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ is a complex orbifold, let $\mathbb{J}$ denote the complex structure of $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$. Let $\rho : f^{-1}(1) \longrightarrow f^{-1}(1)/\mathbb{S}^1 = \mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ denote the natural projection. We define the volume form $\hat{\mu}^1_{\mathrm{Tr}}$ on $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ to be the one such that :

$$\rho^* \hat{\mu}^1_{\mathrm{Tr}} \wedge df \wedge (df \circ \mathbb{J}) = \mu_{\mathrm{Tr}}$$

Our goal in this section is to prove

Proposition 4.4.1 *There exists a constant λ depending on $(\alpha_1, \dots, \alpha_n)$ such that $\hat{\mu}^1_{\mathrm{Tr}} = \lambda \mu_{\mathrm{Hyp}}$.*

This proposition together with Thurston's result implies

Corollary 4.4.2 *The volume of $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ with respect to $\hat{\mu}^1_{\mathrm{Tr}}$ is finite.*

4.4.1 Local formulae for $\hat{\mu}^1_{\mathrm{Tr}}$ and μ_{Hyp}

First, we recall the construction of local charts for $C(\kappa_1, \dots, \kappa_n)$ as presented in [Th], and consequently the definition of μ_{Hyp}.

Given a surface Σ in $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$, we consider Σ as a point in $C(\kappa_1, \dots, \kappa_n)$. Let T be a triangulation of Σ by geodesic segments whose set of vertices is the set of singular points. Choose a singular point of Σ and denote this point x_{last}. We will call all the edges of T which contain x_{last} as an endpoint *followers* . Pick a tree $\tilde{A}$ in T which connects all other singular points of Σ, and call the edges of this tree *leaders*. The remaining edges of T are also called *followers*.

Using a developing map, one can associate to each of the leaders a complex number, there are $n-2$ of them. Let $(z_1, \dots, z_{n-2})$ denote those complex numbers. The same developing map also defines an associated complex number for each of the followers, but these numbers can be calculated from those associated to leaders by complex linear functions. Thus, the complex numbers associated to leaders determine a local coordinate system $\varphi : \mathrm{U} \longrightarrow \mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ for $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ in a neighborhood of $(\Sigma, 1)$, where U is a neighborhood of $(z_1, \dots, z_{n-2})$ in $\mathbb{C}^{n-2}$. Consequently, a neighborhood of Σ in $C(\kappa_1, \dots, \kappa_n)$ is then identified to an open set of $\mathbb{PC}^{n-3}$ which contains $[z_1 : \dots : z_{n-2}]$.

If we add to $\tilde{A}$ a follower which contains x_{last} as an endpoint, then we have an erasing tree A on Σ. We can then construct a local chart $\Psi_{\mathcal{T},\mathcal{A}}$ for $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ from T and A. Recall that $\Psi_{\mathcal{T},\mathcal{A}}$ is defined on an open subset $\mathcal{U}_{\mathcal{T}}$ of $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$, with image in ker $\mathbf{A}_{\mathcal{T}}$, where linear map $\mathbf{A}_{\mathcal{T}} : \mathbb{C}^{N_1} \longrightarrow \mathbb{C}^{N_2}$ is determined by the tree A, and the angles $\alpha_1, \dots, \alpha_n$. By definition, the volume form μ_{Tr} on $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ is identified in this local chart to the volume form on ker $\mathbf{A}_{\mathcal{T}}$ which is induced by the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$.

Now, observe that the following sequence is exact

$$0 \longrightarrow \mathbb{C}^{n-2} \xrightarrow{\Psi_{\mathcal{T},\mathcal{A}}\circ\varphi} \mathbb{C}^{N_1} \xrightarrow{\mathbf{A}_{\mathcal{T}}} \mathbb{C}^{N_2} \longrightarrow 0.$$

Thus, the map $\Psi_{\mathcal{T},\mathcal{A}}\circ\varphi$ is the restriction of an isomorphism between $\mathbb{C}^{n-2}$ and ker $\mathbf{A}_{\mathcal{T}}$ onto an open subset of $\mathbb{C}^{n-2}$. Hence, in the local chart φ, the volume form μ_{Tr} is identified to the volume form $c\lambda_{2(n-2)}$, where $\lambda_{2(n-2)}$ is the Lebesgue measure of $\mathbb{C}^{n-2}$, and c is a constant.

In the local chart φ, the area function f on $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ is expressed as a Hermitian form $\mathbf{H}$. More precisely, if $v \in \mathbb{C}^{n-2}$ is a vector such that $\varphi(v) = (\Sigma, \theta) \in \mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ then $f((\Sigma, \theta)) = \mathbf{Area}(\Sigma) = {}^t\bar{v}\mathbf{H}v$. It is proven in [Th] that $\mathbf{H}$ is of signature $(1, n-3)$. Changing the basis and the sign of $\mathbf{H}$, we can assume that

$$\mathbf{H} = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & -1 \end{pmatrix}$$

Thus we can write

$$f(z_1, \dots, z_{n-2}) = |z_1|^2 + \dots + |z_{n-3}|^2 - |z_{n-2}|^2.$$

Note that by these changes, the vectors of $\mathbb{C}^{n-2}$ representing surfaces in $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ are contained in the set $\mathbf{Q}_1 = f^{-1}(-1)$, and we still have $\mu_{\mathrm{Tr}} = c_0\lambda_{2(n-2)}$ with c_0 a constant.

We use the symbol $\langle , \rangle$ to denote the scalar product defined by Hermitian form $\mathbf{H}$. By definition $f(Z) = \langle Z, Z\rangle$, $\forall Z \in \mathbb{C}^{n-2}$. Let $\mathbb{J}$ denote the natural complex structure of $\mathbb{C}^{n-2}$, that is $\mathbb{J}(z_1, \dots, z_{n-2}) = (\imath z_1, \dots, \imath z_{n-2})$. Let η denote the real symmetric form induced by $\langle , \rangle$, that is

$$\eta(X, Y) = \mathrm{Re}\langle X, Y\rangle.$$

Let Z be a vector in $\mathbf{Q}_1$ which represents a surface in $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$. The tangent space of $\mathbf{Q}_1/\mathbb{S}^1$ at the orbit $\mathbb{S}^1 \cdot Z$ is naturally identified to the orthogonal complement of Z with respect to $\langle , \rangle$. Denote this space $Z^\perp$. The restriction of $\langle , \rangle$ on $Z^\perp$ is a definite positive Hermitian form, which determines the complex hyperbolic metric on $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^* = C(\kappa_1, \dots, \kappa_n)$.

We have

$$df = (\bar{z}_1 dz_1 + \dots + \bar{z}_{n-3}dz_{n-3} - \bar{z}_{n-2}dz_{n-2}) + (z_1 d\bar{z}_1 + \dots + z_{n-3}d\bar{z}_{n-3} - z_{n-2}d\bar{z}_{n-2}),$$

and

$$df \circ \mathbb{J} = \imath(\bar{z}_1 dz_1 + \dots + \bar{z}_{n-3}dz_{n-3} - \bar{z}_{n-2}dz_{n-2}) - \imath(z_1 d\bar{z}_1 + \dots + z_{n-3}d\bar{z}_{n-3} - z_{n-2}d\bar{z}_{n-2}).$$

Note that both df and $df \circ \mathbb{J}$ are invariant by the action of $\mathbb{S}^1$. Put

$$U_k = (0, \dots, 0, \overset{(k)}{\bar{z}_{n-2}}, 0, \dots, \bar{z}_k), \ k = 1, \dots, n-3.$$

and $V_k = \mathbb{J} \cdot U_k = \imath U_k$. One can verify easily that $\{U_1, V_1, \dots, U_{n-3}, V_{n-3}\}$ span $Z^\perp$ as a real vector space. We consider $\{U_1, V_1, \dots, U_{n-3}, V_{n-3}\}$ as a basis of the tangent space of $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ at $\varphi(Z)$.

We know that the restriction of the symmetric form η on $Z^\perp$ defines a Riemannian metric. Let U_k^*, V_k^* denote the $\mathbb{R}$-linear 1-forms dual to U_k and V_k respectively with respect to η. We have :

$$U_k^* = \frac{1}{2}[(z_{n-2}dz_k - z_k dz_{n-2}) + (\bar{z}_{n-2}d\bar{z}_{n-2} - \bar{z}_k d\bar{z}_{n-2})],$$

and

$$V_k^* = \frac{-\imath}{2}[(z_{n-2}dz_k - z_k dz_{n-2}) - (\bar{z}_{n-2}d\bar{z}_{n-2} - \bar{z}_k d\bar{z}_{n-2})].$$

We can consider $\{U_1^*, V_1^*, \ldots, U_{n-3}^*, V_{n-3}^*\}$ as a basis of the cotangent space of $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ at $\varphi(Z)$. Let ρ be the projection from $\mathbf{Q}_1$ to $\mathbf{Q}_1/\mathbb{S}^1$. We define a volume form $\hat{\mu}_{\mathrm{Tr}}^1$ on $\mathbf{Q}_1/\mathbb{S}^1$ by the following condition :

$$\rho^* \hat{\mu}_{\mathrm{Tr}}^1 \wedge df \wedge (df \circ \mathbb{J}) = (\frac{\imath}{2})^{n-2} dz_1 d\bar{z}_1 \ldots dz_{n-2} d\bar{z}_{n-2} = d\lambda_{2(n-2)} \tag{4.4}$$

Since df and $df \circ \mathbb{J}$ are invariant by the action of $\mathbb{S}^1$, the volume form $\hat{\mu}_{\mathrm{Tr}}^1$ is well defined by this condition.

We wish to express $\hat{\mu}_{\mathrm{Tr}}^1(\mathbb{S}^1 \cdot Z)$ in terms of $U_k^*, V_k^*,\ k = 1, \ldots, n-3$.

<u>**Claim 1 :**</u> *We have*

$$\hat{\mu}_{\mathrm{Tr}}^1(\mathbb{S}^1 \cdot Z) = \frac{c_0}{|z_{n-2}|^{2(n-4)}} (U_1^* \wedge V_1^*) \wedge \cdots \wedge (U_{n-3}^* \wedge V_{n-3}^*),$$

where $c_0 = \mu_{\mathrm{Tr}}/\lambda_{2(n-2)}$.

Proof: Consider $U_k^* \wedge V_k^*$, we have

$$\begin{aligned} U_k^* \wedge V_k^* &= \frac{-\imath}{4}(X_k + \overline{X}_k) \wedge (X_k - \overline{X}_k) \\ &= \frac{\imath}{2} X_k \wedge \overline{X}_k \end{aligned}$$

where $X_k = z_{n-2}dz_k - z_k dz_{n-2}$, and $\overline{X}_k = \bar{z}_{n-2}d\bar{z}_k - \bar{z}_k d\bar{z}_{n-2}$.

We can also write

$$df = X + \overline{X}, \text{ and } df \circ \mathbb{J} = \imath(X - \overline{X})$$

with $X = \bar{z}_1 dz_1 + \cdots + \bar{z}_{n-3}dz_{n-3} - \bar{z}_{n-2}dz_{n-2}$, and $\overline{X} = z_1 d\bar{z}_1 + \cdots + z_{n-3}d\bar{z}_{n-3} - z_{n-2}d\bar{z}_{n-2}$.

Hence

$$df \wedge (df \circ \mathbb{J}) = 2\imath X \wedge \overline{X}.$$

Now

$$\begin{aligned}
& (U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^*) \wedge df \wedge (df \circ \mathbb{J}) \\
= & -(\tfrac{\imath}{2})^{n-4} X_1 \wedge \overline{X}_1 \wedge \cdots \wedge X_{n-3} \wedge \overline{X}_{n-3} \wedge X \wedge \overline{X} \\
= & -(\tfrac{\imath}{2})^{n-4} (-1)^{\frac{(n-2)(n-3)}{2}} (X_1 \wedge \cdots \wedge X_{n-3} \wedge X) \wedge (\overline{X}_1 \wedge \cdots \wedge \overline{X}_{n-3} \wedge \overline{X})
\end{aligned}$$

Simple computations give

$$\begin{aligned}
X_1 \wedge \cdots \wedge X_{n-3} \wedge X &= z_{n-2}^{n-4}(|z_1|^2 + \cdots + |z_{n-3}|^2 - |z_{n-2}|^2) dz_1 \ldots dz_{n-2} \\
&= -z_{n-2}^{n-4} dz_1 \ldots dz_{n-2}
\end{aligned}$$

and similarly

$$\overline{X}_1 \wedge \cdots \wedge \overline{X}_{n-3} \wedge \overline{X} = -\bar{z}_{n-2}^{n-4} d\bar{z}_1 \ldots d\bar{z}_{n-2}.$$

Therefore,

$$\begin{aligned}
(X_1 \wedge \cdots \wedge X_{n-3} \wedge X) \wedge (\overline{X}_1 \wedge \cdots \wedge \overline{X}_{n-3} \wedge \overline{X}) &= |z_{n-2}|^{2(n-4)} dz_1 \ldots dz_{n-2} d\bar{z}_1 \ldots d\bar{z}_{n-2} \\
&= 2^{n-2} \imath^{(n-2)(n-4)} |z_{n-2}|^{2(n-4)} d\lambda_{2(n-2)}
\end{aligned}$$

and we get

$$U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^* \wedge df \wedge (df \circ \mathbb{J}) = 4|z_{n-2}|^{2(n-4)} d\lambda_{2(n-2)}.$$

By the definition of $\hat{\mu}_{\mathrm{Tr}}^1$, we obtain

$$\hat{\mu}_{\mathrm{Tr}}^1(\mathbb{S}^1 \cdot Z) = \frac{c_0}{4|z_{n-2}|^{2(n-4)}} U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^*.$$

□

Remark:

- Even though the 1-forms U_k^* and V_k^* are not invariant by the $\mathbb{S}^1$ action, the 2-form $U_k^* \wedge V_k^*$ is. Hence, the $2(n-3)$-form $U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^*$ is invariant by the $\mathbb{S}^1$ action.

- Let μ_{Tr}^1 be the volume form on $\mathbf{Q}_1$ verifying the following condition

$$\mu_{\mathrm{Tr}}^1 \wedge df = \mu_{\mathrm{Tr}}.$$

The tangent vector to the $\mathbb{S}^1$ orbit at a point $Z \in \mathbb{C}^2$ is given by $\imath Z$, and we have

$$df \circ \mathbb{J}(\imath Z) = -df(Z) = -\langle Z, Z \rangle = 1.$$

Therefore, the volume form $\hat{\mu}_{\mathrm{Tr}}^1$ can be considered as the push-forward of μ_{Tr}^1 onto $\mathbf{Q}_1/\mathbb{S}^1$.

Now, we will proceed to compute the volume form defined by η on $Z^{\perp}$ in terms of U_k^*, V_k^*. Let (η_{ij}) with $i, j = 1, \dots, 2(n-3)$ be the (real) matrix of η in the basis $\{U_1, V_1, \dots, U_{n-3}, V_{n-3}\}$. Since the volume form μ_{Hyp} is defined by the metric η, we have

$$\mu_{\mathrm{Hyp}}(\mathbb{S}^1 \cdot Z) = \frac{1}{\sqrt{\det(\eta_{ij})}} U_1^* \wedge V_1^* \wedge \cdots \wedge U_{n-3}^* \wedge V_{n-3}^*.$$

<u>**Claim 2 :**</u> $\det(\eta_{ij}) = |z_{n-2}|^{4(n-4)}$.

Proof: Since η is the real part of $\mathbf{H}$, the matrix (η_{ij}) is the real interpretation of the matrix $(\mathbf{H}_{ij}), i, j = 1, \dots, n-3$, of $\mathbf{H}$ in the complex basis $\{U_1, \dots, U_{n-3}\}$ of $Z^{\perp}$. This implies

$$\det(\eta_{ij}) = |\det(\mathbf{H}_{ij})|^2.$$

We have

$$\mathbf{H}_{ij} = \langle U_i, U_j \rangle = \begin{cases} -z_i \bar{z}_j, & \text{if } i \neq j\,; \\ |z_{n-2}|^2 - |z_i|^2, & \text{if } i = j. \end{cases}$$

Hence

$$\begin{aligned} \det(\mathbf{H}_{ij}) &= \det \begin{pmatrix} |z_{n-2}|^2 - |z_1|^2 & -\bar{z}_1 z_2 & \dots & -\bar{z}_1 z_{n-3} \\ -\bar{z}_2 z_1 & |z_{n-2}|^2 - |z_2|^2 & \dots & -\bar{z}_2 z_{n-3} \\ \dots & \dots & \dots & \dots \\ -\bar{z}_{n-3} z_1 & -\bar{z}_{n-3} z_2 & \dots & |z_{n-2}|^2 - |z_{n-3}|^2 \end{pmatrix} \\ &= |z_{n-2}|^{2(n-3)} \det \begin{pmatrix} 1 - |\varepsilon_1|^2 & -\bar{\varepsilon}_1 \varepsilon_2 & \dots & -\bar{\varepsilon}_1 \varepsilon_{n-3} \\ -\bar{\varepsilon}_2 \varepsilon_1 & 1 - |\varepsilon_2|^2 & \dots & -\bar{\varepsilon}_2 \varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ -\bar{\varepsilon}_{n-3} \varepsilon_1 & -\bar{\varepsilon}_{n-3} \varepsilon_2 & \dots & 1 - |\varepsilon_{n-3}|^2 \end{pmatrix} \end{aligned}$$

where $\varepsilon_k = z_k / z_{n-2},\ k = 1, \dots, n-3$.

