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Mean dimension and spaces of pseudo-holomorphic maps

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DIMENSION MOYENNE ET ESPACES D'APPLICATIONS PSEUDO-HOLOMORPHES

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Contents

Introduction

The work presented in this thesis can be divided in two themes. The first concerns metric geometry in certain Banach spaces (chapter 1) and a variant of entropy, which makes sense for dynamical systems whose entropy is infinite, called mean dimension (chapter 2). The second is about surgery in almost-complex manifolds: to obtain a more precise local expansion when two pseudo-holomorphic map are glued together (chapter 3), to construct pseudo-holomorphic cylinders from spheres (chapter 4) and, under stronger hypothesis, to show an interpolation result on such cylinders (chapter 5). In the last chapter the two themes come together: thanks to interpolation, some spaces of pseudo-holomorphic cylinders have a non-trivial mean dimension.

Imagine that we have gathered a certain quantity of information (an image, a sound track, ...) that we shall represent as an element f of a Banach space X. To communicate that information to someone else, it shows advantageous to find an efficient format, even if a part of that information is lost in the process (lossy compression). Transformation into that format is a map $\phi : X \longrightarrow Y$, where Y is for now any set, that is continuous and such that the fibers $\phi^{-1}(y)$ are small. Then there is a recovery algorithm, a map ψ such that $\phi \circ \psi = \mathrm{Id}$ which allows the recipient to have, with some inaccuracy, the information. Often Y is also a (finite dimensional) Banach space and ϕ is a linear map; the problem is then to find a balance between the size (*e.g.* the diameter) of the fibers (which represent the loss) and the size (*e.g.* the dimension) of Y. One is then lead to introduce

$$\mathrm{wdim}_\varepsilon(X,d) = \inf_{f:X \hookrightarrow K} \dim K,$$

where the infimum is taken on all maps f whose fibers are of diameter less than ε and with value in a polyhedron K. This notion introduced in [19] is actually close to Alexandrov (or Urysohn) widths.

This topic is treated in detail by Donoho in [10] (where the question is expressed in terms of Gel'fand or Kolmogorov widths). In this reference, it is in particular shown that to take ϕ in a certain class of maps (qualified as adaptative maps) rather than linear maps does not give a significant edge to compression. However the problem of knowing which are the optimal non-linear maps remains open. Chapter 1 answers these questions for ℓ^p balls endowed with different metrics (to measure the size of the fibers). The proofs use various methods: the filling radius, the Borsuk-Ulam theorem, Hadamard matrices, and works of Pichugov and Ivanov on Jung's constant in ℓ^p spaces (*cf.* [23]). Among the result obtained, theorem 1.1.4 gives estimates, in the case where the ℓ^p ball is endowed with its usual ℓ^p metric, on the relation between the dimension of the target and the diameter of the fibers. Denote by $\mathrm{wspec}(X,d)$ the set of values taken by $f(\varepsilon) = \mathrm{wdim}_\varepsilon(X,d)$. Here is a summary of theorem 1.1.4 (see the full statement for more precise bounds).

Theorem 1: (*cf.* theorem 1.1.4) Let $p \in [1,\infty)$ and $n > 1$, then $\exists h_n \in \mathbb{Z}$ such that $h_n = n/2$ when n is even, $h_3 = 2$ and $h_n = \frac{n+1}{2}$ or $\frac{n-1}{2}$ when n is odd, such that

$$\{0, h(n), n\} \subset \mathrm{wspec}(B_1^{\ell^p(n)}, \ell^p) \subset \{0\} \cup (\frac{n}{2} - 1, n] \cap \mathbb{Z}.$$

When $p = 2$ or $p = 1$ and there exists a Hadamard matrix of rank $n+1$, then $n-1$ also belongs to $\mathrm{wspec}(B_1^{\ell^p(n)}, \ell^p)$.

The notion of wdim is also useful to define an invariant of certain dynamical systems. Look at the dynamic on $A^{\mathbb{Z}}$ (the set of A valued sequences) with the action of the shift. Suppose that A is a compact metric space and that $A^{\mathbb{Z}}$ is endowed with a metric d which induces the product topology. Introduce the dynamical distance (which is a refinement of the topology) $d_n(a,b) = \sup_{-n<i<n} d(i \cdot a, i \cdot b)$ where $i \cdot a$ denotes the sequence a shifted by i. When A is finite, the entropy is defined as the exponential rate of growth in n of the smallest number of balls of radius ε (for d_n) required to cover $A^{\mathbb{Z}}$, denoted $N(\varepsilon, A, n)$. This can be reinterpreted as the growth of the smallest set F such that there exists a map $A \to F$ whose fibers are of diameter (for d_n) less than 2ε. This vision is close to the definition of wdim: instead of looking at the dimension of the target polyhedron, it is the cardinality of the target which measures the size. The constraint on the fibers remains present. If A is a metric space of positive dimension m, the entropy is infinite. Mean dimension is an alteration, due to Gromov (*cf.* [19]) of the definition of entropy: it is the coefficient of linear growth (in n) of the dimension of the smallest polyhedron to which $A^{\mathbb{Z}}$ can be sent by maps whose fibers are of diameter (for d_n) less than ε.

In chapter 2, it is mostly variants of this definition which will be used. The variant in §2.2 enables to distinguish balls in $\ell^p(\Gamma;\mathbb{R})$ spaces, the space of p-summable functions on a countable group Γ.

Theorem 2: (*cf.* theorem 2.2.4) Let $p, q \in [1,\infty[$, let $B_1^{\ell^p}$ and $B_1^{\ell^q}$ be unit balls of $\ell^p(\Gamma;\mathbb{R}^s)$ and $\ell^q(\Gamma;\mathbb{R}^s)$ respectively, both endowed with the natural action of Γ and metrics of type (2.1.7). When $q > p$, there is no Lipschitz homeomorphism $f : B_1^{\ell^p} \to B_1^{\ell^q}$. Furthermore, if $\alpha q > p$ then f is not even Hölder continuous of exponent α.

However, In the case of equality, such maps exist; it suffices to look at the Mazur map ($x \mapsto \frac{x}{|x|} x^{p/q}$). Also, remark that Lipschitz homeomorphisms are excluded in general due to results of Benyamini.

The second variant is an attempt to define a ℓ^p dimension that extends the ℓ^2 dimension of Von Neumann. It consists of a dimension defined for Γ-invariant linear subspaces of $\ell^p(\Gamma;V)$, where V is a finite vector space and Γ an amenable group. Gromov essentially introduced this variant in [19] where it is shown that it coincides with the usual ℓ^2 dimension. §2.3 presents this definition and some of the useful properties that we would like to see verified. §2.4 is dedicated to a generalization of the Ornstein-Weiss lemma (*cf.* [38]).

Theorem 3: (*cf.* theorem 2.4.7) Let Γ be a discrete amenable group, and let $a : \mathbb{R}_{\geq 0} \times \Gamma \longrightarrow \mathbb{R}_{\geq 0}$ be a function such that, $\forall \Omega, \Omega' \subset \Gamma$ finite and $\forall \varepsilon \in \mathbb{R}_{>0}$

(a)	a is Γ-invariant, *i.e.*	$\forall \gamma \in \Gamma,$	$a(\varepsilon, \gamma\Omega) = a(\varepsilon, \Omega)$
(b)	a is decreasing in ε, *i.e.*	$\forall \varepsilon' \leq \varepsilon,$	$a(\varepsilon', \Omega) \geq a(\varepsilon, \Omega)$
(c)	a is K-sublinear in Ω, *i.e.*	$\exists K \in \mathbb{R}_{>0},$	$a(\varepsilon, \Omega) \leq K\lvert\Omega\rvert$
(d)	a is c-subadditive in Ω, *i.e.*	$\exists c \in]0,1],$	$a(\varepsilon, \Omega \cup \Omega') \leq a(c\varepsilon, \Omega) + a(c\varepsilon, \Omega')$

then, for any Følner sequence $\{\Omega_i\}$,

$$\lim_{\varepsilon \to 0} \limsup_{i \to \infty} \frac{a(\varepsilon, \Omega_i)}{|\Omega_i|} = \lim_{\varepsilon \to 0} \liminf_{i \to \infty} \frac{a(\varepsilon, \Omega_i)}{|\Omega_i|}.$$

In particular, these limits are independent of the sequence $\{\Omega_i\}$ chosen.

Condition (d) is in the usual Ornstein-Weiss lemma subadditivity (*i.e.* $c = 1$); in that case the results hold without the need of taking $\varepsilon \to 0$. This theorem allows us to show in particular that the quantity $\dim_{\ell^p}$ is actually independent of the choice of exhaustion, *cf.* corollary 2.5.1. Some further properties of $\dim_{\ell^p}$ are shown in §2.5.

The second part deals with pseudo-holomorphic maps with value in an almost-complex manifold (M, J) (an unfamiliar reader might consider $\mathbb{C}P^n$ with its usual complex structure). Appendix A quickly introduces basic notations and important results already described in detail by McDuff and Salamon, *cf.* [35]. Given two rational curves in $\mathbb{C}P^n$ (holomorphic maps from $\mathbb{C}P^1$ to $\mathbb{C}P^n$) which intersect at a point, it is possible to describe explicitly another map whose image is as close as needed to the union of the images of these two curves. This result still holds in almost complex manifolds (M, J) given that the structure J satisfies a transversality assumption. The family of maps $u^{\delta,r} : \mathbb{C}P^1 \longrightarrow M$ obtained is not very explicit. If the method used is as described in [35], there is a ring A of $\mathbb{C}P^1$ (whose radii are δr and r/δ) which will be sent close to the point of contact m_0 of the two curves. Unfortunately, the behaviour of u in this ring is vague: $\forall z \in A$, $u(z)$ is at distance $O(r)$ of m_0.

Under stronger assumptions than those used in [35], chapter 3 adapts the method in order to obtain a more precise behaviour.

Theorem 4: (*cf.* theorem 3.1.4) Let (M, J), be an almost complex manifold of real dimension at least 4. Let $u^h : \Sigma \longrightarrow M$, where $h \in \{0, 1\}$, two J-holomorphic curves such that $u^h(0) = m_0$, $\|du^h\|_{L^\infty} \leq C$, J is regular in the sense of definition A.3.4 and D_{u^h} are surjective. If in a chart

$u^h(z) = a^h z + O(|z|^2)$, and that the a^h are linearly independent over $\mathbb{C}$, then $\exists r_0$ t.q. $\forall r \leq r_0$, $\exists u$ a J-holomorphic curve such that in that chart,

$$u(z) = a^0 z + a^1 \frac{r^2}{z} + O(r^{1+\varepsilon})$$

for $z \in A_{r^{4/3}, r^{2/3}} = \{z | r^{4/3} < |z| < r^{2/3}\}$ and where $\varepsilon \in]0, \frac{1}{3}[$; r_0 and c_0 depend on C, on ε, on the a^h, on the second derivatives of u^h, on J (up to its second derivatives) and on the norms of D_{u^h}.

A more precise knowledge of the behaviour will be of use in the proof of proposition 5.3.4. The main effect of this "precise" gluing is that the intersection of a ball (in M) whose radius is of the order of the perturbation with the image of the approximate solution is a disc. When the approximate solution is constant in a ring, as in [35], this property does not hold.

Nevertheless, the main motivation to obtain a better control on the local expansion comes from an article of Donaldson, *cf.* [8]. In this article, Donaldson adapts methods of [47] to get a Runge (approximation) theorem on instantons. Recall that the Runge theorem insures that any holomorphic map defined on a simply connected open U of $\mathbb{C}$ can be approximated on compact sets $K \subset U$ in L^∞ norm by holomorphic maps defined on the whole plane (if U is a disc, this is just a truncation of the local expansion). A method which could extend this result in the pseudo-holomorphic context is sketched in [8]. For a pseudo-holomorphic map $u : U \longrightarrow M$ defined on $U \subset \mathbb{C}P^1$ there always exists a C^∞ extension, $u_0 : \mathbb{C}P^1 \longrightarrow M$. In the neighborhood V of a point z_0 where u_0 is not pseudo-holomorphic (and in an appropriate local chart) $\phi \circ u_0(z) = a_0 z + a_1 \bar{z} + O(|z|^2)$. The crucial remark is that $\bar{z} = \frac{r^2}{z}$ if $|z| = r$. It is thus possible to modify u_0 as follows: outside a disc it remains the same, but inside a smaller disc it is replaced by a pseudo-holomorphic v which possesses the local expansion $\phi \circ v(z) = a_0 z + a_1 \frac{r^2}{z} + O(r^{1+\varepsilon})$ when z belongs to a ring. Introducing cutoff functions to go from v to u_0 and adjusting appropriately the size of the discs and rings, a new map u_1 is obtained. It is holomorphic on a slightly larger region than u_0. If this operation is repeated a large number of times, a map u_N which is pseudo-holomorphic on a very big region is obtained. A fixed point theorem would then be welcome. A considerable number of surgeries take place in regions which are close together making classical techniques unusable. However, Taubes is able to make such an argument work in [47]. Unfortunately, the tools developped there seem deficient precisely when the dimension at the source is 2, the case of our problem. The adaptation in dimension 2 of Taubes' toolbox can be found in appendix B.

However, when the number of surgeries to realise on a curve is bounded, these problems do not arise. For example, if instead of maps which intersect at a point, a family of pseudo-holomorphic curves $u^i : \mathbb{C}P^1 \longrightarrow M$ where $i \in \mathbb{Z}$ is given with the property that the i^{th} crosses the $(i+1)^{\text{th}}$ then theorem 4.1.5 insures (under regularity assumptions for J and compactness of the parameters coming from the u^i) the existence of a pseudo-holomorphic map from the cylinder $\Sigma = \mathbb{C}/2\pi\mathbb{Z}$ whose image is close to the union of the images of these curves. Indeed, even if the number of surgeries is infinite (as in the failed method to prove a Runge theorem) the number of operations on each u^i is finite, this allows the use of a mixed norm $\ell^\infty(L^p)$ (*cf.* (4.1.1)) to which the arguments of [35] or of chapter 3 can be transposed without much difficulty.

Finally, taking again stronger assumptions, chapter 5 gives an interpolation result on such pseudo-holomorphic cylinders .

Theorem 5: (*cf.* theorem 5.1.3) Let (M,J) be an almost complex manifold. Let $u^1,\dots,u^N$ be a finite family of J-holomorphic curves $u^i : \mathbb{C}\mathrm{P}^1 \longrightarrow M$ such that $u^i(\infty) = u^j(0)$ when $j \equiv i+1 \bmod N$. Suppose that J is regular in the sense of definition 4.2.1 and that u^j is deformable. Let $z_* \in \mathbb{C}\mathrm{P}^1 \setminus \{0,\infty\}$ be a marked point and let $m_* = u^j(z_*) \in M$ be its image. There exists a $R \in \mathbb{R}_{>}0$ such that for any sequence $\{r_i\}$ where $r_i \in]0,R[$, there exists a neighborhood V_{m_*} of m_* such that for any sequence of points $\{m_k\}_{k\in\mathbb{Z}}$ in V_{m_*} there exists a J-holomorphic cylinder $u : \Sigma \longrightarrow M$ satisfying the following properties: u is close to the curves u^i and

$$u(z_{k,*}) = m_k,$$

where $r_{\sup} = \sup r_i$, $m_{\sup} = \sup d_M(m_k,m_*)$ and $z_{k,*} = \mu_{j+Nk;r_{j+Nk},r_{j+Nk+1}}(z_*)$.

This interpolation has three consequences. The first concerns the mean dimension of the set $\mathcal{M}$ of J-holomorphic maps from Σ to M. Indeed, automorphisms of the cylinder act naturally on this space. Since the topology of uniform convergence on compact sets can be induced by a distance, mean dimension of this space for the action of the automorphism group can be defined. Corollary 5.3.1 shows that it is positive. Second, for a careful (and not too restrictive) choice of m_k, the resulting J-holomorphic map is simple. Third, suppose the approximate solution used in the proof of the theorem is not injective only at a finite number of points. From proposition 5.3.4 one can conclude, perhaps upon restricting the parameters further, that if two curves obtained by the theorem have the same image then they differ by an automorphism. In particular, most of these curves are not obtained trivially by precomposing a given pseudo-holomorphic map $\Sigma' \longrightarrow M$ by holomorphic maps $\Sigma \longrightarrow \Sigma'$.

Chapter 1

Width of ℓ^p balls

1.1 Introduction

Let (X,d) be a metric space and $\varepsilon \in \mathbb{R}_{>0}$, then we say a map $f : X \to Y$ is an ε-embedding if it is continuous and the diameter of the fibers is less than ε, *i.e.* $\forall y \in Y, \mathrm{Diam} f^{-1}(y) \leq \varepsilon$. We will use the notation $f : X \overset{\varepsilon}{\hookrightarrow} Y$. This type of maps, which can be traced at least to the work of Pontryagin (see [42] or [22]), is related to the notion of Urysohn width (sometimes referred to as Alexandrov width), $a_n(X)$, see [11]. It is the smallest real number such that there exists an ε-embedding from X to a n-dimensional polyhedron. The question of estimating these widths can be seen as trying to minimize the loss of information in a non-linear compression algorithm. The Kolmogorov width (and also the Gel'fand width) plays an important role when one is interested in a linear (or nonadaptive) compression, see [10] for details.

Surprisingly few estimations of $a_n(X)$ can be found, and one of the aims of this chapter is to present some. However, following [19], we shall introduce:

Definition 1.1.1: $\mathrm{wdim}_\varepsilon X$ is the smallest integer k such that there exists an ε-embedding $f : X \to K$ where K is a k-dimensional polyhedron.

$$\mathrm{wdim}_\varepsilon(X,d) = \inf_{X \overset{\varepsilon}{\hookrightarrow} K} \dim K.$$

Thus, it is equivalent to be given all the Urysohn's widths or the whole data of $\mathrm{wdim}_\varepsilon X$ as a function of ε.

Definition 1.1.2: The wdim spectrum of a metric space (X,d), denoted $\mathrm{wspec} X \subset \mathbb{Z}_{\geq 0} \cup \{+\infty\}$, is the set of values taken by the map $\varepsilon \mapsto \mathrm{wdim}_\varepsilon X$.

The $a_n(X)$ obviously form an non-increasing sequence, and the points of $\mathrm{wspec} X$ are precisely the integers for which it decreases. We shall be interested in the widths of the following metric spaces: let $B_1^{\ell^p(n)}$ be the set given by the unit ball in $\mathbb{R}^n$ for the ℓ^p metric ($\|(x_i)\|_{\ell^p} = \left(\sum |x_i|^p\right)^{1/p}$), but look at $B_1^{\ell^p(n)}$ with the ℓ^∞ metric (*i.e.* the sup metric of the product). Then

Proposition 1.1.3: $\mathrm{wspec}(B_1^{\ell^p(n)}, \ell^\infty) = \{0,1,\ldots,n\}$, *and,* $\forall \varepsilon \in \mathbb{R}_{>0}$,

$$\mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^\infty) = \begin{cases} 0 & \text{if} \quad 2 \leq \varepsilon \\ k & \text{if} \quad 2(k+1)^{-1/p} \leq \varepsilon < 2k^{-1/p} \\ n & \text{if} \quad \varepsilon < 2n^{-1/p} \end{cases}.$$

The proof uses an common argument of compression algorithm, namely that the map that keeps only the big coordinates does not loose much data (*i.e.* has small fiber in the appropriate norms). The important outcome of this proposition is that for fixed ε, the $\mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^\infty)$ is bounded from below by $\min(n, m(p,\varepsilon))$ and from above by $\min(n, M(p,\varepsilon))$, where m, M are independent of n. As an upshot high values can only be reached for small ε independently of n. It can be used to show that the mean dimension of the unit ball of $\ell^p(\Gamma)$, for Γ a countable group, with the natural action of Γ and the weak-* topology is zero when $p < \infty$ (see [48]). It is one of the possible ways of proving the non-existence of action preserving homeomorphisms between $\ell^\infty(\Gamma)$ and $\ell^p(\Gamma)$ (with the weak-* or product topology). A simpler argument would be to notice that with the weak-* topology, Γ sends all points of $\ell^p(\Gamma)$ to 0 while $\ell^\infty(\Gamma)$ has many periodic orbits.

The behaviour is quite different when balls are looked upon with their natural metric.

Theorem 1.1.4: *Let* $p \in [1,\infty)$, $n > 1$, *then* $\exists h_n \in \mathbb{Z}$ *satisfying* $h_n = n/2$ *for n even,* $h_3 = 2$ *and* $h_n = \frac{n+1}{2}$ *or* $\frac{n-1}{2}$ *otherwise, such that*

$$\{0, h(n), n\} \subset \mathrm{wspec}(B_1^{\ell^p(n)}, \ell^p) \subset \{0\} \cup (\frac{n}{2} - 1, n] \cap \mathbb{Z}.$$

When $p = 2$ *or when* $p = 1$ *and there is a Hadamard matrix of rank* $n+1$, *then* $n-1$ *also belongs to* $\mathrm{wspec}(B_1^{\ell^p(n)}, \ell^p)$.

More precisely, let $k, n \in \mathbb{N}$ *with* $\frac{n}{2} - 1 < k < n$. *Then there exists* $b_{n;p} \in [1,2]$ *and* $c_{k,n;p} \in [1,2)$ *such that*

$$\begin{array}{llll} \text{if } \varepsilon & \geq 2 & \text{then} & \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^p) = 0 \\ \text{if } \varepsilon & < 2 & \text{then} & \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^p) > \frac{n}{2} - 1 \\ \text{if } \varepsilon & \geq c_{k,n;p} & \text{then} & \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^p) \leq k \\ \text{if } \varepsilon & < b_{k;p} & \text{then} & \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^p) \geq k \end{array}$$

and, for fixed n and p, the sequence $c_{k,n;p}$ *is non-increasing. Furthermore,* $b_{k;p} \geq 2^{1/p'}\left(1 + \frac{1}{k}\right)^{1/p}$ *when* $1 \leq p \leq 2$, *whereas* $b_{k;p} \geq 2^{1/p}\left(1 + \frac{1}{k}\right)^{1/p'}$ *if* $2 \leq p < \infty$.

Additionally, in the Euclidean case ($p = 2$), *we have that* $b_{n;2} = c_{n-1,n;2} = \sqrt{2(1 + \frac{1}{n})}$, *while in the 2-dimensional case* $b_{2;p} \geq \max(2^{1/p}, 2^{1/p'})$ *for any* $p \in [1,\infty]$. *Also, if* $p = 1$, *and there is a Hadamard matrix in dimension* $n+1$, *then* $b_{n;1} = c_{n-1,n;1} = \left(1 + \frac{1}{n}\right)$. *Finally, when* $n = 3$, $\forall \varepsilon > 0, \mathrm{wdim}_\varepsilon B_1^{\ell^p(n)} \neq 1$ *and* $c_{2,3;p} \leq 2(\frac{2}{3})^{1/p}$, *which means in particular that* $c_{2,3;p} = b_{3;p}$ *when* $p \in [1,2]$.

Various techniques are involved to achieve this result; they will be presented in section 1.3. While upper bounds on $\mathrm{wdim}_\varepsilon X$ are obtained by writing down explicit maps to a space of the proper dimension (these constructions use Hadamard matrices), lower bounds are found as consequences of the Borsuk-Ulam theorem, the filling radius of spheres, and lower bounds for the diameter of sets of $n+1$ points not contained in an open hemisphere (obtained by methods very close to those of [23]). We are also able to give a complete description in dimension 3 for $1 \leq p \leq 2$.

1.2 Properties of $\mathrm{wdim}_\varepsilon$

Here are a few well established results; they can be found in [6], [7], [30], and [31].

Proposition 1.2.1: *Let (X,d) and (X',d') be two metric spaces.* $\mathrm{wdim}_\varepsilon$ *has the following properties:*

a. *If X admits a triangulation,* $\mathrm{wdim}_\varepsilon(X,d) \leq \dim X$.

b. *The function* $\varepsilon \mapsto \mathrm{wdim}_\varepsilon X$ *is non-increasing.*

c. *Let X_i be the connected components of X, then* $\mathrm{wdim}_\varepsilon(X,d) = 0 \Leftrightarrow \varepsilon \geq \max_i \mathrm{Diam} X_i$.

d. *If $f:(X,d) \to (X',d')$ is a continuous function such that $d(x_1,x_2) \leq Cd'(f(x_1),f(x_2))$ where $C \in]0,\infty[$, then* $\mathrm{wdim}_\varepsilon(X,d) \leq \mathrm{wdim}_{\varepsilon/C}(X',d')$.

e. *Dilations behave as expected,* i.e. *let $f:(X,d) \to (X',d')$ be an homeomorphism such that $d(x_1,x_2) = Cd'(f(x_1),f(x_2))$; this equality passes through to the* wdim*:* $\mathrm{wdim}_\varepsilon(X,d) = \mathrm{wdim}_{\varepsilon/C}(X',d')$.

f. *If X is compact, then* $\forall \varepsilon > 0, \mathrm{wdim}_\varepsilon(X,d) < \infty$.

Proof: They are brought forth by the following remarks:

a. If $\dim X = \infty$, the statement is trivial. For X a finite-dimensional space, it suffices to look at the identity map from X to a triangulation $T(X)$, which is continuous and injective, thus an ε-embedding $\forall\varepsilon$.

b. If $\varepsilon \leq \varepsilon'$, an ε-embedding is also an ε'-embedding.

c. If $\mathrm{wdim}_\varepsilon X = 0$ then $\exists\phi : X \overset{\varepsilon}{\hookrightarrow} K$ where K is a totally discontinuous space. $\forall k \in K, \phi^{-1}(k)$ is both open and closed, which implies that it contains at least one connected component, consequently $\mathrm{Diam} X_i \leq \varepsilon$. On the other hand, if $\varepsilon \geq \mathrm{Diam} X_i$ the map that sends every X_i to a point is an ε-embedding.

d. If $\mathrm{wdim}_{\varepsilon/C} X' = n$, there exists an $\frac{\varepsilon}{C}$-embedding $\phi : X' \to K$ with $\dim K = n$. Noticing that the map $\phi \circ f$ is an ε-embedding from X to K yields the claimed inequality.

e. This statement is a simple application of the previous one for f and f^{-1}.

f. To show that $\mathrm{wdim}_\varepsilon$ is finite, we will use the nerve of a covering; see [22, §V.9] for example. Given a covering of X by balls of radius less than $\varepsilon/2$, there exists, by compactness, a finite subcovering. Thus, sending X to the nerve of this finite covering is an ε-embedding in a finite dimensional polyhedron. $\square$

Another property worth noticing is that $\lim\limits_{\varepsilon\to 0} \mathrm{wdim}_\varepsilon(X,d) = \dim X$ for compact X; we refer the reader to [6, prop 4.5.1]. Reading [17, app.1] leads to believe that there is a strong relation between wdim and the quantities defined therein (Rad_k and Diam_k); the existence of a relation between wdim and the filling radius becomes a natural idea, implicit in [19, §1.1B]. We shall make a small parenthesis to remind the reader of the definition of this concept, it is advised to look in [17, §1] for a detailed discussion.

Let (X,d) be a compact metric space of dimension n, and let $L^\infty(X)$ be the (Banach) space of real-valued bounded functions on X, with the norm $\|f\|_{L^\infty} = \sup\limits_{x\in X} |f(x)|$. The metric on X yields an isometric embedding of X in $L^\infty(X)$, known as the Kuratowski embedding:

$$\begin{aligned} I_X : X &\to L^\infty(X) \\ x &\mapsto f_x(x') = d(x,x'). \end{aligned}$$

The triangle inequality ensures that this is an isometry:

$$\|f_x - f_{x'}\|_{L^\infty} = \sup_{x''\in X} \left|d(x,x'') - d(x',x'')\right| = d(x,x').$$

Denote by $U_\varepsilon(X)$ the neighborhood of $X \subset L^\infty(X)$ given by all points at distance less than ε from X,

$$\textit{i.e.} \qquad U_\varepsilon(X) = \left\{f \in L^\infty(X) \,\middle|\, \inf_{x\in X} \|f - f_x\|_{L^\infty} < \varepsilon\right\}.$$

Definition 1.2.2: The filling radius of a n-dimensional compact metric space X, written $\mathrm{FilRad}X$, is defined as the smallest ε such that X bounds in $U_\varepsilon(X)$, *i.e.* $I_X(X) \subset U_\varepsilon(X)$ induces a trivial homomorphism in simplicial homology $H_n(X) \to H_n(U_\varepsilon(X))$.

Though FilRad can be defined for an arbitrary embedding, we will only be concerned with the Kuratowski embedding.

Lemma 1.2.3: *Let (X,d) be a n-dimensional compact metric space, $k < n$ an integer, and $Y \subset X$ a k-dimensional closed set representing a trivial (simplicial) homology class in $H_k(X)$. Then*

$$\varepsilon < 2\mathit{FilRad}Y \Rightarrow \mathrm{wdim}_\varepsilon(X,d) > k.$$

If we remove the assumption that $[Y] \in H_k(X)$ be trivial, the inequality is no longer strict: $\mathrm{wdim}_\varepsilon(X,d) \geq k$.

Proof: Let us show that $\mathrm{wdim}_\varepsilon(X,d) \leq k \Rightarrow \varepsilon \geq 2\mathrm{FilRad}Y$. Let us be given an ε-embedding $\phi : X \overset{\varepsilon}{\hookrightarrow} K$. Compactness of X allows us to suppose that ϕ is onto a compact K. Otherwise, we restrict the target to $\phi(X)$.

We will first produce a map $Y \to L^\infty(Y)$ (actually defined on X) whose image is contained in $U_{\varepsilon/2}(Y)$, which is homotopic to I_Y and which factors through ϕ. Let

$$\begin{array}{rcl} Q : K & \to & L^\infty(X) \\ k & \mapsto & g_k(x'') = \varepsilon/2 + \inf_{x' \in \phi^{-1}(k)} d(x'', x') \end{array} \quad \text{, and} \quad \begin{array}{rcl} \rho_Y : L^\infty(X) & \to & L^\infty(Y) \\ f & \mapsto & f\big|_Y . \end{array}$$

First, notice that $g = \rho_Y \circ Q \circ \phi(Y) \subset U_{\delta+\varepsilon/2}(Y), \forall \delta > 0$:

$$\|g - I_Y(y)\|_{L^\infty} = \sup_{y'' \in Y} \left| \frac{\varepsilon}{2} + \left[\inf_{y' \in \phi^{-1}(\phi(y))} d(y'', y') \right] - d(y'', y) \right| = \varepsilon/2,$$

since ϕ is an ε-embedding. Second, $g \sim I_Y$ in $U_{\varepsilon/2}(Y)$, as $L^\infty(Y)$ is a vector space; the homotopy is $h(y,t) = (1-t)g(y) + tI_Y(y)$. Furthermore, $\phi(Y) \subset K$ bounds as a singular chain. That is $\phi_(Y) = c' + \partial c$ where c' is a simplicial chain and c is a singular chain. Since $[Y] = 0$ in $H_k(X)$, $[\phi(Y)] = \phi_*[Y] = [c'] = 0$. Moreover, $\dim K \leq k = \dim Y$, so $c' = 0$. Hence, $\phi_* Y = \partial c$. As singular chains are approximated by simplicial chains arbitrarily closely, $\forall \delta > 0, [g(Y)] = 0$ in $H_k\Big(U_{\delta+\varepsilon/2}(g(Y)), \mathbb{Z}\Big)$. Consequently, $I_Y(Y)$ bounds in $U_{\delta+\varepsilon/2}(Y)$, $\forall \delta > 0$. Y will bound in its $\frac{\varepsilon}{2}$-neighborhood. This will mean that $\varepsilon \geq 2\mathrm{FilRad}\, Y$.

If $[Y] \neq 0 \subset H_k(X)$, the proof still follows by taking K of dimension $k-1$: the homology class $\phi_*[Y]$ is then inevitably trivial, since K has no rank k homology. □

Thus, calculating FilRad is a good starting point. The following lemma gives a lower bound for FilRad:

Lemma 1.2.4: *Let X be a closed convex set in a n-dimensional normed vector space. Suppose it contains a point x_0 such that the convex hull of $n+1$ points on ∂X whose diameter is $< a$ excludes x_0. Then FilRad $\partial X \geq a/2$, and, using lemma 1.2.3, $\varepsilon < a \Rightarrow \mathrm{wdim}_\varepsilon X = n$.*

Proof: Suppose that $Y = \partial X$ has a filling radius less than $a/2$. Then, $\exists \varepsilon > 0$ and $\exists P$ a polyhedron such that Y bounds in P, $P \subset U_{\frac{a}{2}-\varepsilon}(Y)$ and that the simplices of P have a diameter less than ε. To any vertex $p \in P$ it is possible to associate $f(p) \in I_Y(Y)$ so that $\|p - f(p)\|_{L^\infty(Y)} < \frac{a}{2} - \varepsilon$ and $f(p) = p$ if $p \in I_Y(Y)$. Let $p_0, \ldots, p_n$ be a n-simplex of P,

$$\mathrm{Diam}\{f(p_0), \ldots, f(p_n)\} < 2(\frac{a}{2} - \varepsilon) + \varepsilon < a - \varepsilon < a.$$

Since I_Y is an isometry, $f(p_i)$ can be seen as points of Y without changing the diameter of the set they form. Extend f to P by mapping simplices of P to simplices in X (as X is in a vector space). The convex hull of these $f(p_i)$ in X will not contain x_0: the diameter of this set is $< a$. Let π be the projection away from x_0, that is associate to $x \in X$, the point $\pi(x) \in \partial X$ on the half-line joining x_0 to x. Using π, the n-simplex generated by the $f(p_i)$ yields a simplex in Y. Thus $g = \pi \circ f$ maps P to Y.

Furthermore, we claim that $(g \circ I_Y)_*[Y] = [Y]$. Indeed, by definition of f, points of Y are not displaced by more than 2ε, and π is Lipschitz in a sufficiently small neighborhood of Y. Hence for ε sufficiently small $g \circ I_Y : Y \to Y$ will be homotopic to the identity (Y is homeomorphic to a sphere).

Now, let c be a n-chain of P which bounds $I_Y(Y)$, *i.e.* $[I_Y(Y)] = \delta c$. A contradiction becomes apparent: $[I_Y(Y)] = g_*[I_Y(Y)] = g_*(\delta c) = \delta g_* c$. Indeed, if that was to be true, Y, which is $n-1$ dimensional would be bounding an n-dimensional chain in Y. Hence $\mathrm{FilRad}\, Y > a/2$. □

This yields, for example:

Lemma 1.2.5: (cf. *[19, §1.1B]) Let B be the unit ball of a n-dimensional Banach space, then $\forall \varepsilon < 1, \mathrm{wdim}_\varepsilon B = n$.*

Proof: Any set of $n+1$ points on $Y = \partial B$ whose diameter is less than 1 does not contain the origin in its convex hull. So according to lemma 1.2.4, $\mathrm{FilRad}\, Y > 1/2$, and since Y is a closed set of dimension $n-1$ whose homology class is trivial in B, we conclude by applying lemma 1.2.3. □

Let us emphasise this important fact on ℓ^∞ balls in finite dimensional space.

Lemma 1.2.6: *Let $B_1^{\ell^\infty(n)} = [-1,1]^n$ be the unit cube of $\mathbb{R}^n$ with the product (supremum) metric, then*

$$\mathrm{wdim}_\varepsilon B_1^{\ell^\infty(n)} = \begin{cases} 0 & \text{if } \varepsilon \geq 2 \\ n & \text{if } \varepsilon < 2 \end{cases}.$$

This lemma will be used in the proof of proposition 1.1.3. Its proof, which uses the Brouwer fixed point theorem and the Lebesgue lemma, can be found in [31, lem 3.2], [7, prop 2.7] or [6, prop 4.5.4].

Proof of proposition 1.1.3: We first show the lower bound on $\mathrm{wdim}_\varepsilon$. In a k-dimensional space, the ℓ^∞ ball of radius $k^{-1/p}$ is included in the ℓ^p ball: $B_{k^{-1/p}}^{\ell^\infty(k)} \subset B_1^{\ell^p(k)}$, as $\|x\|_{\ell^p(k)} \leq k^{1/p}\|x\|_{\ell^\infty(k)}$. Since $B_1^{\ell^p(k)} \subset B_1^{\ell^p(n)}$, by 1.2.1.d, we can conclude that, if $B_1^{\ell^p(n)}$ is considered with the ℓ^∞ metric, $\varepsilon < 2k^{-1/p}$ implies that $\mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^\infty) \geq k$.

To get the upper bound, we give explicit ε-embeddings to finite dimensional polyhedra. This will be done by projecting onto the union of $(n-j)$-dimensional coordinates hyperplanes (whose points have at least j coordinates equal to 0). Project a point $x \in B_1^{\ell^p(n)}$ by the map π_j as follows: let m be its j^{th} smallest coordinate (in absolute value), set it and all the smaller coordinates to 0, other coordinates are substracted m if they are positive or added m if they are negative.

Denote by $\vec{\varepsilon}$ an element of $\{-1,1\}^n$ and $\vec{\varepsilon}_{\setminus A}$ the same vector in which $\forall i \in A$, ε_i is replaced by 0. The largest fiber of the map π_j is

$$\pi_j^{-1}(0) = \bigcup_{\vec{\varepsilon}, i_1, \ldots, i_{j-1}} \{\lambda_0 \vec{\varepsilon} + \sum_{1 \leq l \leq j-1} \lambda_l \vec{\varepsilon}_{\setminus \{i_1, \ldots, i_l\}} | \lambda_i \in \mathbb{R}_{\geq 0}\} \cap B_1^{\ell^p(n)}.$$

Its diameter is achieved by $s_0 = \big((n-j+1)^{-1/p}, \ldots, (n-j+1)^{-1/p}, 0, \ldots, 0\big)$ and $-s_0$; thus $\mathrm{Diam}\, \pi_j^{-1}(0) = 2(n-j+1)^{-1/p}$. π_j allows us to assert that

$$\varepsilon > 2(n-j+1)^{-1/p} \Rightarrow \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^\infty) \leq n-j,$$

by realising a continuous map in a $(n-j)$-dimensional polyhedron whose fibers are of diameter less than $2(n-j+1)^{-1/p}$. $\square$

As mentionned before, the map to finite dimensional polyhedra is well-known: for example, in compressed sensing one decomposes a signal in some basis (*e.g.* Fourier) then throws aways the small factors without significantly losing information on the initial signal. See also [3, Example 1.5]. This argument for an upper bound also applies to $\mathrm{wdim}_\varepsilon(B^{\ell^p(n)}, \ell^q) \leq k$ if $\varepsilon \geq 2(k+1)^{1/q-1/p}$, but the inclusion of a ℓ^q ball of proper radius in the ℓ^p ball gives a lower bound that does not meet these numbers (see also [48]). Also note that lemma 1.2.3 is efficient to evaluate width of tori, as the filling radius of a product is the minimum of the filling radius of each factor. See example 1.4.1 at the end of this chapter.

1.3 Evaluation of $\mathrm{wdim}_\varepsilon B_1^{\ell^p(n)}$

We now focus on the computation of $\mathrm{wdim}_\varepsilon X$ for unit the ball in finite dimensional ℓ^p. Except for a few cases, the complete description is hard to give. We start with a simple example.

Example 1.3.1*:* Let $B^{\ell^1(2)}$ be the unit ball of $\mathbb{R}^2$ for the ℓ^1 metric, then

$$\mathrm{wdim}_\varepsilon B^{\ell^1(2)} = \begin{cases} 0 & \text{if} \quad \varepsilon \geq 2, \\ 2 & \text{if} \quad \varepsilon < 2. \end{cases}$$

If $B^{\ell^1(2)}$ is endowed with the ℓ^p metric, then $\varepsilon < 2^{1/p} \Rightarrow \mathrm{wdim}_\varepsilon B^{\ell^1(2)} = 2$.

Proof: Given any three points whose convex hull contains the origin, two of them have to be on opposite sides, which means their distance is $2^{1/p}$ in the ℓ^p metric. Hence a radial projection is possible for simplices whose vertices form sets of diameter less than $2^{1/p}$. Invoking lemma 1.2.4, $\mathrm{FilRad}\,\partial B^{\ell^1(2)} \geq 2^{-1+1/p}$. Lemma 1.2.3 concludes. This is specific to dimension 2 and is coherent with lemma 1.2.6, since, in dimension 2, ℓ^∞ and ℓ^1 are isometric. $\square$

An interesting lower bound can be obtained thanks to the Borsuk-Ulam theorem; as a reminder, this theorem states that a map from the n-dimensional sphere to $\mathbb{R}^n$ has a fiber containing two opposite points.

Proposition 1.3.2: *Let $S = \partial B_1^{\ell^p(n+1)}$ be the unit sphere of a $(n+1)$-dimensional Banach space, then*

$$\varepsilon < 2 \Rightarrow \mathrm{wdim}_\varepsilon S > (n-1)/2.$$

In particular, the same statement holds for $B_1^{\ell^p(n+1)}$: $\varepsilon < 2 \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(n+1)} > (n-1)/2$.

Proof: We will show that a map from S to a k-dimensional polyhedron, for $k \leq \frac{n-1}{2}$, sends two antipodal points to the same value. Since radial projection is a homeomorphism between S and the Euclidean sphere $S^n = \partial B_1^{\ell^2(n+1)}$ that sends antipodal points to antipodal points, it will be sufficient to show this for S^n. Let $f : S^n \to K$ be an ε-embedding, where K is a polyhedron,

$\dim K = k \le (n-1)/2$ and $\varepsilon < 2$. Since any polyhedron of dimension k can be embedded in $\mathbb{R}^{2k+1}$, f extends to a map from S^n to $\mathbb{R}^n$ that does not associate the same value to opposite points, because $\varepsilon < 2$. This contradicts Borsuk-Ulam theorem. The statement on the ball is a consequence of the inclusion of the sphere. $\square$

Hence, $\mathrm{wdim}_\varepsilon B_1^{\ell^p(n)}$ always jumps from 0 to at least $\lfloor \frac{n}{2} \rfloor$ if they are equipped with their proper metric.

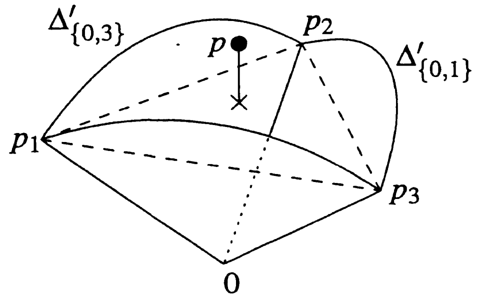
1.3.a A first upper bound. Though this first step is very encouraging, a precise evaluation of wdim can be convoluted, even for simple spaces. It seems that describing an explicit continuous map with small fibers remains the best way to get upper bounds. Denote by $\mathfrak{n} = \{0, \dots, n\}$.

Lemma 1.3.3: *Let B be an unit ball in a normed n-dimensional real vector space. Let $\{p_i\}_{0 \le i \le n}$ be points on the sphere $S = \partial B$ that are not contained in a closed hemisphere. Suppose that $\forall A \subset \mathfrak{n}$ with $|A| \le n-2$, and $\forall \lambda_j \in \mathbb{R}_{>0}$, where $j \in \mathfrak{n}$, if $\|\sum_{i \in A} \lambda_i p_i\| \le 1$, $k \notin A$ and $\|\sum_{i \in A} \lambda_i p_i - \lambda_k p_k\| \le 1$, then $\|\lambda_k p_k\| \le 1$. A set p_i satisfying this assumption gives*

$$\varepsilon \ge \mathrm{Diam}\{p_i\} := \max_{i \ne j} \|p_i - p_j\| \Rightarrow \mathrm{wdim}_\varepsilon B \le n-1.$$

Proof: This will be done by projecting the ball on the cone with vertex at the origin over the $n-2$ skeleton of the simplex spanned by the points p_i. Note that $n+1$ points satisfying the assumption of this lemma cannot all lie in the same open hemisphere, however we need the stronger hyptothesis that they do not belong to a closed hemisphere. Now let Δ_n be the n-simplex given by the convex hull of $p_0, \dots, p_n$. We will project the ball on the various convex hulls of 0 and $n-1$ of the p_i. Call $\mathcal{E}$ the radial projection of elements of the ball (save the origin) to the sphere, and let, for $A \subset \mathfrak{n}$, $P_A = \{p_0, \dots, p_n\} \setminus \{p_i | i \in A\}$. In particular, $P_\varnothing$ is the set of all the p_i. Furthermore, denote by CX the convex hull of X. Given these notations, $\mathcal{E}CP_{\{i\}}$ is the radial projection of the $(n-1)$-simplex $CP_{\{i\}}$ ($CP_{\{i\}}$ does not contain 0 else the points would lie in a closed hemisphere), and $\mathcal{E}CP_{\{i,j\}}$ are parts of the boundary of this projection. Finally, consider, again for $A \subset \mathfrak{n}$, $\Delta'_A = C[\mathcal{E}CP_A \cup 0]$.

Let $s_i : \Delta'_{\{i\}} \to \bigcup_{j \ne i} \Delta'_{\{i,j\}}$ be the projection along p_i. More precisely, we claim that $s_i(p)$ is the unique point of $\Delta'_{\{i,j\}}$ that also belongs to $\Lambda_{p_i}(p) = \{p + \lambda p_i | \lambda \in \mathbb{R}_{\ge 0}\}$. Existence is a consequence of the fact that the points are not contained in an closed hemisphere, *i.e.* $\exists \mu_i \in \mathbb{R}_{>0}$ such that $\sum_{k \in \mathfrak{n}} \mu_k p_k = 0$. Indeed, $p \in \Delta'_{\{i\}}$, if $p \in \Delta'_{\{i,j\}}$ for some j, then there is nothing to show. Suppose that $\forall j \ne i, p \notin \Delta'_{\{i,j\}}$. Then $p = \sum_{k \ne i} \lambda_k p_k$, where $\lambda_k > 0$. Write $p_i = -\frac{1}{\mu_i} \sum_{k \ne i} \mu_k p_k$. It follows that for some λ, $p + \lambda p_i$ can be written as $\sum_{k \in \mathfrak{n} \setminus \{i,j\}} \lambda'_k p_k$ with $0 \le \lambda'_k \le \lambda_k$. Uniqueness comes from a transversality observation. $\Delta'_{\{i,j\}}$ is contained in the plane generated by the set $P_{\{i,j\}}$ and 0 which is of codimension 1. If the line $\Lambda_{p_i}(p)$ was to lie in that plane then the set $P_{\{j\}}$ would lie in the same plane, and $P_\varnothing$ would be contained in a closed hemisphere. Thus $\Lambda_{p_i}(p)$ is transversal to $\Delta'_{\{i,j\}}$. The figure below illustrates this projection in $\Delta'_{\{0\}}$ for $n = 3$.



Our (candidate to be an) ε-embedding s is defined by $s|_{\Delta'_{\{i\}}} = s_i$. Since on $\mathcal{E}CP_{\{i\}} \cap \mathcal{E}CP_{\{j\}} \subset \mathcal{E}CP_{\{i,j\}}$, we see that $s|_{\Delta'_{\{i,j\}}} = \mathrm{Id}$ and that $\underset{i \in n}{\cup} \Delta'_{\{i\}} = B$, this map is well-defined. It remains to check that the diameter of the fibers is bounded by ε. We claim that the biggest fiber is $s^{-1}(0) = \cup_i C\{-p_i, 0\}$, whose diameter is that of the set of vertices of the simplex, $\mathrm{Diam}\{p_i\}$. To see this, note that for $x \in \Delta'_{\{i,j\}}$, the diameter of $s^{-1}(x)$ attained on its extremal points (by convexity of the norm), that is x and points of the form $x - \lambda_k p_k$ (for $k \in A$, where $A \supset \{i, j\}$ and $x \in \Delta'_A \subset \Delta'_{\{i,j\}}$) whose norm is one. However, since $x = \sum \lambda_i p_i$ for $i \notin A$ and $\lambda_i > 0$, $\|x - \lambda_k p_k\| = 1$ implies $\|\lambda_k p_k\| \leq 1$, so a simple translation of $s^{-1}(x)$ is actually included in $s^{-1}(0)$. □

This allows us to have a first look at the Euclidean case.

Theorem 1.3.4: *Let $B_1^{\ell^2(n)}$ be the unit ball of $\mathbb{R}^n$, endowed with the Euclidean metric, and let $b_{n;2} := \sqrt{2(1+\frac{1}{n})}$. Then, for $0 < k < n$,*

$$\begin{array}{rlllll} \mathrm{wdim}_\varepsilon B_1^{\ell^2(n)} & = 0 & \textit{if} & 2 \leq & \varepsilon, & \\ k \leq \mathrm{wdim}_\varepsilon B_1^{\ell^2(n)} & < n & \textit{if} & b_{k+1;2} \leq & \varepsilon & < b_{k;2}, \\ \mathrm{wdim}_\varepsilon B_1^{\ell^2(n)} & = n & \textit{if} & & \varepsilon & < b_{n;2}. \end{array}$$

Proof: First, when $\varepsilon \geq \mathrm{Diam} B_1^{\ell^2(n)} = 2$ this result is a simple consequence of proposition 1.2.1.c; when $n = 1$ it is sufficient, so suppose from now on that $n \geq 2$. Applying lemma 1.2.3 to $\partial B_1^{\ell^2(n)} \subset B_1^{\ell^2(n)}$ yields that $\mathrm{wdim}_\varepsilon B_1^{\ell^2(n)} = n$ if $\varepsilon < 2\mathrm{FilRad}\,\partial B_1^{\ell^2(n)}$, but $\mathrm{FilRad} B_1^{\ell^2(n)} \geq b_{n;2}$ by Jung's theorem (see [13, §2.10.41]), as any set whose diameter is less than $< b_{n;2}$ is contained in an open hemisphere ([24] shows that $\mathrm{FilRad} B_1^{\ell^2(n)} = b_{n;2}$). On the other hand, balls of dimension $k < n$ are all included in $B_1^{\ell^2(n)}$, which means that $\mathrm{wdim}_\varepsilon B_1^{\ell^2(k)} \leq \mathrm{wdim}_\varepsilon B_1^{\ell^2(n)}$, thanks to 1.2.1.d. Hence we have that $\mathrm{wdim}_\varepsilon B_1^{\ell^2(n)} \geq k$ whenever $b_{k+1;2} \leq \varepsilon < b_{k;2}$. This establishes the lower bounds.

The vertices of the standard simplex satisfy the assumption of lemma 1.3.3: thanks to the invariance of the norm under rotation we can assume $p_0 = (1, 0, \ldots, 0)$. The other p_i will all have a negative first coordinate, and so will any positive linear combination. Substracting λp_0 will be norm increasing. As the diameter of this set is $b_{n;2}$, lemma 1.3.3 gives the desired upper bound. □

Let us now give an additional upper bound for the 3-dimensional case:

Proposition 1.3.5: *If $1 \leq p < \infty$, then $\varepsilon \geq 2(\frac{2}{3})^{1/p} \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(3)} \leq 2$.*

Proof: In $\mathbb{R}^3$ there is a particularly good set of points to define our projections. These are $p_0 = 3^{-\frac{1}{p}}(1,1,1)$, $p_1 = 3^{-\frac{1}{p}}(1,-1,-1)$, $p_2 = 3^{-\frac{1}{p}}(-1,1,-1)$ and $p_3 = 3^{-\frac{1}{p}}(-1,-1,1)$. Let $x = \lambda_1 p_1$, where $\lambda \in [0,1]$, and suppose $\|\lambda_1 p_1 - \lambda_2 p_2\|_{\ell^p} \leq 1$ for $\lambda_2 \in \mathbb{R}_{\geq 0}$. We have to check that $\lambda_2 \leq 1$. Suppose $\lambda_2 > 1$, then $1 \geq \|\lambda_1 p_1 - \lambda_2 p_2\|_{\ell^p} = \frac{2}{3}(\lambda_1+\lambda_2)^p + \frac{1}{3}(\lambda_2 - \lambda_1)^p = \lambda_2^p[\frac{2}{3}(1+t)^p + \frac{1}{3}(1-t)^p]$, where $t = \lambda_1/\lambda_2$. The function of t has minimal value 1, which gives $\lambda_2 \leq 1$ as desired.

Suppose now that $x = \lambda_1 p_1 + \lambda_2 p_2$ is of norm less than 1, where without loss of generality we assume $\lambda_2 \geq \lambda_1$, and $\|\lambda_1 p_1 + \lambda_2 p_2 - \lambda_3 p_3\|_{\ell^p} \leq 1$. $\|x\|_{\ell^p} \leq 1$ implies that $1 \geq \frac{1}{3}(\lambda_1+\lambda_2)^p + \frac{2}{3}(\lambda_2-\lambda_1)^p$ so $(\lambda_2-\lambda_1)^p \leq 1 - \frac{1}{3}(\lambda_2+\lambda_1)^p + \frac{1}{3}(\lambda_2-\lambda_1)^p \leq 1$. If $\lambda_3 > 1$, then

$$\begin{aligned} 1 &\geq \|\lambda_1 p_1 + \lambda_2 p_2 - \lambda_3 p_3\|_{\ell^p} \\ &= \tfrac{1}{3}(\lambda_3+\lambda_2+\lambda_1)^p + \tfrac{1}{3}(\lambda_3 - (\lambda_2-\lambda_1))^p + \tfrac{1}{3}(\lambda_3 + (\lambda_2-\lambda_1))^p. \end{aligned}$$

However,

$$\begin{aligned} \lambda_3^p &\leq \tfrac{1}{3}(\lambda_3+\lambda_2+\lambda_1)^p + \tfrac{2}{3}\lambda_3^p \\ &\leq \tfrac{1}{3}(\lambda_3+\lambda_2+\lambda_1)^p + \tfrac{1}{3}(\lambda_3 - (\lambda_2-\lambda_1))^p + \tfrac{1}{3}(\lambda_3 + (\lambda_2-\lambda_1))^p \\ &\leq 1. \end{aligned}$$

Using that $f(t) = (1+t)^p + (1-t)^p$ has minimum 2 for $t \in [0,1]$. These arguments can be repeated for any indices to show that the points p_i, where $i = 0,1,2$ or 3, satisfy the assumption of lemma 1.3.3. The conclusion follows by showing that $\mathrm{Diam}\{p_i\} = 2(\frac{2}{3})^{1/p}$ □

For some dimensions, a set of points that allows us to build projections with small fibers can be found (such as the "particularly good" set of points in the proof above). Their descriptions require the concept of Hadamard matrices of rank N; these are $N \times N$ matrices, that will be denoted H_N, whose entries are ± 1 and such that $H_N \cdot H_N^t = N\mathrm{Id}$. It has been shown that they can only exist when $N = 1,2$ or $4|N$ (see [39]), and it is conjectured that this is precisely when they exist. Up to a permutation and a sign, it is possible to write a matrix H_N so that its first column and its first row consist only of 1s. It is quite easy to see that two other rows or columns of such a matrix have exactly $N/2$ identical elements.

Definition 1.3.6: Let H_N be a Hadamard matrix of rank N, and let, for $0 \leq i \leq N$, h_i be the i^{th} row of the matrix without its first entry (which is a 1). Then the h_i form a Hadamard set in dimension $N-1$.

These N elements, normalised so that $\|h_i\|_{\ell^p(N-1)} = 1$. When so normalised, their diameter (for the ℓ^p metric) is $2^{1-1/p}(1+\frac{1}{N-1})^p$. Since $\sum h_i = 0$, by orthogonality of the columns with the column of 1 that was removed, we see that they are not contained in an open hemisphere. The set of points in the preceding proposition was given by a Hadamard matrix of rank 4, and when $p = 2$ the convex hull of these points is just the standard simplex.

Proposition 1.3.7: *Suppose there exists a Hadamard matrix of rank $n+1$, then*

$$\varepsilon \geq 1 + \frac{1}{n} \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^1(n)} \leq n-1.$$

Proof: Let the h_i be as above, and $N = n+1$. Note that for $i \neq j$, h_i and h_j have $\frac{N}{2}$ opposed coordinates, and $\frac{N}{2} - 1$ identical ones. Thus $\lambda_i h_i - \lambda_j h_j$ has always a bigger ℓ^1 norm than any of its two summands. Indeed, the coefficients c_j of the vector $\sum_{i \in A} \lambda_i h_i$ where the contribution of h_k reduces $|c_j|$ are in lesser number than those that get increased. Since the ℓ^1 norm is linear, the magnitude of the c_j getting smaller is not relevant, only their number.

