Astérisque

Hajer Bahouri
Clotilde Fermanian-Kammerer
Isabelle Gallagher

Phase-space analysis and pseudodifferential calculus on the Heisenberg group

Astérisque, tome 342 (2012)
<http://www.numdam.org/item?id=AST_2012__342__R1_0>

342 ASTÉRISQUE

2012

PHASE-SPACE ANALYSIS AND PSEUDODIFFERENTIAL CALCULUS ON THE HEISENBERG GROUP

Hajer Bahouri & Clotilde Fermanian-Kammerer & Isabelle Gallagher

SOCIÉTÉ MATHÉMATIQUE DE FRANCE
Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Hajer Bahouri
LAMA UMR CNRS 8050, Université Paris-Est Créteil, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France
hbahouri@math.cnrs.fr

Clotilde Fermanian-Kammerer
LAMA UMR CNRS 8050, Université Paris-Est Créteil, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France
Clotilde.Fermanian@u-pec.fr

Isabelle Gallagher
Institut de Mathématiques UMR 7586, Université Paris Diderot (Paris 7), 175, rue du Chevaleret, 75013 Paris, France
Isabelle.Gallagher@math.jussieu.fr

***Classification mathématique par sujet* (2000).** — 35S05, 43A80, 35A27.

Mots-clefs. — Groupe de Heisenberg, calcul pseudo-différentiel, calcul de Weyl-Hörmander.

PHASE-SPACE ANALYSIS AND PSEUDODIFFERENTIAL CALCULUS ON THE HEISENBERG GROUP

Hajer Bahouri, Clotilde Fermanian-Kammerer & Isabelle Gallagher

Abstract. — A class of pseudodifferential operators on the Heisenberg group is defined. As it should be, this class is an algebra containing the class of differential operators. Furthermore, those pseudodifferential operators act continuously on Sobolev spaces and the loss of derivatives may be controled by the order of the operator. Although a large number of works have been devoted in the past to the construction and the study of algebras of variable-coefficient operators, including some very interesting works on the Heisenberg group, our approach is different, and in particular puts into light microlocal directions and completes, with the Littlewood-Paley theory initiated in 2000 by Bahouri, Gérard and Xu, a microlocal analysis of the Heisenberg group.

Résumé **(Analyse dans l'espace des phases, et calcul pseudodifférentiel sur le groupe de Heisenberg).** — Nous définissons une classe d'opérateurs pseudo-différentiels sur le groupe de Heisenberg. Comme il se doit, cette classe constitue une algèbre contenant les opérateurs différentiels. De plus, ces opérateurs pseudo-différentiels sont continus sur les espaces de Sobolev et l'on peut contrôler la perte de dérivée par leur ordre. Si un grand nombre de travaux ont été déjà consacrés à la construction et à l'étude d'algèbres d'opérateurs à coefficients variables, y compris des travaux très intéressants sur le groupe de Heisenberg, notre approche est différente et en particulier elle conduit à la notion de direction microlocale, et complète l'élaboration d'une analyse microlocale sur le groupe de Heisenberg commencée par Bahouri, Gérard et Xu en 2000 par le développement d'une théorie de Littlewood-Paley.

CONTENTS

CHAPTER 1

INTRODUCTION AND MAIN RESULTS

1.1. Introduction

1.1.1. The Heisenberg group. — The Heisenberg group is obtained by constructing the group of unitary operators on $L^2(\mathbb{R}^n)$ generated by the n-dimensional group of translations and the n-dimensional group of multiplications (see for instance the book by M. Taylor [**53**]). It is an unimodular, nilpotent Lie group whose Haar measure coincides with the Lebesgue measure, and its remarkable feature is that its representation theory is rich as well as simple in structure. It is actually the first locally compact group whose infinite-dimensional, irreducible representations were classified (see [**22**]). It can be identified with a subgroup of the group of $(n+2)\times(n+2)$ real matrices with 1's on the diagonal and 0's below the diagonal.

It has a dual nature, in the sense that it may be realized as the boundary of the unit ball in several complex variables (thus extending to several complex variables the role played by the upper half plane and the Hilbert transform on its boundary) as well as being closely tied to quantum theory (via the Heisenberg commutators). We refer to the book by E. Stein [**52**], Chapter XII, for a comprehensive presentation of that duality.

Harmonic analysis on the Heisenberg group is a subject of constant interest, due on the one hand to its rich structure (though simple compared to other noncommutative Lie groups), and on the other hand to its importance in various areas of mathematics, from Partial Differential Equations (see among others [**7**], [**12**], [**16**] [**29**], [**30**], [**44**], [**45**], [**59**], [**60**]) to Geometry (see [**2**], [**18**], [**31**], [**47**]) or Number Theory (see for instance [**42**], [**55**]). Many research articles and monographs have been devoted to harmonic analysis on the Heisenberg group, and we shall give plenty of references as we go along.

1.1.2. Microlocal analysis on $\mathbb{R}^n$. — Microlocal analysis in the euclidian space appeared in the early seventies ([**50**]-[**51**]), and has at its foundation the theory of pseudodifferential operators. The main idea of microlocal analysis is to study a function simultaneously in the space variables of the physical space and in the

Fourier variables. Indeed, some phenomenon need both analysis to be correctly understood. As an example, let us consider the obstuctions to the convergence to zero in $L^2(\mathbb{R}^d)$ of two sequences, one of the form $u_n = h_n^{-d/2}\phi\left(\frac{x-x_0}{h_n}\right)$ and the other of the form $v_n = \exp\left(i\frac{(x\cdot\xi_0)}{h_n}\right)\phi(x)$ where $h_n \to 0$ and ϕ is in the Schwartz class for example. Of course, the point x_0 is a point of *concentration* in the space variables for the sequence u_n and as such, a point of obstruction to strong convergence to zero of the sequence. Similarly the *oscillations* in the direction ξ_0 correspond to *concentration* in Fourier variables for the sequence v_n, and they are also an obstruction to the strong convergence of the sequence.

With this point of view, it appears crucial to be able to use localization operators in space variables *and* in frequencies: the latter are Fourier multipliers. The theory of pseudodifferential operators provides a framework in which both points of view are unified: multiplication operators *and* Fourier multipliers are indeed pseudodifferential operators. More precisely, a pseudodifferential operator is defined by its *symbol* which is a function on the phase space: the symbol of the operator of multiplication by $\phi(x)$ is the function $(x,\xi) \mapsto \phi(x)$ and the symbol of the Fourier multiplier $\chi(D)$ is the function $(x,\xi) \mapsto \chi(\xi)$.

With pseudodifferential operators comes the concept of properties which hold *microlocally.* A function f satisfies a property (P) locally if for all cut-off function χ, the function χf satisfies (P); similarly, replacing the functions χ by a pseudodifferential operator with symbol supported in a given subset Ω of the phase-space, one gets a property satisfied microlocally in Ω. This notion allows a closer perception of the singularities of a function: in the 70's was developed the notion of *wave fronts*, analytic wave front, $\mathscr{C}^\infty$ wave front, etc. The idea is to associate with a given function f a region of the phase space where, microlocally, f is analytic or $\mathscr{C}^\infty$ or whatever else: this region is by definition the complement of the wave front.

One should notice that the phase space corresponds to the space of positions-impulsions of Quantum Mechanics, and thus enjoys nice geometric properties. It can be understood as the cotangent space to $\mathbb{R}^d$ (or to a submanifold if one works on a manifold) and is a symplectic space once endowed with the adapted symplectic form. This geometric aspect has been used successfully in numerous works and is one of the satisfying aspects of microlocal analysis (see for example the development of microlocal defect measures, semi-classical measures and Wigner measures as in [**34**] and [**35**] for example).

Microlocal analysis allowed for a very general study and classification of linear Partial Differential Equations with variable coefficients, using for example Littlewood-Paley operators which select a range of frequencies; such operators are pseudodifferential operators. In the case of nonlinear Partial Differential Equations, the situation is of course much more complicated, but paradifferential calculus ([**13**]) turned out to be a very powerful tool, for instance to analyze the propagation of singularities of solutions to such equations, or to study the associate Cauchy problem (see for instance [**3**], in the case of quasilinear wave equations).

Pseudodifferential operators on the euclidian space form an algebra, which is a very important fact. This algebra contains Fourier multipliers such as differentiation operators, microlocalisation operators, Littlewood-Paley operators, paradifferential operators.

1.1.3. Microlocal analysis on the Heisenberg group. — The development of microlocal tools adapted to the geometric situation at hand is an important issue: we refer for instance to the work of S. Klainerman and I. Rodnianski **[40]** in the case of the Einstein equation, where the construction of an adapted Littlewood-Paley theory is a crucial tool to reach optimal regularity indexes for the initial data. Microlocal theory on $\mathbb{R}^n$ easily passes to submanifolds. Other constructions have been performed on the torus, or more general compact Lie groups (see for instance **[49]**).

A number of articles can be found in the literature, which develop a pseudodifferential calculus on the Heisenberg group. For example, in **[52]**, **[53]**, this question is investigated through the angle of the Weyl correspondence (see also the previous work **[37]**): as recalled above, that correspondence is one of the rich features of the Heisenberg group, and is thoroughly developed in those references. The important work **[33]** consists in constructing an analytic calculus enabling one to obtain parametrices for a class of operators which are analytic hypoelliptic; we also refer to **[43]** and **[10]** as well as **[17]** where a parametrix is constructed for sum-of-squares type operators. One also must mention the series of papers by P. Greiner and his coauthors (see for instance **[9]**, **[32]** and **[36]** and and the references therein) in which in particular symbols of left-invariant vector fields are constructed, from the point of view of Laguerre calculus as well as using the Hermite basis and the recent works **[56]**-**[57]**, where a symbolic calculus on the Heisenberg group is developped, related to contact manifolds. Finally, we refer to the work **[21]** where is constructed a pseudodifferential calculus based on Hörmander calculus, using exclusively the convolution rather than the Fourier transform.

Our approach here is not quite of the same nature as in the works refered to above, as we aim at defining an algebra of operators on functions defined on the Heisenberg group, which contains differential operators and Fourier multipliers, and which has a structure close to that of pseudodifferential operators in the Euclidian space. The difficulty in this approach is that there is no simple notion of symbols as functions on the Heisenberg group $\mathbb{H}^d$, since the Fourier transform is a family of operators on Hilbert spaces depending on a real-valued parameter λ. Those operators are built using the so-called Bargmann representation, or the Schrödinger representation (obtained from the previous one by intertwining operators). One can easily check that what may appear as the symbol associated with a left-invariant vector field is itself a family of operators. This family reads in the Schrödinger representation of $\mathbb{H}^d$ as a family of differential operators belonging to a class of operators of order 1 for the Weyl-Hörmander calculus (see **[38]**) of the harmonic oscillator. That basic observation is the heart of the matter achieved in this paper. Let us point out that in fact symbols on the Heisenberg group cannot depend only on the harmonic oscillator, and this has to do with the dependence on the parameter λ. This induces a number of technical problems that are dealt with by introducing also a specific calculus in the λ direction.

A symbol on the Heisenberg group is thus a function on $\mathbb{H}^d$ valued in the space of families of symbols of the Weyl-Hörmander class associated to the harmonic oscillator, indexed by the parameter λ. Then, to this symbol, one associates a pseudodifferential operator as is usually done by use of the inverse Fourier transform as well as the family of Weyl-quantized operators associated with the symbol.

Once those pseudodifferential operators have been defined, we first prove that they are operators on the Schwartz class, which results from classical Fourier analysis on the Heisenberg group. We then prove that the adjoint of a pseudodifferential operator and the composition of two pseudodifferential operators are also pseudodifferential operators. Our arguments here are deeply inspired by the analysis of the classical case as developped for instance in the book of S. Alinhac and P. Gérard [**1**]. We analyze first the link between the kernel of a pseudodifferential operator and its symbol, using the Fourier transform and its inverse. Then, it is possible to compute the function which could be the symbol of the adjoint of a pseudodifferential operator or of the composition of two pseudodifferential operators and to prove that it actually is a symbol. This comes from the careful analysis of oscillatory integrals. We also give asymptotic formula for the symbol of the adjoint or of the composition. These formulas result from a Taylor formula in the spirit of what is done in the Euclidian space but adapted to the case of the Heisenberg group; in particular, we crucially use functional calculus. The specific feature of these asymptotic formula is that there is no gain on the Heisenberg group: the commutator of two horizontal vector fields is a derivation.

We also study the action of pseudodifferential operators on Sobolev spaces. We prove in particular that zero order operators are bounded on $L^2(\mathbb{H}^d)$ and more generally a pseudodifferential operator is continuous from one Sobolev space to another, the link between the regularity exponents of the Sobolev spaces being controled by the order of the symbol. The arguments of this proof are inspired by the Euclidian proof of R. Coifman and Y. Meyer [**20**] whose approach consists mainly in decomposing the symbol of the pseudodifferential operator on $\mathbb{R}^n$ (which is a function on the phase space $T^*\mathbb{R}^n$) into a convergent series of reduced symbols for which the continuity is a consequence of paradifferential calculus of J.-M. Bony [**13**]. The main interest of this approach is that it requires little regularity on the symbol and that it can be carried out when the pseudodifferential calculus has no gain, which is the case in our situation. Roughly speaking, the proof of R. Coifman and Y. Meyer is done in three steps. In the first step, a symbol is decomposed using a dyadic partition of unity. This reduces the problem to the study of symbols compactly supported in the frequency variable. Next, using a Fourier series expansion, the symbol is expressed as a sum of reduced symbols which are much easier to deal with. Finally, taking advantage of the Littlewood-Paley decomposition on $\mathbb{R}^n$, the continuity on Sobolev spaces of the associate operator is established. To adapt that method to the setting of the Heisenberg group $\mathbb{H}^d$, we begin by decomposing the symbol associated with a given operator (defined as explained above via the Weyl-Hörmander calculus of the harmonic oscillator), using a suitable dyadic partition of unity. Then, we use Fourier series to write the symbol as a convergent series of reduced symbols. But, in contrast to the $\mathbb{R}^n$ setting, the reduced symbols in that case cannot be treated as a sum of Littlewood-Paley operators on the Heisenberg group. To overcome this difficulty, we use Mehler's formula to prove that these operators can be related in some sense to the reduced symbols obtained in the $\mathbb{R}^n$ case. This allows us to finish the proof in more or less the same way as in the $\mathbb{R}^n$ case, up to the fact that an additionnal microlocalization is needed because the spectral parameter is made of two different variables – as pointed out above, this is due to the special structure of the Heisenberg group.

This paper completes, with the Littlewood-Paley theory developed in **[7]** and **[5]**, a microlocal analysis of the Heisenberg group. It calls for developments : a significant application would be the generalization of the concept of wave front set to the setting of the Heisenberg group, in order to obtain results related to the propagation of singularities as in **[58]** for instance. One can also expect a construction of parametrices, as well as the development of a notion of microlocal defect measure (or H-measure). Such studies are postponed to a future work.

Generalizations to other locally compact Lie groups should also be considered. The generalization of the Littlewood-Paley decomposition is in itself a challenge : although it is known (see **[39]**) that a frequency localization process can be defined in general as a convolution product with a function of the Schwartz class, Bernstein inequalities seem very difficult to obtain in general (and these inequalities are the crucial property that allow to construct a Littlewood-Paley theory). Once that difficulty is overcome, the next step should be the understanding of the phase space in more general contexts.

1.1.4. Structure of the paper. — The structure of the paper is the following. The rest of this chapter is devoted to a recollection of the main facts on the Heisenberg group which will be useful for us, as well as to the statement of the main results. More precisely, in Section 1.2.1, we introduce our notation and give the basic definitions and in Section 1.2.2, we recall the definition of the Fourier transform, using irreducible representations. The purpose of the next section of this chapter is to provide the setting for symbols and operators on the Heisenberg group, and it also contains the statement of the main results; for this some elements of Weyl-Hörmander calculus are required, and the necessary definitions are recalled. The main results stated in this chapter (in Section 1.4) concern the continuity of pseudodifferential operators on Sobolev spaces, along with the fact that those classes of operators form an algebra.

The second chapter is devoted to the analysis of examples and to the proof of some fundamental properties of pseudodifferential operators, such as their action on the Schwartz class, the study of their kernel, their composition with differentiation operators.

In the third chapter, we prove that the classes of pseudodifferential operators defined in the previous chapter are stable by adjunction and composition and prove asymptotic expansion of their symbol.

In the fourth chapter we give an outline of the basic elements of Littlewood-Paley theory on the Heisenberg group developed in [7] and [5] recalling in that framework the properties of Besov spaces that we shall need later on. Next, we compare Littlewood-Paley operators with pseudodifferential operators. This is of crucial importance in the next chapter. More precisely, we prove that in some sense, a pseudodifferential operator associated to a truncated symbol, in the Weyl-Hörmander calculus of the harmonic oscillator, is close to a Littlewood-Paley operator.

In the fifth chapter, we prove the continuity on Sobolev spaces, by a (non trivial) adaptation of the technique of R. Coifman and Y. Meyer [20] to the case of the Heisenberg group; in particular an additional microlocalization is required, compared to the classical case.

Finally this paper comprises two appendixes. Appendix A is devoted to the proof of some technical lemmas and formulas concerning the Heisenberg group that are used in the paper. In Appendix B we prove a number of important results used in the proofs of the main theorems of this paper, but for which the arguments are too lengthy or too technical to appear in the main text; they are mainly related to Weyl-Hörmander calculus.

Acknowledgements. — This project originates in a discussion with G. Lebeau, and we are happy to acknowledge his influence in this study. We also thank J.-Y. Chemin , P. Gérard and N. Lerner for numerous fruitful discussions. H. Bahouri gratefully acknowledges the hospitality of the Fondation Sciences Mathématiques de Paris which supported a stay in the Institut de Mathématiques de Jussieu, during which part of this project was accomplished. Finally we extend our thanks to the anonymous referee for a careful reading of the manuscript and fruitful remarks.

1.2. Basic facts on the Heisenberg group $\mathbb{H}^d$

1.2.1. The Heisenberg group. — Before stating the principal results of this paper, let us collect a few well-known definitions and results on the Heisenberg group $\mathbb{H}^d$. We recall that it is defined as the space $\mathbb{R}^{2d+1}$ whose elements $w \in \mathbb{R}^{2d+1}$ can be written $w = (x, y, s)$ with $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, endowed with the following product law:

$$w \cdot w' = (x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' - 2\, x \cdot y' + 2\, y \cdot x'), \tag{1.2.1}$$

where for $x, x' \in \mathbb{R}^d$, $x \cdot x'$ denotes the Euclidean scalar product of the vectors x and x'. Equipped with the standard differential structure of the manifold $\mathbb{R}^{2d+1}$, the set $\mathbb{H}^d$ is a non commutative Lie group with identity $(0,0)$. Note also that

$$\forall\ w = (x, y, s) \in \mathbb{H}^d, \quad w^{-1} = (-x, -y, -s).$$

The Lie algebra of left invariant vector fields (see Section A.1 of the Appendix) is spanned by the vector fields

$$X_j \stackrel{\text{def}}{=} \partial_{x_j} + 2y_j\partial_s\,,\ Y_j \stackrel{\text{def}}{=} \partial_{y_j} - 2x_j\partial_s \quad \text{with } j \in \{1, \dots, d\}, \quad \text{and} \quad S \stackrel{\text{def}}{=} \partial_s = \frac{1}{4}[Y_j, X_j]$$

for $j \in \{1, \dots, d\}$. In the following, we will denote by $\mathcal{X}$ the family of vector fields generated by X_j and by $X_{j+d} = Y_j$ for $j \in \{1, \dots, d\}$. Then for any multi-index $\alpha \in \{1, \dots, 2d\}^k$, we write

$$\mathcal{X}^\alpha \stackrel{\text{def}}{=} X_{\alpha_1} \dots X_{\alpha_k}. \tag{1.2.2}$$

Using the complex coordinate system (z, s) obtained by setting $z_j = x_j + iy_j$, we note that

$$\forall\, ((z,s),(z',s')) \in \mathbb{H}^d \times \mathbb{H}^d, \quad (z,s)\cdot(z',s') = (z+z', s+s'+2\mathrm{Im}(z \cdot \overline{z}')),$$

where $z \cdot \overline{z}' = \sum_{j=1}^d z_j \overline{z}'_j$. Furthermore, the Lie algebra of left invariant vector fields on the Heisenberg group $\mathbb{H}^d$ is generated by the vector fields:

$$Z_j = \partial_{z_j} + i\overline{z}_j \partial_s, \quad \overline{Z}_j = \partial_{\overline{z}_j} - iz_j\partial_s, \quad \text{with } j \in \{1, \dots, d\} \quad \text{and} \quad S = \partial_s = \frac{1}{2i}[\overline{Z}_j, Z_j].$$

Denoting by $\mathcal{Z}$ the family of vector fields generated by Z_j and by $Z_{j+d} = \overline{Z}_j$ for $j \in \{1, \dots, d\}$, we write for any multi-index $\alpha \in \{1, \dots, 2d\}^k$

$$\mathcal{Z}^\alpha \stackrel{\text{def}}{=} Z_{\alpha_1} \dots Z_{\alpha_k}. \tag{1.2.3}$$

One can easily check that for all $j \in \{1, \dots, d\}$,

$$X_j = Z_j + \overline{Z}_j \quad \text{and} \quad Y_j = i(Z_j - \overline{Z}_j). \tag{1.2.4}$$

The space $\mathbb{H}^d$ is endowed with a smooth left invariant measure, the Haar measure, which in the coordinate system (x, y, s) is simply the Lebesgue measure $dw \stackrel{\text{def}}{=} dx\, dy\, ds$. It satisfies the fundamental property:

$$\forall f \in L^1(\mathbb{H}^d),\ \forall w' \in \mathbb{H}^d, \quad \int_{\mathbb{H}^d} f(w)\, dw = \int_{\mathbb{H}^d} f(w' \cdot w)\, dw. \tag{1.2.5}$$

The convolution product of two functions f and g on $\mathbb{H}^d$ is defined by

$$f \star g(w) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w \cdot v^{-1}) g(v) dv = \int_{\mathbb{H}^d} f(v) g(v^{-1} \cdot w) dv.$$

It should be emphasized that the convolution on the Heisenberg group is not commutative. Moreover if P is a left invariant vector field on $\mathbb{H}^d$, then one has

$$P(f \star g) = f \star (P(g)). \tag{1.2.6}$$

Indeed, thanks to the classical differentiation theorem, we have

$$P(f \star g)(w) = \int_{\mathbb{H}^d} f(v) P(g(v^{-1} \cdot w)) dv.$$

Due to (A.1.23), one can write

$$P(g(v^{-1} \cdot w)) = (Pg)(v^{-1} \cdot w),$$

which yields (1.2.6). However in general $f \star (P(g)) \neq (P(f)) \star g$.

Note that the usual Young inequalities are nevertheless valid on the Heisenberg group, namely

$$\forall (p,q,r)\in[1,\infty]^3,\quad \|f\star g\|_{L^r(\mathbb{H}^d)}\leq \|f\|_{L^p(\mathbb{H}^d)}\|g\|_{L^q(\mathbb{H}^d)},\quad 1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}\cdot$$

In fact, Young inequalities are more generally available on any locally compact topological group endowed with a left invariant Haar measure μ which satisfies in addition

$$\mu(A^{-1})=\mu(A)\ \text{ for all borelian sets } A.$$

Let us also point out that on the Heisenberg group $\mathbb{H}^d$, there is a notion of dilation defined for $a>0$ by

$$\delta_a(z,s)\stackrel{\text{def}}{=}(az,a^2s). \tag{1.2.7}$$

Observe that for any real number $a>0$, the dilation δ_a satisfies

$$\delta_a(z,s)\cdot\delta_a(z',s')=\delta_a((z,s)\cdot(z',s'))$$

and that the vector fields Z_j change the homogeneity in the following way:

$$Z_j(f\circ\delta_a)=a(Z_jf)\circ\delta_a. \tag{1.2.8}$$

This fact is crucial in order to obtain Bernstein or Hardy inequalities **[4]** (see Chapter 4).

Let us also remark that the Jacobian of the dilation δ_a is a^N where $N\stackrel{\text{def}}{=}2d+2$ is called the homogeneous dimension of $\mathbb{H}^d$.

Let us now recall how Sobolev spaces on the Heisenberg group are associated with the system of vector fields $\mathcal{X}$ for nonnegative integer indexes.

Definition 1.1. — *Let k be a nonnegative integer. We denote by $H^k(\mathbb{H}^d)$ the inhomogeneous Sobolev space on the Heisenberg group of order k which is the space of functions u in $L^2(\mathbb{H}^d)$ (for the Haar measure) such that*

$$\mathcal{X}^\alpha u\in L^2\quad\textit{for any multi-index}\quad \alpha\in\{1,\dots,2d\}^{\mathbb{N}}\quad\textit{with}\quad|\alpha|\leq k.$$

Moreover, we state

$$\|u\|_{H^k(\mathbb{H}^d)}\stackrel{\text{def}}{=}\Big(\sum_{|\alpha|\leq k}\|\mathcal{X}^\alpha u\|^2_{L^2(\mathbb{H}^d)}\Big)^{\frac12}\cdot \tag{1.2.9}$$

Remark 1.2. — *Equivalently, powers of the Laplacian-Kohn operator defined by*

$$\Delta_{\mathbb{H}^d}\stackrel{\text{def}}{=}\sum_{j=1}^d(X_j^2+Y_j^2)=2\sum_{j=1}^d(Z_j\overline{Z}_j+\overline{Z}_jZ_j)=4\sum_{j=1}^d(Z_j\overline{Z}_j+i\partial_s), \tag{1.2.10}$$

can be used to define those Sobolev spaces, which take into account the different role played by the s-direction. Thus

$$\|u\|_{H^k(\mathbb{H}^d)}\sim\|(\mathrm{Id}-\Delta_{\mathbb{H}^d})^{\frac{k}{2}}u\|_{L^2(\mathbb{H}^d)}$$

where $\sim$ stands for equivalent norms.

Note that homogeneous norms may also be defined, where the summation in (1.2.9) is replaced by a summation over $|\alpha| = k$, *and above* $(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{k}{2}}$ *is replaced by* $(-\Delta_{\mathbb{H}^d})^{\frac{k}{2}}$.

When σ is any nonnegative real number one can, as in the case of classical Sobolev spaces on $\mathbb{R}^n$, define the space $H^\sigma(\mathbb{H}^d)$ by complex interpolation (see for instance [**11**]). As in the euclidian case, other equivalent definitions of Sobolev spaces $H^\sigma(\mathbb{H}^d)$ can be used: the definition using integrals and kernels (see [**48**] and [**52**]), or the definition using Weyl-Hörmander calculus (see [**17**]). Finally, a definition using the Littlewood-Paley theory on the Heisenberg group, in the same spirit as in the Euclidian case and due to [**7**], will be given in Section 4.4.2.

There is a natural Heisenberg distance to the origin defined by

$$\rho(z,s) \stackrel{\text{def}}{=} (|z|^4 + s^2)^{\frac{1}{4}},$$

where $|z|^2 = \sum_{j=1}^d z_j \overline{z}_j$. Similarly, we define the Heisenberg distance by

$$d(w,w') = \rho\left(w^{-1}\cdot w'\right). \tag{1.2.11}$$

The distance d incorporates left translation invariant properties

$$\forall \widetilde{w} \in \mathbb{H}^d, \quad d\left(\widetilde{w}\cdot w, \widetilde{w}\cdot w'\right) = d(w,w'). \tag{1.2.12}$$

To define Hölder spaces on the Heisenberg group, we shall introduce another distance on $\mathbb{H}^d$. Denote by $P = P(X_1,\dots,X_{2d})$ the set of continuous curves which are piecewise integral curves of one of the vectors fields $\pm X_1,\dots,\pm X_{2d}$. To any such curve $\gamma : [0,T] \longrightarrow \mathbb{H}^d$, we associate its length $l(\gamma) \stackrel{\text{def}}{=} T$. It is known (see for instance [**27, 28**]) that, for any couple of points w and w' of $\mathbb{H}^d$, there exists a curve of P joining w to w' and that the function

$$\widetilde{d}(w,w') = \min\Big\{l(\gamma),\ \gamma\in P, \gamma \text{ joining } w \text{ to } w'\Big\} \tag{1.2.13}$$

is a distance on the Heisenberg group, which turns out to be equivalent to the one introduced in (1.2.11).

Now, up to the change of the Euclidean distance into $\widetilde{d}$, the definition of Hölder spaces on the Heisenberg group is similar to the definition of Hölder spaces on $\mathbb{R}^d$.

Definition 1.3. — *Let* $r = k + \sigma$, *where* k *is an integer and* $\sigma \in\,]0,1[$. *The Hölder space* $C^r(\mathbb{H}^d)$ *on the Heisenberg group is the space of functions* u *on* $\mathbb{H}^d$ *such that*

$$\|u\|_{C^r(\mathbb{H}^d)} = \sup_{|\alpha|\le k}\left(\|\mathcal{X}^\alpha u\|_{L^\infty} + \sup_{w\neq w'}\frac{|\mathcal{X}^\alpha u(w) - \mathcal{X}^\alpha u(w')|}{\widetilde{d}(w,w')^\sigma}\right) < \infty,$$

where $\widetilde{d}$ *denotes the distance on the Heisenberg group defined by* (1.2.13).

Remark 1.4. — *Thanks to* (1.2.12) *and the fact that the distances* d *and* $\widetilde{d}$ *are equivalent, the spaces* $C^r(\mathbb{H}^d)$ *are invariant under left translations. It will be useful to point out that Hölder spaces on the Heisenberg group can be also defined using the*

Littlewood-Paley theory on the Heisenberg group, in the same way as in the Euclidian case (see Section 4.4.2).

Finally let us define the Schwartz space.

Definition 1.5. — *The Schwartz space $\mathcal{S}(\mathbb{H}^d)$ is the set of smooth functions u on $\mathbb{H}^d$ such that, for any $k \in \mathbb{N}$, we have*

$$\|u\|_{k,\mathcal{S}} \stackrel{\text{def}}{=} \sup_{\substack{|\alpha|\leq k,\, n\leq k\\ (z,s)\in\mathbb{H}^d}} \left|\mathcal{Z}^\alpha\left((|z|^2 - is)^{2n} u(z,s)\right)\right| < \infty.$$

The Schwartz space on the Heisenberg group $\mathcal{S}(\mathbb{H}^d)$ coincides with the classical Schwartz space $\mathcal{S}(\mathbb{R}^{2d+1})$. This allows to define the space of tempered distributions $\mathcal{S}'(\mathbb{H}^d)$. The weight in (z,s) appearing in the definition above is linked to the Heisenberg distance to the origin ρ defined above.

1.2.2. Irreducible representations and the Fourier transform. — Let us now recall the definition of the Fourier transform. We refer for instance to **[23]**, **[45]**, **[52]**, **[53]** or **[54]** for more details. The Heisenberg group being non commutative, the Fourier transform on $\mathbb{H}^d$ is defined using irreducible unitary representations of $\mathbb{H}^d$. As explained for instance in **[53]** Chapter 2, all irreducible representations of $\mathbb{H}^d$ are unitarily equivalent to one of two representations: the Bargmann representation or the L^2 representation. The representations on $L^2(\mathbb{R}^d)$ can be deduced from Bargmann representations thanks to intertwining operators. The reader can consult J. Faraut and K. Harzallah **[23]** for more details. Both representations will be used here.

1.2.2.1. The Bargmann representations. — They are described by $(u^\lambda, \mathcal{H}_\lambda)$, with $\lambda \in \mathbb{R}\setminus\{0\}$, where $\mathcal{H}_\lambda$ is the space defined by

$$\mathcal{H}_\lambda \stackrel{\text{def}}{=} \{F \text{ holomorphic on } \mathbb{C}^d, \|F\|_{\mathcal{H}_\lambda} < \infty\},$$

with

$$\|F\|^2_{\mathcal{H}_\lambda} \stackrel{\text{def}}{=} \left(\frac{2|\lambda|}{\pi}\right)^d \int_{\mathbb{C}^d} \mathrm{e}^{-2|\lambda||\xi|^2} |F(\xi)|^2 d\xi, \tag{1.2.14}$$

while u^λ is the map from $\mathbb{H}^d$ into the group of unitary operators of $\mathcal{H}_\lambda$ defined by

$$\begin{cases} u^\lambda_{z,s}F(\xi) \stackrel{\text{def}}{=} F(\xi - \overline{z})\mathrm{e}^{i\lambda s + 2\lambda(\xi\cdot z - |z|^2/2)} & \text{for} \quad \lambda > 0,\\ u^\lambda_{z,s}F(\xi) \stackrel{\text{def}}{=} F(\xi - z)\mathrm{e}^{i\lambda s - 2\lambda(\xi\cdot\overline{z} - |z|^2/2)} & \text{for} \quad \lambda < 0. \end{cases} \tag{1.2.15}$$

Let us notice that $\mathcal{H}_\lambda$ equipped with the norm $\|\cdot\|_{\mathcal{H}_\lambda}$ defined in (1.2.14) is a Hilbert space. The monomials

$$F_{\alpha,\lambda}(\xi) \stackrel{\text{def}}{=} \frac{(\sqrt{2|\lambda|}\,\xi)^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^d,$$

constitute an orthonormal basis of $\mathcal{H}_\lambda$.

The Fourier transform of an integrable function of $\mathbb{H}^d$ is given by the following definition.

Definition 1.6. — *For $f \in L^1(\mathbb{H}^d)$, we define*

$$\mathcal{F}(f)(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(w) u_w^\lambda dw.$$

The function $\mathcal{F}(f)$, which takes values in the space of bounded operators on $\mathcal{H}_\lambda$, is by definition the Fourier transform of f.

Note that one has

$$\mathcal{F}(f \star g)(\lambda) = \mathcal{F}(f)(\lambda) \circ \mathcal{F}(g)(\lambda).$$

We recall that an operator $A(\lambda)$ of $\mathcal{H}_\lambda$ such that

$$\sum_{\alpha \in \mathbb{N}^d} |(A(\lambda)F_{\alpha,\lambda}, F_{\alpha,\lambda})_{\mathcal{H}_\lambda}| < +\infty$$

is said to be of *trace-class*. One then sets

$$\operatorname{tr}(A(\lambda)) \stackrel{\text{def}}{=} \sum_{\alpha \in \mathbb{N}^d} (A(\lambda)F_{\alpha,\lambda}, F_{\alpha,\lambda})_{\mathcal{H}_\lambda}. \tag{1.2.16}$$

We recall that if besides the operator $A(\lambda)$ has a kernel, namely that if there exists a function $k_\lambda(\xi, \xi')$ such that

$$\forall F \in \mathcal{H}_\lambda, \;\; A(\lambda)F(\xi) = \int_{\mathbb{C}^d} k_\lambda(\xi, \xi') F(\xi') d\xi', \tag{1.2.17}$$

then its trace is given by

$$\operatorname{tr}(A(\lambda)) = \int_{\mathbb{C}^d} k_\lambda(\xi, \xi) d\xi. \tag{1.2.18}$$

Now if $A(\lambda)^* A(\lambda)$ is trace class, then $A(\lambda)$ is said to be a *Hilbert-Schmidt operator*. The quantity

$$\|A(\lambda)\|_{HS(\mathcal{H}_\lambda)} \stackrel{\text{def}}{=} \Big(\sum_{\alpha \in \mathbb{N}^d} \|A(\lambda)F_{\alpha,\lambda}\|^2 \Big)^{\frac{1}{2}}$$

is then a norm on the vector space of Hilbert-Schmidt operators. The following property on Hilbert-Schmidt norms, which can be found in **[46]** (Volume 1 Chapter VI.6) will be of frequent use in what follows. Let A and B be two bounded operators on $\mathcal{H}_\lambda$, with A Hilbert-Schmidt. Then

$$\|BA\|_{HS(\mathcal{H}_\lambda)} + \|AB\|_{HS(\mathcal{H}_\lambda)} \leq \|B\|_{\mathcal{L}(\mathcal{H}_\lambda)} \|A\|_{HS(\mathcal{H}_\lambda)}. \tag{1.2.19}$$

Similarly if A and B are two Hilbert-Schmidt operators, then AB is trace-class and

$$|\operatorname{tr}(AB)| \leq \|A(\lambda)\|_{HS(\mathcal{H}_\lambda)} \|B(\lambda)\|_{HS(\mathcal{H}_\lambda)}. \tag{1.2.20}$$

These notions are important for stating the Plancherel theorem for the Heisenberg group. The proofs of the two following results can be found for instance in **[23]**.

Theorem 1. — *Let $\mathcal{A}$ denote the Hilbert space of one-parameter families $A = \{A(\lambda)\}_{\lambda\in\mathbb{R}\setminus\{0\}}$ of operators on $\mathcal{H}_\lambda$ which are Hilbert-Schmidt for almost every $\lambda\in\mathbb{R}$, with $\|A(\lambda)\|_{HS(\mathcal{H}_\lambda)}$ measurable and with norm*

$$\|A\| \stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty}\|A(\lambda)\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda\right)^{\frac{1}{2}} < \infty.$$

The Fourier transform can be extended to an isometry from $L^2(\mathbb{H}^d)$ onto $\mathcal{A}$ and we have the Plancherel formulas:

$$\|f\|^2_{L^2(\mathbb{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty}\|\mathcal{F}(f)(\lambda)\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda \quad \text{and} \tag{1.2.21}$$

$$(f|g)_{L^2(\mathbb{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty}\operatorname{tr}\left((\mathcal{F}(g)(\lambda))^*\,\mathcal{F}(f)(\lambda)\right)|\lambda|^d d\lambda. \tag{1.2.22}$$

Remark 1.7. — *If $A = \{A(\lambda)\}_{\lambda\in\mathbb{R}\setminus\{0\}}$ and $B = \{B(\lambda)\}_{\lambda\in\mathbb{R}\setminus\{0\}}$ are two families in $\mathcal{A}$, then*

$$\int|\operatorname{tr}(A(\lambda)B(\lambda))|\,|\lambda|^d\,d\lambda \le \|A\|\,\|B\|.$$

Moreover, the following inversion theorem holds.

Theorem 2. — *If a function f satisfies*

$$\sum_{\alpha\in\mathbb{N}^d}\int_{-\infty}^{\infty}\|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda}|\lambda|^d d\lambda < \infty \tag{1.2.23}$$

then we have for almost every w,

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty}\operatorname{tr}\left(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)\right)|\lambda|^d d\lambda.$$

Remark 1.8. — *The above hypothesis (1.2.23) is satisfied in $\mathcal{S}(\mathbb{H}^d)$ (see for example* **[6]***). Therefore, if we consider for $w_0\in\mathbb{H}^d$, the Dirac distribution in w_0, $\delta_{w_0}(w)$, defined by*

$$\forall f\in\mathcal{S}(\mathbb{H}^d),\ \ <\delta_{w_0}\,,\,f> = f(w_0),$$

we have an expression of δ_{w_0} as a singular integral

$$\delta_{w_0}(w) = \frac{2^{d-1}}{\pi^{d+1}}\int_{-\infty}^{\infty}\operatorname{tr}\left(u^\lambda_{w_0^{-1}w}\right)|\lambda|^d d\lambda. \tag{1.2.24}$$

Now let us study the action of the Fourier transform on derivatives. Straightforward computations (performed in Lemma A.3 page 101 for the convenience of the reader), show that

$$\mathcal{F}(Z_j f)(\lambda) = \mathcal{F}(f)(\lambda)Q_j^\lambda,$$

where Q_j^λ is the operator on $\mathcal{H}_\lambda$ defined by

$$
\begin{aligned}
Q_j^\lambda F_{\alpha,\lambda} &\stackrel{\text{def}}{=} -\sqrt{2|\lambda|}\sqrt{\alpha_j+1}F_{\alpha+\mathbf{1}_j,\lambda} \quad \text{if } \lambda>0\\
&\stackrel{\text{def}}{=} \sqrt{2|\lambda|}\sqrt{\alpha_j}F_{\alpha-\mathbf{1}_j,\lambda} \quad \text{if } \lambda<0
\end{aligned}
\tag{1.2.25}
$$

and in the same way,

$$\mathcal{F}(\overline{Z}_j f)(\lambda)=\mathcal{F}(f)(\lambda)\overline{Q}_j^\lambda,$$

where $\overline{Q}_j^\lambda$ is the operator on $\mathcal{H}_\lambda$ defined by

$$
\begin{aligned}
\overline{Q}_j^\lambda F_{\alpha,\lambda} &\stackrel{\text{def}}{=} \sqrt{2|\lambda|}\sqrt{\alpha_j}F_{\alpha-\mathbf{1}_j,\lambda} \quad \text{if } \lambda>0\\
&\stackrel{\text{def}}{=} -\sqrt{2|\lambda|}\sqrt{\alpha_j+1}F_{\alpha+\mathbf{1}_j,\lambda} \quad \text{if } \lambda<0,
\end{aligned}
\tag{1.2.26}
$$

while we have written $\alpha\pm\mathbf{1}_j=\beta$ where $\beta_k=\alpha_k$ if $k\neq j$ and $\beta_j=\alpha_j\pm 1$.

Observe that $\left(\frac{1}{i}Q_j^\lambda\right)^*=\frac{1}{i}\overline{Q}_j^\lambda$ and that

$$
Q_j^\lambda=\begin{cases} -2|\lambda|\xi_j & \text{if } \lambda>0,\\ \partial_{\xi_j} & \text{if } \lambda<0,\end{cases} \quad \text{and} \quad \overline{Q}_j^\lambda=\begin{cases} \partial_{\xi_j} & \text{if } \lambda>0,\\ -2|\lambda|\xi_j & \text{if } \lambda<0.\end{cases}
\tag{1.2.27}
$$

We therefore can write

$$\mathcal{F}(-\Delta_{\mathbb{H}^d}f)(\lambda)=\mathcal{F}(f)(\lambda)\circ D_\lambda \quad \text{where} \quad D_\lambda \stackrel{\text{def}}{=} 2\sum_j(Q_j\overline{Q}_j+\overline{Q}_jQ_j)$$

Using (1.2.25) and (1.2.26) we notice that

$$\forall\alpha\in\mathbb{N}^d,\quad D_\lambda\, F_{\alpha,\lambda}\stackrel{\text{def}}{=}4|\lambda|(2|\alpha|+d)\,F_{\alpha,\lambda}. \tag{1.2.28}$$

Powers of $-\Delta_{\mathbb{H}^d}$ can therefore be defined in the following way: for any real number ρ,

$$
\begin{aligned}
\mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)&=\mathcal{F}(f)(\lambda)\circ D_\lambda^\rho \quad \text{and}\\
\mathcal{F}((\mathrm{Id}-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)&=\mathcal{F}(f)(\lambda)\circ(\mathrm{Id}+D_\lambda)^\rho.
\end{aligned}
\tag{1.2.29}
$$

Notice that (1.2.28) shows that the quantity $|\lambda|(2|\alpha|+d)$ may be considered as a "frequency" on the Heisenberg group. Finally one sees easily that

$$\mathcal{F}(\partial_s f)(\lambda)=i\lambda\mathcal{F}(f)(\lambda).$$

This explains why the partial derivative ∂_s is usually considered as a second-order operator, though one notices here that its "strength" is somewhat weaker than that of the Laplacian since its action, in Fourier space, corresponds to a multiplication by λ while the Laplacian produces $4|\lambda|(2|\alpha|+d)$.

Finally it will be useful later on to notice that due to formulas (1.2.25), (1.2.26) and (1.2.28), the operators $D_\lambda^{-m/2}\circ(Q_j^\lambda)^m$ and $D_\lambda^{-m/2}\circ(\overline{Q}_j^\lambda)^m$ are uniformly bounded on $\mathcal{H}_\lambda$ for any integer m.

Note that one can also prove, in the same fashion as in the Euclidean case, relations between $\mathcal{F}\left((is-|z|^2)f\right)(\lambda)$ and $\mathcal{F}(f)(\lambda)$; we refer to Proposition 1.11 below for formulas.

Remark 1.9. — *The above computations show that for any function* $f \in \mathcal{S}(\mathbb{H}^d)$,

$$\begin{aligned} Z_j f(w) &= \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda) Q^\lambda_j\right) |\lambda|^d d\lambda, \\ \overline{Z}_j f(w) &= \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda) \overline{Q}^\lambda_j\right) |\lambda|^d d\lambda, \quad \textit{and} \\ -\Delta_{\mathbb{H}^d} f(w) &= \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda) D_\lambda\right) |\lambda|^d d\lambda. \end{aligned}$$

In particular, if we consider the derivatives of the Dirac distribution $\delta_{w_0}(w)$ *defined as usual by duality through*

$$< Z_j \delta_{w_0}, f >= - < \delta_{w_0}, Z_j f >= -Z_j f(w_0) \quad \textit{and}$$

$$< \overline{Z}_j \delta_{w_0}, f >= - < \delta_{w_0}, \overline{Z}_j f >= -\overline{Z}_j f(w_0)$$

for all $f \in \mathcal{S}(\mathbb{H}^d)$ *and for some fixed* $w_0 \in \mathbb{H}^d$, *we obtain an expression of the derivatives of the Dirac distribution as singular integrals*

$$\begin{aligned} Z_j \delta_{w_0}(w) &= -\frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w_0^{-1}w} Q^\lambda_j\right) |\lambda|^d d\lambda, \\ \overline{Z}_j \delta_{w_0}(w) &= -\frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w_0^{-1}w} \overline{Q}^\lambda_j\right) |\lambda|^d d\lambda, \quad \textit{and} \\ -\Delta_{\mathbb{H}^d} \delta_{w_0}(w) &= \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w_0^{-1}w} D_\lambda\right) |\lambda|^d d\lambda. \end{aligned}$$

It turns out that for radial functions on the Heisenberg group, the Fourier transform becomes simplified. Let us first recall the concept of radial functions on the Heisenberg group.

Definition 1.10. — *A function* f *defined on the Heisenberg group* $\mathbb{H}^d$ *is said to be radial if it is invariant under the action of the unitary group* $U(d)$ *of* $\mathbb{C}^d$, *meaning that for any* $u \in U(d)$, *we have*

$$f(z, s) = f(u(z), s), \quad \forall (z, s) \in \mathbb{H}^d.$$

A radial function on the Heisenberg group can then be written under the form

$$f(z, s) = g(|z|, s).$$

Then it can be shown (see for instance **[45]**) that the Fourier transform of radial functions of $L^2(\mathbb{H}^d)$, satisfies the following formula:

$$\mathcal{F}(f)(\lambda) F_{\alpha,\lambda} = R_{|\alpha|}(\lambda) F_{\alpha,\lambda}$$

where

$$R_m(\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} m + d - 1 \\ m \end{pmatrix}^{-1} \int e^{i\lambda s} f(z, s) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} dz ds,$$

and where $L_m^{(p)}$ are Laguerre polynomials defined by

$$L_m^{(p)}(t) \stackrel{\text{def}}{=} \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m+p \\ m-k \end{pmatrix} \frac{t^k}{k!}, \quad t \geq 0, \quad (m,p) \in \mathbb{N}^2 . \tag{1.2.30}$$

Note that in that context, Plancherel and inversion formulas can be stated as follows:

$$\|f\|_{L^2(\mathbb{H}^d)} = \left(\frac{2^{d-1}}{\pi^{d+1}} \sum_m \begin{pmatrix} m+d-1 \\ m \end{pmatrix} \int_{-\infty}^{\infty} |R_m(\lambda)|^2 |\lambda|^d d\lambda \right)^{\frac{1}{2}}$$

and

$$f(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} R_m(\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda. \tag{1.2.31}$$

The context of radial functions allows to compute the Fourier transform of $(is-|z|^2)f$, as stated below (see **[7]** for a proof).

Proposition 1.11. — *For any radial function $f \in \mathcal{S}(\mathbb{H}^d)$, we have for any $m \geq 1$,*

$$\mathcal{F}((is-|z|^2)f)(m,\lambda) = \frac{d}{d\lambda}\mathcal{F}f(m,\lambda) - \frac{m}{\lambda}(\mathcal{F}f(m,\lambda) - \mathcal{F}f(m-1,\lambda)) \textit{ if } \lambda > 0, \textit{ and}$$

$$\mathcal{F}((is-|z|^2)f)(m,\lambda) = \frac{d}{d\lambda}\mathcal{F}f(m,\lambda) + \frac{m+d}{|\lambda|}(\mathcal{F}f(m,\lambda) - \mathcal{F}f(m+1,\lambda)) \textit{ if } \lambda < 0.$$

1.2.2.2. The L^2 representation. — In order to define pseudodifferential operators, it will be useful to use rather the L^2 (or Schrödinger) representations, denoted in the following by $(v_{z,s}^\lambda f)(\xi)$, where ξ belongs to $\mathbb{R}^d$ and f to $L^2(\mathbb{R}^d)$. As recalled above, the representations $v_{z,s}^\lambda$ and $u_{z,s}^\lambda$ are equivalent. The intertwining operator is the Hermite-Weber transform $K_\lambda : \mathcal{H}_\lambda \to L^2(\mathbb{R}^d)$ given by

$$(K_\lambda \phi)(\xi) \stackrel{\text{def}}{=} \frac{|\lambda|^{d/4}}{\pi^{d/4}} \mathrm{e}^{|\lambda|\frac{|\xi|^2}{2}} \phi\left(-\frac{1}{2|\lambda|}\frac{\partial}{\partial\xi}\right) \mathrm{e}^{-|\lambda|\,|\xi|^2}, \tag{1.2.32}$$

which is unitary and intertwines both representations: we have indeed $K_\lambda u_{z,s}^\lambda = v_{z,s}^\lambda K_\lambda$ and

$$v_{z,s}^\lambda f(\xi) = e^{i\lambda(s-2x\cdot y+2y\cdot\xi)} f(\xi - 2x), \quad \forall \lambda \in \mathbb{R}^* . \tag{1.2.33}$$

A short proof of this fact is given in Appendix A.2 for the convenience of the reader (see Proposition A.1 page 98). We also recall that the inverse of K_λ is known as the Segal-Bargmann transform (see for instance **[24]**). Let us denote by h_α the multidimensional Hermite function defined by

$$\forall \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d, \;\; \forall t = (t_1, \ldots, t_d) \in \mathbb{R}^d, \;\; h_\alpha(t) \stackrel{\text{def}}{=} h_{\alpha_1}(t_1) \cdots h_{\alpha_d}(t_d),$$

with

$$h_n(t) \stackrel{\text{def}}{=} \left(2^n\, n!\, \sqrt{\pi}\right)^{-1/2} \mathrm{e}^{-t^2/2} H_n(t) \quad \text{and} \quad H_n(t) \stackrel{\text{def}}{=} \mathrm{e}^{t^2} \left(-\frac{\partial}{\partial t}\right)^n \left(\mathrm{e}^{-t^2}\right).$$

Introducing the scaling operator

$$\forall f \in L^2(\mathbb{R}^d), \ \ T_\lambda f(\xi) \stackrel{\text{def}}{=} |\lambda|^{-d/4} f(|\lambda|^{-1/2}\xi), \tag{1.2.34}$$

and setting $h_{\alpha,\lambda} = T_\lambda^* h_\alpha$ we observe that

$$\forall \alpha \in \mathbb{N}^d, \ \ K_\lambda F_{\alpha,\lambda} = h_{\alpha,\lambda} \tag{1.2.35}$$

where $h_{\alpha,\lambda}$ is an eigenfunction of the rescaled harmonic oscillator $-\Delta_\xi + |\lambda||\xi|^2$. This implies by straightforward computations that

$$\begin{aligned} K_\lambda Q_j^\lambda K_\lambda^* &= \partial_{\xi_j} - |\lambda|\xi_j \quad \text{and} \quad K_\lambda \overline{Q}_j^\lambda K_\lambda^* = \partial_{\xi_j} + |\lambda|\xi_j \ \text{ if } \ \lambda > 0, \\ K_\lambda Q_j^\lambda K_\lambda^* &= \partial_{\xi_j} + |\lambda|\xi_j \quad \text{and} \quad K_\lambda \overline{Q}_j^\lambda K_\lambda^* = \partial_{\xi_j} - |\lambda|y_j \ \text{ if } \ \lambda < 0. \end{aligned}$$

Defining the operator

$$J_\lambda \stackrel{\text{def}}{=} T_\lambda K_\lambda, \tag{1.2.36}$$

and observing that

$$T_\lambda(-\Delta_\xi + |\xi|^2|\lambda|^2)T_\lambda^* = |\lambda|(-\Delta_\xi + |\xi|^2),$$

we infer that

$$\begin{aligned} J_\lambda Q_j^\lambda J_\lambda^* &= \sqrt{|\lambda|}\left(\partial_{\xi_j} - \xi_j\right) \quad \text{and} \quad J_\lambda \overline{Q}_j^\lambda J_\lambda^* = \sqrt{|\lambda|}\left(\partial_{\xi_j} + \xi_j\right) \ \text{ if } \ \lambda > 0 \\ J_\lambda Q_j^\lambda J_\lambda^* &= \sqrt{|\lambda|}\left(\partial_{\xi_j} + \xi_j\right) \quad \text{and} \quad J_\lambda \overline{Q}_j^\lambda J_\lambda^* = \sqrt{|\lambda|}\left(\partial_{\xi_j} - \xi_j\right) \ \text{ if } \ \lambda < 0, \end{aligned} \tag{1.2.37}$$

which finally implies that

$$J_\lambda D_\lambda J_\lambda^* = 4|\lambda|(-\Delta_\xi + |\xi|^2). \tag{1.2.38}$$

In view of Remark 1.9, the Laplacian $-\Delta_{\mathbb{H}^d}$ is associated with the operator D_λ of $\mathcal{H}_\lambda$ in the Bargmann representation; by Equation (1.2.38), it is associated with the harmonic oscillator in the $L^2(\mathbb{R}^d)$ framework.