Since

$$\begin{vmatrix} 1-|\varepsilon_1|^2 & -\bar\varepsilon_1\varepsilon_2 & \dots & -\bar\varepsilon_1\varepsilon_{n-3} \\ -\bar\varepsilon_2\varepsilon_1 & 1-|\varepsilon_2|^2 & \dots & -\bar\varepsilon_2\varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ -\bar\varepsilon_{n-3}\varepsilon_1 & -\bar\varepsilon_{n-3}\varepsilon_2 & \dots & 1-|\varepsilon_{n-3}|^2 \end{vmatrix} = \begin{vmatrix} 1 & -\bar\varepsilon_1\varepsilon_2 & \dots & -\bar\varepsilon_1\varepsilon_{n-3} \\ 0 & 1-|\varepsilon_2|^2 & \dots & -\bar\varepsilon_2\varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ 0 & -\bar\varepsilon_{n-3}\varepsilon_2 & \dots & 1-|\varepsilon_{n-3}|^2 \end{vmatrix} -$$

$$-\varepsilon_1 \begin{vmatrix} \bar\varepsilon_1 & -\bar\varepsilon_1\varepsilon_2 & \dots & -\bar\varepsilon_1\varepsilon_{n-3} \\ \bar\varepsilon_2 & 1-|\varepsilon_2|^2 & \dots & -\bar\varepsilon_2\varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ \bar\varepsilon_{n-3} & -\bar\varepsilon_{n-3}\varepsilon_2 & \dots & 1-|\varepsilon_{n-3}|^2 \end{vmatrix} = \begin{vmatrix} 1-|\varepsilon_2|^2 & -\bar\varepsilon_2\varepsilon_3 & \dots & -\bar\varepsilon_2\varepsilon_{n-3} \\ -\bar\varepsilon_3\varepsilon_2 & 1-|\varepsilon_3|^2 & \dots & -\bar\varepsilon_3\varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ -\bar\varepsilon_{n-3}\varepsilon_2 & -\bar\varepsilon_{n-3}\varepsilon_3 & \dots & 1-|\varepsilon_{n-3}|^2 \end{vmatrix} -$$

$$-\varepsilon_1 \begin{vmatrix} \bar\varepsilon_1 & 0 & \dots & 0 \\ \bar\varepsilon_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \bar\varepsilon_{n-3} & 0 & \dots & 1 \end{vmatrix} = \begin{vmatrix} 1-|\varepsilon_2|^2 & -\bar\varepsilon_2\varepsilon_3 & \dots & -\bar\varepsilon_2\varepsilon_{n-3} \\ -\bar\varepsilon_3\varepsilon_2 & 1-|\varepsilon_3|^2 & \dots & -\bar\varepsilon_3\varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ -\bar\varepsilon_{n-3}\varepsilon_2 & -\bar\varepsilon_{n-3}\varepsilon_3 & \dots & 1-|\varepsilon_{n-3}|^2 \end{vmatrix} - |\varepsilon_1|^2,$$

we deduce

$$\det \begin{pmatrix} 1-|\varepsilon_1|^2 & -\bar\varepsilon_1\varepsilon_2 & \dots & -\bar\varepsilon_1\varepsilon_{n-3} \\ -\bar\varepsilon_2\varepsilon_1 & 1-|\varepsilon_2|^2 & \dots & -\bar\varepsilon_2\varepsilon_{n-3} \\ \dots & \dots & \dots & \dots \\ -\bar\varepsilon_{n-3}\varepsilon_1 & -\bar\varepsilon_{n-3}\varepsilon_2 & \dots & 1-|\varepsilon_{n-3}|^2 \end{pmatrix} = 1-(|\varepsilon_1|^2+\dots+|\varepsilon_{n-3}|^2).$$

It follows that

$$\begin{aligned} \det(\mathbf{H}_{ij}) &= |z_{n-2}|^{2(n-3)}(1-(|\varepsilon_1|^2+\dots+|\varepsilon_{n-3}|^2)) \\ &= |z_{n-2}|^{2(n-4)}(|z_{n-2}|^2-(|z_1|^2+\dots+|z_{n-3}|^2)) \\ &= |z_{n-2}|^{2(n-4)} \end{aligned}$$

Consequently, we have $\det(\eta_{ij}) = |\det(\mathbf{H}_{ij})|^2 = |z_{n-2}|^{4(n-4)}$. The claim is then proved. □

From Claim 1, and Claim 2, we obtain

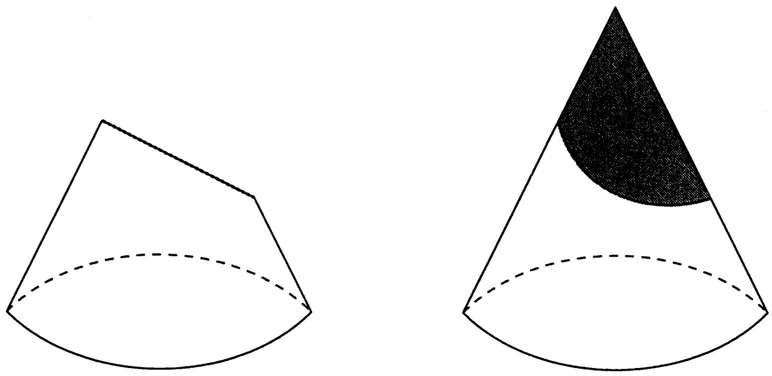
Lemma 4.4.3 *The quotient $\hat\mu^1_{\mathrm{tr}}/\mu_{\mathrm{Hyp}}$ is a locally constant function on $\mathcal{M}_1(\mathbb{S}^2,\bar\alpha)^*$.*

4.4.2 Connectedness of $C(\kappa_1,\dots,\kappa_n)$

To complete the proof of 4.4.1, we will prove

Lemma 4.4.4 *For any $(\alpha_1,\dots,\alpha_n)$, the space $C(\kappa_1,\dots,\kappa_n)$ is connected.*

Proof: To prove this lemma, first, we recall the construction of a surface with $n-1$ singular points from an arbitrary surface Σ in $C(\kappa_1, \dots, \kappa_n)$. Let $x_1, \dots, x_n$ denote the singular points of Σ such that the curvature at x_i is κ_i. Suppose that we have $\kappa_{n-1} + \kappa_n < 2\pi$. Choose a geodesic segment s joining x_{n-1} to x_n which does not pass through any other singular point of Σ (the geodesic segment of minimal length verifies this condition). Slit open Σ along s, and glue to boundary of the surface obtained by this operation a cone so that the points x_{n-1} and x_n become regular. The apex angle of the added cone must be $2\pi - (\kappa_{n-1} + \kappa_n)$. Therefore, after a rescaling, we obtain a flat surface Σ' in $C(\kappa_1, \dots, \kappa_{n-2}, \kappa_{n-1} + \kappa_n)$.



The space $C(\kappa_1, \dots, \kappa_{n-1} + \kappa_n)$ is contained in the metric closure $\overline{C}(\kappa_1, \dots, \kappa_n)$ of $C(\kappa_1, \dots, \kappa_n)$. A neighborhood of $C(\kappa_1, \dots, \kappa_{n-1} + \kappa_n)$ in $\overline{C}(\kappa_1, \dots, \kappa_n)$ looks like $C(\kappa_1, \dots, \kappa_{n-1} + \kappa_n) \times \mathrm{D}^2$. By this construction, we see that any surface in $C(\kappa_1, \dots, \kappa_n)$ can be deformed inside $C(\kappa_1, \dots, \kappa_n)$ into a surface close to the stratum $C(\kappa_1, \dots, \kappa_{n-1} + \kappa_n)$. Hence, if $C(\kappa_1, \dots, \kappa_{n-1} + \kappa_n)$ is connected, so is $C(\kappa_1, \dots, \kappa_n)$.

If $n \geqslant 5$, then there exist $i \neq j \in \{1, \dots, n\}$ such that $\kappa_i + \kappa_j < 2\pi$. Thus, by induction, we only need to prove the lemma for the case $n = 4$. Without loss of generality, we can assume that $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \kappa_4$. We only have two possibilities :

- Case 1 : $\kappa_3 + \kappa_4 < 2\pi$. Since $C(\kappa_1, \kappa_2, \kappa_3 + \kappa_4)$ is only a point, the argument above shows that $C(\kappa_1, \dots, \kappa_4)$ is connected.

- Case 2 : $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \pi$. Every surface in $C(\pi, \pi, \pi, \pi)$ is the quotient of a flat torus by a holomorphic involution which fixes exactly 4 points. This correspondence gives a bijection between $C(\pi, \pi, \pi, \pi)$ and the moduli space of flat tori up to homothety. Since the latter is the modular surface $\mathbb{H}^2/SL(2, \mathbb{Z})$, which is connected, we deduce that $C(\pi, \pi, \pi, \pi)$ is also connected. The lemma is then proved. □

Proposition 4.4.1 follows immediately from Lemma 4.4.3, and Lemma 4.4.4.

Chapitre 5

Finiteness of integrals

5.1 Definitions and main results

Let $\bar{\alpha}, \bar{\beta}$ be as in Chapter 2. Consider the Teichmüller space $\mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$. Let us define

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{T}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta}) & \longrightarrow & \mathbb{R}^+ \\ & ([(\Sigma, \phi)], \xi) & \longmapsto & \exp(-\mathbf{Area}(\Sigma) - \ell^2(\partial\Sigma)) \end{array}$$

where $\ell(\partial\Sigma)$ is the total length of the boundary of Σ.

For surfaces with erasing trees, fix a family of topological trees $\hat{\mathcal{A}} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ and the numbers $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ as in Chapter 3, one can also define a similar function on $\mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ as follows :

$$\begin{array}{rccc} \mathcal{F}^{\mathrm{et}}: & \mathcal{T}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha}) & \longrightarrow & \mathbb{R}^+ \\ & ([(\Sigma, \phi)], \xi) & \longmapsto & \exp(-\mathbf{Area}(\Sigma) - \ell^2(\phi(\hat{\mathcal{A}}))) \end{array}$$

where $\ell(\phi(\hat{\mathcal{A}}))$ is the total length of the trees in $\phi(\hat{\mathcal{A}})$.

Clearly, the function $\mathcal{F}$ (resp. $\mathcal{F}^{\mathrm{et}}$) induces a function on the moduli space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ (resp. $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$), in the sequel of this chapter we will call $\mathcal{F}$ and $\mathcal{F}^{\mathrm{et}}$ *energy functions* on $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, and $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ respectively. The main result of this chapter is the following

Theorem 5.1.1 *a) If the space $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ consists of surfaces with non-empty boundary, then the integral of the energy function $\mathcal{F}$ with respect to the volume form μ_{Tr} is finite*

$$\int_{\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})} \mathcal{F} d\mu_{\mathrm{Tr}} < \infty \tag{5.1}$$

b) If the forest $\hat{A}$ contains trees which are not isolated points, then the integral of the energy function $\mathcal{F}^{\mathrm{et}}$ with respect to the affine volume form μ_{Tr} on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ is finite

$$\int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})} \mathcal{F}^{\mathrm{et}} d\mu_{\mathrm{Tr}} < \infty. \tag{5.2}$$

Recall that $\mathcal{H}_1(k_1, \dots, k_n)$ is the moduli space of closed translation surfaces of area one, or equivalently, the subspace of $\mathcal{H}(k_1, \dots, k_n)$ consisting of pairs (M, ω) such that $\int_M |\omega|^2 = 1$. Even though Theorem 5.1.1 concerns only translation surfaces with boundary, it turns out that one can use this result to prove the classical fact $\mathrm{Vol}_{\mu_0}(\mathcal{H}_1(k_1, \dots, k_n)) < \infty$.

For spherical flat surfaces, using Theorem 5.1.1, we will prove the following

Theorem 5.1.2 *Let μ_{Tr} denote the volume form on $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ defined in Chapter 4, then we have*

$$\int_{\mathcal{M}(\mathbb{S}^2,\bar{\alpha})} \exp(-\mathbf{Area}) d\mu_{\mathrm{Tr}} < \infty \tag{5.3}$$

Consequently, the volume of the set $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})$ is finite.

This result is a generalization of the result of Thurston in [Th], and analogue to a result in [V2] which is proven by a different method.

This chapter is organized as follows : we start by the demonstration of Theorem 5.1.1 for a particular case, where the base surface is a torus, by this example, we introduce the main ideas of the proof for the general case. The proof of Theorem 5.1.1 itself is given in the next two Sections 5.3 and 5.4. In Section 5.5, we show how to obtain the fact that the volume of $\mathcal{H}_1(k_1, \dots, k_n)$ is finite by using 5.1.1. Finally, in Section 5.6, we prove Theorem 5.1.2.

5.2 First example

In this section, we prove Theorem 5.1.1 for the case $g = 1, m = 1, \beta_1 = 2\pi, s_1 = 2$, and $n = 0$. In this case, S is homeomorphic to a torus with an open disk removed. Via this simple case, we would like to introduce the main ideas of the proof for the general case.

Let Σ be a translation surface with boundary homeomorphic to S such that

- $\mathrm{int}(\Sigma)$ contains no singular points,
- the cone angle associated to the unique boundary component of Σ is 2π, and
- there are two points p, q in $\partial\Sigma$ such that $\partial\Sigma \setminus \{p, q\}$ is the union of two geodesic segments.

Let ξ be a normalized parallel vector field on Σ. By definition, the pair (Σ, ξ) represents a point in $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})$. First, we prove

Lemma 5.2.1 *The open surface* $\mathrm{int}(\Sigma)$ *is isometric to a flat torus with a geodesic segment removed.*

Proof: Let a_1, and a_2 denote the two geodesic segments with endpoints p, q which are contained in $\partial\Sigma$. Let η_1, η_2 denote the corner angles at p, and q respectively. We have to show that η_1, η_2 are 2π, and the segments a_1 and a_2 have the same length.

Since the cone angle associated to $\partial\Sigma$ is 2π we have :

$$\eta_1 + \eta_2 = 4\pi. \tag{5.4}$$

Let z_1, z_2 denote the complex numbers associated a_1 and a_2 respectively in a local chart of $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})$ constructed as in the proof of Theorem 2.2.7 for a neighborhood of (Σ, ξ). Assume that a_1 and a_2 are both oriented from p to q, we then have

$$z_1 - z_2 = 0. \tag{5.5}$$

Remark that the numbers z_1 and z_2 are obtained by a developing map, therefore, the angle between z_1 and z_2 is equal to the angle η_1 modulo 2π. Since both η_1, η_2 must be positive, it follows from (5.4) that $\eta_1 = \eta_2 = 2\pi$. Moreover, (5.5) also implies that $|a_1| = |a_2|$, therefore, we can glue the segments a_1, and a_2 together. We then get a flat torus with a marked geodesic segment, and the lemma follows. $\square$

By Lemma 5.2.1, we can identify $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})$ to the moduli space of triples (Σ, I, ξ) where Σ is a flat torus, I is a geodesic segment on Σ, and ξ is a normalized parallel vector field on Σ.

Now, let (Σ, I, ξ) be a triple in $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})$. Let $\psi_t, t \in \mathbb{R}^+$, denote the flow generated by ξ. Let p, q denote the endpoints of I. Let us prove the following lemma

Lemma 5.2.2 *There always exists a pair of parallel simple closed geodesic* γ_p, γ_q *of* Σ *such that* $\gamma_p \cap I = \{p\}$, *and* $\gamma_q \cap I = \{q\}$.

Proof: Assume that I is not parallel to ξ, and let t_0 be the infimum of the set

$$\{t > 0 : \psi_t(I) \cap I \neq \varnothing\}.$$

The value t_0 exists because the stripe which is swept out by $\{\psi_s(I) : 0 \leqslant s \leqslant t\}$ has area λt if $\psi_s(I) \cap I = \varnothing$, $\forall s \in [0, t]$, where $\lambda > 0$ is the transversal measure of I with respect to ξ.

By the definition of t_0, there exists an isometric immersion

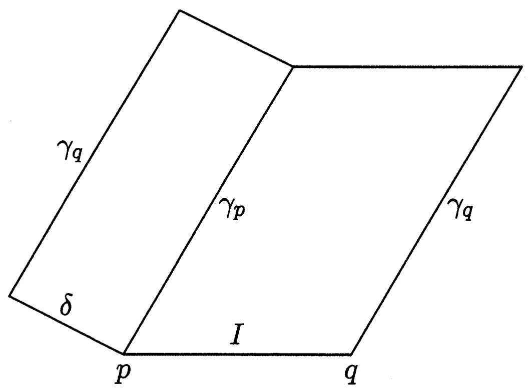
$$\varphi : \mathrm{P} \longrightarrow \Sigma,$$

which is defined on a closed parallelogram P in $\mathbb{R}^2$ with two vertical sides of length t_0, such that the restriction of φ onto $\mathrm{int}(\mathrm{P})$ is an embedding, and φ maps the lower side of P onto I, and the upper side of P onto $\psi_{t_0}(I)$.

Since the segments I and $\psi_{t_0}(I)$ are parallel and have the same length, the intersection set $I \cap \psi_{t_0}(I)$ contains at least one endpoint of I. Without loss of generality, we can assume that $p \in I \cap \psi_{t_0}(I)$. Consequently, $\varphi^{-1}(p)$ contains exactly two points, one in lower side, and the other in the upper side of P.

Let s be the geodesic segment in P joining two points in $\varphi^{-1}(p)$, then $\varphi(s)$ is a closed geodesic in Σ which intersects I at p. We choose γ_p to be $\varphi(s)$, and γ_q the closed geodesic parallel to γ_p which passes through q. By construction, γ_p, and γ_q verify the condition in the statement of the lemma.

In the case where I is parallel to ξ, it suffices to replace ξ by the normalized parallel vector field perpendicular to it, and use the same arguments. The lemma is then proved. □



The closed geodesic γ_p and γ_q cut Σ into two cylinders, the one which contains I will be denoted by C_1, the other one by C_2. Let δ be a geodesic segment joining p and q which is contained in C_2.

The complement in Σ of the set $I \cup \gamma_p \cup \gamma_q \cup \delta$ is the disjoined union of two open parallelograms. By an embedding of $\Sigma \setminus \{I \cup \gamma_p \cup \gamma_q \cup \delta\}$ into $\mathbb{R}^2$ which sends ξ onto the constant vertical vector field $(0, 1)$, we can associate the complex numbers Z, z, w to I, γ_p, and δ respectively. We can choose the orientation of I, γ_p, and δ so that :

$$\theta_1(Z, z, w) = \mathrm{Im}(Z\overline{z}) > 0 \text{ and } \theta_2(Z, z, w) = \mathrm{Im}(z\overline{w}) > 0.$$

Note that the area of the cylinder C_1 equals θ_1, and the area of the cylinder C_2 equals θ_2. Remark that, given (Z, z, w) in $\mathbb{C}^3$ verifying $\theta_1(Z, z, w) > 0$ and $\theta_2(Z, z, w) > 0$, one can construct a flat torus with a marked segment. Set

$$\mathcal{D} = \{(Z, z, w) \in \mathbb{C}^3 : \theta_1(Z, z, w) > 0, \theta_2(Z, z, w) > 0\}.$$

We then get a map :

$$\rho : \mathcal{D} \longrightarrow \mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\}),$$

which is onto and locally homeomorphic. The pull-back of the volume form μ_{Tr} on $\mathcal{D}$ is equal to $\kappa\lambda_6$, where λ_6 is the Lebesgue measure of $\mathbb{C}^3$, and κ is a constant. Clearly, the pull-back of the energy function $\mathcal{F}$ on $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})$ is the following function

$$\hat{\mathcal{F}}(Z, z, w) = \exp(-2|Z|^2 - (\theta_1(Z, z, w) + \theta_2(Z, z, w))).$$

We say that a triple (Σ, I, ξ) is in *special position* if either I is parallel to ξ, or the trajectory $\{\psi_t(p) : t \in \mathbb{R}^+\}$ returns to p without meeting any other point of I. Let $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})^{\mathrm{sp}}$ denote the set of triples in special position in $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})$.