We conclude by applying lemma 1.3.3, as $\mathrm{Diam}_{\ell^1}(h_i) = 1 + \frac{1}{N-1}$. □

Note that in dimension higher than 3 and for $p > 2$, Hadamard sets no longer satisfy the assumption of lemma 1.3.3.

1.3.b Further upper bounds for $\mathrm{wdim}_\varepsilon B_1^{\ell^p(n)}$**.** The projection argument still works for non-Euclidean spheres. It can also be repeated, though inefficiently, to construct maps to lower dimensional polyhedra.

Proposition 1.3.8: *For $1 < p < \infty$, consider the sphere $B_1^{\ell^p(n)}$ with its natural metric. Then, for $\frac{n-1}{2} < k < n$, $\exists c_{k,n;p} \in [1,2)$ such that $c_{k,n;p} \geq c_{k+1,n;p}$, and*

$$\mathrm{wdim}_\varepsilon B_1^{\ell^p(n)} \leq k \text{ if } \varepsilon \geq c_{k,n;p}.$$

Furthermore $c_{n-1,n;2} = b_{n;2}$

Proof: This proposition is also obtained by constructing explicitly maps that reduce dimension (up to $n-j$ for $j < \frac{n+1}{2}$) and whose fibers are small. Unfortunately, nothing indicates this is optimal, and the size of the preimages is hard to determine. We will abbreviate $B := B_1^{\ell^p(n)}$.

We proceed by induction, and keep the notations introduced in the proof of lemma 1.3.3. The p_i that are used here are the vertices of the simplex; they need to be renormalised to be of ℓ^p-norm 1, but note that multiplying them by a constant has actually no effect in this argument. Also note that the sets Δ'_A are not the same for different p, since they are constructed by radial projection to different spheres. The keys to this construction are the maps

$$s_{j;\{i_1,\dots,i_j\}} : \Delta'_{\{i_1,\dots,i_j\}} \longrightarrow \bigcup_{m \notin \{i_1,\dots,i_j\}} \Delta'_{\{i_1,\dots,i_j,m\}}$$

given by projection along the vectors $\sum_{l=1}^{j} p_{i_l}$. Call σ_1 the function s from lemma 1.3.3, then, for $j > 1$,

$$\sigma_j : B \longrightarrow \bigcup_{\{i_1,\dots,i_{j+1}\} \subset n} \Delta'_{\{i_1,\dots,i_{j+1}\}}$$

is obtained by composing, on appropriate domains, $s_{j;\{i_1,\dots,i_j\}}$ with σ_{j-1}. Since $s_{j;i_1,\dots,i_j}$ are equal to the identity when their domain intersect, and their union covers the image of σ_{j-1}, the map is again well-defined. It remains only to calculate the diameter of the fibers. At 0 the fiber is

$$\sigma_j^{-1}(0) = \bigcup_{\{i_1,\dots,i_j\} \subset n} \{-(\lambda_1 + \ldots + \lambda_j)p_{i_1} - (\lambda_1 + \ldots + \lambda_{j-1})p_{i_2} - \ldots - \lambda_1 p_{i_j} | \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

Whereas for a given $x \in \Delta'_A$ in the image (that is A contains at least j elements), x can also be written down as a combination $\sum \lambda_i p_i$, for $i \notin A$ and $\lambda_i \in \mathbb{R}_{>0}$. We have

$$\sigma_j^{-1}(x) = \bigcup_{\{i_1,\ldots,i_j\}\subset A} \{x - (\lambda_1 + \ldots + \lambda_j)p_{i_1} - (\lambda_1 + \ldots + \lambda_{j-1})p_{i_2} - \ldots - \lambda_1 p_{i_j} | \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

If we set $c_{k,n} = \sup_{x\in\sigma_j(B)} \mathrm{Diam}\,\sigma_{n-k}^{-1}(x)$, then when $\varepsilon \geq c_{k,n}$, $\mathrm{wdim}_\varepsilon B_1^{\ell^2(n)} \leq k$. It is possible to determine two simple facts about these numbers. First, they are non-increasing $c_{k,n} \geq c_{k+1,n}$, which is obvious as the construction is done by induction, the size of the fiber of maps to lower dimension is bigger than for maps to higher dimension.

Second, they are meaningful: $c_{k,n} < 2$. Indeed, when $p \neq 1, \infty$, $c_{k,n} = 2$ only if $\sigma_{n-k}^{-1}(x)$ contains opposite points, which is a linear condition. When $x \neq 0$, by convexity of the distance, the points on which the diameter can be attained are at the boundary of $\sigma_j^{-1}(x)$. Say Y is the set of those point except x. The distance from Y to x is at most one, while the diameter of Y is bounded. Indeed, there is a cap of diameter less than 2 that contains all the p_i but one. The biggest diameter of such caps is also less than 2 and bounds $\mathrm{Diam}\,Y$.

Any point of the fiber at 0 is a linear combination of the vertices p_i, and there is only one linear relation between these, namely $\sum p_i = 0$. As long as $j < \frac{n+1}{2}$ (*i.e.* $k > \frac{n-1}{2}$) there are not enough p_i in any two sets that form $\sigma_j^{-1}(0)$ to combine into the required relations, but as soon as j exceeds this bound, opposite points are easily found. □

For $B_1^{\ell^p(n)}$, where $1 < p < \infty$, we used the regular simplex to describe our projections, though nothing indicates that this choice is the most appropriate. In fact, many sets of $n+1$ points allow to build projections to a polyhedron, but it is hard to tell which are more effective: on one hand we need this set to have a small diameter (so that the fiber at 0 is small), while on the other, we need it to be somehow well spread (so as to avoid fibers at x to be too large, as in the assumption of lemma 1.3.3). Furthermore, there is in general no reason for $c_{n-1,n;p}$ to coincide with a lower bound, or even to be different from other $c_{k;p}$, thus we cannot always be sure that $n-1 \in \mathrm{wspec}(B_1^{\ell^p(n)}, \ell^p)$.

1.3.c The lowest non zero element of wspec. Before we return to the general ℓ^p case, notice that together proposition 1.3.2 and theorem 1.3.4 give a good picture of the function $\mathrm{wdim}_\varepsilon B_1^{\ell^2(n)}$. It equals n for $\varepsilon < b_{n;2} = c_{n-1,n;2}$, then $n-1$ for $b_{n;2} \leq \varepsilon < b_{n-1;2}$. Afterwards, I could not show a strict inequality for the $c_{k,n;2}$, but even if they are all equal, $\mathrm{wdim}_\varepsilon B_1^{\ell^2(n)}$ takes at least one value in $(\frac{n}{2}-1, \frac{n}{2}+1) \cap \mathbb{Z}$. Then ,when $\varepsilon \geq 2$, it drops to 0.

For odd dimensional balls, there is a gap between the value given by proposition 1.3.2 and the lowest dimension obtained by the projections introduced above. Say B is of dimension $2l+1$ and ε less than but sufficiently close to 2, then on one hand we know that $\mathrm{wdim}_\varepsilon B \geq l$, while on the other $\mathrm{wdim}_\varepsilon B \leq l+1$. It is thus worth asking whether one of these two methods can be improved, perhaps by using extra homological information on the simplices in the proof of proposition 1.3.2 (*e.g.* if its highest degree cohomology is trivial then a k-dimensional polyhedron is embeddable in $\mathbb{R}^{2k}$, see [14]).

Remark 1.3.9: Such an improvement is actually available when $n = 3$: if the 2-dimensional sphere maps to a 1-dimensional polyhedron (*i.e.* a graph), the map lifts to the universal cover, a tree K. Hence K is embeddable in $\mathbb{R}^2$, and, for $1 < p < \infty$.

$$\varepsilon < 2 \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(3)} \geq 2$$

for otherwise it would contradict Borsuk-Ulam theorem.

Note that estimates obtained in [17, app 1.E5] for Diam_1, can also yield lower bounds for the diameter of fibers for maps to graphs (*i.e.* 1-dimensional polyhedra). Applied to spheres, it becomes a special case of proposition 1.3.2 and of the above remark.

1.3.d Lower bounds for $\mathrm{wdim}_\varepsilon B_1^{\ell^p(n)}$**.** The remainder of this section is devoted to the improvement of lower bounds, using an evaluation of the filling radius as a product of lemma 1.2.4, and a short discussion of their sharpness.

We shall try to find a lower bound on the diameter of $n+1$ points on the ℓ^p unit sphere that are not in an open hemisphere; recall that points f_i are not in an open hemisphere if $\exists \lambda_i$ such that $\sum \lambda_i f_i = 0$. A direct use of Jung's constant (defined as the supremum over all convex M of the radius of the smallest ball that contains M divided by M's diameter) that is cleverly estimated for ℓ^p spaces in [23] does not yield the result like it did in the Euclidean case. This is due to the fact that there are sets of $n+1$ points on the sphere that are not contained in an open hemisphere, but are contained in a ball (not centered at the origin) of radius less than 1. The set of points given by

$$(1.3.10) \qquad (1,\dots,1), \left(-\frac{2}{n-1},\dots,-\frac{2}{n-1},1\right),\dots, \text{ and } \left(1,-\frac{2}{n-1},\dots,-\frac{2}{n-1}\right)$$

is such an example for ℓ^∞, and deforming it a little can make it work for the ℓ^p case, p finite but close to ∞. However, a very minor adaptation of the methods given in [23] is sufficient.

First, we introduce norms for the spaces of sequences (and matrices) taking values in a Banach space E. Let $\alpha_i \in \mathbb{R}_{\geq 0}$ be such that $\sum_{i=0}^{n} \alpha_i = 1$ and denote by α this sequence of $n+1$ real numbers. Let $E_{p,\alpha}$ be the space of sequences made of $n+1$ elements of E and consider the ℓ^p norm weighted by α: $\|x\|_{E_{p,\alpha}} = \left(\sum_i \alpha_i \|x_i\|_E^p\right)^{1/p}$ where $x = (x_0,\dots,x_n)$. On the other hand, E_{p,α^2} shall represent the space of matrices whose entries are in E, with the norm $\left\|(x_{i,j})\right\|_{E_{p,\alpha^2}} = \left(\sum_{i,j} \alpha_i\alpha_j \left\|x_{i,j}\right\|_E^p\right)^{1/p}$. Now define, for E, E' Banach spaces based on the same vector space and for $1 \leq s,t \leq \infty$, the linear operator $T: E_{s,\alpha} \longrightarrow E'_{t,\alpha^2}$ by $(x_i) \mapsto (x_i - x_j)$.

Theorem 1.3.11: *Consider a vector space on which two norms are defined, and denote by E_1, E_2 the Banach space they form. Let $f_i \in E_1^*$, $0 \leq i \leq n$, be such that $\|f_i\|_{E_1^*} = 1$ but that they are not included in an open hemisphere,* i.e. *there exists $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $\sum \lambda_i f_i = 0$ and $\sum \lambda_i = 1$. Let $\mathrm{Diam}_{E_2^*}(f) = \sup_{0\leq i,j\leq n} \left\|f_i - f_j\right\|_{E_2^*}$ be the diameter of this set with respect to the other norm. Then, for $\alpha_i = \lambda_i$,*

$$\mathrm{Diam}_{E_2^*}(f) \geq 2 \sup_{1\leq s,t\leq\infty} \left(1+\frac{1}{n}\right)^{1/t'} \sup_{E_1} \|T\|^{-1}_{(E_1)_{s,\alpha}\to(E_2)_{t,\alpha^2}}.$$

Proof: As the f_i are not in an open hemisphere, real numbers $\lambda_i \in \mathbb{R}_{\geq 0}$ such that $\sum \lambda_i = 1$ and $\sum \lambda_i f_i = 0$ exist. Furthermore, since $\|f_i\|_{E_1^*} = 1$, there also exist $x_i \in E_1$ such that $f_i(x_i) = 1$ and $\|x_i\|_{E_1} = 1$. The remark on which the estimation relies is, as in [23],

$$2 = \sum_{i,j=0}^{n} \lambda_i \lambda_j (f_i - f_j)(x_i - x_j).$$

Choosing $\alpha_i = \lambda_i$, this equality can be rewritten in the form $2 = (Tf)(Tx)$, where $Tx \in (E_2)_{t,\alpha^2}$ and $Tf \in ((E_2)_{t,\alpha^2})^* = (E_2^*)_{t',\alpha^2}$, and thus $2 \leq \|Tf\|_{(E_2^*)_{t',\alpha^2}} \|Tx\|_{(E_2)_{t,\alpha^2}}$. Notice that

$$\sum_{i \neq j} \alpha_i \alpha_j = \sum_{i=0}^{n} \alpha_i (1 - \alpha_i) \leq 1 - \frac{1}{n+1} = \frac{n}{n+1},$$

because $\|\alpha_i\|_{\ell^1(n+1)} = 1 \Rightarrow \|\alpha_i\|_{\ell^2(n+1)} \geq (n+1)^{-1/2}$. We can isolate the required diameter:

$$\begin{aligned} \|Tf\|_{(E_2^*)_{t',\alpha^2}} &= \Big(\sum_{i=0}^{n} \alpha_i \alpha_j \, \|f_i - f_j\|_{E_2^*}^{t'}\Big)^{1/t'} \\ &\leq \operatorname{Diam}_{E_2^*}(f) \Big(\sum_{i \neq j} \alpha_i \alpha_j\Big)^{1/t'} \\ &\leq \operatorname{Diam}_{E_2^*}(f) \left(\tfrac{n}{n+1}\right)^{1/t'}. \end{aligned}$$

On the other hand, $\|x_i\|_{E_1} = 1$, consequently $\|x\|_{(E_1)_{s,\alpha}} = 1$, so we bound

$$\|Tx\|_{(E_2)_{t,\alpha^2}} \leq \|T\|_{(E_1)_{s,\alpha} \to (E_2)_{t,\alpha^2}}.$$

The conclusion is found by substitution of the estimates for the norms of Tf and Tx. □

We only quote the next result, as there is no alteration needed in that part of the argument of Pichugov and Ivanov.

Theorem 1.3.12: *(cf. [23, th 2])*

$$\begin{aligned} &\text{if } 1 \leq p \leq 2, \quad \|T\|_{(\ell^p(n))_{\infty,\alpha} \to (\ell^p(n))_{p,\alpha^2}} \leq 2^{1/p} \left(\tfrac{n}{n+1}\right)^{1/p - 1/p'}, \\ &\text{if } 2 \leq p \leq \infty, \quad \|T\|_{(\ell^p(n))_{\infty,\alpha} \to (\ell^p(n))_{p,\alpha^2}} \leq 2^{1/p'}. \end{aligned}$$

A simple substitution in theorem 1.3.11, with $E_1 = E_2 = \ell^p(n)$, $s = \infty$ and $t = p$, yields the desired inequalities.

Corollary 1.3.13: *Let f_i, $0 \leq i \leq n$, be points on the unit sphere of $\ell^p(n)$ that are not included in an open hemisphere, then*

$$\begin{aligned} &\text{if } 1 \leq p \leq 2, \quad \operatorname{Diam}_{\ell^p(n)}(f) \geq 2^{1/p'} \left(1 + \tfrac{1}{n}\right)^{1/p}, \qquad (*) \\ &\text{if } 2 \leq p \leq \infty, \quad \operatorname{Diam}_{\ell^p(n)}(f) \geq 2^{1/p} \left(1 + \tfrac{1}{n}\right)^{1/p'}. \qquad (**) \end{aligned}$$

Remark 1.3.14*:* Before we turn to the consequences of this result on $\operatorname{wdim}_\varepsilon$, note that there are examples for which the first inequality is attained. These are the Hadamard sets defined in 1.3.6. When normalised to 1, they are not included in an open hemisphere and of the proper diameter.

Hence, when a Hadamard matrix of rank $n+1$ exists, then $(*)$ is optimal. Nothing so conclusive can be said for other dimensions, see the argument in example 1.3.1. I ignore if there are cases for which $(**)$ is optimal, though it is very easy to construct a family $F_n \in (B_1^{\ell^p(n)})^{n+1}$ such that $\mathrm{Diam} F_n \longrightarrow 2^{1/p}$ as $n \longrightarrow \infty$. In particular for $p = \infty$, the points given in (1.3.10) but by substituting $\frac{-1}{n-1}$ instead of the entries with value $\frac{-2}{n-1}$, is a set that is not contained in an open hemisphere and whose diameter is $\frac{n}{n-1}$, which is close to the bound given. Somehow, this case, is also the one where the use of lemma 1.2.4 results in a bound that is quite far from the right value of wdim, *cf.* lemma 1.2.6. This might not be so surprising as sets with small diameter on ℓ^p balls seem, when $p > 2$, to differ from sets satisfying the assumption of lemma 1.3.3.

Still, by lemma 1.2.4 we obtain the following lower bounds on wdim:

Corollary 1.3.15: *Let $b_{k;p}$ be defined by $b_{k;p} = 2^{1/p'}\left(1+\frac{1}{k}\right)^{1/p}$ when $1 \le p \le 2$, whereas $b_{k;p} = 2^{1/p}\left(1+\frac{1}{k}\right)^{1/p'}$ if $2 \le p < \infty$. Then, for $0 < k \le n$,*

$$\varepsilon < b_{k;p} \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(n)} \ge k.$$

Proof: Let $Y = \partial B_1^{\ell^p(n)}$. Since the convex hull of a set of $n+1$ points on the sphere Y will not contain the origin if the diameter of the set is larger than $b_{n;p}$, lemma 1.2.4 gives that $\mathrm{FilRad}\, Y \ge b_{n;p}/2$. We then use lemma 1.2.3 for Y to conclude. □

These inequalities might not be optimal, proposition 1.3.2 for example is always stronger when $k < \lfloor \frac{n}{2} \rfloor$.

In dimension n, $B_{n^{-1/p}}^{\ell^\infty(n)} \subset B_1^{\ell^p(n)}$ yields that $\varepsilon < 2n^{-1/p} \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(n)} = n$ which improves corollary 1.3.15 as long as

$$p \ge \frac{\ln(\frac{2n^2}{n+1})}{\ln(\frac{2n}{n+1})}.$$

However, when $p = 1$, and H_{n+1} is a Hadamard matrix, these estimates are as sharp as we can hope since the lower bound meets the upper bounds.

Corollary 1.3.16: *Suppose there is a Hadamard matrix of rank $n+1$. Then, for $0 \le k < n$,*

$$\begin{array}{rlll} \mathrm{wdim}_\varepsilon B_1^{\ell^1(n)} & = 0 & \textit{if} & 2 \le \varepsilon, \\ \max(\frac{n-1}{2}, k) \le \mathrm{wdim}_\varepsilon B_1^{\ell^1(n)} & < n & \textit{if} & (1+\frac{1}{k+1}) \le \varepsilon < (1+\frac{1}{k}), \\ \mathrm{wdim}_\varepsilon B_1^{\ell^1(n)} & = n & \textit{if} & \varepsilon < (1+\frac{1}{n}). \end{array}$$

Furthermore, in dimension 3, lower bounds of corollary 1.3.15 meet upper bounds of proposition 1.3.5 when $1 \le p \le 2$. In particular, thanks to remark 1.3.9, this gives a complete description of the 3-dimensional case for such p.

Corollary 1.3.17: *Let $p \in [1,2]$, then*

$$\mathrm{wdim}_\varepsilon B_1^{\ell^p(3)} = \begin{cases} 0 & \textit{if} \quad 2 \le \varepsilon, \\ 2 & \textit{if} \quad 2(\frac{2}{3})^{1/p} \le \varepsilon < 2, \\ 3 & \textit{if} \quad \varepsilon < 2(\frac{2}{3})^{1/p}. \end{cases}$$

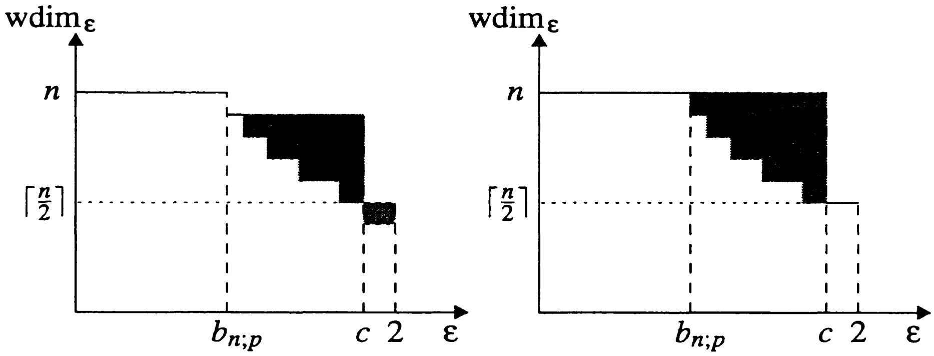
When $p > 2$, all that can be said is that the value of ε for which $\mathrm{wdim}_\varepsilon B_1^{\ell^p(3)}$ drops from 3 to 2 is in the interval $[2(\frac{2}{3})^{1-1/p}, 2(\frac{2}{3})^{1/p}]$.

This last corollary is special to the 3-dimensional case, which happens to be a dimension where there exist a Hadamard set, and where the Borsuk-Ulam argument can be improved to rule out maps to $\frac{n-1}{2}$-dimensional polyhedra. For example, in the 2-dimensional case, a precise description is not so easy. Indeed, thanks to example 1.3.1 and using the inclusion of $B_1^{\ell^1(2)} \subset B_1^{\ell^p(2)}$, we know that $\mathrm{wdim}_\varepsilon B_1^{\ell^p(2)} = 2$ when $\varepsilon \geq 2^{1/p}$. On the other hand, the inclusion of $B_{2^{-1/p}}^{\ell^\infty(2)} \subset B_1^{\ell^p(2)}$ gives $\varepsilon \geq 2^{1/p'} \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(2)} = 2$. Putting these together yields:

$$\varepsilon \geq \max(2^{1/p}, 2^{1/p'}) \Rightarrow \mathrm{wdim}_\varepsilon B_1^{\ell^p(2)} = 2.$$

These simple estimates in dimension 2 are better than corollary 1.3.15 as long as $p \leq 3 - \frac{\ln 3}{\ln 2}$ or $p \geq \ln(\frac{8}{3})/\ln(\frac{4}{3})$. I doubt that any of these estimations actually gives the value of ε where $\mathrm{wdim}_\varepsilon B_1^{\ell^p(2)}$ drops from 2 to 1.

All the results of this section can be summarised to give theorem 1.1.4. Here are two depictions of the situation. Gray areas correspond to possible values, full lines to known values and dotted line to bounds.



When the dimension is odd (but different from 3), the situation is as in the left-hand plot for the euclidean case ($p = 2$) or the case $p = 1$ if there is a Hadamard set. In these cases, a map to a $n-1$-dimensional polyhedron with small fibers can be constructed, but the bounds from the Borsuk-Ulam argument and projections to lower dimensional polyhedron do not meet. The right-hand picture gives the situation in cases where the dimension is even and there is no known projection with small fibers. $c_{\lceil n/2 \rceil, n;p}$ is abbreviated by c. The case of dimension 3 is described in corollary 1.3.17.

It is not expected that $\frac{n-1}{2}$ be in wspec when n is odd, nor is it expected that the lower bounds $b_{k;p}$ be sharp for $B_1^{\ell^p(n)}$ when $k < n$.

1.4 Further results

If a space X is a product of two spaces, then given some information on the widths of its factors, it is possible to gain some insight on the widths of X. The following example considers a solid torus.

Example 1.4.1*:* Let $T \subset \mathbb{R}^3$ be a solid torus of radii $r < R$, *i.e.* the set of all the points at distance $\leq r$ from the a circle of radius R, then

$$\mathrm{wdim}_\varepsilon T = \begin{cases} 0 & \text{if} \quad 2(r+R) \leq \varepsilon \\ 1 & \text{if} \quad 2r \leq \varepsilon < 2(r+R) \\ 2 & \text{if} \quad \sqrt{3}r \leq \varepsilon < 2r \\ 3 & \text{if} \quad \varepsilon < \sqrt{3}r \end{cases}$$

Proof: The first inequality is, as usual, a consequence of 1.2.1.c. As for the second, it suffices to note that the projection on the circle of radius R has fibers of diameter $2r$. The third requires more work. Let Y be the closed set given by the union of the circles of radius $R+r$ and $R-r$. Y has a trivial homology class in T and a filling radius of r. Lemma 1.2.3 implies that $\varepsilon < 2r \Rightarrow \mathrm{wdim}_\varepsilon T > 1$. On the other hand, the torus can be sliced in discs of radius r. The map which sends each of these discs to the cone on the 0-skeleton of the simplex inscribed in these discs has a fiber of radius $\sqrt{3}r$ (as seen in theorem 1.3.4). The last inequality is settled by looking at the usual (empty) torus Y'. It is a homologically trivial closed subset of dimension 2, and its filling radius is $\sqrt{3}r$. $\square$

The following simple lemma will also be of importance when we will look at the widths of an ℓ^p ball with an ℓ^q metric.

Lemma 1.4.2: *Let* $r_{q,p} = \|\mathrm{Id}\|^{-1}_{L^q \to L^p}$. *Then*

$$\begin{array}{lll} si\ q = p & & \Rightarrow r_{q,p} = 1 \\ si\ q < p & \text{and } \mu \text{ is continuous} & \Rightarrow r_{q,p} = 0 \\ & \text{and } \mu \text{ is atomic} & \Rightarrow r_{q,p} = \inf\limits_{v \in \mathrm{supp}\,\mu} \mu(v)^{\frac{1}{q}-\frac{1}{p}} \\ si\ q > p & \text{and } \mu \text{ is infinite} & \Rightarrow r_{q,p} = 0 \\ & \text{and } \mu \text{ is finite} & \Rightarrow r_{q,p} = \mu(V)^{\frac{1}{q}-\frac{1}{p}} \end{array}$$

Proof: The case $p = q$ is simple. If μ is continuous and $q < p$, the existence of functions whose L^q norm is finite but whose L^p norm is infinite implies that a L^q ball will never be contained in an L^p ball. If $q > p$ and $\mu(V) = \infty$, then there also exists function who are of finite L^q norm and of infinite L^p norm.

If $\mu(V) < \infty$ and $q > p$, this is a consequence of Hölder's inequality. Let s, s' be conjugate exponents, *i.e.* $\frac{1}{s} + \frac{1}{s'} = 1$, then

$$\|f\|^q_{L^q} = \|f^q\|_{L^1} \leq \mu(V)^{1/s} \|f^q\|_{L^{s'}} = \mu(V)^{1/s} \|f\|^q_{L^{qs'}}.$$

Hence, $\|f\|_{L^q} \leq \mu(V)^{1/sq} \|f\|_{L^p}$, and $\frac{1}{qs} = \frac{1}{q}(1 - \frac{1}{s'}) = \frac{1}{q} - \frac{1}{p}$, by choosing $qs' = p$.

Suppose now that $q < p$ and μ is atomic. Let $i = \inf\limits_{v \in \mathrm{supp}\,\mu} \mu(v)$, then we must show that $r_{q,p} = i^{\frac{1}{q}-\frac{1}{p}}$. For all $v \in \mathrm{supp}\,\mu$, define the function f_v by $f_v(v) = r_{q,p}\mu(v)^{-1/q}$ and $f_v(v') = 0$ when $v' \neq v$. Then $f_v \in B_1^{L^p} \Rightarrow \mu(v)^{\frac{1}{p}-\frac{1}{q}} r_{q,p} \leq 1$, thus $r_{q,p} \leq i^{\frac{1}{q}-\frac{1}{p}}$. As for the converse inequality, since $\frac{\mu(v)}{i} \geq 1$,

$$\begin{aligned} \|f\|^q_{L^q} = \sum_{v \in \mathrm{supp}\,\mu} \mu(v) |f(v)|^q \leq i^{1-q/p} \quad & \Rightarrow \sum_{v \in \mathrm{supp}\,\mu} \tfrac{\mu(v)}{i} |i^{1/p} f(v)|^q \leq 1 \\ & \Rightarrow |i^{1/p} f|^q \leq 1 \end{aligned}$$

Thus $|if(v)^p| = |i^{1/p}f(v)|^p \leq |i^{1/p}f(v)|^q$. This estimation obtained, the lower bound is easily found:

$$\begin{aligned}
\|f\|^p_{L^p} &= \sum_{v \in \operatorname{supp}\mu} \mu(v)|f(v)|^p \\
&= \sum_{v \in \operatorname{supp}\mu} \tfrac{\mu(v)}{i}|i^{1/p}f(v)|^p \\
&\leq \sum_{v \in \operatorname{supp}\mu} \tfrac{\mu(v)}{i}|i^{1/p}f(v)|^q \leq 1. \quad \square
\end{aligned}$$

Remark 1.4.3*:* Thanks to lemma 1.4.2, the same projections that were used to obtain upper bound in 1.1.3 give, for $k \geq 0$ and $q > p$,

$$\operatorname{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^q) \leq k \text{ when } 2(k+1)^{\frac{1}{q}-\frac{1}{p}} \leq \varepsilon.$$

On the other hand, using 1.2.5, the inclusion $B^{\ell^q(k)}_{k^{\frac{1}{q}-\frac{1}{p}}} \subset B_1^{\ell^p(k)} \subset B_1^{\ell^p(n)}$ yields:

$$\operatorname{wdim}_\varepsilon(B_1^{\ell^p(n)}, \ell^q) \geq k \text{ when } \varepsilon < (k+1)^{\frac{1}{q}-\frac{1}{p}}$$

Proposition 1.2.1.d has an obvious extension that will ne used in the following chapter.

Lemma 1.4.4: *If* $f : (X,d) \to (X',d')$ *is a continuous function such that*

$$d(x_1,x_2) \leq \phi\big(d'(f(x_1),f(x_2))\big)$$

where $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. *Then if* $\widetilde{\phi}(x) = \sup_{y \leq x} \phi(y)$, $\operatorname{wdim}_{\widetilde{\phi}(\varepsilon)}(X,d) \leq \operatorname{wdim}_\varepsilon(X',d')$.

Proof: If $\operatorname{wdim}_\varepsilon X' = n$, there exists an ε-embedding $g : X' \to K$ with $\dim K = n$. Noticing that the map $g \circ f$ is an $\widetilde{\phi}(\varepsilon)$-embedding from X to K leads to the claimed inequality. $\square$

Chapter 2

Mean dimension

This chapter will recall some notions introduced by Gromov in [19], further developped in [31], [30], [7] and [29]; [6] contains a good introduction to mean dimension and its applications in dynamical systems. We will present them here again, and evaluate it for the shift acting on the unit ball of $\ell^p(\Gamma;\mathbb{R})$. Unfortunately, unless $p=\infty$ it is trivial. In §2.2 a first alteration $\mathfrak{meas}_{\mathfrak{f}}$ (a mixed growth factor which depends on the profile $\mathfrak{f}$) of this quantity that enables to characterize different p is introduced. Topological invariance is however lost, but some covariance under Hölder homeomorphisms remains. In §2.3, another variant $\dim_{\ell^p}$ is defined by replacing the metrics by pseudo-metrics. Thanks to [19], Von Neumann dimension coincides with $\dim_{\ell^2}$. Some properties of the Von Neumann dimension turn out to hold for $\dim_{\ell^p}$ for a generic p, but not all; see the discussion in §2.5. In particular, the invariance on the choice of Følner sequece requires an extension of the Ornstein-Weiss lemma presented in §2.4. Finally, a discussion of other possible variants take place in §2.6.

Mean dimension can be introduced as a generalisation of entropy for the shift. Indeed, the dynamical system of the (classical) shift is the set $A^{\mathbb{Z}}$ of A-valued sequence, where A is finite set (the alphabet), with the map $\sigma : (a_i)_{i\in\mathbb{Z}} \mapsto (a_{i+1})_{i\in\mathbb{Z}}$. Endowing A with the metric d_A which induces the discrete topology, the product topology $A^{\mathbb{Z}}$ can also be induced by a metric (typically $d(a,b) = \sum\limits_{i\in\mathbb{Z}} d_A(a_i,b_i)/2^{|i|}$). A definition of entropy which is close to the language of the previous chapter is as follows: let $d_n(a,b) = \sup\limits_{-n<i<n} d(\sigma^{\circ i}a, \sigma^{\circ i}b)$ be the dynamical distance, then define $N(\varepsilon,n,A) = \inf\{|F| \text{ such that } (A,d_n) \xrightarrow{\varepsilon} F\}$. The entropy is then obtained by a limit: $h(A) = \lim\limits_{\varepsilon\to 0}\lim\limits_{n\to\infty} \frac{1}{n}\ln N(\varepsilon,n,A)$ which is, in the present case, equal to $\ln|A|$. If (A,d_A) is a metric space of positive dimension m, mean dimension appears as a modification of this definition. The dynamic of the shift σ on A-valued sequences, $A^{\mathbb{Z}}$, endowed with the metric $d(a,b) = \sum\limits_{i\in\mathbb{Z}} d(a_i,b_i)/2^{|i|}$), will again allow to define metrics d_n. However, since A is of positive dimension, the quantity $N(\varepsilon,n,A)$ will be of the order of $k'/\varepsilon^{m(n+k\ln\varepsilon)}$ and will give an infinite entropy $h(A)$. Thus the necessity to introduce the mean dimension.

2.1 Definition and properties

The definition of mean dimension requires an important remark:

$$\mathrm{wdim}_\varepsilon(X_1 \times X_2, \max(d_1,d_2)) \leq \mathrm{wdim}_\varepsilon(X_1,d_1) + \mathrm{wdim}_\varepsilon(X_2,d_2),$$

in other words, $\mathrm{wdim}_\varepsilon$ is subadditive for the product, if the metric is given by the maximum of the metrics on each factor. This is a consequence of the fact that two ε-embeddings $f_i : (X,d_i) \overset{\varepsilon}{\hookrightarrow} K_i$, give an ε-embedding $f_1 \times f_2$. Recall that equality does not always hold (*cf.* [12] and references therein). Let d_1 and d_2 be two metrics on X. As a metric space endowed with the metric given by the maximum of the two metrics $d = \max(d_1,d_2)$, (X,d) is the diagonal of $(X,d_1) \times (X,d_2)$. Whence

$$\mathrm{wdim}_\varepsilon(X,d) \leq \mathrm{wdim}_\varepsilon(X,d_1) + \mathrm{wdim}_\varepsilon(X,d_2) \tag{2.1.1}$$

We shall use the definition of amenable groups by the existence of Følner sequences (another definition uses the existence of invariant means [38, I.§0]; another reference on amenable groups is [16]).

Definition 2.1.2: Let Γ be a countable group. Γ is amenable if there exists an increasing sequence $\{\Omega_i\}_{i\in\mathbb{N}}$ of finite subsets of Γ such that $\forall g \in \Gamma$

$$\lim_{i\to\infty} \frac{|\Omega_i g \cup \Omega_i \smallsetminus \Omega_i g \cap \Omega_i|}{|\Omega_i|} = 0.$$

Such a sequence is called a Følner sequence for Γ.

Proposition 2.1.3: *Let Γ be a countable amenable group. Let $F \subset \Gamma$ be a finite set, and $\{\Omega_i\}$ be a Følner sequence then*

$$\lim_{i\to\infty} \frac{|\Omega_i F|}{|\Omega_i|} = 1.$$

Proof. It suffices to take $f_0 \in F$ and then to notice by invariance of $|\cdot|$ that

$$|\Omega_i F| \leq |\Omega_i f_0| + \sum_{f_0 \neq f \in F} |\Omega_i f \smallsetminus \Omega_i f_0| = |\Omega_i| + \sum_{f_0 \neq f \in F} |\Omega_i f f_0^{-1} \smallsetminus \Omega_i|.$$

Dividing by $|\Omega_i|$ and using the definition 2.1.2 yields the result. □

Consider metric spaces (X,d) on which amenable groups Γ act (not isometrically). Let μ be a (left) Haar measure on Γ then it is possible to define wdim relative to Γ. Since we are for now on countable groups we will use the notation $\mu(\Omega) = |\Omega|$.

Denote by $d_\gamma(x,x') = d(\gamma x, \gamma x')$ the metric (on X) translated by $\gamma \in \Gamma$. For all $\Omega \subset \Gamma$, a dynamical metric d_Ω on X is defined by:

$$d_\Omega(x,x') = \sup_{\gamma\in\Omega} d_\gamma(x,x') = \left\| d_\gamma(x,x') \right\|_{\ell^\infty(\Omega)}.$$

As a function of $\Omega \subset \Gamma$, $\mathrm{wdim}_\varepsilon(X, d_\Omega)$ is subadditive, see (2.1.1) (for discrete groups). Thus, by amenability of Γ, the limit

$$\mathrm{mdim}_\varepsilon((X,d):\Gamma) := \lim_{i\to\infty} \frac{\mathrm{wdim}_\varepsilon(X, d_{\Omega_i})}{|\Omega_i|}$$

exists for any chosen Følner sequence Ω_i and is independent of this choice thanks to the Ornstein-Weiss lemma, see [38] or [19, §1.3] (see also §2.4 for a generalisation of this lemma).

Definition 2.1.4: The mean dimension of (X,d) for the action of a group Γ is

$$\mathrm{mdim}((X,d):\Gamma) := \lim_{\varepsilon\to 0} \mathrm{mdim}_\varepsilon((X,d):\Gamma) = \lim_{\varepsilon\to 0}\lim_{i\to\infty} \frac{\mathrm{wdim}_\varepsilon(X, d_{\Omega_i})}{|\Omega_i|}.$$

The existence of the last limit is comes from the fact that $\mathrm{wdim}_\varepsilon$ is non-increasing as a function of ε (*cf.* 1.2.1.b). It might be infinite.

Proposition 2.1.5: *(*cf. *[19, §1.5.1]) Let X be a compact space.* $\mathrm{mdim}(X:\Gamma)$ *is a topological invariant,* i.e. *for all pairs d and d' of compatible metrics on X,* $\mathrm{mdim}((X,d):\Gamma) = \mathrm{mdim}((X,d'):\Gamma)$.

Proof. (We write the argument of [19] in details.) Indeed, we look at the modulus of continuity of $\mathrm{Id}:(X,d)\to(X,d')$ which is uniformly continuous, more precisely, there exists an increasing continuous function

$$\omega_{\mathrm{Id}} : [0, \mathrm{Diam}\,(X,d)] \to \mathbb{R}_{\geq 0}$$

such that $\omega_{\mathrm{Id}}(0) = 0$ and $\forall \gamma \in \Gamma$, $d'_\gamma(x,x') \leq \omega_{\mathrm{Id}}(d_\gamma(x,x'))$. This inequality remains valid fot d_Ω and d'_Ω. By lemma 1.4.4,

$$\mathrm{wdim}_{\omega_{\mathrm{Id}}(\varepsilon)}(X, d'_\Omega) \leq \mathrm{wdim}_\varepsilon(X, d_\Omega).$$

Since $\lim_{\varepsilon\to 0}\omega_{\mathrm{Id}}(\varepsilon) = 0$, dividing by $|\Omega|$ passing to the limits yields $\mathrm{mdim}((X,d'):\Gamma) \leq \mathrm{mdim}((X,d):\Gamma)$. The same argument on $\mathrm{Id}:(X,d')\to(X,d)$ gives the result. □

The proof above can be generalized to show that if $i:(X,d)\to(X',d')$ is a Γ-equivariant rough embedding (in the sense of [18]) then $\mathrm{mdim}X \leq \mathrm{mdim}X'$.

The definition of mdim given above is a particular case of a vast family of possible choices, some of which will be introduced in the following sections. In particular, if Γ is not amenable, it is still possible to define mean dimension. However, it is necessary to take a lim sup of an increasing sequence $\{\Omega_i\}$ (which can also be assumed exhausting), but the independence on the choice of sequences is no longer true.

If Γ acts by isometry and X is compact, $\mathrm{mdim}(X:\{\Omega_i\}) = \dim X \lim_{i\to\infty} |\Omega_i|^{-1} = 0$ since $(X, d_{\Omega_i}) = (X,d)$. Consequently actions by isometry are not interesting. We will now focus our attention on non-compact metric space where the action of Γ is isometric. However, compactification will make the action non-isometric. These examples will motivate some of the further developpements. The remainder of this section is contained in [19, §1.6] (with answer to questions).

Let $X = (\mathbb{R}^s)^\Gamma = \{\Gamma \to \mathbb{R}^s\}$ endowed with the natural action of Γ: for $f \in (\mathbb{R}^s)^\Gamma, \gamma f(\cdot) = f(\gamma^{-1}\cdot)$. Let $B_1^{\ell^p(\Gamma;\mathbb{R}^s)}$ the unit ball for the ℓ^p norm (*cf.* lemma 1.4.2), $p \in [1,\infty]$. This set is not compact and Γ acts isometrically.

Before we introduce weak-* topology, recall ℓ^p is the dual of $\ell^{p'}$ when $p \neq 1$ and $\frac{1}{p}+\frac{1}{p'}=1$. As for ℓ^∞, it is the dual of ℓ^1 and ℓ^1 is the dual of c_0 (*cf.* [43, §3 exer.10 p.83 or §4 exer.24 p.109]) where c_0 is the space of sequences which tend to 0 at infinity. This can be interpreted here as the following condition:

$$\forall x \in c_0(\Gamma;\mathbb{R}^s) \subset \ell^\infty(\Gamma;\mathbb{R}^s), \quad \|x\|_{\ell^\infty(\Gamma \setminus F_i)} \to 0$$

for all increasing exhaustive sequences of finite subsets $\{F_i\}$). On a normed vector space E, there are two usual topologies: the one coming from the norm, and the weak topology, which is the roughest topology such that *linear* continuous functions (elements of the dual) are continuous. Its dual E^* possesses three usual topologies: one coming from the norm $\|e^*\| = \sup\limits_{\|e\|=1} \frac{|e^*(e)|}{\|e\|}$ (denoted τ), the second is the weak topology associated to this norm, and the third called the weak topology of the dual (or weak-*, denoted τ^*). This is the roughest topology such that evaluation maps $v_e(e^*) = e^*(e)$ are continuous. If $E = E^{**}$, weak and weak-* topologies are identical. A neighborhood of 0 in the weak-* topology is of the form

$$N^*(0;e_1,\ldots,e_n;\varepsilon) = \big\{e^* \in E^* | \forall j \in \{1,\ldots,n\}, |e^*(e_j)| < \varepsilon\big\} \tag{2.1.6}$$

Alaoglu's theorem states that unit ball of the dual is compact for this topology. This remains true for any weaker topology. In particular, consider the topology τ' on $B_1^{\ell^p} := \{x \in \ell^p(\Gamma;\mathbb{R}^s) | \, \|x\|_{\ell^p} \leq 1\}$, defined by the metric d' defined as follows: let F_i be a sequence of finite sets such that $\cup F_i$ is dense in Γ, let d^{F_i} be defined by

$$d^{F_i}(x_1,x_2) = \|x_1 - x_2\|_{\ell^\infty(F_i)}.$$

Let

$$d'(x_1,x_2) = \sum_{i\geq 1} a_i d^{F_i}(x_1,x_2). \tag{2.1.7}$$

normalized so that if $\sigma_k = \sum_{i\geq k} a_i$, then $\sigma_1 = 1$. Usually, the a_i are chosen to be 2^{-i-1}. The normalisation $\sigma_1 = 1$ is made so that the diameter of the unit ball remains equal to 2. Notice that $d'(x_1,x_2) \leq 2\sigma_{k+1}$ if x_1 and x_2 start to differ on F_{k+1}, but are identical when restricted to the F_j for $j \leq k$.

The topology induced by this metric is weaker than τ^*, since an ε-neighborhood of 0 is the set of x which are small when evaluated on F_i (depending on ε), since d^{F_i} is always smaller than 2 for element of the unit ball. This neighborhood is of the form (2.1.6). Let us show that τ' is Hausdorff: let x_1 and $x_2 \in (X,\ell^p)$, if $x_1 \neq x_2$, then $\exists \gamma \in \Gamma$ such that $x_1(\gamma) \neq x_2(\gamma)$ and since for a i, $\gamma \in F_i$, $d_{F_i}(x_1,x_2) > 0$ and consequently $d'(x_1,x_2) > 0$, this suffices to show that it is Hausdorff. Thus, $B_1^{\ell^p}$ is compact for τ'.

The identity map $(X,\tau^*) \to (X,\tau')$ is continuous and bijective. Since τ' is Hausdorff, this is a homeomorphism of $(B_1^{\ell^p},\tau^*)$ on $(B_1^{\ell^p},\tau')$ (any bijective continuous map between compact spaces is a homeomorphism if the image is Hausdorff).

Remark 2.1.8*:* Since wdim is a topological invariant, the choice of the metric d', which is important for the computations, will not have an impact on the result. It would have been possible to define the d^{F_i} using a ℓ^q norm for any $q \geq p$. However $q = \infty$ simplifies the arguments. Finally, τ' can also be seen as the topology induced on the subspace of elements of ℓ^p norm less than 1 in the space $[-1,1]^\Gamma$ endowed with the product topology.

The next result answers a question of Gromov, it is also shown in [48] but is a consequence of a simple remark (see discussion following the corollary).

Theorem 2.1.9: *(cf. [19, §1.6])*

$$\mathrm{wdim}(B_1^{\ell^p(\Gamma,\mathbb{R}^s)}:\Gamma)\begin{cases} \geq s & \text{if } p=\infty \\ =0 & \text{otherwise.}\end{cases}$$

Proof. Suppose that $p<\infty$. Then, proposition 1.1.3 allows us to bound (independently of Ω) $\mathrm{wdim}_\varepsilon(B_1^{\ell^p},d'_\Omega)$: as said in remark 2.1.8) this argument is done with ℓ^∞ metrics. The restriction $\rho_k:(B_1^{\ell^p},d'_{\Omega_i}) \to (B_1^{\ell^p(|\Omega_i^{-1}F_k|)},\ell^\infty)$ has fibers of diameter σ_{k+1}. Moreover, a ε-embedding of the image of ρ_k becomes upon composition by ρ_k an $(\varepsilon+\sigma_{k+1})$-embedding of the source. Indeed, if $x,y \in B_1^{\ell^p(|\Omega_i^{-1}F_k|)}$ are at ℓ^∞ distance equal to ε then any points in their preimage by ρ_k can only disagree on the sets $\Omega_i^{-1}F_j$ for $j>k$, this contributes at most to an extra distance of σ_{k+1}. Whence

$$\mathrm{wdim}_{\varepsilon+\sigma_{k+1}}(B_1^{\ell^p},d'_{\Omega_i}) \leq \mathrm{wdim}_\varepsilon(B_1^{\ell^p(|\Omega_i^{-1}F_k|)},\ell^\infty). \tag{2.1.10}$$

Supposing that $\varepsilon>\sigma_{k+1}$, the left-hand terms gives a lower bound for $\mathrm{wdim}_{2\varepsilon}$, whereas proposition 1.1.3 gives an upper bound for the right-hand term (by a function of the form $f(p)\varepsilon^{-p}$). The conclusion is direct:

$$\begin{aligned}\mathrm{wdim}(B_1^{\ell^p}:\Gamma) &= \lim_{\varepsilon\to 0}\limsup_{i\to\infty}\mathrm{wdim}_\varepsilon(B_1^{\ell^p},d'_{\Omega_i})|\Omega_i|^{-1}\\ &\leq \lim_{\varepsilon\to 0}\limsup_{i\to\infty} f(p)(\tfrac{\varepsilon}{2})^{-p}|\Omega_i|^{-1}=0.\end{aligned}$$

The lower bound, for $p=\infty$, can be obtained by observing that $B_{a_k}^{\ell^\infty(\Omega_i^{-1}F_k,\mathbb{R}^s)}$ is included (by a map that increases distances) in $(B_1^{\ell^\infty},d'_{\Omega_i})$. Using proposition 1.2.1.d we conclude that $\varepsilon<a_k \Rightarrow \mathrm{wdim}_\varepsilon(B_1^{\ell^\infty},d'_{\Omega_i}) \geq s|\Omega_i^{-1}F_k|$ and

$$\begin{aligned}\mathrm{wdim}(B_1^{\ell^\infty}:\Gamma) &= \lim_{\varepsilon\to 0}\limsup_{i\to\infty}\mathrm{wdim}_\varepsilon(B_1^{\ell^\infty},d'_{\Omega_i})|\Omega_i|^{-1}\\ &\geq \lim_{k\to\infty}\limsup_{i\to\infty} s|\Omega_i^{-1}F_k||\Omega_i|^{-1} \geq s.\end{aligned}$$

□

Remark 2.1.11*:* When Γ is amenable, let us show that $\mathrm{wdim} B_1^{L^\infty(\Gamma;\mathbb{R}^s)}=s$. If $\varepsilon \geq \sigma_k$, the restriction map to F_k is a ε-embedding to a polyhedron. Thus $\mathrm{wdim}_\varepsilon B_1^{\ell^\infty} \leq \dim B_1^{\ell^\infty(F_k,\mathbb{R}^s)} = s\mu(F_k)$ when

$\varepsilon \geq \sigma_k$. Endowed with the metric d'_{Ω_i}, restriction to $\Omega_i^{-1} F_k$ is again a ε-embedding given that $\varepsilon \geq \sigma_k$. Thanks to proposition 2.1.3, $|\Omega_i^{-1} F_k||\Omega_i|^{-1} \longrightarrow 1$, whence

$$\begin{aligned} \mathrm{mdim}(B_1^{\ell^\infty} : \Gamma) &= \lim_{\varepsilon \to 0} \limsup_{i \to \infty} \mathrm{wdim}_\varepsilon(B_1^{\ell^\infty}, d'_{\Omega_i})|\Omega_i|^{-1} \\ &\leq \lim_{k \to \infty} \limsup_{i \to \infty} s|\Omega_i^{-1} F_k||\Omega_i|^{-1} = s. \end{aligned}$$

Corollary 2.1.12: *The unit ball $B_1^{\ell^\infty(\Gamma)}$ is not Γ-equivariantly homeomorphic to $B_1^{\ell^p(\Gamma)}$.*

This is a consequence of proposition 2.1.5. Note that entropy considerations allow to obtain the same result: the action of Γ on a ℓ^p ball endowed with the weak-* topology, for p finite, sends any point to 0 (in other words for any sequence $\gamma_i \in \Omega_i \setminus \Omega_{i-1}$, $\gamma_i x \longrightarrow 0$). There are no periodic orbits (in particular, entropy of such an action is 0). On the other hand, when $p = \infty$ there are many periodic orbits.

2.2 Variation with profile

From the previous section, it appears that mean dimension is not appropriate to distinguish $\ell^p(\Gamma)$ spaces. The next definition is tailored to avoid the problem encountered. Its introduction is explained in §2.6.

Definition 2.2.1: Let $\mathfrak{f} : \mathcal{P}(\Gamma) \longrightarrow \mathbb{R}_{\geq 0}$ a decreasing function (*i.e.* $A \subset B \Rightarrow \mathfrak{f}(A) \geq \mathfrak{f}(B)$) such that $\mathfrak{f}(A) = 0 \Leftrightarrow |A| = \infty$, then the asymptotic measure with profile $\mathfrak{f}$ will be defined for an increasing sequence of compact sets Ω_i (whose measure tends to ∞) by

$$\mathrm{meas}_{\mathfrak{f}}(X : \{\Omega_i\}) = \limsup_{i \to \infty} \frac{\mathrm{wdim}_{\mathfrak{f}(\Omega_i)}(X, d_{\Omega_i})}{|\Omega_i|} \quad \in [0, +\infty].$$

This concept certainly lacks invariance, which explains the change of terminology for an asymptotic measure. It shall nevertheless be useful to detect obstructions.

Proposition 2.2.2: *Let $f : (X, d) \longrightarrow (X', d')$ be a Γ-homeomorphism of metric spaces endowed with an action of Γ whose modulus of continuity is ω_f. If $\exists \psi : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ an increasing function such that $\lim_{\varepsilon \to 0} \psi = 0$, and that $\omega_f(\varepsilon) \leq \psi(\varepsilon)$,*

$$\mathrm{meas}_{\psi \circ \mathfrak{f}}(X' : \{\Omega_i\}) \leq \mathrm{meas}_{\mathfrak{f}}(X : \{\Omega_i\}).$$

Proof. As in the previous proofs, the inequality

$$d'(f(x), f(y)) \leq \omega_f(d(x,y)) \leq \psi(d(x,y))$$

gives the same inequality for d_Ω and d'_Ω. This last inequality yields

$$\mathrm{wdim}_{\psi(\varepsilon)}(X', d'_{\Omega_i}) \leq \mathrm{wdim}_\varepsilon(X, d_{\Omega_i}),$$

by composing any ε-embedding on X by f^{-1}. Choosing $\varepsilon = \mathfrak{f}(\Omega_i)$ and passing to the limit gives the desired inequality. □

The definition of $\mathfrak{meas}_{\mathfrak{f}}$ is made to compensate for the fact that $\mathrm{wdim}_\varepsilon B_1^{\ell^p(n)}$ is quite precisely bounded above and below by some constants times $\varepsilon^{-1/p}$. The next lemma highlights this mechanism.

Lemma 2.2.3: *Let Γ be a countable discrete group, let $\{\Omega_i\}$ an increasing sequence of finite subsets, let $p \in [1,\infty[$, let $B_1^{\ell^p}$ be the unit ball of $\ell^p(\Gamma;\mathbb{R}^s)$ endowed with a metric of the type* (2.1.7) *and of the natural action of Γ. For $c \in \mathbb{R}_{>0}$ and $q \in [1,\infty[$ let $\mathfrak{f}_{c,q}(\Omega) = c|\Omega|^{-1/q}$ be a profile.*

$$\mathfrak{meas}_{\mathfrak{f}_{c,q}}(B_1^{\ell^p} : \{\Omega_i\}) \begin{cases} = \infty & \text{if } p > q \\ \in \mathbb{R}_{>0} & \text{if } p = q \\ = 0 & \text{if } p < q \end{cases}$$

Proof. The arguments that will be used are independent of the choice of the sequence $\{F_i\}$, which is surprising as they enter in the definition of the metric.

As usual, we find a lower and an upper bound. Both use proposition 1.1.3 to know the wdim a ball in $l^p(n)$. This proposition can be rewritten as: for $0 \leq k < n$

$$k < (\varepsilon/2)^{-p} \leq k+1 \Leftrightarrow \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)},\ell^\infty) = k.$$

Whence

$$\min\left((\tfrac{\varepsilon}{2})^{-p} - 1, n\right) \leq \mathrm{wdim}_\varepsilon(B_1^{\ell^p(n)},\ell^\infty) < \min\left((\tfrac{\varepsilon}{2})^{-p}, n\right).$$

This said, we start with the lower bound. This is pertinent only if $p \geq q$. As in theorem 2.1.9 the argument reduces to the construction of an injection which increases distances; here $(B_{a_k}^{\ell^p(\Omega_i^{-1}F_k;\mathbb{R}^s)},\ell^\infty)$ injects in $(B_1^{\ell^p},d'_{\Omega_i})$. Thus giving

$$\forall k \geq 1, \quad \min\left((\tfrac{\varepsilon}{2a_k})^{-p}, |\Omega_i^{-1}F_k|\right) \leq \mathrm{wdim}_\varepsilon(B_{a_k}^{\ell^p(\Omega_i^{-1}F_k,\mathbb{R}^s)},\ell^\infty) \leq \mathrm{wdim}_\varepsilon(B_1^{\ell^p},d'_{\Omega_i}).$$

This inequality being true for all k, the left-hand term can be replaced by the maximum over all k. Next, dividing both sides by $|\Omega_i|$, taking $\varepsilon = c|\Omega_i|^{-1/q}$ and passing to the limit yields

$$\limsup_{i\to\infty} \max_{k\geq 1} \min\left(\left(\frac{2a_k}{c}\right)^p |\Omega_i|^{\frac{p}{q}-1}, \frac{|\Omega_i^{-1}F_k|}{|\Omega_i|}\right) \leq \mathfrak{meas}_{\mathfrak{f}_{c,q}}(B_1^{\ell^p},d').$$

Then, notice that $|\Omega_i^{-1}F_k|/|\Omega_i|$ is not bounded on k, the term $(\frac{2a_k}{c})^p|\Omega_i|^{\frac{p}{q}-1}$ will determine the limit. Indeed, if $p > q$, it tends to infinity. However, when $p = q$, we must evaluate $\max\limits_{k\geq 1} \min\left((\frac{2a_k}{c})^p, \frac{|\Omega_i^{-1}F_k|}{|\Omega_i|}\right)$ knowing that the a_k are bounded. Hence the quantity of interest is larger than $\min\left((\frac{2A}{c})^p, 1\right) > 0$ where $A = \max a_k$ since $\frac{|\Omega_i^{-1}F_k|}{|\Omega_i|} > 1$ (it can even tend to ∞).