These computations indicate that symbolic calculus on $\mathcal{H}_\lambda$ is, via the unitary operator J_λ, equivalent to symbolic calculus on the harmonic oscillator. That theory is well understood: it consists in Weyl-Hörmander calculus associated with a harmonic oscillator metric. This is made precise in the next section.

Before proceeding further, it is instructive to compute the Fourier transform for instance of the function $Z_j\overline{Z}_j f$ for $f \in \mathcal{S}(\mathbb{H}^d)$. Indeed, we notice that with the previous notations, for $\lambda > 0$,

$$\begin{aligned} \mathcal{F}((-iZ_j)(-i\overline{Z}_j)f)(\lambda) &= \mathcal{F}(-i\overline{Z}_j f)(\lambda) J_\lambda^* \sqrt{|\lambda|}(-i\partial_{\xi_j} + i\xi_j) J_\lambda \\ &= \mathcal{F}(f)(\lambda) J_\lambda^* |\lambda| (-i\partial_{\xi_j} - i\xi_j)(-i\partial_{\xi_j} + i\xi_j) J_\lambda \\ &= \mathcal{F}(f)(\lambda) J_\lambda^* |\lambda| (\xi_j^2 - \partial_{\xi_j}^2 + 1) J_\lambda. \end{aligned}$$

This implies that symbols on the Heisenberg group must not only include harmonic oscillator type symbols, but also functions such as powers of λ.

1.3. Weyl-Hörmander calculus

Let us recall in this section some results on the Weyl-Hörmander calculus of the harmonic oscillator which we shall be using. We shall only state the definitions that will be needed in the following, and for further details, we refer for instance to [**14**], [**15**], [**17**], [**19**], [**38**] and [**41**].

1.3.1. Admissible weights and metrics. — Let us denote by $\omega[\Theta,\Theta']$ the standard symplectic form on $T^*\mathbb{R}^d$ (which we shall identify in the following to $\mathbb{R}^{2d}$) : if $\Theta=(\xi,\eta)$ and $\Theta'=(\xi',\eta')$, then $\omega[\Theta,\Theta']\overset{\text{def}}{=}\eta\cdot\xi'-\eta'\cdot\xi$.

For any point $\Theta=(\xi,\eta)$ in $\mathbb{R}^{2d}$, we consider a Riemannian metric g_Θ (which depends measurably on Θ) to which we associate the conjugate metric g^ω_Θ by

$$\forall T\in\mathbb{R}^{2d},\quad (g^\omega_\Theta(T))^{1/2}=\sup_{T'\in\mathbb{R}^{2d}}\frac{|\omega[T,T']|}{g_\Theta(T')^{1/2}}\cdot$$

We also define the *gain factor*

$$\Lambda_\Theta\overset{\text{def}}{=}\inf_T\frac{g^\omega_\Theta(T)}{g_\Theta(T)}\cdot\tag{1.3.1}$$

Definition 1.12. — *We shall say that the metric g is of Hörmander type if it is:*

1. *Uncertain: For all $\Theta\in\mathbb{R}^{2d}$, $\Lambda_\Theta\geq 1$.*
2. *Slowly varying: There is a constant $\overline{C}>0$ such that*

$$g_\Theta(\Theta-\Theta')\leq\overline{C}^{-1}\Rightarrow\sup_{T\in\mathbb{R}^{2d}}\left(\frac{g_\Theta(T)}{g_{\Theta'}(T)}\right)^{\pm1}\leq\overline{C}.$$

3. *Temperate: There are constants $\overline{C}>0$ and $\overline{N}\in\mathbb{N}$ such that for all $(\Theta,\Theta')\in\mathbb{R}^{4d}$,*

$$\sup_{T\in\mathbb{R}^{2d}}\left(\frac{g_\Theta(T)}{g_{\Theta'}(T)}\right)^{\pm1}\leq\overline{C}(1+g^\omega_\Theta(\Theta-\Theta'))^{\overline{N}}.$$

In the following any constant depending only on $\overline{C}$ and $\overline{N}$ will be called a structural constant.

In the definition above we have used the notation

$$\left(\frac{g_\Theta(T)}{g_{\Theta'}(T)}\right)^{\pm1}\overset{\text{def}}{=}\frac{g_\Theta(T)}{g_{\Theta'}(T)}+\frac{g_{\Theta'}(T)}{g_\Theta(T)}\cdot$$

We also define a weight as a positive function on $\mathbb{R}^{2d}$ satisfying the same type of conditions as a Hörmander metric.

Definition 1.13. — *Let g be a metric in the sense of Definition 1.12. A positive function m on $\mathbb{R}^{2d}$ is a g-weight if there are structural constants $\overline{C}'>0$ and $\overline{N}'\in\mathbb{N}$ such that*

1. $g_\Theta(\Theta-\Theta')\leq\overline{C}'^{-1}\Rightarrow\left(\frac{m(\Theta)}{m(\Theta')}\right)^{\pm1}\leq\overline{C}'$.
2. $\left(\frac{m(\Theta)}{m(\Theta')}\right)^{\pm1}\leq\overline{C}'(1+g^\omega_\Theta(\Theta-\Theta'))^{\overline{N}'}$.

It is easy to see that the set of g-weights has a group structure (for the usual product of functions).

For such metrics and weights, one can then define the class $S(m,g)$ of smooth functions a on $\mathbb{R}^{2d}$ such that, for any integer n,

$$\|a\|_{n;S(m,g)} \stackrel{\text{def}}{=} \sup_{\substack{j\leq n,\Theta\in\mathbb{R}^{2d}\\ g_\Theta(T_j)\leq 1}} \frac{|\partial_{T_1}\ldots\partial_{T_j}a(\Theta)|}{m(\Theta)} < \infty, \tag{1.3.2}$$

where $\partial_T a$ denotes the map $\langle da, T\rangle$. Now, if a is a symbol in $S(m,g)$, then its Weyl quantization is the operator which associates to $u \in \mathcal{S}(\mathbb{R}^d)$ the function $\mathrm{op}^w(a)u$ defined by

$$\forall \xi \in \mathbb{R}^d, \quad (\mathrm{op}^w(a)u)(\xi) \stackrel{\text{def}}{=} (2\pi)^{-d}\int_{\mathbb{R}^{2d}} e^{i(\xi-\xi')\cdot\eta} a\left(\frac{\xi+\xi'}{2}, \eta\right) u(\xi')d\xi'd\eta. \tag{1.3.3}$$

The main interest of this quantization is that $\mathrm{op}^w(a)^* = \mathrm{op}^w(\overline{a})$.

Observe also that if $a(\xi,\eta) = \widetilde{a}(\xi)$, the operator $\mathrm{op}^w(a)$ is the operator of multiplication by the function $\widetilde{a}$ and if $a(\xi,\eta) = \widetilde{a}(\eta)$, the operator $\mathrm{op}^w(a)$ is the Fourier multiplier $\widetilde{a}(D)$. In particular one has $\mathrm{op}^w(\eta_j^k) = \left(\frac{1}{i}\partial_{\xi_j}\right)^k$ for any $k \in \mathbb{N}$.

Besides, for all symbols $a \in S(m_1,g)$ and $b \in S(m_2,g)$ where m_1 and m_2 are g-weights, we have the following composition formulas:

$$\mathrm{op}^w(a) \circ \mathrm{op}^w(b) = \mathrm{op}^w(a\#b) \text{ with } a\#b \in S(m_1m_2,g) \quad \text{and}$$

$$(a\#b)(\Theta) = \pi^{-2d}\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} e^{-2i\omega[\Theta-\Theta_1,\Theta-\Theta_2]} a(\Theta_1)b(\Theta_2)d\Theta_1 d\Theta_2. \tag{1.3.4}$$

The (non commutative) bilinear operator # is often referred to as the Moyal product.

This leads to an asymptotic formula

$$a\#b = ab + \frac{1}{2i}\{a,b\} + \cdots + r_N, \tag{1.3.5}$$

where ab belongs to $S(m_1m_2,g)$ and $\frac{1}{2i}\{a,b\}$ belongs to $S(\Lambda^{-1}m_1m_2,g)$, recalling that $\{a,b\}$ is the usual Poisson bracket

$$\{a,b\} \stackrel{\text{def}}{=} \sum_{j=1}^{d} \left(\partial_{\eta_j}a\,\partial_{\xi_j}b - \partial_{\xi_j}a\,\partial_{\eta_j}b\right).$$

Finally for any integer N, the remainder r_N belongs to $S(\Lambda^{-N}m_1m_2,g)$.

Let us mention that the operator $\mathrm{op}^w(a)$ has a kernel $k(\xi,\xi')$ defined by

$$k(\xi,\xi') = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(\xi-\xi')\cdot\eta} a\left(\frac{\xi+\xi'}{2}, \eta\right) d\eta \tag{1.3.6}$$

which is linked to its symbol through

$$a(\xi,\eta) = \int_{\mathbb{R}^d} e^{-i\xi'\eta} k\left(\xi+\frac{\xi'}{2}, \xi-\frac{\xi'}{2}\right) d\xi'. \tag{1.3.7}$$

Let us also point out that a concept of Sobolev space $H(m,g)$ was introduced by R. Beals in [**8**]. We will use the following characterization of those spaces.

Definition 1.14. — *Let g and m be respectively a Hörmander metric and a g-weight, in the sense of Definitions 1.12 and 1.13. We denote by $H(m,g)$ the set of all tempered distributions u on $\mathbb{R}^d$ such that, for any $a \in S(m,g)$, we have $\mathrm{op}^w(a)u \in L^2(\mathbb{R}^d)$. In particular $H(1,g)$ coincides with $L^2(\mathbb{R}^d)$.*

Note that the study of Sobolev spaces associated with a Hörmander metric g and a g-weight has been developed in **[8]**, **[14]**, **[15]**, **[17]** and **[53]** and in particular in **[14]**, it was shown that these spaces are "almost independent" of the metric g. The Weyl quantization defined by (1.3.3) can be extended to an operator on $\mathcal{S}'(\mathbb{R}^d)$ which acts on the Sobolev spaces $H(m,g)$ in the following way.

Proposition 1.15. — *Let g be a Hörmander metric, and let m and m_1 be g-weights. There exists a constant C, depending only on the structural constants of Definitions 1.12 and 1.13, such that the following holds. Let a be in $S(m_1,g)$. Then, there exist an integer n and a constant C such that for any u in $H(m,g)$, we have*

$$\|\mathrm{op}^w(a)u\|_{H(mm_1^{-1},g)} \leq C\|a\|_{n;S(m_1,g)}\|u\|_{H(m,g)}.$$

In particular, there exist an integer n and a constant C such that if $a \in S(1,g)$, then for any $u \in L^2(\mathbb{R}^d)$ one has

$$\|\mathrm{op}^w(a)u\|_{L^2(\mathbb{R}^d)} \leq C\,\|a\|_{n;S(1,g)}\|u\|_{L^2(\mathbb{R}^d)}. \tag{1.3.8}$$

1.3.2. The case of the harmonic oscillator. — As pointed out in Section 1.1.2.2, it is natural to base the quantization of symbols on the Heisenberg group on the calculus related to the harmonic oscillator. In that case one is considering the metric defined by

$$\forall \Theta = (\xi,\eta) \in \mathbb{R}^{2d}, \quad g_\Theta(d\xi,d\eta) \stackrel{\mathrm{def}}{=} \frac{d\xi^2 + d\eta^2}{1+\xi^2+\eta^2} \tag{1.3.9}$$

while the g-weight is

$$\forall \Theta = (\xi,\eta) \in \mathbb{R}^{2d}, \quad m(\Theta) \stackrel{\mathrm{def}}{=} (1+\xi^2+\eta^2)^{\frac{1}{2}}. \tag{1.3.10}$$

It is an exercise to check that g is a Hörmander metric in the sense of Definition 1.12, and that m is a g-weight in the sense of Definition 1.13. This will in fact be performed in the proof of Proposition 1.20 below in a more general setting.

We will be interested in the class of symbols belonging to $S(m^\mu,g)$ for some real number μ, where we notice that (1.3.2) can simply be written equivalently in the following way:

$$\|a\|_{n;S(m^\mu,g)} \stackrel{\mathrm{def}}{=} \sup_{|\beta|\leq n,(\xi,\eta)\in\mathbb{R}^{2d}} (1+\xi^2+\eta^2)^{\frac{|\beta|-\mu}{2}} |\partial^\beta_{(\xi,\eta)} a(\xi,\eta)| < \infty. \tag{1.3.11}$$

It is useful, in particular in the framework of the Littlewood-Paley transformation on the Heisenberg group investigated in Chapter 4, to be able to write the Weyl symbol of functions of the harmonic oscillator on $L^2(\mathbb{R}^d)$. The formula for such symbols is derived using Mehler's formula (see **[26]** for instance)

$$\mathrm{e}^{-t(\xi^2-\Delta_\xi)} = (\operatorname{ch} t)^{-d}\operatorname{op}^w\left(\mathrm{e}^{-(\xi^2+\eta^2)\operatorname{th} t}\right). \tag{1.3.12}$$

More precisely, we have the following result, whose proof is postponed to Appendix B (see page 110).

Proposition 1.16. — *Consider R a smooth function satisfying symbol estimates:*

$$\exists \mu\in\mathbb{R},\ \exists C>0,\ \forall n\in\mathbb{N},\ \left\|(1+|\cdot|)^{n-\mu}\partial^n R\right\|_{L^\infty(\mathbb{R})}\leq C^n. \tag{1.3.13}$$

Then $R(\xi^2-\Delta_\xi)$ is a pseudodifferential operator. Moreover one has formally

$$R(\xi^2-\Delta_\xi)=\operatorname{op}^w(r(\xi^2+\eta^2))$$

with for all $x\neq 0$,

$$r(x)=\frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}}(\cos\tau)^{-d}\mathrm{e}^{i(x\operatorname{tg}\tau-\xi\tau)}R(\xi)d\tau\, d\xi. \tag{1.3.14}$$

Besides $(\xi,\eta)\mapsto r(\xi^2+\eta^2)$ is satisfies the symbol estimates of the class $S(m^\mu,g)$, in the sense of (1.3.11).

Note that r is not well defined at $x=0$ in general, which explains why the relation $R(\xi^2-\Delta_\xi)=\operatorname{op}^w(r(\xi^2+\eta^2))$ is only formal. One also has the inverse formula

$$\operatorname{op}^w\left(r(y^2+\eta^2)\right)=\frac{1}{2\pi}\int\widehat{r}(\tau)\mathrm{e}^{i\,(y^2-\Delta)\operatorname{Arctg}\tau}(1+\tau^2)^{-d/2}d\tau. \tag{1.3.15}$$

This yields that the operator $J_\lambda^*\operatorname{op}^w(r(y^2+\eta^2))J_\lambda$ is diagonal in the basis $(F_{\alpha,\lambda})_{\alpha\in\mathbb{N}^d}$ and thus commutes with operators of the form $\chi(D_\lambda)$ for all continuous bounded functions χ, where $\chi(D_\lambda)$ is the operator

$$\chi(D_\lambda)F_{\alpha,\lambda}=\chi(4|\lambda|(2|\alpha|+d))F_{\alpha,\lambda}. \tag{1.3.16}$$

Remark 1.17. — *Let us note that the operator* $\mathrm{Id}-\Delta_\xi+\xi^2$ *has for symbol m^2, while the symbol of* $4(-\Delta_\xi+\xi^2)$ *is $\widetilde{m}_2(\xi,\eta)$ where $\widetilde{m}_2(\xi,\eta)\stackrel{\text{def}}{=}2(\xi^2+\eta^2)^{\frac12}$.*

Besides, for $\mu\in\mathbb{R}$, Proposition 1.16 shows that there exists a function $m_\mu\in S(m^\mu,g)$ such that $2^\mu(\mathrm{Id}-\Delta_\xi+\xi^2)^{\mu/2}=\operatorname{op}^w(m_\mu)$. In particular, for any $\mu,\mu'\in\mathbb{R}$, $m_\mu\# m_{\mu'}=m_{\mu+\mu'}$.

Finally if $\mu\geq 0$, then there exists a function $\widetilde{m}_\mu\in S(m^\mu,g)$ such that $2^\mu(-\Delta_\xi+\xi^2)^{\mu/2}=\operatorname{op}^w(\widetilde{m}_\mu)$. In particular, for any $\mu,\mu'\in\mathbb{R}$, $\widetilde{m}_\mu\#\widetilde{m}_{\mu'}=\widetilde{m}_{\mu+\mu'}$. Note that the restriction to $\mu\geq 0$ is natural and holds also in the euclidean case.

1.4. Main results: pseudodifferential operators on the Heisenberg group

In this section, motivated by the examples studied in the previous sections of this chapter, we shall give a definition of symbols, and pseudodifferential operators, on the Heisenberg group. Then we will state the main results proved in this paper concerning those operators.

1.4.1. Symbols. — Our approach inspired by the Euclidian strategy of R. Coifman and Y. Meyer **[20]** allows to consider symbols with limited regularity with respect the Heisenberg variable. Therefore, in what follows, we shall define a positive, noninteger real number ρ, which will measure the regularity assumed on the symbols (in the Heisenberg variable). This number ρ is fixed from now on and we emphasize that the definitions below depend on ρ. We have chosen not to keep memory of this number on the notations for the sake of simplicity.

Definition 1.18. — *A smooth function a defined on $\mathbb{H}^d \times \mathbb{R}^* \times \mathbb{R}^{2d}$ is a symbol if there is a real number μ such that for all $n \in \mathbb{N}$, the following semi norm is finite:*

$$\|a\|_{n;S_{\mathbb{H}^d}(\mu)} \stackrel{\text{def}}{=} \sup_{\substack{\lambda\neq 0\\ \Theta\in\mathbb{R}^{2d}}} \sup_{|\beta|+k\leq n} |\lambda|^{-\frac{|\beta|}{2}} \left(1+|\lambda|(1+\Theta^2)\right)^{\frac{|\beta|-\mu}{2}} \|(\lambda\partial_\lambda)^k \partial_\Theta^\beta a(\cdot,\lambda,\Theta)\|_{C^\rho(\mathbb{H}^d)}.$$

Besides, one additionally requires that the function

$$(w,\lambda,\xi,\eta) \mapsto \sigma(a)(w,\lambda,\xi,\eta) \stackrel{\text{def}}{=} a\left(w,\lambda,\operatorname{sgn}(\lambda)\frac{\xi}{\sqrt{|\lambda|}},\frac{\eta}{\sqrt{|\lambda|}}\right) \tag{1.4.1}$$

is smooth close to $\lambda = 0$ for any $(w,\xi,\eta) \in \mathbb{H}^d \times \mathbb{R}^{2d}$. In that case we shall write $a \in S_{\mathbb{H}^d}(\mu)$.

Remark 1.19. — *The additional assumption (1.4.1) is necessary in order to guarantee that pseudodifferential operators associated with those symbols are continuous on $\mathcal{S}(\mathbb{H}^d)$ (see Proposition 2.6). It is also required to obtain that the space of pseudodifferential operators is an algebra.*

In the remainder of this section, we shall discuss two points of view. The first consists in considering the symbol $a \in S_{\mathbb{H}^d}(\mu)$ as a symbol on $\mathbb{R}^{2d}$ depending on the parameters (w,λ) in $\mathbb{H}^d \times \mathbb{R}$ and belonging to a λ-dependent Hörmander class (see Proposition 1.20). The second point of view consists in emphasizing the function $\sigma(a)$ (see Proposition 1.22). Both points of view are in fact interesting, and both will be used in the following.

Let us first analyze the properties of $a \in S_{\mathbb{H}^d}(\mu)$ for a fixed λ. The following proposition is proved in Appendix B (see page 107).

Proposition 1.20. — *The (λ-dependent) metric $g^{(\lambda)}$ defined by*

$$\forall \lambda \neq 0,\ \forall \Theta \in \mathbb{R}^{2d}, \quad g_\Theta^{(\lambda)}(d\xi,d\eta) \stackrel{\text{def}}{=} \frac{|\lambda|(d\xi^2+d\eta^2)}{1+|\lambda|(1+\Theta^2)}$$

is a Hörmander metric in the sense of Definition 1.12, and the function

$$m^{(\lambda)}(\Theta) \stackrel{\text{def}}{=} \left(1+|\lambda|(1+\Theta^2)\right)^{1/2}$$

is a $g^{(\lambda)}$-weight. Moreover the constants $\overline{C}$ and $\overline{N}$ of Definitions 1.12 and 1.13 are independent of λ.

Finally if a is a smooth function defined on $\mathbb{H}^d \times \mathbb{R}^* \times \mathbb{R}^{2d}$, then a belongs to $S_{\mathbb{H}^d}(\mu)$ if and only if (1.4.1) defines a smooth function and for any $k \in \mathbb{N}$, the *function $(\lambda\partial_\lambda)^k a$ is a symbol of order μ in the Weyl-Hörmander class defined by the metric $g^{(\lambda)}$ and the $g^{(\lambda)}$-weight $m^{(\lambda)}$, uniformly with respect to λ.*

Proposition 1.20 has important consequences which are stated below. The first one will be used often in the sequel and states that the continuity constants of Weyl quantizations of symbols are independent of λ and w.

Corollary 1.21. — *Let a be a symbol in $S_{\mathbb{H}^d}(\mu)$. Then for any $w \in \mathbb{H}^d$ and $\lambda \in \mathbb{R}^*$, the operator $\operatorname{op}^w(a(w,\lambda))$ is continuous from $H(m, g^{(\lambda)})$ into $H\left(m(m^{(\lambda)})^{-\mu}, g^{(\lambda)}\right)$ for any $g^{(\lambda)}$-weight m, and the constant of continuity is uniform wih respect to λ and w. In particular for $\mu = 0$, the operator $\operatorname{op}^w(a(w,\lambda))$ maps $L^2(\mathbb{R}^d)$ into itself uniformly with respect to w and λ.*

The second consequence concerns the stability of our class of symbols with respect to the Moyal product (see (1.3.4)): if $a \in S_{\mathbb{H}^d}(\mu_1)$ and $b \in S_{\mathbb{H}^d}(\mu_2)$, then the functions ab and $a \# b$ are symbols in the class $S_{\mathbb{H}^d}(\mu_1+\mu_2)$. Besides, the asymptotic formula can be written

$$a \# b = ab + \frac{|\lambda|}{2i}\sum_{j=1}^{d}\Big(\frac{1}{\sqrt{|\lambda|}}\partial_{\eta_j} a \, \frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j} b - \frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j} a \, \frac{1}{\sqrt{|\lambda|}}\partial_{\eta_j} b\Big) + \cdots$$

Let us also point out that if a belongs to $S_{\mathbb{H}^d}(\mu)$, then for any $j \in \{1,\dots,d\}$ the functions $\frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j} a$ and $\frac{1}{\sqrt{|\lambda|}}\partial_{\eta_j} a$ belong to $S_{\mathbb{H}^d}(\mu-1)$.

Let us now mention some properties of the function $\sigma(a)$ defined in (1.4.1). The following proposition, which is proved in Appendix B (see page 109), will be useful in the proofs of Chapter 3.

Proposition 1.22. — *A function a belongs to $S_{\mathbb{H}^d}(\mu)$ if and only if $\sigma(a) \in \mathscr{C}^\infty(\mathbb{H}^d \times \mathbb{R}^{2d+1})$ satisfies: for all $k, n \in \mathbb{N}$, there exists a constant $C_{n,k} > 0$ such that for any $\beta \in \mathbb{N}^d$ satisfying $|\beta| \leq n$, and for all $(w,\lambda,y,\eta) \in \mathbb{H}^d \times \mathbb{R}^{2d+1}$,*

$$\left\|\partial_\lambda^k \partial_{(\xi,\eta)}^\beta(\sigma(a))\right\|_{\mathscr{C}^\rho(\mathbb{H}^d)} \leq C_{n,k}\left(1+|\lambda|+\xi^2+\eta^2\right)^{\frac{\mu-|\beta|}{2}}(1+|\lambda|)^{-k}. \tag{1.4.2}$$

1.4.2. Operators. — We define pseudodifferential operators as follows.

Definition 1.23. — *To a symbol a of order μ in the sense of Definition 1.18, we associate the pseudodifferential operator on $\mathbb{H}^d$ defined in the following way: for any $f \in \mathscr{S}(\mathbb{H}^d)$,*

$$\forall w \in \mathbb{H}^d, \quad \operatorname{Op}(a)f(w) \stackrel{\text{def}}{=} \frac{2^{d-1}}{\pi^{d+1}}\int_{\mathbb{R}} \operatorname{tr}\left(u_{w^{-1}}^\lambda \mathscr{F}(f)(\lambda) A_\lambda(w)\right)|\lambda|^d\, d\lambda, \tag{1.4.3}$$

where

$$A_\lambda(w) \stackrel{\text{def}}{=} J_\lambda^* \operatorname{op}^w(a(w,\lambda,\xi,\eta))\, J_\lambda \quad \text{if } \lambda \neq 0. \tag{1.4.4}$$

while J_λ is defined in (1.2.36), page 16.

Examples of pseudodifferential operators are provided in Section 2.1 of Chapter 2. Observe that the operator $\mathrm{Op}(a)$ has a kernel

$$k_a(w,w') = \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \mathrm{tr}\left(u^{\lambda}_{w^{-1}w'} A_\lambda(w)\right) |\lambda|^d d\lambda \tag{1.4.5}$$

since by definition of the Fourier transform, one can write

$$\mathrm{Op}(a)f(w) = \int_{\mathbb{H}^d} k_a(w,w')\, f(w')\, dw'. \tag{1.4.6}$$

We shall prove in Chapter 2 an integral formula giving an expression of the kernel in terms of the function $\sigma(a)$ defined in (1.4.1): see Proposition 2.4 page 30.

Let us denote by $m_\mu^{(\lambda)}$ the function

$$m_\mu^{(\lambda)}(\xi,\eta) \stackrel{\mathrm{def}}{=} m_\mu(\sqrt{|\lambda|}\xi, \sqrt{|\lambda|}\eta), \tag{1.4.7}$$

where m_μ is defined in Remark 1.17, page 20.

Then we note that if a is a symbol of order μ, then the operators

$$A_\lambda(\mathrm{Id}+D_\lambda)^{-\mu/2} = J_\lambda^* \mathrm{op}^w(a(w,\lambda) \# m_{-\mu}^{(\lambda)}) J_\lambda \quad \text{and}$$

$$(\mathrm{Id}+D_\lambda)^{-\mu/2} A_\lambda = J_\lambda^* \mathrm{op}^w(m_{-\mu}^{(\lambda)} \# a(w,\lambda)) J_\lambda$$

are uniformly bounded on $\mathcal{H}_\lambda$ (see Corollary 1.21, page 22). More precisely we have, for some integer n,

$$\|A_\lambda(\mathrm{Id}+D_\lambda)^{-\mu/2}\|_{\mathcal{L}(\mathcal{H}_\lambda)} + \|(\mathrm{Id}+D_\lambda)^{-\mu/2}A_\lambda\|_{\mathcal{L}(\mathcal{H}_\lambda)} \leq C_n \|a\|_{n;S_{\mathbb{H}^d}(\mu)}. \tag{1.4.8}$$

1.4.3. Statement of the results. — Let us first state a result concerning the action of pseudodifferential operators on the Schwartz class. This theorem is proved in Chapter 2.

Theorem 3. — *If a is a symbol in $S_{\mathbb{H}^d}(\mu)$ with $\rho = +\infty$, then $\mathrm{Op}(a)$ maps continuously $\mathcal{S}(\mathbb{H}^d)$ into itself.*

Notice that Theorem 3 allows to consider the composition of pseudodifferential operators, as well as their adjoint operators. The following result therefore considers the adjoint and the composition of such operators. It is proved in Chapter 3.

Theorem 4. — *Consider $\mathrm{Op}(a)$ and $\mathrm{Op}(b)$ two pseudodifferential operators on the Heisenberg group of order μ and ν respectively.*

- *If $\rho > 2(2d+1)+|\mu|$, then the operator $\mathrm{Op}(a)^*$ is a pseudodifferential operator of order μ on the Heisenberg group. We denote by a^* its symbol, which is given by (3.1.2).*
- *If $\rho > 2(2d+1)+|\mu|+|\nu|$, then the operator $\mathrm{Op}(a)\circ\mathrm{Op}(b)$ is a pseudodifferential operator of order less or equal to $\mu+\nu$. We denote by $a \#_{\mathbb{H}^d} b$ its symbol.*

We have the following asymptotic formulas for $\lambda \in \mathbb{R}^*$,

$$
(1.4.9) \quad a^* = \overline{a} + \frac{1}{2\sqrt{|\lambda|}} \sum_{1\leq j\leq d} \left(\{Z_j \overline{a}\,,\, \eta_j + i\xi_j\} + \{\overline{Z}_j \overline{a}\,,\, \eta_j - i\xi_j\} \right) + r_1
$$

$$
(1.4.10) \quad \begin{aligned} a \#_{\mathbb{H}^d} b &= b \# a \\ &+ \frac{1}{2\sqrt{|\lambda|}} \sum_{1\leq j\leq d} \left(Z_j b \# \left(\{a\,,\, \eta_j + i\xi_j\} \right) + \overline{Z}_j b \# \left(\{a\,,\, \eta_j - i\xi_j\} \right) \right) + r_2 \end{aligned}
$$

where r_1 *(resp.* r_2*) depends only on* $\mathcal{Z}^\alpha a$ *(resp.* $\mathcal{Z}^\alpha b$*) for* $|\alpha| \geq 2$.

One can find precise formulas for a^* and $a \#_{\mathbb{H}^d} b$ respectively in (3.1.3) and (3.3.3).

The first term appearing in the asymptotic formula for $a \#_{\mathbb{H}^d} b$ is not $a \# b$ as could be expected: this is due to the fact that in Definition (1.4.3) the Fourier transform is composed on the right.

Note that the asymptotic formulas only make sense when the semi norms $\|\cdot\|_{n;S_{\mathbb{H}^d}(\mu)}$ are finite for $\rho > 0$ large enough. Let us also emphasize that due to (1.4.10), the pseudodifferential operator $[\mathrm{Op}(a)\,,\, \mathrm{Op}(b)]$ is of order $\mu + \nu$. Actually the same phenomenon occurs when $\mathrm{Op}(a)$ and $\mathrm{Op}(b)$ are differential operators: there is no gain in the order of the commutators.

It is also important to point out that the asymptotics of (1.4.9) (respectively of (1.4.10)) can be pushed to higher order, as shown in Section 3.4 of Chapter 3. We will discuss in that section in which sense the formula are asymptotic. In fact, in the case where $\mathrm{Op}(a)$ is a differential operator, one obtains a complete description in (1.4.9) and in (1.4.10) since the asymptotic series are in fact finite.

Finally, we point out that even though a is real valued, a^* is generally different from a.

The final result of this paper concerns the action of pseudodifferential operators on Sobolev spaces.

Theorem 5. — *Let* μ *be a real number, and* $\rho > 2(2d+1)$ *be a noninteger real number. Consider a symbol* a *in* $S_{\mathbb{H}^d}(\mu)$ *in the sense of Definition 1.18. Then the operator* $\mathrm{Op}(a)$ *is bounded from* $H^s(\mathbb{H}^d)$ *into* $H^{s-\mu}(\mathbb{H}^d)$, *for any real number* s *such that* $|s - \mu| < \rho$. *More precisely there exists* $n \in \mathbb{N}$ *such that*

$$
\|\mathrm{Op}(a)\|_{\mathcal{L}(H^s(\mathbb{H}^d), H^{s-\mu}(\mathbb{H}^d))} \leq C_n \|a\|_{n;S_{\mathbb{H}^d}(\mu)}.
$$

If $\rho > 0$, *then the result holds for* $0 < s - \mu < \rho$.

Remark 1.24. — *The weaker result for small values of* ρ *is due to the fact that the adjoint of a pseudodifferential operator is also a pseudodifferential operator is only known to be true under the assumption that* ρ *is large enough. A way of overcoming this difficulty would be to have a quantification, stable by adjonction (of the type of the Weyl quantization in the Euclidean space). Unfortunately, the non commutativity of the Heisenberg group seems to make such a quantization difficult to define.*

Theorem 5 is proved in Chapter 5. The idea of the proof consists, as in the classical case, in decomposing the symbol into a series of reduced symbols. The new difficulty here compared to the classical case is that an additionnal microlocalization, in the λ direction, is necessary in order to conclude. This requires significantly more work, as paradifferential-type techniques have to be introduced in order to ensure the convergence of the truncated series (see for instance Proposition 4.15, page 74).

CHAPTER 2

FUNDAMENTAL PROPERTIES OF PSEUDODIFFERENTIAL OPERATORS

The main part of this chapter is devoted to the proof a number of important properties concerning pseudodifferential operators on $\mathbb{H}^d$ defined in Definition 1.23 page 22, which will be crucial in the proof of the main results of this paper. Before stating those properties, we first present several elementary examples of pseudodifferential operators, and analyze their action on Sobolev spaces. Then, we study the action of pseudodifferential operators on the Schwartz space, and prove Theorem 3 stated in the introduction.

2.1. Examples of pseudodifferential operators

Let us give some examples of pseudodifferential operators and their associate symbols. In this section and more generally in this article we will make constant use of functional calculus.

2.1.1. Multiplication operators. — It is easy to see that if b is a smooth function on $\mathbb{H}^d$, then $\mathrm{Op}(b)$ is the multiplication operator by $b(w)$ and clearly maps $H^s(\mathbb{H}^d)$ into itself provided that there exists $\rho > |s|$ and a constant C such that $\|b\|_{C^\rho} \leq C$.

2.1.2. Generalized multiplication operators. — Consider $b(w, \lambda)$ a $C^\rho(\mathbb{H}^d)$ real-valued function depending smoothly on λ so that for some $C \geq 0$,

$$\sup_{\lambda} \|b(\cdot, \lambda)\|_{C^\rho(\mathbb{H}^d)} \leq C.$$

If b is rapidly decreasing in λ in the sense that

$$\forall k \in \mathbb{N}, \quad \sup_{\lambda \in \mathbb{R}} \|(1 + |\lambda|)^k \partial_\lambda^k b(\cdot, \lambda)\|_{C^\rho(\mathbb{H}^d)} < \infty,$$

then b is a symbol of order 0 and the operator $\mathrm{op}^w(b(w, \lambda))$ is the operator of multiplication by the constant $b(w, \lambda)$, which does not depend on (y, η). Therefore,

$A_\lambda(w) = b(w,\lambda)$ is a uniformly bounded operator of $\mathcal{H}_\lambda$. Moreover, if $f \in L^2(\mathbb{H}^d)$ then $\{\mathcal{F}(f)(\lambda) \circ A_\lambda(w)\}_\lambda \in \mathcal{A}$ (as defined in Theorem 1), then

$$\|\mathcal{F}(f)(\lambda) \circ A_\lambda(w)\|_{HS(\mathcal{H}_\lambda)} = |b(w,\lambda)| \, \|\mathcal{F}(f)(\lambda)\|_{HS(\mathcal{H}_\lambda)} \leq C \|\mathcal{F}(f)(\lambda)\|_{HS(\mathcal{H}_\lambda)}$$

which implies that

$$\|\mathrm{Op}(b) f\|_{L^2(\mathbb{H}^d)} \leq C \, \|f\|_{L^2(\mathbb{H}^d)}.$$

Besides, one observes that for all $m \in \mathbb{N}$ and all $j \in \{1, \ldots, d\}$, we have by Lemma A.3,

$$\begin{aligned} \mathcal{F}\left(Z_j^m \left(\mathrm{Op}(b)f\right)\right)(\lambda) &= \mathcal{F}\left(\mathrm{Op}(b)f\right)(\lambda) \circ (Q_j^\lambda)^m \\ &= b(w,\lambda)\, \mathcal{F}\left((-\Delta_{\mathbb{H}^d})^{m/2} f\right)(\lambda) \circ D_\lambda^{-m/2} \circ (Q_j^\lambda)^m \end{aligned}$$

with $D_\lambda^{-m/2} \circ (Q_j^\lambda)^m$ uniformly bounded on $\mathcal{H}_\lambda$. A similar fact occurs for $\overline{Z}_j$. This computation shows that Theorem 5 is easily proved for all s, by interpolation and duality. More precisely, there exists a constant C such that

$$\|\mathrm{Op}(b) f\|_{H^s(\mathbb{H}^d)} \leq C \, \|f\|_{H^s(\mathbb{H}^d)}.$$

2.1.3. Differentiation operators. — Let us prove the following result, which provides the symbols of the family of left-invariant vector fields.

Proposition 2.1. — *We have for* $1 \leq j \leq d$, $\mu \in \mathbb{R}$, $\nu \geq 0$

$$\frac{1}{i} Z_j = \mathrm{Op}\left(\sqrt{|\lambda|}(\eta_j + i \,\mathrm{sgn}(\lambda)\, \xi_j)\right), \quad \frac{1}{i}\overline{Z}_j = \mathrm{Op}\left(\sqrt{|\lambda|}(\eta_j - i \,\mathrm{sgn}(\lambda)\, \xi_j)\right),$$
$$X_j = \mathrm{Op}(2i \,\mathrm{sgn}(\lambda) \sqrt{|\lambda|} \eta_j), \quad Y_j = -\mathrm{Op}(2i \sqrt{|\lambda|} \xi_j),$$
$$S = \mathrm{Op}(i\lambda), \quad -\Delta_{\mathbb{H}^d} = 4 \,\mathrm{Op}\left(|\lambda|(\eta^2 + \xi^2)\right),$$
$$(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{\mu}{2}} = \mathrm{Op}(m_\mu^{(\lambda)}(\xi,\eta)), \quad (-\Delta_{\mathbb{H}^d})^{\frac{\nu}{2}} = \mathrm{Op}(\widetilde{m}_\nu^{(\lambda)}(\xi,\eta)).$$

In particular Z_j, $\overline{Z}_j$, X_j *and* Y_j *are pseudodifferential operators of order 1, while* S *and* $\Delta_{\mathbb{H}^d}$ *are of order 2 and* $(\mathrm{Id} - \Delta_{\mathbb{H}^d})^\mu$ *is of order* 2μ.

Observe that if $\frac{1}{i} Z_j = \mathrm{Op}(d_j)$, $\frac{1}{i}\overline{Z}_j = \mathrm{Op}(\overline{d}_j)$, we have using the map σ defined in (1.4.1) page 21,

$$\sigma(d_j)(\xi,\eta) = \eta_j + i\xi_j \;\; \textit{and} \;\; \sigma(\overline{d}_j)(\xi,\eta) = \overline{\sigma(d_j)(\xi,\eta)} = \eta_j - i\xi_j.$$

Proof. — We perform the proof for Z_j. For $\lambda > 0$, we have from (1.2.37) along with Lemma A.3 stated page 101,

$$\begin{aligned} \mathcal{F}\left(\frac{1}{i} Z_j f\right)(\lambda) &= \frac{1}{i} \mathcal{F}(f)(\lambda) \circ Q_j^\lambda \\ &= \mathcal{F}(f)(\lambda) \circ J_\lambda^* \sqrt{|\lambda|} \left(\frac{1}{i}\partial_{\xi_j} - \frac{1}{i}\xi_j\right) J_\lambda \\ &= \mathcal{F}(f)(\lambda) \circ J_\lambda^* \,\mathrm{op}^w(\sqrt{|\lambda|}(\eta_j + i\xi_j)) J_\lambda. \end{aligned}$$

On the other hand, for $\lambda < 0$,

$$\begin{aligned}\mathcal{F}\Big(\frac{1}{i}Z_j f\Big)(\lambda) &= \mathcal{F}(f)(\lambda)\circ J_\lambda^*\sqrt{|\lambda|}\Big(\frac{1}{i}\partial_{\xi_j}+\frac{1}{i}\xi_j\Big)J_\lambda\\ &= \mathcal{F}(f)(\lambda)\circ J_\lambda^*\mathrm{op}^w(\sqrt{|\lambda|}(\eta_j - i\xi_j))J_\lambda.\end{aligned}$$

The other cases are treated similarly, except for the operators $(\mathrm{Id}-\Delta_{\mathbb{H}^d})^\mu$ and $(-\Delta_{\mathbb{H}^d})^\nu$, for which we refer to Remark 1.17, page 20. This concludes the proof of Proposition 2.1. □

2.1.4. Fourier multipliers. — A Fourier multiplier is an operator K acting on $\mathcal{S}(\mathbb{H}^d)$ such that

$$\mathcal{F}(Kf)(\lambda) = \mathcal{F}(f)(\lambda)\circ U_K(\lambda)$$

for some operator $U_K(\lambda)$ on $\mathcal{H}_\lambda$.

For instance, the differentiation operators Z_j and $\overline{Z}_j$ are Fourier multipliers, and $U_K(\lambda)$ is respectively equal to Q_j^λ and $\overline{Q}_j^\lambda$ as given in formulas (1.2.25) and (1.2.26) page 13. Similarly the Laplacian $-\Delta_{\mathbb{H}^d}$ is a Fourier multiplier, with $U_K(\lambda) = D_\lambda$ according to (1.2.29).

An interesting class of Fourier multipliers consist in the operators obtained from the Laplacian by means of functional calculus: for Ψ bounded and smooth, the operator $\Psi(-\Delta_{\mathbb{H}^d})$ is a bounded operator on $H^s(\mathbb{H}^d)$ for all $s\in\mathbb{R}$, and

$$\forall f\in L^2(\mathbb{H}^d),\ \ \mathcal{F}(\Psi(-\Delta_{\mathbb{H}^d})f)(\lambda) = \mathcal{F}(f)(\lambda)\circ\Psi(D_\lambda).$$

Such operators commute with one another, and so do the operators $\Psi(D_\lambda)$ for different functions Ψ. The Littlewood-Paley truncation operators that we will introduce later (see Chapter 4) are of this type, and we will see that they are pseudodifferential operators (see Proposition 4.18 stated page 79). Observe too that if $\Psi\in\mathcal{C}_0^\infty(\mathbb{R})$, then the operator $\Psi(-\Delta_{\mathbb{H}^d})$ is a smoothing operator which maps $H^s(\mathbb{H}^d)$ into $H^\infty(\mathbb{H}^d)$ for all $s\in\mathbb{R}$.

Another class of Fourier multipliers which are also pseudodifferential operators, is built with functions b in $S(m^\mu, g)$ with $\mu\geq 0$ in the following way.

Proposition 2.2. — *If $a(w,\lambda,\xi,\eta) = b\big(\mathrm{sgn}(\lambda)\sqrt{|\lambda|}\xi, \sqrt{|\lambda|}\eta\big)$ with $b\in S(m^\mu,g)$ and $\mu\geq 0$, then a belongs to $S_{\mathbb{H}^d}(\mu)$, and the operator* $\mathrm{Op}(a)$ *is a Fourier multiplier. Moreover,*

$$\forall u\in H^s(\mathbb{H}^d),\ \ \|\mathrm{Op}(a)u\|_{H^{s-\mu}(\mathbb{H}^d)}\leq C\|b\|_{n;S(m^\mu,g)}\|u\|_{H^s(\mathbb{H}^d)}. \tag{2.1.1}$$

Finally $\sigma(a) = b$ as given in Definition 1.18.

Proof. — The fact that a belongs to $S_{\mathbb{H}^d}(\mu)$ and that the operator $\mathrm{Op}(a)$ is a Fourier multiplier are straightforward. Now let us prove (2.1.1). We have

$$\mathrm{Op}(a)u(w) = \frac{2^{d-1}}{\pi^{d+1}}\int_{\mathbb{R}}\mathrm{tr}\left(u_{w^{-1}}^\lambda\mathcal{F}(u)(\lambda)A_\lambda\right)\ |\lambda|^d\,d\lambda,$$

with $A_\lambda = J_\lambda^*\,\mathrm{op}^w(a)\,J_\lambda$.

In view of the Plancherel formula (1.2.21) recalled page 12, to estimate the $H^{s-\mu}$-norm of $\mathrm{Op}(a)u$, we evaluate the Hilbert-Schmidt norm of $\mathcal{F}\left((\mathrm{Id}-\Delta_{\mathbb{H}^d})^{\frac{s-\mu}{2}}\mathrm{Op}(a)u\right)(\lambda)$. We have

$$\begin{aligned}\mathcal{F}\left((\mathrm{Id}-\Delta_{\mathbb{H}^d})^{\frac{s-\mu}{2}}\mathrm{Op}(a)u\right)(\lambda) &= \mathcal{F}(u)(\lambda)A_\lambda(\mathrm{Id}+D_\lambda)^{\frac{s-\mu}{2}}\\ &= \mathcal{F}\left((\mathrm{Id}-\Delta_{\mathbb{H}^d})^{\frac{s}{2}}u\right)(\lambda)(\mathrm{Id}+D_\lambda)^{-\frac{s}{2}}A_\lambda(\mathrm{Id}+D_\lambda)^{\frac{s-\mu}{2}}.\end{aligned}$$

In light of (1.4.8) page 23, the operators $(\mathrm{Id}+D_\lambda)^{-\frac{s}{2}}A_\lambda(\mathrm{Id}+D_\lambda)^{\frac{s-\mu}{2}}$ are uniformly bounded on $\mathcal{L}(\mathcal{H}_\lambda)$ by $C\|b\|_{n;S(m^\mu,g)}$ which ends the proof of the estimate thanks to property (1.2.19), recalled page 11. This ends the proof of Proposition 2.2. □

More generally, a pseudodifferential operator which is a Fourier multiplier has a symbol which does not depend on w. For this reason, Theorem 4 is easy to prove in that case.

Proposition 2.3. — *Consider a and b two symbols of $S_{\mathbb{H}^d}(\mu)$ which do not depend on the variable w. Then $\mathrm{Op}(a)^* = \mathrm{Op}(\overline{a})$ and $\mathrm{Op}(a)\circ\mathrm{Op}(b) = \mathrm{Op}(b\#a)$.*

Proof. — By the Plancherel formula,

$$(\mathrm{Op}(a)f,g) = \frac{2^{d-1}}{\pi^{d+1}}\int_{\mathbb{R}}\mathrm{tr}\left((\mathcal{F}(g)(\lambda))^*\,\mathcal{F}(f)(\lambda)A_\lambda\right)|\lambda|^d\,d\lambda$$

with $A_\lambda = J_\lambda^*\mathrm{op}^w(a(\lambda))J_\lambda$. Therefore,

$$\mathcal{F}\left(\mathrm{Op}(a)^*g\right)(\lambda) = \mathcal{F}(g)(\lambda)A_\lambda^*.$$

The fact that $A_\lambda^* = J_\lambda^*\mathrm{op}^w(\overline{a}(\lambda))J_\lambda$ gives the first part of the proposition.

Let us now consider $\mathrm{Op}(a)\circ\mathrm{Op}(b)$. We have

$$\mathcal{F}(\mathrm{Op}(a)\circ\mathrm{Op}(b)f)(\lambda) = \mathcal{F}(f)(\lambda)\circ B_\lambda\circ A_\lambda$$

with $B_\lambda = J_\lambda^*\mathrm{op}^w(b(\lambda))J_\lambda$. The fact that $\mathrm{op}^w(b)\circ\mathrm{op}^w(a) = \mathrm{op}^w(b\#a)$ finishes the proof. □

2.2. The link between the kernel and the symbol of a pseudodifferential operator

The kernel of a pseudodifferential operator on the Heisenberg group is given by (1.4.5) page 23. The following proposition provides an integral formula for the kernel of a pseudodifferential operator, as well as a formula enabling one to recover the symbol of an operator, from its kernel.

Proposition 2.4. — *The kernel of the pseudodifferential operator $\mathrm{Op}(a)$ is given by*

$$k(w,w') = \frac{1}{2\pi^{2d+1}}\int e^{2i\lambda(x\cdot y'-y\cdot x')}\sigma(a)(w,\lambda,\xi,\zeta)e^{i\lambda(s'-s)+2iz\cdot(y'-y)-2i\zeta\cdot(x'-x)}d\lambda\,d\xi\,d\zeta,$$

where $\sigma(a)$ is defined in (1.4.1), page 21.

Conversely, one recovers the symbol a through the formula

$$\sigma(a)(w,\lambda,\xi,\eta) = \int_{\mathbb{H}^d} e^{2i\,(y'\cdot\xi - x'\cdot\eta)} e^{i\lambda s'} k(w, w(w')^{-1})\, dw'. \tag{2.2.1}$$

Before proving the proposition, we notice that it allows to obtain directly the symbol of a pseudodifferential operator if one knows its kernel: the following corollary is obtained simply by using Proposition 2.4 and Relation (1.4.1) between a and $\sigma(a)$.

Corollary 2.5. — *Let Q be an operator on $\mathbb{H}^d$ of kernel $k(w,w')$ such that for some $\mu \in \mathbb{R}$, the function defined for $(w,\xi,\eta) \in \mathbb{H}^d \times \mathbb{R}^{2d}$ by*

$$a(w,\lambda,\xi,\eta) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} e^{2i\sqrt{|\lambda|}\,(\operatorname{sgn}(\lambda) y'\cdot\xi - x'\cdot\eta)} e^{i\lambda s'} k(w, w(w')^{-1})\, dw' \tag{2.2.2}$$

belongs to $S_{\mathbb{H}^d}(\mu)$. Then $Q = Op(a)$.

Proof of Proposition 2.4. — Let us start by recalling (1.4.5), which states that

$$k(w,w') = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} J^*_\lambda \operatorname{op}^w\left(a(w,\lambda)\right) J_\lambda\right) |\lambda|^d d\lambda.$$

Note that everywhere in the proof, integrals are to be understood as oscillatory integrals. The Bargmann representation u^λ_w and the Schrödinger representation v^λ_w are linked by the intertwining formula $u^\lambda_w = K^*_\lambda v^\lambda_w K_\lambda$, so using the operator $T_\lambda = J_\lambda K^*_\lambda$ we have

$$k(w,w') = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(v^\lambda_{w^{-1}w'} T^*_\lambda \operatorname{op}^w\left(a(w,\lambda)\right) T_\lambda\right) |\lambda|^d d\lambda.$$

By rescaling it is easy to see that

$$T^*_\lambda \operatorname{op}^w\left(a(w,\lambda)\right) T_\lambda = \operatorname{op}^w\left(a\left(w,\lambda,\sqrt{|\lambda|}\,\cdot, \frac{\cdot}{\sqrt{|\lambda|}}\right)\right), \tag{2.2.3}$$

so we get

$$k(w,w') = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(v^\lambda_{w^{-1}w'} \operatorname{op}^w\left(a\left(w,\lambda,\sqrt{|\lambda|}\,\cdot, \frac{\cdot}{\sqrt{|\lambda|}}\right)\right)\right) |\lambda|^d d\lambda. \tag{2.2.4}$$

In order to compute the trace of the operator $v^\lambda_{w^{-1}w'} \operatorname{op}^w\left(a\left(w,\lambda,\sqrt{|\lambda|}\,\cdot, \frac{\cdot}{\sqrt{|\lambda|}}\right)\right)$, we shall start by finding its kernel $\theta(\xi,\xi')$, and then use the formula (1.2.18) page 11, giving the trace of an operator in terms of its kernel.