Observe that the set $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})^{\mathrm{sp}}$ is of measure 0 with respect to μ_{Tr} as it is the image by ρ of the set

$$\{(Z, z, w) \in \mathcal{D} : \mathrm{Re}(Z) = 0 \text{ or } \mathrm{Re}(z) = 0\},$$

which is obviously of measure zero with respect to the Lebesgue measure λ_6.

Now, let (Σ, I, ξ) be a triple in $\mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\}) \setminus \mathcal{M}_{\mathrm{T}}(\varnothing; \{2\pi, 2\})^{\mathrm{sp}}$. Let (Z, x, w) be the complex numbers associated to I, γ_p, and δ as above. Set $A = \mathrm{Re}(Z), a = \mathrm{Re}(z), b = \mathrm{Re}(w)$ and $B = \mathrm{Im}(Z), x = \mathrm{Im}(z), y = \mathrm{Im}(w)$.

If the closed geodesic γ_p is chosen as in Lemma 5.2.2, then we have $|a| \leqslant |A|$. Remark that, since (Σ, I, ξ) is not in special position, we have $|a| > 0$. Because C_2 is a cylinder, we can choose the segment δ such that $|b| \leqslant |a|$. We deduce that the image by ρ of the set

$$\mathcal{D}_0 = \{(Z, z, w) \in \mathcal{D} : |A| \geqslant |a| \geqslant |b|\}$$

contains the set $\mathcal{M}_{\mathrm{T}}(\varnothing;\{2\pi,2\})\setminus\mathcal{M}_{\mathrm{T}}(\varnothing;\{2\pi,2\})^{\mathrm{sp}}$, and hence, the result of Theorem 5.1.1 for this case will follow from the following proposition :

Proposition 5.2.3 *We have*

$$\mathcal{I}=\int_{\mathcal{D}_0}\hat{\mathcal{F}}(Z,z,w)d\lambda_6=\int_{\mathcal{D}_0}\exp(-2(A^2+B^2)-(\theta_1+\theta_2))dAdBdadbdxdy<\infty.$$

Proof: By definition of the domain $\mathcal{D}_0$, we have

$$\mathcal{I}=\int\int\exp(-2(A^2+B^2))\times[\int_{-|A|}^{|A|}[\int_{-|a|}^{|a|}[\int\int\exp(-\theta_1-\theta_2)dxdy]db]da]dAdB.$$

Consider $\int\int\exp(-\theta_1-\theta_2)dxdy$ for fixed A,B,a,b. By definition we have :

$$\theta_1=Ba-Ax \text{ and } \theta_2=xb-ay.$$

Using the change of variables $(x,y)\longmapsto(\theta_1,\theta_2)$, we have $d\theta_1d\theta_2=|Aa|dxdy$. Since $\theta_1(Z,z,w)>0$, and $\theta_2(Z,z,w)>0$ for every $(Z,z,w)\in\mathcal{D}_0$, it follows

$$\int\int_{(Z,z,w)\in\mathcal{D}_0}\exp(-\theta_1-\theta_2)dxdy=\int_0^{+\infty}\int_0^{+\infty}\frac{e^{-\theta_1}e^{-\theta_2}}{|Aa|}d\theta_1d\theta_2=\frac{1}{|Aa|}.$$

Consequently

$$\mathcal{I}=\int\int\exp(-2A^2-2B^2)[\int_{-|A|}^{|A|}[\int_{-|a|}^{|a|}\frac{1}{|Aa|}db]da]dAdB=4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-2A^2}e^{-2B^2}dAdB=2\pi.$$

This proves the proposition, and hence, Theorem 5.1.1 is proved for the case of $\mathcal{M}_{\mathrm{T}}(\varnothing;\{2\pi,2\})$. $\square$

5.3 Proof of Theorem 5.1.1, Part a)

Let S be the base surface, and $\mathcal{V}$ be the finite subset of S as in Section 2.2. Let $\bar{\alpha}=(\alpha_1,\ldots,\alpha_n)$, and $\bar{\beta}=\{(\beta_1,s_1),\ldots,(\beta_m,s_m)\}$ be the data corresponding to S and $\mathcal{V}$. In this section, we will always assume that $m>0$, which means that the boundary of S is not empty.

Let $\mathcal{T}$ be a triangulation of S whose set of vertices is $\mathcal{V}$. Assume in addition that every edge of T which is contained in the interior of S belongs to the closures of two different triangles (*i.e.* no edges in the interior of S bound the same triangle on both sides). As usual let N_1, and N_2 denote the number of edges, and the number of triangles of $\mathcal{T}$. Set

$$K = \sum_{j=1}^{m} s_m.$$

Recall that we have

$$\dim_{\mathbb{C}} \mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta}) = 2g + n + m - 2 + K = N_1 - N_2.$$

Note that a point in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ is represented by a pair (Σ,ξ), where Σ is a translation surface with geodesic boundary homeomorphic to S, and ξ is a normalized parallel vector field on Σ.

5.3.1 Admissible matrix

Definition 5.3.1 *A matrix* $\mathbf{A}$ *in* $\mathrm{M}_{\mathbb{C}}(N_2, N_1)$ *is said to be* admissible, *if it has the following properties :*

- *Any entry of* $\mathbf{A}$ *belongs to the set* $\{-1, 0, 1\}$.
- *On any row of* $\mathbf{A}$, *there are exactly three non-zero entries.*
- *On any column of* $\mathbf{A}$, *there are either one or two non-zero entries. If a column has two non-zero entries, then one entry equals* 1, *the other equals* -1.

Note that if Σ is a translation surface in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})^*$, and T is an admissible triangulation of Σ, then the normalized matrix associated to T is admissible.

Given an admissible matrix $\mathbf{A}$, we will call *elementary moves* the following transformations of $\mathbf{A}$:

a) interchanging two columns,

b) interchanging two rows,

c) changing the sign of a column.

Two matrices ares said to be *equivalent* if one of them can be obtained from the other by elementary moves.

Remark: If $\mathbf{A}$ is the normalized matrix associated to a triangulation T of a translation surface in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})^*$, then the elementary moves $a), b), c)$ of $\mathbf{A}$ correspond respectively to a renumbering of the edges of T, a renumbering of triangles of T, and a change of orientation of an edge in T.

Let $\mathcal{AD}$ denote the set of equivalence classes of admissible matrices in $\mathbf{M}_{\mathbb{C}}(N_2, N_1)$, for each s in $\mathcal{AD}$, choose a representative $\mathbf{A}_s$ of s, we then get a finite family $\{\mathbf{A}_s,\ s \in \mathcal{AD}\}$.

Let V_s denote the kernel of the linear map from $\mathbb{C}^{N_1}$ onto $\mathbb{C}^{N_2}$ which is defined by the matrix $\mathbf{A}_s$ in the canonical basis of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$.

For any $Z \in \mathrm{V}_s$, let Σ_Z denote the 'surface' which is obtained by the construction described in the proof of Lemma 2.4.2. Let $\mathcal{U}_s$ denote the open subset of V_s, such that Σ_Z is a translation surface homeomorphic to S for any Z in $\mathcal{U}_s$. We define a map from $\mathcal{U}_s$ into $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$ as follows :

$$\begin{array}{rccc} \Phi_s : & \mathcal{U}_s & \longrightarrow & \mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta}) \\ & Z & \longmapsto & (\Sigma_Z, \xi) \end{array}$$

where ξ is the parallel vector field on Σ_Z which is induced by the vertical constant vector field $(0,1)$ of $\mathbb{R}^2$. From the proof of Theorem 2.2.7, we have

Proposition 5.3.2 *For every $s \in \mathcal{AD}$, $\Phi_s(\mathcal{U}_s)$ is an open in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$, and $\{\Phi_s(\mathcal{U}_s), s \in \mathcal{AD}\}$ is a finite open cover of $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})$.*

In the remaining of this section, for any $s \in \mathcal{AD}$, we will assume that, if $Z \in \mathbb{C}^{N_1}$ is a vector in $\mathcal{U}_s$, then the $K = \sum_{j=1}^{m} s_j$ first coordinates of Z correspond to the geodesic segments on the boundary of $\Phi_s(Z)$.

5.3.2 Primary and Auxiliary system of indices

Set

$$N = \dim \mathrm{V}_s = 2g + m + n - 2 + K.$$

Given an equivalence class s in $\mathcal{AD}$, let $(i_1, \dots, i_N)$ be an ordered subset of $\{1, \dots, N_1\}$.

Definition 5.3.3 *We say that $(i_1, \dots, i_N)$ is a* primary system of indices *associated to $\mathbf{A}_s$, if there exist N_1 complex linear functions*

$$f_i : \mathbb{C}^{N_1} \longrightarrow \mathbb{C},\ i = 1, \dots, N_1,$$

such that, if $Z = (z_1, \dots, z_{N_1}) \in \mathrm{V}_s$, then $z_i = f_i(z_{i_1}, \dots, z_{i_N})$.

Given a primary system of indices $(i_1, \dots, i_N)$ associated to $\mathbf{A}_s$, let $(j_K, \dots, j_N)$ be an ordered subset of $\{1, \dots, N_1\}$.

Definition 5.3.4 *We say that* $(j_K, \dots, j_N)$ *is an* auxiliary system *for* $(i_1, \dots, i_N)$ *if, for any* k *in* $\{K, \dots, N\}$, *we have*

i) The function f_{j_k} *depends only on* $z_{i_1}, \dots, z_{i_{k-1}}$.

ii) There is a row in $\mathbf{A}_s$ *whose* j_k*-th and* i_k*-th entries are non-zero.*

Remark: If $(j_K, \dots, j_N)$ is an auxiliary system for $(i_1, \dots, i_N)$, then for any $Z = (z_1, \dots, z_{N_1})$ in $\mathcal{U}_s$, we have

- z_{j_k} can be written as a linear function of $(z_{i_1}, \dots, z_{i_{k-1}})$, $\forall k = K, \dots, N$.
- Let $(\Sigma, \xi) = \Phi_s(Z)$, and let T be the geodesic triangulation of Σ which is obtained from the construction of Φ_s. Recall that each coordinate of Z is the complex number associated to an edge of T. The condition $ii)$ of 5.3.4 implies that z_{i_k} and z_{j_k} correspond to two sides of a triangle in T.

Clearly, the set of triples $(\mathbf{A}_s, (i_1, \dots, i_N), (j_K, \dots, j_N))$, with $s \in \mathcal{AD}$, $(i_1, \dots, i_N)$ a primary system for $\mathbf{A}_s$, and $(j_K, \dots, j_N)$ an auxiliary system for $(i_1, \dots, i_N)$ is finite.

5.3.3 Proof of (5.1)

Let (Σ, ξ) be a point in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, we denote ψ_t, $t \in \mathbb{R}$, the flow generated by ξ on Σ. Recall that on Σ, we have a specified finite subset V corresponding to the subset $\mathcal{V}$ of S, the complement of V contains only regular points of Σ. With a slight abuse of notation, we will call any point in V a singular point of Σ.

Let p be a point in $\mathrm{int}(\Sigma) \setminus V$, if there exists $t_0 > 0$ (resp. $t_0 < 0$) such that $\psi_{t_0}(p) \in V \cup \partial\Sigma$, then, for every $t > t_0$ (resp. $t < t_0$), we consider, by convention, that $\psi_t(p) = \psi_{t_0}(p)$. In other words, we consider that the flow ψ_t is stationary in the set $V \cup \partial\Sigma$. By this convention, $\psi_t(p)$ can be defined for every $t \in \mathbb{R}$,and $p \in \mathrm{int}(\Sigma) \setminus V$.

Let a be a geodesic segment contained in the boundary of Σ with endpoints in V. We can extend the field ξ by continuity onto $\mathrm{int}(a)$. Assume that a is not parallel to the field ξ, then we say that a is an *upper* (resp. *lower*) boundary segment, if the field ξ on $\mathrm{int}(a)$ points outward (resp. inward). Observe that in this case, the image of $\mathrm{int}(a)$ by ψ_t is well defined for all $t \in \mathbb{R}$.

We say that the pair (Σ, ξ) is in *special position* if there exists a geodesic segment on Σ with endpoints in V, and parallel to the field ξ. Let $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$ denote the subset of $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ consisting of pairs (Σ, ξ) which are in special position.

The formula (5.1) is the consequence of the following propositions.

Proposition 5.3.5 *The set* $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$ *is a null set in* $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$ *with respect to* μ_{Tr}.

Proof: For every s in $\mathcal{AD}$, let μ_s denote the volume form on $\mathcal{U}_s$ which is induced by the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ via the linear map $\mathbf{A}_s$. By definition, we have $\mu_s = \Phi_s^* \mu_{\mathrm{Tr}}$.

Let (Σ, ξ) be a pair in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$, let e be a geodesic segment of Σ with endpoint in V which is parallel to the field ξ. There exists an admissible triangulation T of Σ which contains the edge e.

Since e is parallel to ξ, the complex number associated to e in the local chart arising from T is purely real. As a consequence, there exist $s \in \mathcal{AD}$, and $i_0 \in \{1, \dots, N_1\}$ such that $(\Sigma, \xi) = \Phi_s(Z)$, with $Z \in \{(z_1, \dots, z_{N_1}) \in \mathcal{U}_s \ \mathrm{Im}(z_{i_0}) = 0\}$. Remark that the converse assertion is also true.

For every $s \in \mathcal{AD}$, and every $i \in \{1, \dots, N_1\}$, set

$$\mathcal{U}_s^i = \mathcal{U}_s \cap \{(z_1, \dots, z_{N_1}) \in \mathbb{C}^{N_1} \mid \mathrm{Im}(z_i) = 0\}).$$

It follows that

$$\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}} = \bigcup_{s \in \mathcal{AD}} \bigcup_{i=1}^{N_1} \Phi_s(\mathcal{U}_s^i).$$

Clearly, $\mu_s(\mathcal{U}_s^i) = 0, \ \forall s \in \mathcal{AD}, i \in \{1, \dots, N_1\}$, therefore, $\mu_{\mathrm{Tr}}(\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}) = 0$. □

Let (Σ, ξ) be a point in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})$, and T an admissible triangulation of Σ. Let e be an edge of T, we denote $h(e)$ the transversal measure of e with respect to ξ. If we choose an isometric embedding of a neighborhood of e into $\mathbb{R}^2$ such that the vector field ξ is mapped to the constant vertical vector field $(0, 1)$ of $\mathbb{R}^2$, then $h(e)$ is nothing other than the length of the projection into the horizontal axis of the image of e. We call $h(e)$ the horizontal length of e.

A triangle in T whose sides are denoted by e_1, e_2, e_3 is said to be *good* if $h(e_i) > 0, \ \forall i = 1, 2, 3$. Given a good triangle Δ in $\mathbb{R}^2$, we call the unique side of Δ of maximal horizontal length the *base* of Δ. If all of triangles of T are good, T is called a *good triangulation.*

Proposition 5.3.6 *Let* (Σ, ξ) *be a point in* $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta}) \setminus \mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$, *then there exists a good triangulation* T *of* Σ *whose edges are denoted by* $\{e_1, \dots, e_{N_1}\}$ *so that,*

- *The boundary edges of* T *are denoted by* $\{e_1, \dots, e_K\}$.
- *For every* $i \in \{K+1, \dots, N_1\}$, *there exists* $j < i$, *and a triangle* Δ *of* T *which contains both* e_i, e_j *such that* e_j *is the base of* Δ.

Proof: As usual, let V denote the set of distinguished singularities of Σ. We define an admissible triangulation of Σ as follows :

Let $e_1, \dots, e_K$ denote the (closed) geodesic segments with endpoints in V, which are contained in the boundary of Σ. Assume that the segment e_K is of maximal horizontal length among the set $\{e_1, \dots, e_K\}$. Since (Σ, ξ) is not in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$, we have $h(e_K) > 0$. Let p, q denote the two endpoints of e_K. Consider the following procedure :

Assume that e_K is a lower boundary segment. Consider the stripe swept by $\{\psi_t(\mathrm{int}(e_K)),\ t > 0\}$. Since $h(e_K) > 0$, this stripe must meet a singular point in the interior of Σ, or the boundary of Σ, otherwise its area would tend to infinity as t tens to $+\infty$. Remark that, since the horizontal length of e_K is maximal among the set $\{h(e_1), \dots, h(e_K)\}$, for every $t \in \mathbb{R}^+$, $\psi_t(\mathrm{int}(e_K))$ cannot be contained in a geodesic segment (with endpoints in V) in the boundary of Σ. Therefore, there exists $t > 0$ such that $\psi_t(\mathrm{int}(e_K)) \cap V \neq \varnothing$.

Let t_0 be the smallest value of t such that $t_0 > 0$, and $\psi_{t_0}(\mathrm{int}(e_K)) \cap V \neq \varnothing$. Let r denote a point in $\psi_{t_0}(e_K) \cap V$. Let e' and e'' denote the two geodesic segments contained in the stripe $\cup_{0 \leqslant t \leqslant t_0} \psi_t(e_K)$ which join r to p, and to q. It can happen that one of the edge e', e'' is already contained in the boundary of Σ but not both of them, unless Σ is a triangle. We will call e_K the *supporter* of e' and e''.

By construction, we have $h(e_K) \geqslant \max\{h(e'), h(e'')\}$. Clearly, the triangle bounded by e_K, e', e'' is embedded in Σ and e_K is the base of this triangle. Since (Σ, ξ) is not in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$, neither e' nor e'' is parallel to ξ.

In the case where e_K is an upper boundary segment, by considering $\{\psi_t(\mathrm{int}(e_K)),\ t < 0\}$ instead of $\{\psi_t(\mathrm{int}(e_K)),\ t > 0\}$, we get a similar result.

Cut off the triangle bounded by e_K, e', e'' from the surface Σ along the segments e' and e''. The remaining surface is a translation surface with geodesic boundary, which is not necessarily connected.

We can now reapply the same action to the new surface. The assumption that (Σ, ξ) is not in special position allows us to continue until we get a triangulation T of Σ, which is clearly a good triangulation.

