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PABLO PELAEZ

Multiplicative properties of the slice filtration

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MULTIPLICATIVE PROPERTIES
OF THE SLICE FILTRATION

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SOCIÉTÉ MATHÉMATIQUE DE FRANCE

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MULTIPLICATIVE PROPERTIES OF THE SLICE FILTRATION

Pablo PELAEZ

Abstract. — Let S be a Noetherian separated scheme of finite Krull dimension, and $\mathcal{H}(S)$ be the motivic stable homotopy category of Morel-Voevodsky. In order to get a motivic analogue of the Postnikov tower, Voevodsky [25] constructs the slice filtration by filtering $\mathcal{H}(S)$ with respect to the smash powers of the multiplicative group \mathbb{G}_m . We show that the slice filtration is compatible with the smash product in Jardine’s category $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ of motivic symmetric T -spectra [14], and describe several interesting consequences that follow from this compatibility. Among them, we have that over a perfect field all the slices s_q are in a canonical way modules in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ over the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$, and if the field has characteristic zero it follows that the slices s_q are big motives in the sense of Voevodsky, this relies on the work of Levine [16], Røndigs-Østvær [22] and Voevodsky [26]. It also follows that the smash product in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ induces pairings in the motivic Atiyah-Hirzebruch spectral sequence.

Résumé (Les propriétés multiplicatives de la filtration par les tranches). — Soit S un schéma noethérien séparé de dimension de Krull finie, et $\mathcal{H}(S)$ la catégorie homotopique stable de Morel-Voevodsky. Afin d’obtenir un analogue motivique de la tour de Postnikov, Voevodsky [25] définit la filtration par les tranches dans $\mathcal{H}(S)$ considérant les smash-produits itérées de le groupe multiplicatif \mathbb{G}_m . Nous montrons que la filtration par les tranches est compatible avec le smash-produit dans la catégorie de Jardine $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ des T -spectres symétriques motiviques [14]. Cette compatibilité a plusieurs conséquences intéressantes. D’entre eux, sur un corps parfait tous les tranches s_q sont canoniquement modules dans $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ sur le spectre motivique d’Eilenberg-MacLane $H\mathbb{Z}$, et si le corps est de caractéristique zéro les tranches s_q sont motifs grands au sens de Voevodsky, ce utilise les résultats de Levine [16], Røndigs-Østvær [22] et Voevodsky [26]. Nous montrons aussi que le smash-produit dans $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ induit des structures multiplicatives sur la suite spectrale motivique de Atiyah-Hirzebruch.

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INTRODUCTION

Let S be a Noetherian separated scheme of finite Krull dimension, $\mathcal{S}m|_S$ be the category of smooth schemes of finite type over S , \mathcal{M}_* be the category of pointed simplicial presheaves on $\mathcal{S}m|_S$ equipped with the motivic model structure of Morel and Voevodsky [18], and T in \mathcal{M}_* be $S^1 \wedge \mathbb{G}_m$ where \mathbb{G}_m is the multiplicative group $\mathbb{A}_S^1 - \{0\}$ pointed by 1, and S^1 is the simplicial circle. Let $\text{Spt}_T \mathcal{M}_*$ denote the category of T -spectra on \mathcal{M}_* equipped with the motivic stable model structure, and $\mathcal{H}(S)$ its homotopy category, which is triangulated. We consider the following objects in $\text{Spt}_T \mathcal{M}_*$:

$$C_{\text{eff}}^q = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \mid n, r, s \geq 0; s - n \geq q; U \in \mathcal{S}m|_S\}$$

where F_n is the left adjoint to the n -evaluation functor $ev_n : \text{Spt}_T \mathcal{M}_* \rightarrow \mathcal{M}_*$. In order to get a motivic version of the Postnikov tower, Voevodsky [25] constructs a filtered family of triangulated subcategories of $\mathcal{H}(S)$, which we call the *slice filtration*:

$$(1) \quad \dots \subseteq \Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S) \subseteq \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \subseteq \Sigma_T^{q-1} \mathcal{H}^{\text{eff}}(S) \subseteq \dots$$

where $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is the smallest full triangulated subcategory of $\mathcal{H}(S)$ which contains C_{eff}^q and is closed under arbitrary coproducts. The work of Neeman [19], [20], shows that the inclusion:

$$i_q : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \hookrightarrow \mathcal{H}(S)$$

has a right adjoint r_q , and that the functors:

$$f_q, s_q : \mathcal{H}(S) \rightarrow \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$$

are exact, where $f_q = i_q r_q$, and for every X in $\text{Spt}_T \mathcal{M}_*$, $s_q(X)$ fits in the following distinguished triangle:

$$f_{q+1} X \xrightarrow{\rho_q^X} f_q X \xrightarrow{\pi_q^X} s_q X \longrightarrow \Sigma_T^{1,0} f_{q+1} X$$

We say that $f_q(X)$ is the $(q-1)$ -connective cover of X and $s_q(X)$ the q -slice of X .

Let $\text{Spt}_T^\Sigma \mathcal{M}_*$ denote the category of symmetric T -spectra on \mathcal{M}_* equipped with the motivic symmetric stable model structure defined by Jardine [14], and $\mathcal{H}^\Sigma(S)$ its homotopy category, which is triangulated. Since the adjunction:

$$(V, U, \varphi) : \text{Spt}_T \mathcal{M}_* \rightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and forgetful functors is a Quillen equivalence [14], we get a similar filtration for $\mathcal{H}^\Sigma(S)$. Using the slice filtration (1), it is possible to construct a spectral sequence which is an analogue of the classical Atiyah-Hirzebruch spectral sequence in algebraic topology. Namely, let X, Y be in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and $[-, -]_{\text{Spt}}^\Sigma$ be the set of maps between two objects in $\mathcal{H}^\Sigma(S)$. Then the collection of distinguished triangles:

$$\{f_{q+1}^\Sigma X \rightarrow f_q^\Sigma X \rightarrow s_q^\Sigma X \rightarrow \Sigma_T^{1,0} f_{q+1}^\Sigma X\}$$

generates an exact couple $(D_1^{p,q}(Y; X), E_1^{p,q}(Y; X))$, where:

1. $D_1^{p,q}(Y; X) = [Y, \Sigma_T^{p+q,0} f_p^\Sigma X]_{\text{Spt}}^\Sigma$, and
2. $E_1^{p,q}(Y; X) = [Y, \Sigma_T^{p+q,0} s_p^\Sigma X]_{\text{Spt}}^\Sigma$.

Let A be a cofibrant ring spectrum with unit in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and $A\text{-mod}$ be the category of left A -modules in $\text{Spt}_T^\Sigma \mathcal{M}_*$. It follows directly from the work of Jardine [14] and Hovey [9] that the adjunction:

$$(A \wedge -, U, \varphi) : \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow A\text{-mod}$$

induces a model structure $A\text{-mod}(\mathcal{M}_*)$ in $A\text{-mod}$, i.e. a map f in $A\text{-mod}(\mathcal{M}_*)$ is a weak equivalence or a fibration if and only if Uf is a weak equivalence or a fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$. In the rest of this introduction $p, q \in \mathbb{Z}$ will denote arbitrary integers, and $u : \mathbf{1} \rightarrow A$ the unit map of A .

Our main results are the following:

1. There exists a model structure $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ for symmetric T -spectra on \mathcal{M}_* such that its homotopy category $R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$ is triangulated and naturally equivalent as a triangulated category to $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ (see diagrams (2) and (3) at the end of this introduction). Furthermore, the identity:

$$\text{id} : \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a right Quillen functor, and the functor f_q is canonically isomorphic to the following composition of exact functors:

$$\mathcal{H}^\Sigma(S) \xrightarrow{R_\Sigma} R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \xrightarrow{C_q^\Sigma} \mathcal{H}^\Sigma(S)$$

where R_Σ denotes a fibrant replacement functor in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and C_q^Σ denotes a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. For the proof the reader may look at Theorem 3.3.9, Corollary 3.3.17, Theorem 3.3.25, corollary 3.3.5, and Theorem 3.3.22.

2. There exists a model structure $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ for symmetric T -spectra on \mathcal{M}_* such that its homotopy category $S^q \mathcal{H}^\Sigma(S)$ is triangulated and the identity:

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow S^q \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Furthermore, the functor s_q is canonically isomorphic to the following composition of exact functors:

$$\begin{array}{ccccc} \mathcal{H}^\Sigma(S) & & & & \mathcal{H}^\Sigma(S) \\ R_\Sigma \downarrow & & & & \uparrow C_q^\Sigma \\ R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \xrightarrow{C_q^\Sigma} & S^q \mathcal{H}^\Sigma(S) & \xrightarrow{W_{q+1}^\Sigma} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \end{array}$$

where W_{q+1}^Σ denotes a fibrant replacement functor in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. For the proof the reader may look at Theorem 3.3.50, Corollaries 3.3.59, 3.3.5, and Theorem 3.3.68. In the rest of this introduction f_q^Σ , s_q^Σ will denote respectively $C_q^\Sigma \circ R_\Sigma$, $C_q^\Sigma \circ W_{q+1}^\Sigma \circ C_q^\Sigma \circ R_\Sigma$.

3. The smash product of symmetric T -spectra induces the following Quillen bifunctors in the sense of Hovey [10]:

$$R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{-\wedge-} R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$$

$$S^p \text{Spt}_T^\Sigma \mathcal{M}_* \times S^q \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{-\wedge-} S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

For the proof the reader may look at Theorems 3.4.5 and 3.4.12.

4. The homotopy category $\mathcal{H}(A\text{-mod})$ of $A\text{-mod}(\mathcal{M}_*)$ is triangulated. Furthermore, there exist model structures $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, $S^q A\text{-mod}(\mathcal{M}_*)$ in $A\text{-mod}$ such that their homotopy categories $R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod})$, $S^q \mathcal{H}(A\text{-mod})$ are triangulated, and the identity functors:

$$A\text{-mod}(\mathcal{M}_*) \xleftarrow{\text{id}} R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \xrightarrow{\text{id}} S^q A\text{-mod}(\mathcal{M}_*)$$

are left Quillen functors. We will denote by f_q^m , s_q^m the following compositions of exact functors:

$$\mathcal{H}(A\text{-mod}) \xrightarrow{R_m} R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) \xrightarrow{C_q^m} \mathcal{H}(A\text{-mod})$$

$$\begin{array}{ccccc} \mathcal{H}(A\text{-mod}) & & & & \mathcal{H}(A\text{-mod}) \\ R_m \downarrow & & & & \uparrow C_q^m \\ R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & \xrightarrow{C_q^m} & S^q \mathcal{H}(A\text{-mod}) & \xrightarrow{W_{q+1}^m} & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) \end{array}$$

where C_q^m denotes a cofibrant replacement functor in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$; and R_m , W_{q+1}^m denote fibrant replacement functors respectively in $A\text{-mod}(\mathcal{M}_*)$, $S^q A\text{-mod}(\mathcal{M}_*)$. For the proof the reader may look at Theorem 2.8.4, Proposition 3.5.3, Theorems 3.5.8, 3.5.49, and propositions 3.5.12, 3.5.54.

5. If A is cofibrant in $R_{C_{\text{eff}}}^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, then the exact functors:

$$f_q^\Sigma \circ UR_m : \mathcal{H}(A\text{-mod}) \rightarrow \mathcal{H}^\Sigma(S)$$

$$s_q^\Sigma \circ UR_m : \mathcal{H}(A\text{-mod}) \rightarrow \mathcal{H}^\Sigma(S)$$

factor (up to a canonical isomorphism) through $\mathcal{H}(A\text{-mod})$ as follows:

$$\begin{array}{ccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) & & \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \\ \downarrow f_q^m & & \downarrow f_q^\Sigma & & \downarrow s_q^m & & \downarrow s_q^\Sigma \\ \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) & & \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \end{array}$$

This means that for every A -module M , its $(q-1)$ -connective cover $f_q^\Sigma(M)$ and q -slice $s_q^\Sigma(M)$ inherit a natural structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$. For the proof the reader may look at Theorem 3.6.19.

6. If A is cofibrant in $R_{C_{\text{eff}}}^0 \text{Spt}_T^\Sigma \mathcal{M}_*$ and its unit map u is a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, then the functor:

$$s_q^\Sigma : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$$

is canonically isomorphic to the following composition of exact functors:

$$\begin{array}{ccccc} \mathcal{H}^\Sigma(S) & & & & \mathcal{H}^\Sigma(S) \\ \downarrow R_\Sigma & & & & \uparrow UR_m \\ R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) & & & & \mathcal{H}(A\text{-mod}) \\ \downarrow C_q^\Sigma & & & & \uparrow C_q^m \\ S^q \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge P_q^\Sigma -} & S^q \mathcal{H}(A\text{-mod}) & \xrightarrow{W_{q+1}^m} & R_{C_{\text{eff}}}^q \mathcal{H}(A\text{-mod}) \end{array}$$

where P_q^Σ denotes a cofibrant replacement functor in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. This means that for every X in $\text{Spt}_T^\Sigma \mathcal{M}_*$, its q -slice $s_q^\Sigma(X)$ is naturally equipped with a structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$. For the proof the reader may look at Theorem 3.6.20.

7. Let X, X', Y, Y' be in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then the smash product in $\text{Spt}_T^\Sigma \mathcal{M}_*$ induces natural external pairings in the motivic Atiyah-Hirzebruch spectral sequence:

$$E_r^{p,q}(Y; X) \otimes E_r^{p',q'}(Y'; X') \longrightarrow E_r^{p+p',q+q'}(Y \wedge Y'; X \wedge X')$$

For the proof the reader may look at Theorem 3.6.16.

8. Let $H\mathbb{Z}$ denote the motivic Eilenberg-MacLane spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$ [16, Example 8.2.2(2)]. The following result proves a conjecture of Voevodsky [25]:

— If the base scheme S is a perfect field, then for every X in $\text{Spt}_T^\Sigma \mathcal{M}_*$, its q -slice $s_q^\Sigma(X)$ has a natural structure of $H\mathbb{Z}$ -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

For the proof the reader may look at Theorem 3.6.22.

9. If we restrict the field even further to the case of characteristic zero, then we get that the slices may be interpreted as motives in the sense of Voevodsky:

- If k is a field of characteristic zero, then for every X in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ its q -slice $s_q^\Sigma(X)$ is a big motive in the sense of Voevodsky [22, Section 2.3].

For the proof the reader may look at Theorem 3.6.23.

The model category $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is defined as the right Bousfield localization of $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $C_{\mathrm{eff}}^{q,\Sigma}$ -colocal equivalences, where $C_{\mathrm{eff}}^{q,\Sigma}$ denotes the symmetrization of the collection of T -spectra C_{eff}^q described above (see Theorem 3.3.9). On the other hand, $S^q \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is defined as a right Bousfield localization with respect to an auxiliary model structure $L_{<q+1} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$, which is a left Bousfield localization of $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ and its fibrant replacement functor describes the cone of the map $\theta_q^{\Sigma,X} : f_q^\Sigma X \rightarrow X$. The model structures $R_{C_{\mathrm{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, $S^q A\text{-mod}(\mathcal{M}_*)$ are constructed similarly using the left adjoint $A \wedge -$ to define the maps that get inverted in the Bousfield localization. To show that the smash product in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ induces Quillen bifunctors, we use Hovey's approach to symmetric monoidal model categories [10, Chapter 4], and the triangulated structure in the homotopy categories $R_{C_{\mathrm{eff}}^q} \mathcal{H}^\Sigma(S)$, $S^q \mathcal{H}^\Sigma(S)$. The Bousfield localizations are constructed following Hirschhorn's approach [6]. In order to apply Hirschhorn's techniques, it is necessary to check that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ and $A\text{-mod}(\mathcal{M}_*)$ are both *cellular*; for this we rely on Hovey's general approach to spectra [11] and on an unpublished result of Hirschhorn (see Theorem 2.2.4).

We now give an outline of this thesis. In Chapter 1, we just recall some standard results about Quillen model categories. The reader who is familiar with the terminology of model categories may skip this chapter.

In Chapter 2, we review the definitions of the Morel-Voevodsky stable model structure for simplicial presheaves $\mathrm{Spt}_T \mathcal{M}_*$ and Jardine's stable model structure for symmetric T -spectra $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. We also show that these two model structures are cellular, therefore it is possible to apply Hirschhorn's technology to construct Bousfield localizations. In Section 2.8 we recall the construction of the model structures for the categories of A -modules and A -algebras, where A denotes a cofibrant ring spectrum with unit in Jardine's motivic symmetric stable model category. We verify that the category of A -modules equipped with this model structure also satisfies Hirschhorn's cellularity condition. The reader who is familiar with these model structures may either skip this chapter or simply look at Sections 2.2, 2.5, 2.7 and 2.8 where we prove that the cellularity condition holds.

Finally in Chapter 3, we carry out the program sketched above. In Section 3.1, we review Voevodsky's construction for the slice filtration in the setting of simplicial presheaves. In Section 3.2, we apply Hirschhorn's localization techniques to the Morel-Voevodsky stable model structure in order to construct three families of model structures, namely $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T \mathcal{M}_*$, $L_{<q} \mathrm{Spt}_T \mathcal{M}_*$ and $S^q \mathrm{Spt}_T \mathcal{M}_*$. The first family, $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T \mathcal{M}_*$ is constructed by a right Bousfield localization with respect to the

Morel-Voevodsky stable model structure (see Theorem 3.2.1), and it provides a lifting of Voevodsky's slice filtration to the model category level (see Theorem 3.2.23). Moreover, this family has the property that the cofibrant replacement functor C_q provides an alternative description for the functor f_q ($(q-1)$ -connective cover) defined above (see Theorem 3.2.20). In order to get a lifting for the slice functors s_q to the model category level, we need to introduce the model structures $L_{<q}\mathrm{Spt}_T\mathcal{M}_*$ and $S^q\mathrm{Spt}_T\mathcal{M}_*$. The model category $L_{<q}\mathrm{Spt}_T\mathcal{M}_*$ is defined as a left Bousfield localization with respect to the Morel-Voevodsky stable model structure (see Theorem 3.2.29); its main property is that its fibrant replacement functor W_q gives an alternative description for the cone of the natural map $f_q X \rightarrow X$ (see Theorems 3.1.18 and 3.2.52). On the other hand, the model structure $S^q\mathrm{Spt}_T\mathcal{M}_*$ is constructed using right Bousfield localization with respect to the model category $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ (see Theorem 3.2.59), and it gives the desired lifting for the slice functor s_q to the model category level (see Theorem 3.2.80).

In Section 3.3, we promote the model structures defined above (Section 3.2) to the setting of symmetric T -spectra. In this case, Hirschhorn's localization technology applied to Jardine's stable model structure for symmetric T -spectra allows us to introduce three families of model structures which we denote by $R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T^\Sigma\mathcal{M}_*$, $L_{<q}\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ and $S^q\mathrm{Spt}_T^\Sigma\mathcal{M}_*$; where the underlying category is given by symmetric T -spectra (see Theorems 3.3.9, 3.3.26 and 3.3.50). Using the Quillen equivalence [14] given by the symmetrization and the forgetful functors, we are then able to show that these new families of model structures are also Quillen equivalent to the ones introduced in Section 3.2 (see Theorems 3.3.19, 3.3.42 and 3.3.64). Therefore, these model structures give liftings for the functors f_q and s_q to the model category level (see Corollary 3.3.5, and Theorems 3.3.22(3), 3.3.68(3)), with the great technical advantage that the underlying categories are now symmetric monoidal. Hence, we have a natural framework for the study of the multiplicative properties of Voevodsky's slice filtration.

In Section 3.4, we show that the smash product of symmetric T -spectra

$$R_{C_{\mathrm{eff}}^p}\mathrm{Spt}_T^\Sigma\mathcal{M}_* \times R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T^\Sigma\mathcal{M}_* \xrightarrow{-\wedge-} R_{C_{\mathrm{eff}}^{p+q}}\mathrm{Spt}_T^\Sigma\mathcal{M}_*$$

$$S^p\mathrm{Spt}_T^\Sigma\mathcal{M}_* \times S^q\mathrm{Spt}_T^\Sigma\mathcal{M}_* \xrightarrow{-\wedge-} S^{p+q}\mathrm{Spt}_T^\Sigma\mathcal{M}_*$$

is in both cases a Quillen bifunctor in the sense of Hovey (see Theorems 3.4.5 and 3.4.12).

In Section 3.5, we will promote (using the free A -module functor $A \wedge -$) the model structures constructed in Section 3.3 to the category of A -modules, where A is a cofibrant ring spectrum with unit in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$. We will denote these new model structures by $R_{C_{\mathrm{eff}}^q}A\text{-mod}(\mathcal{M}_*)$, $L_{<q}A\text{-mod}(\mathcal{M}_*)$ and $S^qA\text{-mod}(\mathcal{M}_*)$. These new model structures will be used as an analogue of the slice filtration for the motivic stable homotopy category of A -modules, as well as a tool to describe the behavior of the slice functors s_q^Σ when they are restricted to the category of A -modules. We will see that if

one imposes some natural additional conditions on the ring spectrum A , then the free A -module functor $A \wedge -$ induces a strict compatibility between the slice filtration in the categories of symmetric T -spectra and A -modules (see Theorems 3.5.22, 3.5.45, 3.5.68 and 3.5.70).