So let us first compute $\theta(\xi,\xi')$, which we recall is defined by

$$v^\lambda_{w^{-1}w'} \operatorname{op}^w\left(a\left(w,\lambda,\sqrt{|\lambda|}\,\cdot, \frac{\cdot}{\sqrt{|\lambda|}}\right)\right) f(\xi) = \int \theta(\xi,\xi') f(\xi') d\xi'.$$

We also recall that

$$\operatorname{op}^w\left(a\left(w,\lambda,\sqrt{|\lambda|}\,\cdot, \frac{\cdot}{\sqrt{|\lambda|}}\right)\right) f(\xi) = \int A(\xi,\xi') f(\xi')\, d\xi',$$

where as stated in (1.3.6) page 18,

$$A(\xi,\xi') = (2\pi)^{-d} \int e^{i(\xi-\xi')\cdot\Xi} a\left(w,\lambda,\sqrt{|\lambda|}\left(\frac{\xi+\xi'}{2}\right),\frac{\Xi}{\sqrt{|\lambda|}}\right) d\Xi.$$

Finally using Formula (1.2.33) page 15 defining $v^{\lambda}_{w^{-1}w'}$, we get

$$\theta(\xi,\xi') = (2\pi)^{-d} e^{i\lambda(\tilde{s}-2\tilde{x}\cdot\tilde{y}+2\tilde{y}\cdot\xi)} \int a\left(w,\lambda,\sqrt{|\lambda|}\left(\frac{\xi-2\tilde{x}+\xi'}{2}\right),\frac{\Xi}{\sqrt{|\lambda|}}\right) e^{i\Xi\cdot(\xi-2\tilde{x}-\xi')} d\Xi,$$

where $\tilde{w} \stackrel{\text{def}}{=} w^{-1}w'$. Using the relation (1.2.17) given page 11 between the trace and the kernel of an operator and (2.2.4) above, we infer that

$$\begin{aligned} k(w,w') &= \frac{1}{2\pi^{2d+1}} \int e^{i\lambda(\tilde{s}-2\tilde{x}\cdot\tilde{y}+2\tilde{y}\cdot\xi)-2i\Xi\cdot\tilde{x}} a\left(w,\lambda,\sqrt{|\lambda|}(\xi-\tilde{x}),\frac{\Xi}{\sqrt{|\lambda|}}\right) |\lambda|^d d\lambda\, d\Xi\, d\xi \\ &= \frac{1}{2\pi^{2d+1}} \int e^{i\lambda\tilde{s}+2i\tilde{y}\cdot z-2i\zeta\cdot\tilde{x}} a\left(w,\lambda,\frac{z}{\sqrt{|\lambda|}}\operatorname{sgn}(\lambda),\frac{\zeta}{\sqrt{|\lambda|}}\right) d\lambda\, dz\, d\zeta \end{aligned}$$

where we have performed the change of variables $\xi-\tilde{x} = \frac{z}{|\lambda|}\operatorname{sgn}(\lambda)$, and $\Xi = \zeta$.

To end the proof of the proposition, one just needs to notice that

$$k(w,w(w')^{-1}) = \frac{1}{2\pi^{2d+1}} \int e^{-i\lambda s'-2iy'\cdot z+2ix'\cdot\zeta} \sigma(a)(w,\lambda,z,\zeta) dz\, d\zeta\, d\lambda$$

and to apply an inverse Fourier transform (in the Euclidean space). □

2.3. Action on the Schwartz class

The aim of this section is to prove Theorem 3, stating that if a belongs to $S_{\mathbb{H}^d}(\mu)$ and $\rho=+\infty$, then $\operatorname{Op}(a)$ maps continuously $\mathcal{S}(\mathbb{H}^d)$ into $\mathcal{S}(\mathbb{H}^d)$.

Before entering the proof of that result, let us point out that the smoothness condition (1.4.1) (see page 21) is necessary in order for $\operatorname{Op}(a)$ to act on $\mathcal{S}(\mathbb{H}^d)$. A counterexample is provided in the proof of the next statement. Actually one can define $\operatorname{Op}(a)$ without that condition, and typically the counterexample provided below provides an operator which is continuous on all Sobolev spaces.

Proposition 2.6. — *Let μ be an odd integer. There is a function a such that $\|a\|_{n;S_{\mathbb{H}^d}(\mu)}$ is finite for all integers n, and such that the operator $\operatorname{Op}(a)$ is not continuous over $\mathcal{S}(\mathbb{H}^d)$.*

Proof. — Let us define $\mu=2k+1$ and the function $a(w,\lambda,\xi,\eta)=A(\lambda)$, where $A(\lambda)=|\lambda|^{k+\frac{1}{2}}$. Let f be defined by

$$\mathcal{F}(f)(\lambda)F_{0,\lambda} = \phi(\lambda)F_{0,\lambda}, \quad \mathcal{F}(f)(\lambda)F_{\alpha,\lambda}=0 \quad \forall\alpha\neq 0,$$

where ϕ is a nonnegative, smooth, compactly supported function such that $\phi(0)=1$. An easy computation shows that $f\in\mathcal{S}(\mathbb{H}^d)$. Indeed writing

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}}\mathcal{F}(f)(\lambda)\right) |\lambda|^d\, d\lambda$$

and using the definition of the Fourier transform of f given above, a simple computation shows that for some constant C,

$$f(w) = C\int e^{-i\lambda s}\phi(\lambda)\mathrm{e}^{-|\lambda||z|^2}\Big(\int \mathrm{e}^{-2|\lambda||\xi|^2}\,d\xi\Big)\,|\lambda|^{2d}\,d\lambda$$

which gives the result since ϕ is smooth and compactly supported. Now let us consider $\mathrm{Op}(a)f$. A similar computation shows that if N is any integer, then for some fixed constants C' and C'' one has

$$\begin{aligned} s^N\mathrm{Op}(a)f(w) &= C'\int s^N e^{-i\lambda s}\phi(\lambda)A(\lambda)\mathrm{e}^{-|\lambda||z|^2}|\lambda|^d\,d\lambda \\ &= C''\int e^{-i\lambda s}\partial_\lambda^N(\phi(\lambda)|\lambda|^d A(\lambda)\mathrm{e}^{-|\lambda||z|^2})\,d\lambda. \end{aligned}$$

For any fixed z, this is the (real) Fourier transform at the point s of the function

$$\lambda \mapsto \partial_\lambda^N(\phi(\lambda)|\lambda|^d A(\lambda)\mathrm{e}^{-|\lambda||z|^2}).$$

Let us evaluate this integral at the point $z = 0$. Taking N large enough, the result is clearly not bounded in s. □

Proof of Theorem 3. — Consider $f \in \mathcal{S}(\mathbb{H}^d)$, and let us start by proving that $\mathrm{Op}(a)f$ belongs to $L^\infty(\mathbb{H}^d)$. By definition of $\mathrm{Op}(a)$, we need to find a constant C_0 such that for all $w \in \mathbb{H}^d$,

$$\Big|\int \mathrm{tr}\,\big(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)A_\lambda(w)\big)\,|\lambda|^d d\lambda\Big| < C_0. \tag{2.3.1}$$

Consider χ a frequency cut-off function defined by $\chi(r) = 1$ for $|r| \leq 1$ and $\chi(r) = 0$ for $|r| > 2$. We write

$$\int \mathrm{tr}\,\big(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)A_\lambda(w)\big)\,|\lambda|^d d\lambda = I_1 + I_2$$

where

$$I_1 \stackrel{\text{def}}{=} \int \mathrm{tr}\,\big(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)\chi(D_\lambda)A_\lambda(w)\big)\,|\lambda|^d d\lambda$$

and we deal separately with each part.

Let us first observe that for any $k \in \mathbb{N}$ and by Remark 1.7 stated page 12, we have

$$\begin{aligned} |I_1| \ \leq\ & \Big(\int \|u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda\Big)^{\frac12} \\ & \times\Big(\int \|(\mathrm{Id}+D_\lambda)^{-k}\chi(D_\lambda)A_\lambda(w)\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda\Big)^{\frac12}. \end{aligned} \tag{2.3.2}$$

Besides, using (1.2.19) page 11, there exists a constant C such that

$$\|u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|_{HS(\mathcal{H}_\lambda)} \leq C\,\|\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|_{HS(\mathcal{H}_\lambda)},$$

and

$$\begin{aligned}\|(\mathrm{Id}+D_\lambda)^{-k}\chi(D_\lambda)A_\lambda(w)\|_{HS(\mathcal{H}_\lambda)}&\\ &\leq \|(\mathrm{Id}+D_\lambda)^{-\frac{\mu}{2}}A_\lambda(w)\|_{\mathcal{L}(\mathcal{H}_\lambda)}\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\chi(D_\lambda)\|_{HS(\mathcal{H}_\lambda)}\\ &\leq C\,\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\chi(D_\lambda)\|_{HS(\mathcal{H}_\lambda)}\end{aligned}$$

where we have used (1.4.8) (see page 23) for the last bound. We then observe that on the one hand

$$\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k=\mathcal{F}((\mathrm{Id}-\Delta_{\mathbb{H}^d})^k f)(\lambda)$$

so that by the Plancherel formula

$$\frac{2^{d-1}}{\pi^{d+1}}\int\|\mathcal{F}(f)(\lambda)(\mathrm{Id}+D_\lambda)^k\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda=\|(\mathrm{Id}-\Delta_{\mathbb{H}^d})^k f\|^2_{L^2(\mathbb{H}^d)}.$$

On the other hand

$$\begin{aligned}\int\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\chi(D_\lambda)\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d d\lambda&\\ &=\sum_{\alpha\in\mathbb{N}^d}\int\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\chi(D_\lambda)F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda}|\lambda|^d d\lambda\\ &=\sum_{\alpha\in\mathbb{N}^d}\int(1+|\lambda|(2|\alpha|+d))^{\frac{\mu}{2}-k}\,\chi(|\lambda|(2|\alpha|+d))|\lambda|^d d\lambda,\end{aligned}$$

hence

$$\begin{aligned}\int\|(\mathrm{Id}+D_\lambda)^{\frac{\mu}{2}-k}\chi(D_\lambda)\|^2_{HS}|\lambda|^d d\lambda&\\ &\leq C\sum_{m\in\mathbb{N}}(2m+d)^{d-1}\int(1+|\lambda|(2m+d))^{\frac{\mu}{2}-k}\chi(|\lambda|(2m+d))|\lambda|^d d\lambda\end{aligned}$$

where we have used that the number of $\alpha\in\mathbb{N}^d$ such that $|\alpha|=m$ is controlled by m^{d-1}. Then, the change of variables $\beta=(2m+d)\lambda$ gives

$$\int\|(\mathrm{Id}+D_\lambda)^{\mu-k}\chi(D_\lambda)\|^2_{HS}|\lambda|^d d\lambda\leq C\left(\sum_{m\in\mathbb{N}}\frac{1}{1+m^2}\right)\int\chi(|\beta|)(1+|\beta|)^{\frac{\mu}{2}-k+d}d\beta.$$

Therefore, (2.3.2) becomes

$$|I_1|\leq C\,\|(\mathrm{Id}-\Delta_{\mathbb{H}^d})^k f\|_{L^2(\mathbb{H}^d)}\left(\sum_{m\in\mathbb{N}}\frac{1}{1+m^2}\right)\int\chi(|\beta|)(1+|\beta|)^{\frac{\mu}{2}-k+d}d\beta\leq C_0$$

for any k.

A similar argument applies to I_2 and allows to get

$$|I_2|\leq C\|(\mathrm{Id}-\Delta_{\mathbb{H}^d})^k f\|_{L^2(\mathbb{H}^d)}\left(\sum_{m\in\mathbb{N}}\frac{1}{1+m^2}\right)\int\widetilde{\chi}(|\beta|)(1+|\beta|)^{\frac{\mu}{2}-k+d}d\beta$$

where $\widetilde{\chi}$ is a frequency cut-off function defined by $\widetilde{\chi}(r)=1$ for $|r|\geq\frac{3}{4}$, and $\widetilde{\chi}(r)=0$ for $|r|<\frac{1}{2}$. The choice $k>1+d+\frac{\mu}{2}$ achieves the estimate of the term I_2.

The end of the proof of Theorem 3 is a direct consequence of the following lemma. We will emphasize later other formulas of that type which will be useful in the following sections. □

Lemma 2.7. — *For any symbol $a \in S_{\mathbb{H}^d}(\mu)$ and $j \in \{1, \dots, d\}$, there are symbols $b_j^{(1)}, b_j^{(2)}$ belonging to $S_{\mathbb{H}^d}(\mu+1)$ and $c_j^{(1)}, c_j^{(2)} \in S_{\mathbb{H}^d}(\mu-1)$ and $p \in S_{\mathbb{H}^d}(\mu)$ such that*

$$[Z_j\,,\, \mathrm{Op}(a)] = \mathrm{Op}(b_j^{(1)}),\ \ [\overline{Z}_j\,,\, \mathrm{Op}(a)] = \mathrm{Op}(b_j^{(2)}),$$

$$[z_j\,,\, \mathrm{Op}(a)] = \mathrm{Op}(c_j^{(1)}),\ \ [\overline{z}_j\,,\, \mathrm{Op}(a)] = \mathrm{Op}(c_j^{(2)}),$$

$$[is, \mathrm{Op}(a)] = \mathrm{Op}(p).$$

In particular, one has

$$b_j^{(1)} = Z_j a + \sqrt{|\lambda|}\{a, \eta_j + i\,\mathrm{sgn}(\lambda)\xi_j\} \ \text{ and } \ b_j^{(2)} = \overline{Z}_j a + \sqrt{|\lambda|}\{a, \eta_j - i\,\mathrm{sgn}(\lambda)\xi_j\},$$

$$c_j^{(1)} = \frac{1}{2\sqrt{|\lambda|}}\{a, i\xi_j - \mathrm{sgn}(\lambda)\eta_j\} \ \text{ and } \ c_j^{(2)} = \frac{1}{2\sqrt{|\lambda|}}\{a, i\xi_j + \mathrm{sgn}(\lambda)\eta_j\}.$$

Remark 2.8. — *Notice that contrary to the classical case (see* **[1]** *for instance), $[Z_j\,,\, \mathrm{Op}(a)]$ is an operator of order $\mu+1$ instead of μ, due to the additionnal Poisson bracket appearing in the definition of $b_j^{(1)}$ (and the same goes for $[\overline{Z}_j\,,\, \mathrm{Op}(a)]$).*

On the other hand, $[z_j\,,\, \mathrm{Op}(a)]$ and $[\overline{z}_j\,,\, \mathrm{Op}(a)]$ are of order $\mu-1$ as in the classical setting, but $[s, \mathrm{Op}(a)]$ is only of order μ.

Let us now prove Lemma 2.7.

Proof. — Let us consider a function f in $\mathcal{S}(\mathbb{H}^d)$, and a symbol a belonging to $S_{\mathbb{H}^d}(\mu)$. We have for $1 \leq j \leq d$,

$$Z_j \mathrm{Op}(a) f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int \mathrm{tr}\left(Z_j(u^\lambda_{w^{-1}})\mathcal{F}(f)(\lambda)A_\lambda(w) + u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)Z_j A_\lambda(w)\right)|\lambda|^d\, d\lambda$$

with $Z_j A_\lambda(w) = J_\lambda^* \mathrm{op}^w(Z_j a(w,\lambda))J_\lambda$.

Thanks to Lemma A.3 page 101, we have $Z_j u^\lambda_{w^{-1}} = Q_j^\lambda u^\lambda_{w^{-1}}$, recalling that Q_j^λ is defined in (1.2.25) page 13. Therefore, since $\mathcal{F}(Z_j f)(\lambda) = \mathcal{F}(f)(\lambda)Q_j^\lambda$, and using the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we obtain

$$[Z_j\,,\, \mathrm{Op}(a)]f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int \mathrm{tr}\left(u^\lambda_{w^{-1}}\mathcal{F}(f)(\lambda)\left(\left[A_\lambda(w), Q_j^\lambda\right] + Z_j A_\lambda(w)\right)\right)|\lambda|^d\, d\lambda.$$

We then use (1.2.37) page 16 to find, for $\lambda > 0$,

$$\left[A_\lambda(w), Q_j^\lambda\right] = J_\lambda^*\left[\mathrm{op}^w(a(w,\lambda))\,,\, \sqrt{|\lambda|}(\partial_{\xi_j} - \xi_j)\right]J_\lambda$$

and for $\lambda < 0$,

$$\left[A_\lambda(w), Q_j^\lambda\right] = J_\lambda^*\left[\mathrm{op}^w(a(w,\lambda))\,,\, \sqrt{|\lambda|}(\partial_{\xi_j} + \xi_j)\right]J_\lambda.$$

Therefore, by standard symbolic calculus, using in particular the fact that if b is a polynomial of degree one in (ξ, η), then

$$[\mathrm{op}^w(a), \mathrm{op}^w(b)] = \frac{1}{i} op^w(\{a, b\}), \tag{2.3.3}$$

we get

$$\big([A_\lambda(w), Q_j^\lambda] + Z_j A_\lambda(w)\big) = J_\lambda^* \mathrm{op}^w \left(\sqrt{\lambda}\{a(w,\lambda), \eta_j + i\xi_j\} + Z_j a(w,\lambda)\right) \text{ for } \lambda > 0,$$
$$\big([A_\lambda(w), Q_j^\lambda] + Z_j A_\lambda(w)\big) = J_\lambda^* \mathrm{op}^w \left(\sqrt{-\lambda}\{a(w,\lambda), \eta_j - i\xi_j\} + Z_j a(w,\lambda)\right) \text{ for } \lambda < 0,$$

which are the expected formula. We moreover observe that if $a \in S_{\mathbb{H}^d}(\mu)$ and $1 \leq j \leq d$, then $\sqrt{|\lambda|}\partial_{\xi_j} a$ and $\sqrt{|\lambda|}\partial_{\eta_j} a$ are symbols of order $\mu + 1$. Indeed since a is of order μ, there exists a constant C such that, for $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^{2d}$,

$$\begin{aligned} \left|(\lambda\partial_\lambda)^k \partial_{(\xi,\eta)}^\beta (\sqrt{|\lambda|}\partial_{\xi_j} a)\right| &\leq C \sqrt{|\lambda|}^{-2+|\beta|} (1 + |\lambda|(1 + |\xi|^2 + |\eta|^2))^{\frac{\mu-|\beta|-1}{2}} \\ &\leq C \sqrt{|\lambda|}^{-|\beta|} (1 + |\lambda|(1 + |\xi|^2 + |\eta|^2))^{\frac{\mu+1-|\beta|}{2}} . \end{aligned}$$

A similar computation gives the result for $[\overline{Z}_j, \mathrm{Op}(a)]$.

Let us now consider the other types of commutators. For $f \in \mathcal{S}(\mathbb{H}^d)$ and $1 \leq j \leq d$, we have

$$[z_j, \mathrm{Op}(a)] f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int (z_j - z_j') \mathrm{tr}\left(u_{w^{-1}w'}^\lambda A_\lambda(w)\right) f(w') \, |\lambda|^d d\lambda dw'.$$

By Lemma A.2 page 100, we have $z_j u_w^\lambda = \frac{1}{2\lambda}[\overline{Q}_j^\lambda, u_w^\lambda]$. Therefore, setting $\tilde{w} = w^{-1}w' = (\tilde{z}, \tilde{s})$, we get, using (1.2.37) page 16 along with the fact that $A_\lambda(w) = J_\lambda^* \mathrm{op}^w(a(w,\lambda)) J_\lambda$,

$$\begin{aligned} \mathrm{tr}\left(\tilde{z}_j u_{\tilde{w}}^\lambda A_\lambda(w)\right) &= \frac{\sqrt{|\lambda|}}{2\lambda} \mathrm{tr}\left([J_\lambda^*(\partial_{\xi_j} + \mathrm{sgn}(\lambda)\xi_j) J_\lambda, u_{\tilde{w}}^\lambda] A_\lambda(w)\right) \\ &= \frac{\mathrm{sgn}(\lambda)}{2\sqrt{|\lambda|}} \mathrm{tr}\left(J_\lambda^* [\partial_{\xi_j} + \mathrm{sgn}(\lambda)\xi_j, J_\lambda u_{\tilde{w}}^\lambda J_\lambda^*] \mathrm{op}^w(a(w,\lambda)) J_\lambda\right) \\ &= \frac{1}{2\sqrt{|\lambda|}} \mathrm{tr}\left([\mathrm{op}^w(a(w,\lambda)), \mathrm{sgn}(\lambda)\partial_{\xi_j} + \xi_j] J_\lambda u_{\tilde{w}}^\lambda J_\lambda^*\right). \end{aligned}$$

By standard symbolic calculus, this implies that

$$\mathrm{tr}\left(\tilde{z}_j u_{\tilde{w}}^\lambda A_\lambda(w)\right) = \frac{1}{2\sqrt{|\lambda|}} \mathrm{tr}\left(u_{\tilde{w}}^\lambda J_\lambda^* \mathrm{op}^w(\{a, \mathrm{sgn}(\lambda)\eta_j - i\xi_j\}) J_\lambda\right) \tag{2.3.4}$$

which gives the announced formula. Besides, the same argument as before gives that if a is a symbol in $S_{\mathbb{H}^d}(\mu)$ and if $1 \leq j \leq d$, then $\frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j} a$ and $\frac{1}{\sqrt{|\lambda|}}\partial_{\eta_j} a$ are symbols of $S_{\mathbb{H}^d}(\mu - 1)$. Indeed, for $k \in \mathbb{N}$ and $\beta \in \mathbb{N}^{2d}$

$$\left|(\lambda\partial_\lambda)^k \partial_{(\xi,\eta)}^\beta \left(\frac{1}{\sqrt{|\lambda|}}\partial_{\xi_j} a\right)\right| \leq C |\lambda|^{|\beta|} (1 + |\lambda|(1 + |\xi|^2 + |\eta|^2))^{\frac{\mu-|\beta|-1}{2}}$$

A similar argument gives the result for the multiplication by $\overline{z_j}$. In particular, one finds for all $\lambda \in \mathbb{R}^*$,

$$\operatorname{tr}\left(\overline{\tilde{z}}_j u^\lambda_{\tilde{w}} A_\lambda(w)\right) = -\frac{1}{2\sqrt{|\lambda|}} \operatorname{tr}\left(u^\lambda_{\tilde{w}} J^*_\lambda \operatorname{op}^w(\{a, \operatorname{sgn}(\lambda)\eta_j + i\xi_j\}) J_\lambda\right). \tag{2.3.5}$$

Finally, let us consider the last commutator. We have

$$[is, \operatorname{Op}(a)]f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int i(s-s') \operatorname{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right) f(w') \, |\lambda|^d d\lambda dw'$$

Since with $\tilde{w} = w^{-1}w'$, we have $\tilde{s} = s' - s - 2\operatorname{Im}(z\overline{z'})$ and in view of the preceding results, it is enough to observe

$$\begin{aligned} \frac{2^{d-1}}{\pi^{d+1}} \int i\tilde{s} \operatorname{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right) f(w') \, |\lambda|^d d\lambda dw' \\ = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} J^*_\lambda \operatorname{op}^w(g) J_\lambda(w)\right) f(w') \, |\lambda|^d d\lambda dw' \end{aligned}$$

where we have used Lemma A.4 stated page 102 and where g is defined by (A.2.4), whence the fact that $[is, \operatorname{Op}(a)]$ is a pseudodifferential operator of order μ. □

We then observe that the arguments of the proof above give the following proposition.

Proposition 2.9. — *For $j \in \{1, \dots, d\}$ and $a \in S_{\mathbb{H}^d}(\mu)$ in $C^\rho(\mathbb{H}^d)$ with $\rho > 1$, we have*

$$\begin{aligned} Z_j \operatorname{Op}(a) &= \operatorname{Op}\left(Z_j a + a \# \sqrt{|\lambda|}(-\operatorname{sgn}(\lambda)\xi_j + i\eta_j)\right), \\ \operatorname{Op}(a) Z_j &= \operatorname{Op}\left(\sqrt{|\lambda|}(-\operatorname{sgn}(\lambda)\xi_j + i\eta_j) \# a\right), \\ \overline{Z}_j \operatorname{Op}(a) &= \operatorname{Op}\left(\overline{Z}_j a + a \# \sqrt{|\lambda|}(\operatorname{sgn}(\lambda)\xi_j + i\eta_j)\right), \\ \operatorname{Op}(a) \overline{Z}_j &= \operatorname{Op}\left(\sqrt{|\lambda|}(\operatorname{sgn}(\lambda)\xi_j + i\eta_j) \# a\right). \end{aligned}$$

Besides, for $N \in \mathbb{N}$ and $\rho > 2N$, then $(-\Delta_{\mathbb{H}^d})^N \operatorname{Op}(a)$ and $\operatorname{Op}(a)(-\Delta_{\mathbb{H}^d})^N$ are pseudodifferential operators of order $\mu + 2N$. If $k \in \mathbb{R}$ and $\rho > 2k$ then $\operatorname{Op}(a)(\operatorname{Id} - \Delta_{\mathbb{H}^d})^k$ and $(\operatorname{Id} - \Delta_{\mathbb{H}^d})^k \operatorname{Op}(a)$ are pseudodifferential operators of order $\mu + 2k$.

Proof. — The four first relations are by-product of the preceding proof and they directly imply that $(-\Delta_{\mathbb{H}^d})^N \operatorname{Op}(a)$ and $\operatorname{Op}(a)(-\Delta_{\mathbb{H}^d})^N$ are pseudodifferential operators. Then for $k \in \mathbb{R}$, we write

$$\operatorname{Op}(a)(\operatorname{Id} - \Delta_{\mathbb{H}^d})^k f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{R}} \operatorname{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda)(\operatorname{Id} + D_\lambda)^k A_\lambda(w)\right) |\lambda|^d d\lambda.$$

Observing that

$$(\operatorname{Id} + D_\lambda)^k A_\lambda(w) = J^*_\lambda \operatorname{op}^w\left(m^{(\lambda)}_{2k} \# a(w, \lambda)\right) J_\lambda,$$

where $m_{2k}^{(\lambda)}$ is the symbol defined by (1.4.7) page 23, we obtain that $\mathrm{Op}(a)(\mathrm{Id}-\Delta_{\mathbb{H}^d})^k$ is a pseudodifferential operator of order $\mu+2k$. We argue similarly for $\mathrm{Op}(a)(\mathrm{Id}-\Delta_{\mathbb{H}^d})^k$. □

CHAPTER 3

THE ALGEBRA OF PSEUDODIFFERENTIAL OPERATORS

This chapter is devoted to the analysis of the algebra properties of the set of pseudodifferential operators. The two first sections are devoted to the study of the adjoint of a pseudodifferential operator: we first compute what could be its symbol, and then prove that it actually is a symbol. In order to prove that fact, the method consists in writing the formula giving the symbol as an oscillatory integral, and in writing a dyadic partition of unity centered on the stationary point of the phase appearing in that integral. This creates a series of oscillatory integrals which are all individually well defined (since each integral is on a compact set). The convergence of the series is then obtained by multiple integrations by parts using a vector field adapted to the phase, as in a stationary phase method.

The approach is similar for the analysis of the composition of two pseudodifferential operators and this is achieved in the third section. Finally, asymptotic formulas for both the adjoint and the composition are discussed in the last section. These formulas result from a Taylor expansion in the spirit of what is done in the Euclidian space but adapted to the case of the Heisenberg group.

3.1. The adjoint of a pseudodifferential operator

In this section, we prove that the adjoint of a pseudodifferential operator is a pseudodifferential operator. We first observe that if $a \in S_{\mathbb{H}^d}(\mu)$, then $A \stackrel{\text{def}}{=} \operatorname{Op}(a)$ has a kernel $k_A(w,w')$ as given in (1.4.5) page 23, and the kernel of $A^* = \operatorname{Op}(a)^*$ is $k(w,w') = \overline{k_A(w',w)}$, whence

$$
\begin{aligned}
k(w,w') &= \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{R}} \operatorname{tr}\left((u^\lambda_{(w')^{-1}w})^* J^*_\lambda \operatorname{op}^w \left(a(w',\lambda)\right)^* J_\lambda \right) |\lambda|^d \, d\lambda \\
&= \frac{2^{d-1}}{\pi^{d+1}} \int_{\mathbb{R}} \operatorname{tr}\left(u^\lambda_{(w)^{-1}w'} \, J^*_\lambda \operatorname{op}^w \left(\overline{a}(w',\lambda)\right) J_\lambda \right) |\lambda|^d \, d\lambda \qquad (3.1.1)
\end{aligned}
$$

where we have used the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, the formula for the adjoint of a Weyl symbol, and $\overline{\operatorname{tr}(B)} = \operatorname{tr}(B^*)$. Therefore, in view of Corollary 2.5 stated page 31, if $\mathrm{Op}(a)^*$ is a pseudodifferential operator, its symbol a^* will be given for all $(w,\lambda,\xi,\eta)\in\mathbb{H}^d\times\mathbb{R}^*\times\mathbb{R}^{2d}$ by

$$\begin{aligned} a^*(w,\lambda,\xi,\eta) &= \frac{2^{d-1}}{\pi^{d+1}}\int_{\mathbb{R}\times\mathbb{H}^d} e^{2i\sqrt{|\lambda|}(\mathrm{sgn}(\lambda)y'\cdot\xi-x'\cdot\eta)+i\lambda s'} \\ &\quad\times\ \mathrm{tr}\left(u^{\lambda'}_{(w')^{-1}}J^*_{\lambda'}\mathrm{op}^w\left(\overline{a}(w(w')^{-1},\lambda')\right)J_{\lambda'}\right)|\lambda'|^d\,d\lambda'\,dw'. \end{aligned} \tag{3.1.2}$$

It remains to prove that the map $a\mapsto a^*$ which is well defined on $\mathcal{S}(\mathbb{H}^d\times\mathbb{R}^{2d+1})$ can be extended to symbols $a\in S_{\mathbb{H}^d}(\mu)$ and that for such a, their image a^* is also in $S_{\mathbb{H}^d}(\mu)$. Therefore, it is enough to prove the following proposition.

Proposition 3.1. — *The map $a\mapsto a^*$ extends by continuity to $S_{\mathbb{H}^d}(\mu)$ since for all $k\in\mathbb{N}$ there exists $n\in\mathbb{N}$ and $C>0$ such that*

$$\forall a\in S_{\mathbb{H}^d}(\mu),\quad \|a^*\|_{k;S_{\mathbb{H}^d}(\mu)}\leq C\,\|a\|_{n;S_{\mathbb{H}^d}(\mu)}.$$

It is not at all obvious that the formula (3.1.2) for a^* gives the expected result for the examples studied in Section 2.1 of Chapter 2. To see that more clearly, it is convenient to transform the expression of a^* into an integral formula.

Lemma 3.2. — *Let $a\in\mathcal{S}(\mathbb{H}^d\times\mathbb{R}^{2d+1})$, then the symbol a^* of $\mathrm{Op}(a)^*$ given in (3.1.2) can also be written*

$$\begin{aligned} a^*(w,\lambda,\xi,\eta) = \frac{1}{2\pi^{2d+1}}\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d} & e^{2i\sqrt{|\lambda|}(\mathrm{sgn}(\lambda)y'\cdot\xi-x'\cdot\eta)+is'(\lambda-\lambda')-2i\sqrt{|\lambda'|}(\mathrm{sgn}(\lambda')z\cdot y'-\zeta\cdot x')} \\ & \times\ \overline{a}\left(w(w')^{-1},\lambda',z,\zeta\right)|\lambda'|^d d\zeta\,dz\,d\lambda'\,dw'. \end{aligned}$$

The formula given in Lemma 3.2 allows to revisit the examples of Section 2.1, Chapter 2. Indeed if $a=a(\lambda,\xi,\eta)$, then integration in s' gives $\lambda=\lambda'$, then integration in x' (resp. y')) gives $\zeta=\eta$ (resp. $z=y'$); whence $a^*(w,\lambda,\xi,\eta)=\overline{a}(\lambda,\xi,\eta)$.
If $a=a(w)$, then integration in ζ (resp. ξ) gives $x'=0$ (resp. $y'=0$); then integration in s' gives $\lambda=\lambda'$, whence $a^*(w)=\overline{a}(w)$ as expected.

Remark 3.3. — *Let $\sigma(a)$ be defined by (1.4.1) page 21, then $\sigma(a^*)$ and $\sigma(a)$ are related by*

$$\begin{aligned} \sigma(a^*)(w,\lambda,\xi,\eta) &= \frac{1}{2\pi^{2d+1}}\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d} e^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')} \\ &\qquad\times\overline{\sigma(a)}\left(w(w')^{-1},\lambda',z,\zeta\right)d\zeta\,dz\,d\lambda'\,dw'. \end{aligned} \tag{3.1.3}$$

Proof of Lemma 3.2. — The first step consists in computing the trace term using the link between the trace and the kernel stated in (1.2.18) page 11. So let us start by studying the kernel of our operator. Using $J_{\lambda'}=T_{\lambda'}K_{\lambda'}$, we write

$$\mathrm{tr}\left(u^{\lambda'}_{(w')^{-1}}J^*_{\lambda'}\mathrm{op}^w\left(\overline{a}(\tilde{w},\lambda')\right)J_{\lambda'}\right)=\mathrm{tr}\left(K_{\lambda'}u^{\lambda'}_{(w')^{-1}}K^*_{\lambda'}T^*_{\lambda'}\mathrm{op}^w\left(\overline{a}(\tilde{w},\lambda')\right)T_{\lambda'}\right) \tag{3.1.4}$$

where $\tilde{w}=w(w')^{-1}$ and we observe that $K_{\lambda'}u^{\lambda'}_{(w')^{-1}}K^*_{\lambda'}=v^{\lambda'}_{(w')^{-1}}$ where $v^{\lambda'}_{(w')^{-1}}$ is the Schrödinger representation given by (1.2.33) page 15. We shall use the same type of method as for the proof of Proposition 2.4. We recall that if U is an operator on $L^2(\mathbb{R}^d)$ of kernel $k_U(\xi,\xi')$, then the kernel of the operator

$$\tilde{U} \stackrel{\text{def}}{=} v^{\lambda'}_{(w')^{-1}} \circ U$$

is the function $k_{\tilde{U}}$ given by

$$k_{\tilde{U}}(\xi,\xi') = \mathrm{e}^{-i\lambda'(s'+2x'\cdot y'+2y'\cdot\xi)}\, k_U(\xi+2x',\xi').$$

This comes from the definition of the kernel in (1.2.17), page 11, and the definition of $v^{\lambda'}_{(w')^{-1}}$ in (1.2.33), page 15. We take now

$$U = T^*_{\lambda'}\mathrm{op}^w\left(\overline{a}(w(w')^{-1},\lambda',\xi,\eta)\right)T_{\lambda'}.$$

As in (2.2.3) page 31, we have

$$T^*_{\lambda'}\mathrm{op}^w\left(\overline{a}(w(w')^{-1},\lambda',\xi,\eta)\right)T_{\lambda'} = \mathrm{op}^w\left(\overline{a}(w(w')^{-1},\lambda',\sqrt{|\lambda'|}\xi,\frac{\eta}{\sqrt{|\lambda'|}}\right)$$

and using (1.3.6) page 18 this gives

$$k_U(\xi,\xi') = (2\pi)^{-d}\int_{\mathbb{R}^d}\overline{a}\left(w(w')^{-1},\lambda',\sqrt{|\lambda'|}\left(\frac{\xi+\xi'}{2}\right),\frac{\Xi}{\sqrt{|\lambda'|}}\right)\mathrm{e}^{i\Xi\cdot(\xi-\xi')}d\Xi.$$

This implies

$$\begin{aligned}
\mathrm{tr}(\tilde{U}) &= \int_{\mathbb{R}^d} k_{\tilde{U}}(\xi,\xi)d\xi\\
&= \int_{\mathbb{R}^d}\mathrm{e}^{-i\lambda'(s'+2x'\cdot y'+2y'\cdot\xi)}k_U(\xi+2x',\xi)d\xi\\
&= (2\pi)^{-d}\int_{\mathbb{R}^{2d}}\mathrm{e}^{-i\lambda'(s'+2x'\cdot y'+2y'\cdot\xi)+2i\Xi\cdot x'}\overline{a}\left(w(w')^{-1},\lambda',\sqrt{|\lambda'|}(\xi+x'),\frac{\Xi}{\sqrt{|\lambda'|}}\right)d\Xi\,d\xi.
\end{aligned}$$

We finally obtain via (3.1.2) and (3.1.4)

$$\begin{aligned}
a^*(w,\lambda,\xi,\eta) = \frac{1}{2\pi^{2d+1}}\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d} & \mathrm{e}^{2i\sqrt{|\lambda|}(\mathrm{sgn}(\lambda)y'\cdot\xi-x'\cdot\eta)+is'(\lambda-\lambda')-2i\lambda'(x'\cdot y'+y'\cdot\xi)+2ix'\cdot\Xi}\\
&\times\overline{a}\left(w(w')^{-1},\lambda',\sqrt{|\lambda'|}(\xi+x'),\frac{\Xi}{\sqrt{|\lambda'|}}\right)|\lambda'|^d d\Xi\,d\xi\,d\lambda'\,dw'.
\end{aligned}$$

The change of variable $\sqrt{|\lambda'|}(\xi+x') = \mathrm{sgn}(\lambda')z$ and $\Xi = \sqrt{|\lambda'|}\zeta$ gives the formula of the lemma. □

3.2. Proof of Proposition 3.1

To prove Proposition 3.1, we shall use Remark 3.3 and Proposition 1.22. Our aim is to analyze the symbol properties of the oscillatory integral of (3.1.3) in order to prove that what should be the symbol of the adjoint actually is a symbol. More precisely, we want to prove that for all $k \in \mathbb{N}$, there exists a constant $C > 0$ and an integer n such that for any multi-index $\beta \in \mathbb{N}^{2d}$ and for all $m \in \mathbb{N}$, if $m + |\beta| \leq k$, then

$$\forall Y \in \mathbb{R}^{2d},\ \forall \lambda \neq 0,\ \left(1 + |\lambda|(1+Y^2)\right)^{\frac{|\beta|-\mu}{2}} \left\|(\lambda\partial_\lambda)^m \partial^\beta_{(y,\eta)} \sigma(a^*)(\cdot,\lambda,Y)\right\|_{C^\rho(\mathbb{H}^d)} \leq C\|a\|_{n;S_{\mathbb{H}^d}(\mu)}.$$

The first step consists in proving this inequality when $k = 0$, then, in a second step, we will suppose $k \geq 1$ and consider derivatives of the symbol $\sigma(a^*)$.

We follow the classical method of stationary phase, as developed for instance in **[1]**. Noticing that the phase in (3.1.3) is stationary at the point $(0,0,0,\xi,\eta,\lambda)$ in $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, we introduce a partition of unity centered at zero:

$$1 = \tilde{\psi}(u) + \sum_{p \in \mathbb{N}} \psi(2^{-p}u), \quad \forall u \in \mathbb{R}^{4d+2}$$

where ψ is compactly supported in a ring and $\tilde{\psi}$ in a ball. Then decomposing the integral (3.1.3) using that partition of unity, we notice that each integral

$$b_p(w,\lambda,\xi,\eta) \stackrel{\text{def}}{=} \frac{1}{2\pi^{2d+1}} \int_{\mathbb{R}^{2d+1} \times \mathbb{H}^d} \psi\left(2^{-p}x', 2^{-p}y', 2^{-p}s', 2^{-p}(z-\xi), 2^{-p}(\zeta-\eta), 2^{-p}(\lambda'-\lambda)\right)$$
$$\times\ \mathrm{e}^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')} \overline{\sigma(a)}\left(w(w')^{-1},\lambda',z,\zeta\right) d\zeta\, dz\, d\lambda'\, dw'$$

is well defined since it is on a compact set. Notice that this is not the usual Heisenberg change of variables as could be expected, but for technical reasons this change of variables seems more appropriate. The convergence of the series $\sum_{p\in\mathbb{N}} b_p$ will come from integrations by parts which will produce powers of 2^{-p}. Indeed, the change of variables

$$x' = 2^pX,\ y' = 2^pY,\ s' = 2^pS,\ z = \xi + 2^pu,\ \zeta = \eta + 2^pv,\ \lambda' = \lambda + 2^p\Lambda$$

gives with $w(p) \stackrel{\text{def}}{=} w \cdot (2^pX, 2^pY, 2^pS)^{-1}$

$$b_p(w,\lambda,\xi,\eta) = \frac{2^{(4d+2)p}}{2\pi^{2d+1}} \int_{\mathbb{R}^{2d+1} \times \mathbb{H}^d} \psi\left(X,Y,S,u,v,\Lambda\right) \mathrm{e}^{-i2^{2p}(2Y\cdot u - 2X\cdot v + S\Lambda)}$$
$$\times\ \overline{\sigma(a)}\left(w(p), \lambda + 2^p\Lambda, \xi + 2^pu, \eta + 2^pv\right) du\, dv\, dX\, dY\, d\Lambda\, dS.$$

Let us define the differential operator

$$L \stackrel{\text{def}}{=} \frac{1}{i}(X^2+Y^2+S^2+u^2+v^2+\Lambda^2)^{-1}\left(\frac{1}{2}X\partial_v + \frac{1}{2}v\partial_X - \frac{1}{2}Y\partial_u - \frac{1}{2}u\partial_Y - S\partial_\Lambda - \Lambda\partial_S\right),$$

which satisfies

$$L\mathrm{e}^{-i2^{2p}(2Y\cdot u-2X\cdot v+S\Lambda)} = 2^{2p}\mathrm{e}^{-i2^{2p}(2Y\cdot u-2X\cdot v+S\Lambda)}.$$

We remark that the coefficients of $(L^*)^N$ are uniformly bounded on the support of ψ. Performing N integration by parts (here we assume that $\rho > N$) we obtain

$$b_p(w,\lambda,\xi,\eta) = \frac{2^{-p(2N-4d-2)}}{2\pi^{2d+1}} \int_{\mathbb{R}^{2d+1} \times \mathbb{H}^d} \mathrm{e}^{-i2^{2p}(2Y\cdot u-2X\cdot v+S\Lambda)}$$
$$\times\ (L^*)^N \left(\psi\left(X,Y,S,u,v,\Lambda\right) \overline{\sigma(a)}\left(w(p), \lambda + 2^p\Lambda, \xi + 2^pu, \eta + 2^pv\right)\right) du\, dv\, dX\, dY\, d\Lambda\, dS.$$

We then use that $\sigma(a)$ satisfies symbol estimates, so

$$\left|(L^*)^N\overline{\sigma(a)}\,(w(p),\lambda+2^p\Lambda,\xi+2^pu,\eta+2^pv)\right| \\ \leq C2^{pN}\,\|a\|_{N,S_{\mathbb{H}^d}(\mu)}\,\left(1+|\lambda+2^p\Lambda|+|\xi+2^pu|^2+|\eta+2^pv|^2\right)^{\mu/2}.$$

Peetre's inequality

$$\left(1+|\lambda+2^p\Lambda|+|\xi+2^pu|^2+|\eta+2^pv|^2\right)^{\mu/2} \\ \leq \left(1+|\lambda|+\xi^2+\eta^2\right)^{\mu/2}\left(1+|2^p\Lambda|+|2^pu|^2+|2^pv|^2\right)^{|\mu|/2}$$

yields

$$\left(1+|\lambda|+\xi^2+\eta^2\right)^{-\mu/2}\left|(L^*)^N\overline{\sigma(a)}\,(w(p),\lambda+2^p\Lambda,\xi+2^pu,\eta+2^pv)\right| \\ \leq C\,\|a\|_{N,S_{\mathbb{H}^d}(\mu)}\,\left(1+|2^p\Lambda|+|2^pu|^2+|2^pv|^2\right)^{|\mu|/2}.$$

Therefore,

$$\left(1+|\lambda|+\xi^2+\eta^2\right)^{-\mu/2}|b_p(w,\lambda,\xi,\eta)|\leq C\,\|a\|_{N,S_{\mathbb{H}^d}(\mu)}\,2^{p(4d+2+|\mu|-N)},$$

which gives the expected inequality for $k=0$ choosing $N>4d+2+|\mu|$.

Let us now consider derivatives of $\sigma(a^*)$. We observe that by integration by parts,

$$\partial_\lambda\sigma(a^*)(w,\lambda,\xi,\eta) \\ =\frac{i}{2\pi^{2d+1}}\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d}\mathrm{e}^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')}s'\,\overline{\sigma(a)}\left(w(w')^{-1},\lambda',z,\zeta\right)d\zeta\,dz\,d\lambda'\,dw' \\ =\frac{1}{2\pi^{2d+1}}\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d}\mathrm{e}^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')}\partial_{\lambda'}\left(\overline{\sigma(a)}\left(w(w')^{-1},\lambda',z,\zeta\right)\right)d\zeta\,dz\,d\lambda'\,dw'.$$

Since for $m\in\mathbb{N}$, $\partial_\lambda^m\sigma(a)$ satisfies the same symbol estimates as $\sigma(a)$, the arguments developed just above allow to deal with the derivatives in λ. Similarly, integrating by parts

$$2\pi^{2d+1}\xi_j\partial_{\xi_k}\sigma(a^*)(w,\lambda,\xi,\eta) \\ =2i\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d}\mathrm{e}^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')}y'_k\xi_j\,\overline{\sigma(a)}\,(\tilde w,\lambda',z,\zeta)\,d\zeta\,dz\,d\lambda'\,dw' \\ =-\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d}\mathrm{e}^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')}y'_k\left(\partial_{y'_j}-2iz_j\right)\left(\overline{\sigma(a)}\,(\tilde w,\lambda',z,\zeta)\right)d\zeta\,dz\,d\lambda'\,dw' \\ =\frac{i}{2}\int_{\mathbb{R}^{2d+1}\times\mathbb{H}^d}\mathrm{e}^{2iy'\cdot(\xi-z)-2ix'\cdot(\eta-\zeta)+is'(\lambda-\lambda')}\partial_{z_k}\left(\partial_{y'_j}-2iz_j\right)\left(\overline{\sigma(a)}\,(\tilde w,\lambda',z,\zeta)\right)d\zeta\,dz\,d\lambda'\,dw,$$

with $\tilde w=w(w')^{-1}$. So, for $m\in\mathbb{N}$ and $\alpha\in\mathbb{N}^{2d}$, $(\xi_j\partial_{\xi_k})^m\sigma(a)$ satisfies the same symbol estimates as $\sigma(a)$, thus we can treat these derivatives as above with exactly the same arguments. Besides, it is also the case for derivatives in η. This concludes the proof of Proposition 3.1. □

3.3. Study of the composition of two pseudodifferential operators

We consider now two pseudodifferential operators $\mathrm{Op}(a)$ and $\mathrm{Op}(b)$ and study their composition. We shall follow the classical method (see for instance **[1]**) consisting in studying rather $\mathrm{Op}(a)\circ\mathrm{Op}(c)^*$, where c is such that $\mathrm{Op}(c)^*=\mathrm{Op}(b)$.

We recall that if A (resp. B) is an operator of kernel $k_A(w,w')$ (resp. $k_B(w,w')$), then the kernel of $A\circ B$ is

$$k_{A\circ B}(w,w')=\int k_A(w,W)k_B(W,w')dW.$$

If moreover $B=C^*$ with C of kernel $k_C(w,w')$, then

$$k_B(w,w')=\overline{k_C(w',w)}.$$

Those (well-known) results applied to $A=\mathrm{Op}(a)$ and $C=\mathrm{Op}(c)$, imply that the operator $\mathrm{Op}(a)\circ\mathrm{Op}(c)^*$ has a kernel $k(w,w')$ given by

$$k(w,w')=\int_{\mathbb{H}^d}k_A(w,W)\overline{k_C(w',W)}\,dW. \tag{3.3.1}$$

If $\mathrm{Op}(a)\circ\mathrm{Op}(c)^*$ is a pseudodifferential operator of symbol d, then, by Proposition 2.4 page 30, the symbol d is given by its associated function $\sigma(d)$ which satisfies,

$$\sigma(d)(w,\lambda,\xi,\eta)=\int_{\mathbb{H}^d}\mathrm{e}^{2i(y'\cdot\xi-x'\cdot\eta)+i\lambda s'}k(w,w(w')^{-1})\,dw'. \tag{3.3.2}$$

We shall now study the map $(a,c)\mapsto d$ which is well defined for $a,c\in\mathcal{S}(\mathbb{H}^d)$.

Proposition 3.4. — *The map $(a,c)\mapsto d$ extends by continuity to $S_{\mathbb{H}^d}(\mu)\times S_{\mathbb{H}^d}(\mu')$ since for all $k\in\mathbb{N}$ there exist $n\in\mathbb{N}$ and $C>0$ such that*

$$\|d\|_{k;S_{\mathbb{H}^d}(\mu+\mu')}\leq C\,\|a\|_{n;S_{\mathbb{H}^d}(\mu)}\,\|c\|_{n;S_{\mathbb{H}^d}(\mu')}.$$

Note that the Proposition implies that the symbol d of $A\circ B$ satisfies

$$\|d\|_{k;S_{\mathbb{H}^d}(\mu+\mu')}\leq C\,\|a\|_{n;S_{\mathbb{H}^d}(\mu)}\,\|b\|_{n;S_{\mathbb{H}^d}(\mu')}$$

since c is the symbol of B^* and $\|c\|_{n;S_{\mathbb{H}^d}(\mu')}\leq C\,\|b\|_{n;S_{\mathbb{H}^d}(\mu')}$ for all $n\in\mathbb{N}$ by Proposition 3.1.

Proof. — The proof is very similar to the one for the adjoint written in the previous section: one writes the function $\sigma(d)$ as an oscillatory integral that we study with standard techniques. We first obtain, thanks to Proposition 2.4 page 30, (3.3.1) and (1.2.1), that the kernel of $\mathrm{Op}(a)\circ\mathrm{Op}(c)^*$ is

$$k(w,\tilde w)=\frac{1}{(2\pi^{2d+1})^2}\int\sigma(a)(w,\lambda_1,z_1,\zeta_1)\overline{\sigma(c)}(\tilde w,\lambda_2,z_2,\zeta_2)$$
$$\times\mathrm{e}^{i\lambda_1s_1+2iy_1\cdot z_1-2ix_1\cdot\zeta_1-i\lambda_2s_2-2iy_2\cdot z_2+2i\zeta_2\cdot x_2}d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW$$

where $w^{-1}W=(x_1,y_1,s_1)$ and $\tilde w^{-1}W=(x_2,y_2,s_2)$. Therefore, recalling that

$$\sigma(d)(w,\lambda,\xi,\eta)=\int_{\mathbb{H}^d}\mathrm{e}^{2i(y'\cdot\xi-x'\cdot\eta)+i\lambda s'}k(w,w(w')^{-1})\,dw'$$

where k is the kernel given above, we get

$$\sigma(d)(w,\lambda,\xi,\eta) = \frac{1}{(2\pi^{2d+1})^2}\int \sigma(a)(w,\lambda_1,z_1,\zeta_1)\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2) \times e^{i\Phi(W,w',\lambda_1,\lambda_2,z_1,z_2,\zeta_1,\zeta_2)} d\lambda_1\, d\lambda_2\, dz_1\, dz_2\, d\zeta_1\, d\zeta_2\, dW\, dw', \tag{3.3.3}$$

where the phase function Φ (depending on w, λ, ξ and η) is given by

$$\Phi = \lambda s' + \lambda_1 s_1 - \lambda_2 s_2 + 2(y'\cdot\xi + y_1\cdot z_1 - y_2\cdot z_2) - 2(x'\cdot\eta + x_1\cdot\zeta_1 - x_2\cdot\zeta_2) \tag{3.3.4}$$

with $w_1 = (x_1,y_1,s_1) = w^{-1}W$ and $w_2 = (x_2,y_2,s_2) = w'w^{-1}W$; in particular $w_2 = w'w_1$ so writing $W = (X,Y,S)$ and using the group law on $\mathbb{H}^d$, we have

$$\begin{aligned} &x_1 = X - x,\ x_2 = X - x + x', y_1 = Y - y, y_2 = Y - y + y', s_1 = S - s - 2Xy + 2xY,\\ &s_2 = S - s + s' - 2(x'-x)\cdot Y + 2(y'-y)\cdot X + 2x'\cdot y - 2y'\cdot x. \end{aligned} \tag{3.3.5}$$

The function Φ is polynomial of degree 3 in its variables and straightforward computations give

$$\begin{aligned} &\partial_{\lambda_1}\Phi = s_1,\ \ \partial_{\lambda_2}\Phi = -s_2,\ \ \partial_{z_1}\Phi = 2y_1,\ \ \partial_{z_2}\Phi = -2y_2\\ &\partial_{\zeta_1}\Phi = -2x_1,\ \ \partial_{\zeta_2}\Phi = 2x_2,\ \ \partial_{s'}\Phi = \lambda - \lambda_2,\ \ \partial_S\Phi = \lambda_1 - \lambda_2\\ &\partial_{x'}\Phi = -2(\eta - \zeta_2) + 2\lambda_2(Y - y),\ \ \partial_{y'}\Phi = 2(\xi - z_2) - 2\lambda_2(X - x)\\ &\partial_X\Phi = -2(\zeta_1 - \zeta_2) - 2\lambda_2 y' + 2y(\lambda_2 - \lambda_1),\ \partial_Y\Phi = 2\lambda_2 x' - 2x(\lambda_2 - \lambda_1) + 2(z_1 - z_2). \end{aligned}$$

Therefore, one can check easily that the phase Φ satisfies $d\Phi = 0$ if and only if

$$w = W,\ w' = 0,\ \ \lambda = \lambda_1 = \lambda_2,\ z_1 = z_2 = \xi,\ \zeta_1 = \zeta_2 = \eta.$$

In the following we shall denote by $U_0 \in \mathbb{R}^D$ that critical point, with $D = 4(2d+1)$:

$$U_0 \stackrel{\text{def}}{=} (x,y,s,0,\lambda,\lambda,\xi,\xi,\eta,\eta).$$

By a tedious but straightforward computation, we check that $\Phi(U_0) = 0$, $d\Phi(U_0) = 0$ and that $d^2\Phi(U_0)$ is invertible for all (w,λ,ξ,η): computing the Hessian matrix $d^2\Phi(U_0)$ one notices easily that each lign of the matrix has at least one constant term (and the others are either zero or linear in λ, x, y).