We number the edges of T which are contained in the interior of Σ according to their appearing order in the procedure above, the ordering of two edges which appear in the same step is not important. Since every edge of T in the interior of Σ admits a supporter which appears in the procedure before itself, the proposition is then proved. □

Corollary 5.3.7 *If (Σ, ξ) is a point in $\mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta}) \setminus \mathcal{M}_{\mathrm{T}}(\bar{\alpha}; \bar{\beta})^{\mathrm{sp}}$, then there exists an $s \in \mathcal{AD}$, a primary system of indices $\mathrm{Pr} = (i_1, \dots, i_N)$ for $\mathbf{A}_s$, an auxiliary system of indices $\mathrm{Au} = (j_K, \dots, j_N)$ for Pr, and a vector $Z \in \mathcal{U}_s$ such that*

- $|\mathrm{Re}(z_{j_k})| > |\mathrm{Re}(z_{i_k})|$ *for any $k = K, \dots, N$.*
- $(\Sigma, \xi) = \Phi_s(Z)$.

Proof: Let T be the good triangulation of Σ which is obtained from Proposition 5.3.6. Let $\mathbf{A}_{\mathrm{T}}$ be the matrix in $\mathrm{M}_{\mathbb{Z}}(N_2, N_1)$ associated to T, let $Z = (z_1, \dots, z_{N_1})$ be the vector in $\ker \mathbf{A}_{\mathrm{T}}$ whose coordinates are complex numbers associated to edges of T. We can assume that z_i is the complex number associated to e_i.

We choose a primary system of indices Pr and an auxiliary system of indices Au for $\mathbf{A}_{\mathrm{T}}$ as follows :

- The first $K-1$ elements of Pr are $\{1, \dots, K-1\}$.
- Assume that we have chosen k indices $(i_1, \dots, i_k)$ for Pr, and $k+1-K$ indices $(j_K, \dots, j_k)$ for Au. The index i_{k+1} of Pr is the smallest index i such that z_i can not be written as a linear function of $z_{i_1}, \dots, z_{i_k}$, and the index j_{k+1} of Au is the index such that $e_{j_{k+1}}$ is a supporter of $e_{i_{k+1}}$, and $j_{k+1} < i_{k+1}$. From Proposition 5.3.6, j_{k+1} exists, and by assumption, $z_{j_{k+1}}$ is a linear function of $(z_{i_1}, \dots, z_{i_k})$.

By this procedure, we obtain a primary system of indices $(i_1, \dots, i_N)$, and an auxiliary system of indices $(j_K, \dots, j_N)$ associated to $\mathbf{A}_{\mathrm{T}}$. Since for any $k = K, \dots, N$, e_{j_k} is the supporter of e_{i_k}, it follows that

$$|\mathrm{Re}(z_{j_k})| = h(e_{j_k}) > h(e_{i_k}) = |\mathrm{Re}(z_{i_k})|.$$

We know that $\mathbf{A}_{\mathrm{T}}$ is equivalent to a matrix $\mathbf{A}_s$ with s in $\mathcal{AD}$. The transformation of $\mathbf{A}_{\mathrm{T}}$ into $\mathbf{A}_s$ consists of renumbering the coordinates in $\mathbb{C}^{N_1}$, changing their sign. By this transformation, $(i_1, \dots, i_N)$ and $(j_K, \dots, j_N)$, become a primary system and an auxiliary system of indices for $\mathbf{A}_s$, and the vector Z becomes a vector in $\mathcal{U}_s$ which verifies the condition in the statement of the corollary. □

From now on, we call a triple $(\mathbf{A}_s; I; J)$, with $s \in \mathcal{AD}$, $I = (i_1, \dots, i_N)$ a primary system of indices of $\mathbf{A}_s$, and $J = (j_K, \dots, j_N)$ an auxiliary system for I, an *admissible triple*. Given such a triple, set

$$\mathcal{U}_s(I;J) = \{(z_1,\dots,z_{N_1}) \in \mathcal{U}_s \mid |\mathrm{Re}(z_{i_k})| \leqslant |\mathrm{Re}(z_{j_k})|,\ \forall k = K,\dots,N\}.$$

From Corollary 5.3.7, we deduce that the family

$$\{\Phi_s(\mathcal{U}_s(I;J)) |\ (\mathbf{A}_s; I; J) \text{ is admissible }\}$$

covers the set $\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta}) \setminus \mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})^{\mathrm{sp}}$. Since $\mu_{\mathrm{Tr}}(\mathcal{M}_{\mathrm{T}}(\bar{\alpha};\bar{\beta})^{\mathrm{sp}}) = 0$, to prove (5.1), all we need is the following :

Proposition 5.3.8 *Let $(\mathbf{A}_s; I; J)$, where $I = (i_1,\dots,i_N), J = (j_K,\dots,j_N)$, be an admissible triple. Let $\mathcal{F}_s$ denote the pull back of the energy function $\mathcal{F}$ onto $\mathcal{U}_s$ by Φ_s. Then we have :*

$$\int_{\mathcal{U}_s(I;J)} \mathcal{F}_s d\mu_s < \infty,$$

where μ_s is the volume form on $\mathcal{U}_s$ which is induced by the Lebesgue measures of $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$ via the linear map $\mathbf{A}_s$.

Proof: By definition, there are N_1 complex linear functions with real coefficients $f_1,\dots,f_{N_1}$ such that, if $(z_1,\dots,z_{N_1}) \in \mathcal{U}_s$, then $z_i = f_i(z_{i_1},\dots,z_{i_N})$. Note that $f_{i_k} = z_{i_k}$, therefore, we can define a complex linear map

$$\begin{array}{rccc} \mathbf{B}_s : & \mathbb{C}^N & \longrightarrow & \ker \mathbf{A}_s \\ & (z_1,\dots,z_N) & \longmapsto & (f_1(z_1,\dots,z_N),\dots,f_{N_1}(z_1,\dots,z_N)) \end{array}$$

Observe that $\mathbf{B}_s$ is an isomorphism. By definition, we have

$$\mathbf{B}_s^{-1}(\mathcal{U}_s(I;J)) = \{(z_1,\dots,z_N) \in \mathbb{C}^N \mid |\mathrm{Re}(z_k)| \leqslant |\mathrm{Re}(f_{j_k}(z_1,\dots,z_N))|,\ \forall k = K,\dots,N\}.$$

Consider a vector $Z = (z_1,\dots,z_{N_1})$ in $\mathcal{U}_s$, let (Σ,ξ) denote the image of Z by Φ_s. Recall that the map Ψ_s specifies an admissible triangulation T of Σ such that each edge of T corresponds to a coordinate of Z.

By the definition, for any $k = K,\dots,N$, the complex numbers z_{i_k} and z_{j_k} correspond to two edges e_{i_k}, and e_{j_k} which bound a triangle Δ_k of T. With appropriate choices of orientation of e_{i_k}, and e_{j_k}, the area $\hat{\theta}_k$ of Δ_k is given by the function

$$\hat{\theta}_k = \frac{1}{2}(\mathrm{Re}(z_{i_k})\mathrm{Im}(z_{j_k}) - \mathrm{Im}(z_{i_k})\mathrm{Re}(z_{j_k})).$$

Clearly, the triangles $\Delta_k,\ k = K,\dots,N$, are all distinct. Hence, we have

$$\mathbf{Area}(\Sigma) \geqslant \sum_{k=K}^{N} \hat{\theta}_k.$$

Let $\theta_k,\ k = K, \ldots, N$, denote the pull back of the function $\hat{\theta}_k$ by $\mathbf{B}_s$. It follows that $\mathbf{B}_s^{-1}((\mathcal{U}_s(I;J))$ is a subset of a set $\mathcal{W}_s$ where

$$\mathcal{W}_s = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid |\mathrm{Re}(z_k)| \leqslant |\mathrm{Re}(f_{j_k})|,\ \theta_k > 0,\ \forall k = K, \ldots, N\}.$$

Let $\mathcal{G}_s$ denote the pull back of $\mathcal{F}_s$ by $\mathbf{B}_s$. Since the volume form $\mathbf{B}_s^*\mu_s$ equals to $\kappa\lambda_{2N}$, where λ_{2N} is the Lebesgue measure of $\mathbb{C}^N$, and κ is a constant, all we need to show is the following

Lemma 5.3.9 *We have*

$$\int_{\mathcal{W}_s} \mathcal{G}_s d\lambda_{2N} < \infty.$$

Proof: Let $(z_1, \ldots, z_N)$ be a vector in $\mathcal{W}_s$, and (Σ, ξ) be the image of $(z_1, \ldots, z_N)$ by $\Phi_s \circ \mathbf{B}_s$. We can assume that $(z_1, \ldots, z_{K-1})$ are complex numbers associated to geodesic segments in the boundary of Σ.

To simplify the notations, for $k = 1, \ldots, N$, set $x_k = \mathrm{Re}(z_k), y_k = \mathrm{Im}(z_k)$. For $k = K, \ldots, N$, we write f_k in the place of f_{j_k}, and set $a_k = \mathrm{Re}(f_k), b_k = \mathrm{Im}(f_k)$. Recall that, by definition, f_k depends only on $(z_1, \ldots, z_{k-1})$, and since f_k is a linear function with real coefficients, it follows that a_k depends only on $(x_1, \ldots, x_{k-1})$, and b_k depends only on $(y_1, \ldots, y_{k-1})$, for any $k = K, \ldots, N_1$. With these notations, we have

$$\ell^2(\partial\Sigma) \geqslant \sum_{k=1}^{K-1} |z_k|^2, \tag{5.6}$$

$$\theta_k = \frac{1}{2}(x_k b_k - y_k a_k),\ \forall k = K, \ldots, N, \tag{5.7}$$

$$|a_k| \geqslant |x_k|,\ \forall k = K, \ldots, N. \tag{5.8}$$

$$\mathbf{Area}(\Sigma) \geqslant \sum_{k=K}^{N} \theta_k. \tag{5.9}$$

Consequently, we have

$$\mathcal{G}_s \leqslant \exp(-\sum_{k=1}^{K-1} |z_k|^2 - \sum_{k=K}^{N} \theta_k).$$

Therefore, to prove the proposition, it suffices to show that

$$\mathcal{I} = \int_{\mathcal{W}_s} \exp(-\sum_{k=1}^{K-1} |z_k|^2 - \sum_{k=K}^{N} \theta_k) d\lambda_{2N} < \infty. \tag{5.10}$$

Fix $(z_1, \ldots, z_{K-1}) \in \mathbb{C}^{K-1}$ and $(x_K, \ldots, x_N) \in \mathbb{R}^{N-K+1}$, and let

$$\mathcal{W}_s((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N))$$

denote the set

$$\{(y_K, \ldots, y_N) \in \mathbb{R}^{N-K+1} | \, (z_1, \ldots, z_{K-1}, (x_K + \imath y_K), \ldots, (x_N + \imath y_N)) \in \mathcal{W}_s\}.$$

Consider the following integral

$$\mathcal{I}((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) = \int_{\mathcal{W}_s((z_1,\ldots,z_{K-1});(x_K,\ldots,x_N))} \exp(-\sum_{k=K}^{N} \theta_k) dy_K \ldots dy_N.$$

Consider the variable change $(y_K, \ldots, y_N) \longmapsto (\theta_K, \ldots, \theta_N)$. Using (5.7), and the fact that b_k depends only on $(y_1, \ldots, y_{k-1})$, for any $k = K, \ldots, N$, we have :

$$d\theta_K \ldots d\theta_N = \frac{|a_K \ldots a_N|}{2^{N-K+1}} dy_K \ldots dy_N.$$

Since the functions $\theta_k, \; k = K, \ldots, N$, are positive on $\mathcal{W}_s$, it follows

$$\begin{aligned} \mathcal{I}((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) &\leqslant \frac{2^{N-K+1}}{|a_K \ldots a_N|} \int_0^{+\infty} e^{-\theta_K} d\theta_K \ldots \int_0^{+\infty} e^{-\theta_N} d\theta_N \\ &\leqslant \frac{2^{N-K+1}}{|a_K \ldots a_N|}. \end{aligned}$$

Now, set

$$\mathcal{W}_s^* = \{((z_1, \ldots, z_{K-1}); (x_K, \ldots, x_N)) \in \mathbb{C}^{K-1} \times \mathbb{R}^{N-K+1} \mid |a_k| \geqslant |x_k|, \; \forall k = K, \ldots, N\}.$$

We have

$$
\begin{aligned}
\mathcal{I} &= \int_{\mathcal{W}_s^*} \exp(-\sum_{k=1}^{K-1} |z_k|^2)\mathcal{I}((z_1,\dots,z_{K-1});(x_K,\dots,x_N))dx_1dy_1\dots dx_{K-1}dy_{K-1}dx_K\dots dx_N,\\
&\leqslant \int_{\mathcal{W}_s^*} \exp(-\sum_{k=1}^{K-1} |z_k|^2)\frac{2^{N-K+1}}{|a_K\dots a_N|}dx_1dy_1\dots dx_{K-1}dy_{K-1}dx_K\dots dx_N,\\
&\leqslant \int_{\mathbb{C}^{K-1}} \exp(-\sum_{k=1}^{K-1} |z_k|^2)[\int_{-|a_K|}^{|a_K|}[\dots[\int_{-|a_N|}^{|a_N|}\frac{2^{N-K+1}}{|a_K\dots a_N|}dx_N]\dots]dx_K]dx_1dy_1\dots dx_{K-1}dy_{K-1},
\end{aligned}
$$

Using the fact that a_k does not depend on x_j if $k \leqslant j$, $\forall k = K,\dots,N$, we deduce that

$$\mathcal{I} \leqslant 4^{N-K+1}\int_{\mathbb{C}^{K-1}} e^{-(|z_1|^2+\dots+|z_{K-1}|^2)}dx_1dy_1\dots dx_{K-1}dy_{K-1} < \infty.$$

The lemma is then proved. □

The proof of Proposition 5.3.8 is now complete, and (5.1) follows. □

5.4 Proof of Theorem 5.1.1, Part b)

The proof of (5.2) is essentially the same as the proof of (5.1) with some minor modifications.

Assume that the forest $\hat{\mathcal{A}}$ contains m trees denoted by $\mathcal{A}_1,\dots,\mathcal{A}_m$, and the vertices of those trees are $\{p_1,\dots,p_n\}$. Through out this section, we assume that $m < n$, which means that there is at least a tree in $\hat{\mathcal{A}}$ which is not a point, in the sequel, such a tree is called *non-trivial*. Note that the total number of edges of the tree in $\hat{\mathcal{A}}$ is $n-m$. Recall that we have

$$\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha}) = N = \begin{cases} 2g+n-1, & \text{if } \alpha_i \in 2\pi\mathbb{N},\ \forall i=1,\dots,n;\\ 2g+n-2, & \text{otherwise.}\end{cases}$$

A point in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ is a triple $(\Sigma,\hat{A},\xi)$, where Σ is a flat surface homeomorphic to S_g, $\hat{A}$ is an erasing forest isomorphic to $\hat{\mathcal{A}}$, and ξ is a normalized parallel vector field on Σ.

Choose a triple $(\Sigma,\hat{A},\xi)$ in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$, let $\Sigma^\natural$ be the translation surface with boundary obtained by slitting open Σ along the trees in the forest $\hat{A}$. Let T be an admissible triangulation of $\Sigma^\natural$, and let N_1, N_2 denote the number of edges, and the number of triangles in T respectively.

In Section 3.4, we have seen that one can associate to T a system $\mathrm{S}_{\mathrm{T}}^*$ of N_1 unknowns which contains :

- N_2 equations of type (2.3), which will be called *triangle equations*;
- $(n-m)$ equations of type (3.1), which will be called *boundary equations.*

Note that the boundary equations of $\mathrm{S}_{\mathrm{T}}^*$ are determined by the forest $\hat{\mathcal{A}}$, and the angles in $\bar{\alpha}$.

Set $N_2^* = N_2 + (n-m)$. Recall that a matrix is called *normalized* if each of its entries is either 0, or a complex number of module 1. We can now define

Definition 5.4.1 *Let* $\mathbf{A}$ *be a matrix in* $\mathrm{M}_{\mathbb{C}}(N_2^*, N_1)$. *We say that* $\mathbf{A}$ *is* $*$-admissible *if*

i) $\mathbf{A}$ *is normalized.*

ii) Every column of $\mathbf{A}$ *contains exactly two non-zero entries.*

iii) There are N_2 *rows of* $\mathbf{A}$ *which form an admissible matrix defined in Definition 5.3.1. These rows will be called* ordinary.

iv) There exists a bijection from a set of $(n-m)$ *rows of* $\mathbf{A}$ *onto the set of boundary equations of* $\mathrm{S}_{\mathrm{T}}^*$ *such that, each of these rows is the vector of coefficients of the corresponding equation in* $\mathrm{S}_{\mathrm{T}}^*$. *These rows of* $\mathbf{A}$ *will be called* exceptional.

By definition, if $\mathbf{A}_{\mathrm{T}}^*$ is the matrix in $\mathrm{M}_{\mathbb{C}}(N_2^*, N_1)$ associated to the system $\mathrm{S}_{\mathrm{T}}^*$, then $\mathbf{A}_{\mathrm{T}}^*$ is $*$-admissible.

Given a $*$-admissible matrix $\mathbf{A}$, the following transformations of $\mathbf{A}$ will be called *elementary moves*

- interchanging two columns,
- interchanging two rows,
- changing sign of a columns,

Two $*$-admissible matrices are said to be *equivalent*, if one can be obtained from the other by a sequence of elementary moves. Let $\mathcal{AD}^*$ denote the set of equivalence classes of matrices in $\mathrm{M}_{\mathbb{C}}(N_2^*, N_1)$.

For each s in $\mathcal{AD}^*$, choose a matrix $\mathbf{A}_s^*$ in the equivalence class s, we then get a finite family $\{\mathbf{A}_s^*,\ s \in \mathcal{AD}^*\}$ of $*$-admissible matrices in $\mathrm{M}_{\mathbb{C}}(N_2^*, N_1)$.

Given s in $\mathcal{AD}^*$, for any $Z \in \ker \mathbf{A}_s^*$, let Σ_Z be the 'surface' obtained from Z by the construction described in Lemma 3.4.5. Let $\mathcal{U}_s^*$ be the open subset of $\ker \mathbf{A}_s^*$ which is defined by the condition :

$$\mathcal{U}_s^* = \{Z \in \ker \mathbf{A}_s^* \ : \ \Sigma_Z \text{ is a flat surface homeomorphic to } S_g\}.$$

We can then define a map Φ_s^* from $\mathcal{U}_s^*$ into $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ by associating to a vector Z in $\mathcal{U}_s^*$ the triple $(\Sigma_Z, \hat{A}, \xi)$, where $\hat{A}$ is the forest obtained from the exceptional rows in $\mathbf{A}_s^*$, and ξ is the vector field induced from the vertical constant vector field $(0,1)$ of $\mathbb{R}^2$.