The upper bound, which is of interest only if $p \leq q$, is obtained as before by looking at the restriction of $(B_1^{\ell^p},d'_{\Omega_i})$ to $(B_1^{\ell^p(\Omega_i^{-1}F_k,\mathbb{R}^s)},\ell^\infty)$. This gives a 2ε-embedding for $(B_1^{\ell^p},d'_{\Omega_i})$ given any ε-embedding of $(B_1^{\ell^p(\Omega_i^{-1}F_k,\mathbb{R}^s)},\ell^\infty)$, *cf.* (2.1.10). Thus, if $\varepsilon > \sigma_{k+1}$:

$$\mathrm{wdim}_{2\varepsilon}(B_1^{\ell^p},d'_{\Omega_i}) \leq \min(\varepsilon^{-p}, |\Omega_i^{-1}F_k|).$$

For i fixed, $\varepsilon = c|\Omega_i|^{-1/q}$, and k is such that $\varepsilon > \sigma_{k+1}$. In particular, when $i \longrightarrow \infty$, $\varepsilon \longrightarrow 0$ and $k \longrightarrow \infty$. After division by $|\Omega_i|$ and taking the limit, we see that

$$\mathfrak{meas}_{\mathfrak{f}_{c,q}}(B_1^{\ell^p}, d') \leq \limsup_{i \longrightarrow \infty} \min\Big((\tfrac{2}{c})^p |\Omega_i|^{p/q-1}, \frac{|\Omega_i^{-1} F_k|}{|\Omega_i|} \Big).$$

When $p < q$ this tends to 0. But, if $p = q$, things behave differently: the right-hand term is bounded by $(\frac{2}{c})^p$.

In particular, when $p = q$, we have

$$\min\Big(\Big(\frac{2 \max a_k}{c}\Big)^p, 1 \Big) \leq \mathfrak{meas}_{\mathfrak{f}_{c,p}}(B_1^{\ell^p}, d') \leq \Big(\frac{2}{c}\Big)^p. \qquad \square$$

Theorem 2.2.4: *Let $p,q \in [1,\infty[$, Let $B_1^{\ell^p}$ and $B_1^{\ell^q}$ be unit balls of $\ell^p(\Gamma;\mathbb{R}^s)$ and $\ell^q(\Gamma;\mathbb{R}^s)$ respectively, both endowed with the action of Γ and of metrics of the type (2.1.7). When $q > p$, there is no Lipschitz homeomorphism $f : B_1^{\ell^p} \longrightarrow B_1^{\ell^q}$. Moreover, $\alpha q > p$ excludes that f be Hölder continuous of exponent α.*

Proof. If $f : X \longrightarrow X'$ is Hölder continuous of exponent $\alpha \in (0,1)$ or Lipschitz (this corresponds to $\alpha = 1$), $\exists c' \in \mathbb{R}_{>0}$ such that the modulus of continuity ω_f is bounded by $\psi(\varepsilon) = c'\varepsilon^\alpha$. Thus, proposition 2.2.2 gives that $\mathfrak{meas}_{\psi \circ \mathfrak{f}} B_1^{\ell^q} \leq \mathfrak{meas}_{\mathfrak{f}} B_1^{\ell^p}$, in other words, that $\mathfrak{meas}_{\mathfrak{f}_{c'c^\alpha, r/\alpha}} B_1^{\ell^q} \leq \mathfrak{meas}_{\mathfrak{f}_{c,r}} B_1^{\ell^p}$. Taking $p = r$, and thanks to lemma 2.2.3, we must have that $\alpha q \leq r = p$ to avoid a contradiction. $\square$

It is worth noticing that this is expected. Indeed, let $\phi : \ell^p \longrightarrow \ell^q$ be defined for $x \in \ell^p(\Gamma, \mathbb{R}^s)$ by $\phi(x)(\gamma) = \big(x(\gamma)\big)^{p/q}$, where for $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$, $y^{p/q} = (\frac{y_i}{|y_i|}|y_i|^{p/q})_{i=1,\ldots,s}$. Then, $\phi(B_1^{\ell^p}) = B_1^{\ell^q}$ and ϕ is a Γ-equivariant homeomorphism which is Hölder continuous of exponent $\alpha = p/q$ (known as the Mazur map). Also, recall (*cf.* [3, thm 2.3]) that any uniformly continuous map (for the norm topology) from $B_1^{\ell^p} \longrightarrow \ell^q$ is uniformly approximated by Hölder continuous maps.

The result on Lipschitz homoemorphisms can be obtained by another method (without requiring Γ-equivariance). If such an homeomorphism $\phi : B_1^{\ell^p} \longrightarrow B_1^{\ell^q}$ exists, consider the sequence $\{\phi_k\}$ where $\phi_k(x) = k\phi(\frac{1}{k}x)$. It does not converge to a map $\ell^p \longrightarrow \ell^q$, but there exists a subsequence converging on an ultraproduct of ℓ^p spaces. However an ultraproduct of ℓ^p spaces is isometric to an ℓ^p space. Hence we have a map $\phi_\omega : \ell^p \longrightarrow \ell^q$ which is Lipschitz, a contradiction. However, in the Hölderian case, this limit does not converge to a Hölder continuous map, which prevents the use of a similar argument.

2.3 ℓ^p Von Neumann dimension

We shall introduce another variant of mean dimension which coincides with the definition of Von Neumann dimension thanks to an argument of [19, §1.12].

Definition 2.3.1: Let Γ be a countable group and let $\{\Omega_i\}$ be an increasing sequence of finite subsets of Γ. Let (X,τ,δ) a space endowed with a topology τ and a pseudo-metric δ. For such a space, $\mathrm{wdim}_\varepsilon(X,\tau,\delta)$ denotes the smallest integer k such that there exists a polyhedron K of dimension k and a map, continuous for τ, $f : X \to K$ whose fibers are of diameter (measured with δ) less than ε. The $\ell^p(\Gamma)$-measure is

$$\mathfrak{meas}_{\ell^p}(X : \{\Omega_i\}) = \lim_{\varepsilon\to 0} \limsup_{i\to\infty} \frac{\mathrm{wdim}_\varepsilon(X,\tau,\delta_{\ell^p(\Omega_i)})}{|\Omega_i|} \quad \in [0,+\infty].$$

where $\delta_{\ell^p(\Omega)}(x,x') = \big(\sum_{\gamma\in\Omega} \delta(\gamma x,\gamma x')^p\big)^{1/p}$ when $p<\infty$ and $\delta_{\ell^\infty(\Omega)}(x,x') = \sup_{\gamma\in\Omega} \delta(\gamma x,\gamma x')$.

This definition will only be used in a particular context. Namely, we shall be interested in the case of X a subset of $\ell^\infty(\Gamma;V)$, the spaces of bounded sequences (indexed by Γ) with values in a normed vector space V. Of particular interest are linear subspaces of $\ell^p(\Gamma;V)$ or $c_0(\Gamma;V)$. We recall the latter is the space of all $x\in\ell^\infty(\Gamma;V)$ tending to 0 at infinity, *i.e.* $\|x\|_{\ell^\infty(\Gamma\setminus F_i)} \to 0$ for all exhaustive increasing sequence of (finite) subsets $\{F_i\}$.

The topology on $\ell^p(\Gamma;V)$ or $c_0(\Gamma;V)$ will be the product topology (induced by inclusion in V^Γ endowed with the product topology). We will denote it by τ^* (it is equivalent to the weak-* topology when it has a meaning). As for the pseudo-metric, it will be given by evaluation at the neutral element e of Γ: $ev(x,x') = \|x(e)-x'(e)\|_V$.

Definition 2.3.2: Let V be a finite-dimensional normed vector space. Let $Y\subset\ell^\infty(\Gamma;V)$ be a subset invariant by the natural action of Γ, an amenable group. Let Ω_i be a Følner sequence for Γ. Then, the ℓ^p Von Neumann dimension of Y is defined by

$$\dim_{\ell^p}(Y,\{\Omega_i\}) = \sup_{r\in\mathbb{R}_{\geq 0}} \mathfrak{meas}_{\ell^p}(B_r^{Y,p},\tau^*,ev,\{\Omega_i\})$$

where $B_r^{Y,p} = Y \cap B_r^{\ell^p(\Gamma;V)}$.

Note that this definition is valid even if Y is not a linear subspace. However, for such spaces it is not necessary to look at the sup on r: using dilation and a change of variable $\varepsilon\mapsto r\varepsilon$, only the result for the ball of radius 1 needs to be established. Let us begin by some simple examples.

Example 2.3.3*:* Let us show that if $1\leq q<p\leq\infty$, and $Y = B_1^{\ell^q(\Gamma;\mathbb{R})}$ then $\dim_{\ell^p}(Y,\{\Omega_i\}) = 0$ (independently of the choice of sequence $\{\Omega_i\}$). Indeed, $B_r^{Y,p} = B_1^{\ell^q(\Gamma;\mathbb{R})}$ if $r\geq 1$, and $\mathrm{wdim}_\varepsilon(B_r^Y, ev_{\ell^p(\Omega_i)}) = \mathrm{wdim}_\varepsilon(B_1^{\ell^q(n_i)},\ell^p)$ where $n_i = |\Omega_i|$. However using remark 1.4.3, $\mathrm{wdim}_\varepsilon(B^{\ell^q(n_i)},\ell^p)$ is bounded above and below by two functions that do not depend on n_i. Thus,

$$\limsup_{i\to\infty} \frac{\mathrm{wdim}_\varepsilon(B^{\ell^q(n_i)},\ell^p)}{|\Omega_i|} = 0.$$

A similar argument holds when $r<1$. Note that, $\dim_{\ell^p}(Y,\{\Omega_i\}) = \dim_{\ell^q}(Y,\{\Omega_i\}) = 1$ when $Y = B_1^{\ell^q(\Gamma;V)}$ and $p\leq q$.

Finally, let $q\in[1,\infty]$. If $Y' = \ell^q(\Gamma;V)$, then $(B^{Y'}, ev_{\ell^p(\Omega)})$ is "isometric" to $(B_1^{\ell^p(\Gamma;V)}, ev_{\ell^p(\Omega)})$. To be more precise, there is a continuous map whose fibers have "diameter" 0 from $(B^{Y'}, ev_{\ell^p(\Omega)})$ to

$(B_1^{\ell^p(\Gamma;V)}, ev_{\ell^p(\Omega)})$, and vice-versa. Thus, $(B^{Y'}, ev_{\ell^p(\Omega)})$ will have the same $\mathrm{wdim}_\varepsilon$ as $(B_1^{\ell^p(\Gamma;V)}, ev_{\ell^p(\Omega)})$, $\forall \varepsilon$.

When Y is a Γ-invariant linear subspace of $\ell^\infty(\Gamma;V)$,

- P1 (Independence) $\dim_{\ell^p}(Y,\{\Omega_i\})$ is actually independent of the choice of Følner sequence $\{\Omega_i\}$ (*cf.* corollary 2.5.1);
- P2 (Normalisation) $\dim_{\ell^p}\ell^p(\Gamma;\mathbb{R}) = 1$ (*cf.* example 2.3.3);
- P3 If $Y \subset \ell^2(\Gamma;V)$, $\dim_{\ell^2}Y$ coincides with the Von Neumann dimension (*cf.* corollary 2.3.7);

Given P3 and upon reading [4, §1], many other properties for $\dim_{\ell^2}$ can be obtained. One is led to check if they holf for $\dim_{\ell^p}$. These properties are:

- P4 (Invariance) If $f : Y_1 \longrightarrow Y_2$ is an injective Γ-equivariant linear map of finite type, then $\dim_{\ell^p}Y_1 \leq \dim_{\ell^p}Y_2$ (*cf.* proposition 2.5.3 and example 2.5.4);
- P5 (Non-triviality) $Y \subset \ell^p$ is trivial if and only if $\dim_{\ell^p}Y = 0$ (*cf.* proposition 2.5.5 for the ℓ^1 case, but is false for ℓ^∞);
- P6 (Additivity) $\dim_{\ell^p}Y_1 \oplus Y_2 = \dim_{\ell^p}Y_1 + \dim_{\ell^p}Y_2$ (*cf.* remark 2.5.6);
- P7 (Completion) If $\overline{Y}$ is the completion of Y for the ℓ^p norm, then $\dim_{\ell^p}Y = \dim_{\ell^p}\overline{Y}$ (*cf.* proposition 2.5.7);
- P8 (Continuity) If $\{Y_i\}$ is a decreasing sequence of closed linear subspaces then $\dim_{\ell^p}(\cap Y_i) = \lim_{i\to\infty}\dim_{\ell^p}Y_i$ (for $p=1$ this does not hold, *cf.* 2.5.8);
- P9 (Reciprocity) If $\Gamma_1 \subset \Gamma_2$ and if $Y_2 \subset \ell^p(\Gamma_2;V)$ is the subspace induced by $Y_1 \subset \ell^p(\Gamma_1;V)$ then $\dim_{\ell^p}(Y_2,\Gamma_2) = \dim_{\ell^p}(Y_1,\Gamma_1)$ (*cf.* remark 2.5.9);
- P10 (Reduction) If $\Gamma_1 \subset \Gamma_2$ is of finite index, and if $Y \subset \ell^p(\Gamma_2;V)$ is seen by restriction as a subspace of $\ell^p(\Gamma_1;V^{[\Gamma_2:\Gamma_1]})$ then $[\Gamma_2:\Gamma_1]\dim_{\ell^p}(Y,\Gamma_2) = \dim_{\ell^p}(Y,\Gamma_1)$ (*cf.* proposition 2.5.10).

Except for $p=2$ where all these properties are true, properties P5, P6, P8 and P9 are not established (with the exception of P5 for the ℓ^1 case). In order to show P6 and P9 some sort of super-additivity of wdim is needed for balls. Moreover, P5 is false Y for linear subspaces of ℓ^∞ but might hold $Y \subset c_0$.

Though these properties are stated for Γ-invariant linear subspaces, some remain true for more general subsets Y: P1 and P10 hold for any Γ-invariant subset, P4 does not require that Y_2 be a linear subspace if $f : Y_1 \longrightarrow Y_2$ is Lipschitz, and lastly P7 is also true when Y is not Γ-invariant. A weaker property also holds for Hausdorff limits of closed sets.

In [19, §1.12] a link between Von Neumann dimension (in the usual $\dim_{\ell^2}$ context) and wdim of certain balls is shown. Let us first recall a definition.

Let $Y \subset \ell^2(\Gamma;\mathbb{R}^s) \subset (\mathbb{R}^s)^\Gamma$ be a Γ-invariant linear subspace, $\forall \Omega \subset \Gamma$ we define the operator $R_\Omega : Y \longrightarrow (\mathbb{R}^s)^\Omega_{\ell^2}$ by restriction to Ω: $y \longmapsto y_{|\Omega}$. Its adjoint $R^*_\Omega : (\mathbb{R}^s)^\Omega_{\ell^2} \longrightarrow Y$ is the orthogonal projection to Y. To see this, write $R_\Omega(y) = y 1\!\!1_\Omega$ where $1\!\!1_\Omega$ is the characteristic function of Ω, then

$$\langle R^*_\Omega(x), y\rangle := \langle x, R_\Omega y\rangle = \int_\Gamma xy 1\!\!1_\Omega = \int_\Gamma (1\!\!1_\Omega x) y.$$

However this last expression is simply the scalar product of x, extended as a function on all of Γ by 0, with y. Thus, $R^*_\Omega(x)$ is the projection on Y of the extension of x to Γ by 0. In what follows we will omit this inclusion (extension by 0) from $V^\Omega_{\ell^2}$ to $V^{\Omega'}_{\ell^2}$ when $\Omega \subset \Omega'$. Dependence on Ω of R^*_Ω will not be written. A crucial remark is that the invariance of Y by Γ implies that, for $\Omega, \Omega' \subset \Gamma$ finite subsets,

$$\frac{\mathrm{Tr} R_\Omega R^*}{\mathrm{Tr} R_{\Omega'} R^*} = \frac{|\Omega|}{|\Omega'|}.$$

A possible definition of Von Neumann dimension (see [33] or [40]) is

$$\dim_{\ell^2}(Y : \Gamma) := |\Omega|^{-1} \mathrm{Tr} R_\Omega R^*$$

for a $\Omega \subset \Gamma$. This quantity is actually independent of the chosen set. The aim of this section is to retrieve this quantity as the wdim of a certain object.

Theorem 2.3.4: *(cf. [19, 1.12A]) Let $\Omega_i \subset \Gamma$ be a Følner sequence, let $n_i[a,b]$ be the number of eigenvalues of the operator $R_{\Omega_i} R^*$ (defined relative to Y) contained in the interval $[a,b]$. If $0 < a \leq b < 1$, then*

$$\lim_{i \to \infty} \frac{n_i[a,b]}{|\Omega_i|} = 0$$

Proof. (The proof is with minor differences in notation that of [19].) Since R_Ω and R^* are both projections (in ℓ^2), the eigenvalues of $R_\Omega R^*$ will be contained in $[0,1]$. The proof proceeds in three steps.

First, let $x \in \ell^2(\Omega;\mathbb{R}^s)$, it will be called an ε-quasimode of eigenvalue λ for $R_\Omega R^*$ if

$$\|R_\Omega R^* x - \lambda x\|_{\ell^2} \leq \varepsilon \|x\|_{\ell^2}. \tag{2.3.5}$$

If x is such an element, and if its restriction outside Ω is small, more precisely

$$\|R^* x_{|\Gamma \smallsetminus \Omega}\|_{\ell^2} = \|R^* x - R_\Omega R^* x\|_{\ell^2} \leq \delta \|x\|_{\ell^2}, \tag{2.3.6}$$

then $\lambda(1-\lambda) \leq 2\varepsilon + \delta$. Indeed, using (2.3.5) in (2.3.6) yields that $\|R^* x - \lambda x\|_{\ell^2} \leq (\delta + \varepsilon) \|x\|_{\ell^2}$. Since R^* is a projection, $R^* R^* = R^*$ and $\|R^*\| = 1$, whence

$$(1-\lambda) \|R^* x\|_{\ell^2} = \|R^* x - R^* \lambda x\|_{\ell^2} = \|R^*(R^* x - \lambda x)\|_{\ell^2} \leq \|R^* x - \lambda x\|_{\ell^2} \leq (\delta + \varepsilon) \|x\|_{\ell^2},$$

since the eigenvalues of $R_\Omega R^*$ are all contained in $[0,1]$, $|1-\lambda| = 1-\lambda$. Moreover the restriction to Ω can only reduce the norm, $\|R_\Omega R^* x\|_{\ell^2} \leq \|R^* x\|_{\ell^2}$. Using (2.3.5) anew gives,

$$(1-\lambda)\lambda \|x\|_{\ell^2} \leq (1-\lambda)\|R_\Omega R^* x\|_{\ell^2} + (1-\lambda)\varepsilon \|x\|_{\ell^2} \leq (\delta + (2-\lambda)\varepsilon)\|x\|_{\ell^2}$$

Second, denote by $\Omega^{-\rho} \subset \Omega$ the ρ-interior of Ω, *i.e.* the set of points with distance at least ρ from a point point outside of Ω, where the distance on Γ is the word distance for any fixed generating set. The next argument will consist in showing that most of $x \in \ell^2(\Omega^{-\rho}; V)$ have a small projection to $\Gamma \setminus \Omega$. Precisely, let

$$S_\rho = R_{\Gamma\setminus\Omega} R^* = (1-R_\Omega)R^* : \ell^2(\Omega^{-\rho}; \mathbb{R}^s) \to \ell^2(\Gamma\setminus\Omega),$$

then $\mathrm{Tr}\, S_\rho^* S_\rho \leq s\beta_0(\rho)|\Omega^{-\rho}|$ where $\beta_0(\rho)$ tends to 0 when $\rho \to \infty$. The dependence on ρ does not only come from the domain of definition: the operator $S_\rho^* S_\rho$ is

$$\begin{aligned} S_\rho^* S_\rho &= R_{\Omega^{-\rho}}(1-R^*)(1-R_\Omega)R^* &&= (R_{\Omega^{-\rho}} - R_{\Omega^{-\rho}}R^*)(R^* - R_\Omega R^*) \\ &= (R_{\Omega^{-\rho}}R^* - R_{\Omega^{-\rho}}R^*R^* - R_{\Omega^{-\rho}}R_\Omega R^* + R_{\Omega^{-\rho}}R^* R_\Omega R^*) &&= R_{\Omega^{-\rho}}R^*(1-R_\Omega R^*). \end{aligned}$$

Any Dirac mass x_γ with support at a point γ satisfies $\|R^* x_\gamma\|_{\ell^2} \leq 1$ (R^* is a projection). Thus, if $B_\rho(\gamma)$ is the ball of radius ρ, $\|(1-R_{B_\rho(\gamma)})R^* x_\gamma\|_{\ell^2} \leq \beta_0(\rho)$, with $\lim_{\rho\to\infty} \beta_0(\rho) = 0$. Consequently $\|S_\rho x_\gamma\|_{\ell^2} \leq \beta_0(\rho)$ since $\Gamma \setminus \Omega$ is contained in the complement of the union of the $B_\rho(\gamma)$ for $\gamma \in \Omega^{-\rho}$. Since $\|S_\rho^*\| \leq 1$, $\|S_\rho^* S_\rho x_\gamma\|_{\ell^2} \leq \beta_0(\rho)$. The Dirac masses being an orthonormal basis for $\ell^2(\Omega^{-\rho}; \mathbb{R}^s)$, we get that $\mathrm{Tr}\, S_\rho^* S_\rho \leq s\beta_0(\rho)|\Omega^{-\rho}|$.

Last, we shall evaluate $n_i[a,b]$ for $a, b \in]0,1[$ and $b-a = \varepsilon \in]0,1[$. Let X_i be the space generated by eigenvectors of $R_{\Omega_i}R^*$ whose eigenvalue is in $[a,b]$. Then, $\forall\lambda \in [a,b], \forall x \in X_i$, x is an ε-quasimode of eigenvalue λ for $R_{\Omega_i}R^*$. The evaluation of $\dim X_i$ will be done by looking at spaces whose dimension is close. If $X_i^\rho = X_i \cap \ell^2(\Omega_i^{-\rho}, V)$ is the subspace of elements which vanish on the thickened boundary, $\dim X_i - \dim X_i^\rho \leq s|\Omega_i \setminus \Omega_i^{-\rho}|$. The amenability used on Ω_i shows that this difference is negligible, $\lim_{i\to\infty} (\dim X_i - \dim X_i^\rho)/|\Omega_i| = 0$; it will suffice to evaluate $\dim X_i^\rho$.

Unfortunately, neither X_i^ρ nor X_i is *a priori* invariant by $S_\rho^* S_\rho$. Let's nevertheless look at the intersection of X_i^ρ with the space generated by eigenvectors of $S_\rho^* S_\rho$ of eigenvalue $\leq \beta^2$; we will denote this new intersection by $X_i^{\rho,\beta}$. On this space, $\|S_\rho\| \leq \beta$ since

$$\|S_\rho x\|_{\ell^2}^2 = \langle S_\rho x, S_\rho x\rangle = \langle x, S_\rho^* S_\rho x\rangle \leq \beta^2 \|x\|_{\ell^2}^2$$

Yet again, this space is of dimension close to that of X_i^ρ: if $V^{>\beta^2}$ is the space of eigenvectors of $S_\rho^* S_\rho$ whose eignevalue is greater than β^2, then

$$\dim X_i^\rho - \dim X_i^{\rho,\beta} \leq \dim V^{>\beta^2} \leq \beta^{-2}\mathrm{Tr}\, S_\rho^* S_\rho \leq s|\Omega_i^{-\rho}|\beta_0(\rho)/\beta^2.$$

In other words,

$$\forall\beta > 0, \forall\alpha > 0, \exists\rho \text{ such that } \limsup_{i\to\infty} \frac{\dim X_i^\rho - \dim X_i^{\rho,\beta}}{|\Omega_i|} \leq \alpha,$$

Thus, it remains to evaluate $\dim X_i^{\rho,\beta}$. To do so, we use the conclusion of the first part for $\lambda = a$ and $\varepsilon = b-a$: this yields $a(1-a) \leq 2(b-a)+\beta$ given that $\dim X_i^{\rho,\beta} > 0$. Consequently, the inequality $b-a < (a-a^2-\beta)/2$ implies that $\dim X_i^{\rho,\beta} = 0$. Which means that when $\rho \geq \rho_0(\beta,\alpha)$ is sufficiently big, $\limsup_{i\to\infty} \dim X_i^{\rho}/|\Omega_i| \leq \alpha$. It follows that

$$\begin{aligned}\limsup_{i\to\infty} \frac{\dim X_i}{|\Omega_i|} &\leq \limsup_{i\to\infty} \frac{\dim X_i - \dim X_i^{\rho}}{|\Omega_i|} + \limsup_{i\to\infty} \frac{\dim X_i^{\rho}}{|\Omega_i|} \\ &\leq 0 + \limsup_{i\to\infty} \frac{\dim X_i^{\rho} - \dim X_i^{\rho,\beta}}{|\Omega_i|} + \limsup_{i\to\infty} \frac{\dim X_i^{\rho}}{|\Omega_i|} = 0\end{aligned}$$

since $\alpha \to 0$ when $\rho \to \infty$. This proves the theorem for intervals $[a,b]$ satisfying $b-a < a(1-a)/2$, as the size of β in not constained. The conclusion is obtained by noticing that any interval strictly contained in $[0,1]$ can be covered by intervals of this type. □

This property enables us interpret Von Neumann dimension as a wdim for a set with a chosen pseudo-metric.

Corollary 2.3.7: *(cf. [19, cor 1.12.2]) Let $Y \subset \ell^2(\Gamma;V)$ be an invariant subspace, let $B_1^Y = Y \cap B_1^{\ell^2(\Gamma;V)}$ the intersection of the unit ball with Y. Then, for a given Følner sequence $\Omega_i \subset \Gamma$,*

$$\forall \varepsilon \in]0,1[, \qquad \lim_{i\to\infty} \frac{1}{|\Omega_i|} \mathrm{wdim}_\varepsilon(R_{\Omega_i} B_1^Y, \ell^2) = \dim_{\ell^2} Y$$

Proof. (We give the argument of [19] in detail.) To get this result $R_\Omega B_1^Y$ must be seen as an ellipsoid whose semi-axes are related to the eigenvalues of $R_\Omega R^*$. Remark that $B_1^Y = R^* B_1^{\ell^2(\Gamma;V)}$. Then, an ellipsoid can be defined as the image of a ball by an self-adjoint operator, say A; the semi-axis of this ellipsoid are in correspondance with the eigenvaluer of A. It might be worth recalling how this relates to the usual definition of an ellipsoid E (as the set $\{y | \langle y, Py\rangle \leq 1\}$ for a positive definite operator P). The semi-axes of E are of the form $\lambda_i(P)^{-1/2}$ for $\lambda_i(P)$ an eigenvalue of P. Indeed let B^V be a ball in a vector space V, and let $A: V \to V$ be self-adjoint. Restricting to $V' = \mathrm{Im}A = \mathrm{Ker}A^\perp \subset V$, it must be shown that for $x \in V'$ such that $\langle x,x\rangle \leq 1$, there exists $P: V' \to V'$ positive definite such that $\langle Ax, PAx\rangle \leq 1$. Taking $P = A^{-2}$ yields the conclusion: A^{-2} is a positive definite operator on V' whose eigenvalues are $\lambda_i(A)^{-2}$. Thus AB^V is an ellipsoid with semi-axis $\lambda_i(P)^{-1/2} = \lambda_i(A)$.

In our present context, $R_\Omega R^*$ is self-adjoint, thus $R_\Omega R^* B_1^{\ell^2(\Gamma;V)} = R_\Omega B_1^Y$ is an ellipsoid whose semi-axis are the eigenvalues of $R_\Omega R^*$. This ellipsoid contains isometrically the ball obtained by ignoring the semi-axis of length $< \varepsilon$ and replacing the remaining ones by semi-axis of length ε. Thus $\mathrm{wdim}_\varepsilon(R_{\Omega_i} B_1^Y, \ell^2) \geq n_i[\varepsilon, 1]$. On the other hand, $\mathrm{wdim}_\varepsilon(R_{\Omega_i} B_1^Y, \ell^2) \leq n_i[\varepsilon/2, 1]$, as the continuous map obtained by projecting on the sub-ellipsoid formed by the semi-axis of length $> \varepsilon/2$ indicates. When $i \to \infty$, the eigenvalues of $R_{\Omega_i} R^*$ tend to 0 or 1. In particular, when $i \to \infty$ the inequality

$$\frac{1}{|\Omega_i|} n_i[\varepsilon, 1] \leq \frac{1}{|\Omega_i|} \mathrm{wdim}_\varepsilon(R_{\Omega_i} B, \ell^2) \leq \frac{1}{|\Omega_i|} n_i[\varepsilon/2, 1]$$

shows that $\lim_{i\to\infty} \frac{1}{|\Omega_i|} \mathrm{wdim}_\varepsilon(R_{\Omega_i} B_1^Y, \ell^2) = \dim_{\ell^2} Y$, since $n_i[a,1] \to \mathrm{Tr} R_{\Omega_i} R^*$. □

This corollary can be expressed in terms of ℓ^p dimension. Indeed, let $B_1^Y = Y \cap B_1^{\ell^2(\Gamma;V)}$ be endowed with the pseudo-metric of evaluation at $e \in \Gamma$: $ev(x,y) = \|x(e) - y(e)\|_V$. Translation of this pseudo-metric by an element of γ is the evaluation at γ. Thus, $ev_{\ell^2(\Omega)}(x,y) = \|x-y\|_{\ell^2(\Omega)} = \|R_\Omega(x-y)\|_{\ell^2}$. The map $R_\Omega : B_1^Y \to R_\Omega B$ is continuous for the topology of B_1^Y as a subset of ℓ^p (with τ^* or even with the norm topology). The fibers are of "diameter" 0 given that $\Omega' \subset \Omega$. Thus, corollary 2.3.7 can be expressed as follows:

$$\mathfrak{meas}_{\ell^2}(B_1^Y, \tau^*, ev, \{\Omega_i\}) = \dim_{\ell^2} Y.$$

Indeed, $R_{\Omega_i} B_1^Y$ injects isometrically in (B_1^Y, ev_{Ω_i}) and (B_1^Y, ev_{Ω_i}) possesses a map to $R_{\Omega_i} B_1^Y$ whose fiber are of "diameter" 0. Thus $\mathrm{wdim}_\varepsilon(B_1^Y, ev_{\Omega_i}) = \mathrm{wdim}_\varepsilon(R_{\Omega_i} B_1^Y, \ell^2)$. This shows that definition 2.3.2 is equivalent when $p = 2$ to the Von Neumann dimension and this for any Følner sequence $\{\Omega_i\}$ chosen.

It would be surprising that this is not the case in general. An alteration of the Ornstein-Weiss lemma enables to show the independence of the limit on the sequence chosen. The next section is dedicated to its proof and introduces some useful tools to deal with discrete amenable groups.

The choice of τ^* for a topology comes from the fact that τ^* is the roughest topoplogy which is finer than all the topologies induced by the pseudo-metrics $ev_{\ell^p(\Omega)}$.

2.4 Ornstein-Weiss' Lemma

Let us start by some definitions (see also [38], [19] or [28]).

Definition 2.4.1: Let Γ be a group, let $F \subset \Gamma$ be such that $e_\Gamma \in F$ then the F-boundaries of $\Omega \subset \Gamma$ are defined as

$$\begin{array}{llll}
\partial_F^+\Omega & = \{\gamma \notin \Omega \mid \gamma F \cap \Omega \neq \varnothing \text{ and } \gamma F \cap \Omega^c \neq \varnothing\} & = F^{-1}\Omega \cap \Omega^c & \text{(outer } F\text{-boundary)} \\
\partial_F^-\Omega & = \{\gamma \in \Omega \mid \gamma F \cap \Omega \neq \varnothing \text{ and } \gamma F \cap \Omega^c \neq \varnothing\} & = F^{-1}\Omega^c \cap \Omega & \text{(inner } F\text{-boundary)} \\
\partial_F\Omega & = \{\gamma \in \Gamma \mid \gamma F \cap \Omega \neq \varnothing \text{and } \gamma F \cap \Omega^c \neq \varnothing\} & = \partial_F^+\Omega \cup \partial_F^-\Omega & \text{(}F\text{-boundary)} \\
\mathrm{int}_F\Omega & = \{\gamma \in \Gamma \mid \gamma F \subset \Omega\} & = \Omega \setminus \partial_F^-\Omega & \text{(}F\text{-interior)} \\
\mathrm{fer}_F\Omega & = \{\gamma \in \Gamma \mid \gamma F \cap \Omega \neq \varnothing\} & = \Omega \cup \partial_F^+\Omega & \text{(}F\text{-closure).}
\end{array}$$

Moreover, let $|\cdot|$ denote a measure on Γ. The relative amenability function will be defined as $\alpha(\Omega;F) = \frac{|\partial_F\Omega|}{|\Omega|}$, given that these numbers are finite.

Before we move on to technical results, observe that the Følner conditions implies that $\alpha(\Omega_i;F) \to 0$ for any finite set F and any Følner sequence $\{\Omega_i\}$. Another useful property is that if $F' \subset F$, then $\alpha(\Omega;F') \leq \alpha(\Omega;F)$ since $\partial_{F'}\Omega \subset \partial_F\Omega$. We start by showing covering properties of big sets by smaller sets.

Definition 2.4.2: Let $\varepsilon \in]0,1[$. Subsets F_i of finite measure of Γ will be said ε-disjoint if there exists $F_i' \subset F_i$ which are disjoint and such that $|F_i'| \geq (1-\varepsilon)|F_i|$ and $\cup F_i' = \cup F_i$.

A subset of finite measure Ω will be said to admit an ε-quasi-tiling by the subsets F_i if

(a) $F_i \subset \Omega$,

(b) the F_i are ε-disjoint,

Here is a first lemma which studies the proportion of a set Ω covered by an ε-quasi-tiling of translates of another set F.

Lemma 2.4.3: *Let Γ be a discrete group endowed with the counting measure, denoted by $|\cdot|$. Let $\Omega \subset \Gamma$ and $e_\Gamma \in F \subset \Gamma$ both finite sets and such that $\alpha(\Omega;F) < 1$. Let $\{\gamma_i\}_{1\leq i\leq k}$ be a maximal sequence of elements of Γ such that the $\gamma_i F$ form an ε-quasi-tiling of Ω. Let $U_F^i = \bigcup_{j=1}^{i} \gamma_j F$, then*

$$\frac{|U_F^k|}{|\Omega|} \geq \varepsilon(1-\alpha(\Omega;F)).$$

Proof. (This proof corresponds to the first part of the proof of the Ornstein-Weiss lemma in [19, §1.3.1].) We shall use this general fact:

$$\begin{aligned}
\int_\Gamma |G_1 \cap \gamma G_2| d\mu(\gamma) &= \int_\Gamma\int_\Gamma \mathbb{1}_{G_1\cap\gamma G_2}(\gamma') d\mu(\gamma') d\mu(\gamma) \\
&= \int_\Gamma\int_\Gamma \mathbb{1}_{G_1}(\gamma')\mathbb{1}_{G_2}(\gamma'\gamma^{-1}) d\mu(\gamma) d\mu(\gamma') \\
&= \int_\Gamma \mathbb{1}_{G_1}(\gamma') \left(\int_\Gamma \mathbb{1}_{G_2}(\gamma'\gamma^{-1}) d\mu(\gamma)\right) d\mu(\gamma') \\
&= \int_\Gamma \mathbb{1}_{G_1}(\gamma') |G_2| d\mu(\gamma') \\
&= |G_1||G_2|.
\end{aligned}$$

Thus,

$$|\mathrm{int}_F\Omega|^{-1}\int_{\mathrm{int}_F\Omega} |U_F^k \cap \gamma F| d\mu(\gamma) \leq |\mathrm{int}_F\Omega|^{-1}\int_\Gamma |U_F^k \cap \gamma F| d\mu(\gamma) \leq (1-\alpha(\Omega;F))^{-1}|\Omega|^{-1}|U_F^k||F|.$$

Clearly, $|U_F^{i-1} \cap \gamma_i F| \leq \varepsilon|F|$, as the $\gamma_i F$ are ε-disjoint. On the other hand, maximality of k implies that $\forall\gamma \in \mathrm{int}_F\Omega, |U_F^k \cap \gamma F| \geq \varepsilon|F|$. We then observe that

$$|\mathrm{int}_F\Omega|^{-1}\int_{\mathrm{int}_F\Omega} |U_F^k \cap \gamma F| d\mu(\gamma) \geq \varepsilon|F|.$$

Consequently, $\varepsilon(1-\alpha(\Omega;F)) \leq |U_F^k|/|\Omega|$. □

Note that the quasi-tiling can be empty if $\alpha(\Omega;F)$. More precisely, the proof actually works for $\alpha^-(\Omega;F) = \frac{|\partial_F^-\Omega|}{|\Omega|}$ instead of α. It has the advantage that $\mathrm{int}_F\Omega \neq \varnothing$ implies that $\alpha^-(\Omega;F) < 1$ and the quasi-tiling is non-empty. In any case, in the upcoming applications, F will always be contained in Ω. The three following lemmas are technical ingredients which will be used in the proof of the generalisation of the Ornstein-Weiss lemma.

Lemma 2.4.4: *Let $\Omega' \subset \Omega \subset \Gamma$ and $F \subset \Gamma$ be finite. Suppose that there exists ε such that $|\Omega \setminus \Omega'| \geq \varepsilon|\Omega|$, then*

$$\alpha(\Omega \setminus \Omega';F) \leq \frac{\alpha(\Omega';F)+\alpha(\Omega;F)}{\varepsilon}.$$

Proof. Since $|\partial_F(\Omega \smallsetminus \Omega')| \leq |\partial_F \Omega| + |\partial_F \Omega'| = \alpha(\Omega;F)|\Omega| + \alpha(\Omega';F)|\Omega'|$, and that $|\Omega \smallsetminus \Omega'| \geq \varepsilon|\Omega| \geq \varepsilon|\Omega'|$, a substitution yields

$$\alpha(\Omega \smallsetminus \Omega';F) = \frac{|\partial_F(\Omega \smallsetminus \Omega')|}{|\Omega \smallsetminus \Omega'|} \leq \frac{\alpha(\Omega;F)|\Omega|}{\varepsilon|\Omega|} + \frac{\alpha(\Omega';F)|\Omega'|}{\varepsilon|\Omega'|}. \qquad \square$$

Lemma 2.4.5: *Let $F \subset \Gamma$ be finite, and let $\{D_i\}_{1 \leq i \leq n}$ be an ε-disjoint family of subsets. Then*

$$\alpha(\cup D_i;F) \leq \frac{\max(\alpha(D_i;F))}{1-\varepsilon}$$

Proof. Since $\partial_F(\cup D_i) \subset \cup \partial_F D_i$, we obtain that

$$|\partial_F(\cup D_i)| \leq \sum |\partial_F D_i| \leq \sum \alpha(D_i;F)\,|D_i| \leq \max(\alpha(D_i;F)) \sum |D_i|$$

However $(1-\varepsilon)\sum|D_i| \leq |\cup D_i|$ as they are ε-disjoint. Thus

$$\alpha(\cup D_i;F) = \frac{\partial_F(\cup D_i)}{|\cup D_i|} \leq \frac{\max(\alpha(D_i;F))}{1-\varepsilon}. \qquad \square$$

The last lemma is an adaptation of a useful property of $\mathbb{Z}$ to general amenable group: for the typical Følner sequence for $\mathbb{Z}$ $I_i = [-i,i]$, any sufficiently big interval in this family is covered (except for small bits) by translates of the I_i.

Lemma 2.4.6: *Let $\{F_i\}$ be a Følner sequence, let $\delta \in]0,1/2[$. Then there exists a subsequence (which depends on δ) $\{F_{n_i}\}$, an integer $N(\delta)$, and a sequence of integers $\{k_i\}_{1 \leq i \leq N}$ such that for all set Ω which contains F_{n_N} and satisfies $\alpha(\Omega, F_{n_N}) \leq 2\delta^{2N}$ there exists a family $\mathcal{G}$ of δ-disjoint sets such that $|\underset{F \in \mathcal{G}}{\cup} F| \geq (1-\delta)|\Omega|$ and $\mathcal{G}$ consists in k_i translates of the sets F_{n_i}*

Proof. (We write the argument of [19, §1.3.1] in details, this result can also be found in [38]; [28] covers this topic.) In order to better show how the constants enter the proof, we denote $\varepsilon_1 = \delta^{2N}$, $\varepsilon_2 = 2\delta^{2N}$ and $\rho = \delta$. First, $\forall \varepsilon_1 \in]0,1[$, it is possible to refine the sequence $\{F_i\}$ to have

$$\alpha(F_{i+1}, F_i) \leq \varepsilon_1.$$

Now, let $\Omega^{(1)} = \Omega$ so that $\alpha(\Omega^{(1)}, F_n) \leq \varepsilon_2$, where n will be determined later on. We will cover $\Omega^{(1)}$ to a proportion of $1-\delta$ by almost disjoint translates of the F_i, where $1 \leq i \leq n$, in n steps (or less). For any $\rho \in]0, \frac{1}{2}[$, lemma 2.4.3 gives a ρ-quasi-tiling of $\Omega^{(1)}$ by k_n translates of F_n such that $|U_{F_n}^{k_n}| \geq \rho(1-\varepsilon_2)|\Omega^{(1)}|$. Let $\Omega^{(2)} = \Omega^{(1)} \smallsetminus U_{F_n}^{k_n}$, then $|\Omega^{(2)}| \leq (1-\rho+\varepsilon_2\rho)|\Omega^{(1)}|$.

If $|\Omega^{(2)}| \leq \delta|\Omega^{(1)}|$ the goal is achieved and there is no need to continue. Otherwise, lemma 2.4.4 then lemma 2.4.5 shows that

$$\alpha(\Omega^{(2)}, F_{n-1}) \leq \frac{1}{\delta}(\varepsilon_1 + \alpha(U_{F_n}^{k_n}; F_{n-1})) \leq \frac{1}{\delta}(\varepsilon_1 + \frac{\varepsilon_1}{1-\rho}) \leq 3\frac{\varepsilon_1}{\delta}.$$

It is now possible to recover $\Omega^{(2)}$ by a ρ-quasi-tiling of k_{n-1} translates of F_{n-1} in such a way that $|U_{F_{n-1}}^{k_{n-1}}| \geq \rho(1-3\frac{\varepsilon_1}{\delta})|\Omega^{(2)}|$. We now have a set $\Omega^{(3)}$ such that

$$|\Omega^{(3)}| \leq (1-\rho-3\rho\frac{\varepsilon_1}{\delta})|\Omega^{(2)}| \leq (1-\rho+\varepsilon_2\rho)(1-\rho+\rho\frac{3\varepsilon_1}{\delta})|\Omega^{(1)}|$$

We will now take $\varepsilon_2 = 2\varepsilon_1$. Proceeding by induction, as long as $|\Omega^{(i-1)}| \geq \varepsilon|\Omega^{(1)}|$, the set $\Omega^{(i)}$ (for $1 \leq i \leq n$) will have the following properties:

1. $\alpha(\Omega^{(i)}, F_{n-i+1}) \leq (1+i)\varepsilon_1/\delta^{i-1}$
2. $U^{k_{n-i+1}}_{F_{n-i+1}}$ is a ρ-quasi-tiling of $\Omega^{(i)}$ by translates of F_{n-i+1}
3. If $\Omega^{(i+1)} = \Omega^{(i)} \setminus U^{k_{n-i+1}}_{F_{n-i+1}}$ then $|\Omega^{(i+1)}| \leq |\Omega^{(1)}| \prod_{j=1}^{i} (1-\rho(1-(1+j)\varepsilon_1/\delta^{j-1}))$

Since it is not possible to hope that this process terminates before $i = n$, it remains to be checked that if n is big enough, we still get a quasi-tiling that covers $(1-\delta)|\Omega^{(1)}|$ elements. To achieve this, observe that the product in the third property above can be bounded if $i = n$ by

$$\prod_{j=1}^{n}(1-\rho(1-(1+j)\varepsilon_1/\delta^{j-1})) \leq (1-\rho(1-(1+n)\varepsilon_1/\delta^{n-1}))^n.$$

For $\varepsilon_1 = \delta^{2n}$, the right-hand term tends to 0 when n tends to ∞. Thus, $\exists N(\delta,\rho)$ such that if $\varepsilon_1 = \delta^{2N}$ translates of F_j (where $1 \leq j \leq N$) form a ρ-quasi-tiling of any set $\Omega^{(1)}$ such that $\alpha(\Omega^{(1)}; F_N) \leq \delta^{2N}$.

We substitute as promised $\rho = \delta$ to have: for any fixed δ, taking a subsequence whose members satisfy $\alpha(F_{n_{i+1}}, F_{n_i}) \leq \delta^{2N}$ where N is such that $(1-\delta(1-(1+N)\delta^{N+1}))^N < \delta$, then successive applications of lemma 2.4.3 give the required translates of F_{n_i}. □

We are now ready to prove the main result of this section. It might be better to start by reading the proof with $\Gamma = \mathbb{Z}$ in mind ($\Omega_n = [-n,n] \cap \mathbb{Z}$).

Theorem 2.4.7: *Let Γ be a discrete amenable group, and let $a : \mathbb{R}_{\geq 0} \times \Gamma \to \mathbb{R}_{\geq 0}$ be a function such that, $\forall \Omega, \Omega' \subset \Gamma$ are finite and $\forall \varepsilon \in \mathbb{R}_{>0}$*

(a)	*a is Γ-invariant,* i.e.	$\forall \gamma \in \Gamma$,	$a(\varepsilon, \gamma\Omega) = a(\varepsilon, \Omega)$
(b)	*a is decreasing in ε,* i.e.	$\forall \varepsilon' \leq \varepsilon$,	$a(\varepsilon', \Omega) \geq a(\varepsilon, \Omega)$
(c)	*a is K-sublinear in Ω,* i.e.	$\exists K \in \mathbb{R}_{>0}$,	$a(\varepsilon, \Omega) \leq K|\Omega|$
(d)	*a is c-subadditive in Ω,* i.e.	$\exists c \in]0,1]$,	$a(\varepsilon, \Omega \cup \Omega') \leq a(c\varepsilon, \Omega) + a(c\varepsilon, \Omega')$

then, for any Følner sequence $\{\Omega_i\}$,

$$\lim_{\varepsilon \to 0} \limsup_{i \to \infty} \frac{a(\varepsilon, \Omega_i)}{|\Omega_i|} = \lim_{\varepsilon \to 0} \liminf_{i \to \infty} \frac{a(\varepsilon, \Omega_i)}{|\Omega_i|}.$$

In particular, these limits are independent of the chosen sequence $\{\Omega_i\}$.

Proof. Let us first introduce some notations for the functions given by pointwise convergence and their limits. Let$\{\Omega_i^{+,\varepsilon}\}$ and $\{\Omega_i^{-,\varepsilon}\}$ be subsequences of $\{\Omega_i\}$ such that

$$\lim_{i \to \infty} \frac{a(\varepsilon, \Omega_i^{+,\varepsilon})}{|\Omega_i^{+,\varepsilon}|} = \limsup_{i \to \infty} \frac{a(\varepsilon, \Omega_i)}{|\Omega_i|} \quad \text{and} \quad \lim_{i \to \infty} \frac{a(\varepsilon, \Omega_i^{-,\varepsilon})}{|\Omega_i^{-,\varepsilon}|} = \liminf_{i \to \infty} \frac{a(\varepsilon, \Omega_i)}{|\Omega_i|}.$$

Using (c), these limits are respectively real numbers $l^+(\varepsilon)$ and $l^-(\varepsilon)$ belonging to the interval $[0,K]$. Furthermore, let

$$l^+ := \lim_{\varepsilon\to 0} l^+(\varepsilon) \quad \text{and} \quad l^- := \lim_{\varepsilon\to 0} l^-(\varepsilon).$$

Trivially, $l^+(\varepsilon) \geq l^-(\varepsilon)$, but nothing forces $l^\pm(0) = l^\pm$ (in general, equality is not expected). If we try to use the usual argument directly, a problem arises due to the c-subadditivity. Indeed, Taking a sequence which converges to $l^+(\varepsilon)$ and decomposing it using another sequence which converges to $l^-(\varepsilon)$ by subadditivity will fail. A factor of c will appear in front of the ε (see(d)), and this would force to pass from the sequence $\Omega_i^{-,\varepsilon}$ to $\Omega_i^{-,c\varepsilon}$ at each step. Diagonal arguments settle this problem. Let

$$b_i(\varepsilon) = \frac{a(\varepsilon,\Omega_i^{-,1/i})}{|\Omega_i^{-,1/i}|}.$$

This is a sequence of bounded decreasing functions defined for $\varepsilon \in [0,1]$ with value in $[0,K]$. Using (one of) Helly's theorem (*cf.* [27, §36.5 thm 5, p.372]), there exists a subsequence n_i which possesses a limit at each point. We briefly recall how this subsequence is obtained. First, a sequence $\{r_k\}_{k\geq 1}$ of dense rational number in $[0,1]$ is taken. Since the $b_i(\varepsilon)$ are bounded, let $n_i^{(j)}$ be the subsequence which converges at r_k for $1 \leq k \leq j$. The diagonal sequence $n_i = n_i^{(i)}$ converges at each r_k, and since the functions $b_i(\varepsilon)$ are decreasing, the function $l^H(\varepsilon) = \lim_{i\to\infty} b_{n_i}(\varepsilon)$ which is *a priori* only defined for the r_k is also decreasing. It remains to be checked that $l^H(\varepsilon)$ extended at all the points of $[0,1]$ by approximating by a sequence of increasing r_k is the actual limit of the subsequence n_i (see the above reference for details). Let us show that $\lim_{\varepsilon\to 0} l^H(\varepsilon) = l^-$. This follows from

$$\begin{array}{ll}
\forall\delta>0, \exists N_1(\delta) \text{ such that } N_1(\delta)<i & \Rightarrow |b_{n_i}(\tfrac{1}{n_i}) - l^H(\tfrac{1}{n_i})| < \delta; \\
\forall\delta>0, \exists N_2(\delta) \text{ such that } N_2(\delta)<i & \Rightarrow |b_{n_i}(\tfrac{1}{n_i}) - l^-(\tfrac{1}{n_i})| < \delta; \\
\forall\delta>0, \exists N_3(\delta) \text{ such that } N_3(\delta)<i & \Rightarrow |l^-(\tfrac{1}{n_i}) - l^-| < \delta. \\
l^H(\varepsilon) \text{ is decreasing in } \varepsilon & \Rightarrow \lim_{\varepsilon\to 0} l^H(\varepsilon) = \lim_{i\to\infty} l^H(\tfrac{1}{n_i})
\end{array}$$

These four assertions are respectively consequences of the definition if l^H, the choice of $\Omega_i^{-,1/i}$, the definition of l^-, and the fact that a limit that exists (thanks to monotonicity) is achieved by any sequence. We shall now show that

$$\forall\delta>0, l^+(\varepsilon) \leq \lim_{\varepsilon'\to 0} l^H(\varepsilon') + \delta = l^- + \delta.$$

The argument is in essence the same as for subadditive sequences of real numbers: lemma 2.4.6 plays the role of the decomposition $n = kn' + r$ and c-subadditivity (d) forces $\varepsilon \longrightarrow 0$.

Let $\delta \in]0, \frac{1}{2}[$. Denote by $F_i = \Omega_{n_i}^{-,1/n_i}$. It is possible to refine this sequence so that

$$a(\varepsilon, F_i)/|F_i| \leq l^H(\varepsilon) + \delta.$$

Applying lemma 2.4.6 gives an ε-quasi-tiling (which does not cover a set of proportion δ) of any sufficiently big set by translates of the F_i. Since $\{\Omega_i^{+,\varepsilon}\}$ is also a Følner sequence, for i big enough, lemma 2.4.6 applies to each element. Take $\Omega=\Omega_i^{+,\varepsilon}$, denote $\gamma_{F_j;m}F_j$ the k_j translates of F_j obtained ($m=1,\ldots,k_j$), and let i_0 be such that $|\Omega^{(i_0)}|\leq\delta|\Omega^{(1)}|$. Thanks to c-subadditivity (d), we have that

$$\begin{aligned}
a(\varepsilon,\Omega^{(1)}) &\leq \sum_{m=1}^{k_n} a(c^m\varepsilon,\gamma_{F_n;m}F_n)+a(c^{k_n}\varepsilon,\Omega^{(2)})\\
&\leq \ldots\\
&\leq \sum_{i=n-i_0}^{n}\Big(\sum_{m=1}^{k_i} a(c^{\kappa_i+m}\varepsilon,\gamma_{F_i;m}F_i)\Big)+a(c^{\kappa_{i_0}}\varepsilon,\Omega^{(i_0)}),
\end{aligned}$$

where $\kappa_i=\sum_{j=n-i}^{n}k_n$. Using Γ-invariance (a), the fact that these functions are decreasing in ε (b), and the K-sublinear property (c), this inequality yields

$$a(\varepsilon,\Omega^{(1)})\leq\sum_{i=n-i_0}^{n}\Big(\sum_{m=1}^{k_i}a(c^{\kappa_{i_0}}\varepsilon,F_i)\Big)+K|\Omega^{(i_0)}|.$$

On one hand, $|\Omega^{(i_0)}|\leq\delta|\Omega^{(1)}|$ and $\frac{a(c^{\kappa_{i_0}}\varepsilon,F_i)}{|F_i|}\leq l^H(c^{\kappa_{i_0}})+\delta$. Thence,

$$\begin{aligned}
\frac{a(\varepsilon,\Omega^{(1)})}{|\Omega^{(1)}|} &\leq \sum_{i,m}\frac{a(c^{\kappa_{i_0}}\varepsilon,F_i)}{|F_i|}\frac{|\gamma_{F_i;m}F_i|}{|\Omega^{(1)}|}+K\frac{|\Omega^{(i_0)}|}{|\Omega^{(1)}|}\\
&\leq (l^H(c^{\kappa_{i_0}})+\delta)\sum_{i,m}\frac{|\gamma_{F_i;m}F_i|}{|\Omega^{(1)}|}+K\delta
\end{aligned}$$

On the other hand, the $\{\gamma_{F_i;m}F_i\}$ are δ-disjoint. Thus $(1-\delta)\sum|\gamma_{F_i;m}F_i|\leq|\cup\gamma_{F_i;m}F_i|\leq|\Omega^{(1)}|$. This shows that

$$\frac{a(\varepsilon,\Omega_j^{+,\varepsilon})}{|\Omega_j^{+,\varepsilon}|}\leq(l^H(c^{\kappa_{i_0}})+\delta)\sum_{i,l}\frac{|\gamma_{F_i;l}F_i|}{|\Omega^{(1)}|}+K\delta\leq\frac{l^H(c^{\kappa_{i_0}})+\delta}{1-\delta}+K\delta,$$

For all $\Omega_j^{+,\varepsilon}$ big enough, where κ_{i_0} depends on $\Omega_j^{+,\varepsilon}$. Since $l^H(\varepsilon)$ is decreasing and $\lim_{\varepsilon\to 0}l^H(\varepsilon)=l^-$, taking the limit when j and $\kappa_{i_0}\longrightarrow\infty$ is not a problem:

$$l^+(\varepsilon)\leq l^-+\delta(K+l^-+1).$$

We have shown that $l^+=l^-$. To deduce the independance on the choice of sequence, notice that given two Følner sequences $\{\Omega_i\}$ and $\{\Omega_i'\}$, the sequence $\{\widetilde{\Omega}_i\}$ whose elements alternate between thos of the two former sequences will also possess a limit. The limit obtained with $\{\widetilde{\Omega}_i\}$ must be equal to the one taken via $\{\Omega_i\}$ or $\{\Omega_i'\}$. □

Before we close this technical parenthesis, remark that the K-sublinear hypothesis (c) is equivalent to another statement. Indeed, using c-subadditivity (d), Γ-invariance (a) and monotonicity in ε (b), for all Ω, $a(\varepsilon,\Omega)\leq a(c^{|\Omega|}\varepsilon,e)|\Omega|$ where $e\in\Gamma$ is the neutral element. Thus (c) $\Leftrightarrow\lim_{\varepsilon\to 0}a(\varepsilon,e)<\infty$.