We then argue as in the proof for the adjoint by use of a partition of unity centered in the point U_0 where Φ degenerates. For simplicity we denote the new set of variables by

$$V = (X,Y,X,x',y',s',\lambda_1,\lambda_2,z_1,z_2,\zeta_1,\zeta_2) \in \mathbb{R}^D.$$

In the phase Φ there are terms of order 3 and we observe that the only derivatives of order 3 which are non zero are

$$\partial^3_{X,\lambda_2,y'}\Phi = -2 \text{ and } \partial^3_{Y,\lambda_2,x'}\Phi = 2.$$

We write, for any point $U \in \mathbb{R}^D$, $\Phi(U) = \Phi_0(U - U_0) + G(U - U_0)$ where by a direct application of Taylor's formula, one has

$$\forall V \in \mathbb{R}^D,\quad \Phi_0(V) \stackrel{\text{def}}{=} \frac{1}{2}D^2\Phi(U_0)V\cdot V \quad\text{and}\quad G(V) \stackrel{\text{def}}{=} (\lambda_2 - \lambda)\left((Y-y)\cdot x' - (X-x)\cdot y'\right).$$

We are therefore reduced to the study of an integral under the form

$$I = \int_{\mathbb{R}^D} f(U) e^{i\Phi(U)} dU,$$

where we have defined

$$\text{(3.3.6)} \qquad \forall U \in \mathbb{R}^D, \quad f(U) = \sigma(a)(w, \lambda_1, z_1, \zeta_1)\overline{\sigma(c)}(w(w')^{-1}, \lambda_2, z_2, \zeta_2).$$

We shall decompose this integral into a series of integrals by a partition of unity:

$$\begin{aligned} I &= \int_{\mathbb{R}^D} f(U) e^{i\Phi(U)} \tilde{\zeta}(U - U_0) dU + \sum_{q\in\mathbb{N}} \int_{\mathbb{R}^D} f(U) e^{i\Phi(U)} \zeta(2^{-q}(U - U_0)) dU \\ &= \int_{\mathbb{R}^D} f(U) e^{i\Phi(U)} \tilde{\zeta}(U - U_0) dU + \sum_{q\in\mathbb{N}} 2^{qD} \int_{\mathbb{R}^D} f(U_0 + 2^q V)\zeta(V) e^{i2^{2q}\Phi_0(V) + i2^{3q}G(V)} dV, \end{aligned}$$

where $\tilde{\zeta}$ and ζ are functions defining a partition of unity, in the sense that they are nonnegative, smooth compactly supported functions ($\tilde{\zeta}$ in a ball and ζ in a ring) such that

$$\forall U \in \mathbb{R}^D, \quad \tilde{\zeta}(U - U_0) + \sum_{q\in\mathbb{N}} \zeta(2^{-q}(U - U_0)) = 1.$$

Each integral is now well defined, and the main problem consists in proving the convergence of the series in $q \in \mathbb{N}$, as well as in proving symbol estimates. We shall concentrate on the second integral and leave the (easier) computation in the case of $\tilde{\zeta}$ to the reader.

Consider

$$I_q \stackrel{\text{def}}{=} 2^{qD} \int f(U_0 + 2^q V)\zeta(V) e^{i2^{2q}\Phi_0(V) + i2^{3q}G(V)} dV.$$

We shall use a stationary phase method, which will be implemented differently according to whether in the phase $2^{2q}\Phi_0(V) + 2^{3q}G(V)$, the dominant term is the first or the second of the two terms. More precisely, let $\delta \in]0, \frac{1}{2}[$ be any real number and let us cut the integral I_q into two parts depending on whether $|\nabla G(V)| < 2^{-q(1+\delta)}$ or not. For this, we introduce a smooth cut-off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ compactly supported on $[-1, 1]$ and write $I_q = I_q^1 + I_q^2$, where

$$\begin{aligned} I_q^1 &\stackrel{\text{def}}{=} 2^{qD} \int \chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right) f(U_0 + 2^q V)\zeta(V) e^{i2^{2q}\Phi_0(V) + i2^{3q}G(V)} dV \quad \text{and} \\ I_q^2 &\stackrel{\text{def}}{=} 2^{qD} \int (1 - \chi)\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right) f(U_0 + 2^q V)\zeta(V) e^{i2^{2q}\Phi_0(V) + i2^{3q}G(V)} dV. \end{aligned}$$

Let us first analyze I_q^1. We introduce the differential operator

$$L \stackrel{\text{def}}{=} \frac{1}{i} \frac{\nabla\Phi_0(V)}{|\nabla\Phi_0(V)|^2} \cdot \nabla$$

which satisfies

$$L^N \left[e^{i2^{2q}\Phi_0(V)}\right] = 2^{2Nq} e^{i2^{2q}\Phi_0(V)}.$$

Note that the computation of the Hessian mentioned above allows easily to obtain a bound of the following type for $\nabla\Phi_0$:

$$\forall V \in \operatorname{Supp}\zeta, \quad \|\nabla\Phi_0(V)\|^{-1} \leq \frac{C}{1+|\lambda|+|x|+|y|} \tag{3.3.7}$$

where C is a constant. It follows that L is well defined, and its coefficients are at most linear in λ, x and y. One therefore checks easily that on the support of ζ the operator $(L^*)^N$ has uniformly bounded coefficients (the bound is uniform in V as well as in w, λ, ξ and η). Therefore one can write

$$I_q^1 = 2^{qD}2^{-2Nq}\int e^{i2^{2q}\Phi_0(V)}(L^*)^N\left[\zeta(V)\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)e^{i2^{3q}G(V)}f(U_0+2^qV)\right]dV.$$

Using the Leibniz formula, we have

$$\begin{aligned}(3.3.8)\quad &\left|(L^*)^N\left[\zeta(V)\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)e^{i2^{3q}G(V)}f(U_0+2^qV)\right]\right|\\ &\leq C\sum_{|\ell|+|m|+|n|\leq N}|\partial^\ell(f(U_0+2^qV))|\left|\partial^n(e^{i2^{3q}G(V)})\right|\left|\partial^m\left(\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right)\right||\overline{\zeta}(V)|\end{aligned}$$

where ℓ, m, n are multi-indexes in $\mathbb{N}^D$ and where $\overline{\zeta}$ is a function, compactly supported on a ring, defined by

$$\overline{\zeta}(V) = \sup_{|j|\leq N}|\partial^j\zeta(V)|.$$

Now the difficulty consists in estimating each of the three terms containing derivatives on the right-hand side of the above inequality. Recalling that f is defined by (3.3.6), f satisfies the following symbol-type estimate:

(3.3.9)

$$\begin{aligned}|\partial^\ell(f(U_0+2^qV))| \leq C2^{|\ell|q}\sup_{\{j_1,\dots,j_6\}\in\{1,\dots D\}^d}&\left(1+|\lambda+2^qV_{j_1}|+|\xi+2^qV_{j_2}|^2+|\eta+2^qV_{j_3}|^2\right)^{\frac{\mu}{2}}\\ &\times\left(1+|\lambda+2^qV_{j_4}|+|\xi+2^qV_{j_5}|^2+|\eta+2^qV_{j_6}|^2\right)^{\frac{\mu'}{2}}.\end{aligned}$$

Now let us prove an estimate for the second term. We use Faa-di-Bruno's formula, which in general can be stated as follows:

$$\begin{aligned}D^N(e^{F(V)})[h_1,\dots,h_N] = &\sum_{\sigma\in\sigma_N}\sum_{p=1}^N\sum_{r_1+\dots+r_p=N}\frac{1}{r_1!\dots r_p!p!}\\ &\times e^{F(V)}[D^{r_1}F(V)(h_{\sigma(1)},\dots,h_{\sigma(r_1)}),\dots,D^{r_p}F(V)(h_{\sigma(N-r_p+1)},\dots,h_{\sigma(N)})].\end{aligned}$$

But on the support of ζ, the function G is bounded as well as its derivatives, so this implies that on the support of χ,

$$|\partial^n(e^{i2^{3q}G(V)})| \leq C\sum_{p=1}^{|n|}\sum_{r_1+\dots+r_p=|n|}\frac{1}{r_1!\dots r_p!p!}2^{3qp}\left(2^{-q(1+\delta)}\right)^K$$

where $K \stackrel{\text{def}}{=} \operatorname{card}\{j,\ r_j = 1\}$ is the number of integers j in $\{1, \dots, p\}$ such that $r_j = 1$. We notice that the worst situation corresponds to the case when $\{j,\ r_j = 1\} = \varnothing$, which means in particular that $r_j \geq 2$ for all j (in the above summation it is implicitly assumed that the r_j are not zero). The largest possible p for which such a situation may occur is $p = |n|/2$ (or $(|n|-1)/2$ if $|n|$ is odd). But one notices that since $\delta < 1/2$,

$$2^{\frac{3p|n|}{2}} \leq 2^{2|n|p - \delta|n|p}$$

so using the fact that for any $p \leq |n|$ one has clearly $2^{2pq - pq\delta} \leq 2^{2|n|q - |n|q\delta}$ we infer that

$$|\partial^n(e^{i2^{3q}G(V)})| \leq C 2^{2|n|q - |n|q\delta}. \tag{3.3.10}$$

Finally let us consider the last term, namely $\partial^m \left(\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right)$. Taking $|m| = 1$ and writing ∂_j for any derivative in $\mathbb{R}^D$ we have

$$\frac{1}{2}\partial_j\left(\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right) = 2^{2q(1+\delta)}\chi'\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\sum_{i=1}^{D}\partial^2_{ij}G(V)\partial_i G(V)$$

which can be written

$$\frac{1}{2}\partial_j\left(\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right) = 2^{q(1+\delta)}\sum_{i=1}^{D} h_i\left(2^{q(1+\delta)}\nabla G(V)\right)\partial^2_{ij}G(V),$$

where h_i is the smooth, compactly supported function defined by

$$\forall U \in \mathbb{R}^D, \quad h_i(U) \stackrel{\text{def}}{=} U_i \chi'(|U|^2).$$

So, using that the derivatives of G are bounded and by Leibniz formula, one gets

$$\left|\partial_j\left(\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right)\right| \leq C 2^{q(1+\delta)},$$

and arguing in the same way for higher order derivatives one finds finally

$$\left|\partial^m\left(\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right)\right| \leq C 2^{|m|q(1+\delta)}. \tag{3.3.11}$$

Plugging (3.3.9), (3.3.10) and (3.3.11) into (3.3.8), we get

$$\begin{aligned}
&2^{-2qN+qD}\left|(L^*)^N\left[\varsigma(V)\chi\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)e^{i2^{3q}G(V)}f(U_0 + 2^q V)\right]\right| \\
&\leq C \sup_{\{j_1,\dots,j_6\}\in\{1,\dots D\}^d} \sum_{|\ell|+|m|+|n|\leq N} 2^{|\ell|q}2^{2|n|q-|n|q\delta}2^{|m|q(1+\delta)} \\
&\qquad \times \left(1 + |\lambda + 2^q V_{j_1}| + |\xi + 2^q V_{j_2}|^2 + |\eta + 2^q V_{j_3}|^2\right)^{\frac{\mu}{2}} \\
&\qquad \times \left(1 + |\lambda + 2^q V_{j_4}| + |\xi + 2^q V_{j_5}|^2 + |\eta + 2^q V_{j_6}|^2\right)^{\frac{\mu'}{2}}.
\end{aligned}$$

Noticing that

$$2^{-2qN+qD}\sum_{|\ell|+|m|+|n|\leq N} 2^{|\ell|q}2^{2|n|q-|n|q\delta}2^{|m|q(1+\delta)} \leq C2^{qD}(2^{-Nq\delta} + 2^{Nq(\delta-1)})$$

it suffices to choose N large enough and to use Peetre's inequality as in the case of the adjoint to conclude on the summability of the series, and on the symbol estimate on $\sum_q I_q^1$.

Let us now focus on I_q^2. In that case Φ_0 is no longer predominant, so we shall use the full operator

$$L_q(V) \stackrel{\text{def}}{=} \frac{1}{i} \frac{\nabla\Phi_0(V) + 2^q\nabla G(V)}{|\nabla\Phi_0(V) + 2^q\nabla G(V)|^2} \cdot \nabla$$

which is well defined on the support of ζ and satisfies

$$L_q(V)\left[\mathrm{e}^{i2^{2q}\Phi_0(V)+i2^{3q}G(V)}\right] = 2^{2q}\,\mathrm{e}^{i2^{2q}\Phi_0(V)+i2^{3q}G(V)}.$$

This implies that I_q^2 is equal to

$$2^{qD-2Nq}\int (L_q(V)^*)^N\left[(1-\chi)\left(2^{2(1+\delta)q}|\nabla G(V)|^2\right)f(U_0+2^qV)\zeta(V)\right]\mathrm{e}^{i2^{2q}\Phi_0(V)+i2^{3q}G(V)}dV,$$

and it is not difficult to prove by induction that for $N\in\mathbb{N}$, the operator $(L_q^*)^N$ is of the form

$$(L_q^*)^N F(V) = \sum_{k=0}^{N}\sum_{|\alpha|\le N-k}\frac{f_0(V)+2^qf_1(V)+\cdots 2^{kq}f_k(V)}{|\Phi_0(V)+2^q\nabla G(V)|^{2k}}\partial^\alpha F(V),$$

where the f_i are uniformly bounded functions on the support of ζ. As in the case of I_q^1, we apply the Leibniz formula to write

$$\begin{aligned}&\left|\partial^\alpha\left[(1-\chi)\left(2^{2(1+\delta)q}|\nabla G(V)|^2\right)f(U_0+2^qV)\zeta(V)\right]\right|\\ &\le C\sum_{|\ell|+|m|\le|\alpha|}|\partial^\ell(f(U_0+2^qV))|\left|\partial^m\left((1-\chi)\left(2^{2q(1+\delta)}|\nabla G(V)|^2\right)\right)\right||\overline{\zeta}(V)|,\end{aligned}$$

where ℓ and m are multi-indexes in $\mathbb{N}^D$ and where $\overline{\zeta}$ is a function, compactly supported on a ring. The first term of the right-hand side was estimated in (3.3.9), and the second one may be estimated similarly to (3.3.11) since as soon as $|m|\ge 1$, the support of $\partial^m(1-\chi)(V)$ is in a ring far from zero. It follows that

$$\begin{aligned}\left|\partial^\alpha\left[(1-\chi)\left(2^{2(1+\delta)q}|\nabla G(V)|^2\right)f(U_0+2^qV)\zeta(V)\right]\right| &\le C\sum_{|\ell|+|m|\le|\alpha|}2^{|\ell|q}2^{|m|q(1+\delta)}\\ \times\sup_{\{j_1,\dots,j_6\}\in\{1,\dots D\}^d}&\left(1+|\lambda+2^qV_{j_1}|+|\xi+2^qV_{j_2}|^2+|\eta+2^qV_{j_3}|^2\right)^{\frac{\mu}{2}}\\ &\times\left(1+|\lambda+2^qV_{j_4}|+|\xi+2^qV_{j_5}|^2+|\eta+2^qV_{j_6}|^2\right)^{\frac{\mu'}{2}}.\end{aligned}$$

Since on the other hand, on the support of $(1-\chi)\left(2^{2(1+\delta)q}|\nabla G(V)|^2\right)$ and on the support of ζ,

$$\left|\frac{f_0(V)+2^qf_1(V)+\cdots 2^{kq}f_k(V)}{|\Phi_0(V)+2^q\nabla G(V)|^{2k}}\right|\le C2^{-kq}2^{2kq(1+\delta)},$$

this implies that

$$X_q^N \stackrel{\text{def}}{=} (L_q(V)^*)^N \left[(1-\chi)\left(2^{2(1+\delta)q}|\nabla G(V)|^2\right) f(U_0+2^q V)\zeta(V)\right]$$

may be bounded by

$$\begin{aligned} |X_q^N| \quad &\leq \quad C\sum_{k=0}^{N} \sum_{|\alpha|\leq N-k} \sum_{|\ell|+|m|\leq|\alpha|} 2^{-kq}2^{2k(1+\delta)q}2^{|\ell|q}2^{|m|q(1+\delta)} \\ &\times \sup_{\{j_1,\dots,j_6\}\in\{1,\dots D\}^d} \left(1+|\lambda+2^qV_{j_1}|+|\xi+2^qV_{j_2}|^2+|\eta+2^qV_{j_3}|^2\right)^{\frac{\mu}{2}} \\ &\times \left(1+|\lambda+2^qV_{j_4}|+|\xi+2^qV_{j_5}|^2+|\eta+2^qV_{j_6}|^2\right)^{\frac{\mu'}{2}}. \end{aligned}$$

Since

$$\sum_{k=0}^{N} \sum_{|\alpha|\leq N-k} \sum_{|\ell|+|m|\leq|\alpha|} 2^{-kq}2^{2k(1+\delta)q}2^{|\ell|q}2^{|m|q(1+\delta)} \leq C2^{Nq}2^{2N\delta q}$$

we conclude that

$$\begin{aligned} X_q^N \leq C2^{-Nq+2N\delta q+ND} &\sup_{\{j_1,\dots,j_6\}\in\{1,\dots D\}^d} \left(1+|\lambda+2^qV_{j_1}|+|\xi+2^qV_{j_2}|^2+|\eta+2^qV_{j_3}|^2\right)^{\frac{\mu}{2}} \\ &\times \left(1+|\lambda+2^qV_{j_4}|+|\xi+2^qV_{j_5}|^2+|\eta+2^qV_{j_6}|^2\right)^{\frac{\mu'}{2}}. \end{aligned}$$

The choice of $\delta \in \,]0,1/2[$ allows to conclude as in the previous proof via Peetre's inequality.

The analysis of derivatives of $\sigma(d)$ is very similar. Let us for the sake of simplicity only deal with the λ-derivative, and leave the study of the other derivatives to the reader. Taking a partial derivative of $\sigma(d)$, defined in (3.3.3), in the λ direction produces a factor is' in the integral, namely

$$\begin{aligned} \partial_\lambda\sigma(d)(w,\lambda,\xi,\eta) = \frac{1}{(2\pi^{2d+1})^2}\int is'\sigma(a)(w,\lambda_1,z_1,\zeta_1)\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2) \\ \times e^{i\Phi(W,w',\lambda_1,\lambda_2,z_1,z_2,\zeta_1,\zeta_2)}d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'. \end{aligned}$$

But one notices that

$$\begin{aligned} &\partial_{\lambda_2}(e^{i\Phi(W,w',\lambda_1,\lambda_2,z_1,z_2,\zeta_1,\zeta_2)}) \\ &= -i\,(S-s+s'+2xY-2yX-2x'(Y-y)+2y'(X-x))\times e^{i\Phi(W,w',\lambda_1,\lambda_2,z_1,z_2,\zeta_1,\zeta_2)} \end{aligned}$$

which can also be written, using (3.3.5)

$$is'e^{i\Phi} = (-\partial_{\lambda_2}-is_1)e^{i\Phi} - i(-2x'y_1+2y'x_1)e^{i\Phi}.$$

On the other hand an easy computation, using the formula defining Φ in (3.3.4) above, allows to write that

$$is_1e^{i\Phi} = \partial_{\lambda_1}e^{i\Phi}, \quad 2iy_1e^{i\Phi} = \partial_{z_1}e^{i\Phi}, \quad \text{and} \quad -2ix_1e^{i\Phi} = \partial_{\zeta_1}e^{i\Phi}$$

so we find the following identity:

$$is'e^{i\Phi} = (-\partial_{\lambda_2}-\partial_{\lambda_1}+x'\partial_{z_1}+y'\partial_{\zeta_1})e^{i\Phi}.$$

Finally $(2\pi^{2d+1})^2\partial_\lambda\sigma(d)(w,\lambda,\xi,\eta)$ is equal to

$$\int e^{i\Phi}\partial_{\lambda_2}\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2)\sigma(a)(w,\lambda_1,z_1,\zeta_1)d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'$$
$$+\int e^{i\Phi}\partial_{\lambda_1}\sigma(a)(w,\lambda_1,z_1,\zeta_1)\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2)d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'$$
$$+\int e^{i\Phi}\partial_{z_1}\sigma(a)(w,\lambda_1,z_1,\zeta_1)(x-x')\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2)d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'$$
$$-\int e^{i\Phi}x\partial_{z_1}\sigma(a)(w,\lambda_1,z_1,\zeta_1)\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2)d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'$$
$$-\int e^{i\Phi}\partial_{\zeta_1}\sigma(a)(w,\lambda_1,z_1,\zeta_1)(y'-y)\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2)d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'$$
$$-\int e^{i\Phi}y\partial_{\zeta_1}\sigma(a)(w,\lambda_1,z_1,\zeta_1)\overline{\sigma(c)}(w(w')^{-1},\lambda_2,z_2,\zeta_2)d\lambda_1\,d\lambda_2\,dz_1\,dz_2\,d\zeta_1\,d\zeta_2\,dW\,dw'.$$

Since $\sigma(a)$ and $\sigma(c)$ satisfy symbol estimates, the expressions above can be dealt with exactly by the same arguments as those developed above. One proceeds similarly for all the other derivatives. Details are left to the reader. □

3.4. The asymptotic formulas

In this section, we give the asymptotics for the symbol of the adjoint and of the composition, up to one order more than in Theorem 4. The proof that we propose does not use the integral formula obtained for a^* and $a\#_{\mathbb{H}^d}b$ but relies more precisely on functional calculus, which suits more to the Heisenberg properties to our opinion.

Proposition 3.5. — *Let $a\in S_{\mathbb{H}^d}(\mu_1)$ and $b\in S_{\mathbb{H}^d}(\mu_2)$. Then the symbol of the adjoint of* $\mathrm{Op}(a)$ *is given by*

$$a^*=\overline{a}+\frac{1}{2\sqrt{|\lambda|}}\sum_{1\le j\le d}(Z_jT_j+\overline{Z}_jT_j^*)\overline{a}+\frac{1}{8|\lambda|}\sum_{1\le j,k\le d}(Z_jT_j+\overline{Z}_jT_j^*)(Z_kT_k+\overline{Z}_kT_k^*)\overline{a}$$
$$+\frac{1}{i\lambda}\left(-\lambda\partial_\lambda+\frac12\sum_{1\le j\le d}(\eta_j\partial_{\eta_j}+\xi_j\partial_{\xi_j})\right)S\,\overline{a}+\tilde r_1$$

whereas the symbol of the composition $\mathrm{Op}(a)\circ\mathrm{Op}(b)$ *is given by*

$$a\#_{\mathbb{H}^d}b=b\#a+\frac{1}{2\sqrt{|\lambda|}}\sum_{1\le j\le d}\left(Z_jb\#T_ja+\overline{Z}_jb\#T_j^*a\right)$$
$$+\frac{1}{8|\lambda|}\sum_{1\le j,k\le d}(Z_jZ_kb\#T_jT_ka+\overline{Z}_j\overline{Z}_kb\#T_j^*T_k^*a+Z_j\overline{Z}_kb\#T_jT_k^*a+\overline{Z}_jZ_kb\#T_j^*T_ka)$$
$$+\frac{1}{i\lambda}Sb\#\left(-\lambda\partial_\lambda+\frac12\sum_{1\le j\le d}(\eta_j\partial_{\eta_j}+\xi_j\partial_{\xi_j})\right)a+\tilde r_2$$

where S denotes ∂_s, $\tilde{r}_1$ (resp. $\tilde{r}_2$) depends only on $\mathcal{Z}^\alpha a$ (resp. $\mathcal{Z}^\alpha b$) for $|\alpha| \geq 3$ and finally where

$$T_j a \stackrel{\text{def}}{=} \frac{1}{i}\partial_{\eta_j} a - \operatorname{sgn}(\lambda)\partial_{\xi_j} a.$$

Recall that formulas for a^* and $a \#_{\mathbb{H}^d} b$ are provided respectively in (3.1.3) and (3.3.3).

In view of the second term of the asymptotic expansion, one understands better in what sense these formula are asymptotics. Let us comment the development of a^*. The first term is a symbol of order $\mu - 1$, it is of order strictly smaller than a.

The first part of the second term is of order $\mu - 2$; however, the second part of this term is the product of λ^{-1} by a symbol of the same order μ. This is a smaller term only for large values of λ. In view of the proof below, it is easy to see that one could obtain an expansion to any order and that the term of order k will be the sum of terms of the form: λ^{-j} times a symbol of order $\mu - k + 2j$ for $0 \leq 2j \leq k$. It is in this sense that this asymptotic has to be considered.

We shall not discuss here the precise feature of the remainder and will discuss this point in further works for applications where these asymptotic expansions could be useful.

We point out that the asymptotic formula for a^* and $a\#_{\mathbb{H}^d} b$ have their counterpart for $\sigma(a^*)$ and $\sigma\left(a\#_{\mathbb{H}^d} b\right)$. By the definition of the function $\sigma(a)$ associated with a symbol a (see (1.4.1)), the following corollary comes from Proposition 3.5. While the asymptotics of Proposition 3.5 appear as especially useful for large λ, the asymptotics on $\sigma(a)$ seems more pertinent for λ close to 0.

Corollary 3.6. — *Let $a \in S_{\mathbb{H}^d}(\mu_1)$ and $b \in S_{\mathbb{H}^d}(\mu_1)$ then*

$$\begin{aligned}\sigma(a^*) = \overline{\sigma(a)} &+ \frac{1}{2}\sum_{1\leq j\leq d}(Z_j\mathcal{T}_j + \overline{Z}_j\mathcal{T}_j^*)\overline{\sigma(a)}\\ &+ \frac{1}{8|\lambda|}\sum_{1\leq j,k\leq d}(Z_jT_j + \overline{Z}_jT_j^*)(Z_kT_k + \overline{Z}_kT_k^*)\overline{\sigma(a)}\\ &- \frac{1}{i}S\,\overline{\partial_\lambda\sigma(a)} + \sigma(\tilde{r}_1)\end{aligned}$$

and similarly

$$\begin{aligned}\sigma\left(a\#_{\mathbb{H}^d} b\right) &= \sigma(b)\#_\lambda\,\sigma(a) + \frac{1}{2}\sum_{1\leq j\leq d}\big(Z_j\sigma(b)\#_\lambda\,\mathcal{T}_j\sigma(a) + \overline{Z}_j\sigma(b)\#_\lambda\,\mathcal{T}_j^*\sigma(a)\big)\\ &+ \frac{1}{8}\sum_{1\leq j,k\leq d}\Big(Z_jZ_k\sigma(b)\#_\lambda\,\mathcal{T}_j\mathcal{T}_k\sigma(a) + \overline{Z}_j\overline{Z}_k\sigma(b)\#_\lambda\,\mathcal{T}_j^*\mathcal{T}_k^*\sigma(a)\\ &\qquad + Z_j\overline{Z}_k\sigma(b)\#_\lambda\,\mathcal{T}_j\mathcal{T}_k^*\sigma(a) + \overline{Z}_jZ_k\sigma(b)\#_\lambda\,\mathcal{T}_j^*\mathcal{T}_k a\Big)\\ &- \frac{1}{i}S\sigma(b)\#_\lambda\,\partial_\lambda\sigma(a) + \sigma(\tilde{r}_2)\end{aligned}$$

where $\tilde{r}_1$ *(resp.* $\tilde{r}_2$*) depends only on* $\mathcal{Z}^\alpha a$ *(resp.* $\mathcal{Z}^\alpha b$*) for* $|\alpha| \geq 3$ *and where for all functions* $f = f(\xi,\eta)$ *and* $g = g(\xi,\eta)$

$$\forall \Theta \in \mathbb{R}^{2d}, \quad f \,\#_\lambda\, g(\Theta) \stackrel{\text{def}}{=} (\pi\lambda)^{-2d} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-\frac{2i}{\lambda}\omega[\Theta-\Theta_1,\Theta-\Theta_2]} f(\Theta_1) g(\Theta_2) d\Theta_1\, d\Theta_2,$$
$$\mathcal{T}_j f \stackrel{\text{def}}{=} \frac{1}{i}\partial_{\eta_j} f - \partial_{\xi_j} f.$$

The proof of the corollary is straightforward by (1.4.1) and (1.3.4).

Let us now prove Proposition 3.5.

Proof. — It turns out that the proof of the asymptotic formula for the composition and the adjoint are identical, so let us concentrate on the product from now on.

In view of (1.4.5) and (1.4.6) page 23, we can write

$$\big(\mathrm{Op}(a)\circ \mathrm{Op}(b)\big) f(w) = \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \mathrm{tr}\,\big(u^\lambda_{w^{-1}w'} \circ A_\lambda(w)\big)\ \mathrm{tr}\,\Big(u^{\lambda'}_{(w')^{-1}w''} \circ B_{\lambda'}(w')\Big)$$
$$\times f(w'')|\lambda|^d\,|\lambda'|^d d\lambda\, d\lambda'\, dw'\, dw''$$

with

$$A_\lambda(w) = J_\lambda\, \mathrm{op}^w(a(w,\lambda))\, J^*_\lambda \ \text{ and }\ B_\lambda(w) = J_\lambda\, \mathrm{op}^w(b(w,\lambda))\, J^*_\lambda.$$

Now, we shall take into account the framework of the Heisenberg group and use the dilation $\delta_t(w^{-1}w')$, $t\in[0,1]$ (see (1.2.7) page 8) to transform $b(w',\cdot)$ by a Taylor expansion:

$$\begin{aligned} b(w',\lambda,y,\eta) &= b\left(w\delta_1(w^{-1}w'),\lambda,y,\eta\right) \\ &= b(w,\lambda,y,\eta) + \left(\frac{d}{dt}\, b\left(w\delta_t(w^{-1}w'),\lambda,y,\eta\right)\right)_{|t=0} \\ &\quad + \frac{1}{2}\left(\frac{d^2}{dt^2}\, b\left(w\delta_t(w^{-1}w'),\lambda,y,\eta\right)\right)_{|t=0} \\ &\quad + \frac{1}{2}\int_0^1 (1-t)^2 \frac{d^3}{dt^3}\, b\left(w\delta_t(w^{-1}w'),\lambda,y,\eta\right) dt. \end{aligned}$$

Setting $\tilde{w} = (\tilde{z},\tilde{s}) = w^{-1}w'$, we get by the group rule (1.2.1),

$$\frac{d}{dt} b(w\delta_t(\tilde{w})) = 2t\tilde{s}Sb(w\delta_t(\tilde{w})) + \sum_{1\leq j\leq d}\Big[\tilde{x}_j\left(\partial_{x_j} b(w\delta_t(\tilde{w})) + 2y_j\partial_s b(w\delta_t(\tilde{w}))\right)$$
$$+ \tilde{y}_j\left(\partial_{y_j} b(w\delta_t(\tilde{w})) - 2x_j\partial_s b(w\delta_t(\tilde{w}))\right)\Big].$$

This leads by straightforward computations to

$$\Big(\frac{d}{dt} b\left(w\delta_t(w^{-1}w'),\lambda,y,\eta\right)\Big)_{|t=0} = \sum_{1\leq j\leq d} (\tilde z_j Z_j + \overline{\tilde z_j}\,\overline{Z}_j) b(w,\lambda,y,\eta)$$

$$\Big(\frac{d^2}{dt^2} b\left(w\delta_t(w^{-1}w'),\lambda,y,\eta\right)\Big)_{|t=0} = \sum_{1\leq j,k\leq d} \left[(\tilde z_j Z_j + \overline{\tilde z}_j \overline{Z}_j)\circ(\tilde z_k Z_k + \overline{\tilde z}_k \overline{Z}_k)\right] b(w,\lambda,y,\eta) + 2\tilde s\, Sb(w,\lambda,y,\eta).$$

Therefore, we deduce that

$$B_\lambda(w') = C_\lambda(w,w') + R_\lambda(w,w')$$

where R_λ depends only on derivatives of order 3 of b and $C_\lambda(w,w')$ depends polynomially on $\tilde w$:

$$C_\lambda(w,w') \stackrel{\text{def}}{=} B_\lambda(w) + C^{(1)}_\lambda(w)\cdot(\tilde z,\overline{\tilde z}) + C^{(2)}_\lambda(w)(\tilde z,\overline{\tilde z})\cdot(\tilde z,\overline{\tilde z}) + \tilde s\, C^{(3)}_\lambda(w), \tag{3.4.1}$$

where $C^{(1)}_\lambda(w)$ is the $2d$ dimensional vector-valued operator

$$C^{(1)}_\lambda \stackrel{\text{def}}{=} \left(ZB_\lambda(w), \overline{Z}B_\lambda(w)\right),$$

while $C^{(2)}_\lambda(w)$ is the $2d\times 2d$ matrix-valued operator

$$C^{(2)}_\lambda \stackrel{\text{def}}{=} \frac{1}{2}\left[(Z,\overline{Z})\otimes(Z,\overline{Z})\right] B_\lambda(w)$$

and $C^{(3)}_\lambda(w) \stackrel{\text{def}}{=} SB_\lambda(w)$.

To summarize $\big(\mathrm{Op}(a)\circ\mathrm{Op}(b)\big)f(w)$ is the sum of two terms:

$$\big(\mathrm{Op}(a)\circ\mathrm{Op}(b)\big)f(w) = (I) + (J)$$

with

$$(I) = \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \mathrm{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right)\,\mathrm{tr}\left(u^{\lambda'}_{(w')^{-1}w''} C_{\lambda'}(w,w')\right) f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw''.$$

Let us now focus on the term (I) which will give the terms of the asymptotics in which we are interested.

Let us begin by the study of the contribution $(I)_0$ of the term of degree 0 of the polynomial function $C_\lambda(w,w')$. By (3.4.1), we get

$$\begin{aligned}(I)_0 &\stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \mathrm{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right)\,\mathrm{tr}\left(u^{\lambda'}_{(w')^{-1}w''} B_{\lambda'}(w)\right) f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw''\\ &= \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \mathrm{tr}\left(u^\lambda_{w^{-1}w''} u^\lambda_{(w'')^{-1}w'}\,\mathrm{tr}\left(u^{\lambda'}_{(w')^{-1}w''} B_{\lambda'}(w)\right)\,A_\lambda(w)\right)\\ &\qquad\times f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw''.\end{aligned}$$

The change of variables $w' \mapsto w''w'$ turns the integral $(I)_0$ into

$$\left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int_{w'',\lambda} \operatorname{tr}\Big(u^\lambda_{w^{-1}w''}\Big[\int u^\lambda_{w'} \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}} B_{\lambda'}(w)\right) |\lambda'|^d d\lambda' dw'\Big] A_\lambda(w)\Big) f(w'')|\lambda|^d \, d\lambda dw''.$$

By the inverse Fourier formula, we obtain that the term between brackets is

$$\int u^\lambda_{w'} \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}} B_{\lambda'}(w)\right) |\lambda'|^d d\lambda' dw' = \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^{-1} B_\lambda(w),$$

which gives

$$(I)_0 = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w''} B_\lambda(w) A_\lambda(w)\right) f(w'')|\lambda|^d \, d\lambda dw''.$$

We then use classical Weyl symbolic calculus to write

$$\mathrm{op}^w(b(w,\lambda)) \circ \mathrm{op}^w(a(w,\lambda)) = \mathrm{op}^w((b\#a)(w,\lambda)).$$

Thus we have

$$B_\lambda(w) \circ A_\lambda(w) = J^*_\lambda \mathrm{op}^w((b\#a)(w,\lambda)) J_\lambda,$$

whence

$$(I)_0 = \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w''} J^*_\lambda \mathrm{op}^w((b\#a)(w,\lambda)) J_\lambda\right) f(w'')|\lambda|^d \, d\lambda dw'',$$

which gives thanks to (1.4.5) and (1.4.6) the first term in the asymptotic formula for the composition.

Let us now consider the second term of the asymptotic expansion which comes from the term of order 1 of the polynomial function $C_\lambda(w,w')$. To treat this term, we shall use the following relations for $1 \leq j \leq d$,

$$\begin{aligned} \tilde z_j \operatorname{tr}\left(u^\lambda_{\tilde w} J^*_\lambda \mathrm{op}^w(a(w,\lambda)) J_\lambda\right) &= \frac{1}{2\sqrt{|\lambda|}} \operatorname{tr}\left(u^\lambda_{\tilde w} J^*_\lambda \mathrm{op}^w(\{a, -i\xi_j + \mathrm{sgn}(\lambda)\eta_j\}) J_\lambda\right) \\ &= \frac{1}{2\sqrt{|\lambda|}} \operatorname{tr}\left(u^\lambda_{\tilde w} J^*_\lambda \mathrm{op}^w(T_j a(w,\lambda))\right) \qquad (3.4.2) \\ \overline{\tilde z_j} \operatorname{tr}\left(u^\lambda_{\tilde w} J^*_\lambda \mathrm{op}^w(a(w,\lambda)) J_\lambda\right) &= -\frac{1}{2\sqrt{|\lambda|}} \operatorname{tr}\left(u^\lambda_{\tilde w} J^*_\lambda \mathrm{op}^w(\{a, i\xi_j + \mathrm{sgn}(\lambda)\eta_j\}) J_\lambda\right) \\ &= \frac{1}{2\sqrt{|\lambda|}} \operatorname{tr}\left(u^\lambda_{\tilde w} J^*_\lambda \mathrm{op}^w(T^*_j a(w,\lambda))\right) \qquad (3.4.3) \end{aligned}$$

that come respectively from (2.3.4) and (2.3.5) page 37.

This allows to write the second term under the following form

$$(I)_1 \stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''}(\tilde z, \bar{\tilde z}) \cdot C^{(1)}_{\lambda'}(w)\right)$$
$$\times f(w'')|\lambda|^d\, |\lambda'|^d d\lambda d\lambda' dw' dw''$$
$$= \frac{1}{2\sqrt{|\lambda|}} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \sum_{1\leq j\leq d} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} J^*_\lambda \operatorname{op}^w(T_j a(w,\lambda)) J_\lambda\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} Z_j B(w,\lambda')\right)$$
$$\times f(w'')|\lambda|^d\, |\lambda'|^d d\lambda d\lambda' dw' dw''$$
$$+ \frac{1}{2\sqrt{|\lambda|}} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \sum_{1\leq j\leq d} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} J^*_\lambda \operatorname{op}^w(T^*_j a(w,\lambda)) J_\lambda\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \overline{Z}_j B(w,\lambda')\right)$$
$$\times f(w'')|\lambda|^d\, |\lambda'|^d d\lambda d\lambda' dw' dw''.$$

Therefore, arguing as for the first term, we get

$$(I)_1 = \frac{1}{2\sqrt{|\lambda|}} \frac{2^{d-1}}{\pi^{d+1}} \sum_{1\leq j\leq d} \int \operatorname{tr}\Bigg(u^\lambda_{w^{-1}w''} J^*_\lambda \operatorname{op}^w\Big(Z_j b(w,\lambda) \# T_j a(w,\lambda)$$
$$+ \overline{Z}_j b(w,\lambda) \# T^*_j a(w,\lambda)\Big) J_\lambda \Bigg) f(w'')|\lambda|^d\, d\lambda dw'',$$

which leads by (1.4.5) and (1.4.6) to the second term in the asymptotic formula for the composition.

In order to compute the third term of the expansion, we shall consider the terms of order 2 of the polynomial $C_\lambda(w,w')$ and use Lemma A.4 stated page 102. First, let us recall that due to (3.4.1), we have

$$(I)_2 \stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \left(\tilde s C^{(3)}_{\lambda'}(w) + C^{(2)}_{\lambda'}(w)(\tilde z, \bar{\tilde z}) \cdot (\tilde z, \bar{\tilde z})\right)\right)$$
$$\times\, f(w'')|\lambda|^d\, |\lambda'|^d d\lambda d\lambda' dw' dw''$$

where $C^{(3)}_{\lambda'}(w) = SB_\lambda(w)$ and $C^{(2)}_\lambda = \frac{1}{2}\left[(Z,\overline{Z}) \otimes (Z,\overline{Z})\right] B_\lambda(w)$.

We first focus on the term in $C^{(2)}_{\lambda'}$. Let us call $(I)_{2,1}$ its contribution, we have

$$(I)_{2,1} \stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \sum_{1\leq j,k\leq d} \int \operatorname{tr}\left(u^\lambda_{w^{-1}w'} A_\lambda(w)\right)$$
$$\times \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \left((\tilde z_j Z_j + \overline{\tilde z_j}\overline{Z}_j)(\tilde z_k Z_k + \overline{\tilde z_k}\overline{Z}_k) B_{\lambda'}(w)\right)\right) f(w'')|\lambda|^d\, |\lambda'|^d d\lambda d\lambda' dw' dw'.$$

We treat those terms as those of $(I)_1$. We shall explain the argument for one of those terms and leave the analysis of the other terms to the reader. Set

$$(I)_{2,j,k} \stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w'} A_\lambda(w)\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \left(\tilde z_j \overline{\tilde z_k} Z_j \overline{Z}_k\right) B_{\lambda'}\right)\right) \\ \times f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw'.$$

Using (3.4.2) and (3.4.3), we obtain

$$\tilde z_j\, \overline{\tilde z_k} \operatorname{tr}\left(u^{\lambda}_{\tilde w} J^*_\lambda \mathrm{op}^w(a(w,\lambda)) J_\lambda\right) = \operatorname{tr}\left(u^{\lambda}_{\tilde w} J^*_\lambda \mathrm{op}^w(T_j T^*_k a(w,\lambda)) J_\lambda\right)$$

whence, arguing as for $(I)_1$

$$(I)_{2,j,k} = \frac{1}{2\sqrt{|\lambda|}} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w'} J^*_\lambda \mathrm{op}^w(T_j T^*_k a(w,\lambda)) J_\lambda\right) \\ \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} Z_j \overline{Z}_k B(w,\lambda')\right) f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw''$$

$$= \frac{1}{2\sqrt{|\lambda|}} \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w''} J^*_\lambda \mathrm{op}^w\left(Z_j \overline{Z}_k b(w,\lambda) \# T_j T^*_k a(w,\lambda)\right) J_\lambda \right) f(w'')|\lambda|^d\, d\lambda dw''.$$

To deal with the last term

$$\left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w'} A_\lambda(w)\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \tilde s\, C^{(3)}_{\lambda'}(w)\right) f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw''$$

let us apply Lemma A.4 (see page 102) writing

$$\left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w'} A_\lambda(w)\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \tilde s C^{(3)}_{\lambda'}(w)\right) f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw''$$

$$= \frac{1}{i} \left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w'} J^*_\lambda \mathrm{op}^w\left(g(w,\lambda)\right) J_\lambda\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} C^{(3)}_{\lambda'}(w)\right) f(w'')|\lambda|^d\, d\lambda dw''.$$

where g is the symbol of $S_{\mathbb{H}^d}(\mu_1)$ given by (A.2.5) (in particular we have $\sigma(g) = -\partial_\lambda\left(\sigma(a)\right)$).

Finally, arguing as before we get

$$\left(\frac{2^{d-1}}{\pi^{d+1}}\right)^2 \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w'} A_\lambda(w)\right) \operatorname{tr}\left(u^{\lambda'}_{(w')^{-1}w''} \tilde s\, C^{(3)}_{\lambda'}(w)\right) f(w'')|\lambda|^d\,|\lambda'|^d d\lambda d\lambda' dw' dw'' \\ = \frac{1}{i} \frac{2^{d-1}}{\pi^{d+1}} \int \operatorname{tr}\left(u^{\lambda}_{w^{-1}w''} J^*_\lambda \mathrm{op}^w\left(S\, b(w,\lambda) \# g(w,\lambda)\right) J_\lambda\right) f(w'')|\lambda|^d\, d\lambda dw''.$$

This ends the proof of the asymptotic formula for the composition. □

CHAPTER 4

LITTLEWOOD-PALEY THEORY

In this chapter, we shall study various properties related to Littlewood-Paley operators, and their link with various types of pseudodifferential operators.

In the first section, we focus on the Littlewood-Paley theory available on the Heisenberg group. Similarly to the $\mathbb{R}^d$ case, this theory enable us to split tempered distributions into a countable sum of smooth functions frequency localized in a ball or a ring (see Definition 4.1 for more details). In the second section, we recall some basic facts about Besov spaces and introduce paradifferential calculus. Like in the $\mathbb{R}^d$ case, it turns out that Sobolev and Hölder spaces come up as special cases of Besov spaces. The paraproduct algorithm on the Heisenberg group is similar to the paraproduct algorithm on $\mathbb{R}^d$ built by J.-M. Bony **[13]** and allows to transpose to the Heisenberg group a number of classical results (see for instance **[4]**, **[5]** **[6]** and **[7]**). As already mentioned in Section 2.1 of Chapter 2, the Littlewood-Paley truncation operators are Fourier multipliers defined using operators which are functions of the harmonic oscillator. Therefore, it is important for our theory to be able to analyze the Weyl symbol of such operators; this is achieved thanks to Mehler's formula in the third section where we compare Littlewood-Paley operators with pseudodifferential operators; this will be of crucial use for the next chapter. Finally in the last paragraph we introduce another dyadic decomposition, in the variable λ only, which will also turn out to be a necessary ingredient in the proof of Theorem 5.

4.1. Littlewood-Paley operators

In **[7]** and **[5]** a dyadic partition of unity is built on the Heisenberg group $\mathbb{H}^d$, similar to the one defined in the classical $\mathbb{R}^d$ case. A significant application of this decomposition is the definition of Sobolev spaces (and more generally Besov spaces) on the Heisenberg group in the same way as in the classical case.

Let us first define the concept of localization procedure in frequency space, in the framework of the Heisenberg group. We start by giving the definition in the case of smooth functions. The general case follows classically (see **[7]** or **[5]**) by regularizing by convolution, as shown in the remark following the definition. We have defined, for any set B, the operator $\mathbf{1}_{D_\lambda^{-1}B}$ on $\mathcal{H}_\lambda$ by

$$\forall f \in \mathcal{S}(\mathbb{H}^d),\ \forall \alpha \in \mathbb{N}^d, \quad \mathcal{F}(f)(\lambda)\mathbf{1}_{D_\lambda^{-1}B}F_{\alpha,\lambda} \stackrel{\text{def}}{=} \mathbf{1}_{(2|\alpha|+d)^{-1}B}(\lambda)\,\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}.$$

Definition 4.1. — *Let $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0,r_1,r_2)$ be a ring and $\mathcal{B}_r = \mathcal{B}(0,r)$ a ball of $\mathbb{R}$ centered at the origin. A function f in $\mathcal{S}(\mathbb{H}^d)$ is said to be*

— *frequency localized in the ball $2^p\mathcal{B}_{\sqrt{r}}$, if*

$$\mathcal{F}(f)(\lambda) = \mathcal{F}(f)(\lambda)\mathbf{1}_{D_\lambda^{-1}2^{2p}\mathcal{B}_{\sqrt{r}}}(\lambda);$$

— *frequency localized in the ring $2^p\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, if*

$$\mathcal{F}(f)(\lambda) = \mathcal{F}(f)(\lambda)\mathbf{1}_{D_\lambda^{-1}2^{2p}\mathcal{C}_{(r_1,r_2)}}(\lambda).$$

In the case of a tempered distribution u, we shall say that u is frequency localized in the ball $2^p\mathcal{B}_{\sqrt{r}}$ (respectively in the ring $2^p\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$), if

$$u \star f = 0$$

for any radial function $f \in \mathcal{S}(\mathbb{H}^d)$ satisfying $\mathcal{F}(f)(\lambda)\mathbf{1}_{D_\lambda^{-1}2^{2p}\mathcal{B}_{\sqrt{r}}} = 0$ (respectively for any f in $\mathcal{S}(\mathbb{H}^d)$ satisfying $\mathcal{F}(f)(\lambda)\mathbf{1}_{D_\lambda^{-1}2^{2p}\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}} = 0$). In other words u is frequency localized in the ball $2^p\mathcal{B}_{\sqrt{r}}$ (respectively in the ring $2^p\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$), if and only if,

$$u = u \star \phi_p,$$

where $\phi_p = 2^{Np}\phi(\delta_{2^p}\cdot)$, and ϕ is a radial function in $\mathcal{S}(\mathbb{H}^d)$ such that

$$\mathcal{F}(\phi)(\lambda) = \mathcal{F}(\phi)(\lambda)R(D_\lambda),$$

with R compactly supported in a ball (respectively an ring) of $\mathbb{R}$ centered at zero.

Let us now recall the dyadic decomposition and paradifferential techniques introduced in **[7]** and **[5]**, which we refer to for all details and proofs.

Proposition 4.2. — *Let us denote by $\mathcal{B}_0$ and by $\mathcal{C}_0$ respectively the ball $\{\tau \in \mathbb{R},\ |\tau| \leq \frac{4}{3}\}$ and the ring $\{\tau \in \mathbb{R},\ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\}$. Then there exist two radial functions $\widetilde{R}^*$ and R^* the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(\mathcal{B}_0)$ and to $\mathcal{D}(\mathcal{C}_0)$ such that*

$$\forall \tau \in \mathbb{R}, \quad \widetilde{R}^*(\tau) + \sum_{p\geq 0} R^*(2^{-2p}\tau) = 1 \tag{4.1.1}$$

and satisfying the support properties

$$|p-p'| \geq 1 \Rightarrow \operatorname{supp} R^*(2^{-2p}\cdot) \cap \operatorname{supp} R^*(2^{-2p'}\cdot) = \varnothing$$
$$p \geq 1 \Rightarrow \operatorname{supp} \widetilde{R}^* \cap \operatorname{supp} R^*(2^{-2p}\cdot) = \varnothing.$$

Besides, we have

$$\forall \tau \in \mathbb{R}\,,\ \frac{1}{2} \leq \widetilde{R}^*(\tau)^2 + \sum_{p\geq 0} R^*(2^{-2p}\tau)^2 \leq 1. \tag{4.1.2}$$

The dyadic blocks Δ_p and the low frequency cut-off operators S_p are defined as follows similarly to the $\mathbb{R}^d$ case.

Definition 4.3. — *We define the Littlewood-Paley operators associated with the functions $\widetilde{R}^*$ and R^*, for $p \in \mathbb{Z}$, by the following definitions in Fourier variables:*

$$\begin{aligned} \forall p \in \mathbb{N}, \quad \mathcal{F}(S_p f)(\lambda) &= \mathcal{F}(f)(\lambda)\widetilde{R}^*\left(2^{-2p}D_\lambda\right), \\ \forall p \in \mathbb{N}, \quad \mathcal{F}(\Delta_p f)(\lambda) &= \mathcal{F}(f)(\lambda)R^*\left(2^{-2p}D_\lambda\right), \\ \mathcal{F}(\Delta_{-1} f)(\lambda) &= \mathcal{F}(S_0 f)(\lambda), \\ \forall p \leq -2, \quad \mathcal{F}(\Delta_p f)(\lambda) &= 0. \end{aligned}$$

The operator $S_p f$ may be alternately defined by

$$S_p f = \sum_{q \leq p-1} \Delta_q f.$$

Since $\mathcal{F}(\Delta_p f)(\lambda) = \mathcal{F}(f)(\lambda)R^*(2^{-2p}D_\lambda)$, it is clear that the function $\Delta_p f$ is frequency localized in a ring of size 2^p. Along the same lines, one can notice that the function $S_p f$ is frequency localized in a ball of size 2^p.

Moreover, according to the fact that the Fourier transform exchanges convolution and composition, the operators Δ_p and S_p commute with one another and with the Laplacian-Kohn operator $\Delta_{\mathbb{H}^d}$.

Remark 4.4. — *For simplicity of notation, we do not indicate that S_p depends on $\widetilde{R}^*$ and that Δ_p depends on R^*. That is due to the fact that according to Lemma 4.8 below, one can change the basis functions (hence the Littlewood-Paley operators), keeping only the fact that one is supported near zero and the other is supported away from zero and satisfying (4.1.1), while conserving equivalent norms for the function spaces based on those operators.*

It was proved in **[39]**, in the more general context of nilpotent Lie groups, that there are radial functions of $\mathcal{S}(\mathbb{H}^d)$, denoted ψ and φ such that

$$\mathcal{F}(\psi)(\lambda) = \widetilde{R}^*(D_\lambda) \quad \text{and} \quad \mathcal{F}(\varphi)(\lambda) = R^*(D_\lambda).$$

We also refer to **[7]** and **[5]** for a different proof in the case of the Heisenberg group, the ideas of which will be used below to prove Lemma 4.17. Using the scaling of the Heisenberg group, it is easy to see that

$$\Delta_p u = u \star 2^{Np}\varphi(\delta_{2^p}\cdot) \quad \text{and} \quad S_p u = u \star 2^{Np}\psi(\delta_{2^p}\cdot)$$

which implies by Young's inequalities that those operators map L^q into L^q for all $q \in [1, \infty]$ with norms which do not depend on p.

Let us also notice that due to (1.2.8) (see page 8), if P is a left invariant vector fields then

$$P(\Delta_p u) = 2^p(u \star 2^{Np}P(\varphi)(\delta_{2^p}\cdot)).$$

This property is the heart of the matter in the estimate of the action of left invariant vector fields on frequency localized functions (see Lemma 4.7 below).

In view of Mehler's formula (see **[26]**) and Lemma 4.5 in **[25]**, one can prove that the Littlewood-Paley operators on the Heisenberg group are pseudodifferential operators in the sense of Definition 1.23. This is discussed in Section 4.5 below.

4.2. Besov spaces

Along the same lines as in the $\mathbb{R}^d$ case, we can define Besov spaces on the Heisenberg group (see **[7]**).