In Section 3.6, we will rely on all the previous constructions to obtain our main results. We will show that the smash product of symmetric T -spectra induces up to homotopy natural pairings (see Theorem 3.6.9):

$$\begin{aligned} f_p^\Sigma(X) \wedge f_q^\Sigma(Y) &\xrightarrow{\cup_{p,q}^c} f_{p+q}^\Sigma(X \wedge Y) \\ s_p^\Sigma(X) \wedge s_q^\Sigma(Y) &\xrightarrow{\cup_{p,q}^s} s_{p+q}^\Sigma(X \wedge Y) \end{aligned}$$

As a consequence, if A is a ring spectrum in $\mathcal{H}^\Sigma(S)$ (i.e. up to homotopy) and M is an A -module in $\mathcal{H}^\Sigma(S)$, then (see Theorem 3.6.13):

1. The (-1) -connective cover of A , $f_0^\Sigma(A)$ is a ring spectrum (up to homotopy) in $\mathcal{H}^\Sigma(S)$.
2. The $(q-1)$ -connective cover of M , $f_q^\Sigma(M)$ is a module (up to homotopy) over $f_0^\Sigma(A)$.
3. The direct sum of all the connective covers of A , $f^\Sigma(A) = \bigoplus_{n \in \mathbb{Z}} f_n^\Sigma(A)$ is a graded ring (up to homotopy) in $\mathcal{H}^\Sigma(S)$.
4. The direct sum of all the connective covers of M , $f^\Sigma(M) = \bigoplus_{n \in \mathbb{Z}} f_n^\Sigma(M)$ is a graded module (up to homotopy) over $f^\Sigma(A)$.
5. The zero slice of A , $s_0^\Sigma(A)$ is a ring spectrum (up to homotopy) in $\mathcal{H}^\Sigma(S)$.
6. The q -slice of M , $s_q^\Sigma(M)$ is a module (up to homotopy) over $s_0^\Sigma(A)$.
7. The direct sum of all the slices of A , $s^\Sigma(A) = \bigoplus_{n \in \mathbb{Z}} s_n^\Sigma(A)$ is a graded ring (up to homotopy) in $\mathcal{H}^\Sigma(S)$.
8. The direct sum of all the slices of M , $s^\Sigma(M) = \bigoplus_{n \in \mathbb{Z}} s_n^\Sigma(M)$ is a graded module (up to homotopy) over $s^\Sigma(A)$.
9. The smash product of symmetric T -spectra induces (via the external pairings \cup^c and \cup^s) natural external pairings in the motivic Atiyah-Hirzebruch spectral sequence (see definition 3.6.15 and Theorem 3.6.16):

$$\begin{aligned} E_r^{p,q}(Y; X) \otimes E_r^{p',q'}(Y'; X') &\longrightarrow E_r^{p+p',q+q'}(Y \wedge Y'; X \wedge X') \\ (\alpha, \beta) &\longmapsto \alpha \smile \beta \end{aligned}$$

We will also see that if we have a cofibrant ring spectrum A with unit in $R_{C_{\text{eff}}}^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, then for every A -module M in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.6.19):

1. $f_q^\Sigma(M)$ is again an A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (not just up to homotopy, but in a very strict sense).
2. $s_q^\Sigma(M)$ is again an A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (not just up to homotopy, but in a very strict sense).

Furthermore, if the unit map u is a weak equivalence in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, then the free A -module functor $A \wedge -$ induces for every symmetric T -spectrum X (see Theorem 3.6.20), a natural structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (i.e. not just up to homotopy, but in a very strict sense) on its q -slice $s_q^\Sigma(X)$.

Finally, we will be able to prove a conjecture of Voevodsky (see [16, Corollary 11.1.3], [25]), which says that if the base scheme S is a perfect field, then for every symmetric T -spectrum X , its q -slice $s_q^\Sigma(X)$ is naturally equipped with a module structure in $\text{Spt}_T^\Sigma \mathcal{M}_*$ over the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$ (see Theorem 3.6.22). If we restrict the field even further, considering a field of characteristic zero, then as a consequence we will prove that all the slices $s_q^\Sigma(X)$ are big motives in the sense of Voevodsky (see Theorem 3.6.23).

Tower of Quillen model categories for the slice filtration:

$$(2) \quad \begin{array}{ccc} \vdots & & \vdots \\ \text{id} \downarrow & & \downarrow \text{id} \\ R_{C_{\text{eff}}^{q+1}} \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* \\ \text{id} \downarrow & & \downarrow \text{id} \\ R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \\ \text{id} \downarrow & & \downarrow \text{id} \\ R_{C_{\text{eff}}^{q-1}} \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & R_{C_{\text{eff}}^{q-1}} \text{Spt}_T^\Sigma \mathcal{M}_* \\ \text{id} \downarrow & & \downarrow \text{id} \\ \vdots & & \vdots \\ \text{id} \downarrow & & \downarrow \text{id} \\ \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & \text{Spt}_T^\Sigma \mathcal{M}_* \end{array}$$

and its associated diagram of homotopy categories:

$$(3) \quad \begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow \wr & & C_{q+2} \downarrow & & \downarrow C_{q+2}^\Sigma \\ \Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S) & \xleftarrow[\cong]{r_{q+1} C_{q+1}} & R_{C_{\text{eff}}^{q+1}} \mathcal{H}(S) & \xrightarrow[\cong]{VC_{q+1}} & R_{C_{\text{eff}}^{q+1}} \mathcal{H}^\Sigma(S) \\ \downarrow \wr & & C_{q+1} \downarrow & & \downarrow C_{q+1}^\Sigma \\ \Sigma_T^q \mathcal{H}^{\text{eff}}(S) & \xleftarrow[\cong]{r_q C_q} & R_{C_{\text{eff}}^q} \mathcal{H}(S) & \xrightarrow[\cong]{VC_q} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \\ \downarrow \wr & & C_q \downarrow & & \downarrow C_q^\Sigma \\ \Sigma_T^{q-1} \mathcal{H}^{\text{eff}}(S) & \xleftarrow[\cong]{r_{q-1} C_{q-1}} & R_{C_{\text{eff}}^{q-1}} \mathcal{H}(S) & \xrightarrow[\cong]{VC_{q-1}} & R_{C_{\text{eff}}^{q-1}} \mathcal{H}^\Sigma(S) \\ \downarrow \wr & & C_{q-1} \downarrow & & \downarrow C_{q-1}^\Sigma \\ \vdots & & \vdots & & \vdots \\ \downarrow \wr & & \downarrow & & \downarrow \\ \mathcal{H}(S) & \xlongequal{\quad} & \mathcal{H}(S) & \xrightarrow[\cong]{VQ_s} & \mathcal{H}^\Sigma(S) \end{array}$$

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CHAPTER 1

PRELIMINARIES

All the results in this chapter are classical (see [21], [10], [6], [4], [3]) and are included here just to fix notation and to make our discussion relatively self contained.

1.1. Model Categories

Model categories were first introduced by Quillen in [21], his original definition has been slightly modified along the years, we will use the definition introduced in [1].

Definition 1.1.1. — *A model category \mathcal{A} is a category equipped with three classes of maps $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ called weak equivalences, cofibrations and fibrations, such that the following axioms hold:*

MC1 : \mathcal{A} is closed under small limits and colimits.

MC2 : If f, g are two composable maps in \mathcal{A} and two out of $f, g, g \circ f$ are weak equivalences then so is the third one.

MC3 : The classes of weak equivalences, cofibrations and fibrations are closed under retracts.

MC4 : Suppose we have a solid commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration, p is a fibration, and either i or p is a weak equivalence, then the dotted arrow making the diagram commutative exists.

MC5 : Given any arrow $f : A \rightarrow B$ in \mathcal{A} , there exist two functorial factorizations, $f = p \circ i$ and $f = q \circ j$, where p is a fibration and a weak equivalence, i is a cofibration, q is a fibration and j is a cofibration and a weak equivalence.

A map $j : A \rightarrow B$ will be called a *trivial cofibration* (respectively *trivial fibration*) if it is both a cofibration and a weak equivalence (respectively a fibration and a weak equivalence).

If a given category \mathcal{A} has a model structure, then we get immediately the following consequences:

- Remark 1.1.2.** —
1. *The limit axiom MC1 implies that there is an initial and a final object in \mathcal{A} , which we will denote by \emptyset and $*$ respectively. We say that the category \mathcal{A} is pointed if the canonical map $\emptyset \rightarrow *$ is an isomorphism.*
 2. *The axioms for a model category are self dual, therefore the opposite category \mathcal{A}^{op} has also a model structure, where a map $i : A \rightarrow B$ in \mathcal{A}^{op} is a weak equivalence, cofibration or fibration if its dual $i : B \rightarrow A$ is a weak equivalence, fibration or cofibration in \mathcal{A} . This implies in particular that any result we prove about model categories will have a dual version.*
 3. *Let X be an object in \mathcal{A} . Then the category $(\mathcal{A} \downarrow X)$ of objects in \mathcal{A} over X has also a model structure, where the weak equivalences, cofibrations and fibrations are maps which become weak equivalences, cofibrations and fibrations after applying the forgetful functor $(\mathcal{A} \downarrow X) \rightarrow \mathcal{A}$.*
 4. *Similarly the category $(X \downarrow \mathcal{A})$ has a model structure induced from the one in \mathcal{A} . We will denote by \mathcal{A}_* the category $(* \downarrow \mathcal{A})$ of objects under the final object of \mathcal{A} .*
 5. *Let A, X be two objects in \mathcal{A} , then the category $(A \downarrow \mathcal{A} \downarrow X)$ of objects which are simultaneously under A and over X has also a model structure induced from the one in \mathcal{A} .*

Let X be an object in \mathcal{A} . We say that X is *cofibrant* if the natural map $\emptyset \rightarrow X$ is a cofibration. Similarly, we say that X is *fibrant* if the natural map $X \rightarrow *$ is a fibration.

Consider two objects A, X in \mathcal{A} . We say that A is a *cofibrant replacement* for X , if A is cofibrant and there is a map $A \rightarrow X$ which is a weak equivalence in \mathcal{A} . Dually, we say that X is a *fibrant replacement* for A , if X is fibrant and there is a map $A \rightarrow X$ which is a weak equivalence in \mathcal{A} .

Let $i : A \rightarrow B$, $p : X \rightarrow Y$ be two maps in \mathcal{A} . We say that i has the *left lifting property with respect to p* (or that p has the *right lifting property with respect to i*) if for every solid commutative diagram:

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 \downarrow i & \nearrow s & \downarrow p \\
 B & \longrightarrow & Y
 \end{array}$$

the dotted arrow making the diagram commutative exists.

The following are two elementary but extremely useful results about model categories.

Proposition 1.1.3 (Retract Argument, [10]). — *Let \mathcal{A} be a model category and $f = p \circ i$ a factorization of f such that f has the left lifting property with respect to p (respectively f has the right lifting property with respect to i). Then f is a retract of i (respectively f is a retract of p).*

Proof. — By duality it is enough to show the case where f has the left lifting property with respect to p .

Consider the following solid commutative diagram:

$$\begin{array}{ccc}
 A & \xrightarrow{i} & X \\
 f \downarrow & \nearrow j & \downarrow p \\
 B & \xrightarrow{\text{id}} & B
 \end{array}$$

By hypothesis the dotted arrow j making the diagram commutative exists. But then the following commutative diagram shows that f is a retract of i .

$$\begin{array}{ccccc}
 A & \xrightarrow{\text{id}} & A & \xrightarrow{\text{id}} & A \\
 f \downarrow & & \downarrow i & & \downarrow f \\
 B & \xrightarrow{j} & X & \xrightarrow{p} & B
 \end{array}
 \quad \square$$

Lemma 1.1.4 (Ken Brown’s lemma, [10]). — *Let $F : \mathcal{A} \rightarrow \mathcal{D}$ be a functor, where \mathcal{A} is a model category. Assume that there exists a class \mathcal{V} of maps in \mathcal{D} which has the two out of three property, and that $F(i) \in \mathcal{V}$ for all trivial cofibrations $i : A \rightarrow B$ between cofibrant objects A and B in \mathcal{A} . Then $F(g) \in \mathcal{V}$ for all weak equivalences $g : A \rightarrow B$ between cofibrant objects A and B in \mathcal{A} .*

Proof. — Consider the following commutative diagram:

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & A \\
 \downarrow & & \downarrow i_A \\
 B & \xrightarrow{i_B} & A \amalg B \\
 & & \downarrow i \\
 & & C \\
 & & \downarrow p \\
 & & B
 \end{array}$$

\xrightarrow{g} (curved arrow from \emptyset to B)
 $\xrightarrow{\text{id}}$ (curved arrow from B to B)
 $\xrightarrow{(g, \text{id})}$ (curved arrow from B to C)

where we have a factorization of $(g, \text{id}) = p \circ i$, with i a cofibration and p a trivial fibration.

Since A and B are cofibrant, it follows that i_A and i_B are cofibrations. This implies that $i \circ i_A$, $i \circ i_B$ are both cofibrations, and hence C is a cofibrant object in \mathcal{A} .

By the two out of three property in \mathcal{A} , $i \circ i_A$ and $i \circ i_B$ are weak equivalences, since g , p and id_B are weak equivalences. Therefore $i \circ i_A$ and $i \circ i_B$ are both trivial cofibrations. It follows that $F(i \circ i_A)$ and $F(i \circ i_B)$ are both in \mathcal{V} . But then $F(p) \circ F(i \circ i_B) = F(p \circ i \circ i_B) = F(\text{id}) = \text{id}$, and since \mathcal{V} has the two out of three property, we have that $F(p)$ is in \mathcal{V} . Then the two out of three property for \mathcal{V} implies that $F(g) = F(p) \circ F(i \circ i_A)$ is also in \mathcal{V} . \square

By duality we get immediately the following lemma:

Lemma 1.1.5. — *Let $F : \mathcal{A} \rightarrow \mathcal{D}$ be a functor, where \mathcal{A} is a model category. Assume that there exists a class \mathcal{V} of maps in \mathcal{D} which has the two out of three property, and that $F(p) \in \mathcal{V}$ for all trivial fibrations $p : X \rightarrow Y$ between fibrant objects X, Y in \mathcal{A} . Then $F(g) \in \mathcal{V}$ for all weak equivalences $g : X \rightarrow Y$ between fibrant objects X and Y in \mathcal{A} .* \square

The retract argument has the following consequences, which give nice characterizations for the cofibrations and trivial cofibrations (respectively fibrations and trivial fibrations) in terms of a left lifting property (respectively right lifting property).

Corollary 1.1.6. — *The class of cofibrations (respectively trivial cofibrations) in a model category \mathcal{A} is equal to the class of maps having the left lifting property with respect to any trivial fibration in \mathcal{A} (respectively any fibration in \mathcal{A}). The class of fibrations (respectively trivial fibrations) in a model category \mathcal{A} is equal to the class of maps having the right lifting property with respect to any trivial cofibration in \mathcal{A} (respectively any cofibration in \mathcal{A}).*

Proof. — By duality it is enough to prove the case of cofibrations and trivial cofibrations. Suppose that $i : A \rightarrow B$ is a cofibration in \mathcal{A} , then the lifting axiom **MC4** implies that i has the left lifting property with respect to any trivial fibration in \mathcal{A} . Conversely, if $i : A \rightarrow B$ has the left lifting property with respect to any trivial fibration in \mathcal{A} , then the factorization axiom **MC5** implies that $i = ql$ where l is a cofibration in \mathcal{A} and q is a trivial fibration in \mathcal{A} . Since i has the left lifting property with respect to q , the retract argument (see Proposition 1.1.3) implies that i is a retract of l . Therefore, the retract axiom **MC3** implies that i is also a cofibration. The case for trivial cofibrations is similar. \square

Corollary 1.1.7. — *Any isomorphism in a model category \mathcal{A} is a cofibration, a fibration, and a weak equivalence. The class of cofibrations and the class of trivial cofibrations in \mathcal{A} are closed under retracts and pushouts. The class of fibrations and the class of trivial fibrations in \mathcal{A} are closed under retracts and pullbacks.*

Proof. — It follows immediately from the lifting property characterization (Corollary 1.1.6) for cofibrations, trivial cofibrations, fibrations and trivial fibrations. \square

Remark 1.1.8. — *Let \mathcal{A} be a model category. Given any object X in \mathcal{A} , we can apply the factorization axiom **MC5** to the natural map $\emptyset \rightarrow X$ to get a cofibrant replacement for X :*

$$\emptyset \longrightarrow QX \xrightarrow{Q^X} X$$

where QX is cofibrant and Q^X is a trivial fibration. We also get fibrant replacements for X when we factor the natural map $X \rightarrow *$:

$$X \xrightarrow{R^X} RX \longrightarrow *$$

where RX is fibrant and R^X is a trivial cofibration. The factorization axiom **MC5** implies also that these two constructions are functorial.

Definition 1.1.9. — *Let \mathcal{A}, \mathcal{B} be two model categories. A functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called a left Quillen functor if it has a right adjoint $G : \mathcal{B} \rightarrow \mathcal{A}$, and satisfies the following conditions:*

1. *If i is a cofibration in \mathcal{A} , then $F(i)$ is also a cofibration in \mathcal{B} .*
2. *If j is a trivial cofibration in \mathcal{A} , then $F(j)$ is also a trivial cofibration in \mathcal{B} .*

The right adjoint G is called a right Quillen functor, and the adjunction

$$(F, G, \varphi) : \mathcal{A} \longrightarrow \mathcal{B}$$

is called a Quillen adjunction.

Definition 1.1.10. — *Let $(F, G, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ be a Quillen adjunction. We say that F is a left Quillen equivalence if for every cofibrant object X in \mathcal{A} and every fibrant object Y in \mathcal{B} the following condition holds:*

- *A map $f : X \rightarrow GY$ is a weak equivalence in \mathcal{A} if and only if its adjoint $f^\sharp : FX \rightarrow Y$ is a weak equivalence in \mathcal{B} .*

In this case G will be called a right Quillen equivalence, and (F, G, φ) a Quillen equivalence.

Definition 1.1.11. — Let \mathcal{A} be a model category, and let X be an object of \mathcal{A} . We say that \tilde{X} is a cylinder object for X , if we have a factorization of the fold map

$$\begin{array}{ccc} X \amalg X & \xrightarrow{\nabla} & X \\ \downarrow i & \nearrow s & \\ \tilde{X} & & \end{array}$$

where i is a cofibration and s is a weak equivalence.

Definition 1.1.12. — Let \mathcal{A} be a model category, and let X be an object of \mathcal{A} . We say that \hat{X} is a path object for X , if we have a factorization of the diagonal map

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & X \times X \\ \downarrow r & \nearrow p & \\ \hat{X} & & \end{array}$$

where p is a fibration and r is a weak equivalence.

Definition 1.1.13. — Let \mathcal{A} be a model category and consider two maps $f, g : X \rightarrow Y$. We say that f is left homotopic to g ($f \stackrel{l}{\sim} g$) if there exists a cylinder object \tilde{X} for X , together with the following factorization:

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(f,g)} & Y \\ \downarrow i & \nearrow H & \\ \tilde{X} & & \end{array}$$

The map H is called a left homotopy from f to g .

Definition 1.1.14. — Let \mathcal{A} be a model category and consider two maps $f, g : X \rightarrow Y$. We say that f is right homotopic to g ($f \stackrel{r}{\sim} g$) if there exists a path object \hat{Y} for Y , together with the following factorization:

$$\begin{array}{ccc} X & \xrightarrow{(f,g)} & Y \times Y \\ \downarrow H & \nearrow p & \\ \hat{Y} & & \end{array}$$

The map H is called a right homotopy from f to g .

Definition 1.1.15. — Let \mathcal{A} be a model category and consider two maps $f, g : A \rightarrow B$. We say that f is homotopic to g ($f \sim g$) if f and g are both left and right homotopic.

Definition 1.1.16 (cf. [21]). — Let \mathcal{A} be an arbitrary category and \mathcal{W} a class of maps in \mathcal{A} . The localization of \mathcal{A} with respect to \mathcal{W} will be a category $\mathcal{W}^{-1}\mathcal{A}$ together with a functor

$$\gamma : \mathcal{A} \longrightarrow \mathcal{W}^{-1}\mathcal{A}$$

having the following universal property: for every $w \in \mathcal{W}$, $\gamma(w)$ is an isomorphism, and given any functor $F : \mathcal{A} \rightarrow \mathcal{D}$ such that $F(w)$ is an isomorphism for every $w \in \mathcal{W}$, there is a unique functor $\theta : \mathcal{W}^{-1}\mathcal{A} \rightarrow \mathcal{D}$, such that $\theta \circ \gamma = F$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{D} \\ \gamma \downarrow & \nearrow \theta & \\ \mathcal{W}^{-1}\mathcal{A} & & \end{array}$$

Theorem 1.1.17 (Quillen). — Let \mathcal{A} be a model category. Then there exists a category $\text{Ho}\mathcal{A}$, which is the localization of \mathcal{A} with respect to the class \mathcal{W} of weak equivalences, and is called the homotopy category of \mathcal{A} . $\text{Ho}\mathcal{A}$ is defined as follows:

1. The objects of $\text{Ho}\mathcal{A}$ are just the objects in \mathcal{A} .
2. The set of maps in $\text{Ho}\mathcal{A}$ between two objects X, Y is given by the set of homotopy classes between cofibrant-fibrant replacements for X and Y :

$$\text{Hom}_{\text{Ho}\mathcal{A}}(X, Y) = \pi_{\mathcal{A}}(RQX, RQY)$$

and the composition law is induced by the composition in \mathcal{A} .

Let $\text{Ho}\mathcal{A}_c, \text{Ho}\mathcal{A}_f, \text{Ho}\mathcal{A}_{cf}$ be the full subcategories of $\text{Ho}\mathcal{A}$ generated by the cofibrant, fibrant and cofibrant-fibrant objects of \mathcal{A} respectively. In the following diagram, all the functors are equivalences of categories:

$$\begin{array}{ccccc} & & \text{Ho}\mathcal{A}_c & & \\ & \swarrow \sim & & \swarrow \text{Ho}Q & \\ & \text{Ho}R & & \sim & \\ \text{Ho}\mathcal{A}_{cf} & & & & \text{Ho}\mathcal{A} \\ & \swarrow \text{Ho}Q & & \swarrow \sim & \\ & \sim & & \sim & \\ & & \text{Ho}\mathcal{A}_f & & \\ & & \swarrow \text{Ho}R & & \end{array}$$

where the adjoints to the equivalences given above are constructed taking cofibrant, fibrant and cofibrant-fibrant replacements.

Proof. — We refer the reader to [21, I.1 Theorem 1]. □

Theorem 1.1.18 (Quillen). — *Let $(F, G, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ be a Quillen adjunction. Then the adjunction (F, G, φ) descends to the homotopy categories, i.e. we get an adjunction:*

$$(QF, RG, \varphi) : \mathrm{Ho}\mathcal{A} \longrightarrow \mathrm{Ho}\mathcal{B}$$

Furthermore, if (F, G, φ) is a Quillen equivalence, then (QF, RG, φ) is an equivalence of categories.

Proof. — We refer the reader to [21, I.4 Theorem 3]. □

1.2. Cofibrantly Generated Model Categories

In this section we recall the definition of a cofibrantly generated model category.

In order to get the functorial factorizations required in axiom **MC5**, we need to introduce ordinals, cardinals, and regular cardinals. For a definition of these, see [6, Chapter 10]. It will be convenient in some situations to consider an ordinal λ as a small category, with objects equal to the elements of λ , and a unique map from a to b if $a \leq b$.