This understood, the previous theorem is a generalisation of the Ornstein-Weiss lemma. Indeed, taking $a(\varepsilon,\Omega)$ to be constant functions (in ε): then monotonicity (b) always hold, being K-sublinear (c) is automatic, and c-subadditivity (d) is equivalent to usual subadditivity ($c=1$).

2.5 Properties of $\dim_{\ell^p}$

We now show some properties of $\dim_{\ell^p}$.

Corollary 2.5.1: $\dim_{\ell^p}$ *is independent of the choice of Følner sequence.*

Proof. It suffices to show that for any $Y \subset \ell^\infty(\Gamma;V)$ a Γ-invariant set, theorem 2.4.7 can be invoked, where $a(\varepsilon,\Omega) = \mathrm{wdim}_\varepsilon(B_r^Y, ev_{\ell^p(\Omega)})$. Here is why:

(a) By Γ-invariance of Y.

(b) As $\mathrm{wdim}_\varepsilon$ is decreasing in ε (*cf.* 1.2.1.b).

(c) Since $(B_r^Y, ev_{\ell^p(\Omega)})$ can be sent isometrically to $(B_r^{\ell^p(\Omega)}, ev_{\ell^p(\Omega)})$ and $\mathrm{wdim}_\varepsilon(B_r^{\ell^p(\Omega)}, ev_{\ell^p(\Omega)}) \leq |\Omega| \dim V$.

(d) We start with $p < \infty$. If $\mathrm{wdim}_\varepsilon(B_r^Y, ev_{\ell^p(\Omega_i)}) = k_i$, where $i = 1$ or 2, this means there exists $f_i : (B_r^Y, ev_{\ell^p(\Omega_i)}) \overset{\varepsilon}{\hookrightarrow} K_i$ with $\dim K_i = k_i$. The map $f_1 \times f_2$ is a $2^{1/p}\varepsilon$-embedding. Indeed for $x, x' \in (f_1 \times f_2)^{-1}(k_1,k_2)$,

$$ev_{\ell^p(\Omega_1 \cup \Omega_2)}(x,x') \leq \left(ev_{\ell^p(\Omega_1)}(x,x')^p + ev_{\ell^p(\Omega_2)}(x,x')^p \right)^{1/p} \leq 2^{1/p}\varepsilon.$$

Thence, we conclude that $a(\varepsilon,\Omega)$ is $2^{-1/p}$-subadditive. The case $p = \infty$ is dealt with in a similar fashion ($c = 1$). $\square$

Even if proposition 2.5.1 is a very important property, weaker version can be sufficient for some of our needs. The following simple lemma is sufficient to show that $\dim_{\ell^p}$ is preserved under certain maps.

Lemma 2.5.2: *Let Y be as above, and let $\{\Omega_i\}$ and $\{\Omega'_i\}$ be such that*

$$\lim_{i\to\infty} \frac{|\Omega_i \cup \Omega'_i \setminus \Omega_i \cap \Omega'_i|}{|\Omega_i \cup \Omega'_i|} = 0,$$

then $\mathfrak{meas}_{\ell^p}(B_1^Y, ev, \{\Omega_i\}) = \mathfrak{meas}_{\ell^p}(B_1^Y, ev, \{\Omega'_i\})$

Proof. It suffices to note that, when $\Omega \subset \Omega'$,

$$\frac{\mathrm{wdim}_\varepsilon(B_1^Y, ev_{\ell^p(\Omega)})}{|\Omega|}\frac{|\Omega|}{|\Omega'|} \leq \frac{\mathrm{wdim}_\varepsilon(B_1^Y, ev_{\ell^p(\Omega')})}{|\Omega'|} \leq \frac{\mathrm{wdim}_\varepsilon(B_1^Y, ev_{\ell^p(\Omega)})}{|\Omega|}\frac{|\Omega|}{|\Omega'|} + \dim V \frac{|\Omega' \setminus \Omega|}{|\Omega'|}.$$

Furthermore, $\frac{|\Omega|}{|\Omega'|} = 1 - \frac{|\Omega' \setminus \Omega|}{|\Omega'|}$. Thus, computing $\mathfrak{meas}$ with respect to the sequences $\{\Omega_i \cap \Omega'_i\}$, $\{\Omega_i\}$ or $\{\Omega'_i\}$ will yield the same result as a computation made using $\{\Omega_i \cup \Omega'_i\}$. $\square$

Proposition 2.5.3: *Let $Y \subset \ell^\infty(\Gamma;V)$ and $Y' \subset \ell^\infty(\Gamma;V')$ be Γ-invariant linear subspaces. Let $f : Y \to Y'$ be continuous (for the weak-* topology or the product topology), Γ-equivariant and such that there exists a real $c_f \in \mathbb{R}_{>0}$ and a finite subset $D_f \subset \Gamma$ satisfying $ev(x,y) \leq c_f\, ev_{D_f}(f(x), f(y))$ then*

$$\dim_{\ell^p}(Y, \{\Omega_i\}) \leq \dim_{\ell^p}(Y', \{\Omega_i\})$$

Proof. The case $p=\infty$ is simpler, we shall only describe the case $p<\infty$. Let $B_r^{Y'} = Y' \cap B_r^{\ell^p(\Gamma,V')}$. On one hand, since f is continuous (for the product topology or the weak-* topology), $\exists r_f \in \mathbb{R}_{>0}$ such that $f(B_1^Y) \subset B_{r_f}^{Y'}$. Indeed, since the image is weakly-* compact (in particular, weakly-* bounded) it is bounded (*cf.* [43, th 3.18]). On the other hand, the assumption satisfied by f on distances propagates by equivariance to different evaluations:

$$ev_\gamma(x,y) = ev(\gamma x, \gamma y) \leq c_f\, ev_{D_f}(f(\gamma x), f(\gamma y)) = c_f\, ev_{D_f}(\gamma f(x), \gamma f(y)) = c_f\, ev_{\gamma D_f}(f(x), f(y)).$$

This implies that $ev_{\ell^p(\Omega)}(x,y) \leq c_f |D_f|\, ev_{\ell^p(\Omega D_f)}(f(x), f(y))$ (and that f is injective). Lastly, since the image of the ball (of radius 1) is contained in a ball (of radius r_f)

$$\mathrm{wdim}_\varepsilon(B_1^Y, ev_{\ell^p(\Omega_i)}) \leq \mathrm{wdim}_{\varepsilon/c'_f}(B_{r_f}^{Y'}, ev_{\ell^p(\Omega_i D_f)}) \leq \mathrm{wdim}_{\varepsilon/c'_f r_f}(B^{Y'}, ev_{\ell^p(\Omega_i D_f)}),$$

where $c'_f = c_f |D_f|$. The first inequality comes from 1.2.1.d: a map on $B_{r_f}^{Y'}$ which is ε/C-injective, gives rise when composed with f to a ε-injective map defined on B_1^Y. Dividing by $|D_f \Omega_i| = \frac{|D_f \Omega_i|}{|\Omega_i|} |\Omega_i|$ and passing to the limit yields that

$$\mathfrak{meas}_{\ell^p}(B_1^Y, ev, \{\Omega_i\}) \lim_{i\to\infty} \frac{|D_f \Omega_i|}{|\Omega_i|} \leq \mathfrak{meas}_{\ell^p}(B^{Y'}, ev, \{\Omega_i D_f\}).$$

Since $\{\Omega_i\}$ is a Følner sequence, the limit on the left-hand side is 1. Furthermore, the hypothesis of lemma 2.5.2 are satisfied; the right-hand term is just $\dim_{\ell^p}(Y', \{\Omega_i\})$. □

This proof for linear subspaces (where the balls are all similar up to dilatation) can be adapted outside this case if, for example, the map f is Lipschitz. Since the assumptions of the previous proposition are quite abstract, it is good to check that they hold in certain categories of maps. The main constraint is the existence of c_f and D_f.

We recall the construction of maps of finite type. Let $D \subset \Gamma$ be a finite set and let $\underline{g} : W^D \to W'$ be a continuous function. This data enables the definition of a Γ-equivariant continuous function g_D from $Z \subset \ell^p(\Gamma; W)$ to $\ell^p(\Gamma; W')$ as follows

$$g_D(z)(\gamma) = \underline{g}(z(\gamma\delta))_{\delta \in D}.$$

Let f be a map to which proposition 2.5.3 applies. Let $g : Y' = \mathrm{Im} f \to Y$ the inverse of f on its image, then the condition

$$ev(x,y) \leq c_f\, ev_{D_f}(f(x), f(y))$$

can be read as a condition on the modulus of continuity of g. More precisely, $f^{-1} : (Y', ev_{\ell^p(D_f)}) \to (Y, ev)$ must be continuous with a linear modulus of continuity (*i.e.* that f^{-1} be Lipschitz). If the function f^{-1} is continuous for the product topology, weakening the topology on its image is evidently not restrictive. Things are not so direct on the domain. However, let $U \subset (Y, ev)$ be an open set; if Y is seen as a subset of V^Γ, U is an open set on the factor $Y\big|_e$, and all of Y on the other factors. It is then possible possible that on a finite number of factors of $Y' \subset V'^\Gamma$ (the required set

D_f) $f(U)$ will not be all the image of f. For example, for $f = f_D$ of finite type, the condition is that $f : (Y, ev) \longrightarrow (Y, ev_{D^{-1}})$ be open of Lipschitz inverse. If $\overline{f}$ is a linear map, injectivity of f implies that it is open on its image (Banach-Schauder theorem or open mapping theorem). Thus, here is a case where proposition 2.5.3 can be used.

Example 2.5.4*:* Let $F \subset \Gamma$ be finite, Let V and V' be isomorphic vector spaces, ket $\mathrm{Hom}(V,V')[\Gamma]$ the algebra of Γ for the ring $\mathrm{Hom}(V,V')$ (where a fixed isomorphism is chosen to allow composition of maps). Let $f \in \mathrm{Hom}(V,V')[\Gamma]$ be with support on F, it can be associated to a map $\ell^p(\Gamma;V) \longrightarrow \ell^p(\Gamma;V')$ by

$$x \longmapsto f(x) \text{ such that } f(x)(\gamma) = \sum_{\gamma' \in F} a_{\gamma'}\left(x(\gamma'\gamma)\right),$$

where $a_{\gamma'} \in \mathrm{Hom}(V,V')$. This is a linear map of finite type. Suppose that it is injective and possesses an inverse $f^{-1} = g$:

$$x \longmapsto g(x) \text{ such that } g(x)(\gamma) = \sum_{\gamma' \in G} b_{\gamma'}\left(x(\gamma'\gamma)\right),$$

where $b_{\gamma'} \in \mathrm{Hom}(V',V)$. $G \subset \Gamma$ might not be finite. Proposition 2.5.3 can also be used with $D_f = F^{-1}$. As for c_f, it is the Lipschitz constant of $g : (Y', ev_{F^{-1}}) \longrightarrow (Y, ev)$. Thus $c_f \leq \| \oplus_{\gamma \in F^{-1} \cap G} b_\gamma \|$. Consequently, when $f : Y \longrightarrow Y'$ is a Γ-equivariant linear injective map of finite type,

$$\dim_{\ell^p}(Y, \{\Omega_i\}) \leq \dim_{\ell^p}(Y', \{\Omega_i\})$$

We now discuss property P5, that is if Y is non-trivial then $\dim_{\ell^p} Y$ is positive. This question is difficult as an intuitive proof only works for $p = 1$. Before we move to this proof, let us argue that three assumptions seem necessary for it to hold: Y must be a linear subspace, Y must be Γ-invariant, and Y must be contained in $\ell^p(\Gamma;V)$ for finite p or in $c_0(\Gamma;V)$ if $p = \infty$. Here are some cases of non-trivial Y for which one of the assumptions does not hold and where $\dim_{\ell^p}$ is 0.

First, suppose Y is not a linear subspace. Then example 2.3.3 shows the case of ℓ^q balls where $q < p$. Alternatively, if Y is the subset of $\ell^\infty(\Gamma;V)$ given by function with support of cardinality less than k (for a fixed $k \in \mathbb{Z}_{>0}$).

Second, if Y is a linear subspace of $\ell^\infty(\Gamma;V)$ but is not Γ-invariant, it could be of finite dimension, and consequently $\dim_{\ell^p}$ will be trivial.

Last, when p is finite, the existence of a $y \in Y$ whose ℓ^p norm is finite is only garanteed if $Y \subset \ell^p$. Without this assumption, it could happen that $Y \cap B_r^{\ell^p(\Gamma;V)} = \{0\}, \forall r$. On the other hand, if $p = \infty$, take $Y \subset \ell^\infty(\Gamma;V)$ the (Γ-invariant) line generated by a constant function y (*i.e.* such that $\exists v \in V, \forall \gamma \in \Gamma, y(\gamma) = v$). Y is 1-dimensional, and consequently $\dim_{\ell^\infty} Y = 0$. But Y is not trivial. However, the question for a Γ-invariant linear subspace $Y \subset c_0(\Gamma;V)$ remains interesting.

Fortunately, in the ℓ^1 case things can be proved without difficulties. As noted before this method does not extend to $p > 1$.

Proposition 2.5.5: *Let $Y \subset \ell^1(\Gamma;V)$ be a Γ-invariant linear subspace, then* $\dim_{\ell^1}(Y, \{\Omega_i\}) = 0$ *if and only if Y is trivial.*

Proof. If Y is trivial then $\dim_{\ell^1} Y$ is obviously 0. Otherwise, let $0 \neq y \in Y$ and renormalize it so that $\|y\|_{\ell^1(\Gamma)} = 1$. For all $\varepsilon \in]0, 1/2[$, $\exists F \subset \Gamma$ finite (which depends on y and ε) such that $\|y\|_{\ell^1(F)} > 1 - \varepsilon$ (and consequently $\|y\|_{\ell^1(\Gamma \setminus F)} \leq \varepsilon$). Then let $\tilde{y}$ be identical to y on F and 0 elsewhere.

for i sufficiently big, Ω_i contains a non-empty ρ-quasi-tiling by F, since $F \subset \Omega_i$ and $\alpha(\Omega_i; F)$ tends to 0. Applying lemma 2.4.3 to find translates of F which are ρ-disjoint, where $\rho = 1/2|F|$, we obtain a quasi-tiling whose elements are actually disjoint since $\rho < |F|^{-1}$, and the number of such translates is at least $(1 - \alpha(\Omega_i; F))|\Omega_i|/2|F|$.

Let γ_j for $j \in J_i \subset \mathbb{Z}_{>0}$ be the elements by which the sets F are translated for a ρ-quasi-tiling of Ω_i (since the Ω_i form an increasing sequence and that lemma 2.4.3 applies to all maximal ρ-quasi-tiling, it can be assumed that the J_i are increasing). Let $V_i = \langle \gamma_j y | j \in J_i \rangle$ be the linear subspace generated by the corresponding translates of y. Trivially $B_1^{V_i} \subset B_1^Y$, and we will construct a map from a ball to $B_1^{V_i}$. Let

$$\begin{array}{rccc} \pi: & \ell^1(J_i;\mathbb{R}) & \to & V_i \\ & (a_j)_{j\in J_i} & \mapsto & \sum\limits_{j\in J_i} a_j\gamma_j y \end{array} \quad \text{and} \quad \begin{array}{rccc} \tilde{\pi}: & \ell^1(J_i;\mathbb{R}) & \to & V_i \\ & (a_j)_{j\in J_i} & \mapsto & \sum\limits_{j\in J_i} a_j\gamma_j \tilde{y} \end{array}$$

With these notations,

$$\begin{aligned} \|\tilde{\pi}(a)\|_{\ell^1(\Gamma)} &= \sum_{k\in J_i} \Big\| \sum_{j\in J_i} a_j\gamma_j\tilde{y} \Big\|_{\ell^1(\gamma_k F)} &&= \sum_{k\in J_i} \big\| a_k\gamma_k\tilde{y} \big\|_{\ell^1(\gamma_k F)} \\ &= \sum_{k\in J_i} |a_k|\, \|\tilde{y}\|_{\ell^1(F)} &&= \|y\|_{\ell^1(F)} \sum_{k\in J_i} |a_k|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\tilde{\pi}(a) - \pi(a)\|_{\ell^1(\Gamma)} &= \Big\| \sum_{j\in J_i} a_j\gamma_j(\tilde{y} - y) \Big\|_{\ell^1(\Gamma)} &&= \sum_{\gamma\in\Gamma} \Big| \sum_{j\in J_i} a_j\gamma_j(\tilde{y} - y) \Big| \\ &\leq \sum_{\gamma\in\Gamma}\sum_{j\in J_i} |a_j\gamma_j(\tilde{y}(\gamma) - y(\gamma))| &&= \sum_{\gamma\in\Gamma}\sum_{j\in J_i} |a_j||\tilde{y}(\gamma) - y(\gamma)| \\ &= \sum_{j\in J_i} |a_j| \Big(\sum_{\gamma\in\Gamma} |\gamma_j(\tilde{y}(\gamma) - y(\gamma))| \Big) &&= \|y\|_{\ell^1(\Gamma\setminus F)} \sum_{j\in J_i} |a_j|. \end{aligned}$$

The last two computations mean that $\|\tilde{\pi}(a)\|_{\ell^1(\Gamma)} = \|y\|_{\ell^1(F)} \|a\|_{\ell^1(J_i)}$ and $\|\tilde{\pi}(a) - \pi(a)\|_{\ell^1(\Gamma)} \leq \|y\|_{\ell^1(\Gamma\setminus F)} \|a\|_{\ell^1(J_i)}$. Thus

$$\big(\|y\|_{\ell^1(F)} - \|y\|_{\ell^1(\Gamma\setminus F)} \big) \|a\|_{\ell^1(J_i)} \leq \|\pi(a)\|_{\ell^1(\Gamma;V)} \leq \big(\|y\|_{\ell^1(F)} + \|y\|_{\ell^1(\Gamma\setminus F)} \big) \|a\|_{\ell^1(J_i)}$$

This means that $(B_1^Y, ev_{\ell^1(\Omega_i)})$ contains, with a controlled distortion, a ℓ^1 ball (with its ℓ^1 metric) of radius 1 and of dimension $\frac{1}{2|F|}(1 - \alpha(\Omega_i; F))|\Omega_i|$, whence

$$\dim_{\ell^1}(Y, \{\Omega_i\}) = \lim_{\varepsilon'\to 0} \limsup_{i\to\infty} \frac{\mathrm{wdim}_{\varepsilon'}(B_1^Y, ev_{\ell^1(\Omega_i)})}{|\Omega_i|} \geq \lim_{\varepsilon'\to 0} \limsup_{i\to\infty} \frac{1}{2|F|}(1 - \alpha(\Omega_i; F)) = \frac{1}{2|F|}.$$

As required $\dim_{\ell^1}(Y, \{\Omega_i\}) > 0$. $\square$

In a few special cases, this result can be extended to $p > 1$. The first is when Y possesses an element with finite support. The second when Y contains an element in ℓ^1.

We now mention a typical problem when one deals with ℓ^p spaces, for $p \neq 2$, that is the existence of linear subspaces which are not the image of projection (*cf.* [36] and [46]). A characterisation of subspaces of ℓ^p possessing a projection of norm 1 can be found in [32, I.§2]. We shall discuss the case where $Y \subset \ell^p(\Gamma;\mathbb{R})$ is a Γ-invariant linear subspace on which there exists a projection, P_Y. Furthermore, suppose that this projection is Γ-equivariant. Then let $y = P_Y\delta_e$ where δ_e is the Dirac mass at $e \in \Gamma$, and let $q \leq p$ be such that $y \in \ell^q(\Gamma;\mathbb{R})$. For a $x \in \ell^p(\Gamma;\mathbb{R})$, write $x = \sum k_\gamma \delta_\gamma$. By linearity of P_Y,

$$P_Y x = P_Y \sum_{\gamma\in\Gamma} k_\gamma \delta_\gamma = \sum_{\gamma\in\Gamma} k_\gamma \gamma y$$

Thus $(P_Y x)(e) = \sum_{\gamma\in\Gamma} k_\gamma y(\gamma)$. Taking $k_\gamma = |y(\gamma)|^{\frac{q}{p}-1} y(\gamma)$, it appears that $(P_Y x)(e) = \sum |y(\gamma)|^{\frac{q}{p}+1}$. This forces $\frac{q}{p}+1 \leq q$, in other words $q \leq p'$ (where p' is the conjugate exponent to p). When $p > 2$, the existence of such a projection means that there exists in Y an element of $\ell^{p'}(\Gamma;\mathbb{R})$, which is quite restrictive.

Remark 2.5.6: If $Y_1, Y_2 \subset \ell^\infty(\Gamma;V)$ be two linear subspaces, it is possible to construct the subspace given by their direct sum $Y_1 \oplus Y_2 \subset \ell^p(\Gamma; V \oplus V)$. The inequality

$$\dim_{\ell^p}(Y_1 \oplus Y_2, \{\Omega_i\}) \leq \dim_{\ell^p}(Y_1, \{\Omega_i\}) + \dim_{\ell^p}(Y_2, \{\Omega_i\})$$

is easily obtained. On the other hand, the reverse inequality is not so clear. It would require to know if wdim is super-additive for balls in normed vector spaces. In particlular, to know what $\mathrm{wdim}_\varepsilon B_1^E = k$ for a linear subspace $E \subset \ell^p(\Gamma;V)$ could be useful (where E is the image by restriction of Y to function with support on Ω). Trivially, $\dim E \geq k$. If the sequence

$$r_{k;p} = \inf_{E_k \subset \ell^p, \dim E_k = k} \sup\{r \in \mathbb{R}_{>0} | \exists i : B_r^{\ell^p(k)} \hookrightarrow B_1^{E_k} \text{ increase the distances}\}$$

is bounded from below, then the reverse inequality can be obtained. A more general question would be to know if $B^E \times B^{E'}$ contains a "thick" subset when B^E and $B^{E'}$ contain such subsets.

Proposition 2.5.7: *Let $Y \subset \ell^\infty(\Gamma;V)$ be an open linear subspace and let $\overline{Y}$ be its completion in ℓ^p, then* $\dim_{\ell^p}(Y, \{\Omega_i\}) = \dim_{\ell^p}(\overline{Y}, \{\Omega_i\})$.

Proof. The argument is identical to that of example 2.3.3: when restricted to a finite $\Omega \subset \Gamma$, these two spaces cannot be distinguished (being of finite dimension they are close). In other words, there exists a continuous map, given by the restriction R_Ω, and whose fibers have "diameter" equal to 0:

$$R_\Omega : (B_1^{\overline{Y}}, ev_{\ell^p(\Omega)}) \longrightarrow (R_\Omega B_1^Y, ev_{\ell^p(\Omega)}).$$

Thus, $\forall \varepsilon \in [0,1]$, $\mathrm{wdim}_\varepsilon(B_1^{\overline{Y}}, ev_{\ell^p(\Omega)}) \leq \mathrm{wdim}_\varepsilon(R_\Omega B_1^Y, ev_{\ell^p(\Omega)})$. On the other hand, let $s : R_\Omega B_1^Y \longrightarrow B_1^Y$ such that $R_\Omega \circ s = \mathrm{Id}$ be determined by an inverse of $R_\Omega Y \longrightarrow Y$, then s is a continuous map which increases distances. Consequently, $\mathrm{wdim}_\varepsilon(R_\Omega B_1^Y, ev_{\ell^p(\Omega)}) \leq \mathrm{wdim}_\varepsilon(B_1^Y, ev_{\ell^p(\Omega)})$. Finally, by inclusion $Y \subset \overline{Y}$, we have $\mathrm{wdim}_\varepsilon(B_1^Y, ev_{\ell^p(\Omega)}) \leq \mathrm{wdim}_\varepsilon(B_1^{\overline{Y}}, ev_{\ell^p(\Omega)})$. □

Even if we cannot show continuity, the following example is worthy of interest. The sequence of vector subspaces discussed there will not satisfy the continuity property.

Example 2.5.8: Let us consider the space of absolutely convergent sequences, $\ell^1(\mathbb{Z};\mathbb{R})$. Define $\forall k \in \mathbb{Z}_{>0}, \pi_k : \ell^1(\mathbb{Z};\mathbb{R}) \longrightarrow \ell^\infty(\mathbb{Z}/k\mathbb{Z};\mathbb{R})$ in the following way: for $n \in \mathbb{Z}/k\mathbb{Z}$

$$\pi_k(x)(n) = \sum_{i \equiv n \bmod k} x(i).$$

Continuous linear maps between Banach spaces have a closed kernel (for τ, the norm topology in ℓ^1), thus $Y_j = \bigcap_{k=1}^{j} \operatorname{Ker} \pi_k$ is a decreasing sequence of closed sets (for τ). To compute $\dim_{\ell^1}$, choose the Følner sequence $\Omega_i = [-i, i] \cap \mathbb{Z}$. For a $N \in \mathbb{N}$, let $y_N \in Y_1$ be such that $y_N(0) = 1/2$, $y_N(N) = -1/2$ and which is zero elsewhere. Let $N_j = \mathrm{ppcm}(1, 2, \ldots, j)$. For all j, $y_{N_j} \in B_1^{Y_j}$. These elements give a map $(B_{1/2}^{\ell^1(\mathbb{Z};\mathbb{R})}, ev_{\ell^1(\Omega)})$ to $(B_1^{Y_j}, ev_{\ell^1(\Omega)})$ which possesses fibers of "diameter" 0. They are defined as follows, $y \in B_{1/2}^{\ell^1(\mathbb{Z};\mathbb{R})}$ is restricted to Ω then extended by 0 outside Ω. Then, let $k \in \mathbb{Z}_{>0}$ be such that kN_j is bigger than the diameter of $\Omega \subset \mathbb{Z}$, then $\tilde{y}(m) = \sum_{n \in \Omega} 2 y_{kN_j}(m-n) y(n)$ is an element of $B_1^{Y_j}$. Thence $\dim_{\ell^1} Y_j \geq 1$, and as the other inequality is automatic, $\dim_{\ell^1} Y_j = 1$.

We claim that $Y_\infty = \cap Y_j = \{0\}$. If this were false, then a non-trivial element $y \in Y_\infty$ would have the property that

$$\forall i \in \mathbb{Z}, \forall n \in \mathbb{Z}, -y(i) = \sum_{0 \neq k \in \mathbb{Z}} y(i + kn).$$

To get a contradiction, take the limit when $n \longrightarrow \infty$ and show that it is equal to 0. First we normalize y so that it is of norm 1 and suppose that $|y(i)| > \delta$ for some i. As an absolutely convergent sequence, y should be concentrated on some set: there exists n_δ such that $\|y\|_{\ell^1(\Omega_{n_\delta})} \geq 1 - \delta/2$. However when $n > 2n_\delta + 1$

$$|y(i)| = \left| \sum_{0 \neq k \in \mathbb{Z}} y(i + kn) \right| \leq \delta/2,$$

which is a contradiction with the fact that $|y(i)| > \delta$. Thus $\dim_{\ell^1} Y_\infty = 0 \neq 1 = \lim_{n \longrightarrow \infty} \dim_{\ell^1} Y_n$.

Such an eventuality is fortunately confined to ℓ^1; more generally the above construction can be described as follows. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index and $G = \Gamma/\Gamma'$ a set of representatives. Let $\pi : Y \longrightarrow W$ be a Γ'-invariant linear map from $Y \subset \ell^p(\Gamma; V)$ to a finite dimensional vector space W. Then the existence of such a map implies the existence of $\dim W$ elements of $\ell^{p'}(\Gamma; V^*)$ which are invariant by Γ'. This is impossible if $p' \neq \infty$, as such elements would not be decreasing at infinity. For such spaces to exists, p' must be ∞.

When $\Gamma_1 \subset \Gamma_2$ are subgroups, it is possible to obtain from a set $Y_1 \subset \ell^p(\Gamma_1; V)$ an induced set $Y_2 \subset \ell^p(\Gamma_2; V)$. First let $G = \Gamma_2/\Gamma_1$ be a set of representatives then to $y \in \ell^p(\Gamma_2; V)$ one can associate $y_g = (g \cdot y)|_{\Gamma_1}$ which is an element of $\ell^p(\Gamma_1; V)$. With these notations,

$$Y_2 = \{ y \in \ell^p(\Gamma_2; V) | \forall g \in G, y_g \in Y_1 \}.$$

Remark 2.5.9: To show reciprocity (P9), the same problem as for additivity arises. Even if $|G| < \infty$. Indeed, let $\{\Omega_i^{(1)}\}$ be a Følner sequence for Γ_1 and $\{\Omega_i^{(2)}\} = \{\Omega_i^{(1)}G\}$. Then $(B_1^{Y_2}, ev_{\Omega_i^{(2)}})$ is isometric to $(B_1^{Y_1^G}, ev_{\Omega_i^{(1)}})$. Passing from $B_1^{Y_1^G}$ to $(B_1^{Y_1})^G$ can be done as in 2.5.6. However, it is unclear that there exists $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\lim_{\varepsilon\to 0} \phi(\varepsilon) = 0$ and $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ (which could depend on $|G|$) such that

$$\mathrm{wdim}_\varepsilon(B_1^{Y_1}, ev_{\Omega_i^{(1)}}) = k \Rightarrow \mathrm{wdim}_{\phi(\varepsilon)}\left((B_1^{Y_1})^G, ev_{\Omega_i^{(1)}}\right) \geq |G|k - \delta(\varepsilon)$$

If $[\Gamma_2 : \Gamma_1] = |G| < \infty$, a set $Y \subset \ell^p(\Gamma_2; V)$ is also a set of $\ell^p(\Gamma_1; V^G)$. Indeed, to $y \in \ell^p(\Gamma_2; V)$ one can associate $i(y)$ where $i(y)(\gamma) = (y(\gamma g))_{g\in G} \in V^G$. This operation behaves nicely with $\dim_{\ell^p}$.

Proposition 2.5.10: *Let $\Gamma_1 \subset \Gamma_2$ be amenable groups and $G = \Gamma_2/\Gamma_1$ where $|G| < \infty$, if $Y \subset \ell^p(\Gamma_2; V)$ is seen by restriction as a linear subspace of $\ell^p(\Gamma_1; V^G)$ then $|G|\dim_{\ell^p}(Y, \Gamma_2) = \dim_{\ell^p}(Y, \Gamma_1)$.*

Proof. Let $\{\Omega_i^{(1)}\}$ be a Følner sequence for Γ_1 and let $\{\Omega_i^{(2)}\} = \{\Omega_i^{(1)}G\}$ be the corresponding Følner sequence in Γ_2. It is then sufficient to see that $(B_1^{Y_2}, ev_{\Omega_i^{(2)}})$ is by construction isometric to $(B_1^{Y_1}, ev_{\Omega_i^{(1)}})$. □

Finally, for Hausdorff limits, it is possible to invert the order of some limits.

Lemma 2.5.11: *Let $1 < p < \infty$. Let $\{Y_n\}_{n\geq 1}$ be a decreasing sequence of closed convex sets of $\ell^\infty(\Gamma; V)$, let $Y_\infty = \cap Y_n$. If the distance from Y_n to Y_∞ tends to 0, i.e. $\Delta_n = \sup_{y\in B_1^{Y_n}} \inf_{y'\in B_1^{Y_\infty}} \|y - y'\|_{\ell^p(\Gamma)} \to 0$, then*

$$\dim_{\ell^p}(Y_\infty, \{\Omega_i\}) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\limsup_{i\to\infty} \mathrm{wdim}_\varepsilon(B_1^{Y_n}, \{\Omega_i\})/|\Omega_i|.$$

Proof. The inequality, $\dim_{\ell^p}(Y_\infty, \{\Omega_i\}) \leq \dim_{\ell^p}(Y_n, \{\Omega_i\})$ follows from a simple inclusion. Next using the definition of Δ_n, by uniform convexity of ℓ^p and closure and convexity of the Y_n, there exists a map $\pi_n : Y_n \to Y_\infty$ such that each $y_n \in Y_n$ can be written as $y_n = \pi_n(y_n) + y_n'$ where $y_n' \in Y_n \setminus Y_\infty$ (see [3, §2.2]).

If $\mathrm{wdim}_\varepsilon(Y_\infty, ev_{\ell^p(\Omega)}) = k$, then there exists a polyhedron K of dimension k and a map $f : (B_1^{Y_\infty}, ev_{\ell^p(\Omega)}) \xrightarrow{\varepsilon} K$. The map $f \circ \pi_n$ will be our candidate to bound the wdim of the Y_n. Indeed, let $x, y \in (f\circ\pi_n)^{-1}(k)$, write $x = \pi_n(x) + x'$ and $y = \pi_n(y) + y'$. Then

$$ev_{\ell^p(\Omega)}(x,y) = \|x-y\|_{\ell^p(\Omega)} \leq \|\pi_n(x) - \pi_n(y)\|_{\ell^p(\Omega)} + \|x'\|_{\ell^p(\Omega)} + \|y'\|_{\ell^p(\Omega)} \leq \varepsilon + 2\Delta_n.$$

Whence $\mathrm{wdim}_\varepsilon(Y_\infty, ev_{\ell^p(\Omega)}) \geq \mathrm{wdim}_{\varepsilon+2\Delta_n}(Y_n, ev_{\ell^p(\Omega)})$. Let $l_n(\varepsilon) = \limsup_{i\to\infty} \mathrm{wdim}_\varepsilon(Y_n, ev_{\ell^p(\Omega)})$ (and l_∞ the corresponding limit for Y_∞). The inequality on the wdim translates as $l_\infty(\varepsilon) \geq l_n(\varepsilon + 2\Delta_n)$. Thus if we denote the limit of increasing sequences by ${}^{-0}$, $\forall \varepsilon > 0, l_\infty(\varepsilon^{-0}) = \lim_{n\to\infty} l_n(\varepsilon)$, and let $\varepsilon \to 0$, $\dim_{\ell^p} Y_\infty \geq \lim_{\varepsilon\to 0}\lim_{n\to\infty} l_n(\varepsilon)$. □

2.6 Further variants

This section will describe some other variants of wdim that can be introduced and finish by completing the analogy between entropy and mean dimension that begun in the introduction. Consider Γ a topological group that can be continuous; let μ be a left Haar measure and let $p \in [1,\infty[$, define

$$d_{\Omega,p}(x,y) = \left(\frac{\int_\Omega d_\gamma(x,x')^p d\mu(\gamma)}{\int_\Omega d\mu(\gamma)} \right)^{1/p},$$

and $d_{\Omega,\infty}(x,y) = d_\Omega(x,y) = \|d_\gamma(x,y)\|_{L^\infty(\Omega)}$. Inequality (2.1.1) still holds, with the supplementary remark that if $p < \infty$, the intersection of sets must be empty. Let $\Omega_1, \Omega_2 \subset \Omega$ be such that $\Omega_1 \cap \Omega_2 = \varnothing$ and $\Omega_1 \cup \Omega_2 = \Omega$,

$$\mathrm{wdim}_\varepsilon(X, d_{\Omega,p}) \leq \mathrm{wdim}_\varepsilon(X, d_{\Omega_1,p}) + \mathrm{wdim}_\varepsilon(X, d_{\Omega_2,p}). \tag{2.6.1}$$

Indeed, if $f_i : (X, d_{\Omega_i,p}) \overset{\varepsilon}{\hookrightarrow} K_i$ are the functions achieving the wdim of the right-hand side, then $f := f_1 \times f_2$ is a map from X to $K_1 \times K_2$. Furthermore, denote $F := f^{-1}(k_1,k_2) = f_1^{-1}(k_1) \cap f_2^{-1}(k_2)$ and suppose that $x, y \in F$ give the diameter of F, then, for $p \neq \infty$,

$$\begin{aligned} d_{\Omega,p}(x,y) &= \left(\frac{\mu(\Omega_1)}{\mu(\Omega)} d_{\Omega_1,p}(x,y)^p + \frac{\mu(\Omega_2)}{\mu(\Omega)} d_{\Omega_2,p}(x,y)^p \right)^{1/p} \\ &\leq \left(\frac{\mu(\Omega_1)}{\mu(\Omega)} + \frac{\mu(\Omega_2)}{\mu(\Omega)} \right)^{1/p} \varepsilon \\ &\leq \varepsilon, \end{aligned}$$

since $x, y \in f_i^{-1}(k_i)$ bounds their distance by the diameter of these fibers, ε. The inequality is direct when $p = \infty$ and does not require the fact that $\Omega_1 \cap \Omega_2 = \varnothing$.

This understood, it is now possible to extend the definition of mean dimension, without requiring that X be compact.

Definition 2.6.2: Let $\Omega_i \subset \Gamma$ be a sequence of compact subsets such that $\mu(\Omega_i) \to \infty$ and let $p \in [1,\infty]$,

$$\mathrm{wdim}_{L^p}(X : \{\Omega_i\}) = \lim_{\varepsilon \to 0} \limsup_{i \to \infty} \frac{\mathrm{wdim}_\varepsilon(X, d_{\Omega_i,p})}{|\Omega_i|} \in [0,\infty].$$

To avoid questions of convergence, the limits are replaced by lim sup; the independence on the choice of sequence Ω_i is false in general. Under the stronger hypothesis of section 2.1 and for $p = \infty$, this definition is identical to the previous one.

Proposition 2.6.3: *Let (X,d) be a metric space, let Γ be an amenable group acting on X, and let $p, q \in [1,\infty]$.*

(a) $p \leq q \Rightarrow \mathrm{wdim}_{L^p}(X : \{\Omega_i\}) \leq \mathrm{wdim}_{L^q}(X : \{\Omega_i\})$.

(b) *If* $\mathrm{Diam} X < \infty$ *then* $\forall p < \infty, \mathrm{wdim}_{L^p}(X : \{\Omega_i\}) \leq \mathrm{wdim}_{L^1}(X : \{\Omega_i\})$.

In other words, $\forall p < \infty, \mathrm{wdim}_{L^1} X = \mathrm{wdim}_{L^p} X \leq \mathrm{wdim}_{L^\infty} X$ *when* $\mathrm{Diam} X < \infty$.

Proof. First, the function $x \mapsto x^\alpha$ is convex if $\alpha \geq 1$. When $q < \infty$, Jensen's inequality for the function $\phi(x) = x^{q/p}$ shows that

$$\begin{aligned} d_{\Omega,p}(x,y) &= \left(\int_\Omega d_\gamma(x,y)^p d\mu(\gamma) \Big/ \int_\Omega d\mu(\gamma) \right)^{1/p} \\ &\leq \left(\int_\Omega d_\gamma(x,y)^q d\mu(\gamma) \Big/ \int_\Omega d\mu(\gamma) \right)^{1/q} = d_{\Omega,q}(x,y). \end{aligned}$$

Again, for $q = \infty$, this inequality is direct. Consequently, $\mathrm{wdim}_\varepsilon(X, d_{\Omega,p}) \leq \mathrm{wdim}_\varepsilon(X, d_{\Omega,q})$. Passing to the limit yields the inequality of (a).

Secind, remark that the existence of the function

$$\begin{aligned} \omega_p : \quad [0, \mathrm{Diam} X] &\rightarrow [0, \mathrm{Diam} X] \\ t &\mapsto \mathrm{Diam} X^{1/p'} t^{1/p} \end{aligned}$$

which, in addition to being increasing and continuous, is greater than the identity and its p^{th} power is concave (actually, linear). By another application of Jensen's inequality,

$$\begin{aligned} d_{\Omega,p}(x,y) &= \left(\int_\Omega d_\gamma(x,y)^p d\mu(\gamma) \Big/ \int_\Omega d\mu(\gamma) \right)^{1/p} \\ &\leq \left(\int_\Omega \omega_p(d_\gamma(x,y))^p d\mu(\gamma) \Big/ \int_\Omega d\mu(\gamma) \right)^{1/p} \\ &\leq \omega_p \left(\int_\Omega d_\gamma(x,y) d\mu(\gamma) \Big/ \int_\Omega d\mu(\gamma) \right) = \omega_p(d_{\Omega,1}(x,y)). \end{aligned}$$

As a consequence $\mathrm{wdim}_{\omega_p(\varepsilon)}(X, d_{\Omega,p}) \leq \mathrm{wdim}_\varepsilon(X, d_{\Omega,1})$ (thanks to lemma 1.4.4), and passing to the limit $\mathrm{wdim}_{L^p}(X : \{\Omega_i\}) \leq \mathrm{wdim}_{L^1}(X : \{\Omega_i\})$. $\square$

Thus, the presence of L^p norm in the definition of mean dimension will not help to distinguish significantly more spaces when X is compact. Remark that the proof of proposition 2.1.5 remains valid in this context: when X is compact, wdim_{L^p} is a topological invariant. The proof remains the same for $p = \infty$, and when $p = 1$ it suffices to introduce a concave function ψ which is at each point bigger than ω_{Id} so that the inequality on d_γ yields an inequality on d_Ω.

When X is not compact, it seems that the assumption on the diameter of X can be avoided by taking an equivalent metric, *e.g.* $d'(x,x') = \mathrm{arctg} d(x,x')$ or $d'(x,y) = d(x,y)/(1 + d(x,y))$. However topological invariance does not hold for non-compact X, thus the quantity obtained from this metric might not be of interest.

If the normalization by $\mu(\Omega)^{-1/p} = |\Omega|^{-1/p} = \left(\int_\Omega d\mu(\gamma) \right)^{-1/p}$ is not made, some worrying phenomenon can happen: an isometric action of the group could give a positive result, the behaviour for p large would not be close to that of $p = \infty$, and (2.6.1) would not be homogenous in ε, more precisely the left-hand side would contain factors of $2^{1/p}\varepsilon$ instead of ε, which turns subadditivity into $2^{-1/p}$-subadditivity.

The absence of such a normalization would have the advantage to make the difference between balls of $L^p(\Gamma,\mathbb{R}^s)$ for various values of s, a problem which has been discussed in section 2.3 where these objects $d_{\ell^p(\Omega)} = \mu(\Omega)^{1/p} d_{\Omega,p}$ are studied for pseudo-metrics (and discrete groups).

When attempting to prove some invariance for $\mathfrak{meas}_{\ell^p}$ an interesting obstruction arises (which motivates some of the previous choices). Indeed, when (X,d) is compact and d' is a metric equivalent to d, the modulus of continuity of the identity map gives an inequality $d'_\gamma(x,y) \leq \omega_{\mathrm{Id}}(d_\gamma(x,y))$. To go on a function ψ such that $\omega_{\mathrm{Id}} \leq \psi$, $\lim\limits_{\varepsilon\to 0}\psi(\varepsilon) = 0$ and $\overline{\psi}(x) = \psi(x^{1/p})^p$ is convex has to be introduced. These conditions are satisfied by taking, for example, the convex function bounding $\omega_{\mathrm{Id}}(x^p)^{1/p}$, denote it $\overline{\psi}$. The

$$
\begin{aligned}
d'_{\ell^p(\Omega)}(x,y) &\leq \left(\int_\Omega \psi(d_\gamma(x,y))^p\right)^{1/p} \mathrm{d}\gamma \\
&= \mu(\Omega)^{1/p}\left(\mu(\Omega)^{-1}\int_\Omega \overline{\psi}(d_\gamma(x,y)^p)\mathrm{d}\gamma\right)^{1/p} \\
&\leq \mu(\Omega)^{1/p}\overline{\psi}\left(\mu(\Omega)^{-1}\int_\Omega d_\gamma(x,y)^p \mathrm{d}\gamma\right) \\
&= \mu(\Omega)^{1/p}\psi(d_{\ell^p(\Omega)}(x,y)\mu(\Omega)^{-1/p}).
\end{aligned}
$$

If by chance ψ is linear (for example when ω_{Id} is linear) the terms in $\mu(\Omega)$ will cancel out, and it is then possible to conclude, by taking the limit, that for a pair of Lipschitz equivalent metric $\mathfrak{meas}_{\ell^p}(X,d) = \mathfrak{meas}_{\ell^p}(X,d')$ (*cf.* proposition 2.5.3). However, in general this only yields that

$$\mathrm{wdim}_{\mu(\Omega)^{1/p}\psi\left(\varepsilon\mu(\Omega)^{-1/p}\right)}(X, d'_{\ell^p(\Omega)}) \leq \mathrm{wdim}_\varepsilon(X, d_{\ell^p(\Omega)}),$$

where the presence of $\mu(\Omega)$ in front of ε is more conveniently dealt with as a profile $\mathfrak{f}$.

Definition 2.6.2 for $p=\infty$ will be of use later on when we will try to speak of mean dimension for a space of maps with action by the group of automorphisms of the domain. In order to show positivity of this mean dimension, we shall show that the dynamic contains a set of the form $V^{\mathbb{Z}}$ with the action of the shift. In general, discrete groups are easier to deal with. The next lemma is well-known but references for it are hard to find.

Lemma 2.6.4: *Let Γ be an amenable Lie group endowed with the (left) Haar measure μ and (X,d) a metric space whose metric is not invariant under action of Γ. Let Λ be a lattice in Γ, $G=\Gamma/\Lambda$ a set of representants and $\mu(G)<\infty$. Let $\{L_i\}$ be a Følner sequence for Λ and let $\{\Omega_i\} = \{L_iG\}$. Suppose further that there exists a constant c such that*

$$\frac{1}{c}d_{L_i}(x_1,x_2) \leq d_{\Omega_i}(x_1,x_2) \leq c d_{L_i}(x_1,x_2).$$

Then

$$\mathfrak{wdim}(X,\Lambda) = \mu(G)\mathfrak{wdim}(X,\Gamma).$$

Proof. This is a straightforward computation. From proposition 1.2.1.d and the inequality on distances above we get that

$$\mathrm{wdim}_{c\varepsilon}(X,d_{L_i}) \leq \mathrm{wdim}_\varepsilon(X,d_{\Omega_i}) \leq \mathrm{wdim}_{\varepsilon/c}(X,d_{L_i}).$$

Dividing by $\mu(\Omega_i)$, we get

$$\frac{|L_i|}{\mu(\Omega_i)}\frac{\mathrm{wdim}_{c\varepsilon}(X,d_{L_i})}{|L_i|} \leq \frac{\mathrm{wdim}_{\varepsilon}(X,d_{\Omega_i})}{\mu(\Omega_i} \leq \frac{|L_i|}{\mu(\Omega_i)}\frac{\mathrm{wdim}_{\varepsilon/c}(X,d_{L_i})}{|L_i|}.$$

There only remains to evaluate $\lim\limits_{i\to\infty}\frac{|L_i|}{\mu(\Omega_i)}$. To do so, use

$$\mu(\Omega_i) = \mu(L_iG) = |L_i|\mu(G).$$

Thus $\lim\limits_{i\to\infty}\frac{|L_i|}{\mu(\Omega_i)} = \mu(G)^{-1}$. Taking the limit as $i\to\infty$ then as $\varepsilon\to 0$ yields the result. □

Much like proposition 1.2.1.d can be extended in lemma 1.4.4, the above lemma can be extended to the case where the distances are bounded above and below by functions rather than constants $\frac{1}{c}$ and c.

We end this section by developping the idea of the introduction (that is mean dimension as an extension of entropy). First, there exists another definition of entropy which starts by considering separated sets. Instead of counting the smallest number of balls necessary to cover a space, it is also possible to look at $M(\varepsilon,n,B)$ the greatest number of points in $B\subset A^{\mathbb{Z}}$ such that any pair of points is at distance (for d_n) greater than ε.

One could by analogy, define the quantity $\mathrm{Fildim}_{\varepsilon}X = \sup\{\dim K | K\subset X, \mathrm{FilRad}K > \varepsilon\}$. Lemma 1.2.3 states that

$$\mathrm{Fildim}_{\varepsilon/2}X \leq \mathrm{wdim}_{\varepsilon}X.$$

However, it does not seem true that $\mathrm{Fildim}_{\phi_1(\varepsilon)}X \geq \mathrm{wdim}_{\phi_2(\varepsilon)}X + \chi(\varepsilon)$, where ϕ_i and χ depend only on ε and $\lim\limits_{\varepsilon\to 0}\phi_i(\varepsilon) = 0$, the necessary inequality to get an equivalence between these two notions. It would be tempting to replace altogether wdim by Fildim in the definition of mean dimension, but the subadditivity of $\mathrm{Fildim}_{\varepsilon}$ is not obvious. However,

$$\mathrm{Fildim}\,(X_1\times X_2) \geq \mathrm{Fildim}X_1 + \mathrm{Fildim}X_2$$

can easily be shown using the fact that the FilRad of a product is the minimum of the FilRad on each factor, *cf.* [17].

Entropy can also be described as critical exponent, and we will now explain how to view mean dimension in a similar way (though the pratictal interest of such a definition is limited). Let $\mathcal{R}_n(\varepsilon,B)$ be the set of coverings of B by balls of radius ε for a metric d_m where $m\geq n$. In these notations, an element $B_i\in R\in\mathcal{R}_n(\varepsilon,B)$ is then associated to a n_i for which there exists a ball of radius ε in the metric d_{n_i}. Consider now the quantities

$$\begin{aligned} Q_1(n,\varepsilon,s,B) &= \sup_{R\in\mathcal{R}_n(\varepsilon,B)}\sum_{B_i\in R} e^{-n_is}, \\ Q_2(\varepsilon,s,B) &= \lim_{n\to\infty} Q_1(n,\varepsilon,s,B). \end{aligned}$$

Then there is a critical exponent, denote it $s_c(\varepsilon,B)$, such that

$$\begin{aligned} Q_2(\varepsilon,s,B) &= \infty \quad \text{si } s < s_c, \\ Q_2(\varepsilon,s,B) &= 0 \quad \text{si } s > s_c. \end{aligned}$$

Entropy can also be defined as $h(B) = \lim_{\varepsilon \to 0} s_c(\varepsilon, B)$. An analogous formulation of mean dimension exists. Let $\mathcal{F}_n(\varepsilon, B)$ be the set of ε-embeddings, $f : (B, d_n) \overset{\varepsilon}{\hookrightarrow} K$ where K is a polyhedron. Take a polyhedral metric on K. Given such a function, to each $k \in K$ one can associate n_k such that $\mathrm{Diam}\,(f^{-1}(k), d_{n_k}) \leq \varepsilon$ and $\mathrm{Diam}\,(f^{-1}(k), d_{n_k+1}) > \varepsilon$. Let $\mathcal{K}$ be the set of finite coverings of K by open balls, let ϕ be a map which associates to an open sets of K a positive real number, define

$$\begin{aligned} Q_1'(n, \varepsilon, s, B, \phi) &= \sup_{f:(B,d_n) \overset{\varepsilon}{\hookrightarrow} K} \inf_{R \in \mathcal{K}} \sum_{S \in R} \phi(S)^{s \inf_{k \in S} n_k} \\ Q_2'(\varepsilon, s, B, \phi) &= \lim_{n \to \infty} Q_1'(n, \varepsilon, s, B, \phi). \end{aligned}$$

If $\phi(S) = \mathrm{Diam}\, S$, then, having in mind the definition of Hausdorff dimension, there exists a critical exponent, a $s_c(\varepsilon, B)$ such that

$$\begin{aligned} Q_2'(\varepsilon, s, B) &= \infty \quad \text{if } s < s_c, \\ Q_2'(\varepsilon, s, B) &= 0 \quad \text{if } s > s_c. \end{aligned}$$

Mean dimension can also be written as $\mathrm{mdim} B = \lim_{\varepsilon \to 0} s_c(\varepsilon, B)$. A myriad of variants can then be obtained by changing the type of metric space K, the function ϕ or the sets S (*cf.* [13] or [41]).

Chapter 3

Summing J-holomorphic curves

The result described in this chapter was a necessary first step in the proof of a Runge theorem that could not be completed (*cf.* appendix B). It will also be of use later in chapter 5. In short it allows us to show that given two J-holomorphic curves u^0 and u^1 that intersect transversally at a point m_0 there exists a family of curves whose image are close to the union of the images of u^0 and u^1, and whose strangling close to m_0 are different.

It is a refinement of the theorem described by McDuff-Salamon in [35] on the possibility of gluing two curves, *i.e.* to find a family of curves whose images are close to the union of the images of two curves meeting at m_0. This chapter describes how to modify this gluing so as to obtain that (in local charts near m_0 and $0 \in \mathbb{C}\mathrm{P}^1$) in a ring of radius r (around $0 \in \mathbb{C}\mathrm{P}^1$) a local expansion would be of the form: $a^0 z + a^1 \frac{r^2}{z} + O(r^{1+\varepsilon})$, where a^0 and a^1 are the tangents to the curves at m_0, and $\varepsilon \in]0, 1/3[$. This information will be used later in 5 to insure that the intersection of a ball of radius $O(r^{1+\varepsilon})$ with the image of the map gives only discs when non-empty. The method is very close to that of [35, §10], which itself parallels [9, §7.2].

Throughout this chapter Σ will denote a Riemann surface (our interest is restricted to $\mathbb{C}\mathrm{P}^1$) and (M, J) will be an almost complex manifold of real dimension at least 4. The almost complex structure J will be assumed generic in the sense of definition A.3.4 and of class at least C^3. In particular, proposition A.3.1 insures that J-holomorphic maps will be at least C^3.

Before moving to the general setting, here is an example of the summing phenomenon for rational curves in $\mathbb{C}\mathrm{P}^n$.

Example 3.0.1*:* For $h \in \{0, \infty\}$, let $u^h : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ be rational curves of degree k_h that intersect at a point:

$$u^h[z_0 : z_1] = \left[\sum_{i=0}^{k_h} a^h_{i;j} z_0^i z_1^{k_h - i} \right]_{0 \leq j \leq n},$$

with the condition that $[a^0_{0;j}]_{0 \leq j \leq n} = [a^\infty_{k_\infty;j}]_{0 \leq j \leq n}$, as, without loss of generality, we assume the point of intersection to be at $u^0[0:1] = u^\infty[1:0]$. The desired curve v should be close to the two preceding curves (except perhaps at the contact point) and be of degree $k_0 + k_\infty$. Write

$$v[z_0 : z_1] = \left[\sum_{i=0}^{k_0+k_\infty} a'_{i;j} z_0^i z_1^{k_0+k_\infty - i} \right]_{0 \le j \le n} .$$

Asking that v resembles u^0 near $[1:0]$ and u^∞ near $[0:1]$ translates into

$$\begin{array}{rcl} \left[\sum_{i=0}^{k_0+k_\infty} a'_{i;j} \delta^{k_0+k_\infty-i} \right]_{0\le j\le n} & = v[1:\delta] \simeq u^0[1:\delta] = & \left[\sum_{i=0}^{k_0} a^0_{i;j} \delta^{k_0-i} \right]_{0\le j\le n} \\ \left[\sum_{i=0}^{k_0+k_\infty} a'_{i;j} \varepsilon^i \right]_{0\le j\le n} & = v[\varepsilon:1] \simeq u^\infty[\varepsilon:1] = & \left[\sum_{i=0}^{k_\infty} a^\infty_{i;j} \varepsilon^i \right]_{0\le j\le n} \end{array} .$$

Multiplying the right-hand side of the first line by δ^{k_∞}, a natural choice of $a'_{i;j}$ comes to mind:

$$\begin{array}{ll} \text{si } i \le k_\infty \text{ alors } a'_{i;j} = a^\infty_{i;j} & , \\ \text{si } i \ge k_\infty \text{ alors } a'_{i;j} = a^0_{i-k_\infty;j} & . \end{array}$$

However, this identification causes a problem when $i = k_\infty$. This can be straightened out:

$$\begin{array}{rl} u^0[0:1] = u^\infty[1:0] & \Rightarrow [a^0_{0;j}]_{0\le j\le n} = [a^\infty_{k_\infty;j}]_{0\le j\le n} \\ & \Rightarrow \exists \lambda_0, \lambda_\infty \in \mathbb{C}^\times \text{ such that } \forall j, \lambda_0 a^0_{0;j} = \lambda_\infty a^\infty_{k_\infty;j}. \end{array}$$

Thus the following correction solves our problem: $a'_{i;j} = \begin{cases} \lambda_\infty a^\infty_{i;j} & \text{si } i < k_\infty \\ \lambda_\infty a^\infty_{k_\infty;j} = \lambda_0 a^0_{0;j} & \text{si } i = k_\infty \\ \lambda_0 a^0_{i-k_\infty;j} & \text{si } i > k_\infty \end{cases}$.