Definition 4.5. — *Let $s \in \mathbb{R}$ and $(q,r) \in [1,\infty]^2$. The Besov space $B^s_{q,r}(\mathbb{H}^d)$ is the space of tempered distributions u such that*

$$\|u\|_{B^s_{q,r}(\mathbb{H}^d)} \stackrel{\text{def}}{=} \Big\| 2^{ps} \|\Delta_p u\|_{L^q(\mathbb{H}^d)} \Big\|_{\ell^r} < \infty.$$

Remark 4.6. — *It is also possible to characterize these spaces using only the operator S_p : for $s > 0$, we have*

$$\|f\|_{B^s_{q,r}(\mathbb{H}^d)} \sim \Big\| 2^{sp} \|(\mathrm{Id} - S_p) f\|_{L^q(\mathbb{H}^d)} \Big\|_{\ell^r}, \tag{4.2.1}$$

and for $s < 0$,

$$\|f\|_{B^s_{q,r}(\mathbb{H}^d)} \sim \Big\| 2^{sp} \|S_p f\|_{L^q(\mathbb{H}^d)} \Big\|_{\ell^r}, \tag{4.2.2}$$

where $\sim$ stands for equivalent norms.

It is easy to see that for any real number ρ, the operators $(-\Delta_{\mathbb{H}^d})^\rho$ and $(\mathrm{Id} - \Delta_{\mathbb{H}^d})^\rho$ are continuous from $B^s_{q,r}(\mathbb{H}^d)$ to $B^{s-2\rho}_{q,r}(\mathbb{H}^d)$. Note that Besov spaces on the Heisenberg group contain Sobolev and Hölder spaces. Indeed, by (4.1.2) and the Fourier-Plancherel equality (1.2.21), the Besov space $B^s_{2,2}(\mathbb{H}^d)$ coincides with the Sobolev space $H^s(\mathbb{H}^d)$. When $s \in \mathbb{R}^+ \setminus \mathbb{N}$, one can show that $B^s_{\infty,\infty}(\mathbb{H}^d)$ coincides with the Hölder space $C^s(\mathbb{H}^d)$ introduced in Definition 1.3.

Let us point out that a distribution f belongs to $B^s_{q,r}(\mathbb{H}^d)$ if and only if there exists some constant C and some nonnegative sequence $(c_p)_{p\in\mathbb{N}}$ of the unit sphere of $\ell^r(\mathbb{N})$ such that

$$\forall p \in \mathbb{N}, \quad 2^{ps} \|\Delta_p f\|_{L^q(\mathbb{H}^d)} \leq C c_p. \tag{4.2.3}$$

This fact will be useful in what follows.

Arguing as in the classical case, one can prove using this theory many results, such as Sobolev embeddings, refined Sobolev and Hardy inequalities (see **[5]**,**[4]**). This is due to the fact that the dyadic unity decomposition on the Heisenberg group behaves as the classical Littlewood-Paley decomposition. The key argument lies on the following estimates called Bernstein inequalities, proved in **[5]**.

Lemma 4.7. — *Let r be a positive real number. For any nonnegative integer k, there exists a positive constant C_k so that, for any couple of real numbers (a,b) such that $1 \leq$*

$a \leq b \geq \infty$ and any function u of $L^a(\mathbb{H}^d)$ frequency localized in the ball $2^p \mathcal{B}_{\sqrt{r}}$, one has

$$\sup_{|\beta|=k} \|\mathcal{X}^\beta u\|_{L^b(\mathbb{H}^d)} \leq C_k 2^{pN(\frac{1}{a}-\frac{1}{b})+pk} \|u\|_{L^a(\mathbb{H}^d)}, \tag{4.2.4}$$

where $\mathcal{X}^\beta$ denotes a product of $|\beta|$ vectors fields of type (1.2.2), *page 7.*

Let us also point out that the definition of $B^s_{p,r}(\mathbb{H}^d)$ is independent of the dyadic partition of unity chosen to define this space. This is due to the following lemma proved in **[7]**.

Lemma 4.8. — *Let $s \in \mathbb{R}$ and $(p,r) \in [1,\infty]^2$. Let $(u_p)_{p\in\mathbb{N}}$ be a sequence of $L^q(\mathbb{H}^d)$ frequency localized in a ring of size 2^p satisfying*

$$\left\|2^{ps}\|u_p\|_{L^q(\mathbb{H}^d)}\right\|_{\ell^r(\mathbb{N})} < \infty,$$

then $u \stackrel{\text{def}}{=} \sum_{p\in\mathbb{N}} u_p$ belongs to $B^s_{q,r}(\mathbb{H}^d)$ and we have

$$\|u\|_{B^s_{q,r}(\mathbb{H}^d)} \leq C_s \left\|2^{ps}\|u_p\|_{L^q(\mathbb{H}^d)}\right\|_{\ell^r(\mathbb{N})}.$$

Contrary to the $\mathbb{R}^d$ case, there is no simple formula for the Fourier transform of the product of two functions. The following proposition (proved in **[5]**) ensures that spectral localization properties of the classical case are nevertheless preserved on the Heisenberg group after the product has been taken.

Proposition 4.9. — *Let $r_2 > r_1 > 0$ be two real numbers, let p and p' be two integers, and let f and g be two functions of $\mathcal{S}'(\mathbb{H}^d)$ respectively frequency localized in the ring $2^p \mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$ and $2^{p'} \mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$. Then*

- *there exists a ring $\mathcal{C}'$ such that if $p' - p > 1$ then fg is frequency localized in the ring $2^{p'} \mathcal{C}'$.*
- *there exists a ball $\mathcal{B}'$ such that if $|p' - p| \leq 1$, then fg is frequency localized in the ball $2^{p'} \mathcal{B}'$.*

Remark 4.10. — *The proof of this proposition is based on a careful use of the link between the Fourier transform on the Heisenberg group and the standard Fourier transform on $\mathbb{R}^{2d+1}$. For a detailed proof, see* **[5]**.

Proposition 4.9 implies that if two functions are spectrally localized on two rings sufficiently far away one from the other, then their product stays spectrally localized on a ring.

Taking advantage of this result, one can transpose to the Heisenberg group the paraproduct theory constructed by J.-M. Bony **[13]** in the classical case. Let us consider two tempered distributions u and v on $\mathbb{H}^d$. We write

$$u = \sum_p \Delta_p u \quad \text{and} \quad v = \sum_q \Delta_q v.$$

Formally, the product can be written as

$$uv = \sum_{p,q} \Delta_p u \, \Delta_q v$$

Paradifferential calculus is a mathematical tool for splitting the above sum into three parts: the first part concerns the indices (p, q) for which the size of the spectrum of $\Delta_p u$ is small compared to the size of the one of $\Delta_q v$. The second part is the symmetric of the first part and in the last part, we keep the indices (p, q) for which the spectrum of $\Delta_p u$ and $\Delta_q v$ have comparable sizes. This leads to the following definition.

Definition 4.11. — *We shall call paraproduct of v by u and shall denote by $T_u v$ the following bilinear operator:*

$$T_u v \stackrel{\text{def}}{=} \sum_q S_{q-1} u \, \Delta_q v \tag{4.2.5}$$

We shall call remainder of u and v and shall denote by $R(u, v)$ the following bilinear operator:

$$R(u, v) \stackrel{\text{def}}{=} \sum_{|p-q|\leq 1} \Delta_p u \, \Delta_q v \tag{4.2.6}$$

Remark 4.12. — *Just by looking at the definition, it is clear that*

$$uv = T_u v + T_v u + R(u, v). \tag{4.2.7}$$

According to Proposition 4.9, $S_{q-1} u \, \Delta_q v$ is frequency localized in a ring of size 2^q. But, for terms of the kind $\Delta_p u \, \Delta_q v$ with $|p - q| \leq 1$, we have an accumulation of frequencies at the origin. Such terms are frequency localized in a ball of size 2^q.

The way how the paraproduct and remainder act on Besov spaces is similar to the classical case. We refer to [**5**] *for more details.*

Taking advantage of this theory, one can prove the following useful estimates.

Lemma 4.13. — *Let σ be a positive, noninteger real number and consider a real number s such that $|s| < \sigma$. Then, there exists a positive constant C such that for all functions f and g,*

$$\|fg\|_{H^s(\mathbb{H}^d)} \leq C \|f\|_{C^\sigma(\mathbb{H}^d)} \|g\|_{H^s(\mathbb{H}^d)}. \tag{4.2.8}$$

Moreover, for any integer M there exists a positive constant C such that for any function f,

$$\|S_M f\|_{C^\rho(\mathbb{H}^d)} \leq C \|f\|_{C^\rho(\mathbb{H}^d)}, \tag{4.2.9}$$

$$\|(\mathrm{Id} - S_M) f\|_{L^\infty(\mathbb{H}^d)} \leq C 2^{-M\rho} \|f\|_{C^\rho(\mathbb{H}^d)} \tag{4.2.10}$$

and more generally, for $0 < \sigma < \rho$,

$$\|(\mathrm{Id} - S_M) f\|_{C^\sigma(\mathbb{H}^d)} \leq C 2^{-M(\rho-\sigma)} \|f\|_{C^\rho(\mathbb{H}^d)}. \tag{4.2.11}$$

Note that Inequality (4.2.8) is not sharp, but is sufficient for our purposes. The sharper result (proved by the same type of method) would be

$$\|fg\|_{H^s(\mathbb{H}^d)} \leq C(\|f\|_{L^\infty(\mathbb{H}^d)}\|g\|_{H^s(\mathbb{H}^d)} + \|f\|_{C^\sigma(\mathbb{H}^d)}\|g\|_{L^2(\mathbb{H}^d)}).$$

The proof of this lemma is classical: it is the same proof as in $\mathbb{R}^d$ for the classical Littlewood-Paley theory and has no specific feature to the Heisenberg group. We provide it here for the sake of completeness, as it will be used often in the rest of this paper.

Proof. — The first ingredient of the proof of Estimate (4.2.8) is Decomposition (4.2.7) which consists in writing

$$fg = T_f g + T_g f + R(f,g).$$

Let us begin with the study of $T_f g$. By definition of the paraproduct and thanks to Proposition 4.9, one has

$$\Delta_q(T_f g) = \sum_{|p-q|\leq N_0} \Delta_q(S_{p-1}f\,\Delta_p g),$$

where N_0 is a fixed integer, chosen large enough. We deduce thanks to the continuity of Littlewood-Paley operators on Lebesgue spaces, that

$$\begin{aligned} 2^{qs}\|\Delta_q(T_f g)\|_{L^2(\mathbb{H}^d)} &\leq \sum_{|p-q|\leq N_0} 2^{qs}\|\Delta_q(S_{p-1}f\,\Delta_p g)\|_{L^2(\mathbb{H}^d)} \\ &\leq C \sum_{|p-q|\leq N_0} 2^{qs}\|S_{p-1}f\|_{L^\infty(\mathbb{H}^d)}\|\Delta_p g\|_{L^2(\mathbb{H}^d)} \\ &\leq C \ \|f\|_{L^\infty(\mathbb{H}^d)} \sum_{|p-q|\leq N_0} 2^{qs}\|\Delta_p g\|_{L^2(\mathbb{H}^d)}. \end{aligned}$$

Using Littlewood-Paley characterization of Sobolev spaces, we infer that

$$\begin{aligned} 2^{qs}\|\Delta_q(T_f g)\|_{L^2(\mathbb{H}^d)} &\leq C \ \|f\|_{L^\infty(\mathbb{H}^d)} \sum_{|p-q|\leq N_0} 2^{(q-p)s}2^{ps}\|\Delta_p g\|_{L^2(\mathbb{H}^d)} \\ &\leq C \ \|f\|_{L^\infty(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)} \sum_{|p-q|\leq N_0} 2^{(q-p)s}c_p, \end{aligned}$$

where, as in all what follows, (c_p) denotes a generic element of the unit sphere of $\ell^2(\mathbb{N})$. Taking advantage of Young inequalities on series, we obtain

$$2^{qs}\|\Delta_q(T_f g)\|_{L^2(\mathbb{H}^d)} \leq C\|f\|_{L^\infty(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}c_q$$

which ensures the desired estimate for $T_f g$ namely

$$\|T_f g\|_{H^s(\mathbb{H}^d)} \leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}.$$

Let us now consider the second term of the above decomposition of the product fg. Again using spectral localization properties, one can write that

$$\Delta_q(T_g f) = \sum_{|p-q|\leq N_0} \Delta_q(S_{p-1}g\,\Delta_p f).$$

Therefore

$$
\begin{aligned}
2^{qs}\|\Delta_q(T_g f)\|_{L^2(\mathbb{H}^d)} &\leq 2^{qs} \sum_{|p-q|\leq N_0} \|\Delta_q(S_{p-1}g\,\Delta_p f)\|_{L^2(\mathbb{H}^d)} \\
&\leq C\,2^{qs} \sum_{|p-q|\leq N_0} \|S_{p-1}g\|_{L^2(\mathbb{H}^d)}\|\Delta_p f\|_{L^\infty(\mathbb{H}^d)} \\
&\leq C\,\|f\|_{C^\sigma(\mathbb{H}^d)}\,2^{qs} \sum_{|p-q|\leq N_0} \|S_{p-1}g\|_{L^2(\mathbb{H}^d)}2^{-p\sigma}. \qquad (4.2.12)
\end{aligned}
$$

By (4.2.2), we have in the case where $s<0$,

$$
\|S_{p-1}g\|_{L^2(\mathbb{H}^d)} \leq C\|g\|_{H^s(\mathbb{H}^d)}2^{-ps}c_p,
$$

where (c_p) still denotes an element of the unit sphere of $\ell^2(\mathbb{N})$. We deduce in that case that

$$
\begin{aligned}
2^{qs}\|\Delta_q(T_g f)\|_{L^2(\mathbb{H}^d)} &\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{qs} \sum_{|p-q|\leq N_0} 2^{-ps}c_p 2^{-p\sigma} \\
&\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{-q\sigma} \sum_{|p-q|\leq N_0} 2^{-(p-q)(\sigma-|s|)}c_p \\
&\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,c_q.
\end{aligned}
$$

This leads in that case to

$$
\|T_g f\|_{H^s(\mathbb{H}^d)} \leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}.
$$

Let us now estimate $T_g f$ in the case where $s\geq 0$. We have

$$
\begin{aligned}
\|S_{p-1}g\|_{L^2(\mathbb{H}^d)} &\leq C \sum_{p'\leq p-2} \|\Delta_{p'}g\|_{L^2(\mathbb{H}^d)} \\
&\leq C\ \|g\|_{H^s(\mathbb{H}^d)} \sum_{p'\leq p-2} 2^{-p's}c_{p'}.
\end{aligned}
$$

Thus (4.2.12) becomes

$$
\begin{aligned}
2^{qs}\|\Delta_q(T_g f)\|_{L^2(\mathbb{H}^d)} &\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{qs} \sum_{|p-q|\leq N_0} \sum_{p'\leq p-2} 2^{-p\sigma}2^{-p's}c_{p'} \\
&\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{qs} \sum_{|p-q|\leq N_0} 2^{-p\sigma} \\
&\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{-q(\sigma-s)} \\
&\leq C\ \|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,c_q.
\end{aligned}
$$

This obviously ends the estimate of $\|T_g f\|_{H^s(\mathbb{H}^d)}$ for any s satisfying $|s|<\sigma$.

Finally, let us consider the remainder term $R(f,g)$. Taking into account the accumulation of frequencies at the origin, we can write

$$
\Delta_q(R(f,g)) = \sum_{q\leq p+N_0} \sum_{|p-p'|\leq 1} \Delta_q(\Delta_p f\,\Delta_{p'}g).
$$

Thus

$$\begin{aligned} 2^{qs}\|\Delta_q(R(f,g))\|_{L^2(\mathbb{H}^d)} &\leq C2^{qs}\sum_{q\leq p+N_0}\sum_{|p-p'|\leq 1}\|\Delta_p f\|_{L^\infty(\mathbb{H}^d)}\|\Delta_{p'}g\|_{L^2(\mathbb{H}^d)}\\ &\leq C\,\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{qs}\sum_{q\leq p+N_0}\sum_{|p-p'|\leq 1}2^{-p\sigma}2^{-p's}c_{p'}\\ &\leq C\,\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\,2^{qs}\sum_{q\leq p+N_0}2^{-p\sigma}2^{-ps}c_p. \end{aligned}$$

In the case where $s\geq 0$, we infer that

$$2^{qs}\|\Delta_q(R(f,g))\|_{L^2(\mathbb{H}^d)}\leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}\sum_{q\leq p+N_0}2^{-(p-q)s}c_p.$$

Then, thanks to Young inequalities, we get

$$2^{qs}\|\Delta_q(R(f,g))\|_{L^2(\mathbb{H}^d)}\leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}c_q$$

which implies that

$$\|R(f,g)\|_{H^s(\mathbb{H}^d)}\leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}.$$

Now, in the case where $s<0$, we have

$$2^{qs}\|\Delta_q(R(f,g))\|_{L^2(\mathbb{H}^d)}\leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)}2^{-q\sigma}\sum_{q\leq p+N_0}2^{-(p-q)(\sigma-|s|)}c_p.$$

Again, Young inequalities allow to conclude. This achieves the proof of the estimate

$$\|R(f,g)\|_{H^s(\mathbb{H}^d)}\leq C\|f\|_{C^\sigma(\mathbb{H}^d)}\,\|g\|_{H^s(\mathbb{H}^d)},$$

for any $|s|<\sigma$.

Let us now turn to the proof of Inequality (4.2.9). By definition of the C^ρ-norm, we recall that

$$\|S_M f\|_{C^\rho(\mathbb{H}^d)}=\sup_q 2^{q\rho}\|\Delta_q S_M f\|_{L^\infty(\mathbb{H}^d)}.$$

Using commutation properties of Δ_q and S_M, we obtain

$$\begin{aligned} \|S_M f\|_{C^\rho(\mathbb{H}^d)} &= \sup_q 2^{q\rho}\|S_M\Delta_q f\|_{L^\infty(\mathbb{H}^d)}\\ &\leq C\sup_q 2^{q\rho}\|\Delta_q f\|_{L^\infty(\mathbb{H}^d)}\\ &\leq C\,\|f\|_{C^\rho(\mathbb{H}^d)} \end{aligned}$$

thanks to the continuity of Littlewood-Paley operators on Lebesgue spaces, which ends the proof of Estimate (4.2.9). Moreover, it is obvious that

$$\|(\mathrm{Id}-S_M)f\|_{L^\infty(\mathbb{H}^d)}\leq\sum_{q\geq M-N_1}\|\Delta_q f\|_{L^\infty(\mathbb{H}^d)},$$

where N_1 is a fixed integer, chosen large enough. Therefore, according to definition of the C^ρ-norm, we get

$$\begin{aligned}\|(\mathrm{Id}-S_M)f\|_{L^\infty(\mathbb{H}^d)} &\leq C \sum_{q\geq M-N_1} 2^{-q\rho}\|f\|_{C^\rho(\mathbb{H}^d)}\\ &\leq C \ \|f\|_{C^\rho(\mathbb{H}^d)} \sum_{q\geq M-N_1} 2^{-q\rho}\\ &\leq C\|f\|_{C^\rho(\mathbb{H}^d)}2^{-M\rho}.\end{aligned}$$

This achieves the proof of Inequality (4.2.10). Along the same lines, for $0<\sigma<\rho$, one has

$$\|(\mathrm{Id}-S_M)f\|_{C^\sigma(\mathbb{H}^d)} \leq \sum_{q\geq M-N_1} 2^{q\sigma}\|\Delta_q(\mathrm{Id}-S_M)f\|_{L^\infty(\mathbb{H}^d)}.$$

Using again the continuity of Littlewood-Paley operators on Lebesgue spaces, it comes

$$\begin{aligned}\|(\mathrm{Id}-S_M)f\|_{C^\sigma(\mathbb{H}^d)} &\leq C \sum_{q\geq M-N_1} 2^{q\sigma}\|\Delta_q f\|_{L^\infty(\mathbb{H}^d)}\\ &\leq C\|f\|_{C^\rho(\mathbb{H}^d)} \sum_{q\geq M-N_1} 2^{q(\sigma-\rho)}\\ &\leq C\|f\|_{C^\rho(\mathbb{H}^d)}2^{-M(\rho-\sigma)},\end{aligned}$$

thus the desired estimate. This ends the proof of Lemma 4.13. □

4.3. Truncation pseudodifferential operators

In this section we shall compare Littlewood-Paley operators with the pseudodifferential operators $\mathrm{Op}\left(\Phi(2^{-2p}|\lambda|(\xi^2+\eta^2)\right)$, for Φ compactly supported in a unit ring.

We shall see that $\mathrm{Op}\left(\Phi(2^{-2p}|\lambda|(\xi^2+\eta^2)\right)$ is "close" to Δ_p in the sense that the operator $\Delta_q\mathrm{Op}\left(\Phi(2^{-2p}|\lambda|(\xi^2+\eta^2)\right)$ is small in $\mathcal{L}(H^s(\mathbb{H}^d))$ norm if $|p-q|$ is large. This is made precise in the next proposition.

Proposition 4.14. — *Let $\delta_0\in(0,1)$ and Φ be a smooth function, compactly supported in $]0,\infty[$. There is a constant C such that the following result holds. For any $p\geq 0$, define the symbol*

$$a_p(w,\lambda,\xi,\eta)=\Phi_p(|\lambda|(\xi^2+\eta^2)), \quad \textit{where} \quad \Phi_p(r)=\Phi(2^{-2p}r),\ \forall r>0.$$

Then for any integer $q\geq -1$ and any real number s,

$$\|\Delta_q\mathrm{Op}(a_p)\|_{\mathcal{L}(H^s(\mathbb{H}^d))}\leq C2^{-\delta_0|p-q|},$$

where Δ_q is a Littlewood-Paley truncation, as defined in Definition 4.3.

Proof. — We shall start by reducing the problem to the case $s = 0$. Let u belong to $\mathcal{S}(\mathbb{H}^d)$ and let $q \geq 0$ be given (the case $q = -1$ is obvious). The norm $\|\Delta_q \mathrm{Op}(a_p)u\|_{H^s}$ is controlled by the quantity

$$2^{qs}\|\Delta_q \mathrm{Op}(a_p)u\|_{L^2} = 2^{qs}\left(\int \|\mathcal{F}(u)(\lambda)A_\lambda R^*(2^{-2q}D_\lambda)\|^2_{HS(\mathcal{H}_\lambda)}|\lambda|^d\,d\lambda\right)^{1/2}$$

where $A_\lambda = J_\lambda^* \mathrm{op}^w(a_p)J_\lambda$. Defining a smooth, compactly supported (away from zero) function $\mathcal{R}$ such that $\mathcal{R}R^* = R^*$, one has

$$\|\mathcal{F}(u)(\lambda)A_\lambda R^*(2^{-2q}D_\lambda)\|_{HS(\mathcal{H}_\lambda)} = \|\mathcal{F}(u)(\lambda)A_\lambda R^*(2^{-2q}D_\lambda)\mathcal{R}(2^{-2q}D_\lambda)\|_{HS(\mathcal{H}_\lambda)}.$$

But A_λ is a diagonal operator in the diagonalisation basis of D_λ, thus it commutes with the operator $R^*(2^{-2q}D_\lambda)$. So

$$\|\mathcal{F}(u)(\lambda)A_\lambda R^*(2^{-2q}D_\lambda)\mathcal{R}(2^{-2q}D_\lambda)\|_{HS(\mathcal{H}_\lambda)} = \|\mathcal{F}(\widetilde{\Delta}_q u)(\lambda)A_\lambda R^*(2^{-2q}D_\lambda)\|_{HS(\mathcal{H}_\lambda)},$$

where $\widetilde{\Delta}_q$ is the Littlewood-Paley operator associated with $\mathcal{R}(2^{-2q}\cdot)$. Using (1.2.19) stated page 11, we get

$$\|\mathcal{F}(u)(\lambda)A_\lambda R^*(2^{-2q}D_\lambda)\mathcal{R}(2^{-2q}D_\lambda)\|_{HS(\mathcal{H}_\lambda)} \leq \|\mathcal{F}(\widetilde{\Delta}_q u)(\lambda)\|_{HS(\mathcal{H}_\lambda)}\|A_\lambda R^*(2^{-2q}D_\lambda)\|_{\mathcal{L}(\mathcal{H}_\lambda)},$$

and Remark 4.4 gives the expected result: we have reduced the problem to the $L^2(\mathbb{H}^d)$ case, and by the Plancherel formula (1.2.21) and Inequality (1.2.19), it is enough to study the norm as a bounded operator of $L^2(\mathbb{R}^d)$ of the operators

$$R^*(2^{-2q}|\lambda|(\xi^2 - \Delta_\xi))\,\mathrm{op}^w(a_p) \quad \text{and} \quad R^*(2^{-2q}|\lambda|(\xi^2 - \Delta_\xi))\,\mathrm{op}^w(a_p).$$

For this, we use Mehler's formula to turn $\mathrm{op}^w(a_p)$ into an operator given by a function of the harmonic oscillator in order to be able to use functional calculus. From now on we suppose to simplify that $\lambda > 0$.

We will denote, as in Definition 4.3, by $\widetilde{R}^*$ and R^* the basis functions of the truncation Δ_q (with $\widetilde{R}^*$ supported in a unit ball of $\mathbb{R}$ and R^* supported in a unit ring of $\mathbb{R}$).

In view of (1.3.15) (see page 20), one has

$$\mathrm{op}^w\left(\Phi_p(\lambda(\xi^2+\eta^2))\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}(\tau)\frac{e^{i(\xi^2-\Delta)\mathrm{Arctg}(2^{-2p}\lambda\tau)}}{(1+(2^{-2p}\lambda\tau)^2)^{\frac{d}{2}}}\,d\tau.$$

But

$$\left\|R^*\left(2^{-2q}|\lambda|(\xi^2 - \Delta_\xi)\right)\mathrm{op}^w(a_p)\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = \sup_{\alpha,\lambda}|I_p(\alpha,\lambda)|\,R^*(2^{-2q}|\lambda|(2|\alpha|+d))$$

and a similar relation holds for $\widetilde{R^*}$, so we are reduced to estimating, for $\alpha \in \mathbb{N}^d$ and $\lambda 2^{-2q}(2|\alpha|+d)$ in a unit ring (or ball if $q = -1$)

$$I_p(\alpha,\lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}(\tau)\frac{e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\lambda\tau)}}{(1+(2^{-2p}\lambda\tau)^2)^{\frac{d}{2}}}\,d\tau,$$

and we shall argue differently whether $q < p$ or $q > p$.

• *The case when $q > p$.* We argue differently depending on whether $2^{-2p}|\lambda| \leq 2^{q-p}$ or $2^{-2p}|\lambda| \geq 2^{q-p}$. Let us first suppose that $2^{-2p}|\lambda| \leq 2^{q-p}$. Noticing that

$$\frac{d}{d\tau} e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\lambda\tau)} = \frac{i2^{-2p}\lambda(2|\alpha|+d)}{1+(2^{-2p}\lambda\tau)^2} e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\lambda\tau)}$$

we have

$$I_p(\alpha,\lambda) = \frac{i}{(2|\alpha|+d)2^{-2p}\lambda} \int e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\lambda\tau)} \frac{d}{d\tau}\left(\frac{\widehat{\Phi}(\tau)}{(1+(2^{-2p}\lambda\tau)^2)^{\frac{d}{2}-1}}\right) d\tau$$

so using the fact that $2|\alpha| + d \geq 1$,

$$R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)|I_p(\alpha,\lambda)| \leq C2^{2(p-q)}\Big(\int |\widehat{\Phi}'(\tau)|(1+(2^{-2p}\lambda\tau)^2)^{1-\frac{d}{2}} d\tau + \int |\widehat{\Phi}(\tau)| \frac{2^{-4p}\lambda^2\tau}{(1+(2^{-2p}\lambda\tau)^2)^{\frac{d}{2}}} d\tau\Big).$$

Let us consider the first integral. If $d \geq 2$, it is bounded by $\|\widehat{\Phi}'\|_{L^1}$. On the other hand, if $d = 1$, we observe that

$$|\widehat{\Phi}'(\tau)|(1+(2^{-2p}\lambda\tau)^2)^{1-\frac{d}{2}} \leq C|\widehat{\Phi}'(\tau)|\,(1+2^{-2p}\lambda|\tau|).$$

Therefore, since $(1+|\tau|)\,|\widehat{\Phi}'(\tau)| \in L^1$, there exists a constant C such that

$$2^{2(p-q)} \int |\widehat{\Phi}'(\tau)|(1+(2^{-2p}\lambda\tau)^2)^{1-\frac{d}{2}} d\tau \leq C\,2^{2(p-q)}\left(1+2^{q-p}\right) \leq C\,2^{-(q-p)}.$$

Let us now concentrate on the last integral. We have clearly

$$2^{2(p-q)} \int |\widehat{\Phi}(\tau)| \frac{2^{-4p}\lambda^2|\tau|}{(1+(2^{-2p}\lambda\tau)^2)^{\frac{d}{2}}} d\tau \leq 2^{2(p-q)}\,2^{-2p}|\lambda| \int |\widehat{\Phi}(\tau)|\,d\tau,$$

whence a constant C such that

$$2^{2(p-q)} \int |\widehat{\Phi}(\tau)| \frac{2^{-4p}\lambda^2|\tau|}{(1+(2^{-2p}\lambda\tau)^2)^{\frac{d}{2}}} d\tau \leq C\,2^{-(q-p)}.$$

We now suppose that $|\lambda|2^{-2p} \geq 2^{q-p}$ and we perform the change of variables $u = \lambda 2^{-2p}\tau$ in the integral expression of $I_p(\alpha,\lambda)$. We obtain

$$I_p(\alpha,\lambda) = \frac{2^{2p}\lambda^{-1}}{2\pi} \int \widehat{\Phi}\left(2^{2p}\lambda^{-1}u\right)(1+u^2)^{-d/2} e^{i(2|\alpha|+d)\mathrm{Arctg}u}\,du.$$

Using that $|\widehat{\Phi}(\tau)| \leq C\,|\tau|^{-1+\delta_0}$, we get

$$\left|\widehat{\Phi}\left(2^{2p}\lambda^{-1}u\right)\right| \leq C(2^{-2p}|\lambda|)^{1-\delta_0}|u|^{-1+\delta_0}.$$

This yields that there exists a constant C such that

$$|I_p(\alpha,\lambda)| \leq C\,(2^{2p}|\lambda|^{-1})^{\delta_0} \int |u|^{-1+\delta_0}(1+u^2)^{-d/2}du \leq C' 2^{-\delta_0(q-p)}.$$

As a conclusion, we have proved that in that case, for all $\alpha \in \mathbb{Z}^d$,

$$R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)|I_p(\alpha,\lambda)| \leq C\,2^{\delta_0(p-q)}.$$

• *The case when* $q \leq p$. The idea is to compare $I_p(\alpha, \lambda)$ to $\Phi(\lambda 2^{-2p}(2|\alpha| + d))$. Taking the inverse (classical) Fourier transform we can write

$$I_p(\alpha,\lambda) - \Phi(\lambda 2^{-2p}(2|\alpha|+d)) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}(\tau)\left(\frac{e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\tau\lambda)}}{(1+(2^{-2p}\tau\lambda)^2)^{\frac{d}{2}}} - e^{i2^{-2p}\lambda\tau(2|\alpha|+d)}\right) d\tau$$

or again

$$I_p(\alpha,\lambda) - \Phi(\lambda 2^{-2p}(2|\alpha|+d)) = J_p(\alpha,\lambda) + R_p(\alpha,\lambda),$$

with

$$J_p(\alpha,\lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\Phi}(\tau)\left(e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\lambda\tau)} - e^{i2^{-2p}\lambda\tau(2|\alpha|+d)}\right) d\tau.$$

It is easy to see that

$$|R_p(\alpha,\lambda)| \leq C 2^{-2p}\lambda \int_{\mathbb{R}} |\tau\widehat{\Phi}(\tau)|\, d\tau$$

so since $\widehat{\Phi}$ belongs to $\mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned} R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)|R_p(\alpha,\lambda)| &\leq CR^*\left((2|\alpha|+d)\lambda 2^{-2q}\right) 2^{-2p}\lambda \\ &\leq C\,2^{-2(p-q)}, \end{aligned}$$

using the fact that $2|\alpha| + d \geq 1$. Similarly

$$\widetilde{R}^*\left((2|\alpha|+d)\lambda\right)|R_p(\alpha,\lambda)| \leq C\,2^{-2p}.$$

So now we are left with the estimate of J_p, which we shall decompose into two parts:

$$J_p = J_p^1 + J_p^2, \quad \text{with}$$

$$J_p^1(\alpha,\lambda) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{|\tau 2^{-2p}\lambda|\leq 1/2} \widehat{\Phi}(\tau)\left(e^{i(2|\alpha|+d)\mathrm{Arctg}(2^{-2p}\lambda\tau)} - e^{i2^{-2p}\lambda\tau(2|\alpha|+d)}\right) d\tau.$$

The estimate of J_p^2 is very easy, since clearly as above

$$\begin{aligned} |J_p^2(\alpha,\lambda)| &\leq C2^{-2p}\lambda\int_{\mathbb{R}} |\tau\widehat{\Phi}(\tau)|\, d\tau \\ &\leq C\,2^{-2p}\lambda, \end{aligned}$$

so

$$R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)|J_p^2(\alpha,\lambda)| \leq C\,2^{-2(p-q)}$$

and

$$\widetilde{R}^*\left((2|\alpha|+d)\lambda\right)|J_p^2(\alpha,\lambda)| \leq C\,2^{-2p}.$$

Now let us concentrate on J_p^1. We can write

$$J_p^1(\alpha,\lambda) = \frac{1}{2\pi}\int_{|\tau 2^{-2p}\lambda|\leq 1/2} \widehat{\Phi}(\tau)e^{i2^{-2p}\lambda\tau(2|\alpha|+d)}\left(e^{i(2|\alpha|+d)2^{-2p}\lambda h(\tau)} - 1\right) d\tau,$$

with

$$h(\tau) = \tau\sum_{n\geq 1}\frac{(-1)^n(2^{-2p}\lambda\tau)^{2n}}{2n+1},$$

which is well defined, and analytic, for $|\tau 2^{-2p}\lambda| \leq 1/2$. Observe that the function h depends on the integer p and on λ, and that one has to control this dependance. In particular, we notice that $h'(\tau)$ can easily be bounded, by $1/3$, on the domain $|\tau 2^{-2p}\lambda| \leq 1/2$. But

$$\mathrm{e}^{i(2|\alpha|+d)2^{-2p}\lambda h(\tau)} - 1 = i(2|\alpha| + d)2^{-2p}\lambda h(\tau) \int_0^1 \mathrm{e}^{it(2|\alpha|+d)2^{-2p}\lambda h(\tau)}\, dt$$

so

$$J_p^1(\alpha, \lambda) = \frac{i}{2\pi} \int_0^1 \int_{|\tau 2^{-2p}\lambda| \leq 1/2} \widehat{\Phi}(\tau) \mathrm{e}^{i2^{-2p}\lambda(2|\alpha|+d)(\tau+th(\tau))} (2|\alpha| + d)2^{-2p}\lambda h(\tau) dt d\tau.$$

Integrating by parts, we get

$$\begin{aligned} J_p^1(\alpha, \lambda) = &-\frac{1}{2\pi} \int_0^1 \int_{|\tau 2^{-2p}\lambda| \leq 1/2} \mathrm{e}^{i2^{-2p}\lambda(2|\alpha|+d)(\tau+th(\tau))} \partial_\tau \left(\frac{\widehat{\Phi}(\tau)}{1 + th'(\tau)} h(\tau) \right) dt d\tau \\ &+ \frac{1}{2\pi} \int_0^1 \left[\mathrm{e}^{i2^{-2p}\lambda(2|\alpha|+d)(\tau+th(\tau))} \frac{\widehat{\Phi}(\tau)}{1 + th'(\tau)} h(\tau) \right]_{|\tau 2^{-2p}\lambda| = 1/2} dt. \end{aligned}$$

Writing the above formula as $J_p^1 = K_p^1 + K_p^2$, with

$$K_p^2(\alpha, \lambda) = \frac{1}{2\pi} \int_0^1 \left[\mathrm{e}^{i2^{-2p}\lambda(2|\alpha|+d)(\tau+th(\tau))} \frac{\widehat{\Phi}(\tau)}{1 + th'(\tau)} h(\tau) \right]_{|\tau 2^{-2p}\lambda| = 1/2} dt,$$

it is obvious that

$$|K_p^2(\alpha, \lambda)| \leq C \left| \widehat{\Phi}(\frac{1}{2} 2^{2p}\lambda^{-1}) h(\frac{1}{2} 2^{2p}\lambda^{-1}) \right|.$$

Writing

$$h(\tau) = \tau \sum_{n \geq 1} \frac{(-1)^n (2^{-2p}\lambda\tau)^{2n}}{2n + 1} = 2^{-2p}\lambda\tau^2 \sum_{n \geq 1} \frac{(-1)^n (2^{-2p}\lambda\tau)^{2n-1}}{2n + 1},$$

we deduce that

$$\begin{aligned} |K_p^2(\alpha, \lambda)| &\leq C 2^{-2p}\lambda |\widehat{\Phi}(\frac{1}{2} 2^{2p}\lambda^{-1}) 2^{4p}\lambda^{-2}| \\ &\leq C 2^{-2p}\lambda, \end{aligned}$$

where the second estimate comes from the fact that $\widehat{\Phi}$ is a rapidly decreasing function. To bound K_p^1 we just need to notice that

$$\frac{\widehat{\Phi}(\tau)}{1 + th'(\tau)} h(\tau) = \frac{\widehat{\Phi}(\tau)\tau^2}{1 + th'(\tau)} 2^{-2p}\lambda g(\tau), \quad \text{with} \quad g(\tau) = \sum_{n \geq 1} \frac{(-1)^n (2^{-2p}\lambda\tau)^{2n-1}}{2n + 1},$$

so

$$|K_p^1(\alpha, \lambda)| \leq C 2^{-2p}\lambda \int_0^1 \int_{|\tau 2^{-2p}\lambda| \leq 1/2} \left| \partial_\tau \Big(\frac{\widehat{\Phi}(\tau)\tau^2}{1 + th'(\tau)} g(\tau) \Big) \right| d\tau\, dt \leq C 2^{-2p}\lambda.$$

We conclude as previously that

$$R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)|J_p^1(\alpha,\lambda)|\leq C\,2^{-2(p-q)} \quad\text{and}\quad \widetilde{R}^*\left((2|\alpha|+d)\lambda\right)|J_p^1(\alpha,\lambda)|\leq C\,2^{-2p}.$$

Combining those results, we conclude that if $p>q$, then

$$R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)\left|\Phi\left((2|\alpha|+d)\lambda 2^{-2p}\right)-I_p(\alpha,\lambda)\right|\leq C\,2^{-2(p-q)}.$$

But clearly $R^*\left((2|\alpha|+d)\lambda 2^{-2q}\right)\Phi\left((2|\alpha|+d)\lambda 2^{-2p}\right)$ is equal to zero if $|p-q|$ is large enough, so we have proved the expected result if $p>q$.

That concludes the proof of the proposition. □

4.4. λ-truncation operators

We shall use, in the proof of Theorem 5, truncation operators in the variable λ.

Let us consider ψ and ϕ, two smooth radial functions, the values of which are in the interval $[0,1]$, belonging respectively to $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{C})$, where $\mathcal{B}$ is the unit ball of $\mathbb{R}$ and $\mathcal{C}$ a unit ring of $\mathbb{R}$, and such that for $D=1$

$$\forall\zeta\in\mathbb{R}^D,\quad 1=\psi(\zeta)+\sum_{p\geq 0}\phi(2^{-2p}\zeta). \tag{4.4.1}$$

We set

$$\Lambda_p=\mathrm{Op}(\phi(2^{-2p}\lambda))\quad\text{and}\quad \Lambda_{-1}=\mathrm{Op}(\psi(\lambda)).$$

We notice that Λ_p commutes with all operators of the form $\mathrm{Op}(a(\lambda,y,\eta))$, and in particular with powers of $-\Delta_{\mathbb{H}^d}$.

Then the operators Λ_p map continuously $H^s(\mathbb{H}^d)$ into $H^s(\mathbb{H}^d)$ independently of p and we have the following quasi-orthogonality relation: there exists N_0 such that

$$\Lambda_p\Lambda_q=0\quad\text{for}\quad |p-q|\geq N_0, \tag{4.4.2}$$

which implies that

$$\|\Lambda_p u\|_{L^2(\mathbb{H}^d)}\leq c_p\|u\|_{L^2(\mathbb{H}^d)}, \tag{4.4.3}$$

where c_p is an element of the unit sphere of $\ell^2(\mathbb{Z})$. More precisely, there exist constants C_1 and C_2 such that if f belongs to $H^s(\mathbb{H}^d)$, then the following inequality hold:

$$C_1\sum_r\|\Lambda_r f\|^2_{H^s(\mathbb{H}^d)}\leq\|f\|^2_{H^s(\mathbb{H}^d)}\leq C_2\sum_r\|\Lambda_r f\|^2_{H^s(\mathbb{H}^d)}. \tag{4.4.4}$$

Besides, we are able to say something about the Λ_m-localization of a product by an easy adaptation of Lemma 4.1 and of Proposition 4.2 of **[5]**. More precisely, we have the following result which ensures that some Λ_m-spectral localization properties are preserved after the product has been taken.

Proposition 4.15. — *There is a constant $M_1 \in \mathbb{N}$ such that the following holds. Consider f and g two functions of $\mathcal{S}(\mathbb{H}^d)$ such that*

$$\begin{aligned} \mathcal{F}(f)(\lambda) &= \mathbf{1}_{2^{2m}\mathcal{C}}(\lambda)\mathcal{F}(f)(\lambda) \quad \text{and} \\ \mathcal{F}(g)(\lambda) &= \mathbf{1}_{2^{2m'}\mathcal{C}}(\lambda)\mathcal{F}(g)(\lambda) \end{aligned}$$

for some integers m and m'. If $m' - m > M_1$, then there exists a ring $\widetilde{\mathcal{C}}$ such that

$$\mathcal{F}(fg)(\lambda) = \mathbf{1}_{2^{2m'}\widetilde{\mathcal{C}}}(\lambda)\mathcal{F}(fg)(\lambda).$$

On the other hand, if $|m' - m| \leq M_1$, then there exists a ball $\tilde{\mathcal{B}}$ such that

$$\mathcal{F}(fg)(\lambda) = \mathbf{1}_{2^{2m'}\widetilde{\mathcal{B}}}(\lambda)\mathcal{F}(fg)(\lambda).$$

Proof. — The proof of that result follows the lines of the proof of Proposition 4.2 of **[5]**, and is in fact simpler. We write it here for the sake of completeness. By density, it suffices to prove Lemma 4.15 for f, g in $\mathcal{D}(\mathbb{R}^{2d+1})$.

For simplicity, we will only deal with the case where $\lambda > 0$.

By definition of $\mathcal{F}(f)(\lambda)$, we have

$$\begin{aligned} \mathcal{F}(f)(\lambda)F_{\alpha,\lambda}(\xi) &= \int_{\mathbb{H}^d} f(z,s)u^\lambda_{z,s}F_{\alpha,\lambda}(\xi)\,dz\,ds \\ &= \int_{\mathbb{H}^d} f(z,s)F_{\alpha,\lambda}(\xi - \overline{z})e^{i\lambda s + 2\lambda(\xi\cdot z - |z|^2/2)}\,dz\,ds. \end{aligned}$$

Let us write $\xi = \xi_a + i\xi_b$ and $z = z_a + iz_b$, where ξ_a, z_a, ξ_b and z_b are real numbers.

Straightforward computations show that

$$e^{i\lambda s + 2\lambda(\xi\cdot z - |z|^2/2)} = e^{-i\,(-2\lambda\xi_b\cdot z_a - 2\lambda\xi_a\cdot z_b - \lambda s)}e^{-\lambda\,(|\xi-\overline{z}|^2 - |\xi|^2)}.$$

Then we can observe that

$$\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}(\xi) = \widehat{\left(A^\alpha_{\lambda,\xi}f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda), \tag{4.4.5}$$

where $\widehat{h}$ denotes the usual Fourier transform of h on $\mathbb{R}^{2d+1}$ and where

$$A^\alpha_{\lambda,\xi}f(z,s) = F_{\alpha,\lambda}(\xi - \overline{z})e^{-\lambda\,(|\xi-\overline{z}|^2 - |\xi|^2)}f(z,s). \tag{4.4.6}$$

Therefore, one can write

$$\mathcal{F}(fg)(\lambda)F_{\alpha,\lambda}(\xi) = \widehat{\left(A^\alpha_{\lambda,\xi}fg\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda).$$

Noticing that for any multi-index β of $\mathbb{N}^d$ satisfying $\beta \leq \alpha$, we have

$$F_{\alpha,\lambda}(\xi) = C_{\alpha,\beta}\,F_{\alpha-\beta,\lambda}(\xi)\cdot F_{\beta,\lambda}(\xi),$$

with $C_{\alpha,\beta} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}^{-\frac{1}{2}}$, we deduce that $A^\alpha_{\lambda,\xi}fg = B^\beta_{\lambda,\xi}f\cdot A^{\alpha-\beta}_{\lambda,\xi}g$, where

$$B^\beta_{\lambda,\xi}f(z,s) = F_{\beta,\lambda}(\xi - \overline{z})f(z,s)$$

and $\beta \leq \alpha$. Using the fact that the standard Fourier transform on $\mathbb{R}^{2d+1}$ exchanges product and convolution, we get

$$\widehat{\left(A^{\alpha}_{\lambda,\xi}fg\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda) = C_{\alpha,\beta}\,\widehat{\left(B^{\beta}_{\lambda,\xi}f\right)} \star \widehat{\left(A^{\alpha-\beta}_{\lambda,\xi}g\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda),$$

where $\star$ denotes the convolution product in $\mathbb{R}^{2d+1}$ and still for any multi-index β of $\mathbb{N}^d$ satisfying $\beta \leq \alpha$. The question is then reduced to the study of the supports of the functions $\widehat{(B^{\beta}_{\lambda,\xi}f)}$ and $\widehat{(A^{\alpha-\beta}_{\lambda,\xi}g)}$.

According to (4.4.5), the support in λ of the function

$$\widehat{\left(A^{\alpha-\beta}_{\lambda,\xi}g(z,s)\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda)$$

is included in the ring $2^{2m'}\mathscr{C}$. Now, Lemma 4.15 readily follows from the properties of the standard convolution product in $\mathbb{R}^{2d+1}$ for the supports, and from the following lemma, whose proof is given below.

This ends the proof of Lemma 4.15. □

Lemma 4.16. — *Under the hypothesis of Lemma 4.15, we have*

$$\widehat{\left(B^{\beta}_{\lambda,\xi}f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda) = \mathbf{1}_{2^{2m}\mathscr{C}}(\lambda)\,\widehat{\left(B^{\beta}_{\lambda,\xi}f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda).$$

Proof. — By definition of the standard Fourier transform on $\mathbb{R}^{2d+1}$, we have

$$\begin{aligned}\widehat{\left(B^{\beta}_{\lambda,\xi}f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda) &= \int \mathrm{e}^{-i\,(-2\lambda\xi_b\cdot z_a - 2\lambda\xi_a\cdot z_b - \lambda s)} B^{\beta}_{\lambda,\xi}f(z,s)\,dz\,ds\\ &= \int \mathrm{e}^{i\,(2\lambda\xi_b\cdot z_a + 2\lambda\xi_a\cdot z_b + \lambda s)} F_{\beta,\lambda}(\xi - \overline{z})f(z,s)\,dz\,ds\end{aligned}$$

Denoting $2\lambda(\xi_b\cdot z_a + \xi_a\cdot z_b) + \lambda s$ by $J_\lambda(s,z,\xi)$, it follows that

$$\widehat{\left(B^{\beta}_{\lambda,\xi}f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda) = \int \mathrm{e}^{iJ_\lambda(s,z,\xi)}\mathrm{e}^{-\lambda(|\xi-\overline{z}|^2-|\xi|^2)}F_{\beta,\lambda}(\xi-\overline{z})\mathrm{e}^{\lambda(|\xi-\overline{z}|^2-|\xi|^2)}f(z,s)\,dz\,ds.$$

Using that

$$\mathrm{e}^{\lambda|\xi-\overline{z}|^2} = \sum_{\alpha\in\mathbb{N}^d}(\overline{\xi}-z)^\alpha\frac{\lambda^{|\alpha|}(\xi-\overline{z})^\alpha}{\alpha!},$$

and observing that the above series is normally convergent on any compact, we deduce that

$$\mathrm{e}^{\lambda|\xi-\overline{z}|^2}F_{\beta,\lambda}(\xi-\overline{z}) = \sum_{\alpha\in\mathbb{N}^d}(\overline{\xi}-z)^\alpha\Big(\frac{\lambda}{2}\Big)^{\frac{|\alpha|}{2}}\frac{1}{\alpha!}\sqrt{\frac{(\beta+\alpha)!}{\beta!}}F_{\alpha+\beta,\lambda}(\xi-\overline{z}).$$

This leads, since $f \in \mathcal{D}(\mathbb{R}^{2d+1})$, to

$$\widehat{\left(B^{\beta}_{\lambda,\xi} f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda) = \sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-\lambda|\xi|^2}\left(\frac{\lambda}{2}\right)^{\frac{|\alpha|}{2}} \frac{1}{\alpha!}\sqrt{\frac{(\beta+\alpha)!}{\beta!}}$$
$$\times \int \mathrm{e}^{iJ_\lambda(s,z,\xi)}\mathrm{e}^{-\lambda(|\xi-\overline{z}|^2-|\xi|^2)} F_{\alpha+\beta,\lambda}(\xi-\overline{z})(\overline{\xi}-z)^\alpha f(z,s)\,dz\,ds.$$

Recalling that

$$A^{\alpha}_{\lambda,\xi} f(z,s) = F_{\alpha,\lambda}(\xi-\overline{z})\mathrm{e}^{-\lambda(|\xi-\overline{z}|^2-|\xi|^2)} f(z,s),$$

we get

$$\widehat{\left(B^{\beta}_{\lambda,\xi} f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda) = \sum_{\alpha\in\mathbb{N}^d} \mathrm{e}^{-\lambda|\xi|^2}\left(\frac{\lambda}{2}\right)^{\frac{|\alpha|}{2}} \frac{1}{\alpha!}\sqrt{\frac{(\beta+\alpha)!}{\beta!}}$$
$$\times \widehat{\left(A^{\beta+\alpha}_{\lambda,\xi}(\overline{\xi}-z)^\alpha f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda).$$

Let us study separately each term of the above series. By Lemma A.2 and using the fact for $\lambda > 0$, $\overline{Q}^{\lambda}_j = \partial_{\xi_j}$, we obtain

$$\mathcal{F}(z_j f)(\lambda)F = \frac{1}{2\lambda}[\partial_{\xi_j}, \mathcal{F}(f)(\lambda)]F.$$

In particular, for any $\gamma \in \mathbb{N}^d$,

$$\mathcal{F}(z_j f)(\lambda)F_{\gamma,\lambda}(\xi) = \frac{1}{2\lambda}\Big(\partial_{\xi_j}\mathcal{F}(f)(\lambda)F_{\gamma,\lambda}(\xi) - \mathcal{F}(f)(\lambda)\partial_{\xi_j}F_{\gamma,\lambda}(\xi)\Big).$$

The frequency localization of the function f in the ring $2^{2m}\mathcal{C}(\lambda)$ implies then that the support in λ of $\mathcal{F}((\overline{\xi}_i - z_i)f)(\lambda)F_{\gamma,\lambda}(\xi)$ is included in the same ring $2^{2m}\mathcal{C}(\lambda)$. An immediate induction implies that for any multi-index α the support in λ of $\mathcal{F}((\overline{\xi} - z)^\alpha f)(\lambda)F_{\gamma,\lambda}(\xi)$ is still included in the same ring $2^{2m}\mathcal{C}(\lambda)$. Therefore, the support in λ of

$$\widehat{\left(A^{\beta+\alpha}_{\lambda,\xi}(\overline{\xi}-z)^\alpha f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda)$$

is included in the ring $2^{2m}\mathcal{C}(\lambda)$.

As each term of the series is supported in a fixed ring, the same holds for the function

$$\widehat{\left(B^{\beta}_{\lambda,\xi} f\right)}(-2\lambda\xi_b, -2\lambda\xi_a, -\lambda),$$

which ends the proof of the lemma. □

The following results will also be useful in Chapter 5.

Lemma 4.17. — *There exists a constant C such that for any function f,*

$$\|\Lambda_m \Delta_q f\|_{L^\infty(\mathbb{H}^d)} \leq C\|\Delta_q f\|_{L^\infty(\mathbb{H}^d)} \tag{4.4.7}$$

for any integers m and q.
Moreover if ρ is a nonnegative real number, then there exists a constant C such that for any function f

$$\|\Lambda_m f\|_{L^\infty(\mathbb{H}^d)} \leq C 2^{-m\rho} \|f\|_{C^\rho(\mathbb{H}^d)}. \tag{4.4.8}$$

Proof. — Let us first prove (4.4.7). We shall only give the general idea of the proof, as the method follows closely a strategy initiated in **[7]** for the study of Littlewood-Paley operators, and followed also in **[6]** in the analysis of the heat operator.