Definition 1.2.1. — *Let \mathcal{C} be a category that is closed under small colimits, and let \mathcal{V} be a class of maps in \mathcal{C} . If λ is an ordinal, then a λ -sequence in \mathcal{C} is a functor $A : \lambda \rightarrow \mathcal{C}$, i.e. a diagram*

$$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\beta \rightarrow \cdots \quad (\beta < \lambda)$$

such that for every limit ordinal $\gamma < \lambda$ the induced map

$$\mathrm{colim}_{\beta < \gamma} A_\beta \rightarrow A_\gamma$$

is an isomorphism.

The composition of the λ -sequence is the map $A_0 \rightarrow \mathrm{colim}_{\beta < \lambda} A_\beta$.

If $A_\beta \rightarrow A_{\beta+1}$ is in \mathcal{V} for any $\beta < \lambda$, we say that the λ -sequence is a λ -sequence of maps in \mathcal{V} , and the transfinite composition $A_0 \rightarrow \mathrm{colim}_{\beta < \lambda} A_\beta$ is called a transfinite composition of maps in \mathcal{V} .

Proposition 1.2.2. — *Let \mathcal{A} be a model category, then the cofibrations and trivial cofibrations in \mathcal{A} are both closed under transfinite composition.*

Proof. — The cofibrations and trivial cofibrations in \mathcal{A} are characterized by a left lifting property. But the universal property of the colimit clearly preserves this lifting property under transfinite composition. □

Definition 1.2.3. — *Let \mathcal{C} be a category closed under small colimits, and let \mathcal{V} be a class of maps in \mathcal{C} .*

1. If κ is a cardinal, then an object D in \mathcal{C} is κ -small relative to \mathcal{V} , if for every regular cardinal $\lambda \geq \kappa$ and every λ -sequence

$$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\beta \rightarrow \cdots \quad (\beta < \lambda)$$

of maps in \mathcal{V} , we have a bijection of sets:

$$\operatorname{colim}_{\beta < \lambda} \operatorname{Hom}_{\mathcal{C}}(D, A_\beta) \rightarrow \operatorname{Hom}_{\mathcal{C}}(D, \operatorname{colim}_{\beta < \lambda} A_\beta)$$

2. An object D in \mathcal{C} is small relative to \mathcal{V} if it is κ -small relative to \mathcal{V} for some cardinal κ , and it is small if it is small relative to the class of all maps in \mathcal{C} .

Definition 1.2.4. — Let \mathcal{C} be a category, and let I be a set of maps in \mathcal{C} .

1. We define I -inj as the class of maps in \mathcal{C} that have the right lifting property with respect to every map in I .
2. We define I -cof as the class of maps in \mathcal{C} that have the left lifting property with respect to every map in I -inj.

Definition 1.2.5. — Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} , then

1. The relative I -cell complexes are the maps that can be constructed as a transfinite composition of pushouts of elements of I .
2. An object A of \mathcal{C} is an I -cell complex, if the map $\emptyset \rightarrow A$ is a relative I -cell complex.
3. A map is an inclusion of I -cell complexes if it is a relative I -cell complex whose domain is an I -cell complex.

We will denote the class of relative I -cell complexes as I -cells.

Remark 1.2.6. — Since the left lifting property is preserved under pushouts and transfinite compositions we have that I -cells $\subseteq I$ -cof.

Theorem 1.2.7 (Quillen's small object argument). — Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} . Assume that the domains of all the maps in I are small with respect to I -cells. Then for every map $f : X \rightarrow Y$ in \mathcal{C} , there is a functorial factorization

$$X \xrightarrow{i} E_I^f \xrightarrow{p} Y$$

where i is in I -cells, and p is in I -inj.

Proof. — We refer the reader to [21], [6], or [10]. □

Definition 1.2.8. — A model category \mathcal{A} is cofibrantly generated if there exist sets I and J of maps in \mathcal{A} , such that:

1. The domains of all the maps in I are small with respect to the I -cells.

2. The domains of all the maps in J are small with respect to the J -cells.
3. The class $\mathcal{F} \cap \mathcal{W}$ of trivial fibrations in \mathcal{A} is equal to I -inj.
4. The class \mathcal{F} of fibrations in \mathcal{A} is equal to J -inj.

In this situation, I will be called the set of generating cofibrations, and J will be called the set of generating trivial cofibrations.

To work with spectra, we need to start with a pointed model category. The following result will allow us to go from an unpointed cofibrantly generated model category to a pointed one.

Theorem 1.2.9 (Hirschhorn). — *Let \mathcal{A} be a cofibrantly generated model category with set of generating cofibrations I and set of generating trivial cofibrations J . Then the associated pointed model category \mathcal{A}_* (see Remark 1.1.2) is also a cofibrantly generated model category, with set of generating cofibrations $F(I) = I_+$ and set of generating trivial cofibrations $F(J) = J_+$, where F is the functor $F : \mathcal{A} \rightarrow \mathcal{A}_*$ defined on objects A in \mathcal{A} as the pushout in the commutative diagram:*

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & & \downarrow \\ * & \longrightarrow & F(A) = A_+ \end{array}$$

and on maps $i : A \rightarrow B$ in \mathcal{A} as:

$$F(A) = A \amalg * \xrightarrow{i \amalg \text{id}} B \amalg * = F(B)$$

Proof. — We refer the reader to [7, Theorem 2.7]. □

1.3. Cellular Model Categories

In this section we review Hirschhorn's cellularity, which is the main property that a model category has to satisfy if we want to construct Bousfield localizations.

Definition 1.3.1. — *Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} . If $i : A \rightarrow B$ is a relative I -cell complex, then a presentation of i is a pair consisting of a λ -sequence*

$$A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\beta \rightarrow \cdots \quad (\beta < \lambda)$$

for some ordinal λ , and a sequence of ordered triples

$$\{(T^\beta, e^\beta, h^\beta)\}$$

such that:

1. The composition of the λ -sequence is isomorphic to i

2. For every $\beta < \lambda$

(a) T^β is a set.

(b) e^β is a function $e^\beta : T^\beta \rightarrow I$.

(c) If $i \in T^\beta$ and e_i^β is the element $C_i \rightarrow D_i$ of I , then h_i^β is a map $h_i^\beta : C_i \rightarrow A_\beta$, such that there is a pushout diagram

$$\begin{array}{ccc} \coprod_{i \in T^\beta} C_i & \xrightarrow{\coprod e_i^\beta} & \coprod_{i \in T^\beta} D_i \\ \coprod h_i^\beta \downarrow & & \downarrow \\ A_\beta & \longrightarrow & A_{\beta+1} \end{array}$$

Definition 1.3.2. — Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} . If

$$i : A \rightarrow B, A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\beta \rightarrow \cdots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

is a presented relative I -cell complex, then

1. The presentation ordinal of i is λ .
2. The set of cells of i is $\coprod_{\beta < \lambda} T^\beta$.
3. The size of i is the cardinal of the set of cells of i .
4. If e is a cell of i , the presentation ordinal of e is the ordinal β such that $e \in T^\beta$.
5. If $\beta < \lambda$, then the β -skeleton of i is A_β .

The next remark follows directly from the previous definitions.

Remark 1.3.3. — If \mathcal{C} is a category closed under small colimits, and I is a set of maps in \mathcal{C} , then a presented relative I -cell complex is entirely determined by its presentation ordinal λ , and its sequence of triples $\{(T^\beta, e^\beta, h^\beta)\}_{\beta < \lambda}$.

Definition 1.3.4. — Let \mathcal{C} be a category closed under small colimits, and I a set of maps in \mathcal{C} . If

$$i : A \rightarrow B, A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\beta \rightarrow \cdots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

is a presented relative I -cell complex, then a subcomplex of i consists of a presented relative I -cell complex

$$\tilde{i} : A \rightarrow \tilde{B}, A = \tilde{A}_0 \rightarrow \tilde{A}_1 \rightarrow \cdots \rightarrow \tilde{A}_\beta \rightarrow \cdots (\beta < \lambda), \{\tilde{T}^\beta, \tilde{e}^\beta, \tilde{h}^\beta\}_{\beta < \lambda}$$

such that

1. For every $\beta < \lambda$, $\tilde{T}^\beta \subseteq T^\beta$ and \tilde{e}^β is the restriction of e^β to \tilde{T}^β .

2. There is a map of λ -sequences

$$\begin{array}{ccccccc}
 A & \xrightarrow{\text{id}} & \tilde{A}_0 & \longrightarrow & \tilde{A}_1 & \longrightarrow & \tilde{A}_2 \longrightarrow \dots \\
 \text{id} \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A & \xrightarrow{\text{id}} & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 \longrightarrow \dots
 \end{array}$$

such that, for every $\beta < \lambda$ and every $i \in \tilde{T}^\beta$, the map $\tilde{h}_i^\beta : C_i \rightarrow \tilde{A}_\beta$ is a factorization of the map $h_i^\beta : C_i \rightarrow A_\beta$ through the map $\tilde{A}_\beta \rightarrow A_\beta$.

Proposition 1.3.5. — Let \mathcal{C} be a category closed under small colimits, and I a set of maps in \mathcal{C} such that the relative I -cell complexes are monomorphisms, then a subcomplex of a presented relative I -cell complex is entirely determined by its set of cells $\{\tilde{T}^\beta\}_{\beta < \lambda}$.

Proof. — The definition of a subcomplex implies that the maps $\tilde{A}_\beta \rightarrow A_\beta$ are all inclusions of subcomplexes (see definition 1.2.5(3)). Since inclusions of subcomplexes are monomorphisms, there is at most one possible factorization \tilde{h}_i^β of each h_i^β through $\tilde{A}_\beta \rightarrow A_\beta$. \square

Proposition 1.3.6. — Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} such that the relative I -cell complexes are monomorphisms. If

$$i : A \rightarrow B, A = A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_\beta \rightarrow \dots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

is a presented relative I -cell complex, then an arbitrary subcomplex of i can be constructed by the following inductive procedure:

1. Choose an arbitrary subset \tilde{T}^0 of T^0 .
2. If $\beta < \lambda$ and we have defined $\{\tilde{T}^\gamma\}_{\gamma < \beta}$, then we have determined the object \tilde{A}_β and the map $\tilde{A}_\beta \rightarrow A_\beta$. Consider the set

$$\{i \in T^\beta \mid h_i^\beta : C_i \rightarrow A_\beta \text{ factors through } \tilde{A}_\beta \rightarrow A_\beta\}$$

Choose an arbitrary subset \tilde{T}^β of this set. For every $i \in \tilde{T}^\beta$ there is a unique map $\tilde{h}_i^\beta : C_i \rightarrow \tilde{A}_\beta$ that makes the diagram

$$\begin{array}{ccc}
 C_i & & \\
 \tilde{h}_i^\beta \downarrow & \searrow h_i^\beta & \\
 \tilde{A}_\beta & \longrightarrow & A_\beta
 \end{array}$$

commute. Let $\tilde{A}_{\beta+1}$ be defined by the pushout diagram

$$\begin{array}{ccc} \coprod_{i \in \tilde{T}^\beta} C_i & \longrightarrow & \coprod_{i \in \tilde{T}^\beta} D_i \\ \coprod \tilde{h}_i^\beta \downarrow & & \downarrow \\ \tilde{A}_\beta & \longrightarrow & \tilde{A}_{\beta+1} \end{array}$$

Proof. — It follows immediately from the definitions and Proposition 1.3.5. □

Corollary 1.3.7. — *Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} such that the relative I -cell complexes are monomorphisms. Consider an arbitrary*

$$i : A \rightarrow B, A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_\beta \rightarrow \cdots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

presented cell complex. Assume that S is a set and take an arbitrary family $\{A_s\}_{s \in S}$ of subcomplexes of $i : A \rightarrow B$, then there exists a subcomplex $\cup_{s \in S} A_s$ which represents the union of the given family.

Proof. — It follows immediately from Proposition 1.3.6. □

Definition 1.3.8. — *Let \mathcal{C} be a category closed under small colimits, and let I be a set of maps in \mathcal{C} .*

1. *If γ is a cardinal, then an object A of \mathcal{C} is γ -compact relative to I if, for every presented relative I -cell complex $i : X \rightarrow Y$, every map from A to Y factors through a subcomplex of i of size at most γ .*
2. *An object A of \mathcal{C} is compact relative to I if it is γ -compact relative to I for some cardinal γ .*

Definition 1.3.9. — *Let \mathcal{U} be a cofibrantly generated model category with set of generating cofibrations I .*

1. *If γ is a cardinal, then an object X of \mathcal{U} is γ -compact if it is γ -compact relative to I (see definition 1.3.8).*
2. *An object X of \mathcal{U} is compact if there is a cardinal γ for which it is γ -compact.*

To complete the definition of a cellular model category, we need to introduce the concept of *effective monomorphism*.

Definition 1.3.10. — *Let \mathcal{C} be a category that is closed under pushouts. The map $i : A \rightarrow B$ is an effective monomorphism if i is the equalizer of the pair of natural inclusions $B \rightrightarrows B \coprod_A B$.*

Remark 1.3.11. — *In the category of sets, the class of effective monomorphisms is just the class of injective maps.*

Definition 1.3.12 (cf. [6]). — Let \mathcal{A} be a model category. We say that \mathcal{A} is cellular if it satisfies the following conditions:

1. \mathcal{A} is cofibrantly generated (see definition 1.2.8) with set of generating cofibrations I and set of generating trivial cofibrations J .
2. Both the domains and codomains of the maps in I are compact (see definition 1.3.9).
3. The domains of the maps in J are small relative to I (see definition 1.2.3).
4. The cofibrations in \mathcal{A} are effective monomorphisms (see definition 1.3.10).

When we have a cellular model category \mathcal{A} with set of generating cofibrations I , the relative I -cell complexes will be called relative cell complexes.

Theorem 1.3.13 (Hirschhorn). — Let \mathcal{A} be a cellular model category. Then the associated pointed model category \mathcal{A}_* equipped with the model structure considered in Theorem 1.2.9 is also cellular.

Proof. — We refer the reader to [7, Theorem 2.8]. □

1.4. Proper Model Categories

In this section we just recall the definition of proper model categories.

Definition 1.4.1. — Let \mathcal{A} be a model category. We say that \mathcal{A} is left proper if the class of weak equivalences is closed under pushouts along cofibrations, i.e. in any pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow i & & \downarrow \\ B & \xrightarrow{h_*} & Y \end{array}$$

where i is a cofibration and h is a weak equivalence, we then have that h_* is also a weak equivalence.

Definition 1.4.2. — Let \mathcal{A} be a model category. We say that \mathcal{A} is right proper if the class of weak equivalences is closed under pullbacks along fibrations, i.e. in any pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{h^*} & X \\ \downarrow & & \downarrow p \\ B & \xrightarrow{h} & Y \end{array}$$

where p is a fibration and h is a weak equivalence, we then have that h^* is also a weak equivalence.

Definition 1.4.3. — Let \mathcal{U} be a model category. We say that \mathcal{U} is proper if it is both left and right proper.

Theorem 1.4.4 (Hirschhorn). — Let \mathcal{U} be a left proper, right proper, or proper model category. Then the associated pointed model category \mathcal{U}_* (see Remark 1.1.2) is also left proper, right proper, or proper.

Proof. — We refer the reader to [7, Theorem 2.8]. □

1.5. Simplicial Sets

Let Δ denote the category of well ordered finite sets, i.e. the category with objects:

$$\mathbf{n} = \{0 < 1 < \dots < n\}$$

where $n \geq 0$; and maps the weakly order preserving functions, i.e.:

$$\mathrm{Hom}_{\Delta}(\mathbf{m}, \mathbf{n}) = \{f : \mathbf{m} \rightarrow \mathbf{n} \mid i \leq j \Rightarrow f(i) \leq f(j)\}$$

There exists a canonical set of generators for the maps in Δ , called *cofaces* ($\delta^i : \mathbf{n} \rightarrow \mathbf{n} + 1$), and *codegeneracies* ($\sigma^i : \mathbf{n} + 1 \rightarrow \mathbf{n}$), defined as:

$$\delta^i(j) = \begin{cases} j, & \text{if } j < i \\ j + 1, & \text{if } j \geq i \end{cases}$$

$$\sigma^i(j) = \begin{cases} j, & \text{if } j \leq i \\ j - 1, & \text{if } j > i \end{cases}$$

The cofaces and codegeneracies satisfy a list of relations called the *cosimplicial identities*:

$$(4) \quad \begin{aligned} \delta^j \delta^i &= \delta^i \delta^{j-1} && \text{for } i < j \\ \sigma^j \delta^i &= \delta^i \sigma^{j-1} && \text{for } i < j \\ \sigma^i \delta^i &= \mathrm{id} \\ \sigma^i \delta^{i+1} &= \mathrm{id} \\ \sigma^j \delta^i &= \delta^{i-1} \sigma^j && \text{for } i > j + 1 \\ \sigma^j \sigma^i &= \sigma^i \sigma^{j+1} && \text{for } i < j \end{aligned}$$

Definition 1.5.1. — A simplicial set X is a contravariant functor from the category Δ to the category of sets.

We will denote the category of simplicial sets by **SSets**, where the maps between two simplicial sets X and Y are just the natural transformations $\eta : X \rightarrow Y$.

It follows from the cosimplicial identities that to specify a simplicial set X , it is enough to give sets $X_0, X_1, \dots, X_n, \dots$; where $X_i = X(i)$ together with *face maps*

$d_i : X_n \rightarrow X_{n-1}$ ($d_i = X(\delta^i)$) and *degeneracy* maps $s_i : X_n \rightarrow X_{n+1}$ ($s_i = X(\sigma^i)$), satisfying the following relations which are called *simplicial identities* (these are just the duals with respect to the cosimplicial identities):

$$(5) \quad \begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{for } i < j \\ d_i s_j &= s_{j-1} d_i \quad \text{for } i < j \\ d_i s_i &= \text{id} \\ d_{i+1} s_i &= \text{id} \\ d_i s_j &= s_j d_{i-1} \quad \text{for } i > j + 1 \\ s_i s_j &= s_{j+1} s_i \quad \text{for } i < j \end{aligned}$$

There exist three particular interesting families of simplicial sets: Δ^n , $\partial\Delta^n$ and \bigwedge_k^n ; they are defined in the following way:

$$(6) \quad \Delta^n = \text{Hom}_\Delta(-, \mathbf{n})$$

$\partial\Delta^n$ is the subobject of Δ^n characterized by:

$$(7) \quad (\partial\Delta^n)_m = \{f : \mathbf{m} \rightarrow \mathbf{n} \mid f \text{ is not surjective}\}$$

and finally \bigwedge_k^n is the subobject of $\partial\Delta^n$ given by:

$$(8) \quad (\bigwedge_k^n)_m = \{f : \mathbf{m} \rightarrow \mathbf{n} \mid \{0 < 1 < \dots < \hat{k} < \dots < n\} \not\subseteq \text{im}(f)\}$$

where $\{0 < 1 < \dots < \hat{k} < \dots < n\}$ denotes the well ordered set \mathbf{n} with the k element removed.

We also have the dual notion of *cosimplicial set*:

Definition 1.5.2. — *A cosimplicial set X is a covariant functor from the category Δ to the category of sets.*

Given any category \mathcal{C} , we can also define *simplicial* and *cosimplicial* objects in \mathcal{C} , where a simplicial (respectively cosimplicial) object X in \mathcal{C} is just a contravariant (respectively covariant) functor from Δ to \mathcal{C} .

Let **Top** be the category of compactly generated Hausdorff topological spaces. Consider the following family of objects in **Top**:

$$|\Delta^n| = \{(t_0, t_1, \dots, t_n) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbb{R}^{n+1}$$

We get a cosimplicial object $|\Delta^\bullet|$ in **Top** if we define the coface and codegeneracy maps for $|\Delta^n|$ as:

$$(9) \quad \begin{aligned} \delta^i : |\Delta^n| &\longrightarrow |\Delta^{n+1}| \\ (t_0, t_1, \dots, t_n) &\longmapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n) \end{aligned}$$

and

$$(10) \quad \begin{array}{ccc} \sigma^i : |\Delta^{n+1}| & \longrightarrow & |\Delta^n| \\ (t_0, t_1, \dots, t_{n+1}) & \longmapsto & (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_{n+1}) \end{array}$$

Now we are ready to define the *geometric realization* functor:

$$|-| : \mathbf{SSets} \rightarrow \mathbf{Top}$$

Let X be a simplicial set, then its geometric realization $|X|$ is the following topological space:

$$(11) \quad |X| = \varinjlim_{\Delta^n \downarrow X} |\Delta^n|$$

where the indexing category to compute the colimit has objects given by the simplices over X , i.e. maps of simplicial sets $\Delta^n \rightarrow X$; and morphisms given by commutative triangles:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\theta_*} & \Delta^m \\ & \searrow & \swarrow \\ & X & \end{array}$$

for $\theta : \mathbf{n} \rightarrow \mathbf{m}$

The geometric realization functor $|-|$ has a right adjoint:

$$\mathbf{Sing} : \mathbf{Top} \rightarrow \mathbf{SSets}$$

called the *singular functor* and defined in the following way:

$$(12) \quad \begin{array}{ccc} \mathbf{Sing}(T) : \Delta^{\text{op}} & \longrightarrow & \mathcal{S}ets \\ n & \longmapsto & \mathbf{Hom}_{\mathbf{Top}}(|\Delta^n|, T) \end{array}$$

with faces and degeneracies induced by the cofaces and codegeneracies of the cosimplicial object $|\Delta^\bullet|$.

We say that a map of simplicial sets $\theta : X \rightarrow Y$ is a weak equivalence if its geometric realization $|\theta| : |X| \rightarrow |Y|$ is a weak equivalence of topological spaces, i.e. $\pi_i(|\theta|, *)$ is an isomorphism for any $i \geq 0$, and for every choice of base point $* \in |X|$.

With all the previous definitions, we are ready to give a cofibrantly generated model category structure on the category of simplicial sets. Take $I = \{\partial\Delta^n \hookrightarrow \Delta^n\}$ and $J = \{\bigwedge_k^n \hookrightarrow \Delta^n\}$.