Simply said, v is obtained by multiplying u^0 by $\lambda_0 z_0^{k_\infty}$ and u^∞ by $\lambda_\infty z_1^{k_0}$, then by adding their monomials except for the $z_0^{k_\infty} z_1^{k_0}$ monomial which remains that of either curve (after multiplication), as they are equal. For this construction to depend on a parameter that would describe how close the image of the curve is to $u^0(\mathbb{C}\mathrm{P}^1) \cup u^\infty(\mathbb{C}\mathrm{P}^1)$, one needs to reparametrize one of the curves.

Indeed, if we restrict to the case $n = 2$, let $u^0[z_0 : z_1] = [a^0_1 z_0 : a^0_2 z_0 : z_1]$ and $u^1[z_0 : z_1] = [a^1_1 z_0 : a^1_2 z_0 : z_1]$. In local charts ($[a:b] \mapsto a/b$ and $[a:b:c] \mapsto (a/c, b/c)$), these are just affine maps $\vec{a}^h z$ which intersect at 0. In order to fit in the discussion above, it suffices to reparametrize u^1: let $r \in \mathbb{R}_{>0}$ and $u^\infty[z_0 : z_1] = u^1[r^2 z_1, z_0]$. then $u^0[0:1] = [0:0:1]$ and $u^\infty[1:0] = [0:0:1]$, whence $\lambda_0 = \lambda_\infty = 1$ and $v[z_0 : z_1] = [a^0_1 z_0^2 + a^1_1 r^2 z_1^2 : a^0_2 z_0^2 + a^1_2 r^2 z_1^2 : z_0 z_1]$. Going back to local charts, we see that $v(z) = \vec{a}^0 z + \vec{a}^1 r^2 / z$, which is the behaviour we are trying to obtain. This chapter establishes this kind of result for almost-complex manifolds (M, J).

3.1 Definitions and description of the summing map

As we are concerned with local expansions, let us look at the local behaviour of a J-holomorphic map. Let $a \in \mathbb{R}^{2n}$ and $z \in \mathbb{C}$, the product az means $za = (x+iy)a = xa + yJ_0 a$, where $J_0 := \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$.

Note that the local charts will be chosen so that at $0 \in \mathbb{R}^{2n}$ the almost-complex structure induced by J (which will still be denoted by J) will be the usual complex structure, *i.e.* $J(0) = J_0$. Another convention is that the evaluation of J at a point m will be written J_m; note that the confusion that could arise between the usual structure and the evaluation of J at zero in a map is not to be worried about as, by choice of local charts, they will be equal.

Lemma 3.1.1: *Let J be an almost complex structure on $\mathbb{R}^{2n}$ such that $J(0) = J_0 := \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$. Let $u : \mathbb{C} \longrightarrow \mathbb{R}^{2n}$ be a J-holomorphic curve such that $u(0) = 0$. Then $\exists a \in \mathbb{R}^{2n}$ such that $u(z) = az + O(|z|^2)$, for $|z|$ small enough.*

Proof. The notation $(\bar{\partial}_{J_g} f)(z) = \mathrm{d}f(z) + J_{g(z)} \circ \mathrm{d}f(z) \circ j$ will be used to insist on the point at which J is looked at. The first step is to remark that

$$\bar{\partial}_J g = \bar{\partial}_{J''} g + (J_g - J''_g) J'_g (\partial_{J'} - \bar{\partial}_{J'}) g, \tag{3.1.2}$$

for any two complex structures J' and J'', and where $\partial_J = \bar{\partial}_{-J}$. On the other hand, write $u(z) = \sum_{k,l} z^k \bar{z}^l a_{k,l} + O(|z|^3)$, where $k,l \in \{0,1,2\}^2 \setminus O$, $a_{k,l} \in \mathbb{R}^{2n}$ and $(s+it)a = as + (J_0 a)t$. It appears, by choosing $J' = J'' = J_0$ in (3.1.2) or by looking directly at (A.1.9), that $\bar{\partial}_J u = 0$ if and only if

$$\sum_{k,l} a_{k,l} l z^k \bar{z}^{l-1} + O(|z|^2) + (J_u - J_0) J_0 \left(\sum_{k,l} a_{k,l} k z^{k-1} \bar{z}^l - a_{k,l} l z^k \bar{z}^{l-1} + O(|z|^2) \right) = 0. \tag{3.1.3}$$

Furthermore, the coefficients (c_1, c_2) of the matrix of J can be expanded:

$$(J_{\vec{x}})_{c_1,c_2} = (J_0)_{c_1,c_2} + \sum_{\vec{k}} b_{c_1,c_2,\vec{k}} (\vec{x})^{\vec{k}} + O(|\vec{x}|^3),$$

where $\vec{k} \in \{0,1,2\}^{2n} \setminus \{0\}^{2n}$. consequently, there is only one term of order 0 in (3.1.3): $a_{0,1}$. If u is J-holomorphic, it's local expansion must be of the form $u(z) = a_{1,0} z + O(|z|^2)$. □

There are situations where the construction of a J-holomorphic map whose expansion is of the form $a^0 z + a^1 \frac{r^2}{z}$ is easier. Though usually two J-holomorphic curves will be needed, in the following cases, one will be enough:

- if $a^1 = 0$ obviously,
- if $a^0 = 0$ it suffices to reparametrize by $z \mapsto \frac{r^2}{z}$ the domain of a curve whose tangent is a^1,
- if $\exists z_0 \in \mathbb{C}$ such that $a^0 = z_0 a^1$ the reparametrization $z \mapsto z + \frac{r^2 z_0}{z}$ of a map whose local expansion $u(z) = a^0 z + O(|z|^2)$ will do.

Consequently, a^0 and a^1 will be assumed linearly independent (over $\mathbb{C}$; whence the condition $\dim_{\mathbb{R}} M \geq 4$).

Theorem 3.1.4: *Let (M,J) be an almost-complex manifold of (real) dimension at least 4. Let $u^h : \Sigma \longrightarrow M$, where $h \in \{0,1\}$, be two J-holomorphic curves such that $u^h(0) = m_0$, $\left\|du^h\right\|_{L^\infty} \leq C$, J is regular in the sense of definition A.3.4 and D_{u^h} are surjective. If in a local chart $u^h(z) = a^h z + O(|z|^2)$, and that the a^h are linearly independent over $\mathbb{C}$, then $\exists r_0$ such that $\forall r \leq r_0$, $\exists u$ a J-holomorphic curve such that in a local chart,*

$$u(z) = a^0 z + a^1 \frac{r^2}{z} + O(r^{1+\varepsilon})$$

for all $z \in A_{r^{4/3},r^{2/3}} = \{z | r^{4/3} < |z| < r^{2/3}\}$ and where $\varepsilon \in]0, \frac{1}{3}[$; r_0 and c_0 depend on C, ε, a^h, the second derivatives of u^h, J (up to its second derivatives) and on the norm of the inverse to D_{u^h}.

The assumptions are more restrictive than in the gluing procedure of [35, §10] where curves whose differential at m_0 is 0 can be glued. The behaviour of the "summed" curve is however more precise. Indeed in [35, §10] the curve obtained by gluing is a perturbation of a curve which is constant in a ring; this leads to a curve whose behaviour in the given ring is $u(z) = O(r)$. The price to pay to obtain a more precise behaviour is that the approximate solution is no longer constant in a ring. When the approximate solution is constant in a ring A, the almost-complex structure J is also constant for $z \in A$. Section 3.4 describes how to modify the structure J so as to make it constant near the point of intersection, thus allowing to avoid the difficulty that arises.

The main ingredient in the proof remains the implicit function theorem of [35, §3.5]; recall that

$$s_p := \sup_{0 \neq f \in C^\infty(\Sigma)} \frac{\|f\|_{L^\infty}}{\|f\|_{W^{1,p}}} \tag{3.1.5}$$

is the constant of the Sobolev embedding $W^{1,p}(\Sigma,\mathbb{R}) \hookrightarrow L^\infty(\Sigma,\mathbb{R})$, which is finite for $p > \dim\Sigma = 2$ in our case (*cf.* [15, §6.7]).

Theorem 3.1.6: *([35, th 3.5.2]) Let Σ be a complex manifold of dimension 1, let $p > 2$. $\forall c_0$, $\exists \delta > 0$ such that for all volume forms dvol_Σ on Σ, all $u \in W^{1,p}(\Sigma,M)$, all $\xi_0 \in W^{1,p}(\Sigma, u^*\mathrm{T}M)$, and all $Q_u : L^p(\Sigma, \Lambda^{0,1} \otimes_J u^*\mathrm{T}M) \longrightarrow W^{1,p}(\Sigma, u^*\mathrm{T}M)$ satisfying*

$$\begin{array}{lll} s_p(\mathrm{dvol}_\Sigma) \leq c_0, & \|du\|_{L^p} \leq c_0, & \|\xi_0\|_{W^{1,p}} \leq \frac{\delta}{8}, \\ \left\|\bar\partial_J(\exp_u(\xi_0))\right\|_{L^p} \leq \frac{\delta}{4c_0}, & D_u Q_u = \mathbb{1}, & \|Q_u\| \leq c_0, \end{array}$$

there exists an unique ξ such that

$$\bar\partial_J(\exp_u(\xi_0 + \xi)) = 0, \quad \|\xi + \xi_0\|_{W^{1,p}} \leq \delta, \quad \|\xi\|_{W^{1,p}} \leq 2c_0 \left\|\bar\partial_J(\exp_u(\xi_0))\right\|_{L^p}.$$

The proof of this theorem is a minor modification of the proof of proposition 4.1.3. It is a consequence of the implicit function theorem and so requires a bound on the second derivative of $\mathcal{F}_u$ (*cf.* §A.4).

We start by constructing a family of curves u^r whose local expansion is as required, which satisfy the conditions of the above theorem (ξ_0 will be $\equiv 0$) and whose $\bar\partial_J$ is of the order of $O(r^{1+\varepsilon})$.

Then, the ξ obtained (the pertubation of u^r needed to obtain a true solution) will be bounded in L^∞ (since it is bounded in $W^{1,p}$) by $O(r^{1+\varepsilon})$.

Before we describe these maps u^r, we have to define cutoff functions which will be very useful. They will be denoted by β. The definition will not vary much, and, much like the following lemmas, is well-established; see [9] or [35].

Definition 3.1.7: Let $\beta_{\delta,\varepsilon} : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by:

$$\beta_{\delta,\varepsilon}(z) = \begin{cases} 1 & \text{if} \quad |z| < \delta \\ \dfrac{\ln\varepsilon - \ln|z|}{\ln\varepsilon - \ln\delta} & \text{if} \quad \delta < |z| < \varepsilon \\ 0 & \text{if} \quad \varepsilon < |z| \end{cases} .$$

This cutoff function has many useful properties, as can be seen in the following two lemmas:

Lemma 3.1.8: $\displaystyle\int_{\mathbb{R}^2} |\nabla\beta_{\delta,\varepsilon}|^2 = \frac{2\pi}{\ln(\varepsilon/\delta)}$.

Proof. It suffices to note that

$$|\nabla\beta| = \frac{1}{|z|\,|\ln(\varepsilon/\delta)|}$$

to get $\displaystyle\int_{\mathbb{R}^2} |\nabla\beta_{\delta,\varepsilon}|^2 = \int_{A_{\delta,\varepsilon}} \frac{1}{|z|^2\ln(\varepsilon/\delta)^2} = \int_\delta^\varepsilon \frac{2\pi d\rho}{\rho\ln(\varepsilon/\delta)^2} = \frac{2\pi}{\ln(\varepsilon/\delta)}$. □

In particular, this first lemma shows that this family contains a limit case of the Sobolev embeddings. Indeed, for fixed ε and if $\delta \to 0$, the function obtained is in $W^{1,2}$, but not in L^∞. The second lemma is also true when $p < 2$ without even needing to assume that $\xi(0) = 0$.

Lemma 3.1.9: *Let β be as in definition 3.1.7, let $\xi \in W^{1,p}(B_\varepsilon)$ where $p > 2$ be such that $\xi(0) = 0$, then $\exists s_H$ such that: $\displaystyle\|(\nabla\beta)\cdot\xi\|_{L^p(A_{\delta,\varepsilon})} \leq \frac{(2\pi)^{1/p} s_H}{\ln(\varepsilon/\delta)^{1-1/p}} \|\xi\|_{W^{1,p}(B_\varepsilon)}$.*

Proof. Since $\xi \in W^{1,p}$ with $p > 2$, ξ is Hölder continuous of exponent $1-2/p$, *i.e.* $\exists s_H$ t.q. $\forall z_1, z_2 \in B_\varepsilon$, $|\xi(z_1) - \xi(z_2)| \leq s_H \|\xi\|_{W^{1,p}(B_\varepsilon)} |z_1 - z_2|^{1-2/p}$ (see [20, §4.5]). Taking $z_2 = 0$ allows us to bound $\|(\nabla\beta)\cdot\xi\|^p_{L^p(A_{\delta,\varepsilon})}$ as follows:

$$\begin{aligned} \int_{A_{\delta,\varepsilon}} |\nabla\beta|^p |\xi(z)|^p &\leq \frac{s_H^p}{\ln(\varepsilon/\delta)^p} \int_{A_{\delta,\varepsilon}} \frac{1}{|z|^2} \|\xi\|^p_{W^{1,p}(B_\varepsilon)} \\ &\leq \frac{2\pi s_H^p}{\ln(\varepsilon/\delta)^p} \int_\delta^\varepsilon \frac{d\rho}{\rho} \|\xi\|^p_{W^{1,p}(B_\varepsilon)} \\ &\leq \frac{2\pi s_H^p}{\ln(\varepsilon/\delta)^{p-1}} \|\xi\|^p_{W^{1,p}(B_\varepsilon)} . \end{aligned}$$ □

To find a J-holomorphic curve with the desired local behaviour, an intuitive idea would be to add up the local expansions of two curves, namely $u^0(z)$ and $u^1(r^2/z)$, when $|z|$ is close to r and to get back to either map outside and inside the ring. Addition does not exist in manifolds, thus it is necessary to choose local charts in order to achieve this. In the resulting formula, maps should be seen as functions from (an open set of) $\mathbb{C}$ to (an open set of) $\mathbb{R}^{2n}$. The family of maps u^r will be defined as follows:

$$u^r(z) = \begin{cases} u^1(\frac{r^2}{z}) & \text{if} \quad |z| < r^{2-\gamma} \\ \beta(\frac{r^2}{z})u^0(z) + u^1(\frac{r^2}{z}) & \text{if} \quad r^{2-\gamma} < |z| < r^{2-\alpha} \\ u^0(z) + u^1(\frac{r^2}{z}) & \text{if} \quad r^{2-\alpha} < |z| < r^{\alpha} \\ u^0(z) + \beta(z)u^1(\frac{r^2}{z}) & \text{if} \quad r^{\alpha} < |z| < r^{\gamma} \\ u^0(z) & \text{if} \quad r^{\gamma} < |z| \end{cases} \tag{3.1.10}$$

where $0 < \gamma < \alpha < 1$, and $\beta = \beta_{r^\alpha, r^\gamma}$ (see definition 3.1.7).

3.2 Metrics and estimates

Before we can estimate the norms of du^r and of $\bar{\partial}_J u^r$ (in L^p), we need to specify the metric on the domain Σ. When $|z| > r$ the curve defined by u^r will be close to u^0, and when $|z| < r$, $u^r(r^2/z)$ ressembles $u^1(z)$. These two subsets of the domain will play a similar role; it is natural to give them equal weights (at the domain). Intuitively, this also avoids the norm of the differential becoming large by giving to regions whose energy is of the same magnitude, equal weights in the domain. This metric will be the usual (Fubini-Study) metric when $|z| > r$, and the one induced by $z \mapsto \frac{r^2}{z}$ when $|z| < r$ (see Figure 3.1 below). More precisely, the metric will be $g^r := (\theta^r)^{-2}(ds^2 + dt^2)$, where

$$\theta^r(z) = \begin{cases} r^2 + |z|^2/r^2 & \text{si } |z| < r \\ 1 + |z|^2 & \text{si } |z| > r \end{cases}$$

It might seem necessary to work with norms that take into account the two distinct regions, but since the situation is symmetric, estimates valid on a region will hold on the other. A more precise discussion can be found in [35, §10.3]. We will only note that the volume remains bounded $Vol(\Sigma) \leq 2\pi$. The next lemma, taken from [35, §10.3] says that Sobolev constant behaves similarly.

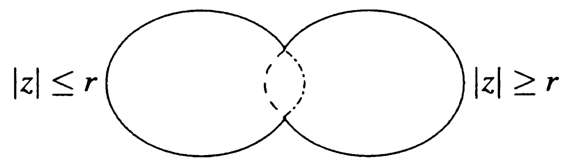


Figure 3.1: A picture of $(\mathbb{C} \cup \{\infty\}, g_r)$

Lemma 3.2.1: *The constant s_p (cf. (3.1.5)) for the metric g^r remains bounded independently of r.*

It is now possible to evaluate the L^p norms of du^r and $\bar{\partial}_J u^r$ in order to satisfy the assumptions of theorem 3.1.6. Our starting point is to bound the norm of powers of z:

Lemma 3.2.2: *Let $r>0$, $l,l'\geq 1$, $0\leq\delta<\varepsilon\leq 1$, and $\|\cdot\|_{L^p(A_{r^\varepsilon,r^\delta})}$ denote the L^p norm restricted to the ring $A_{r^\varepsilon,r^\delta}=\{z|r^\varepsilon<|z|<r^\delta\}$. Then*

$$\begin{array}{lll} \|z^l\|_{L^p(A_{r^\varepsilon,r^\delta})} & = \left(\frac{2\pi(1-r^{(\varepsilon-\delta)(2+lp)})}{2+lp}\right)^{1/p} r^{\delta(l+2/p)} & \sim K_{\varepsilon,\delta,p,l}r^{\delta(l+2/p)} \\ \left\|\frac{r^{l'}}{z^l}\right\|_{L^p,(A_{r^\varepsilon,r^\delta})} & = \left(\frac{2\pi(1-r^{(\varepsilon-\delta)(lp-2)})}{lp-2}\right)^{1/p} r^{l'+\varepsilon(-l+2/p)} & \sim K'_{\varepsilon,\delta,p,l}r^{l'+\varepsilon(-l+2/p)} \end{array}$$

where $K_{\varepsilon,\delta,p,l}$ and $K'_{\varepsilon,\delta,p,l}$ are the limits as $r\longrightarrow 0$ of the terms before the powers of r.

Proof. It is a direct calculation, valid for $l\neq -2/p$:

$$\begin{aligned} \|z^l\|^p_{L^p(A_{r^\varepsilon,r^\delta})} &= \int_{A_{r^\varepsilon,r^\delta}} \rho^{pl+1}d\rho d\theta \\ &= 2\pi\left[\frac{\rho^{2+lp}}{2+lp}\right]_{r^\varepsilon}^{r^\delta} \\ &= \left(\frac{2\pi(1-r^{(\varepsilon-\delta)(2+lp)})}{2+lp}\right) r^{\delta(lp+2)} \end{aligned}$$

A simple manipulation of this equality gives the second estimation. □

Lemma 3.2.3: *Let r_1 be such that $|\ln r^{\alpha-\gamma}|^{-1}<1$, then*

$$\forall r<r_1, \|du^r\|_{L^p}\leq \|du^0\|_{L^p}+\|du^1\|_{L^p}+c_1r^{2/p},$$

where $c_1=(4K'_{\alpha,\gamma,p,2}r^{2(1-\alpha)}+2K'_{1,\alpha,p,2})C$ and $C\geq\max(\|du^0\|_{L^\infty},\|du^1\|_{L^\infty})$.

Proof. In the region $r<|z|<r^\alpha$ this is a simple assertion:

$$\|du^r\|_{L^p(A_{r,r^\alpha})}\leq\|du^0\|_{L^p(A_{r,r^\alpha})}+\left\|\frac{r^2}{z^2}\right\|_{L^p(A_{r,r^\alpha})}\|du^1\|_{C^0}\leq\|du^0\|_{L^p(A_{r,r^\alpha})}+CK'_{1,\alpha,p,2}r^{2/p}.$$

Whereas when $r^\gamma<|z|$, it is trivial since $du^r=du^0$. On A_{r^α,r^γ}, a choice of a local chart and a local expansion for u^1 is needed: if r is small, then $\frac{r^2}{z}$ is also small in the given region, $r^{2-\alpha}>\left|\frac{r^2}{z}\right|>r^{2-\gamma}$. Indeed,

$$\|du^r\|_{L^p(A_{r^\alpha,r^\gamma})}\leq\|du^0\|_{L^p(A_{r^\alpha,r^\gamma})}+\left\|d\Big(\beta(|z|)u^1(\frac{r^2}{z})\Big)\right\|_{L^p(A_{r^\alpha,r^\gamma})},$$

and since $|u^1(z)|\leq C|z|$, the second term can be written as

$$\begin{aligned} \left\|d(\beta(z)u^1(\tfrac{r^2}{z}))\right\|_{L^p} &\leq \left\|\frac{u^1(\frac{r^2}{z})}{|z|\ln r^{\alpha-\gamma}}\right\|_{L^p}+\left\|d(u^1(\tfrac{r^2}{z}))\right\|_{L^p} \\ &\leq \left\|C\frac{r^2}{z^2}\right\|_{L^p}|\ln r^{\alpha-\gamma}|^{-1}+\|du^1\|_{C^0}\left\|\frac{r^2}{z^2}\right\|_{L^p}. \end{aligned}$$

As the terms appearing are of the form $\frac{r^2}{z^2}$, and using lemma 3.2.2,

$$\left\|d(\beta(z)u^1\left(\frac{r^2}{z}\right))\right\|_{L^p}\leq CK'_{\alpha,\gamma,p,2}(1+|\ln r^{\alpha-\gamma}|^{-1})r^{(1-\alpha)(2-2/p)}r^{2/p}.$$

The contribution of the region $r^\alpha<|z|<r^\gamma$ to $\|du^r\|_{L^p}$ tends to zero 0 as $r\longrightarrow 0$ faster than $r^{2/p}$. The final result follows from the symmetry which yields the same conclusion on the region where $|z|<r$. □

Our goal being to give a local expansion at order 1, we have to show that the L^p norm of $\bar{\partial}_J u^r$ (which bounds the $W^{1,p}$ norm and consequently the L^∞ norm of the perturbation necessary to obtain a true solution) is $O(r^{1+\varepsilon})$ when z is of norm close to r.

Lemma 3.2.4: *Take* $2 < p < 4$. *Let* $\alpha \in]\frac{p}{p+2}, \frac{p}{2p-2}[$ *then there exists positive numbers* $\varepsilon < \min(\alpha(1+2/p)-1, 1-\alpha(2-2/p))$, r_2 *and* c_2 *(both depend on the second derivatives of* u^0 *and* u^1*, and on the product of the derivatives of* J *with* C*) such that,* $\forall r < r_2, \|\bar{\partial}_J u^r\|_{L^p} \leq c_2 r^{1+\varepsilon}$.

Proof. Since the situation is symmetric, we will only be concerned with the part where $r < |z|$. We split this region again, as the definition u^r varies.

$$\|\bar{\partial}_J u^r\|^p_{L^p(\{z||z|>r\})} = \|\bar{\partial}_J u^r\|^p_{L^p(A_{r,r^\alpha})} + \|\bar{\partial}_J u^r\|^p_{L^p(A_{r^\alpha,r^\gamma})}.$$

The region where $|z| > r^\gamma$ does not contribute in the equality above since $u^r = u^0$ is J-holomorphic. For the other domains, J_w will be seen as a matrix valued map using a local chart. Again, the notation $(\bar{\partial}_{J_g} f)(z) = \mathrm{d}f(z) + J_{g(z)} \circ \mathrm{d}f(z) \circ j$ will be used to emphasize the point at which J is evaluated. With this understood,

$$\begin{aligned}
\left\|\bar{\partial}_J u^r\right\|_{L^p(A_{r^\alpha,r^\gamma})} &= \left\|\bar{\partial}_J u^r - \bar{\partial}_J u^0\right\|_{L^p(A_{r^\alpha,r^\gamma})} \\
&= \left\|\bar{\partial}_{J_{u^r}}(u^r - u^0) + (\bar{\partial}_{J_{u^r}} - \bar{\partial}_{J_{u^0}})u^0\right\|_{L^p(A_{r^\alpha,r^\gamma})} \\
&= \left\|\bar{\partial}_{J_{u^r}}(\beta(z)u^1(\tfrac{r^2}{z})) + (J_{u^r} - J_{u^0})\mathrm{d}u^0 \circ j\right\|_{L^p(A_{r^\alpha,r^\gamma})} \\
&\leq \left\|\mathrm{d}(\beta(z)u^1(\tfrac{r^2}{z}))\right\|_{L^p(A_{r^\alpha,r^\gamma})} + \|J_\cdot\|_{C^1}\left\|\mathrm{d}u^0\right\|_{C^0}\left\|u^1(\tfrac{r^2}{z})\right\|_{L^p(A_{r^\alpha,r^\gamma})}.
\end{aligned}$$

The bounds obtained in lemma 3.2.3 and the norms computed in lemma 3.2.2 yield the following upper bound:

$$\|\bar{\partial}_J u^r\|_{L^p(A_{r^\alpha,r^\gamma})} \leq CK'_{\alpha,\gamma,p,2}(1 + |\ln r^{\alpha-\gamma}|)r^{2-\alpha(2-2/p)} + \|J_\cdot\|_{C^1} C^2 K'_{\alpha,\gamma,p,1} r^{2-\alpha(1-2/p)}$$

In order to factorize $r^{1+\varepsilon}$, for $l = 1$ or 2 it must be checked that $2 - \alpha(l-2/p) > 1 \Leftrightarrow \alpha < \frac{p}{lp-2}$. This condition is only restrictive for $l = 2$.

To evaluate the other part, we proceed as in lemma 3.1.1. Upon noticing that (A.1.8) implies $|\bar{\partial}_J u| \leq \left|\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t}\right|$, and that the u^h can be written as

$$u^h(z) = a^h z + \sum_{k,l} z^k \bar{z}^l a^h_{k,l} + O(|z|^3) \qquad \text{where } k,l \in \{0,1,2\}^2, k+l \geq 2$$

when $r < |z| < r^\alpha$, the following bound (it can also be seen using (3.1.2)) appears

$$\begin{aligned}
|\bar{\partial}_J u^r| \leq \Big| & a^0_{1,1} z + 2a^0_{0,2}\bar{z} - a^\infty_{1,1}\tfrac{r^4}{z\bar{z}^2} - 2a^\infty_{0,2}\tfrac{r^4}{\bar{z}^3} + O(|z|^2) + O(\tfrac{r^6}{|z|^4}) \\
& + (J_{u^r} - J_0)J_0(a^0_{1,0} + a^\infty_{1,0}\tfrac{r^2}{z^2} + O(|z|) + O(\tfrac{r^4}{|z|^3}))\Big|.
\end{aligned}$$

Thus, the factors $|z|$, $\frac{r^2}{|z|}$ and $\frac{r^4}{|z|^3}$ could endanger our goal, as an expansion of $(J_{u^r} - J_0)$ shows. It also means that our bounds depend on the second derivatives of the u^h, or on a product of the first derivatives of J and u^h. The L^p norm will be made of terms in

$$\begin{aligned} \left\|z^l\right\|_{L^p(A_{r,r^\alpha})} &\sim K_{1,\alpha,p,l} r^{\alpha(l+2/p)} \\ \left\|\frac{r^{l'}}{z^l}\right\|_{L^p(A_{r,r^\alpha})} &\sim K'_{1,\alpha,p,l} r^{l'-l+2/p} \end{aligned}$$

with $l' > l \geq 1$, as computed in lemma 3.2.2. This raises a new condition on α: $\alpha(l+2/p) > 1 \Leftrightarrow \alpha > \frac{p}{lp+2}$. Consequently $\|\bar{\partial}_J u^r\|_{L^p} \leq K r^{1+\varepsilon}$ under the condition that $\alpha \in]\frac{p}{p+2}, \frac{p}{2p-2}[$, which is only possible if $p < 4$. □

Remark 3.2.5*:* For any $p \in]2,4[$, there is an optimal choice of α. Indeed, if $\alpha = \frac{2}{3}$ then $\varepsilon < \frac{1}{3}(\frac{4}{p} - 1)$. Taking p close to 2, enables ε to be close to $1/3$. Theorem 3.1.4 is obtained with this choice of α. However, it is not possible to take $p \longrightarrow 2$ as some constants, *e.g.* s_p, depend on p. The choice of γ is quite secondary, *e.g.* one could choose $\gamma = \frac{5}{6}$.

3.3 Construction of the inverse Q_{u^r}

In this section, we will make the somehow strong assumption that J is constant in a neighborhood of $m_0 \in M$; the reason why such a simplification is possible is explained in §3.4. The whole gluing process is presented in order in §3.5.

Before we apply the implicit function theorem, it is required to have a bounded inverse to the linearization of $\bar{\partial}_J$ at u^r, D_{u^r}. The existence of inverses for D_{u^h} combined with the observation that two maps which are close enough will have close linearization. This will enable the construction of this inverse. First let us show that if u' is close to u in the sense of $W^{1,p}$, then the operators D_u and $D_{u'}$ are close. In order to identify their images, parallel transport is necessary. However, it does not affect significantly the following computation:

$$\begin{aligned} \|D_u\xi - D_{u'}\xi\|_{L^p} &\leq \|(J_u - J_{u'})\nabla\xi\|_{L^p} + \tfrac{1}{2}\left\|J_u \nabla_\xi J_u (du - du')\right\|_{L^p} \\ &\quad + \tfrac{1}{2}\left\|(J_u \nabla_\xi J_u - J_{u'} \nabla_\xi J_{u'}) du'\right\|_{L^p} \\ &\leq \|J_\cdot\|_{C^1} \|u - u'\|_{C^0} \|\nabla\xi\|_{L^p} + \tfrac{1}{2}\left\|J_u \nabla_\xi J_u\right\|_{C^0} \|du - du'\|_{L^p} \\ &\quad + \tfrac{1}{2}\left\|J_\cdot \nabla_\xi J_\cdot\right\|_{C^1} \|u - u'\|_{C^0} \|du'\|_{L^p} \\ &\leq s_p \|J_\cdot\|_{C^1} \|u - u'\|_{W^{1,p}} \|\xi\|_{W^{1,p}} + \tfrac{1}{2}\left\|J_u \nabla_\xi J_u\right\|_{C^0} \|u - u'\|_{W^{1,p}} \\ &\quad + \tfrac{1}{2} s_p \left\|J_\cdot \nabla_\xi J_\cdot\right\|_{C^1} \|u - u'\|_{W^{1,p}} \|du'\|_{L^p} \\ &\leq s_p \|J_\cdot\|_{C^1} \|u - u'\|_{W^{1,p}} \|\xi\|_{W^{1,p}} \\ &\quad + \tfrac{1}{2} s_p \|J_u \nabla J_u\|_{C^0} \|\xi\|_{W^{1,p}} \|u - u'\|_{W^{1,p}} \\ &\qquad + \tfrac{1}{2} s_p^2 \|J_\cdot \nabla J_\cdot\|_{C^1} \|\xi\|_{W^{1,p}} \|u - u'\|_{W^{1,p}} \|du'\|_{L^p} \\ &\leq c_3(\nabla^2 J, du', s_p) \|\xi\|_{W^{1,p}} \|u - u'\|_{W^{1,p}}. \end{aligned} \tag{3.3.1}$$

For the curves we are concerned with, proximity in $\|\cdot\|_{W^{1,p}}$ will be insured as follows: $d(u^r - u^0)$

is zero when $|z| > r^\gamma$, and it is of the order of $\frac{r^2}{z^2}$ when $r < |z| < r^\gamma$, consequently $\|u^r - u^0\|_{W^{1,p}(\{|z|>r\})}$ is of the order of $r^{2/p}$. Thus, D_{u^r} will be close to one of the D_{u^h} inside or outside $|z| = r$.

To be more precise, it is necessary to introduce intermediate curves, denoted by $u^{0,r}$ and $u^{1,r}$. The first will be defined as follows

$$u^{0,r}(z) = \begin{cases} u^0(z) & \text{if} & & |z| & < r^{2-\gamma} \\ u^0(z) + \beta(\frac{r^2}{z})u^1(\frac{r^2}{z}) & \text{if} & r^{2-\gamma} < & |z| & < r^{2-\alpha} \\ u^0(z) + u^1(\frac{r^2}{z}) & \text{if} & r^{2-\alpha} < & |z| & < r^\alpha \\ u^0(z) + \beta(|z|)u^1(\frac{r^2}{z}) & \text{if} & r^\alpha < & |z| & < r^\gamma \\ u^0(z) & \text{if} & r^\gamma < & |z| & \end{cases},$$

and the second in an analogous manner. Since $\|u^{0,r} - u^0\|_{W^{1,p}} \to 0$ as $r \to 0$, the operator $D_{u^{0,r}}$ will be as close as required to D_{u^0} and identical to D_{u^r} when $|z| > r$.

The two inverses Q_{u^0} and Q_{u^1} will be used to construct an inverse to D_{u^r} whose bound is independent of r. First we introduce some notations, *cf.* §A.3. For $u : \Sigma \to M$, let $W^{1,p}_u = W^{1,p}(\Sigma, u^*\mathrm{T}M)$, $L^p_u = L^p(\Sigma, \Lambda^{0,1}\ \mathrm{T}^*\Sigma \otimes_J u^*\mathrm{T}M)$. Given $u^0, u^1 : \Sigma \to M$, such that $u^0(0) = u^1(0)$, denote by

$$W^{1,p}_{u^{0,1}} := \left\{(\xi^0,\xi^1) \in W^{1,p}_{u^0} \times W^{1,p}_{u^1} | \xi^0(0) = \xi^1(0)\right\}.$$

The assumption that $p > 2$ is of importance, since $W^{1,p}$ sections need not be continuous if $p \leq 2$, and their evaluation at a point would not make sense.

Thanks to the assumption made on J (*cf.* lemma A.3.6), the operator

$$\begin{array}{cccc} D_{0,1} : & W^{1,p}_{u^{0,1}} & \to & L^p_{u^0} \times L^p_{u^1} \\ & (\xi^0,\xi^1) & \mapsto & (D_{u^0}\xi^0, D_{u^1}\xi^1) \end{array}$$

is surjective. Thus, $D_{0,1}$ possesses an inverse which depends continuously on the pair (u^0,u^1) and satisfies an uniform bound as (u^0,u^1) varies in $\mathcal{M}^*(C)$. This suffices for our use, but if one would like to stay in a case where "surjectivity" of the gluing map (*cf.* [35, th 10.1.2.iii]) is possible, one needs to show that amongst all the inverses of $D_{0,1}$, the one which is orthogonal to the kernel also has bounded norm. Recall that surjectivity is the property that any J-holomorphic curve which is close to union of the curves u^0 and u^1 is in the image of the gluing map.

More precisely, if $\mathcal{W}_{u^{0,1}} \subset W^{1,p}_{u^{0,1}}$ is the (L^2) orthogonal to the kernel of $D_{0,1}$, then the restriction of this operator to $\mathcal{W}_{u^{0,1}}$ is bijective and bounded. Its inverse will be denoted $Q_{0,\infty}$. It varies continuously with the pair (u^0,u^1) and the bound is uniform as $\mathcal{M}^*(C)$ is compact. To make this explicit, an identification must be made between $W^{1,p}_{u^{0,1}}$ and $W^{1,p}_{v^{0,1}}$ for pairs (u^0,u^1) and (v^0,v^1) sufficiently close (we will not do it here).

Since the maps $u^{0,r}$ and $u^{1,r}$ are small $W^{1,p}$ deformations of u^0 and u^1, the space $W^{1,p}_{0,1}$ may be seen as a limit when $r \to 0$ of spaces $W^{1,p}_{0,1,r}$ corresponding to these slightly altered maps. The operator $D_{0,1,r}$ being a small perturbation of $D_{0,1}$ it will possess a right inverse. To prove surjectivity of the gluing map, it is the inverse $Q_{0,1,r}$ whose image is L^2-orthogonal to the kernel of $D_{0,1,r}$ which must be chosen. A verification must be made to show that the bound on the norm of this operator is independent of r. This argument ([35, lem 10.6.1]) works without need of change in the situation in which we are (the kernel of D_u is finite-dimensional).

Thanks to the operator $D_{u^{0,1,r}}$, an approximate inverse $T_{u^r} : L^p_{u^r} \longrightarrow W^{1,p}_{u^r}$ for D_{u^r} will be obtained. Let $\eta \in L^p_{u^r}$. This 1-form will be cut along the circle $|z| = r$ in two pieces (η^0, η^1):

$$\eta^0(z) = \begin{cases} \eta(z) & \text{if } |z| > r \\ 0 & \text{if } |z| < r \end{cases}, \quad \eta^1(z) = \begin{cases} r^2\eta(r^2 z) & \text{if } |z| < 1/r \\ 0 & \text{if } |z| > 1/r \end{cases}.$$

Since the η^h are only in L^p, the discontinuity is not problematic. Now let $(\xi^0, \xi^1) = Q_{0,1,r}(\eta^0, \eta^1)$. It is worth stressing that $\xi^0(0) = \xi^1(0) =: \xi_{m_0} \in \mathrm{T}_{m_0}M$. Let $\delta \in]0,1[$. This choice is not of importance; it would suffice to take $\delta = \frac{1}{2}$. Let

$$\beta(z) = 1 - \beta_{r^{1+\delta},r}(z) = \begin{cases} 1 & \text{if} \quad r < |z| \\ \dfrac{\ln|z| - \ln(r^{1+\delta})}{-\ln(r^\delta)} & \text{if} \quad r^{1+\delta} < |z| < r \\ 0 & \text{if} \quad |z| < r^{1+\delta} \end{cases}.$$

The approximate inverse is: $T_{u^r}\eta = \xi^r$, where

$$\xi^r(z) = \begin{cases} \xi^0(z) & \text{if} \quad r^{1-\delta} < |z| \\ \xi^0(z) + \beta(\frac{r^2}{z})(\xi^1(\frac{z}{r^2}) - \xi_{m_0}) & \text{if} \quad r < |z| < r^{1-\delta} \\ \xi^0(z) + \xi^1(\frac{z}{r^2}) - \xi_{m_0} & \text{if} \quad r = |z| \\ \xi^1(\frac{z}{r^2}) + \beta(z)(\xi^0(z) - \xi_{m_0}) & \text{if} \quad r^{1+\delta} < |z| < r \\ \xi^1(\frac{z}{r^2}) & \text{if} \quad |z| < r^{1+\delta} \end{cases}. \tag{3.3.2}$$

It remains to show that this is an approximate inverse as claimed, *i.e.* $\|D_{u^r}\xi^r - \eta\|_{L^p} \le \varepsilon \|\eta\|_{L^p}$ for some $\varepsilon \in [0,1[$. By construction the left-hand term is zero outside $r^{1+\delta} < |z| < r^{1-\delta}$. The assumption that the almost-complex structure is constant on that region will now be important. We restrict our attention, thanks to symmetry, to the piece $|z| < r$. By definition $D_{u^r}\xi^1(\cdot/r^2) = \eta(\cdot)$. Hence,

$$\begin{aligned} D_{u^r}\xi^r - \eta &= D_{u^{0,r}}(\beta(\xi^0 - \xi_{m_0})) \\ &= \beta D_{u^{0,r}}(\xi^0 - \xi_{m_0}) + (\xi^0 - \xi_{m_0})\bar{\partial}\beta \\ &= (\xi^0 - \xi_{m_0})\bar{\partial}\beta, \end{aligned} \tag{3.3.3}$$

since $D_{u^{0,r}}\xi^0 = 0$ when $|z| < r$. It remains to bound this norm with repect to the metric (that depends on r). There will be a factor of $\theta^r(z)^{p-2}$ in the norm of the 1-forms, but $\theta^r < \theta^1 \le 2$).

$$\begin{aligned} \|D_{u^r}\xi^r - \eta\|_{L^p(B_r)} &\le 2^{1-2/p}\,\|D_{u^r}\xi^r - \eta\|_{L^p(|z|<r)} \\ &\le 2^{1-2/p}\,\|(\xi^0 - \xi_{m_0})\bar{\partial}\beta\|_{L^p(|z|<r)} \\ &\le 2\pi^{1/p} s_H \frac{\|\xi^0 - \xi_{m_0}\|_{W^{1,p}}}{|\delta \ln r|^{1-1/p}} \\ &\le 2\pi^{1/p} s_H s_p c_4 \frac{(\|\eta^0\|_{L^p} + \|\eta^\infty\|_{L^p})}{|\delta \ln r|^{1-1/p}} \\ &\le 4\pi^{1/p} s_H s_p c_4 \frac{\|\eta\|_{L^p}}{|\delta \ln r|^{1-1/p}}. \end{aligned} \tag{3.3.4}$$

Lemma 3.1.9 is used to go from the 2^{nd} to the 3^{rd} line. The 4^{th} line is obtained from the third using that ξ_{m_0} is bounded by the C^0 norm (and thus by the $W^{1,p}$ norm) of ξ^0, and on the other hand that the $W^{1,p}$ norm of ξ^0 is bounded by a constant (coming from the norm of $Q_{0,1,r}$) multiplied by the L^p norm of η^0.

Lemma 3.3.5: *Let u^r be as defined in* (3.1.10) *then* $\forall \varepsilon \in [0,1[, \exists c_4, \exists r_3$ *(which depend on c_4, s_p, and s_H), such that* $\forall r \leq r_3, \exists T_{u^r}$ *such that* $\|D_{u^r}T_{u^r} - \mathbb{1}\| \leq \frac{1}{2}$ *and* $\|T_{u^r}\| \leq c_4$.

Proof. The only part of the statement which was not proved in the above discussion is the one concerning the norm of T_{u^r}. It requires a bound on $\|\xi^r\|_{W^{1,p}}$ as a function of $\|\eta\|_{L^p}$. Only the cutoff function requires care, the bound being otherwise found thanks to the bound on $Q_{0,1,r}$. However, $\|\nabla\xi^r\|$ remains controlled exactly as in (3.3.4) thanks to lemma 3.1.9. □

Thus the true inverse Q_{u^r} will have the same image as T_{u^r} and will be defined by:

(3.3.6)
$$Q_{u^r} = T_{u^r}(D_{u^r}T_{u^r})^{-1} = T_{u^r}\sum_{k=0}^{\infty}(\mathbb{1} - D_{u^r}T_{u^r})^k$$

It satisfies the relation: $D_{u^r}Q_{u^r} = \mathbb{1}$ and $\|Q_{u^r}\| \leq 2c_4$, where c_4 comme from lemma 3.3.5.

3.4 On the assumption that J is constant near m_0

This section consists in noting that when J is close to J', the operator D_u^J is close to $D_u^{J'}$ for certain u (*e.g.* u^0, u^1 and u^r). In order to speak of a difference between these two operators, we will see their images not as the space of $(0,1)$-forms taking value in TM (since the definition of a $(0,1)$-form depends on the almost-complex structure) but as the space of TM-valued 1-forms. As a consequence their inverses will also be close.

$$\begin{aligned}
\left\|D_u^J\xi - D_u^{J'}\xi\right\|_{L^p} &\leq \tfrac{1}{2}\left\|(J_u - J'_u)\nabla\xi\right\|_{L^p} + \tfrac{1}{2}\left\|(J_u\nabla_\xi J_u - J'_u\nabla_\xi J'_u)du\right\|_{L^p} \\
&\leq \tfrac{1}{2}\left\|\nabla\xi\right\|_{L_p}\left\|J_u - J'_u\right\|_{C^0} + \tfrac{1}{2}\left\|du\right\|_{C^0}\left\|\xi\right\|_{C^0}\left\|J_u\nabla J_u - J'_u\nabla J'_u\right\|_{L_p} \\
&\leq c_5(du)\left\|\xi\right\|_{W^{1,p}}(\left\|J_u - J'_u\right\|_{C^0} + \left\|J_u\nabla J_u - J'_u\nabla J'_u\right\|_{L_p}).
\end{aligned}$$

Thus, it is important to note that the dependance on the differential of u will not be a problem for the maps we consider.

Lemma 3.4.1: $\exists r_4(\alpha,\gamma)$ *such that* $\forall r < r_6$, $\|du^r\|_{C^0} \leq 2C$.

Proof. This proof works in an analogous fashion as the bound of the L^p norm of du^r. When $r < |z| < r^\alpha$ this is a simple thing to check:

$$\|du^r\|_{C^0} \leq \|du^0\|_{C^0} + \left\|\frac{r^2}{z^2}\right\|_{C^0}\|du^1\|_{C^0} \leq 2C.$$

If $r^\gamma < |z|$, then $du^r = du^0$ so the conclusion is direct. Finally on A_{r^α,r^γ}, the computation requires a local chart and a local expansion for u^1: if r is small, then $\frac{r^2}{z}$ is also small on A_{r^α,r^γ}: $r^{2-\alpha} >$

$\left|\frac{r^2}{z}\right| > r^{2-\gamma}$. Thus, $\|du^r\|_{C^0} \leq \|du^0\|_{C^0} + \left\|d(\beta(|z|)u^1(\frac{r^2}{z}))\right\|_{C^0}$, and since $|u^1(w)| \leq C|w| + O(|w|^2)$, the second term can be written as

$$\begin{aligned}\left\|d(\beta(z)u^1(\tfrac{r^2}{z}))\right\|_{C^0} &\leq \left\|\frac{u^1(\frac{r^2}{z})}{|z|\ln r^{\alpha-\gamma}}\right\|_{C^0} + \left\|d(u^1(\tfrac{r^2}{z}))\right\|_{C^0} \\ &\leq \left\|Cr^2/z^2 + O(r^4/z^3)\right\|_{C^0} |\ln r^{\alpha-\gamma}|^{-1} + \|du^1\|_{C^0} \left\|\frac{r^2}{z^2}\right\|_{C^0} \\ &\leq C\left|r^{2-2\alpha}\right| (|\ln r^{\alpha-\gamma}|^{-1} + 1)\end{aligned}$$

The C^0 norm is bounded by the maximum of the bounds on each region: $\|du^r\|_{C^0} \leq \max(2C, C, C + o(1)) \leq 2C$ for all r such that $\left|r^{2-2\alpha}\right| (|\ln r^{\alpha-\gamma}|^{-1} + 1) < 1$. □

This lemma, together with lemma 3.4.3, allows us to choose r arbitrarily small without changing the proximity of $D^J_{u^r}$ and $D^{J'}_{u^r}$. It is important to show that this proximity is valid for the whole family of curves considered. The property required of J' is to be constant in a neighborhood of m_0. Consequently let us define for $R \in]0,1[$ and for $\kappa \in \mathbb{R} > 0$,

$$J'(w) = \begin{cases} J_0 & \text{if} \quad |w| < R(1-R^\kappa) \\ J_{\beta(|w|)w} & \text{if} \quad R(1-R^\kappa) < |w| < R \\ J_w & \text{if} \quad R < |w| \end{cases} \tag{3.4.2}$$

where

$$\beta(x) = \begin{cases} 0 & \text{if} \quad x < R(1-R^\kappa) \\ \dfrac{\ln x - \ln R(1-R^\kappa)}{-\ln(1-R^\kappa)} & \text{if} \quad R(1-R^\kappa) < x < R \\ 1 & \text{if} \quad R < x \end{cases}$$

Then $\|J - J'\|_{C^0} \leq 2\|J - J_0\|_{C^0(\{|w|<R\})} \leq O(R)$. Furthermore, since

$$\begin{aligned}\left|\nabla(J_{\beta(|w|)w})\right| &\leq \left|(\nabla J)_{\beta(|w|)w}\left(\beta(|w|) - \frac{w}{|w|\ln(1-R^\kappa)}\right)\right| \\ &\leq \left|(\nabla J)_{\beta(|w|)w}\right| (1 + |\ln(1-R^\kappa)|^{-1}),\end{aligned}$$

it is possible to obtain a rough bound for $\|J\nabla J - J'\nabla J'\|_{L^p}$, if we suppose that $\|du\| \geq d$ in $B_\rho(0)$, so that the preimage by u of a small ball remains a small ball up to multiplication by a bounded factor. Let R' be such that $B_{R'}(m_0) \cap \operatorname{Im} u^h \subset u^h(B_\rho(0))$. In order to avoid cases where the map sends many subsets of $\mathbb{C}P^1$ to $B_{R'}(m_0)$ (for instance, is non injective), it is possible to introduce almost complex structures that depend on a point of the domain; we shall not go into such details. Thus, the preimage of $|w| < R(1-R^\kappa)$ by u is dilated by at most $k \propto \frac{1}{d}$, whence

$$\|J\nabla J\|_{L^p(u^{-1}(|w|<R(1-R^\kappa)))} \leq \|J\nabla J\|_{L^\infty} (kR^2(1-R^\kappa)^2)^{1/p},$$

and similarly for the ring $R(1-R^\kappa) < |w| < R$, the bound is

$$\left\|J\nabla J - J'\nabla J'\right\|_{L^p(u^{-1}(R(1-R^\kappa)<|w|<R))} \leq \|J\nabla J\|_{L^\infty}\left(2 + \frac{1}{\ln(1-R^\kappa)}\right)(kR^{2+\kappa}(2-R^\kappa))^{1/p}.$$

If $R^{2+\kappa(1-p)} \longrightarrow 0$ (*e.g.* if $\kappa = \frac{1}{p-1}$ and $R \longrightarrow 0$), the operators associated to J and J' will be as close as needed.

There remains to check that the assumption on the lower bound on the differential holds for u^0, u^1 and u^r. For u^0 and u^1 it follows from the fact that a^0 and a^1 are not trivial. As for u^r, it is a consequence of their linear independence over $\mathbb{C}$: let $\mu = \min_z(|a^0 + za^1|, |a^0 z + a^1|)$, then, on A_{r,r^α}, $du^r = du^0 - \frac{r^2}{z^2} du^1$ has norm bounded from below by μ. The cutoff function β is not of importance since it is always multiplied by one of these linearly independent factors (in $u^h \nabla\beta$ or in βdu^h). Thus if the first order terms in local expansions are dominant, the same bounds hold on A_{r^α, r^γ}. In short, we have proved the following lemma.

Lemma 3.4.3: $\exists r_5(a^0, a^1, \nabla^2 u^h), d(a^0, a^1)$ *such that* $\forall z \in \{z \,|\, |z| \leq r_5, |du^h(z)| \geq d$, *and* $\forall r, z$ *satisfying* $\max(|z|, \left|\frac{r^2}{z}\right|) \leq r_5$, $|du^r(z)| \geq d$.

This section is summarized in the next proposition.

Proposition 3.4.4: *Let* u^0 *and* u^1 *be as in the hypothesis of theorem 3.1.4. Let* J' *be the almost complex structure on* M *which is constant on a neighborhood* $B_R(m_0)$ *of* m_0 *defined in* (3.4.2). $\forall u$ *such that* $z \in B_\rho \Rightarrow |du(z)| \geq d$, $\exists c_7(\|du\|_{L^\infty}, \|\nabla J\|_{L^\infty}, \|J\nabla J\|_{L^\infty}, d)$ *which makes the following true:* $\|D^J_u - D^{J'}_u\| \leq c_7 R^{[2+\kappa(1-p)]/p}$.

In particular, the curves u^0, u^1, *and all the* u^r *(as defined in* (3.1.10)*) for* $r < r_5$ *satisfy this condition, for the same constant* c_7 *since their differential is uniformly bounded when* $r < \min(r_6, r_7)$.

In order to justify the assumption that J was constant in a neighborhood of m_0 made in §3.3, it suffices to construct this J'. In the end, the inverse of $D^{J'}_{u^r}$ obtained will be an approximate inverse to $D^J_{u^r}$. The independence of J' with respect to r is crucial for this new structure to be usable.

3.5 Realising the sum

This section presents the proof of theorem 3.1.4; it is a matter of setting up the situation so that theorem 3.1.6 can be applied. First, by hypothesis we are given two curves u^0 and u^1 whose tangents at 0 are linearly independent (over $\mathbb{C}$) and the linearised operators D^J_u are surjective of bounded right inverses. Let $p \in]2,4[$, let u^r be the family of maps introduced in (3.1.10), with paramaters $\alpha = \frac{2}{3}$ and $\gamma = \frac{5}{6}$ (as specified in remark 3.2.5). Thanks to lemma 3.2.3, if $r < r_1 = e^{-6}$

$$\|du^r\|_{L^p} \leq \|du^0\|_{L^p} + \|du^1\|_{L^p} + c_1 r^{2/p}.$$

On the other hand, when $r < r_2(\nabla^2 u^h, C\nabla J)$, lemma 3.2.4 states that

$$\|\bar{\partial}_J u^r\| \leq c_2 r^{1+\varepsilon}.$$

Before we can invoke the implicit function theorem 3.1.6, we must show that there is a bounded (independently of r) tight inverse to D_{u^r}. Two uniform bounds (for r sufficiently small) are obtained by lemmas 3.4.1 and 3.4.3: the first gives an upper bound to $|du^r|$ when $r < r_6$, the second gives a lower bound to the differentials when $r < r_7$ and $r^2/r_7 < |z| < r_7$. Next, when R is small enough so that $B_R(m_0) \cap \partial u^h(B_\rho(0)) = \varnothing$, the operators $D^J_{u^0}$, $D^J_{u^1}$ and $D^J_{u^r}$ are arbitrarily close (by choosing R arbitrarily small) from the one defined by the structure J' of (3.4.2). The difference between these operators is uniform for all choices of parameter r smaller than r_6, r_7 and ρ.

Thus, $D^J_{u^0}$ and $D^J_{u^1}$ have bounded right inverses and so do $D^{J'}_{u^0}$ and $D^{J'}_{u^1}$. From these inverses we construct in section 3.3 and under the assumption that $r < r_5(s_p, s_H, c_4)$, an inverse to $D^{J'}_{u^r}$. The dependence of c_4 on J' will not be fatal since a choice of a smaller r does not increase the difference between $D^J_{u^r}$ and $D^{J'}_{u^r}$. The bounded inverse of the second gives a bounded inverse for the first.

Implicit function theorem can now be used by choosing $\xi_0 = 0$ and $u = u^r$. The result is the J-holomorphic curve $\exp_u \xi$, where

$$\|\xi\|_{W^{1,p}} < cr^{1+\varepsilon}.$$

In particular, thanks to Sobolev imbedding, the sup norm of ξ is bounded, and consequently the difference between the holomorphic map obtained by perturbation of u^r and u^r itself will be of the order of $r^{1+\varepsilon}$.

Remark 3.5.1*:* Let $\varepsilon \in]0, 1/3[$, let $0 < \rho_1 < \rho_2 < r_0(\varepsilon)$, let h^{ρ_1} and h^{ρ_2} be curves obtained by theorem 3.1.4, *i.e.* by applying 3.1.6 to the maps u^{ρ_1} and u^{ρ_2}. It is possible to show that, for a $K \in \mathbb{R}$, if $\rho_1 < \rho_2(1 - K\rho_2^\varepsilon)$ these two curves are at a positive Hausdorff distance. On one hand, theorem 3.1.4 states that the Hausdorff distance from h^{ρ_i} to u^{ρ_i} is bounded by $O(\rho_i^{1+\varepsilon})$. On the other hand, the distance from u^{ρ_1} to u^{ρ_2} is at least $K'(\rho_2 - \rho_1)$.

Furthermore, the implicit function theorem A.4.3 indicates that the dependance of h_ρ on u^ρ is continuous. Thus, there exists a ρ_0 such that the u^ρ for $\rho < \rho_0$ realize all possible strangling.

Chapter 4

Chains of curves

The present chapter is concerned with another alteration of the methods in [35]. It consists in gluing (under certain assumptions) an infinite number of J-holomorphic curves in order to obtain a J-holomorphic cylinder. Although the method applies to more general situations, we shall content ourselves with the following setting : assume three J-holomorphic curves intersect at three points, then there is a J-holomorphic cylinder that curls up around those curves. This construction will be altered again in chapter 5.

4.1 Cylinder and $\ell^\infty(L^p)$ norms

The way the infinite number of gluings will be made is of course important. Let $\Sigma_i = \mathbb{C}\mathrm{P}^1$ be compact Riemann surfaces, let $z_{i;0}$ and $z_{i;\infty} \in \Sigma_i$ be two marked points on each surface, and let $u^i : \Sigma_i \longrightarrow M$ be J-holomorphic maps (for $i \in \mathbb{Z}$) such that $\forall i \in \mathbb{Z}, u^i(z_{i;0}) = u^{i-1}(z_{i-1;\infty})$. Finally, let $\Sigma = \mathbb{R} \times S^1 = \mathbb{C}/2\pi\mathbb{Z}$ be the J-holomorphic cylinder. This chapter will construct a J-holomorphic map $u^{(r_i)} : \Sigma \longrightarrow M$ which is arbitrarily close to the u^i, that is $u^{(r_i)}$ restricted to $[i, i+1] \times S^1$ is close to u^i when $\sup\{r_i\} \longrightarrow 0$.