Recall that

$$\mathcal{F}(\Lambda_m \Delta_q f)(\lambda) = \phi(2^{-2m}\lambda)\mathcal{F}(f)(\lambda)(f)R^*(2^{-2q}D_\lambda).$$

where ϕ and R^* are smooth radial functions with values in the interval $[0,1]$ supported in a unit ring of $\mathbb{R}$. This can be also written

$$\mathcal{F}(\Lambda_m \Delta_q f)(\lambda) = \phi(2^{-2m}\lambda)\mathcal{F}(f)(\lambda)\widetilde{R}^*(2^{-2q}D_\lambda)R^*(2^{-2q}D_\lambda)$$

where $\widetilde{R}^*$ is a smooth radial function compactly supported in a unit ring so that $\widetilde{R}^* R^* = R^*$.

According to the fact that the Fourier transform exchanges convolution and composition, we have

$$\Lambda_m \Delta_q f = \Delta_q f \star h_{m,q},$$

where the function $h_{m,q}$ is defined by

$$\mathcal{F}(h_{m,q})(\lambda) = \phi(2^{-2m}\lambda)\widetilde{R}^*(2^{-2q}D_\lambda).$$

Taking advantage of Young's inequalities, it therefore suffices to prove that the function $h_{m,q}$ belongs to $L^1(\mathbb{H}^d)$ uniformly in m and q.

By rescaling, we are reduced to investigating the function h_j defined by

$$\mathcal{F}(h_j)(\lambda) \stackrel{\text{def}}{=} \phi(2^{-2j}\lambda)\widetilde{R}^*(D_\lambda).$$

By the inversion formula (1.2.31), we get

(4.4.9)

$$h_j(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s}\phi(2^{-2j}\lambda)\widetilde{R}^*((2m+d)\lambda)L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda.$$

In order to prove that h_j belongs to $L^1(\mathbb{H}^d)$ (uniformly in j), the idea (as in **[7]** and **[6]**) consists in proving that the function $(z,s) \mapsto (is-|z|^2)^k h_j(z,s)$ belongs to $L^\infty(\mathbb{H}^d)$ with uniform bounds in j.

Let us start by considering the case $k=0$. It is easy to see that the Laguerre polynomials defined in (1.2.30) page 15 satisfy for all $y \geq 0$

$$|L_m^{(d-1)}(y)\mathrm{e}^{-\frac{y}{2}}| \leq C_d(m+1)^{d-1}$$

Since ϕ is bounded, this gives easily after the change of variables $\beta = (2m+d)\lambda$

$$|h_j(z,s)| \leq C \sum_m \frac{1}{m^2} \int |\widetilde{R}^*(\beta)| d\beta. \tag{4.4.10}$$

To deal with the case $k \neq 0$, we use the result proved in **[7]** (see also Proposition 1.11 recalled in the introduction) stating that for any radial function g, one has

$$\mathcal{F}\left((is - |z|^2)g(z,s)\right)(\lambda)F_{\alpha,\lambda} = Q^\star_{|\alpha|}(\lambda)F_{\alpha,\lambda},$$

where for all $m \geq 1$,

$$\begin{aligned} Q^\star_m(\lambda) &= \frac{d}{d\lambda}Q_m(\lambda) - \frac{m}{\lambda}(Q_m(\lambda) - Q_{m-1}(\lambda)) \quad \text{if } \lambda > 0, \\ Q^\star_m(\lambda) &= \frac{d}{d\lambda}Q_m(\lambda) + \frac{m+d}{|\lambda|}(Q_m(\lambda) - Q_{m+1}(\lambda)) \quad \text{if } \lambda < 0 \end{aligned}$$

while Q_m is given by

$$\mathcal{F}\left(g(z,s)\right)(\lambda)F_{\alpha,\lambda} = Q_{|\alpha|}(\lambda)F_{\alpha,\lambda}.$$

The proof then consists in applying Taylor formulas in the above expressions in order to reduce the problem to an estimate of the same type as (4.4.10). The only difference with the case treated in **[7]** and **[6]** lies in the dependence on j. However it can be noticed that due to the support assumptions on ϕ and $\widetilde{R}^*$, there are two positive constants c_1 and c_2 such that

$$h_j(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_{m\in C_j} \int e^{-i\lambda s}\phi(2^{-2j}\lambda)\widetilde{R}^*((2m+d)\lambda)L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda$$

with $C_j \stackrel{\text{def}}{=} \{m \in \mathbb{N},\ c_1 2^{-2j} \leq 2m+d \leq c_2 2^{-2j}\}$. Now let us decompose h_j into two parts:

$$h_j(z,s) = h_j^1(z,s) + h_j^2(z,s), \quad \text{where}$$

$$h_j^1(z,s) \stackrel{\text{def}}{=} \frac{2^{d-1}}{\pi^{d+1}} \sum_{m\in C_j} \int e^{-i\lambda s}\phi((2m+d)\lambda)\widetilde{R}^*((2m+d)\lambda)L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda.$$

The term h_j^1 is dealt with exactly in the same way as in **[7]** and **[6]**.

For h_j^2 we shall use the Taylor formula

$$\phi(2^{-2j}\lambda) - \phi((2m+d)\lambda) = (2^{-2j} - (2m+d))\lambda \int_0^1 \phi'(t2^{-2j}\lambda + (1-t)(2m+d)\lambda)\, dt.$$

But for any $m \in C_j$, one can find $\alpha_m \in [c_2^{-1}, c_1^{-1}]$ such that

$$2^{-2j} = \alpha_m(2m+d).$$

It follows that one can write

$$\phi(2^{-2j}\lambda) - \phi((2m+d)\lambda) = (\alpha_m - 1)(2m+d)\lambda \int_0^1 \phi'\left([t\alpha_m + (1-t)](2m+d)\lambda\right)\, dt$$

and the change of variables $u = t\alpha_m + (1-t)$ gives

$$\begin{aligned} \widetilde{R}^*((2m+d)\lambda)\left(\phi(2^{-2j}\lambda) - \phi((2m+d)\lambda)\right) &= (2m+d)\lambda\widetilde{R}^*((2m+d)\lambda) \\ &\quad \times \int_{\mathbb{R}} \phi'\left(u(2m+d)\lambda\right) \mathbf{1}_{[1,\alpha_m]} du. \end{aligned}$$

This form is of the same kind that considered in [7], and allows to end the proof of (4.4.7) exactly in the same way.

Let us prove now (4.4.8). On the support of the Fourier transform of $\Delta_p\Lambda_m$, we have $D_\lambda \sim 2^{2p}$ and $|\lambda| \sim 2^{2m}$. Therefore, $2^{2(p-m)}$ has to be greater than or equal to 1. This implies that the only indexes (p,m) that we have to consider are those such that $0 < m \leq p$. So

$$\Lambda_m f = \Lambda_m(\mathrm{Id} - S_{m-1})f.$$

Therefore using (4.4.7), we have

$$\begin{aligned}\|\Lambda_m f\|_{L^\infty(\mathbb{H}^d)} &\leq C \sum_{q\geq m-1} \|\Lambda_m \Delta_q f\|_{L^\infty(\mathbb{H}^d)}\\ &\leq C \sum_{q\geq m-1} \|\Delta_q f\|_{L^\infty(\mathbb{H}^d)}\\ &\leq C \sum_{q\geq m-1} 2^{-q\rho}\|f\|_{C^\rho(\mathbb{H}^d)},\end{aligned}$$

so finally

$$\|\Lambda_m f\|_{L^\infty(\mathbb{H}^d)} \leq C 2^{-m\rho}\|f\|_{C^\rho(\mathbb{H}^d)}.$$

That proves the lemma. □

4.5. The symbol of Littlewood-Paley operators

Applying Proposition 1.16 of Chapter 1 (see its statement page 20) to λ-dependent functions of the harmonic oscillator, we obtain the symbol of our Littlewood-Paley operators, as stated in the next proposition. The proof of the proposition relies heavily on that of Proposition 1.16 which is itself proved in Appendix B. Therefore we postpone the proof also to Appendix B, page 119.

Proposition 4.18. — *The operators Δ_p (resp. S_p) are pseudodifferental operators of order 0. Besides, if we denote by $\Phi_p(\lambda,\xi,\eta)$ (resp. $\Psi_p(\lambda,\xi,\eta)$) their symbols, there exist two functions ϕ and ψ in $\mathscr{C}^\infty(\mathbb{R}^2)$ such that for $\lambda \neq 0$,*

$$\Phi_p(\lambda,\xi,\eta) = \phi(2^{-2p}|\lambda|, 2^{-2p}|\lambda|(\xi^2+\eta^2)) \text{ and } \Psi_p(\lambda,\xi,\eta) = \psi(2^{-2p}|\lambda|, 2^{-2p}|\lambda|(\xi^2+\eta^2)).$$

More precisely one has

$$\forall \lambda \neq 0, \quad \phi(\lambda,\rho) = \frac{sgn\,\lambda}{\lambda}\int (\cos\tau)^{-d} e^{\frac{i}{\lambda}(-r\tau+\rho\,\mathrm{tg}\tau)} R^*(4r)\,d\tau dr, \tag{4.5.1}$$

and a similar formula for ψ.

Remark 4.19. — *The stationary phase theorem (see Appendix B) implies that the function $\phi(\lambda,\rho)$ of (4.5.1) has an asymptotic expansion in powers of λ as λ goes* *to* 0, *the first term of which is* $R^*(\rho)$. *Besides, the change of variables* $\tau \mapsto -\tau$ *gives that* $\phi(-\lambda, \rho) = \phi(\lambda, \rho)$. *Therefore, the function*

$$(y, \eta) \mapsto \Phi_p\left(\lambda, \operatorname{sgn}(\lambda)\frac{\xi}{\sqrt{|\lambda|}}, \frac{\eta}{\sqrt{|\lambda|}}\right)$$

is equal to $\phi(2^{-2p}|\lambda|, 2^{-2p}(\xi^2 + \eta^2))$ *and is smooth close to* $\lambda = 0$.

CHAPTER 5

THE ACTION OF PSEUDODIFFERENTIAL OPERATORS ON SOBOLEV SPACES

In this chapter we shall be giving the proof of Theorem 5. In the first paragraph we reduce the study to the case of operators of order zero, and in the second paragraph we show that it is possible to restrict our attention to a fixed regularity index in a certain range. We then follow the strategy of the proof of continuity of pseudodifferential operators in the $\mathbb{R}^d$ case due to R. Coifman and Y. Meyer **[20]**. The proof is based on the two following ideas: we introduce the notion of reduced symbols (see Section 5.3) of which we prove the continuity. Then, we obtain in Section 5.4 that any symbol a of order 0 on the Heisenberg group is a sum of a convergent series of reduced symbols, and finally deduce the continuity for the operator $\mathrm{Op}(a)$.

Let us mention that the proof below would be much easier if the symbols were only functions of (w, y, η), and not also of λ : in that case, one would not need to use an additional cutoff in λ via the operators Λ_p (see Section 5.5), which will induce some technicalities.

5.1. Reduction to the case of operators of order zero

In this paragraph we shall reduce the study to the case of zero-order operators. Suppose therefore that the result has been proved for any zero-order operator, meaning that for any operator $b \in S_{\mathbb{H}^d}(0)$ of regularity $C^\rho(\mathbb{H}^d)$ and for any $|s| \leq \rho$ if $\rho > 2(2d+1)$ (resp. $0 < s < \rho$ if $\rho > 0$), the operator $\mathrm{Op}(b)$ maps continuously $H^s(\mathbb{H}^d)$ into itself.

Let a be a symbol of order $\mu \in \mathbb{R}$. Then for any $f \in H^s(\mathbb{H}^d)$,

$$\mathrm{Op}(a)f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int \mathrm{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda) A_\lambda(w)\right) |\lambda|^d \, d\lambda$$

with

$$\begin{aligned} \mathcal{F}(f)(\lambda) A_\lambda(w) &= \mathcal{F}(f)(\lambda) J^*_\lambda \mathrm{op}^w(a(w,\lambda)) J_\lambda \\ &= \mathcal{F}((\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{\mu}{2}} f)(\lambda) J^*_\lambda \mathrm{op}^w(m^{(\lambda)}_{-\mu} \# a) J_\lambda . \end{aligned}$$

This can be written

$$\mathrm{Op}(a)f(w) = \mathrm{Op}(b)(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{\mu}{2}} f(w),$$

where $b \stackrel{\text{def}}{=} m_{-\mu}^{(\lambda)} \# a$ is a symbol of order 0. The boundedness of $\mathrm{Op}(b)$ from $H^{s-\mu}$ to $H^{s-\mu}$ for $|s - \mu| < \rho$ (resp. $0 < s < \rho$ if $\rho > 0$) then yields the existence of constants C and C' such that

$$\|\mathrm{Op}(a)f\|_{H^{s-\mu}} \leq C \, \|(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{\mu}{2}} f\|_{H^{s-\mu}} \leq C' \, \|f\|_{H^s}.$$

Therefore it suffices to prove the theorem for symbols of order 0, which we will assume from now on.

5.2. Reduction to the case of a fixed regularity index

In this paragraph, we shall reduce the study of the continuity of pseudodifferential operators of order 0 on Sobolev spaces from arbitrary Sobolev spaces $H^t(\mathbb{H}^d)$, to one Sobolev space $H^s(\mathbb{H}^d)$ with a regularity index s such that $0 < s < \delta_0$, where δ_0 (chosen equal to $\rho - [\rho]$) will be the index entering the assumptions of Proposition 4.14, page 68.

In order to do so, let us suppose that the continuity in $H^s(\mathbb{H}^d)$ is proved for any symbol of order 0 with $0 < s < \delta_0$ (note that $\delta_0 \leq \rho$). Consider a symbol $a(w, \lambda, \xi, \eta)$ of order 0. Let α be a multi-index in $\mathbb{N}^d$ with $|\alpha| \leq [\rho]$ and, using Proposition 2.9, define the $\mathscr{C}^{\delta_0}$ symbol b_α by

$$\mathrm{Op}(b_\alpha) = \mathcal{Z}^\alpha \mathrm{Op}(a)(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{-\frac{|\alpha|}{2}}.$$

Then $\mathrm{Op}(b_\alpha)$ maps $H^t(\mathbb{H}^d)$ into itself for $0 < t < \delta_0$. Therefore, there exists a constant C such that for any $f \in H^{t+[\rho]}(\mathbb{H}^d)$,

$$\begin{aligned} \|\mathrm{Op}(a)f\|^2_{H^{t+[\rho]}(\mathbb{H}^d)} &= \sum_{|\alpha| \leq [\rho]} \|\mathrm{Op}(b_\alpha)(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{|\alpha|}{2}} f\|^2_{H^t(\mathbb{H}^d)} \\ &\leq C \sum_{|\alpha| \leq [\rho]} \|(\mathrm{Id} - \Delta_{\mathbb{H}^d})^{\frac{|\alpha|}{2}} f\|^2_{H^t(\mathbb{H}^d)} = C \, \|f\|^2_{H^{t+[\rho]}(\mathbb{H}^d)}. \end{aligned}$$

Therefore, $\mathrm{Op}(a)$ maps $H^s(\mathbb{H}^d)$ into itself for $s = t + [\rho]$, $t < \delta_0$, whence for $0 < s < \rho$.

Assuming $\rho > 2(2d+1)$ and using the fact that the adjoint of a pseudodifferential operator is a pseudodifferential operator of the same order, we get the continuity on $H^s(\mathbb{H}^d)$ for $0 < |s| < \rho$.

Then $s = 0$ is obtained by interpolation.

5.3. Reduced and reduceable symbols

Let us start by defining the notion of reduced and reduceable symbols.

Definition 5.1. — *Let t be a symbol. Then t is reduceable if it can be decomposed in the following way: for all $(w,\lambda,\xi,\eta)\in\mathbb{H}^d\times\mathbb{R}^*\times\mathbb{R}^{2d}$*

$$\begin{aligned} t(w,\lambda,\xi,\eta) &= \sum_{k\in\mathbb{Z}^{2d}} t^k(w,\lambda,\xi,\eta), \quad \textit{where} \\ t^k(w,\lambda,\xi,\eta) &= b^k_{-1}(w,\lambda)\Psi^k(\lambda,\xi,\eta)+\sum_{p=0}^{\infty} b^k_p(w,\lambda)\Phi^k_p(\lambda,\xi,\eta). \end{aligned}$$

with

$$\Phi^k_p(\lambda,\xi,\eta)\stackrel{\text{def}}{=}\widetilde{\Phi}^k_p(\sqrt{|\lambda|}\xi,\sqrt{|\lambda|}\eta)\ \text{ while }\ \widetilde{\Phi}^k_p(\xi,\eta)\stackrel{\text{def}}{=}e^{ik\cdot(2^{-p}\xi,2^{-p}\eta)}\Phi(2^{-2p}(\xi^2+\eta^2))$$

and Φ is a smooth function with values in $[0,1]$, compactly supported in $]0,\infty[$.

Similarly

$$\Psi^k(\lambda,\xi,\eta)\stackrel{\text{def}}{=}\widetilde{\Psi}^k(\sqrt{|\lambda|}\xi,\sqrt{|\lambda|}\eta)\ \text{ where }\ \widetilde{\Psi}^k(\xi,\eta)\stackrel{\text{def}}{=}e^{ik\cdot(\xi,\eta)}\Psi(\xi^2+\eta^2)$$

and Ψ a smooth function with values in $[0,1]$, compactly supported in $]-1,1[$.

Finally the functions $b^k_p(\cdot,\lambda)$ belong to the Hölder space $C^\rho(\mathbb{H}^d)$ with

$$\sup_{p,\lambda}\|b^k_p(\cdot,\lambda)\|_{C^\rho(\mathbb{H}^d)}=A_k<\infty. \tag{5.3.1}$$

The symbols t^k are called reduced symbols.

It follows from the analysis of the examples of Chapter 2, Section 2.1 that for any $k\in\mathbb{Z}^{2d}$ and $p\in\mathbb{N}$, the operator $\mathrm{Op}(b^k_p(w,\lambda)\Phi^k_p(\lambda,\xi,\eta))$ is bounded in $H^s(\mathbb{H}^d)$ since one can write by easy functional calculus

$$\mathrm{Op}\Big(b^k_p(w,\lambda)\Phi^k_p(\lambda,\xi,\eta)\Big)=\mathrm{Op}\Big(b^k_p(w,\lambda)\Big)\circ\mathrm{Op}\Big(\Phi^k_p(\lambda,\xi,\eta)\Big)$$

where the two operators of the right-hand side are bounded operators on $H^s(\mathbb{H}^d)$ (see Chapter 1, Sections 2.1.2 and 2.1.4 respectively).

The same fact is true for $\mathrm{Op}\Big(b^k_{-1}(w,\lambda)\Psi^k(\lambda,\xi,\eta)\Big)$. Besides, by Proposition 2.2 stated page 29, there is a constant C (independent of k) such that

$$\begin{aligned} \|\mathrm{Op}(b^k_{-1}(w,\lambda)\Psi^k(\lambda,\xi,\eta))\|_{\mathcal{L}(H^s(\mathbb{H}^d))} &\leq C\,A_k\,\|\widetilde{\Psi}^k\|_{n;S(1,g)} \quad \text{and} \\ \|\mathrm{Op}(b^k_p(w,\lambda)\Phi^k_p(\lambda,\xi,\eta))\|_{\mathcal{L}(H^s(\mathbb{H}^d))} &\leq C\,A_k\,\|\widetilde{\Phi}^k_p\|_{n;S(1,g)} \end{aligned} \tag{5.3.2}$$

where we recall that g is the harmonic oscillator metric of Section 1.3.2 in Chapter 1.

The main ingredient in the proof of Theorem 5 is the following result.

Proposition 5.2. — *Let k be fixed in $\mathbb{Z}^{2d}$ and t^k be a reduced symbol as defined in Definition 5.1. The operator $\mathrm{Op}(t^k)$ maps continuously $H^s(\mathbb{H}^d)$ into itself for $0\leq s<\rho$. Its operator norm is bounded by $CA_k(1+|k|)^n$ for some integer n, where C is a constant (independent of k).*

The proof of this proposition is postponed to Section 5.5.

Remark 5.3. — *Due to Proposition 5.2, a reduceable symbol t is the symbol of a bounded operator on $H^s(\mathbb{H}^d)$ as soon as $(A_k(1+|k|)^n)_{k\in\mathbb{Z}^{2d}}$ belongs to $\ell^1(\mathbb{Z}^{2d})$.*

5.4. Decomposition into reduced symbols and proof of the theorem

The aim of this section is to prove the following lemma.

Lemma 5.4. — *Let a be a symbol of order 0. Then a is reduceable and, with the notation of Definition 5.1, for any integer N, there is a constant C_N such that for any $k\in\mathbb{Z}^{2d}$,*

$$A_k \leq \frac{C_N}{(1+|k|)^N}. \tag{5.4.1}$$

In view of Remark 5.3, Lemma 5.4 gives directly Theorem 5 (up to the proof of Proposition 5.2).

Proof. — Let us consider ψ and ϕ defining a partition of unity as in (4.4.1) page 73: one can write

$$\forall(\lambda,\xi,\eta)\in\mathbb{R}^*\times\mathbb{R}^{2d},\quad \psi\left(|\lambda|(\xi^2+\eta^2)\right)+\sum_{p\geq 0}\phi\left(2^{-2p}|\lambda|(\xi^2+\eta^2)\right)=1. \tag{5.4.2}$$

Then

$$\begin{aligned} a(w,\lambda,\xi,\eta) &= a(w,\lambda,\xi,\eta)\psi\left(|\lambda|(\xi^2+\eta^2)\right)+\sum_{p\geq 0}a(w,\lambda,\xi,\eta)\phi\left(2^{-2p}|\lambda|(\xi^2+\eta^2)\right)\\ &= b_{-1}(w,\lambda,\sqrt{|\lambda|}\xi,\sqrt{|\lambda|}\eta)+\sum_{p\geq 0}b_p(w,\lambda,2^{-p}\sqrt{|\lambda|}\xi,2^{-p}\sqrt{|\lambda|}\eta)\end{aligned}$$

with

$$b_{-1}(w,\lambda,\xi,\eta) \stackrel{\text{def}}{=} \widetilde{a}(w,\lambda,\xi,\eta)\psi(\xi^2+\eta^2) \quad\text{and}$$

$$b_p(w,\lambda,\xi,\eta) \stackrel{\text{def}}{=} \widetilde{a}(w,\lambda,2^p\xi,2^p\eta)\phi(\xi^2+\eta^2) \quad \text{for} \quad p\geq 0,$$

where $\widetilde{a}(w,\lambda,\xi,\eta)\stackrel{\text{def}}{=}a(w,\lambda,\frac{\xi}{\sqrt{|\lambda|}},\frac{\eta}{\sqrt{|\lambda|}})$. The functions b_p are compactly supported in (ξ,η), in the ring $\mathcal{C}$ for $p\geq 0$ and in the ball $\mathcal{B}$ for $p=-1$. Moreover, denoting by ∂ a differentiation in ξ or η, we have, for all $p\geq -1$,

$$\partial b_p(w,\lambda,\xi,\eta)=2^p(\partial\widetilde{a})(w,\lambda,2^p\xi,2^p\eta)\phi(\xi^2+\eta^2)+2\,\xi\phi'(\xi^2+\eta^2)\widetilde{a}(w,\lambda,2^p\xi,2^p\eta).$$

We deduce that

$$|\partial b_p(w,\lambda,\xi,\eta)|\leq C\frac{2^p}{\sqrt{1+|\lambda|+(2^p\xi)^2+(2^p\eta)^2}}|\phi(\xi^2+\eta^2)|+C|\xi|\,|\phi'(\xi^2+\eta^2)|\,,$$

and

$$|\lambda\partial_\lambda b_p(w,\lambda,\xi,\eta)|\leq C|\lambda\partial_\lambda\widetilde{a}(w,\lambda,2^p\xi,2^p\eta)|$$

so using the boundedness of the symbol norm of a and the fact that ϕ is compactly supported, and arguing similarly for higher order derivatives, one gets the following uniform norm bound on b_p:

$$\sup_{p,\lambda,\xi,\eta} \|(\lambda\partial_\lambda)^m \partial^\beta_{(\xi,\eta)} b_p(\cdot,\lambda,\xi,\eta)\|_{C^\rho(\mathbb{H}^d)} \leq C_{\beta,m}. \tag{5.4.3}$$

Now, since for $p \geq 0$ the functions b_p are compactly supported in (ξ,η), in a ring $\mathscr{C}$ independent of p, we can write a decomposition in Fourier series:

$$b_p(w,\lambda,\xi,\eta) = \sum_{k\in\mathbb{Z}^{2d}} \mathrm{e}^{ik\cdot(\xi,\eta)} b_p^k(w,\lambda)\widetilde{\phi}(\xi^2+\eta^2),$$

where $\widetilde{\phi}$ is a smooth, radial function, compactly supported in a unit ring, so that $\phi\widetilde{\phi} = \phi$. We have of course

$$b_p^k(w,\lambda) = \frac{1}{(2\pi)^d}\int_{\mathscr{C}} \mathrm{e}^{-ik\cdot(\xi,\eta)} b_p(w,\lambda,\xi,\eta)d\xi d\eta. \tag{5.4.4}$$

Along the same lines, we get

$$b_{-1}(w,\lambda,\xi,\eta) = \sum_{k\in\mathbb{Z}^{2d}} \mathrm{e}^{ik\cdot(\xi,\eta)} b_{-1}^k(w,\lambda)\widetilde{\psi}(\xi^2+\eta^2),$$

where $\widetilde{\psi}$ is a smooth, radial function, compactly supported in a unit ball, so that $\psi\widetilde{\psi} = \psi$.

Defining

$$\Phi^k(\xi,\eta) \stackrel{\text{def}}{=} \mathrm{e}^{ik\cdot(\xi,\eta)}\widetilde{\phi}(\xi^2+\eta^2),$$

it turns out that

$$\begin{aligned} a(w,\lambda,\xi,\eta) &= b_{-1}(w,\lambda,\sqrt{|\lambda|}\xi,\sqrt{|\lambda|}\eta) + \sum_{p,k} b_p^k(w,\lambda)\Phi^k(2^{-p}\sqrt{|\lambda|}\xi, 2^{-p}\sqrt{|\lambda|}\eta) \\ &= b_{-1}(w,\lambda,\sqrt{|\lambda|}\xi,\sqrt{|\lambda|}\eta) + \sum_k t^k(w,\lambda,\xi,\eta). \end{aligned}$$

That concludes the fact that a is reduceable. It remains to prove (5.4.1). From the integral formula (5.4.4), we infer that for any multi-index β and, to simplify, for $p \geq 0$

$$\begin{aligned} \left|k^\beta b_p^k(w,\lambda)\right| &= \left|\frac{1}{(2\pi)^d}\int_{\mathscr{C}} k^\beta \mathrm{e}^{-ik\cdot(\xi,\eta)} b_p(w,\lambda,\xi,\eta)\,d\xi d\eta\right| \\ &\leq C\int_{\mathscr{C}} \left|\partial^\beta_{(\xi,\eta)} b_p(w,\lambda,\xi,\eta)\right| d\xi d\eta \end{aligned}$$

Using (5.4.3), we deduce that

$$\sup_{p,\lambda} \left\|k^\beta b_p^k(\cdot,\lambda)\right\|_{C^\rho(\mathbb{H}^d)} \leq C_\beta \tag{5.4.5}$$

and Lemma 5.4 is proved. □

5.5. Proof of Proposition 5.2

Now it remains to prove Proposition 5.2. We will first give the main steps of the proof and peform some reductions, and then prove the result.

5.5.1. Reductions. — Let us give the main steps of the proof. An easy computation gives that there is a constant C such that for any integer p and any $k \in \mathbb{Z}^{2d}$,

$$\|\widetilde{\Psi}^k\|_{n;S(1,g)} + \|\widetilde{\Phi}^k_p\|_{n;S(1,g)} \leq C\,(1+|k|)^n. \tag{5.5.1}$$

Therefore, in view of (5.3.2), one has

$$\left\|\operatorname{Op}\left(b^k_{-1}(w,\lambda)\Psi^k(\lambda,\xi,\eta)\right)\right\|_{\mathcal{L}(H^s(\mathbb{H}^d))} \leq CA_k(1+|k|)^n.$$

It remains to consider $p \in \mathbb{N}$, and in particular to control the sum over p. The fact that $b^k_p(w,\lambda)$ depends on λ induces a serious difficulty, which we shall deal with by considering a partition of unity in λ. Thus by the same trick as before, we use functions ϕ and ψ such that (4.4.1) holds and we write

$$b^k_p(w,\lambda) = b^k_p(w,\lambda)\psi(\lambda) + \sum_{r\in\mathbb{N}} b^k_p(w,\lambda)\phi(2^{-2r}\lambda).$$

Using the fact that ϕ is compactly supported, we decompose the function $b^k_p(w,2^{2r}\lambda)\phi(\lambda)$ in Fourier series and write

$$b^k_p(w,\lambda) = \sum_{j\in\mathbb{Z}} b^{kj}_{p,-1}(w)\mathrm{e}^{ij\lambda}\widetilde{\psi}(\lambda) + \sum_{r\in\mathbb{N},j\in\mathbb{Z}} b^{kj}_{p,r}(w)\mathrm{e}^{ij2^{-2r}\lambda}\widetilde{\phi}(2^{-2r}\lambda),$$

where

$$b^{kj}_{p,-1}(w) = \int_{\mathscr{B}} \mathrm{e}^{-ij\lambda} b^k_p(w,\lambda)\psi(\lambda)\,d\lambda,\quad b^{kj}_{p,r}(w) = \int_{\mathscr{C}} \mathrm{e}^{-ij\lambda} b^k_p(w,2^{2r}\lambda)\phi(\lambda)\,d\lambda$$

and $\widetilde{\phi}$, $\widetilde{\psi}$ are smooth and compactly supported respectively in $\mathscr{C}$ and $\mathscr{B}$, such that $\widetilde{\phi}\phi = \phi$, and $\widetilde{\psi}\psi = \psi$. We observe that Estimate (5.4.3) satisfied by b_p ensures that for all integers N, there is a constant C_N such that for all indexes p, r, j, k, we have

$$\sup_{p,r}(1+|j|)^N \|b^{kj}_{p,r}\|_{C^\rho(\mathbb{H}^d)} \leq \frac{C_N}{(1+|k|)^N}. \tag{5.5.2}$$

Indeed, by the Leibniz formula

$$\begin{aligned}
\left|j^n k^\beta b^{kj}_{p,r}(w)\right| &\leq C\sum_{m\leq n}\left|\int_{\mathscr{C}} \mathrm{e}^{-ij\lambda}(\lambda 2^{2r})^m k^\beta (\partial^m_\lambda b^k_p)(w,2^{2r}\lambda)\lambda^{-m}(\partial^{n-m}_\lambda\phi)(\lambda)\,d\lambda\right| \\
&\leq C\sup_{\substack{\mu\\ m\leq n}}\left|k^\beta(\mu\partial_\mu)^m b^k_p(w,\mu)\right| \\
&\leq C\sup_{\substack{\lambda\\ m\leq n}}\left|(\lambda\partial_\lambda)^m\partial^\beta_{(\xi,\eta)} b_p(w,\lambda,y,\eta)\right|.
\end{aligned}$$

Owing to (5.4.3), we deduce that (5.5.2) holds. That estimate will ensure the convergence in j of the series. In the following, we therefore consider, for each j and k, the quantities

$$\begin{aligned}\widetilde{t}^{kj}(w,\lambda,\xi,\eta) &\stackrel{\text{def}}{=} \sum_p b^{kj}_{p,-1}(w)\psi^j(\lambda)\Phi^k_p(\lambda,\xi,\eta) \text{ and}\\ t^{kj}(w,\lambda,\xi,\eta) &\stackrel{\text{def}}{=} \sum_{p,r} b^{kj}_{p,r}(w)\phi^j(2^{-2r}\lambda)\Phi^k_p(\lambda,\xi,\eta)\end{aligned}$$

where $\phi^j(\lambda) = \mathrm{e}^{ij\lambda}\phi(\lambda)$, and $\psi^j(\lambda) = \mathrm{e}^{ij\lambda}\psi(\lambda)$. Then we will consider the summation in k and j of t^{kj} and $\widetilde{t}^{kj}$.

The analysis of the convergence of $\widetilde{t}^{kj}$ follows the same lines as that of t^{kj} with great simplifications since the summation is only on one index, namely p. Therefore, we focus on the convergence of t^{kj} and leave to the reader the easy adaptation of the proof to the case of $\widetilde{t}^{kj}$.

Let us therefore now study t^{kj}. We truncate $b^{kj}_{p,r}$ into high and low frequencies, by defining (for some integer M to be chosen large enough later, independently of all the other summation indices),

$$\ell_{pr} \stackrel{\text{def}}{=} S_{p-M} b^{kj}_{p,r} \quad \text{and} \quad h_{pr} \stackrel{\text{def}}{=} (\mathrm{Id} - S_{p-M}) b^{kj}_{p,r}, \tag{5.5.3}$$

where S_p is a Littlewood-Paley truncation operator on the Heisenberg group, as defined in Chapter 4, Section 4.1. Let us notice that by Lemma 4.13, one has the following norm estimates on ℓ_{pr} and h_{pr}:

$$\begin{aligned}\sup_{p,r} \|\ell_{pr}\|_{C^\rho(\mathbb{H}^d)} &\le \sup_{p,r} \|b^{kj}_{p,r}\|_{C^\rho(\mathbb{H}^d)}\\ \sup_r \|h_{pr}\|_{L^\infty(\mathbb{H}^d)} &\le 2^{-p\rho} \sup_{p,r} \|b^{kj}_{p,r}\|_{C^\rho(\mathbb{H}^d)}\\ \sup_r \|h_{pr}\|_{C^\sigma(\mathbb{H}^d)} &\le 2^{-p(\rho-\sigma)} \sup_{p,r} \|b^{kj}_{p,r}\|_{C^\rho(\mathbb{H}^d)},\end{aligned}$$

for $0 < \sigma \le \rho$.

This allows us to write $t^{kj} = \widetilde{t}^\sharp + \widetilde{t}^\flat$, with

$$\begin{aligned}\widetilde{t}^\sharp(w,\lambda,\xi,\eta) &\stackrel{\text{def}}{=} \sum_{p,r} h_{pr}(w)\phi^j(2^{-2r}\lambda)\Phi^k(2^{-p}\sqrt{|\lambda|}\,\xi, 2^{-p}\sqrt{|\lambda|}\,\eta) \quad \text{and}\\ \widetilde{t}^\flat(w,\lambda,\xi,\eta) &\stackrel{\text{def}}{=} \sum_{p,r} \ell_{pr}(w)\phi^j(2^{-2r}\lambda)\Phi^k(2^{-p}\sqrt{|\lambda|}\,\xi, 2^{-p}\sqrt{|\lambda|}\,\eta).\end{aligned}$$

We have dropped the indexes k and j to avoid too heavy notations. Before performing the study of each of those operators, we begin by a remark which will happen to be crucial for our purpose.

5.5.2. Spectral localization. — In this subsection, we take advantage of Proposition 4.14 of Chapter 4 (see page 68) to use spectral localisation. We first observe that

$$\begin{aligned}\Phi_p^k(\lambda,\xi,\eta) &= e^{i\sqrt{|\lambda|}k\cdot(2^{-p}\xi,2^{-p}\eta)}\Phi(2^{-2p}|\lambda|(\xi^2+\eta^2))\\ &= e^{i\sqrt{|\lambda|}k\cdot(2^{-p}\xi,2^{-p}\eta)}\Phi(2^{-2p}|\lambda|(\xi^2+\eta^2))\widetilde{\Phi}(2^{-2p}|\lambda|(\xi^2+\eta^2)),\end{aligned}$$

where $\widetilde{\Phi}$ is a smooth radial function compactly supported in a unit ring so that $\Phi\widetilde{\Phi} = \Phi$.

Symbolic calculus gives that for any $N \in \mathbb{N}$, there exists a symbol $r_{k,p}^{(N)}$ such that

$$\begin{aligned}\mathrm{op}^w(\Phi_p^k) &= \mathrm{op}^w(\Phi_p^k \cdot a_p)\\ &= \mathrm{op}^w(\Phi_p^k)\circ \mathrm{op}^w(a_p) + \mathrm{op}^w(r_{k,p}^{(N)}),\end{aligned}$$

where $a_p(y,\eta) = \widetilde{\Phi}(2^{-2p}|\lambda|(y^2+\eta^2))$ and for any integer n one has

$$\|r_{k,p}^{(N)}\|_{n;S(1,g)} \leq C(1+|k|)^{N+n}2^{-Np}.$$

One obtains that for some integer n,

$$\|\mathrm{op}^w(r_{k,p}^{(N)})\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C\,(1+|k|)^{N+n}\,2^{-Np},$$

and since $\mathrm{Op}(r_{k,p}^{(N)})$ is a Fourier multiplier we get

$$\|\mathrm{Op}(r_{k,p}^{(N)})u\|_{H^s(\mathbb{H}^d)} \leq C2^{-Np}(1+|k|)^{N+n}\|u\|_{H^s(\mathbb{H}^d)}. \tag{5.5.4}$$

Since we deal with Fourier multipliers, we have

$$\mathrm{Op}(\Phi_p^k)u = \mathrm{Op}(a_p)\mathrm{Op}(\Phi_p^k)u + \mathrm{Op}(r_{k,p}^{(N)})u.$$

Finally, by Proposition 4.14 of Chapter 4, we get

$$\begin{aligned}\mathrm{Op}(\Phi_p^k)u &= \Delta_p\mathrm{Op}(a_p)\mathrm{Op}(\Phi_p^k)u + \sum_{q\neq p}\Delta_q\mathrm{Op}(a_p)\mathrm{Op}(\Phi_p^k)u + \mathrm{Op}(r_{k,p}^{(N)})u\\ &= \Delta_p\mathrm{Op}(a_p)\mathrm{Op}(\Phi_p^k)u + \sum_{q\neq p}\Delta_q R_{p,q}\mathrm{Op}(\Phi_p^k)u + \mathrm{Op}(r_{k,p}^{(N)})u, \qquad (5.5.5)\end{aligned}$$

where

$$\|R_{p,q}\|_{\mathcal{L}(H^s(\mathbb{H}^d))} \leq C2^{-\delta_0|p-q|}. \tag{5.5.6}$$

Therefore we can write

$$\mathrm{Op}(t) = \mathrm{Op}(t^\sharp) + \mathrm{Op}(t^\flat) + \mathrm{Op}(t^\natural)$$

with, writing $\phi^j_r(\lambda) = \phi^j(2^{-2r}\lambda)$

$$(5.5.7)\qquad \mathrm{Op}(t^\sharp) = \sum_{p,r} h_{pr}(w)\Lambda_r\Delta_p\mathrm{Op}(a_p)\mathrm{Op}(\phi^j_r\Phi^k_p) + \sum_{\substack{p,r\\ q\neq p}} h_{pr}(w)\Lambda_r\Delta_q R_{p,q}\mathrm{Op}(\phi^j_r\Phi^k_p)$$

$$(5.5.8)\qquad \mathrm{Op}(t^\flat) = \sum_{p,r} \ell_{pr}(w)\Lambda_r\Delta_p\mathrm{Op}(a_p)\mathrm{Op}(\phi^j_r\Phi^k_p) + \sum_{\substack{p,r\\ q\neq p}} \ell_{pr}(w)\Lambda_r\Delta_q R_{p,q}\mathrm{Op}(\phi^j_r\Phi^k_p) \quad \text{and}$$

$$(5.5.9)\qquad \mathrm{Op}(t^\natural) = \sum_{p,r} b^{kj}_{p,r}(w)\Lambda_r\mathrm{Op}(r^{(N)}_{k,p})$$

with $\Lambda_r = \mathrm{Op}(\widetilde{\phi}(2^{-2r}\lambda))$ and $\widetilde{\phi}$ is a compactly supported function in $\mathscr{C}$ such that $\widetilde{\phi}\phi^j = \phi^j$.

In the following, we are going to study each of these three terms, beginning by $\mathrm{Op}(t^\natural)$ which is a remainder term. Besides, in order to simplify the notation we shall write

$$u^{kj}_{pr} \stackrel{\text{def}}{=} \mathrm{Op}(\phi^j_r\Phi^k_p)u,$$

and we recall that due to (5.5.1) and to the fact that $\mathrm{Op}(\phi^j_r\Phi^k_p) = \mathrm{Op}(\phi^j_r)\mathrm{Op}(\Phi^k_p)$ with $\mathrm{Op}(\phi^j_r)$ of norm 1, there is a constant C such that for all indexes p, r, k, j,

$$(5.5.10)\qquad \|u^{kj}_{pr}\|_{H^s} \leq C(1+|k|)^n\|u\|_{H^s}.$$

Moreover, by quasi-orthogonality (see Chapter 4, Subsection 4.4), we have

$$(5.5.11)\qquad \|\Delta_p\Lambda_r u^{kj}_{pr}\|_{L^2} \leq C\,(1+|k|)^n\, c_p\, c_r\, 2^{-ps}\,\|u\|_{H^s}$$

where C is a constant and c_p, c_r denote from now on generic elements of the unit sphere of $\ell^2(\mathbb{Z})$.

5.5.3. The remainder term. — We drop the kj-exponent in $b^{kj}_{p,r}$ for simplicity and decompose $b_{p,r}$ in λ-frequencies: $b_{p,r} = \sum_m \Lambda_m b_{p,r}$ so that $\mathrm{Op}(t^\natural)$ is now a sum on three indices. We decompose this sum into two parts, depending on whether $r \leq m + M_1$ or $r \geq m + M_1$ where M_1 is the threshold of Proposition 4.15 stated page 74.

Let us consider the first case, when $r \leq m + M_1$. We choose σ such that $s < \sigma < \rho$ and by Lemma 4.17 page 76, we find constants C such that

$$\begin{aligned}\|\Lambda_m(b_{p,r})\,\Lambda_r\mathrm{Op}(r^{(N)}_{k,p})u\|_{H^s(\mathbb{H}^d)} &\leq C\,\|\Lambda_m(b_{p,r})\|_{C^\sigma(\mathbb{H}^d)}\|\mathrm{Op}(r^{(N)}_{k,p})u\|_{H^s(\mathbb{H}^d)}\\ &\leq C\,2^{-m(\rho-\sigma)}A_k\|\mathrm{Op}(r^{(N)}_{k,p})u\|_{H^s(\mathbb{H}^d)}\\ &\leq C\,2^{-m(\rho-\sigma)}A_k\,2^{-Np}(1+|k|)^{N+n}\|u\|_{H^s(\mathbb{H}^d)}\end{aligned}$$

where we have used estimates (4.4.8) and (5.5.4). We then obtain

$$\Big\| \sum_{m,p,r\leq m+M_1} \Lambda_m(b_{p,r})\,\Lambda_r \mathrm{Op}(r^{(N)}_{k,p})u \Big\|_{H^s(\mathbb{H}^d)} \leq C \Big(\sum_{m,p} (m+M_1) 2^{-m(\rho-\sigma)} 2^{-Np} \Big) (1+|k|)^{N+n} A_k \|u\|_{H^s(\mathbb{H}^d)}$$

which ends the first step.

We now focus on the sum for $r \geq m+M_1$ and we use that by Proposition 4.15, the function $\Lambda_m(b_{p,r})\,\Lambda_r\mathrm{Op}(r^{(N)}_{k,p})u$ is λ-localized in a ring of size 2^r. Therefore, in view of (4.4.4), it is enough to control the $H^s(\mathbb{H}^d)$-norm of $\sum_{p,m} \Lambda_m(b_{p,r})\,\Lambda_r\mathrm{Op}(r^{(N)}_{k,p})u$ by c_r with $(c_r)\in \ell^2$. We observe that by Lemma 4.17 and (4.4.4), there exists a constant C such that

$$\begin{aligned} \|\Lambda_m(b_{p,r})\,\Lambda_r\mathrm{Op}(r^{(N)}_{k,p})u\|_{H^s(\mathbb{H}^d)} &\leq C\|\Lambda_m(b_{p,r})\|_{C^\sigma(\mathbb{H}^d)}\, c_r\, \|\mathrm{Op}(r^{(N)}_{k,p}u\|_{H^s(\mathbb{H}^d)} \\ &\leq C\, 2^{-m(\rho-\sigma)} A_k c_r 2^{-pN}(1+|k|)^{N+n}\|u\|_{H^s(\mathbb{H}^d)} \end{aligned}$$

where $s<\sigma<\rho$ and where we have used again (4.4.8) and(5.5.4). Therefore, we obtain

$$\Big\| \sum_{m,p} \Lambda_m(b_{p,r})\,\Lambda_r\mathrm{Op}(r^{(N)}_{k,p})u \Big\|_{H^s(\mathbb{H}^d)} \leq c_r \Big(\sum_{m,p} 2^{-m(\rho-\sigma)} 2^{-Np} \Big) (1+|k|)^{N+n} A_k \|u\|_{H^s(\mathbb{H}^d)}$$

which achieves the control of the remainder term.

5.5.4. The high frequencies. — Let us estimate $\mathrm{Op}(t^\sharp)u$ in H^s for any $|s|<\rho$. For any function u belonging to $H^s(\mathbb{H}^d)$, we have

$$\mathrm{Op}(t^\sharp)u = \sum_{p,r} (u^\sharp_{pr} + w^\sharp_{pr}) \quad \text{with}$$

$$u^\sharp_{pr} = h_{pr}\Delta_p\Lambda_r\mathrm{Op}(a_p)u^{kj}_{pr} \quad \text{and} \quad w^\sharp_{pr} = \sum_{q\neq p} h_{pr}\Delta_q\Lambda_r R_{p,q} u^{kj}_{pr}.$$

Let us deal with $u^\sharp_{pr}$. As noticed in Chapter 4 Section 4.4, on the support of the Fourier transform of $\Delta_p\mathrm{Op}(\phi(2^{-2r}\lambda))$ we have $D_\lambda \sim 2^{2p}$ and $|\lambda| \sim 2^{2r}$. Therefore, $2^{2(p-r)}$ has to be greater than or equal to 1. This implies that the only indexes (p,r) that we have to consider are those such that $0<r\leq p$. We will then simply bound the sum of norms of the terms $u^\sharp_{pr}$.

To do so, let us choose σ such that $|s|<\sigma<\rho$. This leads, by Lemma 4.13, to the following estimate

$$\|u^\sharp_{pr}\|_{H^s} \leq C 2^{-p(\rho-\sigma)} \|h_{pr}\|_{C^\rho} \|u^{kj}_{pr}\|_{H^s}.$$

Finally, thanks to (5.5.10) and to the definition of h_{pr} recalled in (5.5.3), we obtain for some integer n (recalling that $0 < r \leq p$)

$$\begin{aligned}\sum_{p,r} \|u^\sharp_{pr}\|_{H^s} &\leq C\,(1+|k|)^n \|u\|_{H^s} \sum_p p\, 2^{-p(\rho-\sigma)} \sup_r \|h_{pr}\|_{C^\rho}\\ &\leq C\,(1+|k|)^n \|u\|_{H^s} \sum_p p\, 2^{-p(\rho-\sigma)} A_k.\end{aligned}$$

Since $\sigma < \rho$ and $p \geq -1$, we infer that $u \mapsto \sum_{p,r} u^\sharp_{pr}$ is bounded in the space $\mathcal{L}(H^s(\mathbb{H}^d))$, by the constant $C(1+|k|)^n A_k$.

Let us now study $w^\sharp_{pr}$. Arguing as before, we restrict the sum on the integers r such that $r \leq q$ and we get

$$\sum_{p,r} \|w^\sharp_{pr}\|_{H^s} \leq C \sum_{p,q\neq p} 2^{-p(\rho-\sigma)} q \sup_r \|h_{pr}\|_{C^\rho} 2^{-\delta_0|p-q|}(1+|k|)^n \|u\|_{H^s}.$$

As before, we get a control by $C(1+|k|)^n A_k$.

So the high frequency part of $t^{k,j}$ satisfies the required estimate.

5.5.5. The low frequencies. — We recall that by (5.5.8), we have for any function u belonging to $H^s(\mathbb{H}^d)$

$$\mathrm{Op}(t^\flat)u = \sum_{p,r} (u^\flat_{pr} + w^\flat_{pr}) \quad \text{with}$$

$$u^\flat_{pr} = \ell_{pr}\Delta_p\Lambda_r \mathrm{Op}(a_p) u^{kj}_{pr} \quad \text{and} \quad w^\flat_{pr} = \sum_{q\neq p} \ell_{pr}\Delta_q\Lambda_r R_{p,q}\, u^{kj}_{pr}.$$

In the following, we are going to use the frequency localization induced by Δ_p in the sense of Definition 4.1. In particular, using Proposition 4.1 of **[5]** (the statement is recalled in Proposition 4.9 page 63), we will be able to say something of the localisation of a product of localised terms. We want to use also the localization in λ induced by Λ_r. For that purpose, we truncate ℓ_{pr} and in doing so, we add a new index of summation. We set $\ell_{pr} = \sum_m \Lambda_m \ell_{pr}$ and we immediately remark that since ℓ_{pr} is a low frequency term, then for $m \geq p$ we have $\Lambda_m \ell_{pr} = 0$. Therefore, the index m is controled by p.

According to (4.4.8), one deduce that

$$\|\Lambda_m \ell_{pr}\|_{L^\infty(\mathbb{H}^d)} \leq C 2^{-m\rho} \sup_{p,r} \|b^{kj}_{p,r}\|_{C^\rho(\mathbb{H}^d)}, \tag{5.5.12}$$

where C is a universal constant.

We can now go into the proof of the proposition for $u^\flat_{pr}$. Let us start by studying

$$u^{kj}_{prm} \stackrel{\text{def}}{=} \Lambda_m \ell_{pr} \Delta_p \Lambda_r \mathrm{Op}(a_p) u^{kj}_{pr}.$$

As soon as the threshold M is large enough, u^{kj}_{prm} is frequency localized, in the sense of Definition 4.1, in a ring of size 2^p due to Proposition 4.9 page 63. So we can use Lemma 4.8 to compute the H^s norm of $\sum_p u^{kj}_{prm}$.

Consider the threshold M_1 given by Proposition 4.15. We shall argue differently depending on whether $r \leq m - M_1$, $r \geq m + M_1$, or $|r - m| < M_1$.

For $r \leq m - M_1$, it is enough (due to Lemmas 4.8 and 4.15) to prove that for any $p, m \in \mathbb{N}$,

$$\sum_{r \leq m - M_1} \|u^{kj}_{prm}\|_{L^2} \leq C\, A_k (1+|k|)^n c_p\, c_m \|u\|_{H^s} 2^{-ps}. \tag{5.5.13}$$

We observe that

$$\begin{aligned} \|u^{kj}_{prm}\|_{L^2} &\leq \|\Lambda_m \ell_{pr}\|_{L^\infty} \|\Delta_p \Lambda_r \mathrm{Op}(a_p) u^{kj}_{pr}\|_{L^2} \\ &\leq C\, \|\Lambda_m \ell_{pr}\|_{L^\infty}\, c_p\, c_r (1+|k|)^n\, 2^{-ps}\, \|u\|_{H^s} \end{aligned}$$

by (5.5.10) and (5.5.11). Therefore, for all integers m we have

$$\begin{aligned} \sum_{r \leq m - M_1} \|u^{kj}_{prm}\|_{L^2} &\leq C\, (1+|k|)^n\, c_p\, 2^{-ps}\, \|u\|_{H^s} \sum_{r \leq m - M_1} c_r \|\Lambda_m \ell_{pr}\|_{L^\infty} \\ &\leq C\, (1+|k|)^n\, c_p\, 2^{-ps}\, \|u\|_{H^s} \sqrt{m} \sup_{p,r} \|\Lambda_m \ell_{pr}\|_{L^\infty} \end{aligned}$$

by the Cauchy-Schwartz inequality. So it is enough to have

$$\left\| \sqrt{m} \sup_{p,r} \|\Lambda_m \ell_{pr}\|_{L^\infty} \right\|_{\ell^2(\mathbb{N})} \leq C A_k \tag{5.5.14}$$

to ensure that (5.5.13) is satisfied, which is implied by (5.5.12).

Let us now consider the indexes $r \geq m + M_1$. This time, it is enough to prove

$$\sum_{m \leq r - M_1} \|u^{kj}_{prm}\|_{L^2} \leq C A_k (1+|k|)^n\, c_p\, c_r\, 2^{-ps} \|u\|_{H^s}. \tag{5.5.15}$$

We have, following the same computations as above,

$$\sum_{m \leq r - M_1} \|u^{kj}_{prm}\|_{L^2} \leq C \sum_{m \leq r - M_1} \|\Lambda_m \ell_{pr}\|_{L^\infty} c_p\, c_r\, (1+|k|)^n \|u\|_{H^s} 2^{-ps}.$$

Therefore, if

$$\sum_m \sup_{p,r} \|\Lambda_m \ell_{pr}\|_{L^\infty} \leq C A_k, \tag{5.5.16}$$

we obtain the expected result, namely (5.5.15). Condition (5.5.16) is obviously ensured by (5.5.12) which achieves the estimate of (5.5.15).