From Lemma 3.4.5, the following proposition is clear,

Proposition 5.4.2 *The family $\{\Phi_s^*(\mathcal{U}_s^*), s \in \mathcal{AD}^*\}$ is an open cover of the space $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}}, \bar{\alpha})$.*

Let us now define the notions of primary and auxiliary system of indices for a matrix $\mathbf{A}_s^*$, $s \in \mathcal{AD}^*$. Set

$$K = \begin{cases} n - m + 1, & \text{if } N = 2g + n - 1\,; \\ n - m, & \text{if } N = 2g + n - 2. \end{cases}$$

Definition 5.4.3 *Given a matrix $\mathbf{A}_s^*$, a* primary system of indices *for $\mathbf{A}_s^*$ is an ordered subset $(i_1, \dots, i_N)$ of $(1, \dots, N_1)$ such that there exist N_1 complex linear functions*

$$f_i : \mathbb{C}^N \longrightarrow \mathbb{C}, \ i = 1, \dots, N_1,$$

such that if $Z = (z_1, \dots, z_{N_1})$ is a vector in $\ker \mathbf{A}_s^*$ *then*

- $z_i = f_i(z_{i_1}, \dots, z_{i_N}),\ \forall i = 1, \dots, N_1.$
- $\forall i = 1, \dots, N_1, \forall k = K, \dots, N$, *the coefficient of z_{i_k} in $f_i(z_{i_1}, \dots, z_N)$ is real.*

Definition 5.4.4 *Given a primary system of indices $I = (i_1, \dots, i_N)$ for $\mathbf{A}_s^*$, an* auxiliary system of indices *for I is an ordered subset $(j_K, \dots, j_N)$ of $\{1, \dots, N_1)$ such that*
- *f_{j_k} depends only on $(z_{i_1}, \dots, z_{i_{k-1}})$;*
- *There exists an ordinary row in $\mathbf{A}_s^*$ whose i_k-th and j_k-th entries are both non-zero.*

Remark: There is a natural way to specify a primary system of indices of $\mathbf{A}_s^*$ as follows : let $\mathbf{A}_s$ be the admissible matrix consisting of the ordinary rows of $\mathbf{A}_s^*$, and let $\tilde{I} = (i_1, \dots, i_{\tilde{N}})$ be a primary system of indices for $\mathbf{A}_s$.

If the i-th column of $\mathbf{A}_s$ has only one non-zero entry, we say that i is a boundary index. Two boundary indices i_1 and i_2 are said to be paired up, if there exists an exceptional row in $\mathbf{A}^*_s$ whose i_1-th and i_2-th entries are non-zero whereas all other entries are zero. By construction, there are $(n-m)$ pairs of boundary indices, they correspond to the edges of the trees in the forest $\hat{\mathcal{A}}$, therefore there are exactly $2(n-m)-1$ boundary indices in the family $\tilde{I}$.

Assume that $(i_1,\dots,i_{2(n-m)-1})$ is the set of boundary indices in $\tilde{I}$, we have two issues :

- If $N = 2g+n-1$, that is $\alpha_i \in 2\pi\mathbb{N}$, $\forall i = 1,\dots,n$, we have $\tilde{N} = N + (n-m) - 1$. In this case, by eliminating one boundary index in each pair if both indices of this pair appear in $\{i_1,\dots,i_{2(n-m)-1}\}$, we obtain a primary system of indices for $\mathbf{A}^*_s$.
- If $N = 2g+n-2$, that is there exists $i \in \{1,\dots,n\}$ such that $\alpha_i \not\in 2\pi\mathbb{N}$, we have $\tilde{N} = N+(n-m)$. In this case, to obtain a primary system for $\mathbf{A}^*_s$, we have to eliminate $(n-m)$ indices from $(i_1,\dots,i_{2(n-m)-1})$ so that any two indices in the remaining family are not paired up.

Let I denote the primary system for $\mathbf{A}^*_s$ which is obtained from $\tilde{I}$ by this method without changing the ordering, observe that an auxiliary system for $\tilde{I}$ is also an auxiliary system for I.

Finally, we say that a triple $(\Sigma;\hat{\mathcal{A}};\xi) \in \mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ is *in special position*, if the pair $(\Sigma^{\natural},\xi)$ is in special position as defined in Section 5.3, where $\Sigma^{\natural}$ is the translation surface with boundary obtained by slitting open Σ along the trees in $\hat{\mathcal{A}}$. Let $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})^{\mathrm{sp}}$ denote the set of triples in special position in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$. With these settings, we have

Proposition 5.4.5 *The set $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})^{\mathrm{sp}}$ is of measure zero with respect to μ_{Tr}.*

Proposition 5.4.6 *For any triple $(\Sigma,\hat{\mathcal{A}},\xi)$ in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ which is not in special position, there exist an $s \in \mathcal{AD}^*$, a primary system of indices $I = (i_1,\dots,i_N)$ for $\mathbf{A}^*_s$, an auxiliary system of indices $J = (j_K,\dots,j_N)$ for I, and a vector $Z \in \mathcal{U}^*_s$ such that*

- *for $k = K,\dots,N$, $|\mathrm{Re}(z_{i_k})| < |\mathrm{Re}(z_{j_k})|$.*
- $\Phi^*_s(Z) = (\Sigma,\hat{\mathcal{A}},\xi)$.

We call a triple $(\mathbf{A}^*_s;I;J)$, with s in $\mathcal{AD}^*$, I a primary system of indices for $\mathbf{A}^*_s$, and J an auxiliary system of indices for I, an $*$-*admissible triple*.

Given an $*$-admissible triple $(\mathbf{A}^*_s;I;J)$, with $I = (i_1,\dots,i_N)$, $J = (j_K,\dots,j_N)$, set

$$\mathcal{U}_s^*(I;J) = \{(z_1,\dots,z_{N_1}) \in \mathcal{U}_s^* \mid |\mathrm{Re}(z_{i_k})| \leqslant |\mathrm{Re}(z_{j_k})|,\ \forall k = K,\dots,N\}.$$

Let $\mathcal{F}_s^{\mathrm{et}}$ denote the pull back of the energy function $\mathcal{F}^{\mathrm{et}}$ by Φ_s^* onto $\mathcal{U}_s^*$.

Proposition 5.4.7 *We have*

$$\int_{\mathcal{U}_s^*(I;J)} \mathcal{F}_s^{\mathrm{et}} d\mu_s < \infty,$$

where μ_s is the volume form on $\mathcal{U}_s^*$ which is induced by the Lebesgue measure of $\mathbb{C}^{N_1}$, and the Lebesgue measure of either $\mathbb{C}^{N_2^*}$, or $\mathbf{W} = \{(z_1,\dots,z_{N_2^*}) \in \mathbb{C}^{N_2^*} \mid z_1 + \dots + z_{N_2^*} = 0\}$ via $\mathbf{A}_s^*$.

The proofs of Propositions 5.4.5, 5.4.6, and 5.4.7 will be omitted since they are completely analogue to the proofs of Proposition 5.3.5, Corollary 5.3.7, and Proposition 5.3.8.

Part b) of Theorem 5.1.1 follows directly from these propositions. □

5.5 Volume of moduli spaces of closed translation surfaces of constant area is finite

In this section, we use Theorem 5.1.1 to prove the well-known fact that the volume of $\mathcal{H}_1(k_1,\dots,k_n)$ is finite. Recall that $\mathcal{H}(k_1,\dots,k_n)$ can be considered as the moduli space of translation surfaces (with parallel vector field) having cone angles $2(k_1+1)\pi,\dots,2(k_n+1)\pi$ at singularities, and $\mathcal{H}_1(k_1,\dots,k_n)$ is the subspace of $\mathcal{H}(k_1,\dots,k_n)$ which contains all surfaces of area one.

On $\mathcal{H}(k_1,\dots,k_n)$, we have a volume form μ_0 which is defined by the period mapping. Let μ_0^1 denote the volume form on $\mathcal{H}_1(k_1,\dots,k_n)$ which is induced by μ_0. Our goal in this section is to prove that

$$\mu_0^1(\mathcal{H}_1(k_1,\dots,k_n)) < \infty. \tag{5.11}$$

First, we remark that (5.11) is equivalent to

$$\int_{\mathcal{H}(k_1,\dots,k_n)} \exp(-\mathbf{Area}) d\mu_0 < \infty.$$

This is because we can identify $\mathcal{H}(k_1,\dots,k_n)$ to $\mathcal{H}_1(k_1,\dots,k_n) \times \mathbb{R}_+^*$, and by this identification, we can write

$$d\mu_0 = t^s d\mu_0^1 dt, \text{ where } s = \dim_{\mathbb{R}} \mathcal{H}_1(k_1,\dots,k_n).$$

Therefore, we have

$$\begin{aligned}\int_{\mathcal{H}(k_1,\dots,k_n)} \exp(-\mathbf{Area})d\mu_0 &= \int_{\mathcal{H}_1(k_1,\dots,k_n)}\int_0^{+\infty} t^s e^{-t^2} dt d\mu_0^1,\\ &= \frac{1}{2}(\frac{s-1}{2})! \int_{\mathcal{H}_1(k_1,\dots,k_n)} d\mu_0^1.\end{aligned}$$

Consequently, all we need to prove is the following

Proposition 5.5.1 *We have*

$$\int_{\mathcal{H}(k_1,\dots,k_n)} \exp(-\mathbf{Area})d\mu_0 < \infty \tag{5.12}$$

Proof: At first glance, it seems that this proposition is a direct consequence of Theorem 5.1.1, Part a), but, unfortunately, the arguments used in the proof of 5.1.1 cannot work without the assumption that the boundary of the surfaces considered is not empty. To overcome this misfortune we will make use of (5.2) in a particular case.

Set $\alpha_i = 2(k_i+1),\ i = 1,\dots,n$. Let $\mathcal{A}_1$ be a topological tree isomorphic to a segment, and for $i = 2,\dots,n$, let $\mathcal{A}_i$ be just a point. Let $\bar{\alpha}$ denote $(2\pi, \alpha_1,\dots,\alpha_n)$, and $\hat{\mathcal{A}}$ denote the family $\{\mathcal{A}_1,\dots,\mathcal{A}_n\}$.

Consider the space $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ with the previous data. In this case, $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ is the moduli space of triples $(\Sigma; (I(x_1,x), x_2,\dots,x_n;\xi)$, where Σ is a closed translation surface, $\{x_1,\dots,x_n\}$ is the set of singularities of Σ with cone angles $\{\alpha_1,\dots,\alpha_n\}$ respectively, and $I(x_1,x)$ is a geodesic segment joining the singular point x_1 to a regular point x.

Let $\tilde{\alpha}$ denote the sequence $\{\alpha_1,\dots,\alpha_n\}$, and let $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$ denote the moduli space of triples $(\Sigma; x_1,\dots,x_n;\xi)$, where Σ is a closed translation surface, $\{x_1,\dots,x_n\}$ is the ordered set of singularities of Σ with cone angles $\{\alpha_1,\dots,\alpha_n\}$ respectively, and ξ is as usual a parallel vector field on Σ. If the angles $\{\alpha_1,\dots,\alpha_n\}$ are pairwise distinct, then $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$ is identified to $\mathcal{H}(k_1,\dots,k_n)$, otherwise $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$ is a finite covering of $\mathcal{H}(k_1,\dots,k_n)$.

Let ϱ denote the map from $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ onto $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$ which is defined by

$$\varrho : (\Sigma; (I(x_1,x), x_2,\dots,x_n);\xi) \longmapsto (\Sigma;(x_1,\dots,x_n);\xi).$$

Let $\hat{\mu}_{\mathrm{Tr}}$ denote the volume form which is defined by using admissible triangulations on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$. Let $\hat{\mu}_0$, and μ_0 denote the volume forms defined by the period mappings on $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$, and $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$ respectively. To prove the proposition, it suffices to show

$$\int_{\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})} \exp(-\mathbf{Area}(\Sigma)) d\mu_0 < \infty \tag{5.13}$$

By Theorem 5.1.1, Part b), we know that

$$\int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})} \exp(-\mathbf{Area}(\Sigma) - \ell^2(I)) d\hat{\mu}_{\mathrm{Tr}} < \infty \tag{5.14}$$

Recall that on each connected component of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ there exists a constant λ such that $\hat{\mu}_{\mathrm{Tr}} = \lambda\hat{\mu}_0$. By a result of Konsevitch-Zorich [KZ], we know that $\mathcal{H}(k_1,\dots,k_n)$ has finitely many connected components. It follows that $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ also so has finitely many connected components. Therefore, (5.14) implies

$$\int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})} \exp(-\mathbf{Area}(\Sigma) - \ell^2(I)) d\hat{\mu}_0 < \infty \tag{5.15}$$

Consider a point $(\Sigma;(x_1,\dots,x_n);\xi)$ in $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$. Fix a tangent vector $v_1 \in T_{x_1}\Sigma$, we can then identify the set of tangent vector of norm one in $T_{x_1}\Sigma$ to the set $\mathbb{R}/\alpha_1\mathbb{Z}$. Any geodesic segment in Σ which contains x_1 as an endpoint is uniquely determined by its tangent vector at x_1, and its length. Consequently, we have an injective map :

$$\varphi : \varrho^{-1}\{(\Sigma;(x_1,\dots,x_n);\xi)\} \longrightarrow (\mathbb{R}/\alpha_1\mathbb{Z}) \times \mathbb{R}^+,$$

Let $\mathcal{U}$ is a neighborhood of $(\Sigma;(x_1,\dots,x_n);\xi)$ in $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$ such that the period mapping Φ defines a local chart on $\mathcal{U}$. For each point $(\Sigma';(x'_1,\dots,x'_n);\xi')$ in $\mathcal{U}$, we choose a tangent vector v'_1 in $T_{x'_1}\Sigma'$ to be the reference vector, we can assume that v'_1 varies continuously as $(\Sigma';(x'_1,\dots,x'_n);\xi')$ varies in $\mathcal{U}$ so that the map φ extended into a map :

$$\hat{\varphi} : \varrho^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times (\mathbb{R}/\alpha_1\mathbb{Z}) \times \mathbb{R}^+,$$

which is continuous and injective.

Let $(\Sigma;(I(x_1,x),x_2,\dots,x_n);\xi)$ be a point in $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ such that

$$\varrho((\Sigma;(I(x_1,x),x_2,\dots,x_n);\xi)) = (\Sigma,(x_1,\dots,x_n),\xi).$$

Let $\hat{\Phi}$ denote the period mapping defining a local chart of $\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\bar{\alpha})$ in a neighborhood of $(\Sigma;(I(x_1,x),x_2,\dots,x_n);$ Suppose that if $\hat{\Phi}(\Sigma;(I(x_1,x),x_2,\dots,x_n);\xi) = (z_1,\dots,z_{N+1})$, then z_{N+1} is the complex number corresponding to the segment $I(x_1,x)$. It follows that in the local charts $\hat{\Phi}$, and Φ the map ϱ can be written as

$$\varrho(z_1,\dots,z_{N+1}) = (z_1,\dots,z_N)$$

and the map $\hat{\varphi}$ verifies

$$\hat{\varphi}(z_1, \dots, z_{N+1}) = ((z_1, \dots, z_N); \arg(z_{N+1}) + c; |z_{N+1}|), \quad \text{with } c \text{ constant},$$

where $N = \dim_{\mathbb{C}} \mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$. Consequently, we can write

$$\hat{\varphi}_* d\hat{\mu}_0 = r d\mu_0 d\theta dr.$$

It follows that

$$\int_{\varrho^{-1}(\mathcal{U})} e^{-\mathbf{Area}(\Sigma)-\ell^2(I)} d\hat{\mu}_0 = \int_{\hat{\varphi}(\varrho^{-1}(\mathcal{U}))} e^{-\mathbf{Area}(\Sigma)-r^2} r d\mu_0 d\theta dr. \tag{5.16}$$

By a well known result (for example, see [MT], Theorem 1.8), we know that on a translation surface, there are no geodesic segments with endpoints in the set of singularities in all directions except a countable set. This implies that there exists a countable subset Θ of $\mathbb{R}/\alpha_1\mathbb{Z}$ such that if θ is not in Θ, then the geodesic ray starting from x_1 in the direction θ can be extended infinitely. It follows immediately that $\hat{\varphi}(\varrho^{-1}(\mathcal{U}))$ is an open dense set of $\mathcal{U} \times (\mathbb{R}/\alpha_1\mathbb{Z}) \times \mathbb{R}^+$. Therefore, we have

$$\begin{aligned}
\int_{\hat{\varphi}(\varrho^{-1}(\mathcal{U}))} e^{-\mathbf{Area}(\Sigma)-r^2} r d\mu_0 d\theta dr &= \int_{\mathcal{U}\times(\mathbb{R}/\alpha_1\mathbb{Z})\times\mathbb{R}^+} e^{-\mathbf{Area}(\Sigma)-r^2} r d\mu_0 d\theta dr, \\
&= \int_0^{+\infty} e^{-r^2} r dr \int_0^{\alpha_1} d\theta \int_{\mathcal{U}} e^{-\mathbf{Area}(\Sigma)} d\mu_0, \\
&= \frac{\alpha_1}{2} \int_{\mathcal{U}} e^{-\mathbf{Area}(\Sigma)} d\mu_0.
\end{aligned}$$

From (5.16), we deduce that

$$\int_{\varrho^{-1}(\mathcal{U})} e^{-\mathbf{Area}(\Sigma)-\ell^2(I)} d\hat{\mu}_0 = \frac{\alpha_1}{2} \int_{\mathcal{U}} e^{-\mathbf{Area}(\Sigma)} d\mu_0 \tag{5.17}$$

Since (5.17) is true for any small neighborhood in $\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})$, we can conclude that

$$\int_{\mathcal{M}_{\mathrm{T}}(\tilde{\alpha})} e^{-\mathbf{Area}(\Sigma)} d\mu_0 = \frac{2}{\alpha_1} \int_{\mathcal{M}^{\mathrm{et}}(\hat{\mathcal{A}},\tilde{\alpha})} e^{-\mathbf{Area}(\Sigma)-\ell^2(I)} d\hat{\mu}_0. \tag{5.18}$$

From (5.15), we know that the right hand side of this equality is finite, hence, so is the left hand side, and (5.13) follows. The proposition is then proved. □

5.6 Volume of $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})$ is finite

In this section, we are interested in the moduli space of spherical flat surfaces. We have defined the volume form μ_{Tr} on the space $\mathcal{M}(\mathbb{S}^2, \bar{\alpha}) = \mathcal{M}(\mathbb{S}^2, \bar{\alpha})^* \times \mathbb{S}^1$, where $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$ is the moduli space of spherical flat surfaces whose singularities have cone angles given by $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$. Recall that $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ is the set of flat surfaces having area 1 in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$, and $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})$ is the product space $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^* \times \mathbb{S}^1$. By Proposition 3.2.3, the space $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$ can be considered as the moduli space of the configurations of n points on the sphere $\mathbb{S}^2$ up to Möbius transformations.