Theorem 1.5.3 (Quillen). — *The category of simplicial sets \mathbf{SSets} has a cofibrantly generated model category structure, where the weak equivalences, the set of generating cofibrations I and the set of generating trivial cofibrations J are defined as above.*

Proof. — The proof is probably one of the most difficult in abstract homotopy theory. We refer the reader to [21, II.3 Theorem 3], [4] or [10]. □

1.6. Simplicial Model Categories

Simplicial model categories were defined by Quillen in [21], we will follow the approach in [4, Chapter 2] and [6, Chapter 9].

Definition 1.6.1. — *Let \mathcal{A} be a category. We say that \mathcal{A} is simplicial if it satisfies the following axioms:*

1. *There exists a functor*

$$\begin{aligned} \mathcal{A}^{\text{op}} \times \mathcal{A} &\longrightarrow \mathbf{SSets} \\ X, Y &\longmapsto \text{Map}(X, Y) \end{aligned}$$

such that

2. *The set of 0-simplices in $\text{Map}(X, Y)$ is equal to the set of maps in \mathcal{A} from X to Y , i.e. $\text{Map}(X, Y)_0 = \text{Hom}_{\mathcal{A}}(X, Y)$.*
3. *For every triple X, Y, Z of objects in \mathcal{A} , there exists a map of simplicial sets called composition law*

$$\circ_{X, Y, Z} : \text{Map}(Y, Z) \times \text{Map}(X, Y) \longrightarrow \text{Map}(X, Z)$$

which is compatible with the composition in \mathcal{A} .

4. *There exists a map of simplicial sets $i_X : * \rightarrow \text{Map}(X, X)$, for every object $X \in \mathcal{A}$.*
5. *There exist three commutative diagrams (see [6, definition 9.1.2]), which give the associativity of the composition law, and right and left unit properties for the map i_X .*

Definition 1.6.2. — *Let \mathcal{A} be a model category, we say that \mathcal{A} is a simplicial model category if it is a simplicial category (see definition 1.6.1) and satisfies the following two axioms:*

- SM0 :** 1. *For every $X \in \mathcal{A}$, the functor*

$$\begin{aligned} \text{Map}(X, -) : \mathcal{A} &\longrightarrow \mathbf{SSets} \\ Y &\longmapsto \text{Map}(X, Y) \end{aligned}$$

has a left adjoint

$$\begin{aligned} X \otimes - : \mathbf{SSets} &\longrightarrow \mathcal{A} \\ K &\longmapsto X \otimes K \end{aligned}$$

such that the adjunction is compatible with the simplicial structure on \mathcal{A} , i.e. $\text{Map}(X \otimes K, Y) \cong \text{Map}(K, \text{Map}(X, Y))$, where the simplicial set on the right hand side is the one defined in Remark 1.6.3(1).

2. For every $Y \in \mathcal{A}$, the functor

$$\begin{aligned} \text{Map}(-, Y) : \mathcal{A}^{\text{op}} &\longrightarrow \mathbf{SSets} \\ X &\longmapsto \text{Map}(X, Y) \end{aligned}$$

has a left adjoint

$$\begin{aligned} Y^- : \mathbf{SSets} &\longrightarrow \mathcal{A}^{\text{op}} \\ K &\longmapsto Y^K \end{aligned}$$

such that the adjunction is compatible with the simplicial structure on \mathcal{A} , i.e. $\text{Map}(X, Y^K) \cong \text{Map}(K, \text{Map}(X, Y))$, where the simplicial set on the right hand side is the one defined in Remark 1.6.3(1).

SM7 : For any cofibration $i : A \rightarrow B$ in \mathcal{A} and fibration $p : X \rightarrow Y$ in \mathcal{A} , the map

$$\text{Map}(B, X) \xrightarrow{(i^*, p_*)} \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is a fibration of simplicial sets, which is trivial if either i or p is a weak equivalence.

Remark 1.6.3. — 1. The category of simplicial sets \mathbf{SSets} has a canonical simplicial model category structure where $\text{Map}(X, Y)$ is the simplicial set having n -simplices

$$\text{Map}(X, Y)_n = \text{Hom}_{\mathbf{SSets}}(X \times \Delta^n, Y)$$

with faces and degeneracies induced from the cosimplicial object Δ^\bullet .

2. The associated category of pointed simplicial sets \mathbf{SSets}_* equipped with the induced model structure from \mathbf{SSets} (see Remark 1.1.2) has a natural simplicial model category structure.

Lemma 1.6.4. — Let \mathcal{A} be a simplicial model category. Suppose that $i : A \rightarrow B$, $p : X \rightarrow Y$ are maps in \mathcal{A} and $j : L \rightarrow K$ is a map of simplicial sets. Then the following are equivalent:

1. For every solid commutative diagram of simplicial sets

$$\begin{array}{ccc} L & \xrightarrow{\quad\quad\quad} & \text{Map}(B, X) \\ j \downarrow & \nearrow \text{---} & \downarrow (i^*, p_*) \\ K & \xrightarrow{\quad\quad\quad} & \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y) \end{array}$$

the dotted arrow making the diagram commutative exists.

2. For every solid commutative diagram in \mathcal{A}

$$\begin{array}{ccc} A \otimes K \amalg_{A \otimes L} B \otimes L & \longrightarrow & X \\ i \square j \downarrow & \nearrow & \downarrow p \\ B \otimes K & \longrightarrow & Y \end{array}$$

the dotted arrow making the diagram commutative exists.

3. For every solid commutative diagram in \mathcal{A}

$$\begin{array}{ccc} A & \longrightarrow & X^K \\ i \downarrow & \nearrow & \downarrow (j^*, p_*) \\ B & \longrightarrow & X^L \times_{Y^L} Y^K \end{array}$$

the dotted arrow making the diagram commutative exists.

Proof. — It follows directly from the existence of the adjunctions in axiom **SM0**. \square

The following is a useful criterion to check axiom **SM7**.

Proposition 1.6.5. — *Let \mathcal{A} be a model category with a simplicial structure (see definition 1.6.1), satisfying axiom **SM0**, then the following are equivalent:*

1. \mathcal{A} satisfies axiom **SM7**.
2. Suppose that $i : A \rightarrow B$ is a cofibration in \mathcal{A} , and $j : L \rightarrow K$ is a cofibration of simplicial sets, then the map

$$A \otimes K \amalg_{A \otimes L} B \otimes L \xrightarrow{i \square j} B \otimes K$$

is a cofibration in \mathcal{A} , which is trivial if either i or j is a weak equivalence.

3. Suppose that $p : X \rightarrow Y$ is a fibration in \mathcal{A} , and $j : L \rightarrow K$ is a cofibration of simplicial sets, then the map

$$X^K \xrightarrow{(j^*, p_*)} X^L \times_{Y^L} Y^K$$

is a fibration in \mathcal{A} , which is trivial if either p or j is a weak equivalence.

Proof. — It follows from Lemma 1.6.4 and Corollary 1.1.6. \square

These characterizations of axiom **SM7**, allow to construct “simplicial” cylinder (respectively path) objects for any cofibrant (respectively fibrant) object A of \mathcal{A} .

Proposition 1.6.6. — *Let \mathcal{U} be a simplicial model category, and let A be a cofibrant object in \mathcal{U} . Then the following diagram represents a cylinder object for A*

$$\begin{array}{ccc} A \otimes \partial\Delta^1 \cong A \amalg A & & \\ \downarrow i & \searrow \nabla & \\ A \otimes \Delta^1 & \xrightarrow{s} & A \otimes * \cong A \end{array}$$

Proof. — Proposition 1.6.5 implies that i is a cofibration. In the following commutative diagram

$$\begin{array}{ccc} A \otimes * \cong A & & \\ \downarrow t & \searrow \text{id} & \\ A \otimes \Delta^1 & \xrightarrow{s} & A \otimes * \cong A \end{array}$$

proposition 1.6.5 implies that t is a trivial cofibration, so by the two out of three property for weak equivalences we have that s is a weak equivalence. It only remains to show that $A \otimes \partial\Delta^1 \rightarrow A \otimes *$ is the fold map $A \amalg A \rightarrow A$, but this follows from the next commutative diagram:

$$\begin{array}{ccccc} & & A \otimes * & & \\ & & \downarrow \text{id} \otimes d_1 & \searrow \text{id} & \\ & & A \otimes \partial\Delta^1 & \xrightarrow{i} & A \otimes \Delta^1 & \xrightarrow{s} & A \otimes * \\ & & \uparrow \text{id} \otimes d_0 & \nearrow \text{id} & & & \\ & & A \otimes * & & & & \end{array}$$

□

The dual statement for path objects is the following.

Proposition 1.6.7. — *Let \mathcal{U} be a simplicial model category, and let X be a fibrant object in \mathcal{U} . Then the following diagram represents a path object for X*

$$\begin{array}{ccc} & & X^{\Delta^1} \\ & \nearrow r & \downarrow p \\ X \cong X^* & \xrightarrow[\Delta]{} & X^{\partial\Delta^1} \cong X \times X \end{array}$$

□

One of the interesting consequences we get when we have a simplicial model category \mathcal{U} , is that we can compute the maps in the homotopy category $\mathbf{Ho}\mathcal{U}$ simplicially.

Proposition 1.6.8. — *Let X, Y be a pair of objects in \mathcal{U} , where X is cofibrant and Y is fibrant. Then $[X, Y] = \pi_0 \text{Map}(X, Y)$, where $[X, Y] = \text{Hom}_{\mathbf{Ho}\mathcal{U}}(X, Y)$.*

Proof. — Since X is cofibrant and Y is fibrant, we have that $[X, Y]$ is just the set of homotopy classes of maps between X and Y . On the other hand, axiom **SM7** implies that $\text{Map}(X, Y)$ is a fibrant simplicial set (Kan complex), so $\pi_0 \text{Map}(X, Y)$ is computed using the simplicial homotopies given by $\Delta^1 \rightarrow \text{Map}(X, Y)$, which by the adjunction are in bijection with the homotopies given by $X \otimes \Delta^1 \rightarrow Y$. But these are just homotopies between X and Y , since Proposition 1.6.6 implies that $X \otimes \Delta^1$ is a cylinder object for X . \square

Corollary 1.6.9. — *Let \mathcal{A} be a simplicial model category, and consider a couple of objects X, Y in \mathcal{A} . Then $[X, Y] = \pi_0 \text{Map}(RQX, RQY)$.*

Proof. — By construction $[X, Y]$ is equal to set of homotopy classes of maps between RQX and RQY . But RQX, RQY are both cofibrant and fibrant objects in \mathcal{A} , so Proposition 1.6.8 implies that this set of homotopy classes of maps is equal to $\pi_0 \text{Map}(RQX, RQY)$. \square

Another simple but very useful consequence of having a simplicial model category \mathcal{A} , is that we can also detect weak equivalences in \mathcal{A} at the level of simplicial sets.

Proposition 1.6.10. — *Let \mathcal{A} be a simplicial model category, and let $h : A \rightarrow B$ be a map between two cofibrant (respectively fibrant) objects in \mathcal{A} . Then h is a weak equivalence if and only if for every fibrant (respectively cofibrant) object X in \mathcal{A} , $h^* : \text{Map}(B, X) \rightarrow \text{Map}(A, X)$ (respectively $h_* : \text{Map}(X, A) \rightarrow \text{Map}(X, B)$) is a weak equivalence of simplicial sets.*

Proof. — By duality, it is enough to consider the case in which A, B are cofibrant objects in \mathcal{A} . Assume that h is a weak equivalence. Since weak equivalences of simplicial sets have the two out of three property, then by Ken Brown's lemma (see Lemma 1.1.4) we can assume that h is a trivial cofibration. The conclusion then follows from axiom **SM7** which implies that for any fibrant object X in \mathcal{A} , $h^* : \text{Map}(B, X) \rightarrow \text{Map}(A, X)$ is a trivial fibration of simplicial sets, so in particular h^* is a weak equivalence.

For the converse, it is enough to show that $h^* : [B, X] \rightarrow [A, X]$ is a bijection for every fibrant object X in \mathcal{A} . But since for every fibrant object X in \mathcal{A} , $h^* : \text{Map}(B, X) \rightarrow \text{Map}(A, X)$ is a weak equivalence of simplicial sets, in particular we have that $h^* : \pi_0 \text{Map}(B, X) \rightarrow \pi_0 \text{Map}(A, X)$ is a bijection, and the result follows from Proposition 1.6.8 since A, B are cofibrant in \mathcal{A} and X is fibrant in \mathcal{A} . \square

Corollary 1.6.11. — *Let \mathcal{A} be a simplicial model category and consider a couple of objects A, B in \mathcal{A} , and a map $h : A \rightarrow B$ between them. Then the following conditions are equivalent:*

1. h is a weak equivalence in \mathcal{A} .

2. For every fibrant object X in \mathcal{A} , $(Qh)^* : \text{Map}(QB, X) \rightarrow \text{Map}(QA, X)$ is a weak equivalence of simplicial sets.
3. For every cofibrant object C in \mathcal{A} , $(Rh)_* : \text{Map}(C, RA) \rightarrow \text{Map}(C, RB)$ is a weak equivalence of simplicial sets.

Proof. — (1) \Leftrightarrow (2). We have that h is a weak equivalence if and only if every (or some) cofibrant approximation $Qh : QA \rightarrow QB$ is also a weak equivalence. Since QA, QB are cofibrant the result follows from Proposition 1.6.10.

(1) \Leftrightarrow (3). We know that h is a weak equivalence if and only if every (or some) fibrant approximation $Rh : RA \rightarrow RB$ is also a weak equivalence. But RA, RB are fibrant, so the result follows from Proposition 1.6.10. \square

1.7. Symmetric Monoidal Model Categories

Symmetric monoidal model categories were introduced by Hovey in [10, Chapter 4]. In this section we just recall some of his definitions and results without proof. This is the language that we will use in Section 3.6 to construct external pairings for the slice filtration.

Definition 1.7.1. — Let \mathcal{C} be a monoidal category. We say that a category \mathcal{D} is a left \mathcal{C} -module if the following conditions are satisfied:

1. There exists a bifunctor $\otimes : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$
2. For every pair of objects X, Y in \mathcal{C} and every object A in \mathcal{D} there exists a natural isomorphism $a : (X \otimes Y) \otimes A \rightarrow X \otimes (Y \otimes A)$.
3. For every object A in \mathcal{D} there exists a natural isomorphism $l : \mathbf{1} \otimes A \rightarrow A$, where $\mathbf{1}$ denotes the unit for the monoidal structure on \mathcal{C} .
4. Three coherence diagrams commute (see [10, definition 4.1.6]).

We also have right modules over a given monoidal category.

Definition 1.7.2. — Given three categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, we define an adjunction of two variables as a bifunctor $\otimes : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ together with two extra functors $\text{Hom}_r : \mathcal{D}^{\text{op}} \times \mathcal{E} \rightarrow \mathcal{C}$ and $\text{Hom}_l : \mathcal{C}^{\text{op}} \times \mathcal{E} \rightarrow \mathcal{D}$, such that there exist the following two adjunctions:

1. $\text{Hom}_{\mathcal{E}}(X \otimes Y, Z) \xrightarrow{\varphi_r} \text{Hom}_{\mathcal{E}}(X, \text{Hom}_r(Y, Z))$
2. $\text{Hom}_{\mathcal{E}}(X \otimes Y, Z) \xrightarrow{\varphi_l} \text{Hom}_{\mathcal{D}}(Y, \text{Hom}_l(X, Z))$

Definition 1.7.3. — We say that a category \mathcal{C} is closed monoidal if it is a monoidal category such that the bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ giving the monoidal structure is an adjunction of two variables.

Definition 1.7.4. — Given model categories \mathcal{A} , \mathcal{B} , \mathcal{D} an adjunction of two variables $\otimes : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D}$ is called a Quillen adjunction of two variables, if given a cofibration $i : A \rightarrow B$ in \mathcal{A} and a cofibration $j : C \rightarrow D$ in \mathcal{B} , the induced map

$$i \square j : A \otimes D \coprod_{A \otimes C} B \otimes C \longrightarrow B \otimes D$$

is a cofibration in \mathcal{D} which is trivial if either i or j is a weak equivalence. In this case, we will refer to the functor \otimes as a Quillen bifunctor.

Lemma 1.7.5 (Hovey). — Let \mathcal{A} , \mathcal{B} , \mathcal{D} be three model categories and let $\otimes : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D}$ be an adjunction of two variables. Then the following conditions are equivalent:

1. \otimes is a Quillen bifunctor.
2. Given a cofibration $j : C \rightarrow D$ in \mathcal{B} and a fibration $p : X \rightarrow Y$ in \mathcal{D} , the induced map

$$(j^*, p_*) : \text{Hom}_r(D, X) \longrightarrow \text{Hom}_r(C, X) \times_{\text{Hom}_r(C, Y)} \text{Hom}_r(D, Y)$$

is a fibration in \mathcal{A} which is trivial if either j or p is a weak equivalence.

3. Given a cofibration $i : A \rightarrow B$ in \mathcal{A} and a fibration $p : X \rightarrow Y$ in \mathcal{D} , the induced map

$$(i^*, p_*) : \text{Hom}_l(B, X) \longrightarrow \text{Hom}_l(A, X) \times_{\text{Hom}_l(A, Y)} \text{Hom}_l(B, Y)$$

is a fibration in \mathcal{B} which is trivial if either i or p is a weak equivalence.

Proof. — It follows immediately from the adjunctions that appear in the definition of an adjunction of two variables (see definition 1.7.2), and the lifting property characterization for cofibrations, fibrations, trivial cofibrations and trivial fibrations. \square

Remark 1.7.6 (cf. [10]). — Let $\otimes : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D}$ be a Quillen bifunctor. Then if A is a cofibrant object in \mathcal{A} , the functor $A \otimes - : \mathcal{B} \rightarrow \mathcal{D}$ is a Quillen functor with right adjoint $\text{Hom}_l(A, -) : \mathcal{D} \rightarrow \mathcal{B}$. Similarly if B is a cofibrant object in \mathcal{B} , we get a Quillen functor $- \otimes B : \mathcal{A} \rightarrow \mathcal{D}$ with right adjoint $\text{Hom}_r(B, -)$. Finally, if X is a fibrant object in \mathcal{D} , we get a Quillen functor $\text{Hom}_r(-, X) : \mathcal{B} \rightarrow \mathcal{A}^{\text{op}}$ with right adjoint $\text{Hom}_l(-, X) : \mathcal{A}^{\text{op}} \rightarrow \mathcal{B}$.

Definition 1.7.7. — A monoidal model category \mathcal{A} is a closed monoidal category with a model category structure, such that the following conditions are satisfied:

1. The bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ giving the monoidal structure is a Quillen bifunctor.
2. Let $q : Q\mathbf{1} \rightarrow \mathbf{1}$ be a cofibrant replacement for the unit $\mathbf{1}$. Then the natural maps $q \otimes \text{id} : Q\mathbf{1} \otimes A \rightarrow \mathbf{1} \otimes A$, $\text{id} \otimes q : A \otimes Q\mathbf{1} \rightarrow A \otimes \mathbf{1}$ are weak equivalences for any cofibrant object A in \mathcal{A} .

We have an analogous definition for symmetric monoidal categories.

Proposition 1.7.8 (Quillen). — *The category of simplicial sets \mathbf{SSets} is a symmetric monoidal model category.*

Proof. — We refer the reader to [21, II.3 Theorem 3]. □

Proposition 1.7.9 (Hovey). — *Let \mathcal{A} be a monoidal model category, with unit $\mathbf{1}$ equal to the terminal object $*$, and assume that $*$ is cofibrant. Then the associated pointed category \mathcal{A}_* (equipped with the model structure described in Remark 1.1.2) is also a monoidal model category, which is symmetric if \mathcal{A} is.*

Proof. — We refer the reader to [10, Proposition 4.2.9]. □

Corollary 1.7.10. — *The category of pointed simplicial sets \mathbf{SSets}_* is a symmetric monoidal model category.*

Proof. — It follows immediately from Propositions 1.7.8 and 1.7.9. □

Definition 1.7.11. — *Let $(F, G, \varphi) : \mathcal{A} \rightarrow \mathcal{B}$ be a Quillen adjunction between two monoidal model categories. We say that (F, G, φ) is a monoidal Quillen adjunction if F is a monoidal functor (see [10, definition 4.1.2]) and the map $F(q_{\mathbf{1}}) : F(Q\mathbf{1}) \rightarrow F\mathbf{1}$ is a weak equivalence. In this situation we say that F is a left Quillen monoidal functor.*

Definition 1.7.12. — *Let \mathcal{A} be a monoidal model category. A \mathcal{A} -model category is a left \mathcal{A} -module \mathcal{B} equipped with a model category structure such that the following conditions hold:*

1. *The action map $- \otimes - : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{B}$ is a Quillen bifunctor.*
2. *If $q : Q\mathbf{1} \rightarrow \mathbf{1}$ is a cofibrant replacement for $\mathbf{1}$ in \mathcal{A} , then the map $q \otimes \text{id} : Q\mathbf{1} \otimes A \rightarrow \mathbf{1} \otimes A$ is a weak equivalence for every cofibrant object A in \mathcal{B} .*

The simplicial model categories discussed in Section 1.6 are just \mathbf{SSets} -model categories.

Proposition 1.7.13 (Hovey). — *Let \mathcal{A} be a monoidal model category where the unit $\mathbf{1}$ is equal to the terminal object $*$. Assume that $*$ is cofibrant. If \mathcal{B} is a \mathcal{A} -model category, then the associated pointed category \mathcal{B}_* has a natural \mathcal{A}_* -model category structure.*

Proof. — We refer the reader to [10, Proposition 4.2.19]. □

Proposition 1.7.14 (Hovey). — *Let $\mathcal{A}, \mathcal{B}, \mathcal{D}$ be three model categories, and let $- \otimes - : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{D}$ be a Quillen bifunctor. Then the total derived functors define an adjunction of two variables $\otimes^{\mathbf{L}} : \text{Ho}\mathcal{A} \times \text{Ho}\mathcal{B} \rightarrow \text{Ho}\mathcal{D}$, with adjoints given by $\mathbf{R}\text{Hom}_l : (\text{Ho}\mathcal{A})^{\text{op}} \times \text{Ho}\mathcal{D} \rightarrow \mathcal{B}$ and $\mathbf{R}\text{Hom}_r : (\text{Ho}\mathcal{B})^{\text{op}} \times \text{Ho}\mathcal{D} \rightarrow \text{Ho}\mathcal{A}$.*

Proof. — We refer the reader to [10, Proposition 4.3.1]. □

Theorem 1.7.15 (Hovey). — *Let \mathcal{A} be a (symmetric) monoidal model category. Then $\mathrm{Ho}\mathcal{A}$ can be given the structure of a closed (symmetric) monoidal category. The adjunction of two variables $(\otimes^{\mathbf{L}}, \mathbf{R}\mathrm{Hom}_l, \mathbf{R}\mathrm{Hom}_r)$ which gives the closed structure on $\mathrm{Ho}\mathcal{A}$ is the total derived adjunction of $(\otimes, \mathrm{Hom}_l, \mathrm{Hom}_r)$ described in Proposition 1.7.14. The associativity and unit isomorphisms (and the commutativity isomorphism in case \mathcal{A} is symmetric) on $\mathrm{Ho}\mathcal{A}$ are derived from the corresponding isomorphisms of \mathcal{A} .*

Proof. — We refer the reader to [10, Theorem 4.3.2]. □

1.8. Localization of Model Categories

In this section we recall some of Hirschhorn's constructions [6, Sections 3.1, 3.2] restricted to the case where all the model categories are simplicial.