Finally, let us consider the case $|r - m| < M_1$. We shall analyze for $j' \in \mathbb{N} \cup \{-1\}$ the quantity $\Lambda_{j'} \left(\Lambda_m \ell_{pr} \Delta_p \Lambda_r \mathrm{Op}(a_p) u^{kj}_p\right)$. We claim that

$$\left\| \sum_{\substack{r,m \\ |r-m| \leq M_1}} \Lambda_{j'} \left(\Lambda_m \ell_{pr} \Delta_p \Lambda_r \mathrm{Op}(a_p) u^{kj}_p\right) \right\|_{L^2} \leq C\, A_k (1+|k|)^n c_{j'}\, c_p \|u\|_{H^s} 2^{-ps}, \tag{5.5.17}$$

which by quasi-orthogonality will prove the result.

We observe indeed that by Proposition 4.15, there exists a constant M_2 such that

$$\sum_{\substack{r,m\\|r-m|\le M_1}} \Lambda_{j'}\left(\Lambda_m \ell_{pr}\Delta_p\Lambda_r \mathrm{Op}(a_p)u^{kj}_{pr}\right) = \sum_{\substack{|r-m|\le M_1\\ r\ge j'-M_2}} \Lambda_{j'}\left(\Lambda_m \ell_{pr}\Delta_p\Lambda_r \mathrm{Op}(a_p)u^{kj}_{pr}\right).$$

Therefore arguing as before,

$$\Bigg\| \sum_{\substack{r,m\\|r-m|\le M_1}} \Lambda_{j'}\left(\Lambda_m \ell_{pr}\Delta_p\Lambda_r \mathrm{Op}(a_p)u^{kj}_{pr}\right)\Bigg\|_{L^2} \le C\,(1+|k|)^n\, c_p\, 2^{-ps}\|u\|_{H^s} \sum_{\substack{j'\le r-M_2\\|r-m|\le M_1}} c_r \sup_{p,r}\|\Lambda_m\ell_{pr}\|_{L^\infty}.$$

The property

$$\exists \mathrm{e}_0>0,\ \sup_{\mathrm{m}}(\sup_{\mathrm{r,p}}\|\Lambda_{\mathrm{m}}\ell_{\mathrm{pr}}\|_{\mathrm{L}^\infty}2^{\mathrm{me}_0}) \le \mathrm{CA}_{\mathrm{k}} \tag{5.5.18}$$

induces that the sequence $\sum_{m\ge j'} 2^{-me_0}c_m$ belongs to $\ell^2_{j'}$, which is enough to prove the claim (5.5.17). Estimate (5.5.12) implies (5.5.18) which concludes the proof of (5.5.17).

Now let us turn to $w^\flat_{pr}$. We shall separate $w^\flat_{pr}$ into three parts, depending on whether $q\gg p$ or $q\ll p$, or $q\sim p$. More precisely, let $N_0\in\mathbb{N}$ be a fixed integer, to be chosen large enough at the end, and let us define

$$v = v^\sharp + v^\flat + v^\natural = \sum_{p,r}(v^\sharp_{pr}+v^\flat_{pr}+v^\natural_{pr}) = \sum_{p,r} w^\flat_{pr} \ \text{ with } \ w^\flat_{pr} = v^\sharp_{pr}+v^\flat_{pr}+v^\natural_{pr} \ \text{ while}$$

$$v^\sharp_{pr} = \sum_{q\ge p+N_0} \ell_{pr}\Delta_q\Lambda_r R_{p,q}u^{kj}_{pr} \quad\text{and}\quad v^\flat_{pr} = \sum_{q+N_0\le p} \ell_{pr}\Delta_q\Lambda_r R_{p,q}u^{kj}_{pr}.$$

Recall that to compute the H^s norm of v, one needs to compute the ℓ^2 norm in j of $2^{js}\|\Delta_j v\|_{L^2}$. We are going to decompose as before $\ell_{pr}=\sum_m \Lambda_m\ell_{pr}$ and consider the cases $m\le r-M_1$, $m\ge r+M_1$ and $|r-m|<M_2$. For each term, we use the same strategy as the one developed before, in the case of $u^\flat_{pr}$. We shall only write the proof for the indexes $m\le r-M_1$ and leave the other cases to the reader.

By quasi-orthogonality, it is enough to prove

$$\|\Delta_j v^*_r\|_{L^2} \le CA_k(1+|k|)^n\, c_j\, c_r\, 2^{-js}\|u\|_{H^s}, \tag{5.5.19}$$

where $v^*_r=\sum_p w^*_{pr}$ and $*$ stands for $\sharp$, $\flat$ or $\natural$.

• *The term* $v^\sharp$: Let $j\ge -1$ be fixed. We recall that ℓ_{pr} is frequency localized in a ball of size 2^{p-M} and $\Delta_q\Lambda_r R_{p,q}u^{kj}_{pr}$ in a ring of size 2^q, so by the frequency localization of the product (see Proposition 4.9 page 63), there is a constant N_1 such that

$$\Delta_j v^\sharp_r = \sum_{m\le r-M_1}\ \sum_{|j-q|\le N_1}\ \sum_{q\ge p+N_0} \Delta_j\left(\Lambda_m\ell_{pr}\Delta_q\Lambda_r R_{p,q}u^{kj}_{pr}\right).$$

Therefore, we have

$$\begin{aligned}
2^{js}\|\Delta_j v_r^\sharp\|_{L^2} &\le 2^{js} \sum_{m\le r-M_1}\sum_{|j-q|\le N_1}\sum_{q\ge p+N_0} \|\Delta_j(\Lambda_m\ell_{pr}\Delta_q\Lambda_r R_{p,q}u_{pr}^{kj})\|_{L^2}\\
&\le C2^{js} \sum_{m\le r-M_1}\sum_{|j-q|\le N_1}\sum_{q\ge p+N_0} \|\Lambda_m\ell_{pr}\|_{L^\infty}\|\Delta_q\Lambda_r R_{p,q}u_{pr}^{kj}\|_{L^2}\\
&\le C \sum_{m\le r-M_1}\sum_{|j-q|\le N_1}\sum_{q\ge p+N_0} 2^{(j-q)s}\|\Lambda_m\ell_{pr}\|_{L^\infty} c_r\, c_q 2^{\delta_0(p-q)}(1+|k|)^n\|u\|_{H^s},
\end{aligned}$$

where we have used the fact that

$$\begin{aligned}
2^{qs}\|\Delta_q\Lambda_r R_{p,q}u_{pr}^{kj}\|_{L^2} &\le Cc_q\, c_r\|R_{p,q}u_{pr}^{kj}\|_{H^s}\\
&\le Cc_q\, c_r 2^{\delta_0(p-q)}\|u_p^{kj}\|_{H^s}
\end{aligned}$$

by (5.5.6), and then (5.5.10). Assuming (5.5.16), the result follows from Young's inequality which ends the proof of (5.5.19) for $v^\sharp$ thanks to (5.5.12).

• *The term* $v^\flat$: Using again the frequency localization of the product, one can write that for some constant N_3,

$$\begin{aligned}
2^{js}\|\Delta_j v^\flat\|_{L^2} &\le C2^{js} \sum_{m\le r-M_1}\sum_{j-p<N_3}\sum_{q+N_0\le p} \|\Lambda_m\ell_{pr}\|_{L^\infty}\|\Delta_q\Lambda_r R_{p,q}u_{pr}^{kj}\|_{L^2}\\
&\le C2^{js} \sum_{m\le r-M_1}\sum_{j-p<N_3}\sum_{q+N_0\le p} \|\Lambda_m\ell_{pr}\|_{L^\infty}2^{-qs}c_r\, c_q\|R_{p,q}u_{pr}^{kj}\|_{H^s}\\
&\le C2^{js} \sum_{m\le r-M_1}\sum_{j-p<N_3}\sum_{q+N_0\le p} \|\Lambda_m\ell_{pr}\|_{L^\infty}2^{-qs}c_r\, c_q 2^{\delta_0(q-p)}\|u_{pr}^{kj}\|_{H^s}\\
&\le C(1+|k|)^n c_r\|u\|_{H^s}\sum_{m\le r-M_1}\|\Lambda_m\ell_{pr}\|_{L^\infty}\sum_{j-p<N_3}2^{(j-p)s}\sum_{q+N_0\le p}c_q 2^{(\delta_0-s)(q-p)}
\end{aligned}$$

thanks to (5.5.6) and (5.5.10).

Applying Young inequality, we thus obtain for $0<s<\delta_0$

$$2^{js}\|\Delta_j v^\flat\|_{L^2} \le C(1+|k|)^n c_r\|u\|_{H^s}\sum_{m\le r-M_1}\|\Lambda_m\ell_{pr}\|_{L^\infty}\sum_{|j-p|<N_3}2^{(j-p)s}c_p. \tag{5.5.20}$$

This ends the proof of the result by Estimate (5.5.12).

• *The term* $v^\natural$: We recall that

$$v^\natural = \sum_{m\le r-M_1}\sum_{|p-q|<N_0}\Lambda_m\ell_{pr}\Delta_q\Lambda_r R_{p,q}u_{pr}^{kj}.$$

It follows that

$$\begin{aligned}
2^{js}\|\Delta_j v^\natural\|_{L^2} &\le C2^{js}\sum_{m\le r-M_1}\sum_{\substack{-1\le j\le q+N_3\\ |p-q|\le N_0}}\|\Lambda_m\ell_{pr}\|_{L^\infty}\|\Delta_q\Lambda_r R_{p,q}u_p^{kj}\|_{L^2}\\
&\le C(1+|k|)^n c_r\|u\|_{H^s}\sum_{m\le r-M_1}\|\Lambda_m\ell_{pr}\|_{L^\infty}\sum_{\substack{j\le q+N_3\\ |p-q|\le N_0}}2^{(j-q)s}c_q 2^{\delta_0(q-p)}, \qquad (5.5.21)
\end{aligned}$$

and we conclude as in the case of $v^\flat$. We point out that it is at this very place that we crucially use that $s > 0$.

The proposition is proved. □

APPENDIX A

SOME USEFUL RESULTS ON THE HEISENBERG GROUP

A.1. Left invariant vector fields

Let us recall that on a Lie group G, a vector field

$$X : G \longrightarrow TG$$

is said to be left invariant whenever the following diagram commutes for all $h \in G$:

$$\begin{array}{ccc} G & \xrightarrow{\tau_h} & G \\ X \downarrow & & \downarrow X \\ TG & \xrightarrow{d\tau_h} & TG \end{array}$$

where τ_h is the *left translate* on G defined by $\tau_h(g) = h \cdot g$. It turns out that for any $h \in G$,

$$X \circ \tau_h = d\tau_h \circ X. \tag{A.1.22}$$

In particular,

$$X(h) = d\tau_h(e)X(e),$$

where e denotes the identity of G. Therefore, as soon as the vector field X is known on e, so is its value everywhere.

Let us mention that this infinitesimal characterization is equivalent to saying that, for all smooth functions f,

$$(Xf_h) = (Xf)_h, \tag{A.1.23}$$

where f_h is the left translate of f on $\mathbb{H}^d$, given by $f_h = f \circ \tau_h$.

To start with the proof of the equivalence of the two characterizations, let us perform differential calculus in (A.1.22). We infer that (A.1.22) is equivalent to

$$(X \circ \tau_h)f = (d\tau_h \circ X)f,$$

for any function $f \in \mathscr{C}^\infty(G)$. This can be written for any h, g belonging to G

$$(Xf)(\tau_h(g)) = df(\tau_h(g))(d\tau_h(g)X(g)) = d(f \circ \tau_h)(g)X(g) = X(f \circ \tau_h)(g).$$

In other words

$$(Xf)\circ\tau_h = X(f\circ\tau_h),$$

for any $h\in G$, which leads to the result.

A.2. Bargmann and Schrödinger representations

In this paragraph we discuss some useful results concerning Bargmann and Schrödinger representations, starting with the formula giving the Schrödinger representation, if the Bargmann representation and the intertwining operator are known.

In a next subsection we prove some useful commutation results.

A.2.1. Connexion between the representations. — In section we shall give a formula for the Schrödinger representation, which is linked to the Bargmann representation by an intertwining operator. This formula is of course classical, but we present it here for the sake of completeness.

We recall that the Bargmann representation is defined by

$$u^\lambda_{z,s}F(\xi) = F(\xi-\overline{z})\mathrm{e}^{i\lambda s+2\lambda(\xi\cdot z-|z|^2/2)} \quad \text{for} \quad \lambda>0,$$
$$u^\lambda_{z,s}F(\xi) = F(\xi-z)\mathrm{e}^{i\lambda s-2\lambda(\xi\cdot\overline{z}-|z|^2/2)} \quad \text{for} \quad \lambda<0,$$

and we also recall the definition of the intertwining operator, as given in (1.2.32) page 15:

$$(K_\lambda\phi)(\xi)\stackrel{\text{def}}{=}\frac{|\lambda|^{d/4}}{\pi^{d/4}}\mathrm{e}^{|\lambda|\frac{|\xi|^2}{2}}\phi\left(-\frac{1}{2|\lambda|}\frac{\partial}{\partial\xi}\right)\mathrm{e}^{-|\lambda|\,|\xi|^2}.$$

Proposition A.1. — *Let v^λ_w be the Schrödinger representation, defined by*

$$\forall F\in\mathcal{H}_\lambda,\ \ K_\lambda u^\lambda_w F = v^\lambda_w K_\lambda F.$$

Then $v^\lambda_{z,s}$ is given by the following formula:

$$v^\lambda_{z,s}f(\xi) = e^{i\lambda(s-2x\cdot y+2y\cdot\xi)}f(\xi-2x), \quad \forall\lambda\in\mathbb{R}^*.$$

Proof. — It turns out to be easier to split the representation u^λ_w into three parts, using the simple fact that

$$w=(x+iy,s)=(0,s+2y\cdot x)\cdot(x,0)\cdot(iy,0).$$

Let us prove the following relations: for $\lambda\in\mathbb{R}$, $x,y\in\mathbb{R}^d$ and $s\in\mathbb{R}$, $\forall F\in\mathcal{H}_\lambda$ and $\eta\in\mathbb{R}^d$:

$$\text{(A.2.1)}\qquad \left(K_\lambda u^\lambda_{(0,s)}F\right)(\eta) = \mathrm{e}^{i\lambda s}\left(K_\lambda F\right)(\eta),$$

$$\text{(A.2.2)}\qquad \left(K_\lambda u^\lambda_{(x,0)}F\right)(\eta) = (K_\lambda F)(\eta-2x),$$

$$\text{(A.2.3)}\qquad \left(K_\lambda u^\lambda_{(iy,0)}F\right)(\eta) = \mathrm{e}^{2i\lambda y\cdot\eta}\left(K_\lambda F\right)(\eta).$$

Notice that those relations give

$$\begin{aligned}
\left(K_\lambda u^\lambda_w F\right)(\eta) &= \left(K_\lambda u^\lambda_{(0,s+2x\cdot y)} u^\lambda_{(x,0)} u^\lambda_{(iy,0)} F\right)(\eta) \\
&= \mathrm{e}^{i\lambda(s+2y\cdot x)} \left(K_\lambda u^\lambda_{(x,0)} u^\lambda_{(iy,0)} F\right)(\eta) \\
&= \mathrm{e}^{i\lambda(s+2y\cdot x)} \left(K_\lambda u^\lambda_{(iy,0)} F\right)(\eta - 2x) \\
&= \mathrm{e}^{i\lambda s+2i\lambda y\cdot\eta-2i\lambda y\cdot x} (K_\lambda F)(\eta-2x).
\end{aligned}$$

which is precisely the expected result.

So it remains to prove the basic relations (A.2.1)–(A.2.3). The first one comes trivially from the fact that $u^\lambda_{(0,s)}$ is the multiplication by the phasis $\mathrm{e}^{i\lambda s}$.

For the two other ones, we write, for any function F in $\mathcal{H}_\lambda$ and using Proposition IV.2 of **[23]**,

$$(K_\lambda F)(\eta) = \left(\frac{|\lambda|}{\pi}\right)^{5d/4} \mathrm{e}^{\frac{|\lambda||\eta|^2}{2}} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda v\cdot(\eta-\eta')-|\lambda||\eta'|^2} F(iv)\, dv\, d\eta'.$$

Therefore, for $\lambda > 0$, we have on the one hand

$$\begin{aligned}
\left(K_\lambda u^\lambda_{(iy,0)}\right) F(\eta) &= \left(\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{\frac{\lambda|\eta|^2}{2}} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda v\cdot(\eta-\eta')-\lambda|\eta'|^2-\lambda|y|^2+2i\lambda y\cdot(iv)} F(i(v+y))\, dv\, d\eta' \\
&= \left(\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{\lambda\frac{|\eta|^2}{2}+2i\lambda y\cdot\eta} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda u\cdot(\eta-\eta'-iy)-2i\lambda y\cdot\eta'+\lambda|y|^2-\lambda|\eta'|^2} F(iu) du d\eta' \\
&= \left(\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{\lambda\frac{|\eta|^2}{2}+2i\lambda y\cdot\eta} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda u\cdot(\eta-\eta'')-\lambda|\eta''|^2} F(iu)\, du\, d\eta'' \\
&= \mathrm{e}^{2i\lambda y\cdot\eta} (K_\lambda F)(\eta).
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\left(K_\lambda u^\lambda_{(x,0)} F\right)(\eta) &= \left(\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{\frac{\lambda|\eta|^2}{2}} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda v(\eta-\eta')-\lambda|\eta'|^2+2\lambda i x\cdot v-\lambda|x|^2} F(iv-x)\, dv\, d\eta' \\
&= \left(\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{\lambda\frac{|\eta|^2}{2}+2\lambda|x|^2-2\lambda\eta\cdot x} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda u(\eta-\eta'-x)-\lambda|\eta'-x|^2} F(iu)\, du\, d\eta' \\
&= \left(\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{\lambda\frac{|\eta-2x|^2}{2}} \int_{\mathbb{R}^{2d}} \mathrm{e}^{-2i\lambda u(\eta-2x-\eta'')-\lambda|\eta''|^2} F(iu)\, du\, d\eta'' \\
&= (K_\lambda F)(\eta-2x).
\end{aligned}$$

Similarly, for $\lambda < 0$,

$$\begin{aligned}
\left(K_\lambda u^\lambda_{(iy,0)}\right) F(\eta) &= \left(-\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{-\frac{\lambda|\eta|^2}{2}} \int_{\mathbb{R}^{2d}} \mathrm{e}^{2i\lambda v\cdot(\eta-\eta')+\lambda|\eta'|^2+\lambda|y|^2+2i\lambda y\cdot(iv)} F(i(v+y))\, dv\, d\eta' \\
&= \left(-\frac{\lambda}{\pi}\right)^{5d/4} \mathrm{e}^{-\lambda\frac{|\eta|^2}{2}+2i\lambda y\cdot\eta} \int_{\mathbb{R}^{2d}} \mathrm{e}^{2i\lambda u\cdot(\eta-\eta'+iy)-2i\lambda y\cdot\eta'-\lambda|y|^2+\lambda|\eta'|^2} F(iu) du d\eta'
\end{aligned}$$

$$= \Big(-\frac{\lambda}{\pi}\Big)^{5d/4} e^{-\lambda\frac{|\eta|^2}{2}+2i\lambda y\cdot\eta} \int_{\mathbb{R}^{2d}} e^{2i\lambda u\cdot(\eta-\eta'')+\lambda|\eta''|^2} F(iu)\,du\,d\eta''$$
$$= e^{2i\lambda y\cdot\eta}(K_\lambda F)(\eta)$$

and

$$\begin{aligned}\Big(K_\lambda u^\lambda_{(x,0)}F\Big)(\eta) &= \Big(-\frac{\lambda}{\pi}\Big)^{5d/4} e^{-\frac{\lambda|\eta|^2}{2}} \int_{\mathbb{R}^{2d}} e^{2i\lambda v(\eta-\eta')+\lambda|\eta'|^2-2\lambda ix\cdot v+\lambda|x|^2} F(iv-x)\,dv\,d\eta'\\
&= \Big(-\frac{\lambda}{\pi}\Big)^{5d/4} e^{-\lambda\frac{|\eta|^2}{2}-2\lambda|x|^2+2\lambda\eta\cdot x} \int_{\mathbb{R}^{2d}} e^{2i\lambda u(\eta-\eta'-x)+\lambda|\eta'-x|^2} F(iu)\,du\,d\eta'\\
&= \Big(-\frac{\lambda}{\pi}\Big)^{5d/4} e^{-\lambda\frac{|\eta-2x|^2}{2}} \int_{\mathbb{R}^{2d}} e^{2i\lambda u(\eta-2x-\eta'')+\lambda|\eta''|^2} F(iu)\,du\,d\eta''\\
&= (K_\lambda F)(\eta-2x).\end{aligned}$$

This proves the estimates, hence the proposition is proved. □

A.2.2. Some useful formulas. — This section is devoted to various properties for Bargmann representation that we collect in the following lemma.

Lemma A.2. — *The following commutation formulas hold true:*

$$\frac{1}{2\lambda}[Q^\lambda_j, u^\lambda_w] = -\overline{z}_j u^\lambda_w \quad \textit{and} \quad \frac{1}{2\lambda}[\overline{Q}^\lambda_j, u^\lambda_w] = z_j u^\lambda_w.$$

for any $\lambda\in\mathbb{R}^*$ and any $w=(z,s)\in\mathbb{H}^d$.

Proof. — In order to prove Lemma A.2, let us first recall formulas (1.2.27) giving the expression of Q^λ_j and $\overline{Q}^\lambda_j$:

$$Q^\lambda_j = \begin{cases} -2|\lambda|\xi_j & \text{if } \lambda>0,\\ \partial_{\xi_j} & \text{if } \lambda<0,\end{cases} \quad \text{and} \quad \overline{Q}^\lambda_j = \begin{cases} \partial_{\xi_j} & \text{if } \lambda>0,\\ -2|\lambda|\xi_j & \text{if } \lambda<0.\end{cases}$$

Let us now prove the first formula, in the case when $\lambda>0$. On the one hand, it is obvious that

$$Q^\lambda_j u^\lambda_w F(\xi) = -2\lambda\xi_j u^\lambda_w F(\xi).$$

On the other hand, an easy computation implies that

$$u^\lambda_w Q^\lambda_j F(\xi) = -2\lambda(\xi_j-\overline{z}_j)e^{i\lambda s+2\lambda(\xi\cdot z-|z|^2/2)}F(\xi-\overline{z}).$$

which implies that $-\overline{z}_j u^\lambda_w = \frac{1}{2\lambda}[Q^\lambda_j, u^\lambda_w]$, for $\lambda>0$. In the case when $\lambda<0$ one has

$$\begin{aligned} Q^\lambda_j u^\lambda_w F(\xi) &= \partial_{\xi_j}(u^\lambda_w F(\xi))\\ &= u^\lambda_w\partial_{\xi_j}F(\xi) - 2\lambda\overline{z}_j e^{i\lambda s-2\lambda(\xi\cdot\overline{z}-|z|^2/2)}F(\xi-z)\\ &= u^\lambda_w\partial_{\xi_j}F(\xi) - 2\lambda\overline{z}_j u^\lambda_w F(\xi)\end{aligned}$$

which ends the proof of the commutation properties $-\overline{z}_j u^\lambda_w = \frac{1}{2\lambda}[Q^\lambda_j, u^\lambda_w]$.

It remains to check the formula for $[\overline{Q}_j^\lambda, u_w^\lambda]$. Arguing as before, one gets for $\lambda > 0$

$$\begin{aligned}\overline{Q}_j^\lambda u_w^\lambda F(\xi) &= \partial_{\xi_j}(u_w^\lambda F(\xi))\\ &= u_w^\lambda \partial_{\xi_j} F(\xi) + 2\lambda z_j \mathrm{e}^{i\lambda s + 2\lambda(\xi\cdot z - |z|^2/2)} F(\xi - \overline{z})\\ &= u_w^\lambda \partial_{\xi_j} F(\xi) + 2\lambda z_j u_w^\lambda F(\xi),\end{aligned}$$

which gives the formula in the case when $\lambda > 0$. Finally, for $\lambda < 0$

$$\overline{Q}_j^\lambda u_w^\lambda F(\xi) = 2\lambda \xi_j u_w^\lambda F(\xi)$$

and

$$u_w^\lambda \overline{Q}_j^\lambda F(\xi) = 2\lambda(\xi_j - z_j) u_w^\lambda F(\xi).$$

This leads easily to the second commutation property. □

Lemma A.2 allows to infer the following result, which is useful in particular to prove Lemma 2.7.

Lemma A.3. — *One has the following properties:*

$$Z_j u_{w^{-1}}^\lambda = Q_j^\lambda u_{w^{-1}}^\lambda \quad \text{and} \quad \overline{Z_j} u_{w^{-1}}^\lambda = \overline{Q_j}^\lambda u_{w^{-1}}^\lambda.$$

for any $\lambda \in \mathbb{R}^*$ *and any* $w = (z, s) \in \mathbb{H}^d$.

Proof. — First, let us compute $Z_j u_{w^{-1}}^\lambda$ in the case when λ is positive. By definition, one has

$$\begin{aligned}Z_j u_{w^{-1}}^\lambda F(\xi) &= (\partial_{z_j} + i\overline{z}_j \partial_s) u_{w^{-1}}^\lambda F(\xi)\\ &= (\partial_{z_j} + i\overline{z}_j \partial_s) F(\xi + \overline{z}) \mathrm{e}^{-i\lambda s + 2\lambda(-\xi\cdot z - |z|^2/2)}\\ &= (-2\lambda\xi_j - \lambda\overline{z}_j + i\overline{z}_j(-i\lambda)) u_{w^{-1}}^\lambda F(\xi)\\ &= -2\lambda \xi_j u_{w^{-1}}^\lambda F(\xi).\end{aligned}$$

Whence the first formula thanks to (1.2.27).

Along the same lines, when λ is negative one can write

$$\begin{aligned}Z_j u_{w^{-1}}^\lambda F(\xi) &= (\partial_{z_j} + i\overline{z}_j \partial_s) u_{w^{-1}}^\lambda F(\xi)\\ &= (\partial_{z_j} + i\overline{z}_j \partial_s) F(\xi + z) \mathrm{e}^{-i\lambda s - 2\lambda(-\xi\cdot\overline{z} - |z|^2/2)}\\ &= (\lambda\overline{z}_j + i\overline{z}_j(-i\lambda)) u_{w^{-1}}^\lambda F(\xi) + u_{w^{-1}}^\lambda \partial_{\xi_j} F(\xi)\\ &= 2\lambda\overline{z}_j u_{w^{-1}}^\lambda F(\xi) + u_{w^{-1}}^\lambda \partial_{\xi_j} F(\xi).\end{aligned}$$

We deduce thanks to (1.2.27) that $Z_j u_{w^{-1}}^\lambda = 2\lambda\overline{z}_j u_{w^{-1}}^\lambda + u_{w^{-1}}^\lambda Q_j^\lambda$. Let us remind that by Lemma A.2, $Q_j^\lambda\, u_w^\lambda - u_w^\lambda\, Q_j^\lambda = -2\lambda\overline{z}_j u_w^\lambda$ which can be also written

$$Q_j^\lambda\, u_{w^{-1}}^\lambda - u_{w^{-1}}^\lambda\, Q_j^\lambda = 2\lambda\overline{z}_j u_{w^{-1}}^\lambda.$$

This implies that $Z_j u_{w^{-1}}^\lambda = Q_j^\lambda\, u_{w^{-1}}^\lambda$, which ends the proof of the first assertion.

Now, let us compute $\overline{Z}_j u^\lambda_{w^{-1}}$. Again, one can write for $\lambda > 0$

$$\begin{aligned}
\overline{Z}_j u^\lambda_{w^{-1}} F(\xi) &= (\partial_{\overline{z}_j} - iz_j\partial_s) u^\lambda_{w^{-1}} F(\xi)\\
&= (\partial_{\overline{z}_j} - iz_j\partial_s) F(\xi+\overline{z}) \mathrm{e}^{-i\lambda s + 2\lambda(-\xi\cdot z - |z|^2/2)}\\
&= u^\lambda_{w^{-1}} \partial_{\xi_j} F(\xi) - (\lambda z_j + iz_j(-i\lambda)) u^\lambda_{w^{-1}} F(\xi)\\
&= u^\lambda_{w^{-1}} \partial_{\xi_j} F(\xi) - 2\lambda z_j u^\lambda_{w^{-1}} F(\xi).
\end{aligned}$$

We point out that, again by (1.2.27), this can be expressed as follows

$$\overline{Z}_j u^\lambda_{w^{-1}} = u^\lambda_{w^{-1}} \overline{Q}^\lambda_j - 2\lambda z_j u^\lambda_{w^{-1}}.$$

But Lemma A.2 states that $\overline{Q}^\lambda_j u^\lambda_w - u^\lambda_w \overline{Q}^\lambda_j = 2\lambda z_j u^\lambda_w$ which can be also written

$$\overline{Q}^\lambda_j u^\lambda_{w^{-1}} - u^\lambda_{w^{-1}} \overline{Q}^\lambda_j = -2\lambda z_j u^\lambda_{w^{-1}}.$$

This ensures that $\overline{Z}_j u^\lambda_{w^{-1}} = \overline{Q}^\lambda_j u^\lambda_{w^{-1}}$ in the case when $\lambda > 0$.

Finally, in the case when $\lambda < 0$, one gets

$$\begin{aligned}
\overline{Z}_j u^\lambda_{w^{-1}} F(\xi) &= (\partial_{\overline{z}_j} - iz_j\partial_s) u^\lambda_{w^{-1}} F(\xi)\\
&= (\partial_{\overline{z}_j} - iz_j\partial_s) F(\xi+z) \mathrm{e}^{-i\lambda s - 2\lambda(-\xi\cdot\overline{z} - |z|^2/2)}\\
&= (2\lambda\xi_j + \lambda z_j - iz_j(-i\lambda)) u^\lambda_{w^{-1}} F(\xi)\\
&= 2\lambda\xi_j u^\lambda_{w^{-1}} F(\xi)\\
&= \overline{Q}^\lambda_j u^\lambda_{w^{-1}} F(\xi)
\end{aligned}$$

where we have used one more time (1.2.27) for the last equality. This ends the proof of the lemma. □

Finally let us state one last result, which provides the symbol of the multiplication operator by s.

Lemma A.4. — *Let $a \in S_{\mathbb{H}^d}(\mu)$, $\tilde{w} = (\tilde{z}, \tilde{s}) \in \mathbb{H}^d$ and $w \in \mathbb{H}^d$, then*

$$\int \mathrm{tr}\left(i\tilde{s} u^\lambda_{\tilde{w}} J^*_\lambda \mathrm{op}^w(a(w,\lambda)) J_\lambda\right) |\lambda|^d \, d\lambda = \int \mathrm{tr}\left(u^\lambda_{\tilde{w}} J^*_\lambda \mathrm{op}^w\left(g(w,\lambda)\right) J_\lambda\right) |\lambda|^d \, d\lambda$$

with $g \in S_{\mathbb{H}^d}(\mu)$ and

$$\sigma(g) = -\partial_\lambda\left(\sigma(a)\right) \tag{A.2.4}$$

or equivalently

$$g = -\partial_\lambda a + \frac{1}{2\lambda} \sum_{1\leq j\leq d} (\eta_j \partial_{\eta_j} + \xi_j \partial_{\xi_j}) a \tag{A.2.5}$$

Proof. — Let us first observe that by Proposition 1.22 page 22, the function g defined by (A.2.4) is a symbol of order μ since

$$(1+|\lambda|+y^2+\eta^2)^{\frac{\mu-|\beta|}{2}} (1+|\lambda|)^{-k-1} \leq (1+|\lambda|+y^2+\eta^2)^{\frac{\mu-|\beta|}{2}} (1+|\lambda|)^{-k}.$$

Besides, by the definition of u_w^λ (see (1.2.15)) we have

$$\begin{aligned}\partial_\lambda u_w^\lambda &= \left(is + 2\xi\cdot z - |z|^2\right) u_w^\lambda \text{ for } \lambda > 0,\\ \partial_\lambda u_w^\lambda &= \left(is - 2\xi\cdot \overline{z} + |z|^2\right) u_w^\lambda \text{ for } \lambda < 0.\end{aligned}$$

Therefore, using Lemma A.2 and using formulas (1.2.27), we have for $\lambda > 0$

$$\begin{aligned}isu_w^\lambda &= \partial_\lambda u_w^\lambda - \sum_{1\leq j\leq d}\Big(-\frac{1}{2\lambda^2}Q_j^\lambda[\overline{Q}_j^\lambda, u_w^\lambda] + \frac{1}{4\lambda^2}\Big[Q_j^\lambda\,,\,[\overline{Q}_j^\lambda, u_w^\lambda]\Big]\Big)\\ &= \partial_\lambda u_w^\lambda - \frac{1}{4\lambda^2}\sum_{1\leq j\leq d}\Big([u_w^\lambda, \overline{Q}_j^\lambda]Q_j^\lambda + Q_j^\lambda[u_w^\lambda, \overline{Q}_j^\lambda]\Big).\end{aligned}$$

Similarly, for $\lambda < 0$, we have

$$\begin{aligned}isu_w^\lambda &= \partial_\lambda u_w^\lambda + \sum_{1\leq j\leq d}\Big(-\frac{1}{2\lambda^2}\overline{Q}_j^\lambda[Q_j^\lambda, u_w^\lambda] + \frac{1}{4\lambda^2}\Big[Q_j^\lambda\,,\,[\overline{Q}_j^\lambda, u_w^\lambda]\Big]\Big)\\ &= \partial_\lambda u_w^\lambda + \frac{1}{4\lambda^2}\sum_{1\leq j\leq d}\Big([u_w^\lambda, Q_j^\lambda]\overline{Q}_j^\lambda + \overline{Q}_j^\lambda[u_w^\lambda, Q_j^\lambda]\Big).\end{aligned}$$

Setting $A_\lambda(w) = J_\lambda^*\mathrm{op}^w(a(w,\lambda))J_\lambda$ and using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ we get

$$\begin{aligned}&\mathrm{tr}\left(i\tilde{s}u_{\tilde{w}}^\lambda A_\lambda(w)\right)\\ &= \mathrm{tr}\left(\partial_\lambda u_{\tilde{w}}^\lambda\, A_\lambda(w)\right) - \frac{1}{4\lambda^2}\sum_{1\leq j\leq d}\mathrm{tr}\Big(u_{\tilde{w}}^\lambda\Big[\overline{Q}_j^\lambda\,,\,A_\lambda(w)Q_j^\lambda + Q_j^\lambda A_\lambda(w)\Big]\Big) \text{ if } \lambda > 0,\end{aligned}$$

$$\begin{aligned}&\mathrm{tr}\left(i\tilde{s}u_{\tilde{w}}^\lambda A_\lambda(w)\right)\\ &= \mathrm{tr}\left(\partial_\lambda u_{\tilde{w}}^\lambda\, A_\lambda(w)\right) + \frac{1}{4\lambda^2}\sum_{1\leq j\leq d}\mathrm{tr}\Big(u_{\tilde{w}}^\lambda\Big[Q_j^\lambda\,,\,A_\lambda(w)\overline{Q}_j^\lambda + \overline{Q}_j^\lambda A_\lambda(w)\Big]\Big) \text{ if } \lambda < 0.\end{aligned}$$

By (1.2.37), using the fact that $\mathrm{op}^w(\eta_j) = -i\partial_{\xi_j}$ and $\mathrm{op}^w(\xi_j) = \xi_j$, along with formula (2.3.3) recalled page 36, we get for $\lambda > 0$,

$$\begin{aligned}&[\overline{Q}_j^\lambda, A_\lambda(w)Q_j^\lambda + Q_j^\lambda A_\lambda(w)]\\ &\qquad = \lambda\, J_\lambda^*\left[\partial_{\xi_j} + \xi_j, \mathrm{op}^w(a(w,\lambda)(\partial_{\xi_j} - \xi_j) + (\partial_{\xi_j} - \xi_j)\mathrm{op}^w(a)\right] J_\lambda\\ &\qquad = 2\lambda\, J_\lambda^*\mathrm{op}^w\left(-2da + \sum_{1\leq j\leq d}(\eta_j + i\xi_j)(i\partial_{\xi_j}a - \partial_{\eta_j}a)\right)J_\lambda.\end{aligned}$$

Similarly, for $\lambda < 0$,

$$\begin{aligned}&[Q_j^\lambda, A_\lambda(w)\overline{Q}_j^\lambda + \overline{Q}_j^\lambda A_\lambda(w)]\\ &\qquad = -2\lambda\, J_\lambda^*\mathrm{op}^w\left(-2da + \sum_{1\leq j\leq d}(\eta_j + i\xi_j)(i\partial_{\xi_j}a - \partial_{\eta_j}a)\right)J_\lambda.\end{aligned}$$

Set

$$b(w,\lambda,y,\eta) = -2da + \sum_{1\le j\le d} (\eta_j + i\xi_j)(i\partial_{\xi_j} a - \partial_{\eta_j} a), \tag{A.2.6}$$

we have obtained

$$\forall \lambda \neq 0,\ \ \mathrm{tr}\left(i\tilde{s} u^\lambda_{\tilde{w}} A_\lambda(w)\right) = \mathrm{tr}\left(\partial_\lambda u^\lambda_{\tilde{w}} A_\lambda(w)\right) - \frac{1}{2\lambda}\mathrm{tr}\left(u^\lambda_{\tilde{w}} J^*_\lambda \mathrm{op}^w(b) J_\lambda\right). \tag{A.2.7}$$

We focus now on the term $\partial_\lambda u^\lambda_{\tilde{w}} A_\lambda(w)$. We have

$$\mathrm{tr}\left(\partial_\lambda u^\lambda_{\tilde{w}} A_\lambda(w)\right) = \partial_\lambda\left(\mathrm{tr}\left(u^\lambda_{\tilde{w}} A_\lambda(w)\right)\right) - \mathrm{tr}\left(u^\lambda_{\tilde{w}} \partial_\lambda A_\lambda(w)\right).$$

This implies, by integration by parts, that

$$\int \mathrm{tr}\left(\partial_\lambda u^\lambda_{\tilde{w}} A_\lambda(w)\right)|\lambda|^d\, d\lambda = -\int \frac{d}{\lambda}\mathrm{tr}\left(u^\lambda_{\tilde{w}} A_\lambda(w)\right)|\lambda|^d\, d\lambda - \int \mathrm{tr}\left(u^\lambda_{\tilde{w}} \partial_\lambda A_\lambda(w)\right)|\lambda|^d\, d\lambda.$$

We claim that

$$\partial_\lambda A_\lambda(w) = J^*_\lambda \mathrm{op}^w\left(\partial_\lambda a(\lambda,w) + \frac{i}{2\lambda}\sum_{1\le j\le d}(\xi_j\partial_{\eta_j} a - \eta_j \partial_{\xi_j} a)\right) J_\lambda. \tag{A.2.8}$$

This yields, with (A.2.6) and (A.2.7),

$$\begin{aligned}\int \mathrm{tr}\left(i\tilde{s} u^\lambda_{\tilde{w}} A_\lambda(w)\right)|\lambda|^d\, d\lambda &= \int \mathrm{tr}\Big(u^\lambda_{\tilde{w}} J^*_\lambda \mathrm{op}^w\Big(-\frac{d}{\lambda}a - \partial_\lambda a - \frac{i}{2\lambda}\sum_{1\le j\le d}(\xi_j\partial_{\eta_j} a - \eta_j\partial_{\xi_j} a)\\ &\qquad + \frac{d}{\lambda}a - \frac{1}{2\lambda}\sum_{1\le j\le d}(\eta_j + i\xi_j)(i\partial_{\xi_j} a - \partial_{\eta_j} a)\Big) J_\lambda\Big)|\lambda|^d\, d\lambda\\ &= \int \mathrm{tr}\left(u^\lambda_{\tilde{w}} J^*_\lambda \mathrm{op}^w\left(-\partial_\lambda a + \frac{1}{2\lambda}\sum_{1\le j\le d}(\eta_j\partial_{\eta_j} + \xi_j\partial_{\xi_j})a\right) J_\lambda\right)|\lambda|^d d\lambda.\end{aligned}$$

We then set

$$g = -\partial_\lambda a + \frac{1}{2\lambda}\sum_{1\le j\le d}(\eta_j\partial_{\eta_j} + \xi_j\partial_{\xi_j})a$$

and observe that a simple computation implies (A.2.4). Therefore, in order to finish the proof of the lemma, it only remains to prove (A.2.8).

Let us now prove (A.2.8). We have, recalling that $A_\lambda(w) = J^*_\lambda \mathrm{op}^w(a(w,\lambda))J_\lambda$ and using the fact that $\partial_\lambda(J_\lambda J^*_\lambda) = 0$,

$$\partial_\lambda A_\lambda(w) = J^*_\lambda \mathrm{op}^w\left(\partial_\lambda a(\lambda,w)\right) J_\lambda + J^*_\lambda\left[\mathrm{op}^w(a(w,\lambda))\,,\,(\partial_\lambda J_\lambda)J^*_\lambda\right] J_\lambda.$$

Besides, for $\alpha \in \mathbb{N}^d$, we have $J_\lambda F_{\alpha,\lambda} = h_\alpha$ whence

$$(\partial_\lambda J_\lambda)F_{\alpha,\lambda} = -J_\lambda(\partial_\lambda F_{\alpha,\lambda}).$$

Let us recall that for $\xi \in \mathbb{C}^d$, $F_{\alpha,\lambda}(\xi) = (\sqrt{|\lambda|})^{|\alpha|}\frac{\xi^\alpha}{\sqrt{\alpha!}}$ so that $\partial_\lambda F_{\alpha,\lambda} = \frac{|\alpha|}{2\lambda}F_{\alpha,\lambda}$. We get

$$\forall \alpha \in \mathbb{N}^d,\ \ (\partial_\lambda J_\lambda)J^*_\lambda h_\alpha = (\partial_\lambda J_\lambda)F_{\alpha,\lambda} = -\frac{|\alpha|}{2\lambda}h_\alpha = -\frac{1}{4\lambda}(\xi^2 - \Delta_\xi - d)h_\alpha.$$

Therefore,

$$(\partial_\lambda J_\lambda)J_\lambda^* = -\frac{1}{4\lambda}(\xi^2 - \Delta_\xi) + \frac{d}{4\lambda}\mathrm{Id}.$$

We then obtain

$$\begin{aligned}[\mathrm{op}^w(a), (\partial_\lambda J_\lambda)J_\lambda^*] &= -\frac{1}{4\lambda}\left[\mathrm{op}^w(a)\ ,\ \xi^2 - \Delta_\xi\right] \\ &= \frac{i}{2\lambda}\sum_{1\le j\le d} \mathrm{op}^w(\xi_j\partial_{\eta_j}a - \eta_j\partial_{\xi_j}a),\end{aligned}$$

which proves the lemma. □

APPENDIX B

WEYL-HÖRMANDER SYMBOLIC CALCULUS ON THE HEISENBERG GROUP

In this appendix, we discuss results of Weyl-Hörmander calculus associated to the Harmonic Oscillator, and in particular we prove Propositions 1.20, 1.22 and 1.16 and stated in the Introduction.

B.1. λ-dependent metrics

This section is devoted to the proof of Proposition 1.20 stated page 21. We therefore consider the λ-dependent metric and weight

$$\forall \lambda \neq 0,\ \forall \Theta \in \mathbb{R}^{2d}, \quad g_\Theta^{(\lambda)}(d\xi, d\eta) \stackrel{\text{def}}{=} \frac{|\lambda|(d\xi^2 + d\eta^2)}{1 + |\lambda|(1 + \Theta^2)}$$

and

$$m^{(\lambda)}(\Theta) \stackrel{\text{def}}{=} \big(1 + |\lambda|(1 + \Theta^2)\big)^{1/2},$$

and we aim at proving that the structural constants, in the sense of Definition 1.12 page 17, may be chosen uniformly of λ; the second point stated in Proposition 1.20 is obvious to check.

It turns out that the proofs for the metric and for the weight are identical, so let us concentrate on the metric from now on, for which we need to prove the uncertainty principle, as well as the fact that the metric is slow and temperate.

The uncertainty principle is very easy to prove, since of course

$$g_\Theta^{(\lambda)\omega}(d\xi, d\eta) = \frac{1 + |\lambda|(1 + \Theta^2)}{|\lambda|}(d\xi^2 + d\eta^2)$$

and

$$|\lambda| \leq 1 + |\lambda|(1 + \Theta^2).$$

The slowness property is also not so difficult to obtain. We notice indeed that, with obvious notation,

$$g_\Theta^{(\lambda)}(\Theta - \Theta') = \frac{|\lambda||\Theta - \Theta'|^2}{1 + |\lambda|(1 + \Theta^2)}$$

and we want to prove that there is a constant $\overline{C}$, independent of λ, such that if

$$|\lambda||\Theta - \Theta'|^2 \leq \overline{C}^{-1}(1 + |\lambda|(1 + \Theta^2)),$$

then

$$\frac{1 + |\lambda|(1 + \Theta^2)}{1 + |\lambda|(1 + \Theta'^2)} + \frac{1 + |\lambda|(1 + \Theta'^2)}{1 + |\lambda|(1 + \Theta^2)} \leq \overline{C}.$$

To do so, we shall decompose the phase space $\mathbb{R}^{2d}$ into regions in terms of the respective sizes of Θ^2 and Θ'^2. In the following we shall write $\Theta^2 \ll \Theta'^2$ if, say $\Theta^2 \leq 10\,\Theta'^2$, and $|\Theta| \sim |\Theta'|$ will mean that, say $\frac{1}{10}\Theta^2 \leq \Theta'^2 \leq 10\,\Theta^2$.

Suppose first that $\Theta^2 \ll \Theta'^2$. Then of course

$$1 + |\lambda|(1 + \Theta^2) \leq 1 + |\lambda|(1 + \Theta'^2),$$

so we assume that $\overline{C} \geq 1$. Moreover, using the obvious algebraic inequality

$$\Theta'^2 \leq 2|\Theta - \Theta'|^2 + 2\Theta^2,$$

we deduce that

$$|\lambda|\Theta'^2 \leq 2|\lambda||\Theta - \Theta'|^2 + 2|\lambda|\Theta^2 \leq (2\overline{C}^{-1} + 2)(1 + |\lambda|(1 + \Theta^2))$$

which leads immediately to the result as soon as

$$2\overline{C}^{-1} + 2 \leq \overline{C}.$$

Conversely if $\Theta^2 \gg \Theta'^2$, then it is clear that

$$1 + |\lambda|(1 + \Theta'^2) \leq 1 + |\lambda|(1 + \Theta^2).$$

Along the same lines as above we get

$$\begin{aligned} |\lambda|\Theta^2 &\leq 2|\lambda|\Theta'^2 + 2\overline{C}^{-1}(1 + |\lambda|(1 + \Theta^2)) \\ &\leq (2\overline{C}^{-1} + 2)(1 + |\lambda|(1 + \Theta^2)), \end{aligned}$$

which choosing $\overline{C}$ large enough (independently of λ) gives the result. Since the estimate is obvious when $|\Theta| \sim |\Theta'|$, the slowness property is proved, with a structural constant independent of λ.

Finally let us prove that the metric is tempered, with uniform structural constants. This is again slightly more technical. We need to find a uniform constant $\overline{C}$ such that

$$\left(\frac{1 + |\lambda|(1 + \Theta^2)}{1 + |\lambda|(1 + \Theta'^2)}\right)^{\pm 1} \leq \overline{C}\left(1 + \frac{1 + |\lambda|(1 + \Theta^2)}{|\lambda|}|\Theta - \Theta'|^2\right).$$

Notice that in the case when $|\Theta| \sim |\Theta'|$, then the estimate is obvious because the left-hand side is bounded by a uniform constant. Let us now deal with the two other types of cases, namely $|\Theta|^2 \ll |\Theta'|^2$, and $|\Theta'|^2 \ll |\Theta|^2$.

Let us start with the case when the left-hand side has power $+1$. If $|\Theta|^2 \ll |\Theta'|^2$, then the left-hand side is uniformly bounded so the result follows with $\overline{C} \geq 1$. Conversely if $|\Theta'|^2 \ll |\Theta|^2$, then we notice that if $0 < |\lambda| \leq 1$, then the left-hand side is bounded by $2 + \Theta^2$ while the right-hand side is larger than $\overline{C}(1 + c\Theta^2(1 + \Theta^2))$ so the estimate is true. On the other hand when $|\lambda| \geq 1$ then factorizing the left-hand side by λ and using the fact that $|\lambda|^{-1} \leq 1$ and $(|\lambda|^{-1} + 1 + \Theta'^2)^{-1} \leq (1 + \Theta'^2)^{-1}$ we get

$$\frac{1 + |\lambda|(1 + \Theta^2)}{1 + |\lambda|(1 + \Theta'^2)} \leq 2\frac{1 + \Theta^2}{1 + \Theta'^2} \leq 2(1 + \Theta^2).$$

Again, since in that case $|\Theta - \Theta'|^2 \geq c\Theta^2$, it comes

$$\left(1 + \frac{1 + |\lambda|(1 + \Theta^2)}{|\lambda|}|\Theta - \Theta'|^2\right) \geq \left(1 + c\Theta^2(1 + \Theta^2)\right)$$

which implies easily the result.

Now let us deal with the case when the left-hand side has power -1. The arguments are similar. Indeed if $|\Theta'|^2 \ll |\Theta|^2$ then the left-hand side is uniformly bounded so the result follows. Conversely if $|\Theta|^2 \ll |\Theta'|^2$ then when $0 < |\lambda| \leq 1$ we use the fact that the left-hand side is bounded by $2 + \Theta'^2$ whereas the right-hand side is larger than $c(1 + \Theta'^2)$. When $|\lambda| \geq 1$ then as above we write

$$\frac{1 + |\lambda|(1 + \Theta'^2)}{1 + |\lambda|(1 + \Theta^2)} \leq 2\frac{1 + \Theta'^2}{1 + \Theta^2} \leq 2(1 + \Theta'^2),$$

and the result follows again from the fact that since in that case $|\Theta - \Theta'|^2 \geq c\Theta'^2$, one has

$$\left(1 + \frac{1 + |\lambda|(1 + \Theta^2)}{|\lambda|}|\Theta - \Theta'|^2\right) \geq \left(1 + c\Theta'^2(1 + \Theta^2)\right) \geq \left(1 + c\Theta'^2\right).$$

The proposition is proved. □

B.2. λ-dependent symbols

In this subsection we shall prove Proposition 1.22 stated page 22, giving an equivalent definition of symbols in terms of the scaling function σ.