The space $\Sigma = \mathbb{R} \times S^1 = \mathbb{C}/2\pi\mathbb{Z}$ will be given a peculiar metric so that each segment $[i, i+1]$ resembles a sphere with two discs removed (see Figure 4.1). Let $(r_i) \in \ell^\infty(\mathbb{Z}; \mathbb{R}_{>0})$. Let $g_{(r_i)}$ be a family of metrics defined as follows. Let $i \in \mathbb{Z}$, then $g_{(r_i)}$ is the metric induced by the map $\mu_{i;r_i,r_{i+1}}$ which embeds $[i, i+1] \times S^1$ into the compact Riemann surface Σ_i with the two discs $B_{r_i}(z_{i;0})$ and $B_{r_{i+1}}(z_{i+1;\infty})$ removed.

The volume of such a space is infinite. Thus, L^p norms are not expeceted to behave nicely. However a slight alteration will do. Let us consider the sup of the L^p norms on annuli around each circle $\{i\} \times S^1$. Let V be a vector bundle over Σ (with a connection and a norm) and let $\xi : \Sigma \longrightarrow V$ be a section, define

$$
\begin{aligned}
||\xi||_{\ell^\infty(L^p)} &= \sup_{n \in \mathbb{Z}} ||\xi||_{L^p([n-\frac{2}{3}, n+\frac{2}{3}] \times S^1)}, \\
||\xi||_{\ell^\infty(W^{1,p})} &= \sup_{n \in \mathbb{Z}} ||\xi||_{W^{1,p}([n-\frac{2}{3}, n+\frac{2}{3}] \times S^1)}.
\end{aligned}
\tag{4.1.1}
$$

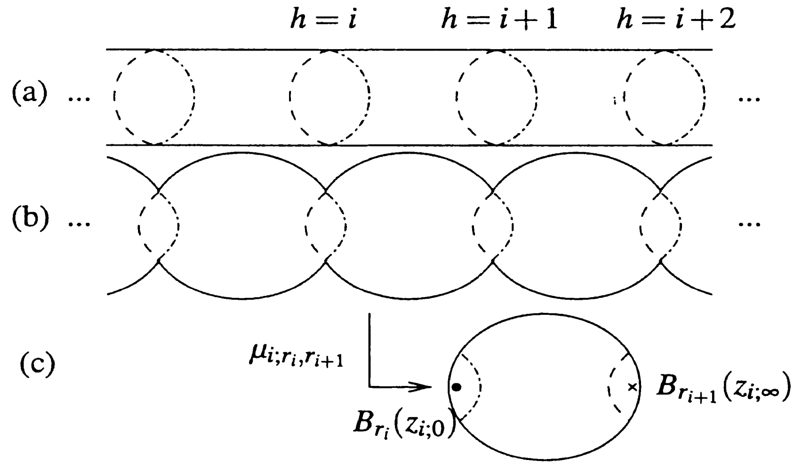


Figure 4.1: (a) the cylinder Σ endowed with its usual metric and the circles $h \in \mathbb{Z}$, (b) Σ endowed with $g_{(r_i)}$, (c) the map $\mu_{i;r_i,r_{i+1}}$ to Σ_i with the discs around $z_{i;0}$ and $z_{i;\infty}$ removed.

A proof identical to that of lemma 3.2.1, allows us to deduce that Sobolev embedding holds with a constant which does not depend on the parameters r_i.

Lemma 4.1.2: *Suppose $d_{\Sigma_i}(z_{i;0}, z_{i;\infty}) \geq c_0$. Given that $\sup_{i\in\mathbb{Z}} r_i < \frac{c_0}{3}$, there exists a constant $s'_p \in \mathbb{R}_{>0}$ such that*

$$s'_p := \sup_{0\neq f\in C^\infty(\Sigma)} \frac{\|f\|_{L^\infty}}{\|f\|_{\ell^\infty(W^{1,p})}}.$$

Proof. Each function can be decomposed as a sequence of functions on $\Sigma_i = \mathbb{C}\mathrm{P}^1$ with $B_{r_n}(z_{n;0})$ and $B_{r_{n+1}}(z_{n+1;\infty})$ removed. The esimates follow from the fact that a ball with the Fubini-Study metric and one (or a fixed finite number of discs) removed has a Sobolev constant that remains bounded as the radius of the discs tends to 0. See [35] for details. □

The main result that will allow us to conclude is an adaptation of theorem 3.1.6 to these norms.

Proposition 4.1.3: *Let Σ be one of the 1-dimensional non-compact complex manifolds described above. Let $p > 2$. $\forall c_0, \exists \delta > 0$ such that for all volume forms dvol_Σ on Σ induced as above by the maps $\mu_{i;r_i,r_{i+1}}$, all continuous map u such that $du \in \ell^\infty(L^p)(\mathrm{T}\Sigma, u^*\mathrm{T}M)$, all $\xi_0 \in \ell^\infty(W^{1,p})(\Sigma, u^*\mathrm{T}M)$, and all $Q_u : \ell^\infty(L^p)(\Sigma, \Lambda^{0,1} \otimes_J u^*\mathrm{T}M) \to \ell^\infty(W^{1,p})(\Sigma, u^*\mathrm{T}M)$ satisfying*

$$s'_p(\mathrm{dvol}_\Sigma) \leq c_0, \qquad \|du\|_{\ell^\infty(L^p)} \leq c_0, \quad \|\xi_0\|_{\ell^\infty(W^{1,p})} \leq \tfrac{\delta}{8},$$
$$\|\bar\partial_J(\exp_u(\xi_0))\|_{\ell^\infty(L^p)} \leq \tfrac{\delta}{4c_0}, \qquad D_u Q_u = \mathbb{1}, \qquad \|Q_u\| \leq c_0,$$

there exists a unique ξ such that

$$\bar\partial_J(\exp_u(\xi_0+\xi)) = 0, \quad \|\xi+\xi_0\|_{\ell^\infty(W^{1,p})} \leq \delta, \quad \|\xi\|_{\ell^\infty(W^{1,p})} \leq 2c_0\|\bar\partial_J(\exp_u(\xi_0))\|_{\ell^\infty(L^p)}.$$

The proof will require the implicit function theorem in Banach spaces, as does the proof of theorem 3.1.6; it will be invoked again in chapter 5.

Let K_1 be a constant such that $\forall x \in M, \forall \xi \in \mathrm{T}_x M$ such that $|\xi| \leq c_0$, and $\forall \eta \in \mathrm{T}_x M$,

$$\|E_x(\xi)\| \leq c_1 \text{ and } \left|\Psi_x(\xi;\xi',\eta)\right| \leq K_1 \, |\xi| \, |\xi'| \, |\eta| \, ,$$

where $\|\cdot\|$ is the norm on $\mathcal{L}(\mathrm{T}_x M, \mathrm{T}_{\exp_x \xi} M)$. The second equality holds since Ψ_x is zero at $\xi = 0$. We conclude that

$$\left|\Phi_u(\xi)^{-1}\Psi_u(\xi;\xi',\mathcal{F}_u(\xi))\right| \leq K_1 \, \|d\exp_u(\xi)\| \, |\xi| \, |\xi'| \, ,$$

$\exp_u(\xi)$ being viewed as a function $\mathbb{C}\mathrm{P}^1 \to M$. Since, by assumption, ξ and du are bounded, there exists a real number K_2 which depends on the metric on M and on c_0, such that $\|d\exp_u \xi\| \leq K_2(|du| + |\nabla\xi|)$. Thus

$$\left\|\Phi_u(\xi)^{-1}\Psi_u(\xi;\xi',\mathcal{F}_u(\xi))\right\|_{\ell^\infty(L^p)} \leq K_1 K_2 (\|du\|_{\ell^\infty(L^p)} + \|\nabla\xi\|_{\ell^\infty(L^p)}) \, \|\xi\|_{L^\infty} \, \|\xi'\|_{L^\infty} \, .$$

Since $\|du\|_{\ell^\infty(L^p)} \leq c_0$, $\|\xi\|_{L^\infty} \leq c_0$ and $\|\xi\|_{L^\infty} \leq c_0 \, \|\xi\|_{\ell^\infty(W^{1,p})}$, taking $K_3 = (1+c_0)c_0^2 K_1 K_2$ yields,

$$\left\|\Phi_u(\xi)^{-1}\Psi_u(\xi;\xi',\mathcal{F}_u(\xi))\right\|_{\ell^\infty(L^p)} \leq K_3 \, \|\xi\|_{\ell^\infty(W^{1,p})} \, \|\xi'\|_{\ell^\infty(W^{1,p})} \, .$$

It remains find an estimate for $\Phi_u(\xi)^{-1} D_{\exp_u \xi}(E_u(\xi)\xi')$. Let $u_\xi := \exp_u \xi$. The starting point of the bound is the equality

$$\begin{aligned} D_{u_\xi}(E_u(\xi)\xi') - \Phi_u(\xi) D_u \xi' &= \Big(\nabla(E_u(\xi)\xi') - \Phi_u(\xi)\nabla\xi'\Big)^{0,1} \\ &\quad -\tfrac{1}{2} J(u_\xi)\Big((\nabla_{E_u(\xi)\xi'} J)(u_\xi) du_\xi - \Phi_u(\xi)(\nabla_{\xi'} J)(u) du\Big)^{0,1} . \end{aligned}$$

Each term on the right-hand side will be bounded separately. For the second, we have a the following bound (J is an isometry of $\mathrm{T}M$):

$$\begin{aligned} \left|(\nabla_{E_u(\xi)\xi'} J)(u_\xi) du_\xi - \Phi_u(\xi)(\nabla_{\xi'} J)(u) du\right| &\leq \left|(\nabla_{E_u(\xi)\xi'} J)(u_\xi) du_\xi - (\nabla_{E_u(\xi)\xi'} J)(u_\xi)\Phi_u(\xi) du\right| \\ &\quad \left|(\nabla_{E_u(\xi)\xi'} J)(u_\xi)\Phi_u(\xi) du - \Phi_u(\xi)(\nabla_{\xi'} J)(u) du\right| \\ &\leq K_1 \, \|\nabla J\|_{L^\infty} \, |\xi'| \, \left|du_\xi - \Phi_u(\xi) du\right| + K_4 \, |du| \, |\xi| \, |\xi'| \\ &\leq K_5(|du| \, |\xi| + |\nabla\xi|) \, |\xi'| \, , \end{aligned}$$

where K_5 depends only on c_0 and on the metric on M. Whereas, for the first term, we obtain

$$\begin{aligned} |\nabla(E_u(\xi)\xi') - \Phi_u(\xi)\nabla\xi'| &\leq |\nabla(E_u(\xi)\xi') - E_u(\xi)\nabla\xi'| + \leq |E_u(\xi)\nabla\xi' - \Phi_u(\xi)\nabla\xi'| \\ &\leq K_6(|\xi| \, |\nabla\xi'| + |du| \, |\xi| \, |\xi'| + |\nabla\xi| \, |\xi'|). \end{aligned}$$

Putting these two bounds together shows

$$\left|D_{u_\xi}(E_u(\xi)\xi') - \Phi_u(\xi) D_u \xi'\right| \leq (K_5 + K_6)(|\xi| \, |\nabla\xi'| + |du| \, |\xi| \, |\xi'| + |\nabla\xi| \, |\xi'|).$$

Taking the $\ell^\infty(L^p)$ norm on the terms in $\nabla\xi$, $\nabla\xi'$ and du, and the L^∞ norm on ξ and ξ', we get

$$\left\|\Phi_u(\xi)^{-1} D_{\exp_u \xi}(E_u(\xi)\xi')\right\|_{\ell^\infty(L^p)} \leq K_7 \, \|\xi\|_{\ell^\infty(W^{1,p})} \, \|\xi'\|_{\ell^\infty(W^{1,p})} \, ,$$

where K_7 depends on c_0, the metric on M, J and j. □

Proof of proposition 4.1.3: (*cf.* [35, p.69]) By hypothesis, D_u has a bounded right inverse Q_u ($\|Q_u\| \leq c_0$). Let c_1 be the constant from lemma 4.1.6, and let $\delta \in]0,1[$ be such that $c_1\delta < 1/2c_0$. Then lemma 4.1.6 insures that $\|d\mathcal{F}_u(\xi) - D_u\| \leq 1/2c_0$ if $\|\xi\| \leq \delta$. The assumptions of theorem A.4.4 are consequently satisfied (with $X = \mathcal{X}_u$, $Y = \mathcal{Y}_u$, $f = \mathcal{F}_u$, $x_0 = 0$, $c_0 = c$ and the same δ). □

4.2 Transversality and right inverse

Before the theorem can be put to good use, it is better to check that the surjectivity of linearized operators holds in a reasonnable class of spaces. The discussion will be similar to that of section §A.3. Recall that $\mathcal{M}^*(A^i,\Sigma_i;J)$ is the space of J-holomorphic maps $\Sigma_i \longrightarrow M$ that are somewhere injective and represent the homology class A^i in $H^2(M)$.

Definition 4.2.1: Let $\forall i \in \mathbb{Z}, A^i \in H^2(M,\mathbb{Z})$, let Σ_i be Riemann surfaces. The structure J will be said regular for $(A^i)_{i\in\mathbb{Z}}$ and $(\Sigma_i)_{i\in\mathbb{Z}}$ if $J \in \cap_{i\in\mathbb{Z}} J_{reg}(\Sigma_i,A^i)$ and if the evaluation map:

$$\begin{array}{rccc} ev_{\mathbb{Z}}: & \ell^\infty(\mathbb{Z};\mathcal{M}^*(A,\Sigma;J)) & \longrightarrow & \ell^\infty(\mathbb{Z};M\times M) \\ & (u^i)_{i\in\mathbb{Z}} & \mapsto & (u^i(z_{i;\infty}),u^{i+1}(z_{i+1;0}))_{i\in\mathbb{Z}} \end{array}$$

is transverse to $\Delta^{\mathbb{Z}} = \ell^\infty(\mathbb{Z};\Delta)$ where $\Delta = \{(m,m) \subset M\times M\}$. The set of structures satisfying these conditions will be written $\mathcal{J}_{reg}((\Sigma_i)_{i\in\mathbb{Z}},(A^i)_{i\in\mathbb{Z}})$ or more simply $\mathcal{J}_{reg}(\Sigma_i,A^i)$.

Although from our point of view the almost complex structure is given, it is wise to show that structures that are regular are abundant. As the intersection of a countable number of dense open subsets is still a set of the second category, to show that $\mathcal{J}_{reg}(\Sigma_i,A^i)$ is of the second category only requires the study of $dev_{\mathbb{Z}}$.

This would require an adaptation of theorem [35, th 6.3.1] (which insures transversality for curves glued according to a finite tree). $\mathbb{Z}$ can be seen as an infinite tree, and so the question can be asked in general for an infinite tree T of bounded degree. It might be tempting to proceed as follows: take an increasing sequence of finite subtrees of T, say $\{T_i\}$. Thanks to theorem [35, th 6.3.1] the set of structures for which the evaluation on the tree T_i is transversal is of the second category. The intersection of these sets should yield a set of the second category.

In what follows we shall suppose that the structure J on M is regular in the sense of definition 4.2.1. This hypothesis is not so strong, especially since, in the cases of interest, the u^i will be a periodic sequence of curves (*i.e.* $\exists n \in \mathbb{Z}_{>0}$ such that $u^i = u^j$ if $i \equiv j \bmod (n)$). Thus the *a priori* infinite condition of definition 4.2.1 are actually finite. Let us assume that each curve $u^i \in \mathcal{M}^*(A^i,\Sigma_i;J)$ is such that dev_0 (its evaluation at $0 \in \Sigma_i = \mathbb{C}\mathrm{P}^1$) is surjective. In other words, for each curve it is possible to choose an infinitesimal perturbation (which is also J-holomorphic) in such a way that this perturbation displaces $u^i(0)$ in any chosen direction (note that we do not make any assumption on the effect of this perturbation at $\infty \in \mathbb{C}\mathrm{P}^1$). Then the evaluation is surjective. Indeed, if we are given a infinitesimal displacement at each point of the gluing, making it equal to the difference between the displacement of $u^i(\infty)$ and of $u^{i+1}(0)$ amounts to solve n equations knowing that in each equation we can fix the value of a term (the one coming from the displacement of $u^i(0)$). Since it is a finite system (by periodicity) it is solvable. Whence the surjectivity of evaluation.

Finally, note that in $\mathbb{C}P^n$ endowed with its usual structure, these hypotheses hold (at least for some A^i) since between any two distinct points of $\mathbb{C}P^n$ there is a line (or a conic).

Let's define the moduli space

$$\mathcal{M}^*(A^i,\Sigma_i;J) = \{(u^i)_{i\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z};\mathcal{M}^*(A,\Sigma;J)) | \forall i \in \mathbb{Z}, u^i(z_{i;\infty}) = u^{i+1}(z_{i+1;0})\}.$$

It is not excluded that the dimension of this space migth be finite. For example, if almost all A^i have a trivial first Chern class, it might happen that the dimension of the modular space is $2n+2\sum c_1(A^i)$. In the present context, we will be interested in a subset of the moduli space when $\Sigma_i = \mathbb{C}P^1$:

$$\mathcal{M}^*_\mathbb{Z}(C) := \mathcal{M}^*(A^i;J;C) := \{(u^i) \in \mathcal{M}^*(A^i,\Sigma_i = \mathbb{C}P^1;J) | \|du^i\|_{L^\infty} \leq C, \forall i \in \mathbb{Z}\}.$$

What matters is that transversality of definition 4.2.1 implies surjectivity of the linearized operator even if it is restricted to vector fields who do not alter the intersection property. Recall that for $u^i : \Sigma_i \longrightarrow M$,

$$\begin{aligned} W^{1,p}_{u^i} &= W^{1,p}(\Sigma_i, u^{i*}\mathrm{T}M) \\ L^p_{u^i} &= L^p(\Sigma_i, \Lambda^{0,1}\mathrm{T}^*\Sigma_i \otimes_J u^{i*}\mathrm{T}M). \end{aligned}$$

Given $u^i : \Sigma_i \longrightarrow M$, such that $u^i(z_{i;\infty}) = u^{i+1}(z_{i+1;0})$, denote by

$$W^{1,p}_{u^\mathbb{Z}} := \left\{ (\xi^i)_{i\in\mathbb{Z}} \in \mathop{\times}_{i\in\mathbb{Z}} W^{1,p}_{u^i} | \xi^i(z_{i;\infty}) = \xi^{i+1}(z_{i+1;0}) \right\}.$$

(The evaluation of $W^{1,p}$ sections makes sense since $p > 2$.)

Lemma 4.2.2: *Suppose J is regular in the sense of definition 4.2.1,* i.e. *the operators $D_{u^i} : W^{1,p}_{u^i} \longrightarrow L^p_{u^i}$ are surjective and $ev_\mathbb{Z}$ is transverse, then the operator*

$$\begin{aligned} D_{u^\mathbb{Z}} : \quad W^{1,p}_{u^\mathbb{Z}} &\longrightarrow \mathop{\times}_{i\in\mathbb{Z}} L^p_{u^i} \\ (\xi^i)_{i\in\mathbb{Z}} &\longmapsto (D_{u^i}\xi^i)_{i\in\mathbb{Z}} \end{aligned}$$

is surjective.

Proof. Let $\eta^i \in L^p_{u^i}$ (where $i \in \mathbb{Z}$). Each of the D_{u^i} being surjective, there exist $\xi^i \in W^{1,p}_{u^i}$ such that $D_{u^i}\xi^i = \eta^i$. Since the evaluation is transverse to the diagonal, choose $\zeta^i \in \mathrm{T}_{u^i}\mathcal{M}^*(A^i;J)$ so that

$$dev_\mathbb{Z}((u^i)_{i\in\mathbb{Z}})((\zeta^i)_{i\in\mathbb{Z}}) = (\zeta^i(z_{i;\infty}), \zeta^{i+1}(z_{i+1;0}))_{i\in\mathbb{Z}} \in \mathop{\times}_{i\in\mathbb{Z}} \left((\xi^i(z_{i;\infty}), \xi^{i+1}(z_{i+1;0})) + \mathrm{T}_{(m_i,m_i)}\Delta \right)$$

where $ev_\mathbb{Z}$ is the map defined in 4.2.1, $m_i = u^i(z_{i;\infty}) = u^{i+1}(z_{i+1;0})$ and $\Delta \subset M \times M$ is the diagonal. Then $(\xi^i - \zeta^i)_{i\in\mathbb{Z}}$ is an element of $W^{1,p}_{u^\mathbb{Z}}$ whose image by $D_{u^\mathbb{Z}}$ is also $(\eta^i)_{i\in\mathbb{Z}}$. □

In order to use proposition 4.1.3 we have to describe an approximate solution and show that it has bounded right inverse. To do so, two choices are possible: either the method of [35, §10] or the one from chapter 3 (if the curves u^i are transversal at their point of intersection, so in particular $\dim_{\mathbb{R}} M \geq 4$). The second is of interest since by remark 3.5.1 it could allow to prescribe different characteristics of curves (strangling), an idea that will be used again in chapter 5. These two situations are dealt with in an identical fashion. The main point is to notice that in the constructions the approximate solutions differ from the initial curves only in a neighborhood of the points of intersection. Similarly, the approximate inverses only differ from a true inverse in those neighborhoods.

Suppose that we are trying to use the construction of chapter 3, some problem might arise from the fact that a subsequence of $\{m_i\}$ may be arbitrarily close. Modifying the almost complex structure J would then be a problem. To avoid this, it is again necessary to introduce structures which depend on a point of the domain. We will not detail this argument. Furthermore, the construction used a reparametrization of one of the curves; here is how what this means in the present context.

Suppose we are in the case where $z_{i;0} = [0:1] = 0$ and $z_{i;\infty} = [1:0] = \infty \in \Sigma_i = \mathbb{C}P^1$. Let $\phi_r : \mathbb{C}P^1 \to \mathbb{C}P^1$ be defined by $\phi_r(z) = r^2/z$. Then, the condition of intersection is $u^i \circ \phi_1(0) = u^{i+1}(0) = m_i$, and the local expansion in a chart $\psi_i : M \to \mathbb{C}^m$ which sends m_i to 0 and such that $(\psi_i^* J)(0) = i$ is

$$\psi_i \circ u^{i+1}[z:1] = a_{i+1;0} z + O(|z|^2) \quad \text{and} \quad \psi_i \circ u^i[1:z] = \psi_i \circ u^i \circ \phi_1[z:1] = a_{i;\infty} z + O(|z|^2).$$

The ring $A_{r_{i+1}^{4/3}, r_{i+1}^{2/3}}$ corresponds on Σ to the $z \in [i, i+1] \times S^1$ such that $\phi_1 \circ \mu_{i;r_i,r_{i+1}}(z) < r_{i+1}^{2/3}$ and to the $z \in [i+1, i+2] \times S^1$ such that $\mu_{i+1;r_{i+1},r_{i+2}}(z) < r_{i+1}^{2/3}$.

The arguments used for the L^p norm will be adapted without pain to the $\ell^\infty(L^p)$ context: in this norm there is at most one gluing to consider at a time. It suffices to check that the curves u^i and the points $m_i = u^i(z_{i;\infty}) = u^{i+1}(z_{i+1;0})$ belong to a compact family.

We now transpose the methods of §3.3 to conclude.

Proof of theorem 4.1.5: With a small deformation the J-holomorphic curves u^i can be modified into maps $u^{i;r_i,r_{i+1}}$; this deformation is identical to the one which changes u^0 into $u^{0,r}$ except it takes places at two points of $\mathbb{C}P^1$. Thus, the operators D_{u^i}, and their inverses, are close to $D_{u^{i;r_i,r_{i+1}}}$. The opertor $D_{u^{\mathbb{Z}}}$ also is also close (in the norm of linear maps $\ell^\infty(W^{1,p}) \to \ell^\infty(L^p)$) to an operator $D_{u^{\mathbb{Z};(r_i)}}$ where the $D_{u^{i;r_i,r_{i+1}}}$ take place of the D_{u^i}. It is also surjective and their inverses are close.

We describe the map $u^{(r_i)} : \Sigma \to M$ (we will write $u = u^{(r_i)}$ for short in this paragraph) which will be an approximate solution, in the sense that $\|\bar{\partial}_J u\|_{\ell^\infty(L^p)}$ is small, and that D_u will have a bounded right inverse. It will be defined by composing the maps $\mu_{i;r_i,r_{i+1}} : [i, i+1] \times S^1 \to \Sigma_i$ with the maps $u^{i;r_i,r_{i+1}} : \Sigma_i \to M$. Then a $(0,1)$-form, say η, along u can be cut in pieces to give rise to η^i along each $u^{i;r_i,r_{i+1}}$ (by extending by 0, that is in an analogous way as (3.3.3) where η^0 and η^1 were obtained from η). From these η^i, the inverse of $D_{u^{\mathbb{Z};(r_i)}}$ will give vector fields along the

$u^{i;r_i,r_{i+1}}$, say ξ^i. These vector fields can be glued by a surgery (which copies the definition of ξ^r in (3.3.2)) to get a vector field $\xi^{(r_i)}$ along u. The computation made in (3.3.4) still works out in an identical fashion at each point of intersection. By definition of the $\ell^\infty(L^p)$ norm, this construction produces an approximate inverse to D_u. By the technique used in (3.3.6), a true bounded inverse is then found. This allows the use of proposition 4.1.3 and finishes the proof. $\square$

Chapter 5

Spaces of pseudo-holomorphic maps

In this chapter, we give an example of a space of pseudo-holomorphic maps which is of positive mean dimension. As before the cylinder will be noted $\Sigma = \mathbb{R} \times S^1 = \mathbb{C}/2\pi\mathbb{Z}$. We will assume that the of curves u^i is of finite type (periodic), in the sense that only a finite number of distinct maps are described as i runs over $\mathbb{Z}$. The theorem 5.1.3 could also be proven using the gluing introduced in [35]. However, in order to prove proposition 5.3.4, it is necessary to have approximate solutions which are injective (with a discrete set of exceptions). This is incompatible with an approximate solution which is constant on a whole ring.

5.1 Mean dimension and parametrization

In order to speak of mean dimension, we must first state the action of the group. On spaces of pseudo-holomorphic maps $\Sigma \to M$, the group of automorphisms of Σ acts naturally. The action is given by reparametrization at the source. For compact Riemann surfaces this does not present much interest. However the automorphisms of the plane or the cylinder form a (continuous) amenable group. Since chapter 4 establishes gluing to get maps from the cylinder, we will focus on that case. Let $\mathcal{M}_\Sigma$ be the space of J-holomorphic maps from the cylinder to (M,J). One can then ask if it is of positive mean dimension.

This question requires some precision. The first is that the group of automorphisms of the cylinder is continuous, so the mean dimnesion is defined according to definition 2.6.2 and not 2.1.4 (which was restricted to countable groups). The second concerns the metric to be used on $\mathcal{M}_\Sigma$. The topology of uniform convergence on compacts makes the space compact. Thus, for $u, u' : \Sigma \to M$, we will use the distance

$$d(u,u') = \sup_{k \in \mathbb{Z}_{>0}} 2^{-k} \sup_{z \in [-k,k] \times S^1} d_M(u(z), u'(z)) \tag{5.1.1}$$

which induces an equivalent topology.

This understood, the question possesses two simple affirmative answers. First, suppose there exists a pseudo-holomorphic $u : \mathbb{C}\mathrm{P}^1 \to M$ or $u' : \Sigma \to M$. Then one can precompose u or u'

by holomorphic maps $\Sigma \to \mathbb{C}\mathrm{P}^1$ or $\Sigma \to \Sigma$. This would suffice to generate a family of pseudo-holomorphic which is sufficiently big. However, they would all have their image contained in the image of u or u'.

Furhtermore, we could be tempted to use directly theorem 4.1.5. To do so, suppose there are J-holomorphic maps $u^i : \mathbb{C}\mathrm{P}^1 \to M$ where $i = 1, \dots, N$ such that u^i has a point of intersection with u^j if $j \equiv i \pm 1 \bmod N$. Being finite, this family gives rise to $\{u^i\}_{i\in\mathbb{Z}}$ which satisfies the hypothesis of theorem 4.1.5 by defining $u^k = u^i$ when $k \equiv i \bmod N$ (theorem 4.1.5 will again be used on such a finite family of curves). Before gluing those curves $\{u^i\}$ in a cylinder, it is however possible to precompose them by an automorphism fixing the two points which links u^i to its neighbour in the chain, u^{i-1} and u^{i+1}. Let θ_i be these automorphisms fixing $z_{i;0}$ and $z_{i;\infty}$, the maps $\{u^i \circ \theta_i\}$ are another family of maps satisfying the hypothesis of theorem 4.1.5. By taking all possible θ_i this will give rise to a familly of positive mean dimension. It is unfortunately hard to show that the members of this family have disctinct images.

So the question of whether the images of pseudo-holomorphic maps from the cylinder to M is rich remains complete. Indeed, all the curves obtained above could have the same image. It would be tempting to introduce a distance on $\mathcal{M}_\Sigma$, but then the action of the group of automorphisms would be trivial on such a distance. Consequently, under hypothesis to be precised, we shall prove that there exists a family of positive mean dimension (and infinite usual dimension), and that if two curves obtained by this process have the same image they differ only by an automorphism.

Here are the hypothesis we shall need. As in the above discussion, we will assume that there exists a finite family $u^i : \mathbb{C}\mathrm{P}^1 \to M$ of J-holomorphic curves where $i \in \{1, \dots, N\}$ such that $u^i(\infty) = u^j(0)$ when $j \equiv i+1 \bmod N$. In order to be able to glue them, the evaluation map $ev_\mathbb{Z}$ will be assumed transversal as in 4.2.1. Furthermore, we will suppose one of these curves is deformable.

More precisely, for a fixed $j \in \{1, 2, \dots N\}$, there exists a point $z_* \in \mathbb{C}\mathrm{P}^1$ and a family of $W^{1,p}$ vector fields along u^j which belong to the kernel of D_{u^j} and such that the map defined by $\xi \mapsto \exp_{u^j(z_*)} \xi(z_*)$ is surjective on a neighborhood of $m_* = u^j(z_*)$. Thus, to $x \in \mathrm{T}_{m_*}M$ we will associate the vector field $X_x \in \mathrm{Ker}\, D_{u^j} \subset W^{1,p}(\mathbb{C}\mathrm{P}^1, (u^j)^*\mathrm{T}M)$ such that $X_x(z_*) = x$. In other words, we need to make the hypothesis that the differential of the map given by evaluation at z_* is surjective on the kernel of D_{u^j}.

Remark 5.1.2*:* Remark that this hypothesis is close the transversality of the evaluation map in definition 4.2.1. Let $ev_{j+N\mathbb{Z}}$ be the evaluation at (u^i) in z_* when $i \equiv j \bmod N$. To ask that

$$D_{u^\mathbb{Z}} \oplus \mathrm{d}ev_{j+N\mathbb{Z}} : \underset{i\in\mathbb{Z}}{\times} W^{1,p}_{u^i} \to (\underset{i\in\mathbb{Z}}{\times} L^p_{u^i}) \oplus (\underset{i\in\mathbb{Z}}{\times} \mathrm{T}_{u^{j+Ni}(z_*)}M)$$

is surjective is, given that J is already regular in the sense of definition 4.2.1, equivalent to ask that the restriction of $\mathrm{d}ev_{u^{j+Ni}}$ to the subspace $\mathrm{Ker}\, D_{u^{j+Ni}}$ be surjective. This condition is naturally expressed in the vocabulary of transversality. Indeed, if in the construction of chapter 4, it is required, in addition to the gluings between the curves in the chain, to glue another curve at a point

z_* (*e.g.* a constant curve), then regularity for this gluing scheme (which obtained from $\mathbb{Z}$ as a tree, by adding a leaf to the integers $j+N\mathbb{Z}$) implies the surjectivity of $dev_{u^{j+Ni}}|_{\mathrm{Ker}D_{u^{j+Ni}}}$. This way of stating the hypothesis indicates that it is not significantly stronger the one made in the preceding chapter, particularly for a finite family of curves. For example, it holds for a finite number of curves with appropriate intersection in $\mathbb{C}\mathrm{P}^n$ with its usual complex structure.

Finally, in order to remain in the setting of a compact family of maps $\{u^i\}$, it will be necessary to restrict to a sufficiently small ball $B_{m_*} \subset \mathrm{T}_{m_*}M$ such that for all $x \in B_{m_*}$ the curves $\exp_{u^j} X_x$ form a compact family.

Theorem 5.1.3: *Let (M,J) be an almost-complex manifold. Let $u^1,\dots,u^N$ be a finite family of J-holomorphic curves $u^i : \mathbb{C}\mathrm{P}^1 \longrightarrow M$ such that $u^i(\infty) = u^j(0)$ when $j \equiv i+1 \bmod N$. Suppose that J is regular in the sense of definition 4.2.1 and that u^j is deformable. Let $z_* \in \mathbb{C}\mathrm{P}^1 \setminus \{0,\infty\}$ be a marked point and let $m_* = u^j(z_*) \in M$ be its image. Suppose that the curves are uniformly transverse. There exists c and $R \in \mathbb{R}_{>}0$ such that for any sequence $\{r_i\}$ satisfying $r_{\sup} = \sup r_i \leq R$, the exists a neighborhood V_{m_*} of m_* such that for all sequence of points $\{m_k\}_{k\in\mathbb{Z}}$ in V_{m_*} there exists a J-holomorphic cylinder $u : \Sigma \longrightarrow M$ satisfying the following properties:*

R1 - u passes by the prescribed points, i.e.

$$u(z_{k,*}) = m_k,$$

where $z_{k,*} = \mu_{j+Nk;r_{j+Nk},r_{j+Nk+1}}(z_*)$,

R2 - the behaviour of u near the gluing points is as follows:

$$\begin{aligned} &\forall z \in (\phi_1 \circ \mu_{i;r_i,r_{i+1}})^{-1}\{A_{r_{i+1},r_{i+1}^{2/3}}(\infty)\}, && \psi_i \circ u \circ \phi_1 \circ \mu_{i;r_i,r_{i+1}}(z) = a_{i;\infty} z + a_{i+1;0}\frac{r_{i+1}^2}{z} + O(r_{\sup}^{1+\varepsilon}) \\ \text{and } &\forall z \in \mu_{i+1;r_{i+1},r_{i+2}}^{-1}\{A_{r_{i+1},r_{i+1}^{2/3}}(0)\}, && \psi_i \circ u \circ \mu_{i+1;r_{i+1},r_{i+2}}(z) = a_{i+1;0} z + a_{i;\infty}\frac{r_{i+1}^2}{z} + O(r_{\sup}^{1+\varepsilon}) \end{aligned}$$

where $A_{r_1,r_2}(z_0) = B_{r_2}(z_0) \setminus B_{r_1}(z_0)$, and $\psi_i : M \longrightarrow \mathbb{C}^m$ is a local chart that maps m_i to 0 and such that $(\psi_i^* J)(0) = J_0$.

R3 - u is close to the curves u^i (or $\exp_{u^j} X$ if it is the deformable curve):

$$\forall z \in \mu_{i;r_i,r_{i+1}}^{-1}(\Sigma_i), \quad d_M(u^{(r_i)}(z), u^i(\mu_{i;r_i,r_{i+1}}(z))) \leq c(r_{\sup}^{1+\varepsilon} + \delta_i)$$

where $\delta_i = d_M(m_k, m_*)$ if $i = j+kN$ and $\delta_i = 0$ else.

(The maps $\mu_{i;r_i,r_{i+1}}$ are the same as the one introduced in §4.1.)

5.2 Local inversion

As before we are trying to find a solution to an equation containing a non linear term by an implicit function theorem. However, in addition to $\bar{\partial}_J u = 0$, we have to satisfy a sequence of punctual constraints. As we shall see these will not have a significative impact on the argments. In order to describe the situation, note $z_{k;*}$ the marked points on the cylinder Σ (they will be chosen to be equal to $\mu_{j+Nk;r_{j+Nk},r_{j+Nk+1}}(z_*)$ later on), and let $ev_* : \mathcal{M}_\Sigma \longrightarrow \ell^\infty(\mathbb{Z};M)$ be the evaluation map at these points $z_{k;*}$. Even if ev_* takes value in M, we are in a situation where only the curve u and its perturbation by a vector field will intervene.

Since we need the vector fields to be of ℓ^∞ norm smaller than the injectivity radius so that the evaluation of $\exp_u \xi$ makes sense, it is better to see the target space of ev_* as a product of balls in the tangent plane. Let $\mathrm{T}_{\mathbb{Z};*}M = \underset{i\in\mathbb{Z}}{\times} \mathrm{T}_{u(z_{i;*})}M$, the elements $w \in \mathrm{T}_{\mathbb{Z};*}M$ will sometime be written as $w = (w_i)_{i\in\mathbb{Z}}$.

This understood, ev_* defined in a neighborhood of u takes values in $\mathrm{T}_{\mathbb{Z};*}M$. This is actually a linear map if we look at the neighborhood of u as given by vector fields along u. The equations to solve are $\bar{\partial}_J u = 0$ and $ev_* u = w$ for some $w \in \mathrm{T}_{\mathbb{Z};*}M$. This said, it remains to use theorem A.4.4.

Proposition 5.2.1: *Let Σ be the cylinder and let $p > 2$. $\forall c_0, \exists \delta > 0$ such that for any volume form* dvol_Σ *on Σ induced by the $\mu_{i;r_i,r_{i+1}}$ (cf. Figure 4.1), any continuous map u and such that $du \in \ell^\infty(L^p)(\mathrm{T}\Sigma, u^*\mathrm{T}M)$, all $\xi_0 \in \ell^\infty(W^{1,p})(\Sigma, u^*\mathrm{T}M)$, and all $T_u : \ell^\infty(L^p)(\Sigma, \Lambda^{0,1} \otimes_J u^*\mathrm{T}M) \oplus \mathrm{T}_{\mathbb{Z},*}M \longrightarrow \ell^\infty(W^{1,p})(\Sigma, u^*\mathrm{T}M) \oplus \mathrm{T}_{\mathbb{Z};*}M$ satisfying*

$$\begin{gathered} s'_p(\mathrm{dvol}_\Sigma) \le c_0, \quad \|du\|_{\ell^\infty(L^p)} \le c_0, \quad \|\xi_0\|_{\ell^\infty(W^{1,p})} \le \tfrac{\delta}{8}, \\ D_u T_u = \mathbb{1}, \qquad\qquad \|T_u\| \le c_0, \\ \|\bar{\partial}_J(\exp_u(\xi_0))\|_{\ell^\infty(L^p)} \le \tfrac{\delta}{8c_0}, \quad \|\xi_0(z_{k;*}) - w_k\|_{\ell^\infty} \le \tfrac{\delta}{8c_0} \end{gathered}$$

there exists a unique ξ such that

$$\begin{gathered} \bar{\partial}_J(\exp_u(\xi_0 + \xi)) = 0, \qquad \xi_0(z_{k;*}) + \xi(z_{k;*}) = w_k, \qquad \|\xi + \xi_0\|_{\ell^\infty(W^{1,p})} \le \delta, \\ \|\xi\|_{\ell^\infty(W^{1,p})} \le 2c_0 \big(\|\bar{\partial}_J(\exp_u(\xi_0))\|_{\ell^\infty(L^p)} + \|\xi_0(z_{k;*}) - w_k\|_{\ell^\infty} \big). \end{gathered}$$

Proof. We proceed in the same fashion as in the proof of theorem 4.1.3. Let c_1 be the constant of lemma 4.1.6, and let $\delta \in]0,1[$ be such that $c_1\delta < 1/2c_0$. Then lemma 4.1.6 insures that $\|d\mathcal{F}_u(\xi) - D_u\| \le 1/2c_0$ when $\|\xi\| \le \delta$. The map $ev_* : U \longrightarrow (\mathrm{T}_{m_*}M)^{\mathbb{Z}}$ is defined for $U \subset \mathcal{X}_u$ the open set of vector fields whose ℓ^∞ norm is less than the injectivity radius. With these notations, $dev_*(\xi) = dev_*(0)$, and no estimate on the second derivative is required.

Theorem A.4.4 will be used with the following notations: w is an element of $(\mathrm{T}_{m_*}M)^{\mathbb{Z}}$ contained in the image of ev_*,

$$X = \mathcal{X}_u, \quad Y = \mathcal{Y}_u \oplus (\mathrm{T}_{m_*}M)^{\mathbb{Z}}, \quad f = (\mathcal{F}_u, ev_* - w), \quad x_0 = 0, \quad c_0 = c$$

and with δ the minimum of the δ above and of the real number δ' such that $\|\xi\|_{L^\infty}$ is less than the injectivity radius of de M. Note that $df_x - df_{x_0}$ is bounded since by lemma 4.1.6 and as $dev_*(\xi) = dev_*(0)$. □

In order to prove theorem 5.1.3, the approximate solution to $f = 0$ must be made and the operator $D_u \oplus ev_*$ must be shown to have a bounded right inverse. Let us go back to $D_{u\mathbb{Z}} \oplus ev_{j+N\mathbb{Z}}$. In chapter 4, it was important that, under the hypothesis of the regularity of J, the map $D_{u\mathbb{Z}}$ possesses a bounded right inverse (for the $\ell^\infty(L^p)$ and $\ell^\infty(W^{1,p})$ norms). Call this inverse $Q_u\mathbf{z}$. Let us show that this allows us to construct an inverse to $D_u\mathbf{z} \oplus \mathrm{d}ev_{j+N\mathbb{Z}}$.

Lemma 5.2.2: *Under the hypothesis discussed in remark 5.1.2, a bounded inverse to $D_u\mathbf{z} \oplus \mathrm{d}ev_{j+N\mathbb{Z}}$ exists.*

Proof. The hypothesis was designed so that, for the structure J given, $\mathrm{d}ev_{u^j}|_{\mathrm{Ker} D_{u^j}}$ is surjective on $\mathrm{T}_{u^j(z_*)}M$. Thus there exists a map $q_j : \mathrm{T}_{u^j(z_*)}M \to \mathrm{Ker} D_{u^j}$ such that $\mathrm{d}ev_{u^j} \circ q_j = \mathrm{Id}$. This map is bounded since its domain is finite dimensional. Let us introduce $(\mathrm{T}_{m_*}M)^{\mathbb{Z}} = \underset{i\in\mathbb{Z}}{\times} \mathrm{T}_{u^{j+Ni}(z_*)}M$. Recall that $u^{j+Ni} = u^j$ and $u^j(z_*) = m_*$. Thus, the map

$$q : (\mathrm{T}_{m_*}M)^{\mathbb{Z}} \to \mathrm{Ker} D_{u^{j+Ni}}$$

which reproduce q_j on each factor is bounded from $\ell^\infty(|\cdot|) \to \ell^\infty(W^{1,p})$ where $|\cdot|$ denotes a norm on $\mathrm{T}_{u^j(z_*)}M$ and $\ell^\infty(|\cdot|)$ is the supremum of these norm on the product. Let $\eta \in W^{1,p}_{u\mathbf{z}}$ and $w \in (\mathrm{T}_{m_*}M)^{\mathbb{Z}}$, define $T : L^p_{u\mathbf{z}} \oplus (\mathrm{T}_{m_*}M)^{\mathbb{Z}} \to W^{1,p}_{u\mathbf{z}}$ by

$$T(\eta, w) = Q_u\mathbf{z}\eta + q(w - \mathrm{d}ev_{u^{j+N\mathbb{Z}}} Q_u\mathbf{z}\eta).$$

Since $D_u\mathbf{z}q = 0$, $D_u\mathbf{z}T(\eta, w) = \eta$ and $\mathrm{d}ev_{u^{j+N\mathbb{Z}}}T(\eta, w) = w$. T is the required right inverse to $D_u\mathbf{z} \oplus \mathrm{d}ev_{u^{j+N\mathbb{Z}}}$. □

The following proof is a small modification of the proof of theorem 4.1.5.

Proof of theorem 5.1.3: We start by describing the approximate solution $u^{(r_i);(w_k)} : \Sigma \to M$. The points $z_{*,k}$ will be chosen *a posteriori* as they will depend on the parameters (r_i). This dependance could certainly be avoided by perturbing again the approximate solution, but this would require unnecessary estimates. As described at the end of chapter 4, $u^{(r_i);(w_k)}$ will be defined by composing the maps $\mu_{i;r_i,r_{i+1}} : [i, i+1] \times S^1 \to \Sigma_i$ with $u^{i;r_i,r_{i+1}} : \Sigma_i \to M$ if $i \not\equiv j \bmod N$. When $i = j + Nk$, we will firts deform u^j by the vector field X_{w_k} in a map $\exp_{u^j} X_{w_k}$, before it is deformed into the a map $u^{j;r_j,r_{j+1}} : \Sigma_j \to M$ (those are the small $W^{1,p}$ defomations defined similarly to $u^{0,r}$ from u^0 in §3.3).

The $(0,1)$-form η along u will be splitted into η^i along the $u^{i;r_i,r_{i+1}}$ by extending with 0 where it is not defined. From these η^i, we obtain the vector fields ξ^i along the $u^{i;r_i,r_{i+1}}$ thanks to the inverse of $D_u\mathbf{z}_{;(r_i)} \oplus \mathrm{d}ev_{j+N\mathbb{Z}}$. The ξ^i will have the property that $D_{u^{i;r_i,r_{i+1}}}\xi^i = \eta^i$ and that if $i = j + Nk$ then $\xi^i(z_*) = 0$. To obtain a vector field $\xi^{(r_i);(w_k)}$ along u, it remains to glue these fields as in §3.3 and in the proof of theorem 4.1.5. Indeed, if the (r_i) are sufficiently small so that the point z_* will not be contained in the region where the gluing of the vector fields take place, this method gives a $\xi^{(r_i);(w_k)}$ which is 0 at certain points, which means that the curve displaced by this vector field will take the prescribed value.

The points $z_{k;*}$ are determined only at this step. First, the (r_i) are chosen so as to satisfy the constraints mentionned so far but also so that theorem 4.1.5 applies. Then, we fix

$$z_{k;*} = \mu^{-1}_{j+Nk;r_{j+Nk},r_{j+Nk+1}}(z_*).$$

Thus, $\xi^{(r_i);(w_k)}(z_{k;*}) = 0$. Furthermore, the computation of (3.3.4) remains identical. This construction is an exact inverse for ev_*. It also an approximate inverse to D_u (for the $\ell^\infty(L^p)$ norm) since the (r_i) have been so chosen. Consequently, this is an approximate inverse for $D_u \oplus ev_*$. The proper inverse is then obtained and proposition 5.2.1 applies to yield theorem 5.1.3. $\square$

5.3 Non-triviality

What theorem 5.1.3 allows is the existence of a family of pseudo-holomorphic maps which can be parametrzed as follows. Note by $\mathcal{R} : \mathbb{R}^{\mathbb{Z}}_{>0} \times B^{\mathbb{Z}}_{m_*} \longrightarrow \mathcal{M}_\Sigma$ the map obtained by theorem 5.1.3. In a neighborhood V_{m_*} of a point $m_* = u^j(z_*)$ the map $\mathcal{R}\big((r_i);(w_k)\big) : \Sigma \longrightarrow M$ is characterized by the value it takes at $z_{k;*}$. In order to alleviate notations, it is possible to choose the r_i to be all equal since the constraints they must satitsfy are finite. When $r_i = r$, we write for short $\mathcal{R}_r(w) = \mathcal{R}\big((r_i);(w_k)\big)$. This choice also gives that $z_{k;*}$ is the point $z_{0;*}$ translated by iNk. This understood, for fixed r, the maps obtained by theoreme 5.1.3 are characterized by $m_k = \exp_{m_*} w_k \in V_{m_*}$.

5.3.a Mean dimension. Thus to a sequence $\{w_k\}$ of vectors in a small ball B_{m_*} around $\mathrm{T}_{m_*}M$ we can associate a curve. The distance (5.1.1) between curves associated to $\{w_k\}$ and $\{w'_k\}$ is bounded from below by

$$d'(w,w') = \sup_{k\in\mathbb{Z}_{>0}} 2^{-|k|} \left\|w_k - w'_k\right\|.$$

If on $B^{\mathbb{Z}}_{m_*}$ the metric d' above is used, $\mathcal{R}_r : B^{\mathbb{Z}}_{m_*} \longrightarrow \mathcal{M}_\Sigma$ does not reduces the distances. Furthermore, $\mathbb{Z}$ acts on $B^{\mathbb{Z}}_{m_*}$ by shifting and this action is (up to some identification) equivariant if $\mathbb{Z}$ is seen as a subgroup of the automorphisms of the cylinder (by translation). More precisely, since deformations only happen on the curve u^j amongst the N curves which form the chain, the shift of an integer in $B^{\mathbb{Z}}_{m_*}$ will correspond to the translation by iN on the cylinder (here $\Sigma = \mathbb{C}/2\pi\mathbb{Z}$).

The mean dimension of $(\mathrm{T}_{m_*}M)^{\mathbb{Z}}$ for the topology induced by d' (this is the product topology) and the action of $\mathbb{Z}$ is $\dim \mathrm{T}_{m_*}M = \dim M$. To see this, take the Følner sequence $\Omega_i = [-i,i] \cap \mathbb{Z}$. Then,

$$d'_{\Omega_i} = \sup_{k\in\mathbb{Z}_{>0}} 2^{-\psi_i(k)} \left\|w_k - w'_k\right\|.$$

where $\|\cdot\|$ is the norm coming from $\mathrm{T}_{m_*}M$ and $\psi_i(k)$ is the distance of k to the interval $[-i,i]$, *i.e.* $\psi_i(k) = 0$ if $k \in [-i,i]$ and $\psi_i(k) = \max(|k-i|,|k+i|)$ otherwise. Since $\dim B_{m_*} = \dim \mathrm{T}_{m_*}M = \dim M$, the bound

$$\mathrm{wdim}\,(B^{\mathbb{Z}}_{m_*}, d'_{\Omega_i}) \leq \dim M(1+2i+2\lceil\log_2 \varepsilon\rceil)$$

is obtained by mapping $B^{\mathbb{Z}}_{m_*} \twoheadrightarrow B^{[-i-\log_2\varepsilon, i+\log_2\varepsilon]\cap\mathbb{Z}}_{m_*}$. Whence

$$\mathrm{mdim}(B^{\mathbb{Z}}_{m_*}, \mathbb{Z}) \leq \lim_{\varepsilon\to 0}\lim_{i\to\infty} \dim M(1+2i+2\lceil\log_2\varepsilon\rceil)/(2i+1) = \dim M$$

On the other hand, for ε sufficiently small $\mathrm{wdim}_\varepsilon(B_{m_*}, \|\cdot\|) = \dim M$ (*cf.* 1.2.5). For that same, ε, $(B^{\mathbb{Z}}_{m_*}, d'_{\Omega_i})$ will contain isometrically $(B^{2i+1}_{m_*}, \ell^\infty(\|\cdot\|))$. Lemma 1.2.6 indicates that

$$\mathrm{wdim}\,(B^{\mathbb{Z}}_{m_*}, d'_{\Omega_i}) \geq \dim M(1+2i)$$

which implies the opposite inequality.

Thus, theorem 5.1.3 gives a map $\mathcal{R}$ which does not reduce distances, equivariant for the subgroup $iN\mathbb{Z}$ whose co-volume is $2\pi N$. As a consequence of lemma 2.6.4, $\mathfrak{wdim}\mathcal{M}_\Sigma \geq \dim M/2\pi N$. The next corollary summarizes this result.

Corollary 5.3.1: *If there exists an almost complez structure J and a family of curves u^i satisfying the hypothesis of 5.1.2, then the mean dimension of $\mathcal{M}_\Sigma$ for the action of the automorphism group of the cylinder is at least* $\dim M/2\pi N > 0$.

5.3.b Simple maps. Before we look at curves obtained by $\mathcal{R}$ which possess the same image, we make a digression to show that a careful choice of parameters allows us to show that the map u does not factorize by a quotient of the cylinder. Recall that $u^{r;w}$ is the approximate solution constructed in order to obtain $\mathcal{R}_r(w)$. The map $u^{r;0}$ is periodic, in the sense that it factorizes as $u^{r;0} : \Sigma \twoheadrightarrow \Sigma/iN\mathbb{Z} \longrightarrow M$.

Lemma 5.3.2: *Let $u : \Sigma \longrightarrow M$ be a pseudo-holomorphic map sufficiently C^0 close to a map $u_0 : \Sigma \longrightarrow M$ which is $iN\mathbb{Z}$-periodic. Let $\phi : \Sigma \longrightarrow \Sigma'$ and $u' : \Sigma' \longrightarrow M$ be such that $u = u' \circ \phi$. Then ϕ is periodic: it is the quotient of the cylinder by a discrete subgroup (without fixed points) of the automorphisms of Σ.*

Proof. Let $w \in \Sigma'$ then $\phi^{-1}(w) \subset P_w := u^{-1}(u'(w))$. P_w is contained in a ball and its translates. Let $B_w \subset \Sigma/i\mathbb{Z}$ such that $B_w := \bigcap_{z \in P_w} \pi(B_\rho(z))$ where $\rho = \|u - u_0\|_{C^0}$ and $\pi : \Sigma \longrightarrow \Sigma/i\mathbb{Z}$ is the projection on the torus. In particular, $\phi^{-1}(w) \subset \pi^{-1}(B_w)$.

We wish to show that ϕ is periodic. In order to avoid an accumulation, the number of elements of $\phi^{-1}(w)$ in a component of B_w has to be bounded. Let

$$I_{x;k} = i]x - k - \tfrac{1}{2}, x + k + \tfrac{1}{2}[\oplus \mathbb{R}/2\pi\mathbb{Z},$$

where $x \in \mathbb{R}$ and $k \in \mathbb{Z}_{>0}$, a piece of the cylinder containing $2k+1$ connected components of $\pi^{-1}(B_w)$. We wish to construct a J-holomorphic map which associates to $w \in \Sigma'$ the mean of its preimages.

In order to do so, let us first describe this for a proper and non-constant holomorphic function $f : U \longrightarrow \Sigma'$ where $U \subset \Sigma$. A problem might occur at critical points of f. However, locally is $f(z) = a_d z^d + O(z^{d+1})$, there exists a function g such that $f = g^d$ and $g'(0) \neq 0$. In particular, g is invertible. Furthermore, $f(z) = w \Leftrightarrow g(z) \in \{x | x^d = w\}$. Thus, if $h(z) = \sum_{j \geq 0} a_j z^j$ is a polynomial and $g^{-1}(x)^j = \sum_{k \geq 0} b_{j,k} x^k$ the local expansion of g, the sum of the values of h on the preimages of

w will be written as

$$\begin{aligned}\sum_{z\in f^{-1}(w)} h(z) &= \sum_{z\in g^{-1}(w^{1/d})} h(z) &&= \sum_{z\in g^{-1}(w^{1/d})}\sum_{j\geq 0} a_j z^j \\ &= \sum_{j\geq 0} a_j \sum_{k\geq 0}\sum_{x\in w^{1/d}} b_{j,k} x^k &&= \sum_{j\geq 0}\sum_{k\geq 0} a_j d b_{j,dk} w^k\end{aligned}$$

and is a holomorphic function of w.

In the case of interest to us, when ϕ is restricted to $I_{x;k}$, the function which takes the sum of the premiages is well-defined. Let $F_{x;k}(w) = \phi|_{I_{x;k}}^{-1}(w)$. Let $\psi_{x;k} : \Sigma' \longrightarrow \Sigma/i\mathbb{Z}$ be the sequence of function given by

$$\psi_{x;k}(w) = \frac{1}{2k+1} \sum_{z\in F_{x;k}(w)} \pi(z).$$

In a neighborhood of w these function are holomorphic. However they present some discontinuities when moving w makes point of the preimage $\phi^{-1}(w)$ leave or enter $I_{x;k}$. In particular, $\psi_{w;k}$ is holomorphic in a neighborhood of w whose size is bounded from below (by the distance from B_w to the boundary of $I_{x;k}$). Furthermore, Since the $\psi_{x;k}$ are holomorphic outside these jumps, it is possible to extract convergent subsequences for each x. Note that the size of these discontinuities is bounded by K_1/k for some $K_1 \in \mathbb{R}_{>0}$. This bound enables to get that, for all k,

$$|\psi_{x;k} - \psi_{x';k}| \leq K_2 \frac{|x-x'|}{k}.$$

We start by choosing a sequence of points $\{w_i\}$ of Σ' which is dense (but as the $\psi_{w;k}$ do not present jumps near w, it would suffice to take a sufficiently small net). Next choose a subsequence $\{n_k^{(1)}\}$ for which $\psi_{w_1;k}$ converges. Then this subsequence $\{n_k^{(l)}\}$ is refined again in another subsequence $\{n_k^{(l+1)}\}$ which makes $\psi_{w_l;n_k^{(l)}}$ converge. The sequence $\{n_k^{(k)}\}$ will make all the $\psi_{w_i;n}$ converge to holomorphic functions. Furthermore, the difference between the jumps tends to 0, thus the limit of these sequence will not depend on the point w_i chosen. Denote this limit by ψ; this is the desired averaging function.