Definition 1.8.1. — *Let \mathcal{A} be a model category and let \mathcal{V} be a class of maps in \mathcal{A} . A left localization of \mathcal{A} with respect to \mathcal{V} is a model category $L_{\mathcal{V}}\mathcal{A}$ equipped with a left Quillen functor $\lambda : \mathcal{A} \rightarrow L_{\mathcal{V}}\mathcal{A}$ satisfying the following properties:*

1. *The total left derived functor $\mathbf{L}\lambda : \mathrm{Ho}\mathcal{A} \rightarrow \mathrm{Ho}L_{\mathcal{V}}\mathcal{A}$ takes the images in $\mathrm{Ho}\mathcal{A}$ of the elements in \mathcal{V} into isomorphisms in $\mathrm{Ho}L_{\mathcal{V}}\mathcal{A}$.*
2. *If \mathcal{B} is a model category and $\tau : \mathcal{A} \rightarrow \mathcal{B}$ is a left Quillen functor such that $\mathbf{L}\tau : \mathrm{Ho}\mathcal{A} \rightarrow \mathrm{Ho}\mathcal{B}$ takes the images in $\mathrm{Ho}\mathcal{A}$ of the elements of \mathcal{V} into isomorphisms in $\mathrm{Ho}\mathcal{B}$, then there exists a unique left Quillen functor $\sigma : L_{\mathcal{V}}\mathcal{A} \rightarrow \mathcal{B}$ with $\sigma\lambda = \tau$.*

Definition 1.8.2. — *Let \mathcal{A} be a model category and let \mathcal{V} be a class of maps in \mathcal{A} . A right localization of \mathcal{A} with respect to \mathcal{V} is a model category $R_{\mathcal{V}}\mathcal{A}$ equipped with a right Quillen functor $\rho : \mathcal{A} \rightarrow R_{\mathcal{V}}\mathcal{A}$ satisfying the following properties:*

1. *The total right derived functor $\mathbf{R}\rho : \mathrm{Ho}\mathcal{A} \rightarrow \mathrm{Ho}R_{\mathcal{V}}\mathcal{A}$ takes the images in $\mathrm{Ho}\mathcal{A}$ of the elements in \mathcal{V} into isomorphisms in $\mathrm{Ho}R_{\mathcal{V}}\mathcal{A}$.*
2. *If \mathcal{B} is a model category and $\tau : \mathcal{A} \rightarrow \mathcal{B}$ is a right Quillen functor such that $\mathbf{R}\tau : \mathrm{Ho}\mathcal{A} \rightarrow \mathrm{Ho}\mathcal{B}$ takes the images in $\mathrm{Ho}\mathcal{A}$ of the elements of \mathcal{V} into isomorphisms in $\mathrm{Ho}\mathcal{B}$, then there exists a unique right Quillen functor $\sigma : R_{\mathcal{V}}\mathcal{A} \rightarrow \mathcal{B}$ with $\sigma\rho = \tau$.*

From the universal property, we immediately get the following uniqueness statement.

Remark 1.8.3. — *Let \mathcal{A} be a model category and \mathcal{V} a class of maps in \mathcal{A} . If a left or right localization of \mathcal{A} with respect to \mathcal{V} exists, then it is unique up to a unique isomorphism.*

Definition 1.8.4. — Let \mathcal{A} be a model category and \mathcal{V} a class of maps in \mathcal{A} .

1. An object A of \mathcal{A} is \mathcal{V} -local if A is fibrant and for every map $f : C \rightarrow D$ in \mathcal{V} , the induced map of simplicial sets $\text{Map}(QD, A) \rightarrow \text{Map}(QC, A)$ is a weak equivalence.
2. A map $f : C \rightarrow D$ in \mathcal{A} is a \mathcal{V} -local equivalence if for every \mathcal{V} -local object A , the induced map of simplicial sets $\text{Map}(QD, A) \rightarrow \text{Map}(QC, A)$ is a weak equivalence.

Definition 1.8.5. — Let \mathcal{A} be a model category and \mathcal{V} a class of maps in \mathcal{A} .

1. An object A of \mathcal{A} is \mathcal{V} -colocal if A is cofibrant and for every map $f : C \rightarrow D$ in \mathcal{V} , the induced map of simplicial sets $\text{Map}(A, RC) \rightarrow \text{Map}(A, RD)$ is a weak equivalence.
2. A map $f : C \rightarrow D$ in \mathcal{A} is a \mathcal{V} -colocal equivalence if for every \mathcal{V} -colocal object A , the induced map of simplicial sets $\text{Map}(A, RC) \rightarrow \text{Map}(A, RD)$ is a weak equivalence.

The following definition will be necessary for the construction of right Bousfield localizations.

Definition 1.8.6. — Let \mathcal{A} be a model category and let K be a set of objects in \mathcal{A} .

1. A map $g : X \rightarrow Y$ is a K -colocal equivalence if for every object A in K the induced map of simplicial sets $(Rg)_* : \text{Map}(QA, RX) \rightarrow \text{Map}(QA, RY)$ is a weak equivalence.
2. If \mathcal{V} is the class of K -colocal equivalences, then a \mathcal{V} -colocal object will be called K -colocal.

Proposition 1.8.7 (Hirschhorn). — Let \mathcal{A} be a model category and let \mathcal{V} be a class of maps in \mathcal{A} .

1. The class of \mathcal{V} -local equivalences satisfies the two out of three property (see MC2 in definition 1.1.1).
2. The class of \mathcal{V} -colocal equivalences satisfies the two out of three property.

Proof. — We refer the reader to [6, Proposition 3.2.3]. □

1.9. Bousfield Localization

In this section we review Hirschhorn's construction of Bousfield localizations [6, Section 3.3] in the restricted situation where all the model categories are simplicial. These constructions will be the main technical ingredient in our approach to produce a lifting of the slice filtration to the model category setting (see Chapter 3).

Definition 1.9.1. — Let \mathcal{A} be a model category and let \mathcal{V} be a class of maps in \mathcal{A} . The left Bousfield localization of \mathcal{A} with respect to \mathcal{V} (in case it exists) is a model category structure $L_{\mathcal{V}}\mathcal{A}$ on the underlying category of \mathcal{A} such that

1. the class of weak equivalences of $L_{\mathcal{V}}\mathcal{A}$ is defined as the class of \mathcal{V} -local equivalences of \mathcal{A} (see definition 1.8.4).
2. the class of cofibrations of $L_{\mathcal{V}}\mathcal{A}$ is the same as the class of cofibrations of \mathcal{A} .
3. the class of fibrations of $L_{\mathcal{V}}\mathcal{A}$ is defined as the class of maps that have the right lifting property with respect to the maps which are cofibrations and \mathcal{V} -local equivalences.

We will also need the dual notion of right Bousfield localization.

Definition 1.9.2. — Let \mathcal{A} be a model category and let \mathcal{V} be a class of maps in \mathcal{A} . The right Bousfield localization of \mathcal{A} with respect to \mathcal{V} (in case it exists) is a model category structure $R_{\mathcal{V}}\mathcal{A}$ on the underlying category of \mathcal{A} such that

1. the class of weak equivalences of $R_{\mathcal{V}}\mathcal{A}$ is defined as the class of \mathcal{V} -colocal equivalences of \mathcal{A} (see definition 1.8.5).
2. the class of fibrations of $R_{\mathcal{V}}\mathcal{A}$ is the same as the class of fibrations of \mathcal{A} .
3. the class of cofibrations of $R_{\mathcal{V}}\mathcal{A}$ is defined as the class of maps that have the left lifting property with respect to the maps which are fibrations and \mathcal{V} -colocal equivalences.

Proposition 1.9.3 (Hirschhorn). — Let \mathcal{A} be a model category and \mathcal{V} a class of maps in \mathcal{A} . Let $L_{\mathcal{V}}\mathcal{A}$ be the left Bousfield localization of \mathcal{A} with respect to \mathcal{V} , then

1. every weak equivalence in \mathcal{A} is a weak equivalence in $L_{\mathcal{V}}\mathcal{A}$.
2. the class of trivial fibrations of $L_{\mathcal{V}}\mathcal{A}$ equals the class of trivial fibrations of \mathcal{A} .
3. every fibration of $L_{\mathcal{V}}\mathcal{A}$ is a fibration of \mathcal{A} .
4. every trivial cofibration of \mathcal{A} is a trivial cofibration of $L_{\mathcal{V}}\mathcal{A}$.

Proof. — We refer the reader to Proposition 3.3.3 in [6]. □

We then get the dual version for right Bousfield localizations.

Proposition 1.9.4 (Hirschhorn). — Let \mathcal{A} be a model category and \mathcal{V} a class of maps in \mathcal{A} . Let $R_{\mathcal{V}}\mathcal{A}$ be the right Bousfield localization of \mathcal{A} with respect to \mathcal{V} , then

1. every weak equivalence in \mathcal{A} is a weak equivalence in $R_{\mathcal{V}}\mathcal{A}$.
2. the class of trivial cofibrations of $R_{\mathcal{V}}\mathcal{A}$ equals the class of trivial cofibrations of \mathcal{A} .
3. every cofibration of $R_{\mathcal{V}}\mathcal{A}$ is a cofibration of \mathcal{A} .
4. every trivial fibration of \mathcal{A} is a trivial fibration of $R_{\mathcal{V}}\mathcal{A}$.

Proof. — We refer the reader to Proposition 3.3.3 in [6]. □

Proposition 1.9.5 (Hirschhorn). — *Let \mathcal{A} be a model category and \mathcal{V} a class of maps in \mathcal{A} .*

1. *If $L_{\mathcal{V}}\mathcal{A}$ is the left Bousfield localization of \mathcal{A} with respect to \mathcal{V} , then the identity functor $\text{id} : \mathcal{A} \rightarrow L_{\mathcal{V}}\mathcal{A}$ is a left Quillen functor with right adjoint $\text{id} : L_{\mathcal{V}}\mathcal{A} \rightarrow \mathcal{A}$.*
2. *If $R_{\mathcal{V}}\mathcal{A}$ is the right Bousfield localization of \mathcal{A} with respect to \mathcal{V} , then the identity functor $\text{id} : R_{\mathcal{V}}\mathcal{A} \rightarrow \mathcal{A}$ is a left Quillen functor with right adjoint $\text{id} : \mathcal{A} \rightarrow R_{\mathcal{V}}\mathcal{A}$.*

Proof. — It follows immediately from Propositions 1.9.3 and 1.9.4. □

Theorem 1.9.6 (Hirschhorn). — *Let \mathcal{A} be a model category and let \mathcal{V} be a class of maps in \mathcal{A} .*

1. *If $L_{\mathcal{V}}\mathcal{A}$ is the left Bousfield localization of \mathcal{A} with respect to \mathcal{V} , then the identity functor $\text{id} : \mathcal{A} \rightarrow L_{\mathcal{V}}\mathcal{A}$ is a left localization of \mathcal{A} with respect to \mathcal{V} (see definition 1.8.1).*
2. *If $R_{\mathcal{V}}\mathcal{A}$ is the right Bousfield localization of \mathcal{A} with respect to \mathcal{V} then the identity functor $\text{id} : \mathcal{A} \rightarrow R_{\mathcal{V}}\mathcal{A}$ is a right localization of \mathcal{A} with respect to \mathcal{V} .*

Proof. — We refer the reader to [6, Theorem 3.3.19]. □

CHAPTER 2

MOTIVIC UNSTABLE AND STABLE HOMOTOPY THEORY

In this chapter we review the construction of the Morel-Voevodsky motivic stable model structure and the construction of Jardine's motivic symmetric stable model structure (see Sections 2.4 and 2.6). We also show that these two model structures satisfy Hirschhorn's cellularity condition (see Sections 2.5 and 2.7). Therefore, it is possible to apply Hirschhorn's localization techniques to get Bousfield localizations with respect to these two model structures. Finally, in Section 2.8 we recall the construction of the model structures for the categories of A -modules and A -algebras, where A denotes a cofibrant ring spectrum in Jardine's motivic symmetric stable model category. We will see that the category of A -modules equipped with this model structure also satisfies Hirschhorn's cellularity condition.

2.1. The Injective Model Structure

Let S be a Noetherian separated scheme of finite Krull dimension, and consider the category $\mathcal{S}m|_S$ of smooth schemes of finite type over S . $(\mathcal{S}m|_S)_{\text{Nis}}$ will denote the site with underlying category $\mathcal{S}m|_S$ equipped with the Nisnevich topology. We are interested in the category $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ of presheaves of simplicial sets on $\mathcal{S}m|_S$. The objects in $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ can also be described as simplicial presheaves on $\mathcal{S}m|_S$. The work of Jardine (see [13]) shows in particular that $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ has the structure of a proper simplicial cofibrantly generated model category.

We will denote by Δ_U^n the representable simplicial presheaf corresponding to the objects U in $\mathcal{S}m|_S$ and \mathbf{n} in Δ , i.e.

$$\begin{array}{ccc} \Delta_U^n : (\mathcal{S}m|_S \times \Delta)^{\text{op}} & \longrightarrow & \mathcal{S}ets \\ (V, m) & \longmapsto & (\text{Hom}_{\mathcal{S}m|_S}(V, U)) \times (\Delta^n)_m \end{array}$$

The following functor gives a fully faithful embedding of $\mathcal{S}m|_S$ into $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$:

$$\begin{aligned} Y : \mathcal{S}m|_S &\longrightarrow \Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}} \\ U &\longmapsto \Delta_U^0 \end{aligned}$$

we will abuse notation and write U instead of Δ_U^0 . Given any simplicial set K we can consider the associated constant presheaf of simplicial sets which we also denote by K , i.e.

$$\begin{aligned} K : (\mathcal{S}m|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longrightarrow K_n \end{aligned}$$

The category of simplicial presheaves $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ inherits a natural simplicial structure from the one on simplicial sets.

Given a simplicial presheaf X , the tensor objects for the simplicial structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ are defined as follows:

$$X \otimes - : \mathbf{S}\mathbf{Sets} \longrightarrow \Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$$

where $X \otimes K$ is the following simplicial presheaf:

$$\begin{aligned} X \otimes K : (\mathcal{S}m|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto X_n(U) \times K_n \end{aligned}$$

The simplicial functor in two variables is:

$$\text{Map}(-, -) : (\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}})^{\text{op}} \times \Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}} \longrightarrow \mathbf{S}\mathbf{Sets}$$

where $\text{Map}(X, Y)$ is the simplicial set given by:

$$\begin{aligned} \text{Map}(X, Y) : \Delta^{\text{op}} &\longrightarrow \mathcal{S}ets \\ \mathbf{n} &\longmapsto \text{Hom}_{\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}}(X \otimes \Delta^n, Y) \end{aligned}$$

and finally for any simplicial presheaf Y we have the following functor

$$Y^- : \mathbf{S}\mathbf{Sets} \longrightarrow (\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}})^{\text{op}}$$

where Y^K is the simplicial presheaf given as follows:

$$\begin{aligned} Y^K : (\mathcal{S}m|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto \text{Hom}_{\mathbf{S}\mathbf{Sets}}(K \times \Delta^n, Y(U)) \end{aligned}$$

Let t be a point in $(\mathcal{S}m|_S)_{\text{Nis}}$. Denote by θ_t the fibre functor which assigns to every simplicial presheaf its stalk at t :

$$\begin{aligned} \theta_t : \Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}} &\longrightarrow \mathbf{S}\mathbf{Sets} \\ X &\longmapsto \theta_t(X) = X_t \end{aligned}$$

Now we proceed to define the model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ constructed by Jardine. A map $f : X \rightarrow Y$ in $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ is defined to be a weak equivalence, if f induces a weak equivalence of simplicial sets in all the stalks on $(\mathcal{S}m|_S)_{\text{Nis}}$, i.e. if for every point t in $(\mathcal{S}m|_S)_{\text{Nis}}$ the map

$$\theta_t(X) \xrightarrow{\theta_t(f)} \theta_t(Y)$$

is a weak equivalence of simplicial sets.

The set I of generating cofibrations is given by all the subobjects of Δ_U^n for U in $\mathcal{S}m|_S$ and $n \geq 0$, i.e.

$$I = \{Y \hookrightarrow \Delta_U^n | U \in (\mathcal{S}m|_S), n \geq 0\}$$

it is easy to see that a map $i : X \rightarrow Y$ is in I -cell if and only if it is a monomorphism, i.e. $i_n(U) : X_n(U) \rightarrow Y_n(U)$ is an injective map of sets, for every U in $\mathcal{S}m|_S$, $n \geq 0$.

Let λ be a cardinal, and X a simplicial presheaf on $\mathcal{S}m|_S$. We say that X is λ -bounded if the cardinal of all the simplices of X is bounded by λ , i.e. $|X_n(U)| < \lambda$ for every U in $\mathcal{S}m|_S$, $n \geq 0$. The site $(\mathcal{S}m|_S)_{\text{Nis}}$ is essentially small, so we can find a cardinal κ such that κ is greater than 2^α , where α is the cardinality of the set $\text{Map}(\mathcal{S}m|_S)$ of maps in $\mathcal{S}m|_S$.

We say that a map $j : X \rightarrow Y$ of simplicial presheaves in $\mathcal{S}m|_S$ is a *trivial cofibration*, if it is both a cofibration and a weak equivalence. The set J of generating trivial cofibrations is given by all the trivial cofibrations where the codomain is bounded by the cardinal κ described above, i.e.

$$J = \{j : X \rightarrow Y | j \text{ is a trivial cofibration and } Y \text{ is } \kappa\text{-bounded}\}$$

Theorem 2.1.1 (Jardine). — *The category $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ of simplicial presheaves on the Nisnevich site $(\mathcal{S}m|_S)_{\text{Nis}}$, has the structure of a proper simplicial cofibrantly generated model category where the class \mathcal{W} of weak equivalences, and the sets I, J of generating cofibrations and generating trivial cofibrations are defined as above.*

Proof. — We refer the reader to [13, Theorem 2.3]. □

The model structure defined above will be called the *injective model structure* for $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$.

Remark 2.1.2. — *The cofibrations for the injective model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ have the following properties:*

1. *The class of cofibrations coincides with the class of relative I -cell complexes, therefore a map is a cofibration if and only if it is a monomorphism.*
2. *If a map $i : A \rightarrow B$ in $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ is a cofibration then for every point t in $(\mathcal{S}m|_S)_{\text{Nis}}$ the associated map $\theta_t(i) : \theta_t(A) \rightarrow \theta_t(B)$ is a cofibration of simplicial sets.*

3. Every object A in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is an I -cell complex, therefore every object in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is cofibrant.

The category $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ of simplicial presheaves on the smooth Nisnevich site $(\mathcal{M}|_S)_{\text{Nis}}$ also has a closed symmetric monoidal structure which is compatible with the injective model structure, i.e. $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the injective structure is a symmetric monoidal model category in the sense of Hovey (see definition 1.7.7).

The closed symmetric monoidal structure is defined as follows:

$$\begin{aligned} \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} \times \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} \\ (X, Y) &\longmapsto X \times Y \end{aligned}$$

where $X \times Y$ is the presheaf of simplicial sets defined as follows:

$$\begin{aligned} X \times Y : (\mathcal{M}|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto X_n(U) \times Y_n(U) \end{aligned}$$

and the functor that gives the adjunction of two variables is the following:

$$\mathbf{Hom}_{\text{Pre}}(-, -) : (\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \times \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$$

where $\mathbf{Hom}_{\text{Pre}}(X, Y)$ is the simplicial presheaf given by:

$$\begin{aligned} \mathbf{Hom}_{\text{Pre}}(X, Y) : (\mathcal{M}|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto \mathbf{Hom}_{\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}}(X \times \Delta_U^n, Y) \end{aligned}$$

Proposition 2.1.3. — Let X, Y, Z be simplicial presheaves on $(\mathcal{M}|_S)_{\text{Nis}}$.

1. There is a natural isomorphism of simplicial sets

$$\text{Map}(X \times Y, Z) \xrightarrow{\cong} \text{Map}(X, \mathbf{Hom}_{\text{Pre}}(Y, Z))$$

2. There is a natural isomorphism of simplicial presheaves on $(\mathcal{M}|_S)_{\text{Nis}}$

$$\mathbf{Hom}_{\text{Pre}}(X \times Y, Z) \xrightarrow{\cong} \mathbf{Hom}_{\text{Pre}}(X, \mathbf{Hom}_{\text{Pre}}(Y, Z))$$

Proof. — (1). To any n -simplex α in $\text{Map}(X \times Y, Z)$

$$(X \otimes \Delta^n) \times Y \xrightarrow{\cong} (X \times Y) \otimes \Delta^n \xrightarrow{\alpha} Z$$

associate the n -simplex α_* in $\text{Map}(X, \mathbf{Hom}_{\text{Pre}}(Y, Z))$

$$\alpha_* : X \otimes \Delta^n \rightarrow \mathbf{Hom}_{\text{Pre}}(Y, Z)$$

coming from the adjunction between $- \times Y$ and $\mathbf{Hom}_{\text{Pre}}(Y, -)$.