The volume form μ_{Tr} induces a volume form μ^1_{Tr} on $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})$. Pushing forward μ^1_{Tr} by imposing the condition that the volume of each $\mathbb{S}^1$ fiber is 2π, we get a volume form $\hat{\mu}^1_{\mathrm{Tr}}$ on $\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*$. The goal in this section is to prove Theorem 5.1.2. Note that a direct consequence of Theorem 5.1.2, is the following

Corollary 5.6.1 $\hat{\mu}^1_{\mathrm{Tr}}(\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*)$ *is finite.*

Remark: A similar result was proved in [V2], Section 18,19.

Proof: Since we have

$$\int_{\mathcal{M}(\mathbb{S}^2,\bar{\alpha})} \exp(-\mathbf{Area})d\mu_{\mathrm{Tr}} = C\int_{\mathcal{M}_1(\mathbb{S}^2,\bar{\alpha})} d\mu^1_{\mathrm{Tr}},$$

where C is a constant depending only on the dimension of $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$, Theorem 5.1.2 implies that

$$\mu^1_{\mathrm{Tr}}(\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})) < \infty.$$

It follows immediately that

$$\hat{\mu}^1_{\mathrm{Tr}}(\mathcal{M}_1(\mathbb{S}^2, \bar{\alpha})^*) < \infty.$$

□

5.6.1 The function δ

Let Σ be a flat surface in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$. Let $x_1, \dots, x_n$ denote the singular points of Σ so that the cone angle at x_i is α_i. Let $\mathbf{d}$ denote the distance defined by the metric on Σ.

For any subset I of $\{1, \dots, n\}$, let $\mathbf{diam}_I(\Sigma)$ denote the diameter of the set $\{x_i,\ i \in I\}$. We define

$$\delta_I(\Sigma) = \min\{\mathbf{d}(x_i, x_j) : \ i \in I, \ j \notin I\}$$

and

$$\delta_I^+(\Sigma) = \begin{cases} \delta_I(\Sigma) & \text{if } \delta_I(\Sigma) \geqslant 3\mathbf{diam}_I(\Sigma)\,; \\ 0 & \text{otherwise.} \end{cases}$$

A subset I of $\{1, \dots, n\}$ is called *essential* if we have

$$\sum_{i \in I} \alpha_i \notin 2\pi\mathbb{Z}.$$

We define a function δ on the space $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$ as follows

$$\forall \Sigma \in \mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*, \ \delta(\Sigma) = \max\{\delta_I^+(\Sigma) : \ I \subset \{1, \dots, n\}, I \text{ is essential }\}.$$

The function δ is always positive since when $I = \{i\}$, $\delta_I^+(\Sigma) = \min\{\mathbf{d}(x_i, x_j), \ j \neq i\} > 0$, and there always exists $i \in \{1, \dots, n\}$ such that $\alpha_i \notin 2\pi\mathbb{Z}$.

To simplify the notations, we also denote δ the composition of δ and the natural projection pr_1 from $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ onto $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$.

The proof of Theorem 5.6.1 splits naturally into two propositions :

Proposition 5.6.2 *We have*

$$\int_{\mathcal{M}(\mathbb{S}^2, \bar{\alpha})} \exp(-\mathbf{Area} - \delta^2) d\mu_{\mathrm{Tr}} < \infty,$$

and

Proposition 5.6.3 *There exists a constant $C(\bar{\alpha})$ depending on $\bar{\alpha}$ such that for any surface Σ in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$ we have*

$$\delta^2(\Sigma) < C(\bar{\alpha})\mathbf{Area}(\Sigma).$$

5.6.2 Good tree and good forest

Let Σ be a surface in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$. Let $x_1, \dots, x_n$ denote the singular points of Σ so that the cone angle at x_i is α_i. Let V denote the set $\{x_1, \dots, x_n\}$, and as usual let $\mathbf{d}$ be the distance defined by the metric on Σ. Set

$$\delta = \delta(\Sigma).$$

For any geodesic tree A on Σ, we denote $\mathrm{Ver}(A)$ the vertex set of A, $\max(A)$ the length of the longest edge of A, and $\mathrm{R}(A)$ the distance from $\mathrm{Ver}(A)$ to the set $\mathrm{V} \setminus \mathrm{Ver}(A)$.

Definition 5.6.4 *Let A be a geodesic tree in Σ whose set of vertices is a subset of V. Let k be the number of edges of A. The tree A is said to be* good, *if either A is a singular point with cone angle in $2\pi\mathrm{N}$, or $k \geqslant 1$ and we have*

- $\max(A) \leqslant 4^{k-1}\delta$,
- $\mathbf{diam}(\mathrm{Ver}(A)) \leqslant 4^{k-1}\delta$,
- *The index set corresponding to the vertex set of A is non essential, that is the sum of all cone angles at the vertices of A belongs to the set $2\pi\mathrm{N}$.*
- *Either $\mathrm{Ver}(A) = V$, or $\mathrm{R}(A) \geqslant 3.4^{k-1}\delta$.*

Let us start by

Lemma 5.6.5 *There always exists a good tree on Σ.*

Proof: First, let e be a geodesic segment which realizes the distance

$$\min\{\mathbf{d}(x_i, x_j),\ \alpha_i \notin 2\pi\mathrm{N} \text{ and } i \neq j\}.$$

By definition, we have

$$\mathbf{leng}(e) \leqslant \delta.$$

Let A^1 denote the tree which contains only the segment e. By assumption, we have

$$\max(A^1) = \mathbf{diam}(\mathrm{Ver}(A^1)) = \mathbf{leng}(e_1) \leqslant \delta.$$

Consider the following procedure, which will be called the *points adding procedure* :

Suppose that we already have a geodesic tree A^k connecting $k+1$ points in $\{x_1, \ldots, x_n\}$ verifying the following condition :

$$(*) \left\{ \begin{array}{c} \max(A^k) \leqslant 4^{k-1}\delta, \\ \mathbf{diam}(\mathrm{Ver}(A^k)) \leqslant 4^{k-1}\delta. \end{array} \right.$$

Let I be the subset of $\{1, \ldots, n\}$ corresponding to the vertex set of A^k. We have two cases :

- Case 1 : I is essential. In this case, let e_{k+1} be a segment realizing the distance $\delta_I(\Sigma)$, and let x_j be the endpoint of e_{k+1} which does not belong to $\mathrm{Ver}(A^k)$.

By definition, we have either $\mathbf{leng}(e_{k+1}) \leqslant 3\mathbf{diam}(\mathrm{Ver}(A^k))$, or $\mathbf{leng}(e_{k+1}) \leqslant \delta$. Since we have $\mathbf{diam}(\mathrm{Ver}(A^k)) \leqslant 4^{k-1}\delta$, we deduce that, in both cases

$$\mathbf{leng}(e_{k+1}) \leqslant 3.4^{k-1}\delta.$$

Slit open the surface Σ along the tree A^k, and denote the new surface Σ'. The vertex set $\mathrm{Ver}(A^k)$ gives rise to a finite subset V^k of the boundary of Σ'. Let us prove that the distance from V^k to the point x_j, with respect to the distance in Σ', is at most $4^k\delta$.

Consider e_{k+1} as a ray exiting from x_j, and let y be the first intersection point between e_{k+1} and the tree A^k. Since we have $\max(A^k) \leqslant 4^{k-1}\delta$, there exists a path on Σ joining x_j to an endpoint of the edge containing y without crossing the tree A^k, whose length is at most $3.4^{k-1}\delta + 4^{k-1}\delta = 4^k\delta$. Because this path does not cross the tree A^k, it represents a path on Σ' joining x_j to a point in V^k. Thus, we deduce that the distance between x_j and V^k in Σ' is at most $4^k\delta$.

Let a' be the path realizing the distance from x_j to V^k in Σ'. The path a' corresponds to a path a in Σ which is piecewise geodesic, and meets the tree A^k at one of its vertices. Note that $\mathbf{leng}(a) = \mathbf{leng}(a') \leqslant 4^k\delta$.

Adding a to A^k, we obtain a new tree which will be denoted by A^{k+r}, where r is the number of geodesic segments contained in a. Let us prove that this new tree also verifies the condition $(*)$.

- If $r = 1$ then $\mathrm{Ver}(A^{k+1}) = \mathrm{Ver}(A^k) \cup \{x_j\}$. Since $\mathbf{diam}(A^k) \leqslant 4^{k-1}\delta$, and the distance from x_j to $\mathrm{Ver}(A^k)$ is at most $3.4^{k-1}\delta$, we deduce that

$$\mathbf{diam}(\mathrm{Ver}(A^{k+1})) \leqslant 4^{k-1}\delta + 3.4^{k-1}\delta = 4^k\delta.$$

  By assumption we know that $\max(A^k) \leqslant 4^{k-1}\delta$, and we have proved that the length of the added edge is at most $4^k\delta$, hence we have $\max(A^{k+1}) \leqslant 4^k\delta$.

- If $r > 1$, it means that the path a contains some singular points in its interior. The distance from those points to the set $\mathrm{Ver}(A^k)$ is bounded by the length of a which is at most $4^k\delta$. Hence, the diameter of the set $\mathrm{Ver}(A^{k+r})$ is at most

$$4^{k-1}\delta + 4^k\delta \leqslant 4^{k+r-1}\delta.$$

  As for $\max(A^{k+r})$, we have

$$\max(A^{k+r}) = \max\{\max(A^k), \mathbf{leng}(a)\} \leqslant 4^k\delta.$$

We can now restart the procedure with A^{k+r} in the place of A^k.

- Case 2 : I is non-essential. In this case, if $\mathrm{Ver}(A^k) = \mathrm{V}$, or $\mathrm{R}(\mathrm{Ver}(A^k)) \geqslant 3.4^{k-1}\delta$, then the procedure stops. Otherwise, by the same arguments as in Case 1, we can add to A^k some edges so that the new tree also verifies the condition $(*)$, and we restart the procedure.

Since we only have finitely many singular points in Σ, the points adding procedure must stop, and we obtain a good tree. □

Definition 5.6.6 *A union of disjoint geodesic trees with vertices in* V *is called a* good forest *if every tree in this union is good.*

Lemma 5.6.7 *There exists a good forest in Σ whose set of vertices is* V.

Proof: By Lemma 5.6.5, we know that there exists a good tree A_1 in Σ. Clearly, A_1 itself is a good forest. If $\mathrm{Ver}(A_1) = \mathrm{V}$, or every point in the set $\mathrm{V} \setminus \mathrm{Ver}(A_1)$ has cone angle in $2\pi\mathbb{N}$, then we are done. Otherwise, there exists a point x_i in $\mathrm{V} \setminus \mathrm{Ver}(A_1)$, with cone angle not in the set $2\pi\mathbb{N}$.

In this case, we would like to construct a good tree A_2 containing x_j by the points adding procedure. However, this procedure can not be carried out straightly because of the presence of the tree A_1. Namely, it may happen that we have $\mathrm{R}(\mathrm{Ver}(A_2)) \leqslant 3.4^{k_2-1}\delta$, where k_2 is the number of edges of A_2, but the segment realizing this distance intersects the tree A_1. We will call this the *blocking situation*.

Let us consider the following procedure, which will be called the *trees joining procedure* :

Assume that we already have l disjoint geodesic trees $A_1, \dots, A_l$ with the following properties :

a) A_j is a good tree $\forall j = 1, \dots, l-1$.

b) A_l satisfies the condition $(*)$.

c) $\mathrm{d}(A_l, \sqcup_{j=1}^{l-1} A_j) \leqslant 4^{k_l}\delta$.

Let $k_1, \dots, k_l$ be the numbers of edges of $A_1, \dots, A_l$ respectively. Let c be a path of length less than $4^{k_l}\delta$ joining a point in A_l to a point in $\sqcup_{j=1}^{l-1} A_j$.

Without loss of generality, we can assume that c joins a point in A_l to a point in A_{l-1}. Since both A_{l-1} and A_l verify the condition $(*)$, we deduce that there exists a path $\hat{c}$ joining a vertex of A_{l-1} to a vertex of A_l without intersecting the set $\sqcup_{j=1}^{l-1} A_j$ (except at the endpoints) whose length is at most

$$4^{k_l-1}\delta + 4^{k_l}\delta + 4^{k_{l-1}-1}\delta \leqslant 4^{k_l+k_{l-1}}\delta.$$

Consider the surface with boundary obtained by slitting open Σ along the trees $A_1, \dots, A_l$. The path $\hat{c}$ represents a path in this new surface, joining a point in the boundary component corresponding to A_{l-1} to a point in the component corresponding to A_l.

Consider a path of minimal length joining these two points in the new surface. This path contains a piecewise geodesic path c_0 in Σ joining a vertex of A_{l-1} to a vertex of A_l without crossing the edges of $A_1, \dots, A_l$. Note that the endpoints of the geodesic segments in c_0 are singular points of Σ. The union of c_0 and all the trees in $\{A_1, \dots, A_l\}$ which have at least a common point with c_0 is a geodesic tree. This new tree contains obviously A_{l-1} and A_l as subtrees.

Denote the remaining trees, ones that have no common points with c_0, $A'_1, \dots, A'_{l'-1}$, and the new tree $A'_{l'}$. Note that $l' < l$ and the tree $A'_{l'}$ contains at least $k_{l-1} + k_l + 1$ edges.

It is a routine to verify that the family $\{A'_1, \dots, A'_{l'}\}$ also satisfies the conditions a), and b). If the condition c) still holds, then we can restart the procedure. Therefore the procedure can be repeated until we get a family $\tilde{A}_1, \dots, \tilde{A}_{\tilde{l}}$ of disjoint geodesic trees, verifying a), and b), and in addition we have :

$$\mathbf{d}(\tilde{A}_{\tilde{l}}, (\tilde{A}_1 \sqcup \cdots \sqcup \tilde{A}_{\tilde{l}-1})) \geqslant 4^{k_{\tilde{l}}}\delta,$$

where $k_{\tilde{l}}$ is the number of edges of $\tilde{A}_{\tilde{l}}$.

It is clear that, if we have a blocking situation, then the hypothesis of the trees joining procedure are satisfied, we can then use the trees joining procedure to get out of the blocking situation, and reapply the points adding procedure until we get to a blocking situation again. Since the number of singular points in Σ is finite, this algorithm must stop, and we obtain a good forest. □

Corollary 5.6.8 *There exists a constant κ, such that for any Σ in $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$, there exists an erasing forest $\hat{A}$ in Σ which verifies*

$$\ell(\hat{A}) \leqslant \kappa\delta.$$

Proof: By Lemma 5.6.7, we know that there exists a good forest $\hat{A} = \sqcup_{j=1}^{m} A_j$ in Σ. By definition, $\hat{A}$ is an erasing forest. Since every tree in $\hat{A}$ verifies the condition $(*)$, we have $\ell(A_j) \leqslant k_j 4^{k_j}\delta$, where k_j is the number of edges of A_j, $\forall j = 1, \dots, m$.

Observe that $k_1 + \cdots + k_m = n - m \leqslant n - 1$, therefore we have

$$\ell(\hat{A}) = \sum_{j=1}^{m} \ell(A_j) \leqslant (n-1)4^{n-1}\delta,$$

and the corollary follows. □

5.6.3 Proof of Proposition 5.6.2

Let $\mathcal{A}_{\text{ad}}(\bar{\alpha})$ denote the set of all families $\hat{\mathcal{A}} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\}$ of m $(0 < m < n)$ topological trees, whose vertices are labelled by $\{1, \ldots, n\}$, verifying the following condition : if I_j, $j = 1, \ldots, m$, is the subset of $\{1, \ldots, n\}$ corresponding to the vertices of the tree $\mathcal{A}_j$, then

$$\sum_{i \in I_j} \alpha_i \in 2\pi\mathbb{Z}.$$

For each $\hat{\mathcal{A}} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m\} \in \mathcal{A}_{\text{ad}}(\bar{\alpha})$, let $\mathcal{U}_{\hat{\mathcal{A}}}$ be the subset of $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ consisting of all triples $(\Sigma, \hat{A}, \xi)$ satisfying the following condition :

$$\ell(\hat{A}) \leqslant \kappa\delta(\Sigma),$$

where $\hat{A} = \sqcup_{j=1}^{m} A_j$ is a geodesic erasing forest of Σ, with A_j isomorphic to $\mathcal{A}_j$, and κ is the constant in Corollary 5.6.8.

Let $\rho_{\hat{\mathcal{A}}}$ denote the projection from $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$ onto $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$, which associates to every triple $(\Sigma, \hat{A}, \xi)$ the surface Σ. From Corollary 5.6.8, we know that the family

$$\{\mathcal{V}_{\hat{\mathcal{A}}} = \rho_{\hat{\mathcal{A}}}(\mathcal{U}_{\hat{\mathcal{A}}}) : \hat{\mathcal{A}} \in \mathcal{A}_{\text{ad}}(\bar{\alpha})\}$$

covers the space $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$. Let ρ_1 be the natural projection from $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ onto $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})^*$, it follows that the family

$$\{\rho_1^{-1}(\mathcal{V}_{\hat{\mathcal{A}}}) : \hat{\mathcal{A}} \in \mathcal{A}_{\text{ad}}(\bar{\alpha})\}$$

covers the space $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$.