For any multi-index β satisfying $|\beta| \leq n$, we have

$$\begin{aligned} \left|\partial^\beta_{(y,\eta)}\left(\sigma(a)(w,\lambda,\xi,\eta)\right)\right| &= \left||\lambda|^{-\frac{|\beta|}{2}}\left(\partial^\beta_{(\xi,\eta)}a\right)\left(w,\lambda,\operatorname{sgn}(\lambda)\frac{\xi}{\sqrt{|\lambda|}},\frac{\eta}{\sqrt{|\lambda|}}\right)\right| \\ &\leq \|a\|_{n;S_{\mathbb{H}^d}(\mu)}\left(1 + |\lambda| + \xi^2 + \eta^2\right)^{\frac{\mu-|\beta|}{2}}. \end{aligned} \tag{B.2.1}$$

Besides, there exists a constant $C > 0$ such that for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \left|(\lambda\partial_\lambda)^k\left(\sigma(a)(w,\lambda,\xi,\eta)\right)\right| &\leq C\left|\left((\lambda\partial_\lambda)^k a\right)\left(w,\lambda,\operatorname{sgn}(\lambda)\frac{\xi}{\sqrt{|\lambda|}},\frac{\eta}{\sqrt{|\lambda|}}\right)\right| \\ &\quad + C\sum_{|\beta|=k}|\lambda|^{-\frac{k}{2}}(\xi^2+\eta^2)^{\frac{k}{2}}\left|\left(\partial^\beta_{(\xi,\eta)}a\right)\left(w,\lambda,\operatorname{sgn}(\lambda)\frac{\xi}{\sqrt{|\lambda|}},\frac{\eta}{\sqrt{|\lambda|}}\right)\right| \\ &\leq C\|a\|_{k,S_{\mathbb{H}^d}(\mu)}\left(1 + |\lambda| + \xi^2 + \eta^2\right)^{\frac{\mu}{2}}. \end{aligned}$$

The converse inequalities come easily: one has $a \in S_{\mathbb{H}^d}(\mu)$ if and only if for all $k, n \in \mathbb{N}$, there exists a constant $C_{n,k}$ such that for any $\beta \in \mathbb{N}^d$ satisfying $|\beta| \leq n$ and for all (w, λ, y, η) belonging to $\mathbb{H}^d \times \mathbb{R}^{2d+1}$,

$$\text{(B.2.2)} \qquad \left\| (\lambda \partial_\lambda)^k \partial^\beta_{(\xi,\eta)} (\sigma(a)) \right\|_{\mathscr{C}^\rho(\mathbb{H}^d)} \leq C_{n,k} \left(1 + |\lambda| + \xi^2 + \eta^2\right)^{\frac{\mu - |\beta|}{2}}.$$

We then remark that if $|\lambda| \leq 1$, the smoothness of $\sigma(a)$ yields that (B.2.1) implies on the compact $\{|\lambda| \leq 1\}$,

$$(1 + |\lambda|)^k \left\| \partial^k_\lambda \partial^\beta_{(\xi,\eta)} (\sigma(a)) \right\|_{\mathscr{C}^\rho(\mathbb{H}^d)} \leq C_{n,k} \left(1 + |\lambda| + \xi^2 + \eta^2\right)^{\frac{\mu - |\beta|}{2}}.$$

Besides, for $|\lambda| \geq 1$, (B.2.2) gives

$$\left\| \partial^k_\lambda \partial^\beta_{(\xi,\eta)} (\sigma(a)) \right\|_{\mathscr{C}^\rho(\mathbb{H}^d)} \leq C_{n,k} \left(1 + |\lambda| + \xi^2 + \eta^2\right)^{\frac{\mu - |\beta|}{2}} (1 + |\lambda|)^{-k}.$$

Conversely, if (1.4.2) holds, then one gets (B.2.2) since the function $\frac{|\lambda|^p}{(1+|\lambda|)^k}$ is bounded for any integer $p \in \{0, \ldots, k\}$. This ends the proof of the proposition. □

B.3. Symbols of functions of the harmonic oscillator

In this section we aim at proving that an operator $R(\xi^2 - \Delta_\xi)$ given as a function of the harmonic oscillator by functional calculus is a pseudodifferential operator, and at computing its (formal) symbol. We refer to Proposition 1.16 stated page 20 for a precise statement. Taking the inverse Fourier transform, we have by functional calculus

$$R(\xi^2 - \Delta_\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tau(\xi^2 - \Delta_\xi)} \widehat{R}(\tau)\, d\tau.$$

We then use Mehler's formula as in **[25]**, which gives (1.3.14) after an obvious change of variables.

We therefore have formally

$$\text{(B.3.1)} \qquad r(x) = \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (\cos \tau)^{-d} e^{i(x \operatorname{tg}\tau - y\tau)} R(y) d\tau\, dy,$$

and let us now prove that the function r is well defined outside $x = 0$, and that the map $(\xi, \eta) \mapsto r(\xi^2 + \eta^2)$ satisfies the symbol estimates of the class $S(m^\mu, g)$.

If $x \in \mathbb{R}^*$ is fixed, then (B.3.1) defines $r(x)$ as an oscillatory integral. Indeed the change of variables $u = \operatorname{tg}\tau$ performed on each interval of the form $\left]-\frac{\pi}{2} + k\pi, k\pi + \frac{\pi}{2}\right[$ for $k \in \mathbb{Z}$ turns the integral into a series of oscillatory integrals: we have $r(x) = \sum_{k \in \mathbb{Z}} r_k(x)$ with

$$\begin{aligned} r_k(x) &\stackrel{\text{def}}{=} \frac{1}{2\pi} (-1)^{kd} \int_{\mathbb{R}} e^{ixu} \widehat{R}\left(k\pi + \operatorname{Arctg} u\right) (1 + u^2)^{\frac{d}{2} - 1} du \\ &= \frac{1}{2\pi} (-1)^{kd} \int_{\mathbb{R} \times \mathbb{R}} e^{ixu - iy \operatorname{Arctg} u - iyk\pi} R(y) (1 + u^2)^{\frac{d}{2} - 1} du\, dy. \end{aligned}$$

We remark that these integrals have a non stationary phase for $|k| \geq 1$. This fact will be used below. We also observe that for $N_0 \in \mathbb{N}$, by integrations by parts,

$$\begin{aligned} k^{N_0} r_k(x) &= \frac{1}{2\pi} k^{N_0} (-1)^{kd} \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ixu - iy\mathrm{Arctg}u - iyk\pi} R(y)\,(1+u^2)^{\frac{d}{2}-1} du\,dy \\ &= \frac{1}{2\pi}\frac{(-i)^{N_0}}{\pi^{N_0}} (-1)^{kd} \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ixu - iyk\pi} (1+u^2)^{\frac{d}{2}-1} \partial_y^{N_0} \left(R(y)\mathrm{e}^{-iy\mathrm{Arctg}u}\right) du\,dy \\ &= \frac{1}{2\pi}\frac{(-i)^{N_0}}{\pi^{N_0}} (-1)^{kd} \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ixu - iyk\pi - iy\mathrm{Arctg}u} (1+u^2)^{\frac{d}{2}-1} f_{N_0}(y,u) du\,dy \end{aligned}$$

where $f_{N_0}(y,u) = \mathrm{e}^{iy\mathrm{Arctg}u}\partial_y^{N_0}\left(R(y)\mathrm{e}^{-iy\mathrm{Arctg}u}\right)$. The fact that the integrals $r_k(x)$ are well defined away from zero and that the series in k converges then comes from the following lemma.

Lemma B.1. — *Let f and g be two smooth functions on $\mathbb{R}$ such that*

$$\forall n \in \mathbb{N},\ \exists C > 0,\ \forall u \in \mathbb{R},\ |\partial^n g(u)| \leq C(1+u^2)^{\frac{\nu-n}{2}}$$
$$\forall n \in \mathbb{N},\ \exists C > 0,\ \forall y \in \mathbb{R},\ |\partial^n f(y)| \leq C(1+y^2)^{\frac{\mu-n}{2}},$$

for some $\mu, \nu \in \mathbb{R}$. Then for any $a > 0$, there exists a constant $C_0 > 0$ such that the function

$$I(f,g)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ixu - iy\mathrm{Arctg}u - iyk\pi} f(y)g(u)dy\,du$$

satisfies

$$\forall |x| \geq a,\ |I(f,g)(x)| \leq C_0(1+x^2)^{\frac{\mu}{2}}.$$

Before proving this lemma, let us show how to use it. The function $f_{N_0}(y,u)$ above writes as a sum of terms satisfying the assumptions of the Lemma. Therefore, $(1+|x|)^{-\mu} k^{N_0} r_k(x)$ is uniformly bounded in k and x whence the convergence of the series. To prove the symbol estimate, we notice that two integrations by parts give

$$\begin{aligned} xr'(x) &= i\,x \int_{\mathbb{R}\times\mathbb{R}} (\cos\tau)^{-d} \mathrm{tg}\tau \mathrm{e}^{ix\mathrm{tg}\tau - iy\tau} R(y) dy d\tau \\ &= x \int_{\mathbb{R}\times\mathbb{R}} (\cos\tau)^{-d} \frac{\mathrm{tg}\tau}{\tau} \mathrm{e}^{ix\mathrm{tg}\tau - iy\tau} R'(y) dy d\tau \\ &= -i \int_{\mathbb{R}\times\mathbb{R}} (\cos\tau)^{-d} \frac{\mathrm{tg}\tau}{\tau} (1+(\mathrm{tg}\tau)^2)^{-1} \partial_\tau \left(\mathrm{e}^{ix\mathrm{tg}\tau}\right) \mathrm{e}^{-iy\tau} R'(y) dy d\tau \\ &= i \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{-iy\tau + ix\mathrm{tg}\tau} \Bigg[-iy \left((\cos\tau)^{-d} \frac{\mathrm{tg}\tau}{\tau} (1+(\mathrm{tg}\tau)^2)^{-1} \right) \\ &\qquad + \partial_\tau \left((\cos\tau)^{-d} \frac{\mathrm{tg}\tau}{\tau} (1+(\mathrm{tg}\tau)^2)^{-1} \right) \Bigg] R'(y) dy d\tau. \end{aligned}$$

This last integral is an oscillatory integral of the same kind as the one defining $r(x)$, and can also be studied using Lemma B.1. This allows to obtain the symbol bounds, by iteration of the argument to any order of derivatives.

Now let us prove Lemma B.1. The idea, as is often the case in this paper, is to use a stationary phase method. The variable x may be seen as a parameter in the problem, and one notices easily that x may be factorized out of the phase after having the change of variable $y = x(1+t)$. Moreover one notices that the phase is stationary at the point $t = u = 0$, when $k = 0$. This implies that one should use a dyadic partition of unity centered at that stationary point. One furthermore notices that if $|u|^2 \ll |t|$, then the u-derivative of the phase is bounded from below, so it is enough to use a ∂_u vector field in the integrations by parts. As it produces naturally negative powers of t, one can deduce the convergence of the dyadic series. In the case $|t| \leq |u|^2$ however that vector field cannot work since the u-derivative of the phase may vanish. One must then use the whole vector field (in both u and t directions), and gaining negative powers of u turns out to be more difficult.

So let us start by performing the change of variables $y = x(1+t)$ so that $I(f,g)$ writes

$$I(f,g)(x) = x\, \mathrm{e}^{-ixk\pi} \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ix\Phi_k(u,t)} f(x(1+t))g(u)dtdu,$$

where

$$\Phi_k(u,t) \stackrel{\text{def}}{=} (u - \mathrm{Arctg} u) - t\,(\mathrm{Arctg} u + k\pi)\,.$$

The phase Φ_k satisfies

$$\partial_t \Phi_k = -\mathrm{Arctg} u - k\pi \quad \text{and} \quad \partial_u \Phi_k = \frac{u^2 - t}{1+u^2}\cdot$$

When $k \neq 0$, Φ_k is therefore non stationary, whereas when $k = 0$, Φ_0 has a non-degenerate stationary point in $(0,0)$. Therefore, we introduce a partition of unity on the real line:

$$\forall z \in \mathbb{R}, \quad 1 = \sum_{p \in \mathbb{N} \cup \{-1\}} \zeta_p(z)$$

with ζ_{-1} compactly supported in a ball and for $p \in \mathbb{N}$, $\zeta_p(z) = \zeta(2^{-p}z)$ where ζ is compactly supported in a ring. We get

$$I(f,g) = \mathrm{e}^{-ixk\pi} \sum_{p,q \in \mathbb{N} \cup \{-1\}} I_{p,q}(f,g)$$

with

$$I_{p,q}(f,g)(x) \stackrel{\text{def}}{=} x \int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ix\Phi_k(u,t)} \zeta_p(t)\zeta_q(u) f(x(1+t))g(u)dtdu.$$

These integrals are now well-defined because they are integrals of smooth compactly supported functions. We have to prove the convergence of the series in p and q. As explained above, we shall argue differently whether $|u|^2 \ll |t|$ or not. So let us fix a parameter $\varepsilon < 1/3$, to be chosen appropriately below, and let us separate the study into two subcases, depending whether $2^p > 2^{2q(1+\varepsilon)}$ (which corresponds to the case $|u|^2 \ll |t|$) or $2^p \leq 2^{2q(1+\varepsilon)}$.

Let us suppose $p > 2q(1+\varepsilon)$. We observe that in that case one has $u^2 - t \neq 0$ on the support of $\zeta_p(t)\zeta_q(u)$, so as explained above one can use integrations by parts with the vector field

$$\ell \stackrel{\text{def}}{=} (i\partial_u \Phi_k)^{-1}\,\partial_u. \tag{B.3.2}$$

Of course one has

$$\ell\,(\exp(ix\Phi_k)) = x\exp(i\Phi_k)\,.$$

Performing N integrations by parts for $N \in \mathbb{N}$, we find

$$I_{p,q}(f,g)(x) \;=\; x^{1-N}\int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ix\Phi_k}(\ell^*)^N\Big(f(x(1+t))g(u)\zeta_p(t)\zeta_q(u)\Big)dtdu.$$

We then write

$$\ell^* = -\ell + ic$$

where

$$c \stackrel{\text{def}}{=} -\frac{\partial_u^2\Phi_k}{(\partial_u\Phi_k)^2} = -2\,\frac{u(1+t)}{(1+u^2)^2}\,\frac{(1+u^2)^2}{(u^2-t)^2} = -2\,\frac{u(1+t)}{(u^2-t)^2}\cdot$$

Let us analyze the properties of ℓ^*. If (u,t) belongs to the support of $\zeta_q(u)\zeta_p(t)$, we have for $p > 2q(1+\varepsilon)$

$$c_2\,2^p \leq c_1 2^p - C_1 2^{2q} \leq |t-u^2| \leq C_1\,2^p(1+2^{2q-p}) \leq C_2\,2^p.$$

We infer that

$$|\partial_u\Phi_k|^{-1} \leq C\,2^{-p+2q} \;\text{ and }\; |c| \leq C\,2^{-p+q}.$$

Using $q < \frac{p}{2(1+\varepsilon)}$, we have

$$-p+2q < \Big(-1+\frac{1}{1+\varepsilon}\Big)p = -\frac{\varepsilon}{1+\varepsilon}p < -\frac{\varepsilon}{1+\varepsilon}\,\frac{p}{2} - \varepsilon q$$

so that there exists some $\delta > 0$ such that on the integration domain

$$|\partial_u\Phi_k|^{-1} + |c| \leq C\,2^{-\delta(p+q)}. \tag{B.3.3}$$

By induction one actually also can prove that

$$\forall m \in \mathbb{N}, \quad |\partial_u^m c| \leq C2^{-m\delta(p+q)}. \tag{B.3.4}$$

Now we shall use the Leibniz formula in order to evaluate $(\ell^*)^N\Big(f(x(1+t))g(u)\zeta_p(t)\zeta_q(u)\Big)$. This generates three typical terms:

$$\begin{aligned}
(1) &\stackrel{\text{def}}{=} (\partial_u\Phi_k)^{-N}\partial_u^N\left(\zeta_q(u)g(u)\right)f(x(1+t))\zeta_p(t),\\
(2) &\stackrel{\text{def}}{=} c^N\zeta_q(u)g(u)f(x(1+t))\zeta_p(t) \quad\text{and}\\
(3) &\stackrel{\text{def}}{=} \sum_{\substack{n+m+p=N\\ n,m,p<N}} c^n\,\partial_u^m c\,(\partial_u\Phi_k)^{-p}\partial_u^p\left(\zeta_q(u)g(u)\right)f(x(1+t))\zeta_p(t).
\end{aligned}$$

Due to the estimates (B.3.3) and (B.3.4), it turns out that the term (3) is an intermediate case between (1) and (2) so we shall only study the two first types of terms here.

We observe that defining $\widetilde{\zeta}(u) = \sup_{n\leq N} |\zeta^{(n)}(u)|$ and using the symbol estimate on g, we have

$$|\partial_u^N (\zeta_q(u)g(u))| \leq C\,(1+|u|)^\nu\, 2^{-qN}\widetilde{\zeta}_q(u)$$

so by (B.3.3) and using the symbol estimate on f we obtain that

$$|(1)| \leq C\, 2^{-qN} 2^{-\delta N(p+q)}\,(1+|u|)^\nu\,(1+|x(1+t)|)^\mu\,\zeta_p(t)\widetilde{\zeta}_q(u).$$

Using Peetre's inequality

$$(1+|x(1+t)|)^\mu \leq C\,(1+|x|)^\mu (1+|xt|)^{|\mu|},$$

we therefore conclude that (recalling that x is away from zero)

$$x^{1-N}\int_{\mathbb{R}\times\mathbb{R}} |(1)|dtdu \leq C\,|x|^\mu |x|^{1-N+|\mu|} 2^{-\delta N(p+q)+q\nu+p|\mu|+p+q-qN}. \tag{B.3.5}$$

A similar argument allows to deal with the second term. Indeed we have

$$|(2)| \leq C\, 2^{-\delta N(p+q)}\,(1+|u|)^\nu\,(1+|x(1+t)|)^\mu\,\zeta_p(z)\zeta_q(u) \tag{B.3.6}$$

By integration we obtain

$$x^{1-N}\int_{\mathbb{R}\times\mathbb{R}} |(2)|\,dudz \leq C|x|^\mu |x|^{1-N+|\mu|} 2^{-\delta N(p+q)+q\nu+p|\mu|+p+q}.$$

Therefore, choosing $N > \delta^{-1}\mathrm{Max}(\nu+1, |\mu|+1)$, we obtain the convergence in p and q of the series, uniformly with respect to k and x in the set $\{|x| \geq a\}$, with the expected bound $|x|^\mu$.

Let us now suppose $p \leq 2q(1+\varepsilon)$. The objective is now to gain negative powers of 2^q. The difficulty then comes from the fact that $\partial_u\Phi_k$ may vanish. We observe that for this range of indexes p and q, we have $q \geq 0$ so that the integral is supported far from $u=0$. For this reason, if χ is a smooth cut-off function, compactly supported in the unit ball and identically equal to one near zero, then the function

$$(t,u) \mapsto \chi\left(\frac{t-u^2}{u^\kappa}\right)$$

is a smooth function for any $\kappa \in \mathbb{R}$. The value of κ will be chosen later.

We now cut $I_{p,q}$ into two parts, writing $I_{p,q} = I^1_{p,q} + I^2_{p,q}$ with

$$I^1_{p,q}(x) \stackrel{\text{def}}{=} x\int_{\mathbb{R}\times\mathbb{R}} e^{ix\Phi_k}\left(1-\chi\left(\frac{t-u^2}{u^\kappa}\right)\right) f(x(1+t))g(u)\zeta_p(t)\zeta_q(u)dt\,du.$$

Let us study first $I^1_{p,q}$. We notice that on the domain of integration, one has $|t-u^2| \geq C|u|^\kappa$, so on the support of ζ_q we have $|t-u^2| \geq C\,2^{\kappa q}$. It follows that

$$\left|\frac{t-u^2}{1+u^2}\right| \geq C\,2^{(\kappa-2)q},$$

which leads to

$$|\partial_u\Phi_k|^{-1} \leq C\,2^{-(\kappa-2)q}. \tag{B.3.7}$$

Therefore the u-derivative of the phase does not vanish in this case, so we may use again the vector field ℓ defined in (B.3.2). The coefficients of that vector field are now of order $2^{-(\kappa-2)q}$ and one has

$$|c| = \left|-2\frac{u(1+t)}{(u^2-t)^2}\right| \leq C\frac{2^q(1+2^p)}{2^{2\kappa q}} \leq C\,2^{-2\kappa q+3q(1+\varepsilon)}. \tag{B.3.8}$$

We therefore choose κ such that $2\kappa > 3(1+\varepsilon)$. By induction, one sees that

$$\forall m \in \mathbb{N}, \quad |\partial^m c| \leq C\,2^{-mq-2\kappa q+3q(1+\varepsilon)}. \tag{B.3.9}$$

We can write

$$I^1_{p,q}(x) = x^{1-N}\int_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{ix\Phi_k}\,(\ell^*)^N\left[\left(1-\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)g(u)\zeta_q(u)\right]f(x(1+t))\zeta_p(t)dt\,du.$$

Compared to the case studied above, the terms generated by $(\ell^*)^N$ are of the form

$$(1') \quad \stackrel{\text{def}}{=} \quad (\partial_u\Phi_k)^{-N}\partial_u^N\left(\left(1-\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\zeta_q(u)g(u)\right)f(x(1+t))\zeta_p(t),$$

$$(2') \quad \stackrel{\text{def}}{=} \quad c^N\left(1-\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\zeta_q(u)g(u)f(x(1+t))\zeta_p(t) \quad \text{and}$$

$$(3') \quad \stackrel{\text{def}}{=} \sum_{\substack{n+m+p=N\\ n,m,p<N}} c^n\partial_u^m c(\partial_u\Phi_k)^{-p}\partial_u^p\left(\left(1-\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\zeta_q(u)g(u)\right)f(x(1+t))\zeta_p(t).$$

As in the previous case and due to (B.3.8) and (B.3.9), it is enough to control the two first terms.

Thanks to (B.3.8), the term $(2')$ is bounded exactly as before, assuming that $2\kappa > 3(1+\varepsilon)$. Now let us study $(1')$. As above we apply the Leibniz formula, which compared to the previous case generates derivatives of χ. However they produce negative powers of 2^q, as one differentiation gives the term

$$\chi'\left(\frac{t-u^2}{u^\kappa}\right)\left[-\frac{2}{u^{\kappa-1}}-\frac{\kappa}{u}\left(\frac{t-u^2}{u^\kappa}\right)\right]$$

which may easily be bounded by

$$\left|\chi'\left(\frac{t-u^2}{u^\kappa}\right)\left[-\frac{2}{u^{\kappa-1}}-\frac{\kappa}{u}\left(\frac{t-u^2}{u^\kappa}\right)\right]\right| \leq C(2^{-q(\kappa-1)}+2^{-q}) \leq C2^{-q(\kappa-1)}$$

assuming moreover that $\kappa \leq 2$, which is possible since e $< 1/3$. Similarly m derivatives produce $2^{-q(\kappa-1)m}$, and it is easy to conclude that $(1')$ may be dealt with as above, hence can also be summed over q and p (recalling that $p \leq 2q(1+\varepsilon)$, so that decay in 2^q is enough to conclude to both summations).

Now let us study $I^2_{p,q}$, which is more challenging as the u-derivative of the phase can now vanish. We therefore need to use the full vector field

$$L_k \stackrel{\text{def}}{=} \frac{1}{i}|\nabla\Phi_k|^{-2}\nabla\Phi_k\cdot\nabla$$

which satisfies

$$L_k\left(\exp\left(ix\Phi_k\right)\right) = x\exp\left(ix\Phi_k\right).$$

Let us check that this vector field is well defined: on the one hand if $k = 0$, then the assumption $q \geq \frac{p}{2(1+\varepsilon)}$ implies $q \geq 0$, thus u is supported on a ring and $|\text{Arctg} u| \geq c_0$ on the support of $\zeta_q(u)$. On the other hand one notices that $|\nabla\Phi_k|^2 \geq (\text{Arctg} u + k\pi)^2 \geq c_0^2$ for $k \geq 1$. It follows that there is a universal constant such that for any $k \geq 0$ and on the domain of integration, the following bound holds:

$$|\nabla\Phi_k|^{-1} \leq C.$$

Moreover we have

$$L_k^* = -L_k + c_k$$

with

$$\begin{aligned} c_k &\stackrel{\text{def}}{=} -\frac{1}{i}\nabla\cdot\left(|\nabla\Phi_k|^{-2}\nabla\Phi_k\right) \\ &= -\frac{1}{i}\left[\frac{\partial_u^2\Phi_k}{|\nabla\Phi_k|^2} - 2\frac{\partial_u\Phi_k}{|\nabla\Phi_k|^4}\left(\partial_u^2\Phi_k\partial_u\Phi_k + \partial_{ut}^2\Phi_k\partial_t\Phi_k\right) - 2\frac{\partial_t\Phi_k}{|\nabla\Phi_k|^4}\partial_{tu}^2\Phi_k\partial_u\Phi_k\right] \\ &= -\frac{1}{i}\left[\frac{\partial_u^2\Phi_k}{|\nabla\Phi_k|^2} - \frac{2}{|\nabla\Phi_k|^4}\left((\partial_u\Phi_k)^2\partial_u^2\Phi_k + 2\partial_{tu}^2\Phi_k\partial_t\Phi_k\partial_u\Phi_k\right)\right]. \end{aligned}$$

In view of

$$\partial_u^2\Phi_k = 2\,\frac{u(1+t)}{(1+u^2)^2} \quad \text{and} \quad \partial_{ut}^2\Phi_k = -\frac{1}{1+u^2}$$

we have

$$|c_k| \leq C\,|\nabla\Phi_k|^{-2}\left(2^{p-3q} + 2^{-2q}\right) \leq C\,2^{-(1-2\varepsilon)q}. \tag{B.3.10}$$

An easy induction left to the reader actually shows that

$$\forall\alpha\in\mathbb{N}^2, \quad |\partial_{(u,t)}^\alpha c_k| \leq C\,2^{-(|\alpha|+1)(1-2\varepsilon)q}. \tag{B.3.11}$$

We then write for $N \in \mathbb{N}$

$$I_{p,q}^2 = x^{1-N}\int \mathrm{e}^{ix\Phi_k}\,(L_k^*)^N\left[\chi\left(\frac{t-u^2}{u^\kappa}\right)f(x(1+t))g(u)\zeta_p(t)\zeta_q(u)\right]dt\,du.$$

Now we need to understand the action of the operator $(L_k^*)^N$. The main difficulty will come from the t-derivative, which does not produce directly negative powers of u. However we notice that on the domain of integration, one has

$$t = u^2 + Zu^\kappa \quad \text{with } |Z| \leq 1,$$

so since κ has been chosen smaller than 2, there is a constant $c > 0$ such that

$$|t| \geq |u|^2 - |Zu^\kappa| \geq c|u|^2.$$

This means that the domain of summation is actually essentially restricted to

$$2q \leq p \leq 2q(1+\mathrm{e}) \tag{B.3.12}$$

so it suffices to gain negative powers of t to conclude to convergence.

The constant term c_k has already been computed and estimated in (B.3.10)-(B.3.11). Moreover following similar computations to above, for any given function F one may write that

$$
\begin{aligned}
|(L_k^*)^N F| \quad &\leq \quad C \sup_{|\alpha|=N} |\partial^\alpha_{(u,t)} F| + |c_k^N F| \\
&+ \quad C \sum_{\substack{|\alpha+\beta|+m=N \\ |\alpha|,|\beta|,m<N}} |\partial^\alpha_{(u,t)} c_k| \, |c_k|^m \, |\partial^\beta_{(u,t)} F|.
\end{aligned} \tag{B.3.13}
$$

The first step of the analysis therefore consists in estimating, for any $|\beta| \leq N$, the quantity

$$\sum_{m+m'=|\beta|} \partial_u^m \partial_t^{m'} \left(\chi\left(\frac{t-u^2}{u^\kappa}\right) \zeta_q(u) g(u) \zeta_p(t) f(x(1+t))\right).$$

Let us start by studying the action of the u-differentiations on $\chi\left(\frac{t-u^2}{u^\kappa}\right) g(u)\zeta_q(u)$. On the one hand one has, using the symbol estimate on g,

$$|\partial_u^m (\zeta_q(u) g(u))| \leq C 2^{q(\nu-m)} \widetilde{\zeta}_q(u)$$

where $\widetilde{\zeta}_q(u) \stackrel{\text{def}}{=} \sup_{m\leq N} |\partial_u^m \zeta_q(u)|$. This can in turn be written

$$|\partial_u^m (\zeta_q(u) g(u))| \leq C 2^{q(\nu-m)} \overline{\zeta}(2^{-q} u) \tag{B.3.14}$$

where $\overline{\zeta}$ is a nonnegative, smooth compactly supported function such that $\overline{\zeta} = 1$ on the support of ζ.

On the other hand, as we have seen above one has the following identity:

$$\partial_u \left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right) = \chi'\left(\frac{t-u^2}{u^\kappa}\right) \left[-\frac{2}{u^{\kappa-1}} - \frac{\kappa}{u} \left(\frac{t-u^2}{u^\kappa}\right)\right]$$

so since the support of χ' does not touch zero, one has on the support of ζ_q the following estimate:

$$\left|\partial_u \left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\right| \leq C(2^{-q(\kappa-1)} + 2^{-q}) \leq C 2^{-q(\kappa-1)},$$

as soon as $\kappa \leq 2$. Actually by induction one also has

$$\forall m \in \mathbb{N}, \quad \left|\partial_u^m \left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\right| \leq C 2^{-q(\kappa-1)m}. \tag{B.3.15}$$

The Leibniz formula yields for any $m \leq N$

$$\left|\partial_u^m \left(\chi\left(\frac{t-u^2}{u^\kappa}\right) \zeta_q(u) g(u)\right)\right| \leq C \sum_{m'\leq m} \binom{m}{m'} \left|\partial_u^{m'} (\zeta_q(u) g(u))\right| \left|\partial_u^{m-m'} \left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\right|$$

whence by (B.3.14) and (B.3.15) the estimate

$$\left|\partial_u^m \left(\chi\left(\frac{t-u^2}{u^\kappa}\right) \zeta_q(u) g(u)\right)\right| \leq C 2^{q(\nu-(\kappa-1)m)} \overline{\zeta}(2^{-q} u). \tag{B.3.16}$$

Now let us consider t-derivatives. The Leibniz formula again implies that for any $m' \leq N$

$$\text{(B.3.17)}\quad \partial_t^{m'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\zeta_p(t)f(x(1+t))\right) = \chi\left(\frac{t-u^2}{u^\kappa}\right)\partial_t^{m'}\left(\zeta_p(t)f(x(1+t))\right) + \sum_{0<n'\leq m'}\begin{pmatrix} m' \\ n' \end{pmatrix}\partial_t^{n'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\partial_t^{m'-n'}\left(\zeta_p(t)f(x(1+t))\right).$$

For the second term in the right-hand side of (B.3.17), one uses the fact that on the support of ζ_q, one has the estimate

$$\text{(B.3.18)}\quad \begin{aligned}\left|\partial_t^{n'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\right| &= \frac{1}{|u|^{n'\kappa}}\left|\chi^{(n')}\left(\frac{t-u^2}{u^\kappa}\right)\right| \\ &\leq C2^{-qn'\kappa}.\end{aligned}$$

In order to also control the action of multiple differentiations in the t and u directions of $\partial_u\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)$, it is useful to notice that

$$\partial_u\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right) = -\frac{2}{u^{\kappa-1}}\chi'\left(\frac{t-u^2}{u^\kappa}\right) + \frac{\kappa}{u}\widetilde{\chi}\left(\frac{t-u^2}{u^\kappa}\right)$$

where $\widetilde{\chi}$ is a smooth compactly supported function. So t-derivatives of $\partial_u\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)$ are controled exactly like $\partial_t\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)$.

Estimate (B.3.18) gives, along with the symbol estimate satisfied by f, for any $n' \leq m'$,

$$\left|\partial_t^{n'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\partial_t^{m'-n'}\left(\zeta_p(t)f(x(1+t))\right)\right| \leq C2^{-qn'\kappa}2^{-p(m'-n')}(1+|x(1+t)|)^\mu\,\overline{\zeta}(2^{-p}t),$$

where again $\overline{\zeta}$ is a nonnegative, smooth compactly supported function such that $\overline{\zeta}=1$ on the support of ζ.

Peetre's inequality allows finally to write that for any $m' \leq N$ and any $0 < n' \leq m'$,

$$\left|\partial_t^{n'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\right)\partial_t^{m'-n'}\left(\zeta_p(t)f(x(1+t))\right)\right| \leq C2^{-q\kappa}2^{-p(m'-n')}(1+|x|)^\mu(1+|xt|)^{|\mu|}\overline{\zeta}(2^{-p}t),$$

hence for any $m' \leq N$, we get

$$\text{(B.3.19)}\quad \begin{aligned}&\left|\sum\nolimits_{0<n'\leq m'}\begin{pmatrix} m' \\ n' \end{pmatrix}\partial_t^{n'}\left(\chi\left(\tfrac{t-u^2}{u^\kappa}\right)\right)\partial_t^{m'-n'}\left(\zeta_p(t)f(x(1+t))\right)\right| \\ &\qquad\leq C2^{-q\kappa}2^{-p(m'-n')}(1+|x|)^\mu(1+|xt|)^{|\mu|}\overline{\zeta}(2^{-p}t) \\ &\qquad\leq C2^{-q\kappa+p|\mu|}(1+|x|)^{\mu+|\mu|}\overline{\zeta}(2^{-p}t).\end{aligned}$$

Finally let us deal with the first term on the right-hand side of (B.3.17). We write, using Peetre's inequality again, that

$$\text{(B.3.20)}\quad \left|\chi\left(\frac{t-u^2}{u^\kappa}\right)\partial_t^{m'}\left(\zeta_p(t)f(x(1+t))\right)\right| \leq C2^{-p(m'-|\mu|)}(1+|x|)^{\mu+|\mu|}\overline{\zeta}(2^{-p}t),$$

and plugging (B.3.19) and (B.3.20) into (B.3.17) therefore gives

$$\left|\partial_t^{m'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\zeta_p(t)f(x(1+t))\right)\right| \leq C\overline{\zeta}(2^{-p}t)(2^{-q\kappa+p|\mu|}+2^{-p(m'-|\mu|)})(1+|x|)^{\mu+|\mu|}.$$

Putting the above estimate together with (B.3.16) allows to obtain that

$$\sum_{m+m'=|\beta|}\partial_u^m\partial_t^{m'}\left(\chi\left(\frac{t-u^2}{u^\kappa}\right)\zeta_q(u)g(u)\zeta_p(t)f(x(1+t))\right)$$
$$\leq C\overline{\zeta}(2^{-p}t)\overline{\zeta}(2^{-q}u)\sum_{m+m'=|\beta|}2^{q(\nu-(\kappa-1)m)}(2^{-q\kappa+p|\mu|}+2^{-p(m'-|\mu|)})(1+|x|)^{\mu+|\mu|},$$

hence, bounding p by $2q(1+\mathrm{e})$, we get

$$(1+|x|)^{-\mu-|\mu|}\sum_{m+m'=|\beta|}\partial_u^m\partial_t^{m'}\left(\chi\left(\tfrac{t-u^2}{u^\kappa}\right)\zeta_q(u)g(u)\zeta_p(t)f(x(1+t))\right)$$
$$\leq C\overline{\zeta}(2^{-p}t)\overline{\zeta}(2^{-q}u)\sum_{m+m'=|\beta|}2^{q(\nu+2|\mu|(1+\mathrm{e})-(\kappa-1)\mathrm{m})}(2^{-q\kappa}+2^{-pm'}). \tag{B.3.21}$$

Finally let us go back to (B.3.13). Denoting $\widetilde{\mu}\stackrel{\text{def}}{=}2|\mu|(1+\mathrm{e})$ and choosing

$$F\stackrel{\text{def}}{=}\chi\left(\frac{t-u^2}{u^\kappa}\right)f(x(1+t))g(u)\zeta_p(t)\zeta_q(u),$$

one has the following estimate:

$$(1+|x|)^{-\mu-|\mu|}|(L_k^*)^N F| \leq C\overline{\zeta}(2^{-p}t)\overline{\zeta}(2^{-q}u)2^{q(\nu+\widetilde{\mu})}$$
$$\times\sum_{\substack{|\alpha|+|\beta|+n=N\\|\alpha|,|\beta|,n<N}}\sum_{m\leq|\beta|}2^{-(|\alpha|+1)(1-2\mathrm{e})\mathrm{q}-\mathrm{n}(1-2\mathrm{e})\mathrm{q}-\mathrm{q}(\kappa-1)\mathrm{m}}(2^{-q\kappa}+2^{-p(|\beta|-m)})$$
$$+C2^{-N(1-2\mathrm{e})\mathrm{q}}+C\overline{\zeta}(2^{-p}t)\overline{\zeta}(2^{-q}u)2^{q(\nu+\widetilde{\mu})}\sum_{m+m'=N}2^{-q(\kappa-1)m}(2^{-q\kappa}+2^{-pm'}).$$

using the above estimate along with (B.3.11) and (B.3.21).

The conclusion comes from (B.3.12). This ends the proof of the proposition. □

B.4. The symbol of Littlewood-Paley operators on the Heisenberg group

In this section we shall prove Proposition 4.18 stated in Chapter 4.1, giving the symbol of the Littlewood-Paley truncation operators. The proof relies on the arguments of the previous section, proving Proposition 1.16.

Recall that as defined in Definition 4.3,

$$\mathcal{F}(\Delta_p f)(\lambda)=\mathcal{F}(f)(\lambda)R^*(2^{-2p}D_\lambda)=\mathcal{F}(f)(\lambda)J_\lambda^*R^*(2^{-2p}4|\lambda|(-\Delta_\xi+\xi^2))J_\lambda.$$

If χ is a smooth cut-off function compactly supported on $\mathbb{R}$ and such that $\chi(\lambda)=1$ for $|\lambda|\leq 4$ and $\chi(\lambda)=0$ for $|\lambda|>5$, then

$$\mathcal{F}(\Delta_p f)(\lambda)=\mathcal{F}(f)(\lambda)J_\lambda^*R^*(2^{-2p}4|\lambda|(-\Delta_\xi+\xi^2))\chi(2^{-2p}\lambda)J_\lambda.$$

It will be important in the following to notice that for fixed p, we are only concerned with bounded frequencies λ.

We now apply Proposition 1.16 and write

$$R^*(2^{-2p}4|\lambda|(-\Delta_\xi + \xi^2)) = \mathrm{op}^w\left(\Phi_p(\lambda,\xi,\eta)\right)$$

with

$$\Phi_p(\lambda,\xi,\eta) = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} (\cos\tau)^{-d}\, \mathrm{e}^{i((\xi^2+\eta^2)\mathrm{tg}\tau - r\tau)} R^*(2^{-2p+2}|\lambda|r)dr\, d\tau. \tag{B.4.1}$$

For $\lambda \neq 0$, a change of variable shows that $\Phi_p(\lambda,\xi,\eta) = \phi(2^{-2p}|\lambda|, 2^{-2p}|\lambda|(\xi^2+\eta^2))$ as stated in Proposition 4.18.

Let us prove now that $\Phi_p \in S_{\mathbb{H}^d}(0)$. Actually due to the comment above, it is enough to prove that the function $(\lambda,\xi,\eta) \mapsto \Phi_p(\lambda,\xi,\eta)\chi(2^{-2p}\lambda)$ is a symbol in $S_{\mathbb{H}^d}(0)$. It is moreover enough to prove it for $p = 0$.

We first observe that by Proposition 1.16, $\Phi_0\left(\lambda, \mathrm{sgn}(\lambda)\frac{\xi}{\sqrt{|\lambda|}}, \frac{\eta}{\sqrt{|\lambda|}}\right) = \phi(|\lambda|, \xi^2 + \eta^2)$ is well defined for $\lambda \neq 0$ and is a symbol in $S(1,g)$ for any λ. Besides, Remark 4.19 gives that Φ_0 has the required regularity close to $\lambda = 0$, and as noted above one can also restrict our attention to a compact set in λ. All those observations imply that to prove that the function $\Phi_0(\lambda,\xi,\eta)$ belongs to the symbol class $S_{\mathbb{H}^d}(0)$, it is enough due to Proposition 1.20 to prove the following estimate: for any compact set K of $\mathbb{R}^*$,

(B.4.2)

$$\forall k,n \in \mathbb{N},\ \exists C_{k,n} > 0,\ \forall \rho \in \mathbb{R},\ \forall \lambda \in K,\ \left|(1+\rho^2)^{\frac{n}{2}}(\lambda\partial_\lambda)^k \partial_\rho^n \phi(\lambda,\rho)\right| \leq C_{n,k}.$$

We point out that by Proposition 1.16, we already now that this estimate is true for λ fixed in $\mathbb{R}^*$. Moreover since λ belongs to a compact set, it is enough to consider the $\lambda\partial_\lambda$ derivatives and to prove that $(\lambda\partial_\lambda)\phi(\lambda,\rho)$ may be bounded independently of λ.

In fact we shall prove that $\lambda\partial_\lambda\phi(\lambda,\rho)$ has the same integral form as ϕ, which by a direct induction will allow to conclude the proof of the proposition. So let us compute $\lambda\partial_\lambda\phi(\lambda,\rho)$. We have

$$\lambda\partial_\lambda\phi(\lambda,\rho) = \frac{1}{2\lambda\pi}\int (\cos\tau)^{-d}\mathrm{e}^{\frac{i}{\lambda}(\rho\mathrm{tg}\tau - r\tau)}\left(-\frac{i}{\lambda}(\rho\mathrm{tg}\tau - r\tau) - 1\right) R^*(4r)dr\, d\tau,$$

so integrating by parts we get

$$\lambda\partial_\lambda\phi(\lambda,\rho) = -\frac{1}{2\lambda\pi}\int (\cos\tau)^{-d}\mathrm{e}^{\frac{i}{\lambda}(\rho\mathrm{tg}\tau - r\tau)}\left[\partial_r\left((\rho\frac{\mathrm{tg}\tau}{\tau} - r)R^*(4r)\right) + R^*(4r)\right] dr\, d\tau,$$

which gives finally

$$\lambda\partial_\lambda\phi(\lambda,\rho) = -\frac{1}{2\lambda\pi}\int (\cos\tau)^{-d}\mathrm{e}^{\frac{i}{\lambda}(\rho\mathrm{tg}\tau - r\tau)}\left[4\frac{\rho\,\mathrm{tg}\tau - r\tau}{\tau}\,(R^*)'(4r)\right] dr\, d\tau.$$

One then notices that

$$\rho\mathrm{e}^{\frac{i}{\lambda}(\rho\mathrm{tg}\tau)} = \frac{\lambda}{i}(1+(\mathrm{tg}\tau)^2)^{-1}\partial_\tau\left(\mathrm{e}^{\frac{i}{\lambda}\rho\mathrm{tg}\tau}\right),$$

which allows to transform the integral into

$$\begin{aligned}\lambda\partial_\lambda\phi(\lambda,\rho) &= \frac{2}{\lambda\pi}\int(\cos\tau)^{-d}e^{\frac{i}{\lambda}(\rho\mathrm{tg}\tau-r\tau)}(R^*)'(4r)\,dr\,d\tau\\ &- \frac{2}{i\pi}\int(\cos\tau)^{-d}\frac{\mathrm{tg}\tau}{\tau(1+(\mathrm{tg}\tau)^2)}e^{-ir\tau}\partial_\tau\left(e^{\frac{i}{\lambda}\rho\mathrm{tg}\tau}\right)(R^*)'(4r)\,dr\,d\tau.\end{aligned}$$

The first integral on the right-hand side is exactly of the same form as ϕ, so to conclude we need to prove that the second integral can also be written in a similar way. Let us perform an integration by parts in the τ variable. This produces the following identity:

$$\begin{aligned}&\int(\cos\tau)^{-d}\frac{\mathrm{tg}\tau}{\tau(1+(\mathrm{tg}\tau)^2)}e^{-ir\tau}\partial_\tau\left(e^{\frac{i}{\lambda}\rho\mathrm{tg}\tau}\right)dr\,d\tau\\ &\qquad=\int e^{-ir\tau+\frac{i}{\lambda}\rho\mathrm{tg}\tau}\left(ir-\partial_\tau\left((\cos\tau)^{-d}\frac{\mathrm{tg}\tau}{\tau(1+(\mathrm{tg}\tau)^2)}\right)\right)(R^*)'(4r)\,dr\,d\tau\end{aligned}$$

which again is of a similar form that can be dealt with as in the proof of Proposition 1.16.

The proof of Proposition 4.18 is complete. □

BIBLIOGRAPHY

[1] S. ALINHAC & P. GÉRARD – *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels, InterÉditions, 1991.

[2] L. AMBROSIO & S. RIGOT – "Optimal mass transportation in the Heisenberg group", *J. Funct. Anal.* **208** (2004), p. 261–301.

[3] H. BAHOURI & J.-Y. CHEMIN – "Équations d'ondes quasilinéaires et estimations de Strichartz", *Amer. J. Math.* **121** (1999), p. 1337–1377.

[4] H. BAHOURI, J.-Y. CHEMIN & I. GALLAGHER – "Refined Hardy inequalities", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5** (2006), p. 375–391.

[5] H. BAHOURI & I. GALLAGHER – "Paraproduit sur le groupe de Heisenberg et applications", *Rev. Mat. Iberoamericana* **17** (2001), p. 69–105.

[6] ———, "The heat kernel and frequency localized functions on the Heisenberg group", in *Advances in phase space analysis of partial differential equations*, Progr. Nonlinear Differential Equations Appl., vol. 78, Birkhäuser, 2009, p. 17–35.

[7] H. BAHOURI, P. GÉRARD & C.-J. XU – "Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg", *J. Anal. Math.* **82** (2000), p. 93–118.

[8] R. W. BEALS – "Weighted distribution spaces and pseudodifferential operators", *J. Analyse Math.* **39** (1981), p. 131–187.

[9] R. W. BEALS, B. GAVEAU, P. C. GREINER & J. VAUTHIER – "The Laguerre calculus on the Heisenberg group. II", *Bull. Sci. Math.* **110** (1986), p. 225–288.

[10] R. W. BEALS & P. C. GREINER – *Calculus on Heisenberg manifolds*, Annals of Math. Studies, vol. 119, Princeton Univ. Press, 1988.

[11] J. BERGH & J. LÖFSTRÖM – *Interpolation spaces. An introduction*, Grundl. Math. Wiss., vol. 223, Springer, 1976.

[12] I. BIRINDELLI & E. VALDINOCI – "The Ginzburg-Landau equation in the Heisenberg group", *Commun. Contemp. Math.* **10** (2008), p. 671–719.

[13] J.-M. BONY – "Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires", *Ann. Sci. École Norm. Sup.* **14** (1981), p. 209–246.

[14] J.-M. BONY & J.-Y. CHEMIN – "Espaces fonctionnels associés au calcul de Weyl-Hörmander", *Bull. Soc. Math. France* **122** (1994), p. 77–118.

[15] J.-M. BONY & N. LERNER – "Quantification asymptotique et microlocalisations d'ordre supérieur. I", *Ann. Sci. École Norm. Sup.* **22** (1989), p. 377–433.

[16] R. W. BROCKETT – "Control theory and singular Riemannian geometry", in *New directions in applied mathematics (Cleveland, Ohio, 1980)*, Springer, 1982, p. 11–27.

[17] C. E. CANCELIER, J.-Y. CHEMIN & C.-J. XU – "Calcul de Weyl et opérateurs sous-elliptiques", *Ann. Inst. Fourier (Grenoble)* **43** (1993), p. 1157–1178.

[18] L. CAPOGNA, D. DANIELLI, S. D. PAULS & J. T. TYSON – *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Progress in Math., vol. 259, Birkhäuser, 2007.

[19] J.-Y. CHEMIN & C.-J. XU – "Inclusions de Sobolev en calcul de Weyl-Hörmander et champs de vecteurs sous-elliptiques", *Ann. Sci. École Norm. Sup.* **30** (1997), p. 719–751.

[20] R. R. COIFMAN & Y. MEYER – *Au delà des opérateurs pseudo-différentiels*, Astérisque, vol. 57, Soc. Math. France, 1978.

[21] S. CORÉ & D. GELLER – "Hörmander type pseudodifferential calculus on homogeneous groups", preprint arXiv:0802.3452.

[22] L. J. CORWIN & F. P. GREENLEAF – *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Math., vol. 18, Cambridge Univ. Press, 1990.

[23] J. FARAUT & K. HARZALLAH – "Deux cours d'analyse harmonique", in *École d'été d'analyse harmonique de Tunis*, Progress in Math., Birkhäuser, 1984.

[24] J. FARAUT & L. SAAL – "The Wigner semi-circle law and the Heisenberg group", in *Noncommutative harmonic analysis with applications to probability*, Banach Center Publ., vol. 78, Polish Acad. Sci. Inst. Math., Warsaw, 2007, p. 133–143.

[25] C. FERMANIAN-KAMMERER – "Semiclassical analysis of generic codimension 3 crossings", *Int. Math. Res. Not.* **2004** (2004), p. 2391–2435.

[26] R. P. FEYNMAN & A. R. HIBBS – "Quantum mechanics and path integrals", International Series in Pure and Applied Physics, McGraw-Hill, 1965.

[27] B. Franchi, S. Gallot & R. L. Wheeden – "Sobolev and isoperimetric inequalities for degenerate metrics", *Math. Ann.* **300** (1994), p. 557–571.

[28] B. Franchi & E. Lanconelli – "Une métrique associée à une classe d'opérateurs elliptiques dégénérés", *Rend. Sem. Mat. Univ. Politec. Torino* **Special Issue** (1983), p. 105–114.

[29] G. Furioli, C. Melzi & A. Veneruso – "Strichartz inequalities for the wave equation with the full Laplacian on the Heisenberg group", *Canad. J. Math.* **59** (2007), p. 1301–1322.

[30] G. Furioli & A. Veneruso – "Strichartz inequalities for the Schrödinger equation with the full Laplacian on the Heisenberg group", *Studia Math.* **160** (2004), p. 157–178.

[31] N. Garofalo & D. Vassilev – "Regularity near the characteristic set in the non-linear Dirichlet problem and conformal geometry of sub-Laplacians on Carnot groups", *Math. Ann.* **318** (2000), p. 453–516.

[32] B. Gaveau, P. C. Greiner & J. Vauthier – "Intégrales de Fourier quadratiques et calcul symbolique exact sur le groupe d'Heisenberg", *J. Funct. Anal.* **68** (1986), p. 248–272.

[33] D. Geller – *Analytic pseudodifferential operators for the Heisenberg group and local solvability*, Mathematical Notes, vol. 37, Princeton Univ. Press, 1990.

[34] P. Gérard – "Microlocal defect measures", *Comm. Partial Differential Equations* **16** (1991), p. 1761–1794.

[35] P. Gérard, P. A. Markowich, N. J. Mauser & F. Poupaud – "Homogenization limits and Wigner transforms", *Comm. Pure Appl. Math.* **50** (1997), p. 323–379.

[36] P. C. Greiner – "On the Laguerre calculus of left-invariant convolution (pseudodifferential) operators on the Heisenberg group", in *Goulaouic-Meyer-Schwartz Seminar, 1980–1981*, École Polytech., 1981, exp. n° XI, 40.

[37] A. Grossmann, G. Loupias & E. M. Stein – "An algebra of pseudodifferential operators and quantum mechanics in phase space", *Ann. Inst. Fourier (Grenoble)* **18** (1968), p. 343–368.

[38] L. Hörmander – *The analysis of linear partial differential operators*, Grundl. math. Wiss., vol. 256, 1983.

[39] A. Hulanicki – "A functional calculus for Rockland operators on nilpotent Lie groups", *Studia Math.* **78** (1984), p. 253–266.

[40] S. KLAINERMAN & I. RODNIANSKI – "A geometric approach to the Littlewood-Paley theory", *Geom. Funct. Anal.* **16** (2006), p. 126–163.

[41] N. LERNER – *Metrics on the phase space and non-selfadjoint pseudo-differential operators*, Pseudo-Differential Operators. Theory and Applications, vol. 3, Birkhäuser, 2010.

[42] Y. I. MANIN – "Theta functions, quantum tori and Heisenberg groups", *Lett. Math. Phys.* **56** (2001), p. 295–320.

[43] A. MELIN – "Parametrix constructions for some classes of right-invariant differential operators on the Heisenberg group", *Comm. Partial Differential Equations* **6** (1981), p. 1363–1405.

[44] D. MÜLLER & E. M. STEIN – "L^p-estimates for the wave equation on the Heisenberg group", *Rev. Mat. Iberoamericana* **15** (1999), p. 297–334.

[45] A. I. NACHMAN – "The wave equation on the Heisenberg group", *Comm. Partial Differential Equations* **7** (1982), p. 675–714.

[46] M. REED & B. SIMON – *Methods of modern mathematical physics. I*, second ed., Academic Press Inc., 1980.

[47] S. RIGOT – "Transport de masse optimal et géométrie sous-riemannienne: le cas du groupe de Heisenberg", in *Séminaire: Équations aux dérivées partielles. 2006–2007*, École Polytech., 2007, exp. n° XIX, 16.

[48] L. P. ROTHSCHILD & E. M. STEIN – "Hypoelliptic differential operators and nilpotent groups", *Acta Math.* **137** (1976), p. 247–320.

[49] M. RUZHANSKY & V. TURUNEN – "On the Fourier analysis of operators on the torus", in *Modern trends in pseudo-differential operators*, Oper. Theory Adv. Appl., vol. 172, Birkhäuser, 2007, p. 87–105.

[50] M. SATO, T. KAWAI & M. KASHIWARA – "Microfunctions and pseudo-differential equations", in *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, Lecture Notes in Math., vol. 287, Springer, 1973, p. 265–529.

[51] ______, "The theory of pseudodifferential equations in the theory of hyperfunctions", *Sūgaku* **25** (1973), p. 213–238.

[52] E. M. STEIN – *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton Univ. Press, 1993.

[53] M. E. TAYLOR – *Noncommutative harmonic analysis*, Mathematical Surveys and Monographs, vol. 22, Amer. Math. Soc., 1986.

[54] S. Thangavelu – *Harmonic analysis on the Heisenberg group*, Progress in Math., vol. 159, Birkhäuser, 1998.

[55] R. Tolimieri – "The theta transform and the Heisenberg group", *J. Functional Analysis* **24** (1977), p. 353–363.

[56] E. Van Erp – "The Atiyah-Singer index formula for subelliptic operators on contact manifolds, Part I", preprint arXiv:0804.2490.

[57] ______, "The index of hypoelliptic operators on foliated manifolds", preprint arXiv:0811.1969.

[58] C.-J. Xu – "Propagation au bord des singularités pour des problèmes de Dirichlet non linéaires d'ordre deux", *J. Funct. Anal.* **92** (1990), p. 325–347.

[59] J. Zienkiewicz – "Schrödinger equation on the Heisenberg group", *Studia Math.* **161** (2004), p. 99–111.

[60] C. Zuily – "Existence globale de solutions régulières pour l'équation des ondes non linéaire amortie sur le groupe de Heisenberg", *Indiana Univ. Math. J.* **42** (1993), p. 323–360.