- (2). To any n -simplex α in $\mathbf{Hom}_{\text{Pre}}(X \times Y, Z)_n(U)$

$$(X \times \Delta_U^n) \times Y \xrightarrow{\cong} (X \times Y) \times \Delta_U^n \xrightarrow{\alpha} Z$$

associate the n -simplex α_* in $\mathbf{Hom}_{\mathbf{Pre}}(X, \mathbf{Hom}_{\mathbf{Pre}}(Y, Z))_n(U)$

$$\alpha_* : X \times \Delta_U^n \rightarrow \mathbf{Hom}_{\mathbf{Pre}}(Y, Z)$$

coming from the adjunction between $- \times Y$ and $\mathbf{Hom}_{\mathbf{Pre}}(Y, -)$. \square

Lemma 2.1.4 (cf. [14]). — *The category $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ of simplicial presheaves on the smooth Nisnevich site $(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the injective model structure is a symmetric monoidal model category (see definition 1.7.7).*

Proof. — We need to check that the conditions (1)-(2) in definition 1.7.7 are satisfied. Since every object is cofibrant in $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$, condition (2) is trivially satisfied. To check condition (1), we need to show that if we take two cofibrations $i : A \rightarrow B$ and $j : C \rightarrow D$ for the injective model structure on $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$, then the induced map

$$i \square j : A \times D \coprod_{A \times C} B \times C \rightarrow B \times D$$

is also a cofibration, which is trivial if either i or j is a weak equivalence. To see that $i \square j$ is a cofibration, it is enough to show that $i \square j(U)$ is a cofibration of simplicial sets for every U in $\mathcal{M}|_S$, but this is true since the category of simplicial sets is a symmetric monoidal model category.

Now we show that $i \square j$ is a trivial cofibration if either i or j is a weak equivalence. The definition of weak equivalences for the injective model structure implies that is enough to prove it at the level of the stalks, so let t be any point in $(\mathcal{M}|_S)_{\text{Nis}}$, and consider the induced map of simplicial sets

$$\theta_t(i \square j) : \theta_t(A) \times \theta_t(D) \coprod_{\theta_t(A) \times \theta_t(C)} \theta_t(B) \times \theta_t(C) \rightarrow \theta_t(B) \times \theta_t(D)$$

Now since the category of simplicial sets is in particular a symmetric monoidal model category, we have that $\theta_t(i \square j)$ is a trivial cofibration if either i or j is a weak equivalence. Since this holds for every point t in $(\mathcal{M}|_S)_{\text{Nis}}$, we have that $i \square j$ is a cofibration in $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ which is trivial if either i or j is a weak equivalence, hence the result follows. \square

Lemma 2.1.5 (Morel-Voevodsky, cf. [18]). — *Let X, Y be two fibrant simplicial presheaves in the injective model structure, and consider a map $f : X \rightarrow Y$. The following are equivalent:*

1. f is a weak equivalence in the injective model structure for $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$.
2. For every U in $\mathcal{M}|_S$ the map

$$f(U) : X(U) \rightarrow Y(U)$$

is a weak equivalence of simplicial sets.

Proof. — Assume that f is a weak equivalence in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$. Since X, Y are fibrant and weak equivalences of simplicial sets satisfy the two out of three property, by Ken Brown's lemma (see Lemma 1.1.4) we can assume that f is a trivial fibration in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$. Now consider U as an element of $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$. Since every object in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is cofibrant, axiom **SM7** for simplicial model categories implies that: $f_* : \text{Map}(U, X) \rightarrow \text{Map}(U, Y)$ is a trivial fibration of simplicial sets, but this is just equal to $f(U) : X(U) \rightarrow Y(U)$.

Conversely, suppose now that for every U in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$, $f(U) : X(U) \rightarrow Y(U)$ is a weak equivalence of simplicial sets. Let t be an arbitrary point in $(\mathcal{M}|_S)_{\text{Nis}}$. We know that t is associated to a pro-object $\{U_\alpha\}$ in $(\mathcal{M}|_S)_{\text{Nis}}$. Therefore $\theta_t(f) : \theta_t(X) \rightarrow \theta_t(Y)$ is a filtered colimit of weak equivalences of simplicial sets, hence a weak equivalence of simplicial sets. But this implies that f is a weak equivalence in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$. \square

Definition 2.1.6. — Let X be a simplicial presheaf on $(\mathcal{M}|_S)_{\text{Nis}}$. We say that X satisfies the B.G. property if any elementary Cartesian square

$$\begin{array}{ccc} U \times_W V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & W \end{array}$$

of smooth schemes over S with p étale, i an open immersion and $p^{-1}(W-U) \cong W-U$ (both equipped with the reduced scheme structure) maps to a homotopy Cartesian diagram of simplicial sets after applying X

$$\begin{array}{ccc} X(W) & \longrightarrow & X(V) \\ \downarrow & & \downarrow \\ X(U) & \longrightarrow & X(U \times_W V) \end{array}$$

Theorem 2.1.7 (Jardine). — Let X be a simplicial presheaf on $(\mathcal{M}|_S)_{\text{Nis}}$. Then X satisfies the B.G. property if and only if any fibrant replacement $X \rightarrow GX$ in the injective model structure for $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is a sectionwise weak equivalence of simplicial sets, i.e. for any U in $\mathcal{M}|_S$,

$$X(U) \xrightarrow{g_X(U)} GX(U)$$

$g_X(U)$ is a weak equivalence of simplicial sets.

Proof. — We refer the reader to [14, Theorem 1.3]. \square

Definition 2.1.8. — Consider $U \in \mathcal{M}|_S$ with structure map $\phi : U \rightarrow S$, i.e. ϕ is a smooth map of finite type. Then we have the following adjunction (see [5, proposition I.5.1]):

$$(\phi^{-1}, \phi_*, \varphi) : \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_U)_{\text{Nis}}$$

where ϕ^{-1} and ϕ_* are defined as follows:

$$\begin{array}{ccc} \phi^{-1} : \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} & \longrightarrow & \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_U)_{\text{Nis}} \\ X \vdash & \longrightarrow & \phi^{-1}X \end{array}$$

with $\phi^{-1}X$ defined as the composition of ϕ and X :

$$\begin{array}{ccc} (\mathcal{M}|_U \times \Delta)^{\text{op}} & \xrightarrow{\phi \times \text{id}} & (\mathcal{M}|_S \times \Delta)^{\text{op}} \\ & \searrow \phi^{-1}X & \swarrow X \\ & \mathcal{S}\text{ets} & \end{array}$$

and the right adjoint ϕ_* is given by:

$$\begin{array}{ccc} \phi_* : \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_U)_{\text{Nis}} & \longrightarrow & \Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}} \\ X \vdash & \longrightarrow & \phi_*X \end{array}$$

where ϕ_*X is the following simplicial presheaf:

$$\begin{array}{ccc} \phi_*X : (\mathcal{M}|_S \times \Delta)^{\text{op}} & \longrightarrow & \mathcal{S}\text{ets} \\ (V, n) \vdash & \longrightarrow & X_n(V \times_S U) \end{array}$$

Remark 2.1.9. — Let $\phi : U \rightarrow S$ be a smooth map of finite type, and let Y be an arbitrary simplicial presheaf on $(\mathcal{M}|_U)_{\text{Nis}}$. It follows immediately from the description of the functors ϕ^{-1} and ϕ_* that the counit of the adjunction $(\phi^{-1}, \phi_*, \varphi)$

$$\phi^{-1}\phi_*Y \xrightarrow{\cong} Y$$

is an isomorphism which can be naturally identified with the identity map on Y , in particular $\phi^{-1}\phi_*Y$ is canonically isomorphic to Y .

Proposition 2.1.10. — Let $\phi : U \rightarrow S$ be a smooth map of finite type, and let X be an arbitrary simplicial presheaf on $(\mathcal{M}|_S)_{\text{Nis}}$. Then we have a canonical isomorphism:

$$\phi_*\phi^{-1}X \xrightarrow{\cong} \mathbf{Hom}_{\text{Pre}}(U, X)$$

Proof. — To any n -simplex α in $(\phi_*\phi^{-1}X)_n(V) = X_n(V \times_S U)$

$$\Delta_V^n \times U \cong \Delta_{V \times_S U}^n \xrightarrow{\alpha} X$$

associate the n -simplex α_* in $\mathbf{Hom}_{\text{Pre}}(U, X)_n(V)$

$$\Delta_V^n \xrightarrow{\alpha_*} \mathbf{Hom}_{\text{Pre}}(U, X)$$

coming from the adjunction between $- \times U$ and $\mathbf{Hom}_{\mathbf{Pre}}(U, -)$. \square

Proposition 2.1.11. — *Let $\phi : U \rightarrow S$ be a smooth map of finite type, let X be a simplicial presheaf on $(\mathcal{M}|_S)_{\mathbf{Nis}}$ and Y a simplicial presheaf on $(\mathcal{M}|_U)_{\mathbf{Nis}}$. Then we have the following enriched adjunctions:*

1. *There is a natural isomorphism of simplicial sets*

$$\mathrm{Map}(\phi^{-1}X, Y) \xrightarrow{\cong} \mathrm{Map}(X, \phi_*Y)$$

2. *There is a natural isomorphism of simplicial presheaves on $(\mathcal{M}|_S)_{\mathbf{Nis}}$*

$$\mathbf{Hom}_{\mathbf{Pre}}(X, \phi_*Y) \xrightarrow{\cong} \phi_*(\mathbf{Hom}_{\mathbf{Pre}}(\phi^{-1}X, Y))$$

3. *There is a natural isomorphism of simplicial presheaves on $(\mathcal{M}|_U)_{\mathbf{Nis}}$*

$$\phi^{-1}(\mathbf{Hom}_{\mathbf{Pre}}(X, \phi_*Y)) \xrightarrow{\cong} \mathbf{Hom}_{\mathbf{Pre}}(\phi^{-1}X, Y)$$

Proof. — (1): To any n -simplex α in $\mathrm{Map}(\phi^{-1}X, Y)$

$$\phi^{-1}(X \otimes \Delta^n) \cong \phi^{-1}X \otimes \Delta^n \xrightarrow{\alpha} Y$$

associate the n -simplex α_* in $\mathrm{Map}(X, \phi_*Y)$

$$X \otimes \Delta^n \xrightarrow{\alpha_*} \phi_*Y$$

coming from the adjunction between ϕ^{-1} and ϕ_* .

- (2): To any n -simplex α in $\mathbf{Hom}_{\mathbf{Pre}}(X, \phi_*Y)_n(V)$ (where $V \in (\mathcal{M}|_S)$)

$$X \times \Delta_V^n \xrightarrow{\alpha} \phi_*Y$$

associate the n -simplex α^* in $\phi_*(\mathbf{Hom}_{\mathbf{Pre}}(\phi^{-1}X, Y))_n(V)$

$$\phi^{-1}X \times \Delta_{V \times_S U}^n \cong \phi^{-1}(X \times \Delta_V^n) \xrightarrow{\alpha^*} Y$$

coming from the adjunction between ϕ^{-1} and ϕ_* .

- (3): To any n -simplex α in $\phi^{-1}(\mathbf{Hom}_{\mathbf{Pre}}(X, \phi_*Y))_n(V)$ (where $V \in (\mathcal{M}|_U)$)

$$X \times \Delta_V^n \xrightarrow{\alpha} \phi_*Y$$

associate the n -simplex α^* in $\mathbf{Hom}_{\mathbf{Pre}}(\phi^{-1}X, Y)_n(V)$

$$\phi^{-1}X \times \Delta_V^n \cong \phi^{-1}(X \times \Delta_V^n) \xrightarrow{\alpha^*} Y$$

coming from the adjunction between ϕ^{-1} and ϕ_* . \square

Definition 2.1.12 (cf. [14]). — *Let X be a simplicial presheaf on $(\mathcal{M}|_S)_{\mathbf{Nis}}$. We say that X is flasque if:*

1. X is a presheaf of Kan complexes.

2. Every finite collection $V_i \hookrightarrow V$, $i = 1, \dots, n$ of subschemes of a scheme V induces a Kan fibration

$$X(V) \cong \text{Map}(V, X) \xrightarrow{i^*} \text{Map}(\cup_{i=1}^n V_i, X)$$

Remark 2.1.13. — 1. Let X be a simplicial presheaf on $(\mathcal{S}m|_S)_{\text{Nis}}$ which is fibrant in the injective model structure for $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$. Then X is flasque and satisfies the B.G. property.

2. The class of flasque simplicial presheaves is closed under filtered colimits.
 3. The B.G. property is stable under filtered colimits.
 4. The functors ϕ^{-1} and ϕ_* preserve flasque simplicial presheaves.
 5. The functors ϕ^{-1} and ϕ_* preserve the B.G. property.

2.2. Cellularity of the Injective Model Structure

In this section we prove that the injective model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$ is cellular (see definition 1.3.12). This is an unpublished result due to Hirschhorn, which also appears in [8, Theorem 1.4]. The author would like to thank Jens Hornbostel for the discussion related to Hirschhorn's cellularity results.

Lemma 2.2.1. — Let A be a simplicial presheaf on the smooth Nisnevich site $(\mathcal{S}m|_S)_{\text{Nis}}$. Then A is small (see definition 1.2.3).

Proof. — Let μ be the cardinal of the set S_A of simplices of A , i.e.

$$S_A = \coprod_{V \in (\mathcal{S}m|_S), n \geq 0} A_n(V)$$

and let κ be the successor cardinal of μ . Since κ is a successor cardinal, we have that κ is a regular cardinal (see [6, Proposition 10.1.14]).

We claim that A is κ -small with respect to the class of all maps in $\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}$. In effect, consider an arbitrary λ -sequence where λ is a regular cardinal greater than κ ,

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_\beta \rightarrow \dots \quad (\beta < \lambda)$$

we need to show that the map $\text{colim}_{\beta < \lambda} \text{Hom}_{\Delta^{\text{op}}\text{Pre}(\mathcal{S}m|_S)_{\text{Nis}}}(A, X_\beta) \rightarrow \text{Hom}(A, X_\lambda)$ is a bijection. To check the injectivity, we just take sections on every $U \in (\mathcal{S}m|_S)$, and use the fact that every simplicial set is small (see [10, Lemma 3.1.1]). To check the surjectivity, consider an arbitrary map $f : A \rightarrow X_\lambda$, now the restriction of f to every simplex of A ($\Delta_U^n \rightarrow A$), factors through some X_β with $\beta < \lambda$. Since λ is a regular cardinal and there are fewer than κ simplices in A (where $\kappa < \lambda$), there exists X_α with $\alpha < \lambda$ such that the restriction of f to every simplex of A factors through X_α . But this implies that f factors through X_α , and therefore we get the surjectivity. \square

Lemma 2.2.2. — Consider the category $\Delta^{\text{op}}\text{Pre}(\mathcal{J}m|_S)_{\text{Nis}}$ of simplicial presheaves on the smooth Nisnevich site $(\mathcal{J}m|_S)_{\text{Nis}}$ equipped with the injective model structure. Then all the cofibrations in $\Delta^{\text{op}}\text{Pre}(\mathcal{J}m|_S)_{\text{Nis}}$ are effective monomorphisms.

Proof. — A map $i : A \rightarrow B$ is an effective monomorphism if and only if for every $U \in (\mathcal{J}m|_S)$, $n \geq 0$ the induced map $i_n(U) : A_n(U) \rightarrow B_n(U)$ is an effective monomorphism of sets, this is true since all small limits and colimits are computed termwise. Now in the injective model structure for $\Delta^{\text{op}}\text{Pre}(\mathcal{J}m|_S)_{\text{Nis}}$ the class of cofibrations coincides with the class of monomorphisms. But this implies that all the cofibrations are effective monomorphisms in $\Delta^{\text{op}}\text{Pre}(\mathcal{J}m|_S)_{\text{Nis}}$, since in the category of sets any injective map is an effective monomorphism (see Remark 1.3.11). \square

The next proposition is an unpublished result due to Hirschhorn, which also appears in [8, Lemma 1.5], nevertheless the proof given here is slightly different since it also handles the case of a relative I -cell complex, which is necessary according to Hirschhorn's definition of compactness (see definition 1.3.12).

Proposition 2.2.3. — Let I be the set of generating cofibrations for the injective model structure in the category of simplicial presheaves $\Delta^{\text{op}}\text{Pre}(\mathcal{J}m|_S)_{\text{Nis}}$ (see Theorem 2.1.1). The domains and codomains of the maps in I are compact relative to I .

Proof. — Let μ be the cardinal of the set S_I of simplices corresponding to all the domains and codomains of the maps in I , i.e.

$$S_I = \coprod_{(i:A \rightarrow B) \in I} \coprod_{U \in (\mathcal{J}m|_S), n \geq 0} A_n(U) \sqcup B_n(U)$$

and let κ be the successor cardinal of μ . Since κ is a successor cardinal, we have that it is a regular cardinal (see [6, Proposition 10.1.14]).

If X is a presented I -cell complex with presentation ordinal λ ,

$$i : \emptyset \rightarrow X, \emptyset = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

we claim that every cell e of X is contained in a subcomplex X_e of X of size less than κ . This follows from a transfinite induction argument over the presentation ordinal of e (see definition 1.3.2). If the presentation ordinal of e is 0, then the cell e defines a subcomplex of X of size 1, this gets the induction started. Now assume that the result holds for every cell of presentation ordinal less than $\beta < \lambda$, and consider an arbitrary cell e of presentation ordinal β . The attaching map h_e of this cell has image contained in the union of fewer than κ simplices $\{s^e\}$ of X (since the domain of h_e is also a domain for a map in I), now each such simplex s^e is contained in a cell e_s of presentation ordinal less than β and the induction hypothesis implies that each such cell e_s is contained in a subcomplex X_s of size less than κ , thus taking the union of

all these subcomplexes X_s (which is possible by Corollary 1.3.7 since all the I -cells are monomorphisms in this case) we get a subcomplex X'_e of size less than κ (since κ is regular) which contains the image of the attaching map h_e . Therefore if we define X_e as the subcomplex obtained from X'_e after attaching the cell e via h_e , we get a subcomplex of size less than κ containing the given cell e . This proves the claim.

Now if A is a simplicial presheaf on $(\mathcal{J}m|_S)$ which is a domain or codomain of a map in I , we have that A has less than κ simplices. Consider a map $j : A \rightarrow X$ where X is a presented I -cell complex,

$$i : \emptyset \rightarrow X, \emptyset = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

then the image of j has less than κ simplices $\{s_j\}$, each such simplex s_j is contained in some cell e_s of X which by the previous argument is contained in a subcomplex X_s of X of size less than κ . We take now the union of all these subcomplexes X_s to get a subcomplex X_j of X of size less than κ (since κ is regular) which contains the image of j . Therefore j factors through the subcomplex X_j which has size less than κ .

Finally, we consider a relative cell complex $f : X \rightarrow Y$,

$$f : X \rightarrow Y, X = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots (\beta < \lambda), \{T^\beta, e^\beta, h^\beta\}_{\beta < \lambda}$$

Take any map $j : A \rightarrow Y$ where A is a domain or codomain of a map in I . Since all the inclusions are I -cells for the injective model structure, we have that X is a cell complex,

$$i : \emptyset \rightarrow X, \emptyset = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_\beta \rightarrow \cdots (\beta < \nu), \{T^\beta, e^\beta, h^\beta\}_{\beta < \nu}$$

Combining this presentation of X with the presentation of f we get a presentation for Y as a cell complex, where X is a subcomplex. The previous argument shows that the image of j is contained in a subcomplex W' of Y where the size of W' is less than κ . Taking the union of W' and X we get a subcomplex X_f of f having the same size as W' (as a subcomplex of f) which contains the image of j . Therefore j factors through X_f where X_f is a subcomplex of f of size less than κ , and this shows that A is κ -compact relative to I . \square

Finally we are ready to prove Hirschhorn's cellularity theorem.

Theorem 2.2.4. — *The category $\Delta^{\text{op}}\text{Pre}(\mathcal{J}m|_S)_{\text{Nis}}$ of simplicial presheaves on the smooth Nisnevich site $(\mathcal{J}m|_S)_{\text{Nis}}$ is a cellular model category when it is equipped with the injective model structure, the sets of generating cofibrations and generating trivial cofibrations are the ones considered in Theorem 2.1.1.*

Proof. — We have to check that the conditions (1)-(4) of definition 1.3.12 hold. (1) follows from Theorem 2.1.1 which shows that the injective model structure is cofibrantly

generated. (2) follows from proposition 2.2.3 and (3) follows from Lemma 2.2.1 which says that every simplicial presheaf is small. Finally (4) follows from Lemma 2.2.2. \square

Theorem 2.2.4 will be used to show that the category $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ of T -spectra on $\mathcal{M}|_S$ equipped with the motivic stable model structure is cellular (see Theorem 2.5.4). This will allow us to apply all the localization technology of Hirschhorn [6] to construct new model structures for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.

2.3. The Motivic Model Structure

Let \mathbb{A}_S^1 be the affine line over S . Consider the following set of maps

$$\mathcal{V}_M = \{\pi_U : U \times \mathbb{A}_S^1 \rightarrow U \mid U \in (\mathcal{M}|_S)\}$$

In [18] Morel and Voevodsky show in particular that for simplicial sheaves on $(\mathcal{M}|_S)_{\text{Nis}}$ the left Bousfield localization for the injective model structure with respect to \mathcal{V}_M exists, and furthermore they show it is a proper simplicial model structure. Their work was extended to the case of simplicial presheaves by Jardine in [14, Section 1]. Following Jardine we call this localized model structure the *motivic model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$* .

Theorem 2.3.1 (Morel-Voevodsky, Jardine). — *Consider the category of simplicial presheaves on the smooth Nisnevich site $(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the injective model structure. Then the left Bousfield localization (see Section 1.9) with respect to the set of maps \mathcal{V}_M defined above exists. This model structure will be called motivic, and the category $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the motivic model structure will be denoted by \mathcal{M} . Furthermore \mathcal{M} is a proper and simplicial model category.*

Proof. — We refer the reader to [14, Theorem 1.1]. \square

The following theorem gives explicit sets of generating cofibrations and trivial cofibrations for \mathcal{M} ; and it also shows that with this choice of generators, \mathcal{M} has the structure of a cellular model category. In [8, Corollary 1.6] it is also proved that \mathcal{M} is cellular.

Theorem 2.3.2. — *\mathcal{M} is a cellular model category, where the set I_M of generating cofibrations and the set J_M of generating trivial cofibrations are defined as follows:*

1. $I_M = I$ where I is the set of generating cofibrations for the injective model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ (see Theorem 2.1.1).
2. $J_M = \{j : A \rightarrow B\}$ such that:
 - (a) j is a monomorphism.
 - (b) j is a \mathcal{V}_M -local equivalence.