Since the set $\mathcal{A}_{\text{ad}}(\bar{\alpha})$ is finite, it is enough to show that, for every $\hat{\mathcal{A}}$ in $\mathcal{A}_{\text{ad}}(\bar{\alpha})$, we have

$$\int_{\rho_1^{-1}(\mathcal{V}_{\hat{\mathcal{A}}})} \exp(-\mathbf{Area} - \delta^2) d\mu_{\text{Tr}} < \infty. \tag{5.19}$$

Since the space $\mathcal{M}(\mathbb{S}^2, \bar{\alpha})$ can be locally identified to $\mathcal{M}^{\text{et}}(\hat{\mathcal{A}}, \bar{\alpha})$, we have

$$\int_{\rho_1^{-1}(\mathcal{V}_{\hat{A}})} \exp(-\mathbf{Area} - \delta^2) d\mu_{\mathrm{Tr}} = \int_{\mathcal{U}_{\hat{A}}} \exp(-\mathbf{Area} - \delta^2) d\mu_{\mathrm{Tr}}$$

By definition, for every $(\Sigma, \hat{A}, \xi)$ in $\mathcal{U}_{\hat{A}}$, we have $\ell(\hat{A}) \leqslant \kappa\delta(\Sigma)$. It follows

$$\int_{\mathcal{U}_{\hat{A}}} \exp(-\mathbf{Area} - \delta^2) d\mu_{\mathrm{Tr}} \leqslant \int_{\mathcal{U}_{\hat{A}}} \exp(-\mathbf{Area} - \frac{1}{\kappa^2}\ell^2(\hat{A})) d\mu_{\mathrm{Tr}} \tag{5.20}$$

By Theorem 5.1.1, Part b), we know that the right hand side of (5.20) is finite. Consequently, (5.19) is true, and the proposition follows. □

5.6.4 Proof of Proposition 5.6.3

Let I_0 be a subset of $\{1, \dots, n\}$ such that $\delta^+_{I_0}(\Sigma) = \delta(\Sigma) = \delta$. Let s be a geodesic segment joining a point x_{i_0} with $i_0 \in I_0$ and a point x_{i_1} with $i_1 \notin I_0$ such that $\mathbf{leng}(s) = \delta$. Let p denote the midpoint of s. As usual we denote $\mathbf{d}$ the distance induced by the flat metric of Σ.

First, we have

Lemma 5.6.9 $B(p, \delta/2) = \{x \in \Sigma : \mathbf{d}(p, x) < \delta/2\}$ *does not contain any singular point of* Σ.

Proof: Suppose on the contrary that a singular point x_k, with $k \notin \{i_0, i_1\}$, is contained in $B(p, \delta/2)$, then we have $\mathbf{d}(x_{i_0}, x_k) < \delta$, and $\mathbf{d}(x_{i_1}, x_k) < \delta$, but this would imply that $\delta_{I_0}(\Sigma) < \delta$, and we have a contradiction. □

Let $D(\delta/2)$ denote the open disk with center $(0, 0)$ and radius $\delta/2$ in the Euclidean plane $\mathbb{E}^2 = \mathbb{R}^2$. Let f be the isometric immersion from $D(\delta/2)$ to Σ, which maps the horizontal diameter of $D(\delta/2)$ to the segment s, and the origin $(0, 0)$ to the point p. The immersion f can be defined because the smallest distance from p to a singular point of Σ is $\delta/2$.

Let ϵ be the maximal value such that the restriction of f on the disk $D(\epsilon\delta)$ with center $(0, 0)$ and radius $\epsilon\delta$ is an embedding. If $\epsilon \geqslant 1/4$ then there is an embedded Euclidean disk of radius $\delta/4$ in Σ, which means that $\mathbf{Area}(\Sigma) \geqslant (\pi\delta^2)/16$. In the sequel, we will suppose that $\epsilon < 1/4$, consequently, the set $f^{-1}(p)$ contains points other than $(0, 0)$. Let p_1 be the point in $f^{-1}(p) \setminus \{(0, 0)\}$ closest to $(0, 0)$, and c_1 be the segment joining $(0, 0)$ to p_1 in $D(\delta/2)$.

For any subset I of $\{1, \dots, n\}$, we denote α_I the sum

$$\alpha_I = \sum_{i \in I} \alpha_i,$$

and $||\alpha_I||$ the distance from α_I to the set $\pi\mathbb{Z}$ in $\mathbb{R}$. Set

$$\alpha_0 = \min\{||\alpha_I|| : I \subset \{1, \dots, n\}, ||\alpha_I|| \neq 0\}.$$

Choose a number ϵ_0 such that $\epsilon_0 < \min\{1/6, \sin(\alpha_0)/4\}$. We will prove that there exists an embedded disk of radius $\epsilon_0\delta$ in Σ, which is enough to prove the proposition.

Let d_0 denote the horizontal diameter of $D(\delta/2)$, and d_1 denote the lift of s passing through p_1. Let c denote the image of c_1 by f, then c is a geodesic loop in Σ with base point p. Let θ be angle between d_0 and d_1, by this we mean the angle in $[0;\pi/2]$ between the two lines supporting d_0 and d_1. Let us prove

Lemma 5.6.10 *We have, either $\theta = 0$, or $\epsilon > \epsilon_0$.*

Proof: Remark that θ equals the rotation angle of the holonomy of c modulo π. Suppose that $\theta \neq 0$, then, by the definition of α_0, we have $\theta \geqslant \alpha_0$.

If $\epsilon < \epsilon_0$, then the distance from $(0, 0)$ to d_1 is less than $2\epsilon_0\delta < \sin(\alpha_0)\delta/2$. Together with the fact that $\theta > \alpha_0$, this implies that d_1 intersects d_0, in other words, the segment s has self-intersection, which is impossible. Therefore, we can conclude that either $\theta = 0$, or $\epsilon > \epsilon_0$. □

If $\epsilon > \epsilon_0$, then we are done. Therefore, we only have to consider the case $\theta = 0$, and we have

Lemma 5.6.11 *If $\theta = 0$, then the rotation angle of the holonomy of c is 0 modulo 2π.*

Proof: If it is not the case, then this angle equals π modulo 2π, and hence, the holonomy of c is the composition of a rotation of angle π and a translation which maps $(0, 0)$ to p_1.

Such a transformation must fix the midpoint q_1 of the segment joining $(0, 0)$ to p_1. It follows that q_1 is mapped by f into a singular point of Σ, which is impossible because q_1 is contained in the disk $D(\delta/2)$.□

From Lemma 5.6.11, we deduce that the set $f(D(\delta/2))$ contains a cylinder C with length $(1 - 2\epsilon)\delta$ and width bounded by $2\epsilon\delta$.

Remark that c is then a closed geodesic in C which cuts Σ into two flat surfaces with geodesic boundary, each of which is homeomorphic to a topological closed disk. We denote Σ_0 the flat disk that contains x_{i_0}.

Lemma 5.6.12 *For any i in I_0, x_i is contained in Σ_0.*

Proof: Recall that by the definition of δ, we have

$$\mathbf{diam}\{x_i,\ i \in I_0\} < \delta/3,$$

which implies that $\mathbf{d}(x_{i_0}, x_i) < \delta/3,\ \forall i \in I_0$.

If there exists $i \in I_0$ such that $x_i \notin \Sigma_1$, then the path realizing the distance $\mathbf{d}(x_{i_0}, x_i)$ must intersect the closed geodesic c, therefore it crosses C. Consequently,

$$\mathbf{d}(x_{i_0}, x_i) \geqslant (1 - 2\epsilon)\delta > 2/3\delta,$$

which is impossible. □

The rotation angle of the holonomy of c equals the sum of all cone angles at singular points in Σ_0 modulo 2π. By assumption, we know that $\alpha_{I_0} \notin 2\pi\mathbb{Z}$, it means that Σ_0 contains singular points which do not belong to $\{x_i,\ i \in I_0\}$. Note that we have

$$\min\{\mathbf{d}(x_i, x_j\},\ i \in I_0,\ j \notin I_0,\ x_j \in \Sigma_0\} \geqslant \delta_{I_0}(\Sigma) = \delta.$$

Since Σ_0 is a flat surface with geodesic boundary which contains no singularities on the boundary, we can restrict ourselves into Σ_0 and restart the whole procedure. This procedure can be continued as long as the rotation angle of the loop c is zero.

Since we only have finitely many singular points in Σ, the procedure must stop, and we get a point in Σ whose injectivity radius is at least $\epsilon_0\delta$. Proposition 5.6.3 is then proved. □

Appendices

Annexe A

Curves and Isotopies

Throughout this chapter, S will be a fixed compact surface whose Euler characteristic is negative. Our goal in this section, is to prove the following lemma

Lemma A.0.1 *Let $c_1, \ldots, c_k$ be a family of curves in S verifying the following conditions :*

i) For every $i = 1, \ldots, k$, the curve c_i is either a simple arc, or a simple loop if its two endpoints coincide lying in the interior of S except its endpoints when the later are contained in the boundary.

ii) If $i \neq j$ then c_i and c_j are not in the same homotopy class with fixed endpoints. If c_i is a loop then c_i is not homotopic to the constant loop, and if the endpoints of c_i are contained in the boundary, c_i is not homotopic with fixed endpoints to a subsegment of a boundary component.

iii) If $i \neq j$, then c_i and c_j intersect at most at their common endpoints.

The union of $c_1, \ldots, c_k$ will be denoted by C.

Let φ be a homeomorphism of S which is isotopic to the identity by an isotopy which is identity on the boundary of S, and fixes every endpoint of the arcs $c_1, \ldots, c_k$. Suppose that $\varphi(c_i) = c_i$, $\forall i = 1, \ldots, k$, then there exists an isotopy from φ to Id_S which is identity on the boundary, and leaves the set C invariant.

It seems to the author that this lemma is classical, but he could not find a good reference for it. Fortunately, it turns out that one can prove this lemma by a combination of classical theorems, and Epstein-Zieschang, and eventually the theorem of Alexander on homeomorphisms of the closed disk which is identity on the boundary.

In the sequel, we call a homeomorphism φ of S a 1-homeomorphism if it is isotopic to the identity by an isotopy which is identity on the boundary of S. If A is a subset of S, then a $A-1$-homeomorphism is a homeomorphism which is isotopic to the identity by an isotopy fixing every point in the set $\partial S \cup A$.

A.1 Basic Theorems

We recall here some important theorems which are useful for the proof of Lemma A.0.1.

The following theorem follows from results of Epstein-Zieschang (see [B], Theorem A.4, Theorem A.5 page 411).

Theorem A.1.1 (Epstein-Zieschang) *Let $\{c_1, \ldots, c_k\}$ be a family of curves with the properties described in Lemma A.0.1. Assume in addition that all the endpoints of $c_1, \ldots, c_k$ lie on the boundary of S.*

Let $\{\gamma_1, \ldots, \gamma_k\}$ be another family of curves verifying the same properties such that γ_i and c_i are homotopic with fixed endpoints, then there exists a homeomorphism ϕ of S such that

- *ϕ is isotopic to the identity by an isotopy which is identity on the boundary of S, and fixes all the endpoints of $c_1, \ldots, c_k$.*
- *$\phi(c_i) = \gamma_i, \ \forall i = 1, \ldots, k$.*

Next, we also need the following theorem of Alexander

Theorem A.1.2 (Alexander) *Any homeomorphism of the unity disk $\mathbb{D}$ of $\mathbb{R}^2$ is isotopic to $\mathrm{Id}_{\mathbb{D}}$.*

A direct consequence of A.1.2 is the following

Corollary A.1.3 *Let $\{a_1, \ldots, a_n\}$ be a family of curves in S verifying the properties in Lemma A.0.1 such that $\mathrm{int}(S) \setminus (\cup_{i=1}^n a_i)$ is a disjoint union of topological open disks. Let ϕ be a homeomorphism of S which is identity on ∂S, fixes all the endpoints of the curves $a_1, \ldots, a_n$, and preserves the set $\cup_{i=1}^n a_i$. Then ϕ is a 1-homeomorphism of S.*

Proof: By assumption, we have $\phi(a_i) = a_i, \ \forall i = 1, \ldots, n$. For each $i = 1, \ldots, n$, let $h_i : a_i \times [0,1] \longrightarrow a_i$ be an isotopy from $\phi_{|a_i}$ to Id_{a_i}. Since the curves $a_1, \ldots, a_n$ cut $\mathrm{int}(S)$ into open disk, we can extend the isotopies $h_i, \ i = 1, \ldots, n$ to an isotopy from ϕ to a homeomorphism ϕ' which is identity on the set $\partial S \cup (\cup_{i=1}^n a_i)$. Note that this isotopy is identity on the boundary of S.

Now, applying Theorem A.1.2 to the closure of each of the disks in the set $\mathrm{int}(S) \setminus (\cup_{i=1}^n a_i)$, we deduce that the homeomorphism ϕ' is isotopic to the identity of S by an isotopy which is identity on the set $\partial S \cup (\cup_{i=1}^n a_i)$, and the corollary follows. $\square$

A.2 Proof of Lemma A.0.1

First, we add to the family $\{c_1, \ldots, c_k\}$ the simple curves $c_{k+1}, \ldots, c_n$ such that the family $\{c_1, \ldots, c_n\}$ verify the same conditions as $\{c_1, \ldots, c_k\}$, and $c_1, \ldots, c_n$ cut $\mathrm{int}(S)$ into a union of open disks.

By cutting off a small disk around each endpoint of the curves $c_1, \ldots, c_n$ in the interior of S, we can assume that all the endpoints of $c_1, \ldots, c_n$ are contained in the boundary of S. Equip S with a hyperbolic metric such that ∂S become a union of closed geodesics. The universal cover $\tilde{S}$ of S is then a domain of $\mathbb{H}^2$ bounded by geodesic lines and a subset of $\partial\mathbb{H}^2 = \mathbb{S}^1$.

For $i = k+1, \ldots, n$, let γ_i denote the image of c_i by φ. Recall that by assumption $\varphi(c_i) = c_i, \forall i = 1, \ldots, k$. Let S' denote the surface we obtain by cutting S along $c_1, \ldots, c_k$. We will show that, for all $i = k+1, \ldots, n$, c_i is homotopic to γ_i in S'.

Fix an i in $\{k+1, \ldots, n\}$, consider a lift $\tilde{c}_i$ of c_i, and a lift $\tilde{\gamma}_i$ of γ_i such that $\tilde{c}_i$ and $\tilde{\gamma}_i$ have the same endpoints in $\tilde{S}$. Note that, by assumption, for every $j = 1, \ldots, k$, $\mathrm{int}(c)_i \cap \mathrm{int}(c_j) = \varnothing$, and $\mathrm{int}(c_j) \cap \mathrm{int}(\gamma_i) = \varnothing$, consequently $\tilde{c}_i$ and $\tilde{\gamma}_i$ do not intersect any lift of c_j.

Now, let r be the number of intersection points between $\tilde{c}_i$ and $\tilde{\gamma}_i$ except their common endpoints. It follows that there exists $r+1$ disks in $\tilde{S}$ each of which is bounded by a sub-arc of $\tilde{c}_i$ and a sub-arc of $\tilde{\gamma}_i$.

Let D be one of those disks. For any $j \in \{1, \ldots, k\}$, let $\tilde{c}_j$ be a lift of c_j, observe that $D \cap c_i = \varnothing$. Suppose on the contrary that $D \cap \tilde{c}_j \neq \varnothing$, then, since $\tilde{c}_i$ and $\tilde{\gamma}_i$ cannot intersect $\mathrm{int}(\tilde{c}_j)$, the disk D must contain both endpoints of $\tilde{c}_j$. By assumption, the endpoints of $\tilde{c}_j$ are contained in a geodesic line of the boundary of $\tilde{S}$, it follows that there is a geodesic line in $\partial\tilde{S}$ that intersects the disk D, but this would imply that either $\tilde{c}_i$ or $\tilde{\gamma}_i$ is not contained inside $\tilde{S}$, which is impossible.

Now, the observation above implies that $\tilde{c}_i$ is homotopic to $\tilde{\gamma}_i$ by an isotopy which does not meet any lift of c_j, $\forall j = 1, \ldots, k$. We deduce that c_i is homotopic to γ_i in S'.

Theorem A.1.1 shows that there exist a 1-homeomorphism φ' of S' such that $\varphi'(c_i) = \gamma_i$, $\forall i = k+1, \ldots, n$. The homeomorphism φ' can be interpreted as a homeomorphism of S which is identity in the set $\partial S \cup C$. Hence, we deduce that φ is isotopic to a homeomorphism $\hat{\varphi}$ of S by an isotopy fixing every point in the set $\partial S \cup C$, such that $\hat{\varphi}(c_i) = \gamma_i$, $\forall i = k+1, \ldots, n$. Since the curves $c_1, \ldots, c_n$ cut $\mathrm{int}(S)$ into a disjoint union of open disks, Corollary A.1.3 allows us to conclude. $\square$

Annexe B

Flat surfaces and Teichmüller space

Throughout this chapter, S_g will be a fixed flat surface, without boundary, having n singularities, denoted by $p_1, \dots, p_n$, with cone angles $\alpha_1, \dots, \alpha_n$ respectively. Recall that the Teichmüller space $\mathcal{T}(g,n)$ can be interpreted as the space of all pairs (Σ, ϕ), where Σ is a Riemann surface, and ϕ is a homeomorphism from S_g on to Σ, modulo isotopy relative to $\{p_1, \dots, p_n\}$.

Our goal in this chapter is to prove the following

Proposition B.0.1 *Let Σ_0 be a flat surface of genus g, without boundary, having n singularities, denoted by $x_1, \dots, x_n$, with cone angles $\alpha_1, \dots, \alpha_n$ respectively. Let $\phi_0 : S_g \longrightarrow \Sigma_0$ be a homeomorphism which sends the set of singularities of S_g onto the set of singularities of Σ_0 respecting cone angles. Let T_0 be a geodesic triangulation of Σ_0 such that the set of vertices of T_0 coincides with the set of singularities of Σ_0. The pair (Σ_0, ϕ_0) represents an element of the Teichmüller space $\mathcal{T}(g,n)$ which is denoted as usual by $[(\Sigma_0, \phi_0)]$.*

Suppose that there exists a closed curve γ in $\Sigma_0 \setminus \{x_1, \dots, x_n\}$ such that $\mathbf{orth}(\gamma) \neq \mathrm{Id}$. Then, every element of $\mathcal{T}(g,n)$ close enough to $[(\Sigma_0, \phi_0)]$ is represented by a pair $(\Sigma, f_\Sigma \circ \phi_0)$, where

- *Σ is a flat surface with cone singularities of angles $\alpha_1, \dots, \alpha_n$;*
- *The map $f_\Sigma : \Sigma_0 \longrightarrow \Sigma$ is a homeomorphism sending T_0 onto a geodesic triangulation of Σ, whose vertex set coincides with the set of singularities of Σ.*

B.1 Preliminaries

Set $n_1 = 4(2g+n-1) - 3$ and $n_2 = 3(2g+n-1) - 2$. First, we show that the surface Σ_0 can be associated to a vector in $\mathbb{C}^{n_1}$ satisfying a system of n_2 linear equations.