The map $\psi\circ\phi : \Sigma \longrightarrow \Sigma/i\mathbb{Z}$ has the property that $|\psi\circ\phi(z) - \pi(z)| < \rho$ since the two points belong to B_w. If ρ is less than the injectivity radius of Σ, this enables to define a function $f(z) = \psi\circ\phi(z) - \pi(z) \in \mathbb{C}$ which extends to X and is bounded. Consequently, $\exists c \in \mathbb{C}$ such that $\psi\circ\phi(z) = z + c$ in $\Sigma/i\mathbb{Z}$. In particular, ϕ is a covering map.

Hence Σ' is a quotient of $\mathbb{C}$ by a discrete subgroup without fixed points of the automorphism group of $\mathbb{C}$. This subgroup necessarily contains translations, and thus, contains only translations. Since the fundamental group of Σ' is abelian, the covering map $\phi : \Sigma \longrightarrow \Sigma'$ is the quotient by some group action. □

By taking B_{m_*} smaller if necessary, lemma 5.3.2 can be aplied to curves obtained by $\mathcal{R}$. Suppose that $u = \mathcal{R}_r(w)$ can be written $u = u'\circ\phi$, where $\phi : \Sigma \longrightarrow \Sigma'$ is a quotient map by a discrete subgroup without fixed points.

These subgroups of the automorphisms of $\Sigma = \mathbb{C}/2\pi\mathbb{Z}$ possess at most a finite generator (of the form $2\pi/n$ where $n \in \mathbb{Z}$) and an infinite generator (if it possesses an imaginairy part). This is seen by looking at the corresponding discrete subgroup without fixed point of automorphisms of $\mathbb{C}$, *i.e.* which is a lattice of rank 1 or 2.

Thus, if ϕ is the quotient by an infinite subgroup, then there exists $c \in \mathbb{C} \setminus \mathbb{R}$ such that $u(z+c) = u(z)$. Let $\pi : \mathbb{C} \twoheadrightarrow \mathbb{C}/(2\pi\mathbb{Z} \oplus iN\mathbb{Z})$, for all δ there exists an integer $n_c(\delta)$ such that $\pi(n_c(\delta)c) < \delta$. In other words, if u is periodic, the w_k must be almost periodic: $w_k - w_{k+n_c} < \rho + O(\delta)$ (where $\rho = O(r^{1+\varepsilon})$ is the distance from $u = \mathcal{R}_r((w_k))$ to the approximate solution $u^{r;(w_k)}$). For r is sufficiently small and one of the w_k is apart from the others, u will not be periodic.

Suppose now that ϕ is a finite quotient map. Consider a segment $I_k = i[k, k+1] \oplus \mathbb{R}/2\pi\mathbb{Z}$ where u is close to the map u^k. u is again periodic, but this time in the sense that there exists $c \in \mathbb{R}$ such that $u(z+c) = u(z)$. Note that $\mu_{k;r_k,r_{k+1}}(z) \mapsto \mu_{k;r_k,r_{k+1}}(z+c)$ will correspond to an automorphism ϕ' of $\mathbb{C}\mathrm{P}^1$ which fixes 0 and ∞. Hence $u^k \circ \phi'(z) - u^k(z) \leq \rho$ for $z \in \mu_{k;r_k,r_{k+1}}(I_k)$. Consequently if one of the curves u^k is simple and its image is not contained in a small neighborhood, that is of size $O(\rho) = O(r^{1+\varepsilon})$. Thus if a curve is of positive energy, this situation cannot happen.

Recall that a map $u : \Sigma \to M$ is said simple if when $u = u' \circ \phi$ where $\phi : \Sigma \to \Sigma'$ and $u' : \Sigma' \to M$ then ϕ is a degree 1 covering map of Riemann surfaces (an automorphism). The previous discussion can be summarized as follows.

Corollary 5.3.3: *If for a $k \in \{1, \dots, N\}$, u^k is a simple curve of positive energy, there exists a ball B_{m_*} and a r_0 such that for all $r < r_0$ and for all $w \in \ell^\infty(\mathbb{Z}; B_{m_*})$ of which a coordinate is at distance at least $\frac{1}{10}\mathrm{Diam} B_{m_*}$ from the others, $u = \mathcal{R}_r(w)$ is a simple map from Σ to M.*

5.3.c Distinct images. Another interesting property of this family of J-holomorphic applications resides in the fact that, under appropriate hypothesis, two curves obtained from $\mathcal{R}$ have the same image only if they differ by an automorphism.

Proposition 5.3.4: *Let u^i where $i = 1, \dots, N$ such that there exists r_0 and $B_{m_*} \subset \mathrm{T}_{m_*}M$ such that $\forall r < r_0$ and $\forall w \in \ell^\infty(B_{m_*})$ the number of points where $u^{r;w}$ is not injective is finite. Then there exists $r_1 < r_0$ and $C \in \mathbb{R}_{>0}$ such that for all $r < r_1$ and all $w_1, w_2 \in \ell^\infty(B_{m_*})$ such that $\|u^{r;w_1} - u^{r;w_2}\|_{C^0} < Cr_1$, if $u_1 = \mathcal{R}_r(w_1)$ and $u_2 = \mathcal{R}_r(w_2)$ possess the same image then they differ by the precomposition of an automorphism.*

Proof. Introduce

$$\Gamma = \{(z_1, z_2) \subset \Sigma \times \Sigma | u_1(z_1) = u_2(z_2)\}.$$

This is an analytic set (*cf.* [44, proposition 5 and its remark]), which is moreover complex, and, since the two curves have same image, of (complex) dimension 1. Note $\rho(r) = O(r^{1+\varepsilon})$ the maximum of the C^0 distances between $u^{r;w_k}$ and $u_k = \mathcal{R}_r(w_k)$. Choose r_1 so that $B_{4\rho(r)}(u^{r;w_k}(z)) \cap u^{r;w_k}(\Sigma)$ be isomorphic to discs for all $r \leq r_1$. Next, take C so that $Cr_1 < 4\rho(r_1) - 2\rho(r)$ (for example $C = 2\rho(r_1)/r_1$). Then, $\|u_1(z) - u_2(z)\|_{C^0} \leq 2\rho(r) + Cr_1 < 4\rho(r_1)$.

Let $\Delta \subset \Sigma \times \Sigma$ be the diagonal and let $U_\rho\Delta$ a ρ-neighborhood of the latter. Then Γ is close to $\Delta' = \cup_{z\in Z}(\Delta + z + iN\mathbb{Z})$ where Z is the set of points where the $u^{r;w_k}$ are not injective and $\Delta + c$ is a short notation for the diagonal translated along one of its factors in the product (*i.e.* the set of pairs $(z+c,z)$). These choices made, Γ is contained in a neighborhood of these translated diagonals, $U_{4\rho(r_1)}\Delta'$.

The map $s : (z_1,z_2) \mapsto (z_1, z_2 - z_1)$ is an isomorphism of $\Sigma \times \Sigma$ on itself which sends the neighborhood $U_\rho\Delta$ on $\Sigma \times D_\rho$ (where D_ρ is the disc of radius ρ). Let $\pi_k : \Gamma \to \Sigma$ be the projections on each factors. Given that the curves have the same image, these maps are surjective. Thus, $\pi_1 \circ s(\Gamma) \subset \Sigma$ and $\pi_2 \circ s(\Gamma) \subset D_\rho$. Let Γ_0 be a connected component of Γ; this is a closed analytic complex set of dimension 1. So is $s(\Gamma_0)$ which is contained in $\Sigma \times D_{4\rho(r_1)}$. This analytic set lifts to a subset of $\mathbb{C} \times D_{4\rho(r_1)}$. Describing this set by equations with holomorphic coefficients, one sees that the coefficient must be constant. Consequently this is a line. Thus, Γ_0 is contained in a translate of the diagonal; in other words for $z_1 \in \pi_1(\Gamma_0)$, $u_1(z_1) = u_2(z_1 + c)$. As $\pi_1(\Gamma_0)$ is Σ (note that by the uniqueness of the extension of J-holomorphic maps, it would suffice to have it non empty), this means that $u_1(z) = u_2(z+c)$. $\square$

There is a natural action of $\mathbb{C}$ on the maps $\Sigma \to M$ which is given by translation at the source. For a fixed r (less than the r_1 above), identifying all the curves given by theorem 5.1.3 which have the same image will not reduce the dimension significantly.

Indeed, let $I : \mathcal{M}_\Sigma \to \mathcal{P}(M)$ be defined by taking the image of curve, $I(u) = u(\Sigma)$. $\mathbb{C}$ acts by reparametrisation on $\mathcal{R}_r(w)$; this leads us to look at the quotient $\mathcal{R}_r(\ell^\infty(\mathbb{Z};B_{m_*}))/\mathbb{C}$. Proposition 5.3.4 insures us that I is locally injective on the quotient.

In order to construct a family of curve with different images, it might also be possible to proceed differently. For example, in the case a single gluing, remark 3.5.1 indicates that letting the parameter vary gives curves of different images. However, it is not possible to obtain this result directly from the implicit function theorem. Indeed, if at a point the parameter is r_i, the true solution obtained by proposition 4.1.3 is a perturbation in the $\ell^\infty(W^{1,p})$ norm of the order of $\sup\limits_{i\in\mathbb{Z}} r_i$. It is also difficult to introduce a measure of the strangling which can be defined on a curve u which is of class $W^{1,p}$. For these reasons, evaluation at a point have been used.

Appendix A

J-holomorphic curves

This appendix contains the basic definitions and concepts needed to deal with J-holomorphic curves. Then we briefly summarize properties of the operator $\bar{\partial}_J$, describe its linearisation and discuss its surjectivity. We will only lightly get into this subject; it is suggested to read [35], [34], [2], [21] and some of the references therein to see it treated in all due details. We follow the exposition contained in McDuff-Salamon.

A.1 Basic structures

Definition A.1.1: An almost symplectic form on $2n$ dimensional manifold is a non degenerate 2-form ω. If it is closed, it is called symplectic.

Whether the form is symplectic or almost symplectic, it will be denoted ω, and since this is not the main object of interest here, closedness will not be necessary.

Definition A.1.2: Let M be a manifold of even dimension. An almost complex structure J on M is a section of the vector bundles of the endomorphisms of the tangent plane, that is $J_m \in \mathrm{End}T_mM$, such that, on each fiber, the endomorphism is an anti-involution: $J_m^2 = -\mathbb{1}$.

J is said tamed by ω if $\forall v \in \mathrm{T}M, v \neq 0 \Rightarrow \omega(v, Jv) > 0$.

If $\omega(Jv, Jw) = \omega(v, w)$, then J is said compatible with ω.

Remark A.1.3*:* The (almost) symplectic structure will come in handy to have a special metric associated to J. As soon as J is tamed by ω, a riemannian metric can be associated to these two elements by

$$g_J(v,w) := \langle v,w\rangle_J := \frac{1}{2}\big(\omega(v,Jw) + \omega(w,Jv)\big). \tag{A.1.4}$$

If furthermore, J and ω are compatible, this expression is equal to $\omega(w, Jv)$.

The almost complex structure enables us to see the tangent bundle as a complex vector bundle; $(a+ib)v = av + bJv$.

Definition A.1.5: A map $\phi : (M,J) \longrightarrow (M',J')$ between two almost complex manifold will be called (J,J')-holomorphic if its differential $d\phi(x) : T_xM \longrightarrow T_{\phi(x)}M'$ is a linear map between complex vector spaces, *i.e.* $d\phi(x) \circ J(x) = J'(\phi(x)) \circ d\phi(x)$.

J is said integrable if it comes from a complex structure on M.

Remark A.1.6: Integrability of J is equivalent to the existence of an atlas of (J,i)-holomorphic charts, where i is the standard complex structure on $\mathbb{C}^n$.

In dimension 2, any almost complex manifold is complex. This result is often presented as a consequence of a theorem which characterize integrable structures as those for which the Nijenhuis tensor vanishes; [37] or [9, §2]. The Nijenhuis tensor of J is defined as

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y].$$

Since $N_J(X,JX) \equiv 0$, and that in dimension 2 the pair (X,JX) is pointwise a basis of the tangent plane, it always vanishes. An elementary proof of the integrability of almost complex surfaces can be found in [34, §4.2]. Another reference on the topic is [26, §9 and app.8].

Definition A.1.7: A J-holomorphic curve (or map) is a (j,J)-holomorphic map $u : \Sigma \longrightarrow M$ where Σ is a Riemann surface endowed with its complex structure j.

J-holomorphic curves are taken as parametrized. Let $u : \Sigma \longrightarrow M$ be a J-holomorphic curve, the condition $du \circ j = J \circ du$ is more convenient written as the equation

$$\bar{\partial}_J(u) := \frac{1}{2}(du + J \circ du \circ j) = 0.$$

In fact, the 1-form $\bar{\partial}_J(u) \in \Omega^{0,1}(\Sigma, u^*TM)$ is nothing else than the antilinear complex part (with respect to the structure of complex vector bundle defined by J) of the differential du.

In order to clarify the meaning of this equation, we will express this operator in a local chart. Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}^{2n}$, $j = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $J : \mathbb{R}^{2n} \longrightarrow GL(2n,\mathbb{R})$ such that $J(w)^2 = -\mathbb{1}$. Then du can be written as $\frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt$. Since $ds \circ j = -dt$ and $dt \circ j = ds$,

$$\bar{\partial}_J u = \frac{1}{2}\left(\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t}\right) ds + \frac{1}{2}\left(-J(u)\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right) dt. \tag{A.1.8}$$

It suffices to notice that the coefficient (with value in $\mathbb{R}^{2n}$) of the factor dt is $-J$ times the coefficient in front of ds. Thus

$$\bar{\partial}_J u = 0 \Leftrightarrow \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0; \tag{A.1.9}$$

in other words, this is a first-order non-linear equation. If J is the constant structure, then it can always be taken to be the standard structure thanks to a change of variables. Write $J = J_0 = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$, and let $f,g : \mathbb{R}^2 \longrightarrow \mathbb{R}^{2n}$ be such that $u = f + Jg$ and that they are identically zero on the n last coordinates, *i.e.* $\forall k \in \{n+1,\dots,2n\}, f_k = g_k = 0$ (this corresponds to functions taking real value), then the previous condition can be rewritten as the usual Cauchy-Riemann equations:

$$\frac{\partial f}{\partial s} + J\frac{\partial f}{\partial t} + J\frac{\partial g}{\partial s} - \frac{\partial g}{\partial t} = 0 \Leftrightarrow \frac{\partial f}{\partial s} = \frac{\partial g}{\partial t} \text{ and } \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial s}.$$

A.2 Linearisation of $\bar{\partial}_J$

As non-linear operators are hard to deal with, we will be lead to look at an approximation by a linear operator at a given curve. To do so introduce the moduli space $\mathcal{M}(A,\Sigma;J)$ of J-holomorphic curves representing the homology class $A \in H_2(M,\mathbb{Z})$. This is a subset of $\mathcal{B} \subseteq C^\infty(A,\Sigma,M)$, the space of all smooth maps $u : \Sigma \longrightarrow M$ which represent A. When it seen as a differential manifold (of infinite dimension), the tangent plane to $\mathcal{B}$ at u is the space of vector fields $\xi(z) \in \mathrm{T}_{u(z)}M$ along u, *i.e.* $\mathrm{T}_u\mathcal{B} = \Omega^0(\Sigma, u^*\mathrm{T}M)$. Let's consider the vector bundle $\mathcal{E}$ on $\mathcal{B}$ whose fiber at u is the space $\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*\mathrm{T}M)$ of $(0,1)$-forms with value in $u^*\mathrm{T}M$. Then $\bar{\partial}_J$ gives rise to a section of this bundle by associating to u the pair $S(u) = (u, \bar{\partial}_J u)$. The moduli space $\mathcal{M}(A,\Sigma;J)$ is the zero locus of this section.

Some curves will however be avoided in this moduli space. A J-holomorphic curve $u : \Sigma \longrightarrow M$ will be called simple if there does not exist a covering map $\phi : \Sigma \longrightarrow \Sigma'$ of degree > 1, such that $u = u' \circ \phi$ for another J-holomorphic map $u' : \Sigma' \longrightarrow M$. As we shall not consider structures depending on the point at the source, some properties are only true for simple curves. The moduli space of simple curves $\mathcal{M}^*(A,\Sigma;J)$ is defined as the intersection of $\mathcal{M}(A,\Sigma;J)$ with $\mathcal{B}^*$, the open set of $\mathcal{B}$ given by maps that are somewhere injective.

The linearised operator at a J-holomorphic u is usually obtained by considering a part of the differential of the section S mentionned above (see [35, §3.1]). However, we will mostly need it for curves that are not J-holomorphic. In that case, we must choose a connection which preserves the complex structure J so that the fibers of $\mathcal{E}$ are invariant by parallel transport.

Remark A.2.1*:* When $\nabla J = 0$, where ∇ is the Levi-Civita connection induced by the metric defined in (A.1.4), J is integrable. To see this, one can use $[X,Y] = \nabla_Y X - \nabla_X Y$ in the definition of N_J and realise that some terms cancel out as soon as J is parallel for ∇. Consequently, $\nabla J = 0$ is a very restrictive condition and we will often need to choose another connection $\widetilde{\nabla}$ for which J is parallel

A section $\alpha_\lambda \in \Omega^{0,1}(\Sigma, u_\lambda^*\mathrm{T}M)$ of $\mathcal{E}$ along a curve $\mathbb{R} \longrightarrow \mathcal{B} : \lambda \longmapsto u_\lambda$ is said parallel with respect to a connection $\widetilde{\nabla}$ on $\mathrm{T}M$ if the vector field $\lambda \longmapsto \alpha_\lambda(z;\zeta) \in \mathrm{T}_{u_\lambda(z)}M$ along the curve $\lambda \longrightarrow u_\lambda(z)$ is parallel for all $\zeta \in \mathrm{T}_z\Sigma$. Furthermore, if $\xi(\lambda) \in \mathrm{T}_{x(\lambda)}M$ is a vector field along a curve u_λ in M, $\widetilde{\nabla}_{\partial_\lambda u_\lambda}\xi$ means its covariant derivative. The (linear complex) connection which will be used is defined by

$$\widetilde{\nabla}_v X = \nabla_v X - \frac{1}{2} J(\nabla_v J) X,$$

where ∇ is the connection induced by g_J. For this new connection, the metric and the almost complex structure are parallel:

$$
\begin{aligned}
(\widetilde{\nabla}_v g)(X,Y)) &= \tfrac{1}{2}(g(J(\nabla_v J)X,Y)+g(X,J(\nabla_v J)Y)) \\
&= -\tfrac{1}{2}(g((\nabla_v J)JX,Y)+g(JX,(\nabla_v J)Y)=0 \\
(\widetilde{\nabla}_v J)(X) &= \widetilde{\nabla}_v(JX)-J\widetilde{\nabla}_v X \\
&= \nabla_v(JX)-\tfrac{1}{2}J(\nabla_v J)JX-J\nabla_v X+\tfrac{1}{2}JJ(\nabla_v J)X \\
&= \nabla_v(JX)-(\nabla_v J)X-J\nabla_v X=0,
\end{aligned}
$$

where $X,Y\in\Omega^0(\Sigma,\mathrm{T}M)$, $v\in\mathrm{T}M$. To conclude the first computation, the identity $g((\nabla J)Z,Y)+g(Z,(\nabla J)Y)=0$ is used. It can be obtained by differentiating $g(JZ,Y)+g(Z,JY)=0$, a relation that $g=g_J$ satisfies by definition, see (A.1.4). As for the second, one can use the fact that $J(\nabla J)+(\nabla J)J=0$ by differentiating $J^2=-\mathbb{1}$. However, the torsion of this connection is not zero; it is related to the Nijenhuis tensor.

Let $\xi\in\Omega^0(\Sigma,u^*\mathrm{T}M)$ be a vector field, and let, for $\|\xi\|_{L^\infty}$ less than the injectivity radius of M, $\Phi_u(\xi):u^*\mathrm{T}M\longrightarrow\exp_u(\xi)^*\mathrm{T}M$ be the isomorphism of complex vector bundles given by parallel transport along the geodesics $s\mapsto\exp_{u(z)}(s\xi(z))$ (with respect to $\widetilde{\nabla}$). Let

$$
\begin{array}{rccc}
\mathcal{F}_u: & \Omega^0(\Sigma,u^*\mathrm{T}M) & \longrightarrow & \Omega^{0,1}(\Sigma,u^*\mathrm{T}M) \\
& \xi & \longmapsto & \Phi_u(\xi)^{-1}\bar{\partial}_J(\exp_u(\xi)).
\end{array}
\tag{A.2.2}
$$

Its differential at 0 is the linearisation of the operator $\bar{\partial}_J$ at u; it will be denoted D_u^J or D_u when there is no ambiguity on the almost complex structure.

Proposition A.2.3: *([35, prop 3.1.1]) Let $u:\Sigma\longrightarrow M$ be a smooth map, and let D_u^J be the operator defined by*

$$
\begin{array}{rccc}
D_u^J: & \Omega^0(\Sigma,u^*\mathrm{T}M) & \longrightarrow & \Omega^{0,1}(\Sigma,u^*\mathrm{T}M) \\
& \xi & \longmapsto & \mathrm{d}\mathcal{F}_u(0)\xi,
\end{array}
$$

then $\forall\xi\in\Omega^0(\Sigma,u^*\mathrm{T}M)$,

$$
D_u^J\xi=\frac{1}{2}(\nabla\xi+J(u)\circ\nabla\xi\circ j)-\frac{1}{2}J(u)\circ(\nabla_\xi J)(u)\circ\partial_J u.
$$

Proof. Define the path $\mathbb{R}\longrightarrow C^\infty(\Sigma,M)$ by $\lambda\mapsto u_\lambda:=\exp_u(\lambda\xi)$. Since $\Phi_u(\lambda\xi)\mathcal{F}_u(\lambda\xi)=\bar{\partial}_J u_\lambda$ and that $\Phi_u(\lambda\xi)$ represent parallel transport along the geodesics $\lambda\mapsto u_\lambda(z)$, D_u can then be written as:

$$
\begin{aligned}
D_u\xi &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{F}_u(\lambda\xi)\Big|_{\lambda=0} \\
&= \widetilde{\nabla}_{\partial_\lambda u_\lambda}\bar{\partial}_J u_\lambda\Big|_{\lambda=0} \\
&= \tfrac{1}{2}(\widetilde{\nabla}_{\partial_\lambda u_\lambda}\mathrm{d}u_\lambda+J(u)\widetilde{\nabla}_{\partial_\lambda u_\lambda}\mathrm{d}u_\lambda j)\Big|_{\lambda=0} \\
&= \Big[\tfrac{1}{2}(\nabla_{\partial_\lambda u_\lambda}\mathrm{d}u_\lambda+J(u)\nabla_{\partial_\lambda u_\lambda}\mathrm{d}u_\lambda j)-\tfrac{1}{4}(J(u_\lambda)(\nabla_{\partial_\lambda u_\lambda}J)\mathrm{d}u_\lambda-(\nabla_{\partial_\lambda u_\lambda}J)\mathrm{d}u_\lambda j)\Big]\Big|_{\lambda=0} \\
&= \tfrac{1}{2}(\nabla\xi+J(u)\nabla\xi j)-\tfrac{1}{2}J(u)(\nabla_\xi J)\partial_J u,
\end{aligned}
$$

the last inequality holds since ∇ is without torsion, and thus $\nabla_{\partial_\lambda w}\partial_t w=\nabla_{\partial_t w}\partial_\lambda w$ for any smooth map $w:\mathbb{R}^2\longrightarrow M$. Taking $w(\lambda,t)=u_\lambda(z(t))$ and $\partial_t z(t)=V\in\mathrm{T}\Sigma$ yields

$$
\nabla_{\partial_\lambda u_\lambda}\mathrm{d}u_\lambda(V)|_{\lambda=0}=\nabla_{\partial_\lambda w}\partial_t w|_{\lambda=0}=\nabla_{\partial_t w}\partial_\lambda w|_{\lambda=0}=\nabla_V\xi. \qquad\square
$$

A.3 Surjectivity of D_u

Another important aspect that has to be checked in order to apply a Newton(-Picard) method is the existence of an (right) inverse for the linearised operator. In this section, the arguments and hypothesis that enable to show the existence of this operator will not be explained in full details, these can be found in [35].

First, even if we assumed so far that u was a smooth map, the definition of $\mathcal{F}_u$ (and that of D_u) can be extended to the case where u is of class $W^{k,p}$, for $k \geq 1$ and $p > 2$. Thus $\mathcal{F}_u$ is a map between the completions (with respect to the Sobolev norms):

$$\mathcal{F}_u : W^{k,p}(\Sigma, M) \longrightarrow W^{k-1,p}(\Sigma, \Lambda^{0,1}\mathrm{T}^*\Sigma \otimes_J u^*\mathrm{T}M).$$

Note that $\mathcal{F}_u$ has the same degree of differentiability as J. It is a good moment to mention an important property of J-holomorphic maps (elliptic regularity):

Proposition A.3.1: (cf. *[35, prop 3.1.9]*) *Suppose that J is an almost complex structure of class C^l, where $l \geq 1$. If $u : \Sigma \longrightarrow M$ is J-holomorphic of class $W^{1,p}$, where p>2, then u is of class $W^{l+1,p}$, and in particular $u \in C^l$. Moreover, if J is smooth the u is also smooth.*

Furthermore, proposition A.2.3 shows that D_u is a real linear Cauchy-Riemann operator (*cf.* [25] and [35, §C.1]), and consequently that this is a Fredholm operator (*i.e.* its image is closed, and its kernel and cokernel are finite dimensional). Its Fredholm index (the difference between the dimension of the kernel and of the cokernel) is stable under perturbation compact of order 0, and thanks to the Riemann-Roch theorem it is equal to

$$\mathrm{Ind}\, D_u = n(2-2g) + 2c_1(u^*\mathrm{T}M)$$

where g is the genus of Σ. As for $c_1(u^*\mathrm{T}M)$, the first Chern class of the induced bundle $u^*\mathrm{T}M$, it is important to note that it does not depend on J. This comes from the fact that the space of almost complex strucutre tamed by ω, denoted by $\mathcal{J}_d(M, \omega)$, is contractible. Thus, two structures $J, J' \in \mathcal{J}_d(M, \omega)$ give rise to isomorphic complex vector bundles since $\mathcal{J}_d(M, \omega)$ is path connected. The Chern classes $c_i(\mathrm{T}M)$ are consequently independent of J. This argument is also valid for the space of almost complex structures compatible with ω, $\mathcal{J}_c(M, \omega)$.

The index formula can also be deduced upon noticing that the order 0 part in D_u is a compact perturbation. In particular it is negligible from the point of view of the index. The principal part (of order 1) of D_u defines a complex linear Cauchy-Riemann operator, and thus a holomorphic structure on the bundle $u^*\mathrm{T}M$. The index is then the Euler characteristic of the Dolbeault cohomology and is given by the Riemann-Roch formula.

It remains to be shown that $\mathcal{M}^*(A, \Sigma; J)$ is a manifold of the "appropriate" dimension:

$$\dim \mathcal{M}^*(A, \Sigma; J) = n(2-2g) + 2c_1(u^*\mathrm{T}M)$$

for a generic J. For this result to hold, a sufficiently big space of almost complex structures $\mathcal{J}$ must be taken into account. This is actually true for any open space of smooth almost complex structures (endowed with the C^∞ topology), *e.g.* $\mathcal{J}_d(M,\omega)$ or $\mathcal{J}_c(M,\omega)$. $\mathcal{J}^l$ will denote the space of such structures but that are only l times differentiable.

One of the key remarks is that the universal moduli space of simple curves $\mathcal{M}^*(A,\Sigma;\mathcal{J}^l) = \{(u,J)|J \in \mathcal{J}^l, u \in \mathcal{M}^*(A,\Sigma;J)\}$ is a Banach manifold for l sufficiently large. Furthermore, the projection $\pi : \mathcal{M}^*(A,\Sigma;\mathcal{J}^l) \to \mathcal{J}^l$ is a Fredholm operator since its differential at (u,J) is essentially D_u. In particular, it possesses the same index ans is surjective precisely when D_u is. Using the implicit function theorem (since we are in a Banach manifold), $\mathcal{M}^*(A,\Sigma;J)$ is indeed a manifold (of finite dimension) whose tangent space at u is the kernel of D_u when J is a regular value of π. Thanks to the Sard-Smale theorem [45] the set of these regular values is of the second category in $\mathcal{J}^l$. In conclusion, $\mathcal{M}^*(A,\Sigma;J)$ is of the "appropriate" dimension for a generic J in $\mathcal{J}^l$.

Taubes gave an argument which allows us to pass to smooth structures: the space of simple J-holomorphic curves is the reunion of a countable set of compacts, for each of which the set of regular J is an open set, and consequently the set of regular J is the countable intersection of open sets. It remains to be shown that these open sets are dense. Here is a summary of the results we shall use:

Definition A.3.2: Let Σ be a compact Riemann surface and let $A \in H^2(M,\mathbb{Z})$ a homology class. An almost complex structure J is said regular (with respect to A and Σ) if the linear operator D_u is surjective for all $u \in \mathcal{M}^*(A,\Sigma;J)$. $J_{reg}(\Sigma,A)$ will denote the set of $J \in \mathcal{J}$ regular with respect to A and Σ

Theorem A.3.3: *([35, thm 3.1.5]) Let $\mathcal{J} = \mathcal{J}_c(M,\omega)$ or $\mathcal{J}_d(M,\omega)$, Let Σ be a Riemann surface, and let $A \in H^2(M,\mathbb{Z})$ ne a homology class. If $J \in J_{reg}(\Sigma,A)$ then $\mathcal{M}^*(A,\Sigma;J)$ is a (oriented) smooth manifold of dimension $n(2-2g)+2c_1(A)$ where g is the genus of Σ and $c_1(A) := \langle c_1(\mathrm{T}M),A\rangle$ if the evaluation of A on the first Chern class of $\mathrm{T}M$. Furthermore, the set $J_{reg}(\Sigma,A)$ is of the second category in $\mathcal{J}$,* i.e. *it is a countable intersection of open dense sets in $\mathcal{J}$.*

In the chapter 3, it will be necessary to look at a pair of curves $S^2 = \mathbb{C}\mathrm{P}^1 \to M$ passing through a same point; $0 \in S^2$ denotes $[0,1] \in \mathbb{C}\mathrm{P}^1$, whereas $\infty \in S^2$ is $[1,0] \in \mathbb{C}\mathrm{P}^1$. Shorten $\mathcal{M}(A,S^2;J)$ by $\mathcal{M}(A;J)$.

Definition A.3.4: Let $A^0,A^1 \in H^2(M,\mathbb{Z})$, the structure J will be said regular for (A^0,A^1) and S^2 if $J \in J_{reg}(S^2,A^0) \cap J_{reg}(S^2,A^1)$ and the evaluation map

$$\begin{array}{rccc} ev: & \mathcal{M}^*(A^0;J) \times \mathcal{M}^*(A^1;J) & \to & M \times M \\ & (u^0,u^1) & \mapsto & (u^0(0),u^1(0)) \end{array}$$

is transverse to the diagonal. The set of structures satisfying these conditions will be written $\mathcal{J}_{reg}(S^2,A^{0,1})$.

The intersection of two sets of the second category being of the second category, showing that the set $\mathcal{J}_{reg}(S^2,A^{0,1})$ is of the second category only requires the study of dev. Again, If the set of J-holomorphic structures considered $\mathcal{J}$ is big enough, this map will actually be surjective (*cf.* [35, th 6.3.1]).

Proposition A.3.5: *For a generic J, the evaluation map is transverse to the diagonal. In particular, $\mathcal{J}_{reg}(S^2, A^{0,1})$ is a set of the second category in $\mathcal{J}$. Furthermore, the moduli space*

$$\mathcal{M}^*(A^{0,1};J) = \{(u^0,u^1) \in \mathcal{M}^*(A^0;J) \times \mathcal{M}^*(A^1;J) | u^0(0) = u^1(0)\}$$

is a manifold of finite dimension $2n + 2c_1(A^0 + A^1)$.

Intuitively, the dimension of the moduli space can be found by taking the sum of the dimension of $\mathcal{M}^*(A^0;J)$ and $\mathcal{M}^*(A^1;J)$, minus $2n$ since the condition $u^0(0) = u^1(0)$ represents $2n$ equations. From now on, a compact subset

$$\mathcal{M}^*(C) := \mathcal{M}^*(A^{0,1};J;C) := \{(u^0,u^1) \in \mathcal{M}^*(A^{0,1};J) | \|du^h\|_{L^\infty} \leq C, h = 0 \text{ ou } 1\}$$

will be considered.

Before we finish these reminders, let us show that transversality in the sense of definition A.3.4, implies surjectivity of an operator which will be of importance later on. For $u : \Sigma \longrightarrow M$, let

$$\begin{aligned} W^{1,p}_u &= W^{1,p}(\Sigma, u^*\mathrm{T}M) \\ L^p_u &= L^p(\Sigma, \Lambda^{0,1}\mathrm{T}^*\Sigma \otimes_J u^*\mathrm{T}M) \end{aligned}$$

Given $u^0, u^1 : \Sigma \longrightarrow M$, such that $u^0(0) = u^1(0)$, denote

$$W^{1,p}_{u^{0,1}} := \left\{(\xi^0,\xi^1) \in W^{1,p}_{u^0} \times W^{1,p}_{u^1} | \xi^0(0) = \xi^1(0)\right\}.$$

For the evaluation of sections $W^{1,p}$ to make sense, it is necessary that $p > 2$.

Lemma A.3.6: *Suppose ev is transverse to the diagonal. The operator*

$$\begin{array}{cccc} D_{0,1}: & W^{1,p}_{u^{0,1}} & \longrightarrow & L^p_{u^0} \times L^p_{u^1} \\ & (\xi^0,\xi^1) & \mapsto & (D_{u^0}\xi^0, D_{u^1}\xi^1) \end{array}$$

is surjective when D_{u^0} and D_{u^1} are.

Proof. Let $\eta^h \in L^p_{u^h}$ (where $h = 0,1$). Each of the D_{u^h} being surjective, there exists $\xi^h \in W^{1,p}_{u^h}$ so that $D_{u^h}\xi^h = \eta^h$. Since the evaluation is transverse to the diagonal, choose $\zeta^h \in \mathrm{T}_{u^h}\mathcal{M}^*(A^h;J)$ so that

$$dev(u^0,u^1)(\zeta^0,\zeta^1) = (\zeta^0(0), \zeta^1(0)) \in (\xi^0(0), \xi^1(0)) + \mathrm{T}_{(m_0,m_0)}\Delta$$

where $ev(u,u')$ is the map defined in A.3.4, $m_0 = u^0(0) = u^1(0)$ and $\Delta \subset M \times M$ is the diagonal. Then $(\xi^0 - \zeta^0, \xi^1 - \zeta^1)$ is an element of $W^{1,p}_{u^{0,1}}$ whose image by $D_{0,1}$ is also (η^0,η^1). □

A.4 Implicit function theorem.

In order to obtain a solution to a non-linear problem, we will construct an approximate solution and then deform it to an actual solution using an implicit function theorem. This section sketches the method that yield this theorem (following [35, §A.3 and §3.5]).

Let X and Y be two Banach spaces and $f : X \to Y$ a map of class C^1. For all $x \in X$, the differential of f, denoted $\mathrm{d}f_x : X \to Y$, is defined by

$$\mathrm{d}f_x(x') = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x') - f(x)}{\varepsilon}.$$

In this linearisation of f at x is bijective, its inverse $\mathrm{d}f_x^{-1}$ is also a continuous linear map (by the Banach-Schauder or open mapping theorem). The first step to get to the implicit function theorem is a lemma on maps $\psi : X \to X$ whose differential is close to identity. We will often write $B_r(x;X)$ the ball of radius r centered at x in X, this will be shortened to $B_r(x)$ when there is no ambiguity on the space X and to B_r when $x = 0$.

Lemma A.4.1: *Let $\gamma < 1$ and R be real positive numbers, let X be a Banach space, let $x_0 \in X$ and let $\psi : B_R(x_0) \to X$ be a map such that $\forall x \in B_R(x_0), \|\mathbb{1} - \mathrm{d}\psi(x)\| \leq \gamma$. Then ψ is injective and sends $B_R(x_0)$ in an open set so that $B_{(1-\gamma)R}(\psi(x_0)) \subset \psi(B_R(x_0)) \subset B_{(1+\gamma)R}(x_0)$. Furthermore, $\psi^{-1} : \psi(B_R(x_0)) \to B_R(x_0)$ is differentiable and $\mathrm{d}(\psi^{-1})_y = \mathrm{d}\psi^{-1}_{\psi^{-1}(y)}$.*

Theorem A.4.2: *Let X and Y be two Banach spaces, let $U \subset X$ an open subset, and let $f : U \to Y$ be differentiable of continuous differential. Let $x_0 \in U$ and suppose that $\mathrm{d}f_{x_0} : X \to Y$ is bijective. Then there exists an open set $U_0 \subset U$ around x_0 such that the restriction of f to U_0 is injective, that $V_0 \subset f(U_0)$ is open, that $f^{-1} : V_0 \to U_0$ has continuous differential, and that $\forall y \in V_0, \mathrm{d}(f^{-1})_y = (\mathrm{d}f_{f^{-1}(y)})^{-1}$. Thus if f is of class C^l, so is f^{-1}.*

The statement of the implicit function theorem in Banach spaces requires the notion of regular value and Fredholm map. Recall that a map between Banach spaces $f : X \to Y$ is Fredholm if $\forall x \in X$, its differential at x (the linear operator $\mathrm{d}f_x$) is a Fredholm operator. That is, $\mathrm{d}f_x$ is of closed image, and of finite dimensional kernel and cokernel. Since the index of a Fredholm operator D, $\operatorname{ind} D = \dim \operatorname{Ker} D - \dim \operatorname{Coker} D$, is invariant under small perturbations of D, the index of $\mathrm{d}f_x$ is independent of the choice of x; it will subsequently be denoted $\operatorname{ind} f$.

Whether a map is Fredholm or not, a $y \in Y$ is said to be a regular value for f if $\forall x \in f^{-1}(y)$ the differential $\mathrm{d}f_x$ possesses a right inverse. A crude description of the implicit function theorem could say that $f^{-1}(y)$ is a (Banach) manifold when y is a regular value and that for Fredholm maps it is of dimension equal to the index of f.

Theorem A.4.3: *Let X and Y be Banach spaces, let $U \subset X$ be an open set and $l \in \mathbb{Z}_{>0}$ be a positive integer. If $f : U \to Y$ is of class C^l and y is a regular value of f, then $\mathcal{M} = f^{-1}(y) \subset X$ is a (Banach) manifold of class C^l and its tangent space at $x \in \mathcal{M}$ is the kernel of the differential at x,* i.e. $\forall x \in \mathcal{M}, \mathrm{T}_x\mathcal{M} = \operatorname{Ker} \mathrm{d}f_x$. *In particular, when f is a Fredholm map, $\mathcal{M}$ is of finite dimension, $\dim \mathcal{M} = \operatorname{ind} f$.*

A quantitative version of this theorem will be of use. In particular, this version is can be used to show the qualitative result above. To do so, we first describe theorem A.4.3 in an analytical language. Take $y = 0$ and recall that if $D = \mathrm{d}f_{x_0}$ possesses a right inverse, *i.e.* a $Q : Y \to X$ such that $DQ = \mathbb{1}_Y$, then it is surjective. The existence of this inverse Q is equivalent to the existence of a decomposition of X as $X = \operatorname{Ker} D \oplus \operatorname{Im} Q$. In other words, the (non-linear) space of solutions to the equation $f(x) = 0$ ressembles to the kernel of D close to x_0. To say that the kernel of D is the tangent plane at x_0 expresses the fact that there is a function $\phi : \mathrm{Ker} D \longrightarrow X$ such that $\mathrm{d}\phi_0 = 0$ and for all x sufficiently close to x_0, $\exists \xi \in \mathrm{Ker} D$ satisfying $x = x_0 + \xi + Q\phi(\xi)$. Thus, the problem reduces to that of finding a solution to $f(x_1 + \eta) = 0$ where $x_1 = x_0 + \xi$ and $\eta \in \mathrm{Im}\, Q$, only knowing that $f(x_1)$ is "close" to 0 (that is without supposing that the solution x_0 is known). The theorem that follows insures us of the existence of such a solution to $f(x) = 0$ in $x_1 + \mathrm{Im}\, Q$ when $f(x_1)$ is small.

Theorem A.4.4: *Let X and Y be two Banach spaces, let $U \subset X$ be an open set, and let $f : U \longrightarrow Y$ be a continously differentiable map. Let $x_0 \in U$ be such that $D = \mathrm{d}f_{x_0} : X \longrightarrow Y$ is surjective and has a right inverse $Q : Y \longrightarrow X$ (a bounded linear map). Suppose there exists δ and c two real positive numbers such that $\|Q\| \leq c$, $B_\delta(x_0;X) \subset U$ and $\forall x \in B_\delta(x_0;X)$, $\|\mathrm{d}f_x - D\| \leq 1/2c$. Suppose these exists $x_1 \in B_{\delta/8}(x_0;X)$ such that $\|f(x_1)\| < \delta/4c$, then there exists $x \in B_\delta(x_0;X)$ such that*

$$f(x) = 0, \qquad x - x_1 \in \mathrm{Im}\, Q, \qquad \textit{et } \|x - x_1\| \leq 2c \, \|f(x_1)\| \, .$$

This is the statement of the implicit function that will be used in chapters 3, 4 and 5.

Appendix B

Elliptic analysis à la Taubes

This appendix contains an adaptation of Taubes "toolbox" [47] in dimension 2. The aim of this adaptation was to attempt to prove a Runge theorem for J-holomorphic curves under quite strong assumptions. The heuristic idea can be found in Donaldson's paper [8]. Given a J-holomorphic map $f_0 : D \longrightarrow M$ from a disc to an almost-complex manifold (M,J), it is always possible to extend it by a C^∞ map $f : S^2 \longrightarrow M$ defined on the sphere and identical to the former when restricted to a compact subset of the disc. There is a set H, presumably quite large, where this map is not J-holomorphic. In order to get an holomorphic map from this one, the idea is to change the definition of the function on small discs. On these discs one would like to replace it by a J-holomorphic map having a behaviour on the boundary of the disc close to that of the rough C^∞ extension of f.

In an almost-complex manifold (M,J), the idea is to proceed as follows. Let us be at a point where $\bar{\partial}_J(f) \neq 0$, and let us consider local charts at the source and the image so that the almost complex structure induces the endomorphism i on $\mathbb{C}^n$. The rough extension f can be written as $f(z) = az + b\bar{z} + O(|z|^2)$. It is of course impossible to approximate this by a holomorphic map. However, suppose there is a J-holomorphic function g such that $g(z) = az + b\frac{r^2}{z} + o(|z|)$ around $|z| = r$. This is a possible approximation of $az + b\bar{z}$ when $|z| \simeq r$. The strategy is to graft g to f along this circle, and to repeat this operation until the set of points where f is not J-holomorphic is small. Chapter 3 shows in particular that under certain conditions a function with local expansion similar to that of g can be obtained.

There a some differences between the case of instantons (on the sphere) and the J-holomorphic problem: the non-linearity is quadratic in the former, whereas it does not seem to have any particular behaviour in the latter. The scenario is closer to that of anti-self-dual metrics in dimension 4, studied by Taubes in [47]; it is also easier as we are dealing with a first order equation rather than one of order 2 and the symmetry group is finite dimensional rather than infinite dimensional.

The goal is to solve the non-linear equation $\bar{\partial}_J f = 0$. To do so, we look at the linearization. In our case, this is a linear elliptic operator. But Taubes' norm prove to be useful through their clever use of the Laplacian. For example, in [47, §5], even if the linearization is not elliptic, the method still applies.

The approach discussed in chapter A cannot be used in this situation. Here are a few reasons. First, surjectivity of the linearization is not guaranteed, the problem would need to be solved up to the kernel. Second, L^p norms are not convenient: an arbitrarily large number of graftings will happen in a given region. In particular Sobolev's constant s_p is no longer bounded. Third, the L^p norm of du is not bounded. Indeed on each disc where a surgery occurs, this norm increases by a quantity which is *a priori* significative and the number of these surgeries is not bounded. This seems to indicate that new norms are required ; norms which depend on a sup rather than an integral over the whole surface. Unfortunately, Taubes' norm do not behave as nicely in dimension 2 than in higher dimensions: Green's kernel has a logarithmic singularity, the bound obtained in theorem B.5.3 contains a logarithm which becomes a constant in higher dimension. The aim of this appendix is to put forth the appearance of this logarithmic term.

B.1 Definitions and properties of the norms

We start by recalling the following lemma.

Lemma B.1.1: *Let $E > 0$ and $\eta \in C^\infty(V)$ be given. $\exists c_1 > 0$ such that there exists an unique $u \in (\Pi_E L^2(V)) \cap W^{2,2}(V)$ satisfying $\nabla^*\nabla u = \Pi_E \eta$. Moreover, $u \in C^\infty(V)$ and*

$$\|\nabla u\|_{L^2}^2 + E\,\|u\|_{L^2}^2 \leq (1+\frac{c_1}{E})\left|\int_M \langle u,\eta\rangle\right|.$$

As said above Sobolev norms are unfortunately not appropriate for our problem. Still it is important to have norms which take into account the pointwise behaviour of maps. The first norms we introduce look like an L^∞ norm but applied to the inverse of the Laplacian (the convolution with Green's kernel).

Definition B.1.2: Let $\rho \in]0, e^{-1}[$. Let $x \in M$, $B_\rho(x)$ be the open ball of radius ρ centered at x. Define

$$\begin{aligned}
\|u\|_\infty &= \sup_{x\in M} |u(x)|, \\
\|u\|_{*,\rho} &= \sup_{x\in M} \int_{B_\rho(x)} \ln(d(x,y)^{-1})\,|u(y)|\,dy \\
\|u\|_{2*,\rho} &= \sup_{x\in M} \left[\int_{B_\rho(x)} \ln(d(x,y)^{-1})\,|u(y)|^2\,dy\right]^{1/2} \\
\|u\|_{\mathcal{L}^0,\rho} &= \|u\|_\infty + \|\nabla u\|_{2*,\rho}
\end{aligned}$$

These norms will not be sufficient for our needs, a seminorm $\mathcal{L}^1$ will arise naturally; it can be seen as an "integration by parts" norm: although derivatives do not appear explicitly, they are nevertheless measured in it. A parenthesis is necessary for their introduction.

Denote by $\mathcal{S}(\mathrm{T}^*\Sigma \otimes V) \subset C^\infty(\mathrm{T}^*\Sigma \otimes V)$ the subset of elements of $\mathcal{L}^0$ norm equal to 1. Furthermore, given local charts around x, then for ρ sufficiently small, $B_\rho(x)$ identifies to an usual ball of $\mathbb{R}^2$. In these coordinates, a section of $T^*\mathbb{R}^2$ can be written as a map $\mathbb{R}^2 \to \mathbb{R}^2$. Next, notice that maps from the circle $\mathbb{R}^2 \supset S^1 \to \mathbb{R}^2$ extend to maps independent of the radial coordinate $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$. Last, denote by $\Gamma = \{f \in C^\infty(S^1, \mathbb{R}^2) \mid \|f\|_{L^2} = 1\}$.

Definition B.1.3: Let $\rho \in]0, e^{-1}[$ be less than the injectivity radius, the seminorm $\mathcal{L}^1$ associated to $u \in C^\infty(V)$

$$\|u\|_{\mathcal{L}^1,\rho} = \sup_{x\in M} \sup_{v\in\mathcal{S}(T^*\Sigma\otimes V)} \sup_{\phi\in\Gamma} \int_{B_\rho(x)} \frac{\langle v, \phi\otimes u\rangle\,(y)}{d(x,y)} dy$$

and enters in the definition of the following two norms:

$$\begin{aligned} \|u\|_{\mathcal{L},\rho} &= \|u\|_{\mathcal{L}^0,\rho} + \|\nabla u\|_{\mathcal{L}^1,\rho}, \\ \|u\|_{\mathcal{H},\rho} &= \|u\|_{2*,\rho} + \|u\|_{\mathcal{L}^1,\rho}. \end{aligned}$$

We begin with elementary properties of these norms.

Proposition B.1.4: *Suppose that* $\rho \in]0, e^{-1}[$.

a. $\|ab\|_{*,\rho} \leq \|a\|_{2*,\rho}\,\|b\|_{2*,\rho}$

b. $\|\cdot\|_{L^1(B_\rho(x))} \leq |\ln\rho|^{-1}\,\|\cdot\|_{*,\rho}$ *and* $\|\cdot\|_{L^2(B_\rho(x))} \leq |\ln\rho|^{-1/2}\,\|\cdot\|_{2*,\rho}$.

c. *If* $k \in \mathbb{R}_{>1}$ *and if* X *denotes* $*, 2*, \mathcal{L}^0, \mathcal{L}^1, \mathcal{L}$, *and* $\mathcal{H}$ *then* $\|\cdot\|_{X,\rho} \leq \|\cdot\|_{X,k\rho} \leq 4k^2\,\|\cdot\|_{X,\rho}$

d. *The norm* $\mathcal{L}^0$ *is submultiplicative:* $\|v\otimes w\|_{\mathcal{L}^0,\rho} \leq \|v\|_{\mathcal{L}^0,\rho}\,\|w\|_{\mathcal{L}^0,\rho}$.

Proof. The first of these properties is a direct consequence of Hölder's inequality.

Whereas the second one follows from

$$\|u\|_{L^1(B_\rho(x))} \leq |\ln\rho|^{-1} \int_{B_\rho(x)} |u(y)|\,|\ln\rho|\,dy \leq |\ln\rho|^{-1} \int_{B_\rho(x)} |u(y)|\,|\ln d(x,y)|\,dy;$$

the L^2 case being identical.

As for the third, the norms $*, 2*$, ou $\mathcal{L}^1$ are obtained by the sup of integrals on balls, the ratio of areas allows us to bound the integral taken on a large ball by those computed on smaller balls. Since the $\mathcal{L}^0, \mathcal{L}$, are $\mathcal{H}$ combinations of $*, 2*, \mathcal{L}^1$, or L^∞ norms, the inequality also holds.

The last property is again a simple calculation:

$$\begin{aligned} \|v\otimes w\|_{\mathcal{L}^0,\rho} &\leq \|v\|_{L^\infty}\,\|w\|_{L^\infty} + \|\nabla v\otimes w + v\otimes\nabla w\|_{2*,\rho} \\ &\leq \|v\|_{L^\infty}\,\|w\|_{L^\infty} + \|\nabla v\otimes w\|_{2*,\rho} + \|v\otimes\nabla w\|_{2*,\rho} \\ &\leq \|v\|_{L^\infty}\,\|w\|_{L^\infty} + \|\nabla v\|_{2*,\rho}\,\|w\|_{L^\infty} + \|v\|_{L^\infty}\,\|\nabla w\|_{2*,\rho} \\ &\leq \|v\|_{\mathcal{L}^0,\rho}\,\|w\|_{\mathcal{L}^0,\rho} \end{aligned}$$ □

Before we move on to the estimates these norms enable to find, the following lemma describes the difference between Green's kernel for the Laplacian (with a singularity at x) and the function $\ln d(x,\cdot)^{-1}$.

Lemma B.1.5: *Given $x \in \Sigma$, let $G(x,\cdot) : C^\infty(\Sigma \setminus \{x\})$ be Green's function for $\nabla^*\nabla + 1 : C^\infty(\Sigma) \to C^\infty(\Sigma)$. $\exists c_2 \in \mathbb{R}_{>0}$ depending on the diameter of Σ such that*

$$\begin{aligned} \left|G(x,\cdot) + (2\pi)^{-1}\ln(d(x,\cdot))\right| &\leq c_2\left|d(x,\cdot)^2\ln(d(x,\cdot)\right| \\ \left|\nabla G(x,\cdot) + (2\pi d(x,\cdot))^{-1}\nabla d(x,\cdot)\right| &\leq c_2\left|d(x,\cdot)\ln(d(x,\cdot)\right| \\ \left|\nabla^*\nabla G(x,\cdot)\right| &\leq c_2 d(x,\cdot)^{-2}. \end{aligned}$$

Proof. For this proof, it is recommended to (re)read the important results on Green's function; we cite [1, ch4 §2.1-§2.3]. Start by writing the Laplacian for a function depending only on polar (geodesic) coordinates (*cf.* [1, 4.9]):

$$\nabla\phi(r) = \phi'' + \frac{1}{r}\phi' + \phi'\partial_r \ln\sqrt{|g|},$$

where g is the metric; an useful bound of the term where it plays a role is $\partial_r \ln\sqrt{|g|} \leq K_1 r$ for $K_1 \in \mathbb{R}_{>0}$, see [1, thm 1.53]. Let $f(r) : \mathbb{R}_{\geq 0} \to [0,1]$ be a smooth function which is 0 if $r > \mathrm{injrad}\Sigma$ and equal to 1 when $r < \mathrm{injrad}\Sigma/2$. Furthermore, take $r = d(x,y)$ and define the parametrix

$$H(x,y) = -(2\pi)^{-1} f(r)\ln r.$$

A direct calculation shows that

$$\Delta_y H(x,y) = f''(r)\ln r + f'(r)r^{-1}(2+\ln r) + (f'(r)\ln r + f(r)r^{-1})\partial_r \ln\sqrt{|g|}.$$

Thanks to the bound on the last term and since $f'(r) = f''(r) = 0$ when $r < \mathrm{injrad}\Sigma/2$, there exists a constant K_2 (depending on the injectivity radius and the choice of f) such that

$$\left|\Delta_y H(x,y)\right| \leq K_2.$$

This said, the first inequality follows from equation [1, (4.17)]; let $\Gamma_1(x,y) = -\Delta_y H(x,y)$, let $\Gamma_{i+1} = \int_\Sigma dvol(z)\Gamma_i(x,z)\Gamma_1(z,y)$ and let $F_k(x,y)$ be defined by $\Delta_y F(x,y) = \Gamma_k(x,y) - (\int_\Sigma dvol)^{-1}$. With these notations, $\forall k \in \mathbb{N}_{\geq 2}$,

$$G(x,y) = H(x,y) + \sum_{i=1}^{k}\int_\Sigma dvol(z)\Gamma_i(x,z)H(z,y) + F_{k+1}(x,y).$$

The term $i = 1$ will have the most singular behaviour at 0. However, since $\Gamma_1(x,y) = -\Delta_y H(x,y)$ is bounded and since $H(x,y)$ is essentially a logarithm of the distance, a positive real number K_3 which depends on the diameter exists so that

$$\left|\int_\Sigma dvol(z)\Gamma_1(x,z)H(z,y)\right| \leq K_3 r^2 |\ln r|.$$

The estimations of the derivatives are obtained likewise. □

B.2 Estimation on the solutions of $\delta^*\delta u = \chi$.

Let V and W be vector bundles on Σ having the same dimension. Let $\delta : C^\infty(V) \longrightarrow C^\infty(W)$ be an elliptical operator of order 1. Let $\sigma \in \mathrm{Hom}(\mathrm{T}^*\Sigma, \mathrm{Hom}(V,W))$ the symbol of δ, defined by the relation $\delta = \sigma\nabla + l$, where $l \in \mathrm{Hom}(V,W)$ is the term of order 0. Ellipticity of δ means that $\sigma(z)$ is an isomorphism when $z \neq 0$. Moreover, if σ^* is the symbol of δ^*, the following relation will be assumed: $\forall z \in \mathrm{T}^*\Sigma, \sigma^*(z)\sigma(z) = |z|^2 \,\mathrm{Id}_V$.