- (c) *The size of B as an I -cell complex (see definition 1.3.2) is less than κ , where κ is the cardinal defined by Hirschhorn in [6, definition 4.5.3].*

Proof. — By Theorem 2.2.4 the injective model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is cellular. Therefore we can use Hirschhorn’s techniques (see Section 1.9) to construct the left Bousfield localization with respect to the set of maps \mathcal{V}_M defined above. This model structure is identical to the motivic model structure of Theorem 2.3.1 since both are left Bousfield localizations with respect to the same set of maps. Now using [6, Theorem 4.1.1] we have that the motivic model structure is cellular. So it only remains to show that the sets of generating cofibrations and trivial cofibrations are the ones described above. For the set of generating cofibrations it is clear. Theorem 4.1.1 in [6] implies that the generating trivial cofibrations are the maps $j : A \rightarrow B$ where j is an inclusion of I -cell complexes and a \mathcal{V}_M -local equivalence, and the size of B is less than κ . The result follows from the fact that in the injective model structure for $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$, I -cell is just the class of monomorphisms and that every object in $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is an I -cell complex (see Remark 2.1.2). \square

Following Jardine we say that a simplicial presheaf X is *motivic fibrant* if X is \mathcal{V}_M -local.

Proposition 2.3.3. — *The following conditions are equivalent:*

1. X is motivic fibrant.
2. X is fibrant in the injective structure and for every U in $\mathcal{M}|_S$ the map induced by $U \times \mathbb{A}_S^1 \rightarrow U$

$$\text{Map}(U, X) \longrightarrow \text{Map}(U \times \mathbb{A}_S^1, X)$$

is a weak equivalence of simplicial sets.

3. X is fibrant in the injective structure and for every U in $\mathcal{M}|_S$ the map induced by $U \times * \rightarrow U \times \mathbb{A}_S^1$

$$\text{Map}(U \times \mathbb{A}_S^1, X) \longrightarrow \text{Map}(U \times *, X)$$

is a weak equivalence of simplicial sets, where $ \rightarrow \mathbb{A}_S^1$ is any rational point for \mathbb{A}_S^1 .*

4. X is fibrant in the injective structure and for every U in $\mathcal{M}|_S$ the map induced by $U \times * \rightarrow U \times \mathbb{A}_S^1$

$$\text{Map}(U \times \mathbb{A}_S^1, X) \longrightarrow \text{Map}(U \times *, X)$$

is a trivial fibration of Kan complexes, where $ \rightarrow \mathbb{A}_S^1$ is any rational point for \mathbb{A}_S^1 .*

5. X is fibrant in the injective structure and for every U in $\mathcal{M}|_S$ the map induced by $U \times * \rightarrow U \times \mathbb{A}_S^1$

$$\mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X) \longrightarrow \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X)$$

is a trivial fibration between fibrant objects in the injective model structure for $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$, where $* \rightarrow \mathbb{A}_S^1$ is any rational point for \mathbb{A}_S^1 .

Proof. — The claim that (1) and (2) are equivalent follows from the definition of \mathcal{V}_M -local and the fact that every simplicial presheaf is cofibrant in the injective model structure. (2) and (3) are equivalent since the following diagram is commutative

$$\begin{array}{ccc} U \cong U \times * & \longrightarrow & U \times \mathbb{A}_S^1 \\ \text{id} \downarrow & \swarrow & \\ U & & \end{array}$$

and weak equivalences of simplicial sets satisfy the two out of three property. (3) and (4) are equivalent since the injective structure is in particular a simplicial model category.

(4) \Rightarrow (5): Since $\Delta^{\text{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the injective model structure is a symmetric monoidal model category we have that

$$\mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X) \xrightarrow{p} \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X)$$

is a fibration between fibrant objects in the injective structure. It only remains to show that p is a weak equivalence in the injective model structure. Lemma 2.1.5 implies that it is enough to show that

$$\mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X)(V) \xrightarrow{p(V)} \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X)(V)$$

is a weak equivalence of simplicial sets for every V in $(\mathcal{M}|_S)$. But for any simplicial presheaf Z we have a natural isomorphism of simplicial sets $Z(V) \cong \text{Map}(V, Z)$, therefore $p(V)$ is just

$$\text{Map}(V, \mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X)) \xrightarrow{p(V)} \text{Map}(V, \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X))$$

Now using the enriched adjunctions of Proposition 2.1.3, $p(V)$ becomes

$$\text{Map}(V \times U \times \mathbb{A}_S^1, X) \xrightarrow{p(V)} \text{Map}(V \times U \times *, X)$$

and by hypothesis we know that this map is a weak equivalence of simplicial sets.

- (5) \Rightarrow (4): Since the injective model structure is simplicial, we have that

$$\text{Map}(U \times \mathbb{A}_S^1, X) \xrightarrow{f} \text{Map}(U \times *, X)$$

is a fibration between Kan complexes. So it only remains to show that f is a weak equivalence of simplicial sets. By hypothesis we have that

$$\mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X) \xrightarrow{p} \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X)$$

is a trivial fibration between fibrant objects in the injective model structure. Lemma 2.1.5 implies that if we take global sections at S :

$$\mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X)(S) \xrightarrow{p(S)} \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X)(S)$$

we get a weak equivalence of simplicial sets. But $p(S)$ is natural isomorphic to

$$\mathrm{Map}(U \times \mathbb{A}_S^1, X) \xrightarrow{f} \mathrm{Map}(U \times *, X)$$

so this proves the result. \square

Proposition 2.3.4. — *Let X be a motivic fibrant simplicial presheaf on the smooth Nisnevich site $(\mathcal{M}|_S)_{\mathrm{Nis}}$. Then for any Y in $\Delta^{\mathrm{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\mathrm{Nis}}$, the simplicial presheaf $\mathbf{Hom}_{\mathbf{Pre}}(Y, X)$ is also motivic fibrant.*

Proof. — Since the injective structure is a symmetric monoidal model category (see Lemma 2.1.4) we have that $\mathbf{Hom}_{\mathbf{Pre}}(Y, X)$ is a fibrant object for the injective model structure. Proposition 2.3.3(5) implies that for every U in $\mathcal{M}|_S$, the map

$$\mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X) \xrightarrow{p} \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X)$$

is a trivial fibration between fibrant objects in the injective model structure for $\Delta^{\mathrm{op}}\mathbf{Pre}(\mathcal{M}|_S)_{\mathrm{Nis}}$, and since the injective model structure is simplicial we have that

$$\mathrm{Map}(Y, \mathbf{Hom}_{\mathbf{Pre}}(U \times \mathbb{A}_S^1, X)) \xrightarrow{p_*} \mathrm{Map}(Y, \mathbf{Hom}_{\mathbf{Pre}}(U \times *, X))$$

is a trivial fibration of Kan complexes. Now using the enriched adjunctions of proposition 2.1.3, p_* becomes

$$\mathrm{Map}(Y \times U \times \mathbb{A}_S^1, X) \xrightarrow{p_*} \mathrm{Map}(Y \times U \times *, X)$$

and finally

$$\mathrm{Map}(U \times \mathbb{A}_S^1, \mathbf{Hom}_{\mathbf{Pre}}(Y, X)) \xrightarrow{p_*} \mathrm{Map}(U \times *, \mathbf{Hom}_{\mathbf{Pre}}(Y, X))$$

therefore Proposition 2.3.3(4) implies that $\mathbf{Hom}_{\mathbf{Pre}}(Y, X)$ is motivic fibrant since p_* is a trivial fibration of Kan complexes for every U in $(\mathcal{M}|_S)$. \square

Since the motivic and the injective model structures have the same class of cofibrations and the same set of generating cofibrations, it follows that the cofibrations for the motivic model structure also have the properties described in Remark 2.1.2.

Corollary 2.3.5. — *\mathcal{M} is a symmetric monoidal model category.*

Proof. — The cofibrations for the motivic and injective model structures coincide, therefore it only remains to show that if we have two cofibrations $i : A \rightarrow B, j : C \rightarrow D$ where j is a motivic weak equivalence, the induced map

$$i \square j : A \times D \coprod_{A \times C} B \times C \rightarrow B \times D$$

is a trivial cofibration in \mathcal{M} . Since every simplicial presheaf is cofibrant in the motivic model structure, it is enough to prove the following claim: For any trivial cofibration $j : C \rightarrow D$ in \mathcal{M} and for any simplicial presheaf A , the induced map $j \times \text{id} : C \times A \rightarrow D \times A$ is a trivial cofibration in \mathcal{M} . Since the injective model structure for $\Delta^{\text{op}}\text{Pre}(\mathcal{M}|_S)_{\text{Nis}}$ is a symmetric monoidal model category (see Lemma 2.1.4) we have that $j \times \text{id}$ is a cofibration, so it only remains to show that it is a weak equivalence in \mathcal{M} . Let X be any motivic fibrant simplicial presheaf, Proposition 2.3.4 implies that $\mathbf{Hom}_{\text{Pre}}(A, X)$ is also motivic fibrant, therefore since j is a weak equivalence in \mathcal{M} , the map

$$\text{Map}(D, \mathbf{Hom}_{\text{Pre}}(A, X)) \xrightarrow{j^*} \text{Map}(C, \mathbf{Hom}_{\text{Pre}}(A, X))$$

is a weak equivalence of simplicial sets. Now using the enriched adjunctions of Proposition 2.1.3, j^* becomes

$$\text{Map}(D \times A, X) \xrightarrow{j^*} \text{Map}(C \times A, X)$$

and this implies that $j \times \text{id} : C \times A \rightarrow D \times A$ is a weak equivalence in \mathcal{M} , hence the result follows. \square

Remark 2.3.6. — Proposition 1.7.9 implies that the associated pointed category $\Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_S)_{\text{Nis}}$ of pointed simplicial presheaves is also closed symmetric monoidal, we denote by $X \wedge Y$ the functor giving the monoidal structure, and by $\mathbf{Hom}_{\mathcal{M}_*}(X, Y)$ the adjunction of two variables.

Proposition 2.3.7. — Let \mathcal{M}_* denote the pointed category associated to \mathcal{M} (see Remark 1.1.2), i.e. the category with pointed simplicial presheaves as objects and base point preserving maps. The model structure on \mathcal{M}_* induced from the model structure on \mathcal{M} is cellular, proper, simplicial and symmetric monoidal. Furthermore, \mathcal{M}_* is a \mathbf{SSets}_* -model category (see definition 1.7.12). The sets $I_{\mathcal{M}_*}, J_{\mathcal{M}_*}$ of generating cofibrations and trivial cofibrations respectively, are defined as follows:

1.

$$I_{\mathcal{M}_*} = \{i_+ : Y_+ \hookrightarrow (\Delta_U^n)_+\}$$

where $i : Y \hookrightarrow \Delta_U^n$ is a generating cofibration for \mathcal{M} (see Theorem 2.3.2(1)).

2.

$$J_{\mathcal{M}_*} = \{j_+ : A_+ \rightarrow B_+\}$$

where j is a map in the set J defined in theorem 2.3.2(2), i.e. j is a generating trivial cofibration for \mathcal{M} .

Proof. — Theorems 2.3.1, 2.3.2 together with Corollary 2.3.5 imply that \mathcal{M} is cellular, proper, simplicial and symmetric monoidal. Then theorem 1.3.13 and theorem 1.4.4 imply that the associated pointed category \mathcal{M}_* with the induced model structure is cellular (with the sets of generating cofibrations and trivial cofibrations as defined above) and proper. Now Proposition 1.7.13 implies that \mathcal{M}_* is a \mathbf{SSets}_* -model category, and this induces a simplicial model structure in \mathcal{M}_* , since the natural functor $\mathbf{SSets} \rightarrow \mathbf{SSets}_*$ which adds a disjoint base point is a left Quillen monoidal functor. Finally Proposition 1.7.9 implies that \mathcal{M}_* is symmetric monoidal. \square

Definition 2.3.8 (cf. [14]). — Let $X \in \mathcal{M}$ be a simplicial presheaf. We say that X is motivic flasque if:

1. X is flasque (see definition 2.1.12).
2. For every $U \in \mathcal{M}|_S$ the map

$$X(U) \cong \mathrm{Map}(U, X) \longrightarrow \mathrm{Map}(U \times \mathbb{A}_S^1, X) \cong X(U \times \mathbb{A}_S^1)$$

induced by the projection $U \times \mathbb{A}_S^1 \rightarrow U$ is a weak equivalence of simplicial sets.

Remark 2.3.9. — 1. The class of motivic flasque simplicial presheaves is closed under filtered colimits.

2. The functors ϕ^{-1} and ϕ_* (see definition 2.1.8) preserve motivic flasque simplicial presheaves.
3. If X is fibrant in the motivic model structure for $\Delta^{\mathrm{op}}\mathrm{Pre}_*(\mathcal{M}|_S)_{\mathrm{Nis}}$ then X is also motivic flasque.

Definition 2.3.10 (cf. [14]). — Let $X \in \mathcal{M}_*$ be a pointed simplicial presheaf. We say that X is compact if:

1. All inductive systems $Z_1 \rightarrow Z_2 \rightarrow \dots$ of pointed simplicial presheaves induce isomorphisms

$$\mathbf{Hom}_{\mathcal{M}_*}(X, \varinjlim Z_i) \cong \varinjlim \mathbf{Hom}_{\mathcal{M}_*}(X, Z_i)$$

2. If Z is motivic flasque, then $\mathbf{Hom}_{\mathcal{M}_*}(X, Z)$ is also motivic flasque.
3. The functor

$$\mathbf{Hom}_{\mathcal{M}_*}(X, -) : \mathcal{M}_* \longrightarrow \mathcal{M}_*$$

takes sectionwise weak equivalences of motivic flasque pointed simplicial presheaves to sectionwise weak equivalences.

Proposition 2.3.11. — Let $X \in \mathcal{M}_*$ be a pointed simplicial presheaf, and let

$$Z_1 \longrightarrow Z_2 \longrightarrow \dots$$

be an inductive system of pointed simplicial presheaves. If X is compact in the sense of Jardine (see definition 2.3.10) then:

$$[X, \varinjlim Z_i] \cong \varinjlim [X, Z_i]$$

where $[-, -]$ denotes the set of maps in the homotopy category associated to \mathcal{M}_* .

Proof. — Let R denote a functorial fibrant replacement in \mathcal{M}_* , such that the natural map $R_Y : Y \rightarrow RY$ is always a trivial cofibration. Consider the following commutative diagram:

$$\begin{array}{ccccccc} Z_1 & \longrightarrow & Z_2 & \longrightarrow & \cdots & \longrightarrow & \varinjlim Z_i & \xrightarrow{j} & R(\varinjlim Z_i) \\ \downarrow & & \downarrow & & & & \downarrow i & & \downarrow i_R \\ RZ_1 & \longrightarrow & RZ_2 & \longrightarrow & \cdots & \longrightarrow & \varinjlim RZ_i & \xrightarrow{j_R} & R(\varinjlim RZ_i) \end{array}$$

Since all the maps $Z_i \rightarrow RZ_i$ are trivial cofibrations, it follows that the induced map $i : \varinjlim Z_i \rightarrow \varinjlim RZ_i$ is also a trivial cofibration. Therefore:

$$(13) \quad [X, \varinjlim Z_i] \cong [X, \varinjlim RZ_i] \cong [X, R(\varinjlim RZ_i)]$$

We have that the pointed simplicial presheaves RZ_i are motivic fibrant, then Remark 2.1.13(3) implies that $\varinjlim RZ_i$ satisfies the B.G. property. Therefore using Theorem 2.1.7 we get that the map $j_R : \varinjlim RZ_i \rightarrow R(\varinjlim RZ_i)$ is a sectionwise weak equivalence. On the other hand $\varinjlim RZ_i$ and $R(\varinjlim RZ_i)$ are both motivic flasque (see Remark 2.3.9(1)), and since X is compact we have that

$$\varinjlim \mathbf{Hom}_{\mathcal{M}_*}(X, RZ_i) \cong \mathbf{Hom}_{\mathcal{M}_*}(X, \varinjlim RZ_i) \longrightarrow \mathbf{Hom}_{\mathcal{M}_*}(X, R(\varinjlim RZ_i))$$

is a sectionwise weak equivalence of simplicial sets. Taking global sections at S we get the following weak equivalence of simplicial sets:

$$\varinjlim \text{Map}(X, RZ_i) \rightarrow \text{Map}(X, R(\varinjlim RZ_i))$$

Therefore

$$(14) \quad [X, R(\varinjlim RZ_i)] \cong \pi_0 \text{Map}(X, R(\varinjlim RZ_i)) \\ \cong \pi_0 \varinjlim \text{Map}(X, RZ_i) \cong \varinjlim \pi_0 \text{Map}(X, RZ_i)$$

On the other hand:

$$(15) \quad \varinjlim \pi_0 \text{Map}(X, RZ_i) \cong \varinjlim [X, RZ_i]$$

Hence equations (13), (14) and (15) imply that

$$[X, \varinjlim Z_i] \cong [X, R(\varinjlim RZ_i)] \cong \varinjlim [X, RZ_i] \cong \varinjlim [X, Z_i]$$

as we wanted. □

Definition 2.3.12. — Let $A \in \mathcal{M}_*$ be an arbitrary pointed simplicial presheaf. We define the functor of A -loops as follows:

$$\begin{aligned} \Omega_A : \mathcal{M}_* &\longrightarrow \mathcal{M}_* \\ X &\longmapsto \mathbf{Hom}_{\mathcal{M}_*}(A, X) \end{aligned}$$

Remark 2.3.13. — 1. The functor of A -loops Ω_A has a left adjoint given by smash product with A , i.e.

$$\begin{aligned} - \wedge A : \mathcal{M}_* &\longrightarrow \mathcal{M}_* \\ X &\longmapsto X \wedge A \end{aligned}$$

2. The adjunction

$$(- \wedge A, \Omega_A, \varphi) : \mathcal{M}_* \longrightarrow \mathcal{M}_*$$

is a Quillen adjunction.

2.4. The Motivic Stable Model Structure

In [14] Jardine constructs a stable model structure for the category of T -spectra on $\mathcal{S}m|_S$. In order to define this stable model structure, he constructs two auxiliary model structures called projective and injective. In this section we recall Jardine's definitions for these three model structures on the category of T -spectra.

Let S^1 denote the constant presheaf associated to the pointed simplicial set $\Delta^1/\partial\Delta^1$, let S^n denote $S^1 \wedge \cdots \wedge S^1$ (n -factors) and let \mathbb{G}_m denote the multiplicative group over the base scheme S , i.e. $\mathbb{G}_m = \mathbb{A}_S^1 - \{0\}$ pointed by the unit e for the group operation. Let $T = S^1 \wedge \mathbb{G}_m$.

Proposition 2.4.1. — 1. $T = S^1 \wedge \mathbb{G}_m$ is compact in the sense of Jardine (see definition 2.3.10).

2. Consider $U \in \mathcal{S}m|_S$ and $r, s \geq 0$. Then the pointed simplicial presheaf $S^r \wedge \mathbb{G}_m^s \wedge U_+$ is compact in the sense of Jardine, where \mathbb{G}_m^s denotes $\mathbb{G}_m \wedge \cdots \wedge \mathbb{G}_m$ (s -factors).

Proof. — It follows immediately from [14, Lemma 2.2]. □

Definition 2.4.2. — 1. A T -spectrum X is a collection of pointed simplicial presheaves $(X^n)_{n \geq 0}$ on the smooth Nisnevich site $\mathcal{S}m|_S$, together with bonding maps

$$T \wedge X^n \xrightarrow{\sigma^n} X^{n+1}$$

2. A map $f : X \rightarrow Y$ of T -spectra is a collection of maps

$$X^n \xrightarrow{f^n} Y^n$$

in \mathcal{M}_* which are compatible with the bonding maps, i.e. the following diagram:

$$\begin{array}{ccc} T \wedge X^n & \xrightarrow{\text{id} \wedge f^n} & T \wedge Y^n \\ \sigma^n \downarrow & & \downarrow \sigma^n \\ X^{n+1} & \xrightarrow{f^{n+1}} & Y^{n+1} \end{array}$$

commutes for all $n \geq 0$.

3. With the previous definitions we get a category called the category of T -spectra which will be denoted by $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.

The category of T -spectra has a natural simplicial structure induced from the one on pointed simplicial presheaves.

Given a T -spectrum X , the tensor objects are defined as follows:

$$\begin{array}{ccc} X \wedge - : \mathbf{SSETS} & \longrightarrow & \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \\ K \longmapsto & & X \wedge K \end{array}$$

where $(X \wedge K)^n = X^n \wedge K_+$ and the bonding maps are

$$T \wedge (X^n \wedge K_+) \xrightarrow{\cong} (T \wedge X^n) \wedge K_+ \xrightarrow{\sigma^n \wedge \text{id}_{K_+}} X^{n+1} \wedge K_+$$

The simplicial functor in two variables is:

$$\begin{array}{ccc} \text{Map}(-, -) : (\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \times \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} & \longrightarrow & \mathbf{SSETS} \\ (X, Y) \longmapsto & & \text{Map}(X, Y) \end{array}$$

where $\text{Map}(X, Y)_n = \text{Hom}_{\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}}(X \wedge \Delta_+^n, Y)$, and finally for any T -spectrum Y we have the following functor

$$\begin{array}{ccc} Y^- : \mathbf{SSETS} & \longrightarrow & (\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \\ K \longmapsto & & Y^K \end{array}$$

where $(Y^K)^n = (Y^n)^{K_+}$ with bonding maps

$$T \wedge (Y^n)^{K_+} \xrightarrow{\alpha} (T \wedge Y^n)^{K_+} \xrightarrow{(\sigma^n)^*} (Y^{n+1})^{K_+}$$

where for $U \in (\mathcal{M}|_S)$, $\alpha(U)$ is adjoint to

$$T(U) \wedge (Y^n(U))^{K_+} \wedge K_+ \xrightarrow{\text{id}_{T(U)} \wedge \text{ev}_{K_+}} T(U) \wedge Y^n(U)$$

Remark 2.4.3. — 1. In fact there exists an adjunction of two variables (see definition 1.7.2):

$$- \wedge - : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \times \mathbf{S}\text{Sets}_* \longrightarrow \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$$

which induces the simplicial structure for T -spectra described above via the monoidal functor $\mathbf{S}\text{Sets} \rightarrow \mathbf{S}\text{Sets}_*$ which adds a disjoint base point.