We begin by choosing $2g+n-1$ edges $\{b_1, \dots, b_{2g+n-1}\}$ of T_0 such that $\Sigma_0 \setminus (\cup_{j=1}^{2g+n-1} b_j)$ is an open disk, we call such a set of edges a *family of primitive edges*. Remark that such families always exist. To see this, consider the dual graph of T_0 on Σ_0. Since this graph is connected, we can find a maximal tree contained inside it, by *maximal tree* we mean a tree which contains all the vertices of the graph. The complement of a maximal tree is a set of $2g+n-1$ (open) edges of the dual graph. These edges correspond to a family of primitive edges in T_0.

Cut open the surface Σ_0 along the edges $b_1, \dots, b_{2g+n-1}$, we obtain a flat surface D_0 with geodesic boundary, homeomorphic to a closed disk. Note that the boundary of D_0 contains $2(2g+n-1)$ geodesic segments.

Let b'_j and b''_j, $j = 1, \dots, 2g+n-1$, denote the two geodesic segments on the boundary of D_0 which are identified to the edge b_j of T_0. The triangulation T_0 of Σ_0 induces a geodesic triangulation of D_0 which contains n_1 edges. To simplify notations, this triangulation of D_0 is also denoted by T_0. We choose an orientation for each edge of T_0. Assume that the edges on the boundary of D_0 are oriented coherently with the orientation of D_0.

Using a developing map of D_0, we can associate to each oriented edge e of T_0 a complex number $z(e)$. Let Z_0 denote the vector in $\mathbb{C}^{n_1}$ whose coordinates are the complex numbers associated to the edges of T_0. We assume that the first coordinate z_1^0 of Z_0 corresponds to the edge b'_1.

Since the developing map is defined up to a rotation, the vector Z_0 is defined up to a multiplication by $e^{\imath\theta}$ with θ in $[0; 2\pi]$. Hence, we can assume that $\mathrm{Im} z_1^0 = 0$.

As we have seen previously in the proof of 3.1.10, the coordinates of Z_0 must verify a system of linear equations $\mathrm{S}_{\mathrm{T}_0}$ which contains $2(2g+n-1)-2$ equations of type (2.3), and $2g+n-1$ equations of type (3.1). Observe that $(2(2g+n-1)-2)+(2g+n-1) = 3(2g+n-1)-2 = n_2$.

Let $\mathrm{V}_{\mathrm{T}_0}$ denote the subspace of $\mathbb{C}^{n_1}$ consisting of solutions of the system $\mathrm{S}_{\mathrm{T}_0}$. Clearly, we have $Z_0 \in \mathrm{V}_{\mathrm{T}_0}$.

For the dimension of $\mathrm{V}_{\mathrm{T}_0}$ we have

Lemma B.1.1

$$\dim_{\mathbb{C}} \mathrm{V}_{\mathrm{T}_0} = n_1 - n_2 = 2g+n-2.$$

Proof: Let us consider in more detail the equations of type (3.1) of $\mathrm{S}_{\mathrm{T}_0}$. The equations of type (3.1) in $\mathrm{S}_{\mathrm{T}_0}$ are of the form :

$$z(b''_j) = -e^{\imath\theta_j} z(b'_j),$$

with $j = 1, \dots, 2g+n-1$.

For each j in $\{1, \dots, 2g+n-1\}$, let c_j be a path in D_0 joining the midpoint of b'_j to the midpoint of b''_j. By construction, there exists a map $h_0 : \mathrm{D}_0 \longrightarrow \Sigma_0$ which is isometric in the interior of D_0, and maps $\partial \mathrm{D}_0$ on to the set $(\cup_{j=1}^{2g+n-1} b_j)$. The image of c_j by h_0, denoted by $\tilde{c}_j$, is a closed curve in Σ_0 which intersects the set $(\cup_{j=1}^{2g+n-1} b_j)$ at only one point. Observe that θ_j is the angle of the rotation $\mathbf{orth}(\tilde{c}_j)$. It is worth noticing that the closed curves $\{\tilde{c}_1, \dots, \tilde{c}_{2g+n-1}\}$ form a basis of the group $H_1(\Sigma_0 \setminus \{x_1, \dots, x_n\}, \mathbb{Z})$.

By assumption, there exists a closed curve γ on $\Sigma_0 \setminus \{x_1, \dots, x_n\}$ such that $\mathbf{orth}(\gamma) \neq \mathrm{Id}$, it follows that there exists $j \in \{1, \dots, 2g+n-1\}$ such that $\theta_j \notin 2\pi\mathbb{Z}$. Now, using the arguments in the proof of Lemma 3.4.6, we conclude that $\dim_{\mathbb{C}} V'_{\mathrm{T}_0} = n_1 - n_2 = 2g+n-2$. □

Let $\mathbf{H}_{\mathrm{T}_0}$ denote the Hermitian form determined by the area of Σ_0. Let $\mathrm{W}_{\mathrm{T}_0}$ denote the set $\{Z = (z_1, \dots, z_{n_1}) \in \mathrm{V}_{\mathrm{T}_0} \mid \overline{Z}^t \mathbf{H}_{\mathrm{T}_0} Z = 1, \mathrm{Im} z_1 = 0\}$. Observe that $\mathrm{W}_{\mathrm{T}_0}$ is a real sub-manifold of $\mathbb{C}^{n_1}$ of real dimension $2(2g+n-2)-2$.

By assumption Z_0 is contained in $\mathrm{W}_{\mathrm{T}_0}$. Let U_0^1 denote an open subset of $\mathrm{W}_{\mathrm{T}_0}$ containing Z_0 and homeomorphic to a ball in $\mathbb{R}^{2(2g+n-2)-2}$. We can then define a map

$$\Phi_{\mathrm{T}_0} : \mathrm{U}_0^1 \longrightarrow \mathcal{T}(g,n),$$

such that for every $Z \in \mathrm{U}_0^1$, $\Phi_{\mathrm{T}_0}(Z)$ is represented by a pair $(\Sigma, f_\Sigma \circ \phi_0)$, where Σ is a flat surface, and f_Σ is a homeomorphism, which sends T_0 onto a geodesic triangulation of Σ whose vertices are the singularities. This map is constructed in the same way as the one defined in the proof of Lemma 3.4.5. We have

Lemma B.1.2 *The map Φ_{T_0} is continuous and injective.*

Proof: For injectivity, suppose that $\Phi_{\mathrm{T}_0}(Z_1) = \Phi_{\mathrm{T}_0}(Z_2)$. Let (Σ_i, ϕ_i); $i = 1,2$ be the pair representing $\Phi_{\mathrm{T}_0}(Z_i)$, which is obtained by the construction of Φ_{T_0}. By definition, we can write $\phi_i = f_i \circ \phi_0$, where f_i is a homeomorphism mapping T_0 onto a geodesic triangulation of Σ_i.

By definition, there exists a conformal homeomorphism h from Σ_1 to Σ_2 such that $\phi_2^{-1} \circ h \circ \phi_1$ is an element of $\mathrm{Homeo}_0^+(S_g, \{p_1, \dots, p_n\})$. Using Proposition 3.2.3, we deduce that h is an isometry from Σ_1 onto Σ_2. Lemma 2.3.8 then implies that h maps the triangulation $f_1(\mathrm{T}_0)$ of Σ_1 onto the triangulation $f_2(\mathrm{T}_0)$ of Σ_2. As a consequence, we see that $Z_1 = Z_2$.

For the continuity, we use the same arguments as in the proof of Proposition 2.5.3. □

Since the Teichmüller space $\mathcal{T}(g,n)$ is of real dimension $6g+2n-6$, to prove B.0.1, we have to extend the map Φ_{T_0} to a continuous and injective map from a ball in $\mathbb{R}^{6g+2n-6}$ into $\mathcal{T}(g,n)$. To get such a map, we introduce small perturbations of the system $\mathbf{S}_{\mathrm{T}_0}$. First, we observe that the angles θ_j, $j=1,\dots,2g+n-1$, are not independent. Choose n edges among $b_1,\dots,b_{2g+n-1}$ which form a tree A_0 connecting the singular points $x_1,\dots,x_n$. Such edges exist because any two points in $\{x_1,\dots,x_n\}$ are joined by a path in $(\cup_{j=1}^{2g+n-1}b_j)$. Without loss of generality we can assume that A_0 contains the edges $b_{2g+1},\dots,b_{2g+n-1}$.

Lemma B.1.3 *For every $j\in\{2g+1,\dots,2g+n-1\}$, we have*

$$\theta_j=\eta_j(\alpha_1,\dots,\alpha_n,\theta_1,\dots,\theta_{2g}),$$

where η_j is a linear function with integer coefficients.

Proof: The curves $\{\tilde{c}_1,\dots,\tilde{c}_{2g}\}$ form a basis of the group $H_1(\Sigma_0\setminus A_0,\mathbb{Z})$. Note that since the group $SO(2)$ is Abelian, if the closed curves γ_1 and γ_2 are homologous in $\Sigma_0\setminus\{x_1,\dots,x_n\}$, then $\mathbf{orth}(\gamma_1)=\mathbf{orth}(\gamma_2)$.

For each j in $\{2g+1,\dots,2g+n-1\}$, the curve $\tilde{c}_j$ is homologous to the curve $l_{i_1}\circ\cdots\circ l_{i_k}\circ\tilde{c}'_j$, where $i_s\in\{1,\dots,n\}$, l_{i_s} is a curve homologous to a small loop about x_{i_s}, and $\tilde{c}'_j$ is a closed curve in $\Sigma_0\setminus A_0$.

The curve $\tilde{c}'_j$ represents an element of the group $H_1(\Sigma_0\setminus A_0,\mathbb{Z})$, hence the rotation $\mathbf{orth}(\tilde{c}'_j)$ is determined by the rotations $\mathbf{orth}(\tilde{c}_1),\dots,\mathbf{orth}(\tilde{c}_{2g})$. We deduce that, for every j in $\{2g+1,\dots,2g+n-1\}$, the rotation $\mathbf{orth}(\tilde{c}_j)$ is determined by the angles $\alpha_1,\dots,\alpha_n$ and the rotations $\mathbf{orth}(\tilde{c}_1),\dots,\mathbf{orth}(\tilde{c}_{2g})$. The lemma is then proved. □

B.2 Proof of Proposition B.0.1

Let ϵ be a small positive real number. Set

$$\Lambda=\{\bar{\lambda}=(\lambda_1,\dots,\lambda_{2g})\in\mathbb{R}^{2g}\ :\ |\lambda_j|<\epsilon,\ \forall j=1,\dots,2g\}.$$

For each $\bar{\lambda}=(\lambda_1,\dots,\lambda_{2g})$ in Λ, set $\theta_j(\bar{\lambda})=\theta_j+\lambda_j$, for $j=1,\dots,2g$, and $\theta_j(\bar{\lambda})=\eta_j(\alpha_1,\dots,\alpha_n,\theta_1+\lambda_1,\dots,\theta_{2g}+\lambda_{2g})$, for $j=2g+1,\dots,2g+n-1$. Let $\mathbf{S}_{\mathrm{T}_0}(\bar{\lambda})$ denote the system obtained by replacing θ_j by $\theta_j(\bar{\lambda})$ into $\mathbf{S}_{\mathrm{T}_0}$. Let $\mathrm{V}_{\mathrm{T}_0}(\bar{\lambda})$ denote the sub-space of $\mathbb{C}^{n_1}$ consisting of solutions of $\mathbf{S}_{\mathrm{T}_0}(\bar{\lambda})$.

Since there exists $j \in \{1, \ldots, 2g+n-1\}$ such that $\theta_j \notin \{2k\pi : k \in \mathbb{Z}\}$, if ϵ is small enough, then $\theta_j(\bar{\lambda}) \notin \{2k\pi : k \in \mathbb{Z}\}$, for all $\bar{\lambda} \in \Lambda$. It follows that $\dim_{\mathbb{C}} \mathrm{V}_{\mathrm{T}_0}(\bar{\lambda}) = 2g+n-2$, for all $\bar{\lambda}$ in Λ.

Let $\mathrm{W}_{\mathrm{T}_0}(\bar{\lambda})$ denote the set $\{Z = (z_1, \ldots, z_{n_1}) \in \mathrm{V}_{\mathrm{T}_0}(\bar{\lambda}) |\ \overline{Z}^t \mathbf{H}_{\mathrm{T}_0} Z = 1,\ \mathrm{Im} z_1 = 0\}$. Obviously, we have $\mathrm{V}_{\mathrm{T}_0}(0) = \mathrm{V}_{\mathrm{T}_0}$ and $W_{\mathrm{T}_0}(0) = W_{\mathrm{T}_0}$. Therefore, we can find, for each $\bar{\lambda}$ in Λ, an open subset $\mathrm{U}^1(\bar{\lambda})$ of $\mathrm{W}_{\mathrm{T}_0}(\bar{\lambda})$ homeomorphic to a ball in $\mathbb{R}^{2(2g+n-2)-2}$ such that $\mathrm{U}^1(0) = \mathrm{U}^1_0$, and the set $\mathrm{U}^1(\bar{\lambda})$ varies continuously as $\bar{\lambda}$ varies in Λ.

Let Ω denote the set $\{(Z, \bar{\lambda}) \in \mathbb{C}^{n_1} \times \Lambda \ | Z \in \mathrm{U}^1(\bar{\lambda})\}$. It is now clear that Ω is homeomorphic to an open ball in $\mathbb{R}^{2(2g+n-2)-2} \times \mathbb{R}^{2g} \simeq \mathbb{R}^{6g+2n-6}$. Note that Ω can be realized as a subset of $\mathbb{C}^{n_1}$ such that $\mathrm{U}^1(\bar{\lambda}) = \mathrm{V}_{\mathrm{T}_0}(\bar{\lambda}) \cap \Omega$. We define a map

$$\widetilde{\Phi}_{\mathrm{T}_0} : \Omega \longrightarrow \mathcal{T}(g, n),$$

in the same way as the map Φ_{T_0}, that is, for each $(Z, \bar{\lambda})$ in Ω, we construct a flat surface Σ by forming triangles and gluing them together using T_0 as pattern. Recall that, by this construction, we obtain a pair $(\Sigma, f_\Sigma \circ \phi_0)$, where $f_\Sigma : \Sigma_0 \longrightarrow \Sigma$ is a homeomorphism which sends T_0 onto a geodesic triangulation of Σ.

Using the same arguments as in Lemma B.1.2, we can show that $\widetilde{\Phi}_{\mathrm{T}_0}$ is continuous and injective. Since Ω is homeomorphic to a ball in $\mathbb{R}^{6g+2n-6}$, and the Teichmüller space $\mathcal{T}(g, n)$ is of the same real dimension, the map $\widetilde{\Phi}_{\mathrm{T}_0}$ is a homeomorphism. This implies that $\widetilde{\Phi}_{\mathrm{T}_0}(\Omega)$ is a neighborhood of $[(\Sigma_0, \phi_0)]$, and the proposition is then proved. $\square$

Bibliographie

[B] P. Buser *Geometry and Spectra of Compact Riemann Surfaces*. Progress in Mathematics, Birkhäuser (1992).

[BG] C. Bavard, E. Ghys : Polygones du plan et polyèdres hyperboliques. Geom. Dedicata **43**, No. 2, 207-224 (1992).

[BS] A.I. Bobenko, B.A. Springborn : A discrete Laplace-Beltrami operator for simplicial surfaces. Discrete Comput. Geom., **38**, No. 4, 740-756 (2007).

[EMZ] A. Eskin, H. Masur, A. Zorich : Moduli spaces of abelian differentials : The principal boundary, counting problems, and the Siegel-Veech constants. Publ. Math. Inst. Hautes Études Sci. No. 97, 61-179 (2003).

[EO] A. Eskin, A. Okounkov : Asymptotics of number of branched coverings of a torus and volume of moduli spaces of holomorphic diferentials. Invent. Math., **145 :1**, 59-104 (2001).

[Ga] F. Gardiner : *Teichmüller Theory and Quadratic Differentials*. In collecion : Pure and Applied Mathematics-A Wiley-Interscience series (1987).

[Go] W.M. Goldman : *Complex Hyperbolic Geometry*. In collection : Oxford mathematical monographs, Oxford University Press (1999).

[K] M. Konsevich : Lyapunov exponents and Hodge theory. "The mathematical beauty of physics" (Saclay, 1996), (in Honor of C. Itzykson) 318-332, Adv. Ser. Math. Phys., 24. World Sci. Publishing, River Edge, NJ(1997).

[KMS] S. Kerckhoff, H. Masur, J. Smillie : Ergodicity of billiard flows and quadratic differentials. Ann. of Math. (2) 124, 293-311(1986).

[KZ] M. Konsevich, A. Zorich : Connected components of the moduli spaces of Abelian differentials. Invent. Math., **153 :3**, 631-678 (2003).

[M] H. Masur : Inteval exchange transformations and measured foliations. Annals of Math. **115**, 169-200 (1982).

[MT] H. Masur, S. Tabachnikov : Rational billards and flat structures. In : B. Hasselblatt and A. Katok (ed) : Handbook of Dynamical Systems, Vol. 1A, Elsevier Sience B.V., 1015-1089 (2002).

[Th] W.P. Thurston : Shape of polyhedra and triangulations of the sphere. In : "The Epstein Birthday Schrift", Geom. Topo. Monogr., **1**, Geom. Topo. Pub., Coventry 511-549 (1998).

[Tr1] M. Troyanov : Prescribing curvature on compact surfaces with conical singularities. Trans. Amer. Math. Soc. **324 :2**, 793-821 (1991).

[Tr2] M. Troyanov : On the moduli space of singular Euclidean surfaces. In : "Handbook of Teichmüller theory." Vol. I, 507-540. IRMA Lect. Math. Theor. Phys., **11**, Eur. Math. Soc., Zürich (2007).

[V1] W.A. Veech : Moduli spaces of quadratic differentials. Journal d'Analyse Math., **55**, 117-171 (1990).

[V2] W.A. Veech : Flat surfaces. Amer. Journal of Math., **115**, 589-689 (1993).

[V3] W.A. Veech : Gauss measures for transformations on the space of interval exchange maps. Ann. Math., **115**, 201-242 (1982).

[Z1] A. Zorich : Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents. Ann. Inst. Fourier, **46**, 325-370 (1996).

[Z2] A. Zorich : Flat surfaces. In collection "Frontiers in Number Theory, Physics and Geometry", Vol. 1 : On random matrices, zeta functions and dynamical systems, Ecole de Physique des Houches, France, March 9-21 2003, Springer-Verlag (2006).