Fix $E > 0$ and $\rho \in]0, e^{-1}[$, the latter being small. Suppose that $\chi \in C^\infty(V)$ is orthogonal to eigenspaces corresponding to small eigenvalues of the Laplacian, *i.e.* $(1 - \Pi_E)\chi = 0$. It will frequently be decomposed as:

$$\chi = q + b_1 \nabla b_2$$

where b_2 is a section of a vector bundle $Y \longrightarrow M$, b_1 a section of $C^\infty(\mathrm{Hom}(Y \otimes \mathrm{T}^*, V))$, and $q \in C^\infty(V)$.

Proposition B.2.1: *Let E, ρ, $\exists c_3(\mathrm{Diam}\Sigma)$ and $c_4(vol\Sigma, \mathrm{Diam}\Sigma)$ two real positive numbers so that given $\chi = q + b_1\nabla b_2$ as above and for $u \in \Pi_E C^\infty(V)$ the unique solution to $\delta^*\delta u = \chi$, then*

$$(a) \quad \|u\|_{\mathcal{L}^o,\rho} \;\leq\; c_3(\rho^{-1}|\ln\rho|\,\|u\|_{L^2} + \|q\|_{*,\rho} + \|b_1\|_{\mathcal{L}^0,\rho}\,\|b_2\|_{\mathcal{H},\rho}).$$

If moreover $(1 - \Pi_E)\chi = 0$

$$(b) \quad \|u\|_{\mathcal{L}^o,\rho} \;\leq\; c_4(1 + \rho^{-4}|\ln\rho|E^{-1})(\|q\|_{*,\rho} + \|b_1\|_{\mathcal{L}^0,\rho}\,\|b_2\|_{\mathcal{H},\rho}).$$

Proof. Introduce a smooth function $\alpha : [0,\infty) \longrightarrow [0,1]$ equal to 1 on $[0,1]$ and 0 on $[2,\infty)$. For a fixed x, this function enables to define a function which is constant on $B_\rho(x)$ and with support in $B_{2\rho}(x)$:

$$\alpha_x(y) = \alpha(\rho^{-1}d(x,y)).$$

The equality

$$\nabla^*\nabla\,|u|^2 = 2\,\langle u, \nabla^*\nabla u\rangle_g - 2\,|\nabla u|^2$$

allows, together with

$$\delta^*\delta u = \nabla^*\nabla u + \sigma'\nabla u + Ru$$

which comes from the relation $\sigma^*(\cdot)\sigma(\cdot) = |\cdot|^2$ satisfied by the symbol σ of δ, to write $\langle u, \delta^*\delta u\rangle_g = \langle u, \chi\rangle_g$ as

$$\frac{1}{2}(\nabla^*\nabla\,|u|^2 + |u|^2) + |\nabla u|^2 + \langle u, \sigma'\nabla u\rangle + \left\langle u, Ru - \frac{1}{2}u\right\rangle = \langle u, \chi\rangle.$$

Both sides of this equality are then multiplied by $\alpha_x(\cdot)G(x,\cdot)$ then integrated over Σ. Here is what the first term gives:

$$\begin{aligned}
&\int_\Sigma \alpha_x(\cdot)G(x,\cdot)(\nabla^*\nabla|u(\cdot)|^2+|u|^2)\\
&\quad=\int_\Sigma G(x,\cdot)(\nabla^*\nabla|u|^2+|u|^2)-\int_\Sigma(1-\alpha_x(\cdot))G(x,\cdot)(\nabla^*\nabla|u|^2+|u|^2)\\
&\quad=|u(x)|^2-\int_\Sigma(1-\alpha_x(\cdot))G(x,\cdot)\,|u(\cdot)|^2-\int_\Sigma(1-\alpha_x(\cdot))G(x,\cdot)\nabla^*\nabla|u(\cdot)|^2\\
&\quad=|u(x)|^2-\int_\Sigma(1-\alpha_x(\cdot))G(x,\cdot)\,|u(\cdot)|^2-\int_\Sigma|u(\cdot)|^2\,\nabla^*\nabla[(1-\alpha_x(\cdot))G(x,\cdot)].
\end{aligned}$$

Thanks to B.1.5, for a constant c_2, $|\nabla^*\nabla[(1-\alpha_x(y))G(x,y)]|$ is bounded above by $K_1\rho^{-2}|\ln\rho|$ when $y\in B_{2\rho}(x)\setminus B_\rho(x)$ and zero elsewhere. It then follows that

$$\begin{aligned}
&|u(x)|^2+\int_{B_\rho(x)}|\nabla u(\cdot)|^2\ln(d(x,\cdot)^{-1})\\
&\quad\le|u(x)|^2+\int_\Sigma|\nabla u(\cdot)|^2\alpha_x(\cdot)G(x,\cdot)\\
&\quad\le K_2\Bigg(\int_\Sigma(1-\alpha_x(\cdot))G(x,\cdot)\,|u(\cdot)|^2+\rho^{-2}|\ln\rho|\int_{A_{\rho,2\rho}}|u(\cdot)|^2\\
&\qquad+K_3\int_{B_{2\rho}}|u(\cdot)|^2\ln(d(x,\cdot)^{-1})+K_4\int_{B_{2\rho}}\alpha_x(\cdot)G(x,\cdot)\,|u|\,|\nabla u|\\
&\qquad+\int_{B_{2\rho}}\alpha_x(\cdot)G(x,\cdot)\,\langle u,\chi\rangle\Bigg)
\end{aligned}\tag{B.2.2}$$

The sup of the left-hand term on $x\in M$ bounds $\frac{1}{2}\|u\|^2_{\mathcal{L}^0,\rho}$; thus in order to bound the right-hand term, a factor of $\|u\|_{\mathcal{L}^0,\rho}$ will always have to be removed. We proceed differently for each of the five term on the right-hand side of (B.2.2).

First term. The integrand is of support in $\Sigma\setminus B_\rho(x)$, a rough bound allows us to rewrite it in a shape close to that of the second term, that is,

$$\int_\Sigma(1-\alpha_x(\cdot))G(x,\cdot)\,|u(\cdot)|^2\le\|u\|^2_{L^2}\,\|(1-\alpha_x(\cdot))G(x,\cdot)\|_{L^\infty},$$

and, since $\|(1-\alpha_x(\cdot))G(x,\cdot)\|_{L^\infty}<K_5|\ln\rho|$, this term is bounded if we wish to show (a) by $K_5|\ln\rho|\,\|u\|_{L^\infty}\|u\|_{L^2}$ (K_5 depends on DiamΣ and $vol\Sigma$). As for the bound that gives (b), the work is to be done as in the tratment of the second term. We immediatly explain how.

Second term. To obtain (a), it suffices to notice that $\|u\|_{L^2(A_{\rho,2\rho})}\le\sqrt{3\pi}\rho\,\|u\|_{L^\infty}$. Thus, the first and second term are bounded by $(K_5+\sqrt{3\pi}\rho^{-1})|\ln\rho|\,\|u\|_{L^\infty}\|u\|_{L^2}$.

However, in order to get (b), we first write, thanks to B.1.1,

$$\|u\|^2_{L^2}\le c_1E^{-1}\int_\Sigma\langle u,\chi\rangle\,dy.$$

That last term, after decomposing χ and integration by parts, is bounded by

$$\int_\Sigma\langle u,\chi\rangle\,dy\le\|u\|_{L^\infty}(\|q\|_{L^1}+\|\nabla b_1\|_{L^2}\|b_2\|_{L^2})+\|\nabla u\|_{L^2}\|b_1\|_{L^\infty}\|b_2\|_{L^2}$$

Covering Σ by $K_6\rho^{-2}$ balls, where K_6 is function of the volume of Σ, the L^p norms are bounded by Taubes norm:

$$\begin{aligned}
\|\cdot\|_{L^1}&\le K_6\rho^{-2}\,\|\cdot\|_{*,\rho}\\
\|\cdot\|_{L^2}&\le\sqrt{K_6}\rho^{-1}\,\|\cdot\|_{2*,\rho}
\end{aligned}$$

Finally, these inequalities give

$$\rho^{-2}|\ln\rho|\,\|u\|_{\mathcal{L}^2}^2 \leq K_6\rho^{-4}|\ln\rho|E^{-1}\|u\|_{\mathcal{L}^0}(\|q\|_{*,\rho}+\|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{2*,\rho})$$

Third term. This one is bounded quite simply, as the singularity is integrable:

$$\int_{B_{2\rho}} |u(\cdot)|^2\ln(d(x,\cdot)^{-1}) \leq 8\rho^2|\ln\rho|\,\|u\|_{\mathcal{L}^\infty}^2 \leq 8\rho^2|\ln\rho|\,\|u\|_{\mathcal{L}^0}^2$$

This term is thus destined to disappear: for ρ small enough, it can be substracted to both sides of the inequality.

Fourth term. As the preceding one, this term will only be negligible for ρ small. We bound it by

$$\begin{aligned}\int_{B_{2\rho}} \alpha_x(\cdot)G(x,\cdot)\,|u|\,|\nabla u| &\leq \|u\|_{\mathcal{L}^\infty}\|\nabla u\|_{2*,\rho}\int_{B_{2\rho}}(\alpha_x(\cdot)G(x,\cdot))^2/\ln(d(x,\cdot)^{-1})\\ &\leq K_7\rho^2|\ln\rho|\,\|u\|_{\mathcal{L}^\infty}\|\nabla u\|_{2*,\rho}\\ &\leq K_7\rho^2|\ln\rho|\,\|u\|_{\mathcal{L}^0}^2,\end{aligned}$$

where K_7 does not depend on the gluing map since $\sup_{x\in\Sigma}\|G(x,\cdot)/|\ln d(x,\cdot)|\|_{\mathcal{L}^\infty(B_{2\rho}(x))} \leq (2\pi)^{-1} + 4c_2\rho^2|\ln 2\rho|$.

Last term. First decompose $\chi = q + b_1\cdot\nabla b_2$. The part containing q is bounded simply thanks to lemma B.1.5 by $c_2\|u\|_{\mathcal{L}^\infty}\|q\|_{*,\rho}$. The rest requires more care. First we intergrate by parts:

$$\begin{aligned}\int_{B_{2\rho}} \alpha_x(\cdot)G(x,\cdot)\,\langle u,\chi\rangle &= -\int_{B_{2\rho}}\Big[\alpha_x(\cdot)G(x,\cdot)\big((\nabla u, b_1\cdot b_2)+(u,\nabla b_1\cdot b_2)\big)\\ &\qquad +(\mathrm{d}(\alpha_x(\cdot)G(x,\cdot))\otimes u, b_1\cdot b_2)\Big].\end{aligned}$$

Apart from the last term, lemma B.1.5 and proposition B.1.4 ($\|ab\|_{*,\rho}\leq\|a\|_{2*,\rho}\|b\|_{2*,\rho}$) allows us to bound this by

$$c_2\|u\|_{\mathcal{L}^0,\rho}\|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{2*,\rho}.$$

As for the ultimate remaining term, we again use lemma B.1.5 to bound the difference between $\mathrm{d}(\alpha_x(\cdot)G(x,\cdot))$ and $(2\pi)^{-1}d(x,\cdot)\nabla d(x,\cdot)$. However, $\phi := \nabla d(x,\cdot)\in C^\infty(S^1;\mathbb{R}^2)$ whence we find the following bound for this remaining term:

$$K(\|u\|_{\mathcal{L}^0,\rho}\|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{2*,\rho}+\|u\otimes b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{\mathcal{L}^1,\rho})$$

Using B.1.4 yields:

$$\int_{B_{2\rho}} \alpha_x(\cdot)G(x,\cdot)\,\langle u,\chi\rangle \leq K\|u\|_{\mathcal{L}^0,\rho}\|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{\mathcal{H},\rho}$$

The bounds found for the five terms enables (when $2K_7\rho^2|\ln\rho| < 1/2$ so that the third and fourth do not weigth on the right-hand side) to show that

$$\|u\|_{\mathcal{L}^0,\rho}^2 \leq c_4\|u\|_{\mathcal{L}^0,\rho}(1+\rho^{-4}|\ln\rho|E^{-1})(\|q\|_{*,\rho}+\|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{\mathcal{H},\rho}).\qquad\square$$

When δ is without order 0 term, if $\eta \in C^\infty(W)$ and $u \in W^{1,2}(V)$ are related by

$$\delta u = \eta,$$

The results of B.2.1 apply using that $\delta^*\delta u = \delta^*\eta$. Indeed, since $\delta^* = -\sigma^*\nabla + l^*$, it suffices to take $q = l^*\eta$ $b_1 = -\sigma^*$ and $b_2 = \eta$ so as to have the following corollary.

Corollary B.2.3: *Let ρ be a small positive number and $E > 0$. Let $c_4 > 0$ as above, if $\eta \in C^\infty(W)$ and $u \in \Pi_E C^\infty(V)$ are so that $\delta u = \eta$, then*

$$\|u\|_{\mathcal{L}^0,\rho} \le c_3(\rho^{-1}|\ln\rho|\,\|u\|_{L^2} + \|\eta\|_{*,\rho}\,\|\eta\|_{\mathcal{H},\rho}) \le c_4(1+\rho^{-6}E^{-1})\,\|\eta\|_{\mathcal{H}}).$$

If $\eta \equiv 0$, it is still possible to get a bound on the norm of u, using standard results.

Lemma B.2.4: *$\forall k \in \mathbb{N}\exists c_{5,k}$ such that $\xi \in C^\infty(V)$ and $\xi \in \operatorname{Ker}\delta$, i.e. $\delta\xi = 0$, then*

$$\|\nabla^{\otimes k}\xi\|_{L^\infty} \le c_{5,k}\,\|\xi\|_{L^2}\,.$$

B.3 Estimating the $\mathcal{L}^1$ norm.

We will now try to get information on the equation $\delta^*\eta = \chi$, with δ^* elliptic and again the decomposition of χ as $q + b_1\nabla b_2$.

Lemma B.3.1: *Let $\rho \in]0,1[$ be sufficiently small, $\exists c_6 > 0$ such that if $\delta^*\eta = \chi$ then*

$$\|\eta\|_{\mathcal{L}^1,\rho} \quad \le c_6(\|q\|_{*,\rho} + |\ln\rho|\,\|b_1\|_{\mathcal{L}^0,\rho}\,\|b_2\|_{\mathcal{H},\rho} + |\ln\rho|\,\|\eta\|_{2*,\rho})$$

The proof being far from obvious, it requires a preparatory lemma and a few extra notations. We first describe a test function which will be multiplied to the equation $\delta^*\eta = \chi$ in order to conclude by integration by parts.

Let V_0 and W_0 be vector spaces of equal dimensions and let $\sigma \in \operatorname{Hom}(\mathbb{R}^2, \operatorname{Hom}(V_0, W_0))$ be such that $\sigma_0(z)$ is an isomorphism $\forall 0 \neq z \in \mathbb{R}^2$. Recall that $\sigma^* \in \operatorname{Hom}(\mathbb{R}^2, \operatorname{Hom}(W_0, V_0))$, thus for $z \in \mathbb{R}^2$, $\sigma_0^*(z)\sigma_0(z) \in \operatorname{End}(V_0)$. Suppose further that $\sigma_0^*(z)\sigma_0(z) = |z|^2\,\mathrm{Id}$.

Let ∇_0 be the euclidean covariant derivative in $\mathbb{R}^2$, then $\delta_0 = \sigma_0(\nabla_0)$ is an elliptic operator of order 1 on $\mathbb{R}^2$ which sends maps with value in V_0 to maps with value in W_0. Similarly, it sends sections of $(V_0 \otimes W_0)$ on sections of $(W_0 \otimes W_0)$.

Finally, since W_0 is an Euclidean vector space, $\operatorname{End}(W_0)$ identifies to $(W_0 \otimes W_0)$, and $1 \in (W_0 \otimes W_0)$ will mean identity as an endomorphism.

Lemma B.3.2: *Let $\psi \in C^\infty(S^1)$, be seen as function on $\mathbb{R}^2 \setminus \{0\}$ which is radially constant. $\exists t_1 \in C^\infty(V_0 \otimes W_0)|_{S^1}$ unique (seen as a section independent of the norm) and t_2 a constant such that for $s(\cdot) = t_1(\cdot) + t_2\ln|\cdot|$*

$$\delta_0(s) = \frac{\psi \otimes 1}{|\cdot|}$$

and, for c_7 a universal constant

$$|t_2| + \|t_1\|_{L^\infty(S^1)} + \|t_1\|_{W^{3,2}(S^1)} \le c_7\,\|\psi\|_{L^2(S^1)}$$

Proof. The operator δ_0 has a (Green's) kernel defined by

$$\mathfrak{p}_p(\cdot) = K_1 \frac{\sigma_0^*(y-p)}{|y-p|^2} \in \mathrm{Hom}(W_0, V_0)$$

Let $\hat{y} = y/|y|$. Let $\psi_L(y) = \frac{\sigma_0(\hat{y})}{2\pi}\int_{S^1}\sigma_0^*(\hat{x})\psi(\hat{x})\mathrm{d}\hat{x}$ be a section of W_0 on $\mathbb{R}^2 \setminus \{O\}$. Then $t_2 = \sigma_0^*(\hat{y})\psi_L(\hat{y}) = \frac{1}{2\pi}\int_{S^1}\sigma_0^*(\hat{x})\psi(\hat{x})\mathrm{d}\hat{x}$ is an element of V_0. Let $\psi_N = \psi - \psi_L$. A formal solution to the equation can be written as

$$t_2 \ln|p| + K_1 \int_{\mathbb{R}^2} \frac{\sigma_0^*(y-p)}{|y-p|^2}\frac{\psi_N(\hat{y})}{|y|}\mathrm{d}y.$$

Let $t_1(p)$ be the expression corresponding to the integral. If $\hat{p} = p/|p|$ and by making a change of variables $y \longrightarrow |p|y$, it appears that $t_1(p) = t_1(\hat{p})$. Consequently, if it converges, the integral defines a section on the circle. Let us now write y in polar coordinates $(|y|, \hat{y})$, then

$$\begin{aligned}
\int_{S_1}\sigma_0^*(\hat{y})\psi_N(\hat{y})\mathrm{d}\hat{y} &= \int_{S_1}\sigma_0^*(\hat{y})\psi(\hat{y})\mathrm{d}\hat{y} - \int_{S_1}\sigma_0^*(\hat{y})\psi_K(\hat{y})\mathrm{d}\hat{y}\\
&= \int_{S_1}\sigma_0^*(\hat{y})\psi(\hat{y})\mathrm{d}\hat{y} - \int_{S_1}\frac{|\hat{y}|^2}{2\pi}\left(\int_{S^1}\sigma^*(\hat{x})\psi(\hat{x})\mathrm{d}\hat{x}\right)\mathrm{d}\hat{y}\\
&= 0.
\end{aligned}$$

Whence, using $\sigma_0^*(\hat{y} - p/|y|) = \sigma_0^*(\hat{y}) - \sigma_0^*(p)/|y|$,

$$\begin{aligned}
t_1(p) &= \int_{\mathbb{R}^2}\frac{|y|}{|y-p|^2}\sigma_0^*(\hat{y} - p/|y|)\psi_N(\hat{y})\mathrm{d}|y|\mathrm{d}\hat{y}\\
&= \int_{\mathbb{R}^2}\frac{|y|}{|y-p|^2}\left(\sigma_0^*(\hat{y})\psi_N(\hat{y}) - \sigma_0^*(p)\psi_N(\hat{y})/|y|\right)\mathrm{d}|y|\mathrm{d}\hat{y}\\
&= \int_{\mathbb{R}^2}\frac{|y|}{|y-p|^2}\sigma_0^*(\hat{y})\psi_N(\hat{y})\mathrm{d}|y|\mathrm{d}\hat{y} + \int_{\mathbb{R}^2}\frac{1}{|y-p|^2}\sigma_0^*(-p)\psi_N(\hat{y})\mathrm{d}|y|\mathrm{d}\hat{y}.
\end{aligned}$$

Thus, the second integral is convergent. There remains to show that the first also converges. The eventuality of divergence could come from large values of $|y|$. Choose p such that $|p| = 1$, when $|y| > 1$, the expansion

$$|y-p|^{-2} = |y|^{-2}(1 + \langle\hat{y}, p\rangle/|y| + |p|^2/|y|^2)^{-1} = |y|^{-2}(1 + o(|y|^{-1}))$$

enables to write

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus B_1(O)}\frac{|y|}{|y-p|^2}\sigma_0^*(\hat{y})\psi_N(\hat{y})\mathrm{d}|y|\mathrm{d}\hat{y} &= \int_{\mathbb{R}^2\setminus B_1(O)}|y|^{-1}\sigma_0^*(\hat{y})\psi_N(\hat{y})\mathrm{d}|y|\mathrm{d}\hat{y}\\
&\quad + \int_{\mathbb{R}^2\setminus B_1(O)} o(|y|^{-1})|y|^{-1}\sigma_0^*(\hat{y})\psi_N(\hat{y})\mathrm{d}|y|\mathrm{d}\hat{y}.
\end{aligned}$$

Integrating first on the angular coordinate, the first integral is shown to be zero, whereas the second converges. Thus we conclude that the integral $t_1(p)$ is also convergent.

The promised bounds on the norms of these function remain to be found.

$$|t_2| \le (2\pi)^{-1}\left\|\sigma_0^*\right\|_{L^2(S^1)}\left\|\psi\right\|_{L^2(S^1)} \le K_2\left\|\psi\right\|_{L^2(S^1)}.$$

As for t_1, it satisfies a first order ordinary differential equation, the norm of its derivative is bounded by that of ψ (the difference between ψ and ψ_N is bounded by $\|\psi\|_{L^2(S^1)}$). Thus,

$$\|\nabla t_1\|_{L^2(S^1)} \leq K_2 \|\psi\|_{L^2(S_1)} .$$

By compactness of S^1, $\|t_1\|_{L^\infty(S^1)} \leq K_3 \|\psi\|_{L^2(S_1)}$ and consequently $\|t_1\|_{L^2(S^1)} \leq \sqrt{2\pi}K_3 \|\psi\|_{L^2(S_1)}$.
□

Proof of lemma B.3.1: Let $x \in \Sigma$ be fixed, and $\rho < \operatorname{injrad}\Sigma$. We will work in a Gaussian coordinate system around x. The metric that comes up in the evaluations of the norms will be replaced by an euclidean metric: indeed, the expressions $\int_{B_\rho(0)} \frac{(v,\phi\otimes w)_{g_E}}{|\cdot|_{g_E}}$ and $\int_{B_\rho(x)} \frac{(v,\phi\otimes w)_g}{|\cdot|_g}$ do not differ by much, the ratio between an euclidean metric and the metric of Σ is a power of $(1+\rho^2)$. Since $\|v\|_{L^\infty} = 1$ and $\|\phi\|_{L^2(S^1)} \leq 1$, this difference is bounded by $K_1\rho^2 \|w\|_{2*,\rho}$ where K_1 is the absolute value of $\rho^{-2}\int_\rho^{2\rho} r^{-2}\left((1+r^2)^k - 1\right)|\ln r|^{-1} r dr$.

Gaussian coordinates give a local trivialization of the cotangent bundle $\mathrm{T}^*|_{B_\rho(x)}$ by associating it to the cotangent bundle of $B_\rho(0) \subset \mathbb{R}^2$. We write the local coordinates of the latter as dy_i, $i = 1$ or 2,and let $v = \sum v_i \otimes dy_i$. In a similar fashion, a local trivialisation of $V|_{B_\rho(x)}$ and $W|_{B_\rho(x)}$ over $B_\rho(0) \times V_0$ and $B_\rho(0) \times W_0$, where $V_0 = V|_x$ and $W_0 = W|_x$, is given by these local coordinates.

Consider now $\sigma_0 = \sigma|_x$ where σ is the principal symbol of the operator δ. Then $\delta_o = \sigma_0(\nabla_0)$ is defined as in lemma B.3.2. This lemma applies on the components ϕ_1 and ϕ_2 of ϕ to give two functions s_1 and s_2. Let $\mathfrak{s}$ be the section (in coordinates) of $V|_{B_\rho(x)}$ defined by

$$\mathfrak{s} = s_1 \otimes v_1 + s_2 \otimes v_2.$$

Multiplying both sides of the equation $\delta^*\eta = \chi$ by $\gamma_x \mathfrak{s}$ (where γ_x is the cutoff function introduced before), an integration by parts reveals

$$\int_{B_{2\rho}(x)} \langle \delta(\alpha_x \mathfrak{s}), \eta\rangle_g = \int_{B_{2\rho}(x)} \langle \alpha_x \mathfrak{s}, \chi\rangle_g \tag{B.3.3}$$

Decomposing $\delta = \delta_0 + d(x,\cdot)\delta'$, yields

$$\delta\mathfrak{s} = \sum(\delta_0 s_i) \otimes v_i + \sum s_i \otimes \delta v + d(x,\cdot)\sum \delta' s_i \otimes v_i.$$

Thus the left-hand side of (B.3.3) can be rewritten as

$$\begin{aligned}\int_{B_{2\rho}(x)} \langle \delta(\alpha_x \mathfrak{s}), \eta\rangle_g &= \int_{A_{\rho,2\rho}(x)} \langle (\delta\alpha_x)\mathfrak{s}, \eta\rangle_g + \int_{B_{2\rho}(x)} \left\langle \Sigma \tfrac{\phi_i v_i}{|\cdot|_{g_E}}, \eta\right\rangle_g \\ &+ \int_{B_{2\rho}(x)} \langle \alpha_x s \otimes \delta v, \eta\rangle_g + \int_{B_{2\rho}(x)} \langle \alpha_x d(x,\cdot)\delta' s \otimes v, \eta\rangle_g\end{aligned}$$

In other words, the term we are interested in is

$$\begin{aligned}\int_{B_{2\rho}(x)} \left\langle \Sigma \tfrac{\phi_i v_i}{|\cdot|_{g_E}}, \eta\right\rangle_g &= \int_{B_{2\rho}(x)} \langle \alpha_x \mathfrak{s}, \chi\rangle_g - \int_{A_{\rho,2\rho}(x)} \langle (\delta\alpha_x)\mathfrak{s}, \eta\rangle_g + \\ &- \int_{B_{2\rho}(x)} \langle \alpha_x s \otimes \delta v, \eta\rangle_g - \int_{B_{2\rho}(x)} \langle \alpha_x d(x,\cdot)\delta' s \otimes v, \eta\rangle_g .\end{aligned}$$

Recall that $s_i(\cdot) = t_{1,i}(\cdot) + t_{2,i} \ln|\cdot|$. The last three terms are bounded as follows:

$$\begin{aligned}
\left| \int_{A_{\rho,2\rho}(x)} \langle (\delta\alpha_x)\mathfrak{s}, \eta \rangle_g \right| &\leq K_3 \|v\|_{L^\infty} \big(K_2 \|t_1\|_{L^\infty} \|\eta\|_{L^2(B_{2\rho})} \\
&\quad + K_2 |\ln\rho| \, |t_2| \, \|\eta\|_{2*,2\rho} \big) \\
\left| \int_{B_{2\rho}(x)} \langle \alpha_x s \otimes \delta v, \eta \rangle_g \right| &\leq K_3 \|t_1\|_{L^\infty} \|\nabla v\|_{L^2(B_{2\rho})} \|\eta\|_{L^2(B_{2\rho})} \\
&\quad + K_3 |t_2| \, \|\nabla v\|_{2*,2\rho} \|\eta\|_{2*,2\rho} \\
\left| \int_{B_{2\rho}(x)} \langle \alpha_x d(x,\cdot) \delta' s \otimes v, \eta \rangle_g \right| &\leq 4\rho^2 K_3 K_4 \|v\|_{L^\infty} \big(\rho \|\nabla t_1\|_{L^2(S^1)} \|\eta\|_{L^2(B_{2\rho})} \\
&\quad + 2\pi |t_2| \, \|v\|_{L^\infty} \|\eta\|_{L^2(B_{2\rho})} \big),
\end{aligned}$$

where $K_2 = 2\pi \int_\rho^{2\rho} r^{-1} dr = 2\pi \ln 2$, K_3 depends on the symbol of δ and $K_4 = \left\| \frac{d(x,\cdot)}{|\cdot|} \right\|_{L^\infty(B_{2\rho})}$. Proposition B.1.4 will be used to find the usual norms: $\|\cdot\|_{L^2(B_{2\rho})} \leq |\ln\rho|^{-1/2} \|\cdot\|_{2*,2\rho}$ given that $\rho < e^{-1}$. Using $\chi = q + b_1 \nabla b_2$, the first term becomes

$$\begin{aligned}
\int_{B_{2\rho}(x)} (\alpha_x \mathfrak{s}, \chi)_g &= \int_{B_{2\rho}(x)} \langle \alpha_x \mathfrak{s}, q \rangle_g + \int_{B_{2\rho}(x)} \langle \alpha_x \mathfrak{s}, b_1 \nabla b_2 \rangle_g \\
&= \int_{B_{2\rho}(x)} \langle \alpha_x \mathfrak{s}, q \rangle_g - \int_{B_{2\rho}(x)} \langle \alpha_x \mathfrak{s}, (\nabla b_1) b_2 \rangle_g - \int_{B_{2\rho}(x)} \langle \nabla(\alpha_x \mathfrak{s}), b_1 b_2 \rangle_g,
\end{aligned}$$

where an integration by parts took place in order to obtain the last line. The first of these three terms can simply be bounded by

$$\|t_1\|_{L^\infty} \|v\|_{L^\infty} \|q\|_{L^1(B_{2\rho})} + |t_2| \, \|v\|_{L^\infty} \|q\|_{*,2\rho}.$$

As for the second, it is bounded by

$$\|t_1\|_{L^\infty} \|v\|_{L^\infty} \|\nabla b_1\|_{L^2(B_{2\rho})} \|b_2\|_{L^2(B_{2\rho})} + |t_2| \, \|v\|_{L^\infty} \|\nabla b_1\|_{2*,2\rho} \|b_2\|_{2*,2\rho}.$$

The third can be written as:

$$\begin{aligned}
\int_{B_{2\rho}(x)} (\nabla(\alpha_x \mathfrak{s}), b_1 b_2)_g &= \int_{A_{\rho,2\rho}(x)} \langle (\nabla\alpha_x)\mathfrak{s}, b_1 b_2 \rangle_g + \int_{B_{2\rho}(x)} \langle \alpha_x (\nabla t_1 \otimes v), b_1 b_2 \rangle_g \\
&\quad + \int_{B_{2\rho}(x)} \left\langle \alpha_x t_2 \frac{\nabla|\cdot|}{|\cdot|} \otimes v, b_1 b_2 \right\rangle_g + \int_{B_{2\rho}(x)} \langle \alpha_x s \otimes \nabla v, b_1 b_2 \rangle_g.
\end{aligned}$$

The bounds are obtained as follows:

$$\begin{aligned}
\left| \int_{A_{\rho,2\rho}(x)} \langle (\nabla\alpha_x)\mathfrak{s}, b_1 b_2 \rangle_g \right| &\leq K_2 \|v\|_{L^\infty} \|b_1\|_{L^\infty} \big(\|t_1\|_{L^\infty} \|b_2\|_{L^2(B_{2\rho})} \\
&\quad + |\ln\rho| \, |t_2| \, \|b_2\|_{2*,2\rho} \big) \\
\left| \int_{B_{2\rho}(x)} \langle \alpha_x \nabla t_1 \otimes v, b_1 b_2 \rangle_g \right| &\leq 4\rho^2 \|\nabla t_1\|_{L^2(S^1)} \|v\|_{L^\infty} \|b_1\|_{L^\infty} \|b_2\|_{L^2(B_{2\rho})} \\
\left| \int_{B_{2\rho}(x)} \left\langle \alpha_x t_2 \frac{\nabla|\cdot|}{|\cdot|} \otimes v, b_1 b_2 \right\rangle_g \right| &\leq \|t_2\|_{L^\infty} \|v\|_{L^\infty} \|b_1\|_{L^\infty} \|b_2\|_{L^1} \\
\left| \int_{B_{2\rho}(x)} \langle \alpha_x s \otimes \nabla v, b_1 b_2 \rangle_g \right| &\leq \|t_1\|_{L^\infty} \|\nabla v\|_{L^2(B_{2\rho})} \|b_1\|_{L^\infty} \|b_2\|_{L^2(B_{2\rho})} \\
&\quad + |t_2| \, \|\nabla v\|_{2*,2\rho} \|b_1\|_{L^\infty} \|b_2\|_{2*,2\rho}.
\end{aligned}$$

□

A remark on the cutoff function might be relevant. In our situation, such a function will always be written as

$$\gamma(r) = \begin{cases} 1 & \text{si } r \in [0,\rho]) \\ \frac{g(2\rho)-g(r)}{g(2\rho)-g(\rho)} & \text{si } r \in (\rho, 2\rho) \\ 0 & \text{si } r \in [2\rho, \infty) \end{cases}$$

Since for ρ sufficiently small $g(2\rho) - g(\rho) \longrightarrow (\nabla g)\rho$, the L^p norm with density $f(r)$ of $\nabla\gamma$ in a ball will be $2\pi\rho^{-p}\int_\rho^{2\rho} rf(r)dr$. Thus, for $f(r) = |\ln r|$, this norm is bounded (it tends to 0) only if $p < 2$.

B.4 The kernel of Π_E.

We now prove some bounds on the part that has so far been neglected. Our goal is to get a bound on $(1-\Pi_E)\chi$ in terms of the norms of q, b_1, and

b_2. To alleviate notations, π_E will denote the projection on small eigenvalues of $\nabla^*\nabla$: $\pi_E = 1 - \Pi_E$. Let $N(E)$ be the number of such eigenvalues and let $\{v_i\}_{i=1}^{N(E)}$ a basis of the image of π_E:

$$\pi_E\chi = \sum_{i=1}^{N(E)} \left[\int_\Sigma \langle v_i, \chi\rangle_g\right] v_i.$$

The main result of this section is to bound $\|\pi_E\chi\|_{*,\rho}$ by $\|q\|_{*,r}$, $\|b_1\|_{\mathcal{L}^0,r}$ and $\|b_2\|_{2*,r}$ but with a parameter $r \neq \rho$. We begin with some preparatory lemmas.

Lemma B.4.1: *Let $\varepsilon \in \mathbb{R}_{>0}$, there exists constants $c_{8,n}$ depending on ε and on the metric on Σ, such that if $\rho \leq \operatorname{injrad}\Sigma$ and $\rho^{-\varepsilon} \geq |\ln\rho|$, then for v an eigenvector of $\nabla^*\nabla$ whose eigenvalue is λ and whose norm $\|v\|_{L^2} = 1$*

$$\left\|\nabla^{\otimes n} v\right\|_{L^\infty} \leq c_{8,n} \max(1, \lambda^{(1+\varepsilon)/(2-\varepsilon)})$$

Proof. When $n = 0$, this bound is a consequence of B.2.1.(a). Indeed, taking $\delta = \nabla$, $E < \lambda$, $u = v$ and $\chi = \lambda v$, yields

$$\begin{aligned} \|v\|_{L^\infty} \leq \|v\|_{\mathcal{L}^0} &\leq c_3(\rho^{-1}|\ln\rho| + \lambda\|v\|_{*,\rho}) \\ &\leq c_3(\rho^{-1}|\ln\rho| + \lambda\|v\|_{L^\infty} K_1\rho^2|\ln\,\rho|), \end{aligned}$$

where $K_1 \leq 8\pi$ comes from the integral $\int_{B_{2\rho}} |\ln r|dr$. Thus, if $\lambda K_1\rho^2|\ln\,\rho| \leq \lambda K_1\rho^{2-\varepsilon} < 1/2$, we have

$$\|v\|_{L^\infty} \leq 2c_3\rho^{-1}|\ln\rho| \leq 2c_3\rho^{-(1+\varepsilon)} \leq 2c_3\max(K_2, (2K_1\lambda)^{(1+\varepsilon)/(2-\varepsilon)}),$$

since ρ can be as large as allowed. We then continue by induction. Suppose that the statement is true for any integer $\leq k$. Then, applying B.2.1.(a) to $u = \nabla^2(\nabla^{\otimes k+1}v)$ and $\chi = \lambda\nabla^{\otimes k+1}v + \sum_{0\leq i\leq k} \mathcal{R}_i\nabla^{\otimes k-i}v_i$, the conclusion follows by the same argument (the $\mathcal{R}_i$ depend only on the metric).

□

Lemma B.4.2: *Let $N(E)$ be the rank of π_E, then $N(E) \leq c_9(E+1)^2$*

A proof of this result can be found in [5].

Lemma B.4.3: *There exists a constant E_0 which depends on the metric such that $\forall x \in \Sigma$ if r_x : $\pi_{E_0}C^\infty(V) \longrightarrow V|_x$ is the restriction at x and $r_x \circ \nabla : \pi_{E_0}C^\infty(V) \longrightarrow (\mathrm{T}\otimes V)|_x$ is the restriction of the derivatices, then r_x and $r_x \circ \nabla$ are surjective.*

Proof. Let s be s smooth section of V such that $\|s\|_{L^2} = 1$. Since Σ is compact, $\|\nabla s\|_{L^2} \leq K_1$. Thus, the expression of s in terms of eigenfunctions converges pointwise. Thus, for any basis of $V|_x$ there exists a E_x such that this basis can be approximated by elements of π_{E_x}. This surjectivity remains valid for points close to x, and by compactness of Σ the conclusion is achieved. The same argument works for $r_x \circ \nabla$. □

We are now ready to show the main result of this section.

Lemma B.4.4: *There exists a constant c_{10} such that for $E \in \mathbb{R}_{>0}$, and $r, \rho < R_{10}$, then for $\chi \in C^\infty(V)$ which can be written as $\chi = q + b_1 \nabla b_2$*

$$\|\pi_E \chi\|_{*,\rho} \leq c_{10} \frac{\rho^2 |\ln \rho|}{r^2 |\ln r|}(1 + r^2 E^{8/3})(\|q\|_{*,r} + \|b_1\|_{L^0} \|b_2\|_{2*,r}).$$

Proof. For two integers n, m big enough, it is possible to choose a set Ω such that

- $\bigcup_{x\in\Omega} B_r(x) = \Sigma$,
- $B_r(x) \cap B_r(x') = \varnothing$ if $x \neq x'$ are two points of Ω,
- for $\Omega' \subset \Omega$, $|\Omega'| \geq m \Rightarrow \bigcap_{x\in\Omega'} B_{nr}(x) = \varnothing$.

This set is easily realized in Euclidean space. Since Σ can be isometrically embedded in $\mathbb{R}^k$, this remains true up to a small perturbation. Consider again the cutoff function α_x defined this time with parameter nr rather than ρ. Furthermore let $\gamma_x(\cdot) = \alpha_x(\cdot)/\sum_{y\in\Omega}\alpha_y(\cdot)$ be the partition of unity associated to the covering of Σ by $\{B_{nr}(x)\}_{x\in\Omega}$. Moreover, the gradient of γ_x behaves nicely: $|\nabla\gamma_x(\cdot)| \leq K_1 r^{-1}$.

As the projection π_E is a linear operator, the bound on χ can be obtained thanks to $\chi = \sum_{x\in\Omega}\gamma_x(\cdot)\chi(\cdot)$. Using lemma B.4.3, for each point $x \in \Omega$, there exists a L^2-orthonormal basis $\{v_i\}_{i=1}^{N(E)}$ of $\pi_E C^\infty(V)$ such that $v_i \in C^\infty(V)$ and, when $i > N(E_0)$, $r_x v_i = 0 = r_x \nabla v_i$. Again, upon integrating by parts, the following expression for the projection of χ on v_i can be obtained

$$\text{(B.4.5)} \qquad \begin{aligned}\int_\Sigma \langle v_i, \gamma_x \chi\rangle_g &= \int_\Sigma \langle v_i, \gamma_x q\rangle_g - \int_\Sigma \langle v_i, \gamma_x(\nabla b_1) b_2\rangle \\ &\quad - \int_\Sigma \langle \nabla v_i, \gamma_x b_1 b_2\rangle - \int_\Sigma \langle v_i, (\nabla\gamma_x) b_1 b_2\rangle .\end{aligned}$$

We begin by the projection of $\pi_E \gamma_x \chi$ on v_i when $i \leq N(E_0)$. In that case, lemma B.4.1 enables us to bound v_i and ∇v_i uniformly by $c_{8,1}(1+E_0)^{2/3}$, thus the right-hand terms in (B.4.5) are bounded respectively by

$$|\ln r|^{-1}\|q\|_{*,2nr}+|\ln r|^{-1}\|\nabla b_1\|_{2*,2nr}\|b_2\|_{2*,2nr}$$
$$+r|\ln r|^{-1}\|b_1\|_{L^\infty}\|b_2\|_{2*,2nr}+K_1^2|\ln r|^{-1}\|b_1\|_{L^\infty}\|b_2\|_{2*,2nr}.$$

All these norms can be put together to give $\left|\int_\Sigma \langle v_i,\gamma_x\chi\rangle_g\right| \leq |\ln r|^{-1}K_2(\|q\|_{*,r}+\|b_1\|_{\mathcal{L}^0,r}\|b_2\|_{2*,r})$, where B.1.4 is used to pass from the parameter $2nr$ to r. Also, $\|v_i\|_{*,\rho}\leq c_{8,0}(1+E_0)^{2/3}\rho^2|\ln\rho|$, which yields:

$$\left\|v_i\int_\Sigma \langle v_i,\gamma_x\chi\rangle_g\right\|_{*,\rho}\leq K_3\rho^2|\ln\rho||\ln r|^{-1}(\|q\|_{*,r}+\|b_1\|_{\mathcal{L}^0,r}\|b_2\|_{2*,r}).$$

Now, if $i>N(E_0)$ the choice of the v_i gives that $|v_i|\leq c_{8,2}r^2E^{2/3}$ and $|\nabla v_i|\leq c_{8,2}rE^{2/3}$. This time the right-hand terms of (B.4.5) are bounded as follows:

$$c_{8,2}E^{2/3}r^2|\ln r|^{-1}\big[\|q\|_{*,2nr}+\|\nabla b_1\|_{2*,2nr}\|b_2\|_{2*,2nr}$$
$$+\|b_1\|_{L^\infty}\|b_2\|_{2*,2nr}+K_1^2\|b_1\|_{L^\infty}\|b_2\|_{2*,2nr}\big].$$

Since $\|v_i\|_{*,\rho}\leq c_{8,0}(1+E)^{2/3}\rho^2|\ln\rho|$, we have

$$\left\|v_i\int_\Sigma \langle v_i,\gamma_x\chi\rangle_g\right\|_{*,\rho}\leq K_4\rho^2|\ln\rho|r^2|\ln r|^{-1}(\|q\|_{*,r}+\|b_1\|_{\mathcal{L}^0,r}\|b_2\|_{2*,r}).$$

As $N(E)-N(E_0)\leq K_5E^2$, the decomposition $\pi_E\gamma_x\chi=\sum_{x\in\Omega}v_i\int_\Sigma\langle v_i,\gamma_x\chi\rangle_g$ enables to conclude that

$$\|\pi_E\gamma_x\chi\|_{*,\rho}\leq K_6\rho^2|\ln\rho||\ln r|^{-1}(1+r^2E^{8/3})(\|q\|_{*,r}+\|b_1\|_{\mathcal{L}^0,r}\|b_2\|_{2*,r}).$$

The finishing touch consists in noticing that the cardinality of Ω is bounded by K_7r^{-2}, where K_7 depends on the metric. □

B.5 Existence and a priori bound on solutions

It will be necessary to introduce

$$\langle\chi\rangle_\rho=\|q\|_{*,\rho}+\|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{\mathcal{H},\rho}$$

The linearized operator of $\bar\partial_J$ at f is the operator D_f introduces in §A.2. Even if for many structures it is invertible when f is J-holomorphic, we shall need to deal with this operator for a function which is precisely not J-holomorphic (in the complement of a disc). The projection Π_E enables to avoid problems that arise from a lack of surjectivity.

Define first $\chi'(u)$ by

$$\delta^*\delta u=\nabla^*\nabla u+\sigma'\nabla u+Ru=\nabla^*\nabla u+\chi'(u), \tag{B.5.1}$$

where σ' is the symbol of a first-order operator. A wise use of lemma B.1.1 will give the existence of a $u\in\Pi_EC^\infty$ such that $\Pi_E\delta^*\delta u=\Pi_E\chi$.

Lemma B.5.2: *Let δ be an elliptic operator as above, there exists a constant c_{11} (which depends on δ) such that when $E > c_{11}$, the equation $\Pi_E\delta^*\delta u = \Pi_E\chi$ admits an unique solution $u \in \Pi_E C^\infty(V)$. Moreover this solution depends continuously and linearly on χ.*

Proof. Write $\nabla^*\nabla u = \Pi_E(\chi - \chi'(u))$. Lemma B.1.1 insures the existence of a section u_χ such that $\nabla^*\nabla u_\chi = \Pi_E\chi$ and of $\psi(u)$ solution to $\nabla^*\nabla\psi(u) = -\Pi_E\chi'(u)$. Thus, the problem can be expressed as the existence of a fixed point for

$$u = \psi(u) + u_\chi.$$

It suffices to show that $u \mapsto \psi(u)$ is contracting as a map from $\Pi_E W^{1,2}(V)$ to itself. First, since $\|\chi'\|_{L^2} \leq K_1 \|u\|_{W^{1,2}}$ lemma B.1.1 shows that if $E > c_1$

$$\begin{aligned} & E\,\|\psi(u)\|_{L^2}^2 && \leq 2\left|\textstyle\int \langle \psi(u), \chi'\rangle\right| && \leq 2K_1 \|\psi(u)\|_{L^2}\|u\|_{W^{1,2}} \\ \Rightarrow\quad & \|\psi(u)\|_{L^2} && \leq 2K_1E^{-1}\|u\|_{W^{1,2}}. \end{aligned}$$

Using this inequality, a second application of the same lemma gives

$$\begin{aligned} \|\nabla\psi(u)\|_{L^2}^2 &\leq 2\left|\textstyle\int \langle \psi(u), \chi'\rangle\right| \\ &\leq 2K_1\|\psi(u)\|_{L^2}\|u\|_{W^{1,2}} \\ &\leq 4K_1^2E^{-1}\|u\|_{W^{1,2}}^2. \end{aligned}$$

Thus, $\|\nabla\psi(u)\|_{W^{1,2}} \leq 4K_1E^{-1/2}\|u\|_{W^{1,2}}$, that is the linear map $\psi : \Pi_E W^{1,2}(V) \to \Pi_E W^{1,2}(V)$ in question is contracting given that $E > \max(16K_1^2, 1, c_1)$. In other words, $u = \psi(u) + u_\chi \Leftrightarrow (\mathrm{Id} - \psi)(u) = u_\chi$. However $\mathrm{Id} - \psi$ can be inverted by a formal serie (which converges since $\|\psi\| < 1$). The solution to our fixed point equation is $u = (\mathrm{Id} - \psi)^{-1}(u_\chi)$. Thus, linearity of the dependence on u comes from the linear dependence of u_χ on χ. Arguments of ellipticity enables us to conclude that $u \in \Pi_E C^\infty(V)$. □

Theorem B.5.3: *Let E and ρ be such that $E\rho^4|\ln\rho| > 10^{-6}$. The equation $\Pi_E\delta^*\delta u = \Pi_E\chi$ admits an unique solution $u \in \Pi_E C^\infty(V)$ which depends continuously and linearly on χ and such that*

$$\|u\|_{\mathcal{L},\rho} \leq c_{12}(1 + |\ln\rho|)\big(\|q\|_{*,\rho} + \|b_1\|_{\mathcal{L}^0,\rho}\|b_2\|_{\mathcal{H},\rho}\big)$$

Proof. The previous lemma covers all the assertions of the theorem with the exception of the bound on $\|u\|_{\mathcal{L}}$. This is done using lemma B.2.1:

$$\|u\|_{\mathcal{L}^0,\rho} \leq c_4(1 + E^{-1}\rho^{-4}|\ln\rho|)\,\langle\chi\rangle_\rho \tag{B.5.4}$$

The $\mathcal{L}^0$ norm of ∇u requires more work. First observe that u satisfies the following system of equations

$$\begin{aligned} \nabla^*\nabla u &= \Pi_E(\chi + \chi'(u)) \\ \nabla\nabla u &= R^\nabla u \end{aligned}$$

where R^∇ is the curvature tensor. The operator $\nabla^* \oplus \nabla : C^\infty(V) \to C^\infty(V) \oplus C^\infty(\mathrm{T}^*\Sigma \times \mathrm{T}^*\Sigma \times V)$ is elliptic of the first order. Lemma B.3.1 can be used on $(\nabla^* \oplus \nabla)(\nabla u) = \Pi_E(\chi + \chi'(u)) \oplus R^\nabla u$ to get that

$$\|\nabla u\|_{\mathcal{L}^1} \leq c_6\big(\|q\|_{*,\rho} + \left\|\Pi_E \chi'(u)\right\|_{*,\rho} + \left\|R^\nabla u\right\|_{*,\rho} + |\ln\rho|\, \|b_1\|_{\mathcal{L}^0,\rho}\, \|b_2\|_{\mathcal{H},\rho} + |\ln\rho|\, \|\nabla u\|_{2*,\rho}\big)$$

where q, b_1 and b_2 come from the decomposition $\Pi_E\chi = q + b_1 \nabla b_2$. For a constant K_1 which depends of the terms of order less than 2 in $\delta^*\delta$,

$$\left\|\Pi_E \chi'(u)\right\|_{*,\rho} \leq K_1(\|u\|_{*,\rho} + \|\nabla u\|_{2*,\rho}) \leq K_1 \|u\|_{\mathcal{L}^0,\rho}.$$

Moreover, there exists another constant such that $\left\|R^\nabla u\right\|_{*,\rho} \leq K_2 \|u\|_{*,\rho}$. Thus,

$$\|\nabla u\|_{\mathcal{L}^1,\rho} \leq K_3\big(\|q\|_{*,\rho} + |\ln\rho|\, \|b_1\|_{\mathcal{L}^0,\rho}\, \|b_2\|_{\mathcal{H},\rho} + \|u\|_{\mathcal{L}^0,\rho} + |\ln\rho|\, \|\nabla u\|_{2*,\rho}\big).$$

Adding this inequality to (B.5.4) multiplied by $(2 + |\ln\rho|)K_3$ gives

$$\|u\|_{\mathcal{L},\rho} \leq K_4(1 + |\ln\rho|)\big(\|q\|_{*,\rho} + \|b_1\|_{\mathcal{L}^0,\rho}\, \|b_2\|_{\mathcal{H},\rho}\big) \qquad \square$$

Note that if for some reason the operator δ is surjective, it is no longer necessary to project on large eigenvalues of the Laplacian. Thus, it is possible to obtain the same estimates. Here is a case of interest.

Corollary B.5.5: *Let $\Sigma = \mathbb{C}\mathrm{P}^1$, and let h be a solution of $D_u D_u^* h = \chi$, then*

$$\|\xi\|_{\mathcal{L},10^{-1}} \leq 4c_{12}\big(\|q\|_{*,10^{-1}} + \|b_1\|_{\mathcal{L}^0,10^{-1}}\, \|b_2\|_{\mathcal{H},10^{-1}}\big)$$

Proof. The proof is identical to the one of the previous theorem with the exception that it is needed to take $\rho = 10^{-1} < e^{-1}$ and E the smallest eigenvalue of the Laplacian. $\square$

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Dimension moyenne et espaces d'applications pseudo-holomorphes

Résumé

Le présent texte s'articule en deux thèmes. Le premier commence par l'évaluation des largeurs de boules unités dans des espaces de Banach. Ces évaluations peuvent être perçues comme un problème de compression: on s'intéresse à des applications non linéaires de fibre aussi petites que possible qui envoient ces boules unités vers des polyèdres de dimension fixée. Des bornes pour ces quantités sont obtenues, le cas des boules l^p (de dimension finie ou infinie) avec leur métrique y est plus particulièrement étudié. Les largeurs interviennent aussi dans la définition de la dimension moyenne, une adaptation de l'entropie à des cas où elle est infinie. Cependant, cet invariant dynamique est insuffisant pour différencier les systèmes donnée par la boule unité de $l^p(\Gamma;\mathbb{R})$ avec action naturelle de Γ où p est fini et Γ est un groupe (typiquement $\mathbb{Z}$). Une modification de la dimension moyenne est ainsi introduite pour s'occuper de ces cas, elle n'est cependant plus un invariant topologique mais est Hölder covariante. Ceci est encore suffisant pour obtenir des obstructions. Une autre variante, $\dim_{l^p}$, qui est reliée à la dimension de Von Neumann est aussi introduite s'inspirant de résultats de Gromov. Quelques unes des propriétés de $\dim_{l^p}$ sont démontrées (requérant une généralisation du lemme d'Ornstein-Weiss). Cependant, des propriétés importantes restent en suspens.

Le second thème traite des courbes pseudo-holomorphes. Un résultat sur le recollement de deux courbes pseudo-holomorphes est d'abord démontré. Celui-ci permet d'avoir une idée plus précise du comportement de la courbe recollée près du point où les deux courbes d'origines se touchent. Ensuite, nous nous intéressons à former des cylindres pseudo-holomorphes depuis une chaîne de courbes pseudo-holomorphes, et sous de fortes hypothèses, un résultat d'interpolation est obtenu. L'interpolation permet entre autres de montrer que les cylindres obtenus sont simples, d'images distinctes, et forment une famille de dimension infinie. Les deux thèmes se rejoignent étant donné que la famille d'applications obtenue est de dimension moyenne positive.

Un appendice contient une adaptation de la "boîte à outils" de Taubes (des méthodes d'analyse elliptique introduite dans "The existence of anti-self-dual structures") au cas de dimension 2. Cependant, à cause de la spécificité du noyau de Green en dimension 2, celles-ci n'a pu être appliquée à la démonstration d'un théorème de Runge pour les courbes pseudo-holomorphes.

Mots-clefs : largeur, dimension moyenne, lemme d'Ornstein-Weiss, dimension de Von Neumann, applications pseudo-holomorphe, chirurgie dénombrable, interpolation, analyse elliptique à la Taubes.

MEAN DIMENSION AND
SPACES OF PSEUDO-HOLOMORPHIC MAPS

Abstract

This thesis covers two themes. The first begins by evaluating the width of unit balls in Banach spaces. Evaluation of width can be seen as a problem arising from compressed sensing: we look at nonlinear maps with small fiber diameters that send these unit balls to polyhedra of given dimension. Bounds for these quantities are found, focusing on the case of l^p balls (of finite or infinite dimension) with their proper metric. Widths are also related to mean dimension, an adaptation of entropy to cases where it would be infinite. However, this dynamical invariant turns out to be inefficient if one wishes to distinguish between the dynamical systems given by the unit ball of $l^p(\Gamma;\mathbb{R})$ with natural action of Γ for finite p and Γ a discrete group (typically $\mathbb{Z}$). A alteration of mean dimension is thus introduced to deal with this case, but it is no longer a topological invariant but Hölder covariant. This is still sufficient to obtain obstructions. Another variant which relates to Von Neumann dimension is also introduced, following Gromov, and some properties are then proved (requiring in particular an extension of the Orstein-Weiss lemma). However, important properties are left unproven.

The second theme deals with pseudo-holomorphic curves. We first modify a result on the gluing of two pseudo-holomorphic curves so as to have a precise behaviour of the glued curve close to the point of intersection of the two curves it comes from. Then pseudo-holomorphic cylinders are constructed from a chain of pseudo-holomorphic curves. Under strong assumptions, we obtain an interpolation result on these cylinders. This interpolation result has many consequences, in particular, that thedifferent cylinders obtained are simple, have different images, and form a family of infinite dimension. This theme is reunited with the first as this family has also positive mean dimension.

An appendix contains an adaptation of "Taubes toolbox" (methods of elliptic analysis developed by Taubes in "The existence of anti-self-dual structures") to the 2-dimensional case. However, due to the specificity of Green's kernel in dimension 2, these could not be applied to the proof of a Runge theorem for pseudo-holomorphic curves.

Keywords : largeur, dimension moyenne, lemme d'Ornstein-Weiss, dimension de Von Neumann, applications pseudo-holomorphe, chirurgie dénombrable, interpolation, analyse elliptique à la Taubes.