2. For any two given spectra X, Y , the simplicial set $\text{Map}(X, Y)$ is just $\text{Map}_*(X, Y)$ (i.e. the pointed simplicial set coming from the adjunction of two variables described above) after forgetting its base point

$$\omega_0 : X \longrightarrow * \longrightarrow Y$$

We have the following family of shift functors between T -spectra defined for every $n \in \mathbb{Z}$

$$\begin{aligned} s_n : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \\ X &\longmapsto X[n] \end{aligned}$$

where $X[n]$ is defined as follows:

$$(X[n])^m = \begin{cases} * & \text{if } m + n < 0. \\ X^{m+n} & \text{if } m + n \geq 0. \end{cases}$$

with the obvious bonding maps induced by X . It is clear that $s_0 = \text{id}$ and that for $n \geq 0$, s_n is right adjoint to s_{-n} , i.e.

$$\text{Hom}_{\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}}(X, Y[n]) \cong \text{Hom}_{\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}}(X[-n], Y)$$

We define the projective model structure as follows.

Definition 2.4.4. — Consider the following family of functors from the category of pointed simplicial presheaves to the category of T -spectra:

$$(16) \quad \begin{aligned} F_n : \mathcal{M}_* &\longrightarrow \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \\ X &\longmapsto (\Sigma_T^\infty X)[-n] \end{aligned}$$

where $\Sigma_T^\infty X$ is defined as follows:

$$(\Sigma_T^\infty X)^k = T^k \wedge X$$

where the bonding maps are the canonical isomorphisms $T \wedge (T^k \wedge X) \xrightarrow{\cong} T^{k+1} X$ and $T^0 \wedge X$ is just X .

We also have the following evaluation functors from the category of T -spectra to the category of pointed simplicial presheaves:

$$\begin{aligned} Ev_n : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \mathcal{M}_* \\ X &\longmapsto X^n \end{aligned}$$

where $n \geq 0$. It is clear that F_0 is left adjoint to Ev_0 . This implies that for every $n > 0$, F_n is left adjoint to Ev_n and F_{-n} is left adjoint to $\Omega_T^n \circ Ev_0$.

We say that a map of T -spectra $f : X \rightarrow Y$ is a *level equivalence* if for every $n \geq 0$, $f^n : X^n \rightarrow Y^n$ is a weak equivalence in \mathcal{M}_* .

Let I_{M_*} and J_{M_*} denote the sets of generating cofibrations and trivial cofibrations for \mathcal{M}_* (see Proposition 2.3.7).

Theorem 2.4.5 (Jardine). — *There exists a cofibrantly generated model structure for the category $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ of T -spectra with the following choices:*

1. *The weak equivalences are the level weak equivalences defined above.*
2. *The set I of generating cofibrations is*

$$I = \bigcup_{n \geq 0} F_n(I_{M_*})$$

3. *The set J of generating trivial cofibrations is*

$$J = \bigcup_{n \geq 0} F_n(J_{M_*})$$

This model structure will be called the projective model structure for T -spectra. Furthermore, the projective model structure is proper and simplicial.

Proof. — We refer the reader to [14, Lemma 2.1]. □

Remark 2.4.6. — *Let $f : A \rightarrow B$ be a map of T -spectra.*

1. *f is a cofibration in the projective model structure if and only if $f^0 : A^0 \rightarrow B^0$ and the induced maps*

$$T \wedge B^n \coprod_{T \wedge A^n} A^{n+1} \xrightarrow{(\sigma^n, f^{n+1})} B^{n+1}$$

are all cofibrations in \mathcal{M}_ .*

2. *f is a fibration in the projective model structure if and only if f is a level motivic fibration, i.e. for every $n \geq 0$, $f^n : A^n \rightarrow B^n$ is a fibration in \mathcal{M}_* .*

Proposition 2.4.7. — *Let $n \geq 0$. Consider \mathcal{M}_* and $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the projective model structure (see Theorem 2.4.5). Then the adjunction*

$$(F_n, Ev_n, \varphi) : \mathcal{M}_* \longrightarrow \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$$

is a Quillen adjunction.

Proof. — It is enough to show that Ev_n is a right Quillen functor. Let $p : X \rightarrow Y$ be a fibration in the projective model structure for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$, then p is a level motivic fibration. In particular, $Ev_n(p) = p^n : X^n \rightarrow Y^n$ is a fibration in \mathcal{M}_* .

Now let $q : X \rightarrow Y$ be a trivial fibration in the projective model structure for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$. Then q is a level motivic trivial fibration. In particular, $Ev_n(q) = q^n : X^n \rightarrow Y^n$ is a trivial fibration in \mathcal{M}_* . \square

We now proceed to define the injective model structure for the category of T -spectra.

We say that a map of T -spectra $i : A \rightarrow B$ is a *level cofibration* (respectively *level trivial cofibration*) if for every $n \geq 0$, $i^n : A^n \rightarrow B^n$ is a cofibration (respectively trivial cofibration) in \mathcal{M}_* . Notice that a map $i : A \rightarrow B$ is a level cofibration if and only if it is a monomorphism in the category of T -spectra.

Let A be an arbitrary T -spectrum. We say that A is λ -bounded if for every $n \geq 0$, the presheaf of pointed simplicial sets A^n is λ -bounded.

Theorem 2.4.8 (Jardine). — *Let κ be a regular cardinal larger than 2^α where α is the cardinality of the set $\text{Map}(\mathcal{M}|_S)$ of maps in $\mathcal{M}|_S$. There exists a cofibrantly generated model structure for the category $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ of T -spectra with the following choices:*

1. *The weak equivalences are the level weak equivalences.*
2. *The set I of generating cofibrations is*

$$I = \{i : A \rightarrow B\}$$

where i satisfies the following conditions:

- (a) *i is a level cofibration.*
 - (b) *The codomain B of i is κ -bounded.*
3. *The set J of generating trivial cofibrations is*

$$J = \{j : A \rightarrow B\}$$

where j satisfies the following conditions:

- (a) *j is a level trivial cofibration.*
- (b) *The codomain B of j is κ -bounded.*

This model structure will be called the injective model structure for T -spectra. Furthermore, the injective model structure is proper and simplicial.

Proof. — We refer the reader [14, Lemma 2.1]. \square

Remark 2.4.9. — 1. *Let $f : A \rightarrow B$ be a map of T -spectra. Then f is a cofibration in the injective model structure for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ if and only if f is a level cofibration.*

2. *The identity functor on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ induces a left Quillen functor from the projective model structure to the injective model structure.*

Proposition 2.3.7 implies in particular that \mathcal{M}_* is a closed symmetric monoidal category. The category of T -spectra $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ has the structure of a closed

\mathcal{M}_* -module, which is obtained by extending the symmetric monoidal structure for \mathcal{M}_* levelwise.

The bifunctor giving the adjunction of two variables is defined as follows:

$$\begin{aligned} - \wedge - : \mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}} \times \mathcal{M}_* &\longrightarrow \mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}} \\ (X, A) &\longmapsto X \wedge A \end{aligned}$$

with $(X \wedge A)^n = X^n \wedge A$ and bonding maps given by

$$T \wedge (X^n \wedge A) \xrightarrow{\cong} (T \wedge X^n) \wedge A \xrightarrow{\sigma^n \wedge \mathrm{id}_A} X^{n+1} \wedge A$$

The adjoints are given by:

$$\begin{aligned} \Omega_- - : \mathcal{M}_*^{\mathrm{op}} \times \mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}} &\longrightarrow \mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}} \\ (A, X) &\longmapsto \Omega_A X \end{aligned}$$

$$\begin{aligned} \mathbf{hom}_r(-, -) : (\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}})^{\mathrm{op}} \times \mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}} &\longrightarrow \mathcal{M}_* \\ (X, Y) &\longmapsto \mathbf{hom}_r(X, Y) \end{aligned}$$

where $(\Omega_A X)^n = \Omega_A X^n$ and the bonding maps $T \wedge (\Omega_A X^n) \rightarrow \Omega_A X^{n+1}$ are adjoint to

$$T \wedge (\Omega_A X^n) \wedge A \xrightarrow{\mathrm{id} \wedge \mathrm{ev}_A} T \wedge X^n \xrightarrow{\sigma^n} X^{n+1}$$

and $\mathbf{hom}_r(X, Y)$ is the following pointed simplicial presheaf on $\mathcal{M}|_S$:

$$\begin{aligned} \mathbf{hom}_r(X, Y) : (\mathcal{M}|_S \times \Delta)^{\mathrm{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto \mathrm{Hom}_{\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}}(X \wedge (\Delta_U^n)_+, Y) \end{aligned}$$

Proposition 2.4.10. — 1. Let $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ denote the category of T -spectra equipped with the projective model structure. Then $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ is a \mathcal{M}_* -model category (see definition 1.7.12).

2. Let $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ denote the category of T -spectra equipped with the injective model structure. Then $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ is a \mathcal{M}_* -model category.

Proof. — In both cases we need to check that conditions (1) and (2) in definition 1.7.12 are satisfied. Condition (2) is automatic since the unit $* \coprod *$ is cofibrant in \mathcal{M}_* .

(1): To check condition (1) in definition 1.7.12 we use Lemma 1.7.5(3) which implies that it is enough to prove the following claim: Given a cofibration $i : A \rightarrow B$ in \mathcal{M}_* and a fibration $p : X \rightarrow Y$ in $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ then $(i^*, p_*) : \Omega_B X \rightarrow \Omega_A X \times_{\Omega_A Y} \Omega_B Y$ is a fibration of T -spectra (in the projective model structure), which is trivial if either i or p is a weak equivalence. But fibrations in the projective model structure are level motivic fibrations, by Proposition 2.3.7 we have that \mathcal{M}_* is a symmetric monoidal model category, so in particular (i^*, p_*) is a level motivic fibration which is trivial if either i or p is a weak equivalence. This proves the claim.

(2): We will prove directly that we have a Quillen bifunctor, i.e. given a cofibration $i : A \rightarrow B$ in \mathcal{M}_* and a level cofibration $j : C \rightarrow D$ of T -spectra, we will show that $i \square j : D \wedge A \coprod_{C \wedge A} C \wedge B \rightarrow D \wedge B$ is a level cofibration (i.e a cofibration in the injective model structure) which is trivial if either i or j is a weak equivalence. But cofibrations in the injective model structure are level cofibrations, and since \mathcal{M}_* is a symmetric monoidal model category, we have that $i \square j$ is a level cofibration which is trivial if either i or j is a weak equivalence. This finishes the proof. \square

If we fix A in \mathcal{M}_* , we get an adjunction

$$(- \wedge A, \Omega_A, \varphi_A) : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$$

Proposition 2.4.11. — *Let A in \mathcal{M}_* be an arbitrary presheaf of pointed simplicial sets on $\mathcal{M}|_S$.*

1. *The adjunction $(- \wedge A, \Omega_A, \varphi_A)$ defined above is a Quillen adjunction for the projective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.*
2. *The adjunction $(- \wedge A, \Omega_A, \varphi_A)$ defined above is a Quillen adjunction for the injective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.*

Proof. — Since every object A in \mathcal{M}_* is cofibrant, the result follows immediately from Proposition 2.4.10. \square

Proposition 2.4.12. — *Let X, Y be two arbitrary T -spectra and let A in \mathcal{M}_* be an arbitrary presheaf of pointed simplicial sets. Then we have the following enriched adjunctions:*

$$\begin{aligned} \text{Map}(A, \mathbf{hom}_r(X, Y)) &\xrightarrow[\cong]{\alpha} \text{Map}(X \wedge A, Y) \xrightarrow[\cong]{\beta} \text{Map}(X, \Omega_A Y) \\ \mathbf{Hom}_{\mathcal{M}_*}(A, \mathbf{hom}_r(X, Y)) &\xrightarrow[\cong]{\delta} \mathbf{hom}_r(X \wedge A, Y) \xrightarrow[\cong]{\epsilon} \mathbf{hom}_r(X, \Omega_A Y) \end{aligned}$$

where the maps in the first row are isomorphisms of simplicial sets and the maps in the second row are isomorphisms in \mathcal{M}_* .

Proof. — We consider first the simplicial adjunctions: To any n -simplex t in $\text{Map}(A, \mathbf{hom}_r(X, Y))$

$$A \otimes \Delta^n \xrightarrow{t} \mathbf{hom}_r(X, Y)$$

associate the following n -simplex in $\text{Map}(X \wedge A, Y)$:

$$X \wedge A \otimes \Delta^n \xrightarrow{\alpha(t)} Y$$

corresponding to the adjunction between $X \wedge -$ and $\mathbf{hom}_r(X, -)$.

To any n -simplex t in $\text{Map}(X \wedge A, Y)$

$$\Delta^n \otimes X \wedge A \xrightarrow{\cong} X \wedge A \otimes \Delta^n \xrightarrow{t} Y$$

associate the following n -simplex in $\text{Map}(X, \Omega_A Y)$:

$$X \otimes \Delta^n \xrightarrow{\cong} \Delta^n \otimes X \xrightarrow{\beta(t)} \Omega_A Y$$

corresponding to the adjunction between $- \wedge A$ and Ω_A .

We consider now the isomorphisms of simplicial presheaves: To any simplex s in $\mathbf{Hom}_{\mathcal{M}_*}(A, \mathbf{hom}_r(X, Y))$

$$A \wedge \Delta_U^n \xrightarrow{s} \mathbf{hom}_r(X, Y)$$

we associate the following simplex in $\mathbf{hom}_r(X \wedge A, Y)$

$$X \wedge A \wedge \Delta_U^n \xrightarrow{\delta(s)} Y$$

corresponding to the adjunction between $X \wedge -$ and $\mathbf{hom}_r(X, -)$.

To any simplex s in $\mathbf{hom}_r(X \wedge A, Y)$

$$X \wedge \Delta_U^n \wedge A \xrightarrow{\cong} X \wedge A \wedge \Delta_U^n \xrightarrow{s} Y$$

we associate the following simplex in $\mathbf{hom}_r(X, \Omega_A Y)$

$$X \wedge \Delta_U^n \xrightarrow{\epsilon(s)} \Omega_A Y$$

corresponding to the adjunction between $- \wedge A$ and Ω_A . \square

We now proceed to define the stable model structure for the category of T -spectra. Consider the functor Ω_T of T -loops in $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$. There is another way to promote the T -loops functor from the category of pointed simplicial presheaves to the category of T -spectra.

Definition 2.4.13. — We define the functor Ω_T^ℓ as follows:

$$\begin{array}{ccc} \Omega_T^\ell : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} & \longrightarrow & \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \\ X & \longmapsto & \Omega_T^\ell X \end{array}$$

where $(\Omega_T^\ell X)^n = \Omega_T X^n$ and the bonding maps $T \wedge \Omega_T X^n \rightarrow \Omega_T X^{n+1}$ are given by the adjoints to

$$\Omega_T X^n \xrightarrow{\Omega_T(\sigma_*^n)} \Omega_T(\Omega_T X^{n+1})$$

where $\sigma_*^n : X^n \rightarrow \Omega_T X^{n+1}$ is adjoint to the bonding map

$$X^n \wedge T \xrightarrow{\cong} T \wedge X^n \xrightarrow{\sigma^n} X^{n+1}$$

Following Jardine we call the functor Ω_T^ℓ the *fake T -loops functor*.

Remark 2.4.14. — *The fake T -loops functor Ω_T^ℓ has a left adjoint Σ_T^ℓ called the fake T -suspension functor defined as follows:*

$$\begin{array}{ccc} \Sigma_T^\ell : \text{Spt}_T(\mathcal{A}m|_S)_{\text{Nis}} & \longrightarrow & \text{Spt}_T(\mathcal{A}m|_S)_{\text{Nis}} \\ X & \longmapsto & \Sigma_T^\ell X \end{array}$$

where $(\Sigma_T^\ell X)^n = T \wedge X^n$ and the bonding maps are

$$\text{id} \wedge \sigma^n : T \wedge (T \wedge X^n) \rightarrow T \wedge X^{n+1}$$

We will denote by Σ_T the left adjoint $(- \wedge T)$ to Ω_T .

For any T -spectrum X , the adjoints $\sigma_*^n : X^n \rightarrow \Omega_T X^{n+1}$ of the bonding maps are the levelwise components of a map $\sigma_* : X \rightarrow \Omega_T^\ell X[1]$. Consider the following inductive system of T -spectra:

$$X \xrightarrow{\sigma_*} \Omega_T^\ell X[1] \xrightarrow{\Omega_T^\ell \sigma_*[1]} (\Omega_T^\ell)^2 X[2] \xrightarrow{(\Omega_T^\ell)^2 \sigma_*[2]} \dots$$

and denote its colimit by $Q_T X$. The functor Q_T is called the *stabilization functor*.

Following Jardine, J will denote a fibrant replacement functor for the projective model structure and I will denote the corresponding fibrant replacement functor for the injective model structure on $\text{Spt}_T(\mathcal{A}m|_S)_{\text{Nis}}$. The tranfinite composition $X \rightarrow Q_T X$ will be denoted η_X , and we define $\tilde{\eta}_X$ as the composition

$$X \xrightarrow{\eta_X} Q_T X \xrightarrow{Q_T(j_X)} Q_T JX$$

We say that a map $f : X \rightarrow Y$ of T -spectra is a *stable equivalence* if it becomes a level equivalence after taking a fibrant replacement and applying the stabilization functor, i.e. if $Q_T J(f) : Q_T JX \rightarrow Q_T JY$ is a level equivalence of T -spectra.

Remark 2.4.15. — *Let $f : X \rightarrow Y$ be a map of T -spectra.*

1. *f is a stable equivalence if and only if the map*

$$IQ_T J(f) : IQ_T JX \rightarrow IQ_T JY$$

is a level equivalence of T -spectra.

2. *If f is a level motivic equivalence, then f is also a stable equivalence.*

Theorem 2.4.16 (Jardine). — *Let $\text{Spt}_T(\mathcal{A}m|_S)_{\text{Nis}}$ be the category of T -spectra equipped with the projective model structure (see Theorem 2.4.5). Then the left Bousfield localization of $\text{Spt}_T(\mathcal{A}m|_S)_{\text{Nis}}$ with respect to the class of stable equivalences exists, and furthermore it is proper and simplicial. This model structure will be called motivic stable, and the category of T -spectra $\text{Spt}_T(\mathcal{A}m|_S)_{\text{Nis}}$, equipped with the motivic stable model structure will be denoted by $\text{Spt}_T \mathcal{M}_*$.*

Proof. — We refer the reader to [14, Theorem 2.9]. □

Proposition 2.4.17. — *Let $n \geq 0$. Consider the adjunction*

$$(F_n, Ev_n, \varphi) : \mathcal{M}_* \longrightarrow \mathrm{Spt}_T \mathcal{M}_*$$

described in Proposition 2.4.7. Then (F_n, Ev_n, φ) is a Quillen adjunction.

Proof. — It follows immediately from Proposition 2.4.7 and the following fact:

- The identity functor on $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ is a left Quillen functor from the projective model structure to the motivic stable model structure.

□

Lemma 2.4.18 (Jardine). — *Let $p : X \rightarrow Y$ be a map of T -spectra. Then p is a fibration in $\mathrm{Spt}_T \mathcal{M}_*$ (we then say that p is a stable fibration) if the following conditions are satisfied:*

1. *p is a fibration in the projective model structure for $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$, i.e. p is a level motivic fibration.*
2. *The following diagram is level homotopy Cartesian:*

$$\begin{array}{ccc} X & \xrightarrow{\tilde{\eta}_X} & Q_T JX \\ p \downarrow & & \downarrow Q_T J(p) \\ Y & \xrightarrow{\tilde{\eta}_Y} & Q_T JY \end{array}$$

Proof. — We refer the reader to [14, Lemma 2.7].

□

Lemma 2.4.19 (Jardine). — *Let X be a T -spectrum. The following are equivalent:*

1. *X is a fibrant object in $\mathrm{Spt}_T \mathcal{M}_*$ (we then say that X is stably fibrant).*
2. *X is a fibrant object in the projective model structure for T spectra (i.e. X is level motivic fibrant) and the adjoints to the bonding maps $\sigma_*^n : X^n \rightarrow \Omega_T X^{n+1}$ are weak equivalences in \mathcal{M}_* .*
3. *X is a fibrant object in the projective model structure for T -spectra and the adjoints to the bonding maps are sectionwise weak equivalences of simplicial sets, i.e. for any $U \in (\mathcal{M}|_S)$ the induced map $\sigma_*^n(U) : X^n(U) \rightarrow \Omega_T X^{n+1}(U)$ is a weak equivalence of simplicial sets.*

Proof. — We refer the reader to [14, Lemma 2.8].

□

We say that a T -spectrum X is *stably fibrant injective*, if X is a fibrant object in both the motivic stable and the injective model structures for $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$.

Corollary 2.4.20. — *Let X be a T -spectrum. Then $IQ_T JX$ is a stably fibrant injective replacement for X , i.e. the natural map*

$$X \xrightarrow{r_X} IQ_T JX$$

is a stable weak equivalence and $IQ_T JX$ is stably fibrant injective.

Proof. — It is clear that r_X is a stable weak equivalence and that $IQ_T JX$ is fibrant in the injective model structure for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$, so we only need to show that $IQ_T JX$ is stably fibrant. Since the identity functor on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ is a left Quillen functor from the projective to the injective model structure (see remark 2.4.9(2)), we have that $IQ_T JX$ is in particular a fibrant object in the projective model structure for T spectra. Lemma 2.4.19(3) implies that it is enough to show that the adjoints to the bonding maps for $IQ_T JX$, $\sigma_*^n : (IQ_T JX)^n \rightarrow \Omega_T(IQ_T JX)^{n+1}$ are all sectionwise weak equivalences of simplicial sets. We will prove that using the following commutative diagram, and showing that the top row and the vertical maps are all sectionwise weak equivalences of simplicial sets:

$$\begin{array}{ccc} (Q_T JX)^n & \xrightarrow{\tau_*^n} & \Omega_T(Q_T JX)^{n+1} \\ \downarrow & & \downarrow \\ (IQ_T JX)^n & \xrightarrow{\sigma_*^n} & \Omega_T(IQ_T JX)^{n+1} \end{array}$$

A cofinal argument implies that the adjoints of the bonding maps for $Q_T JX$:

$$(Q_T JX)^n \xrightarrow{\tau_*^n} \Omega_T(Q_T JX)^{n+1}$$

are isomorphisms, so in particular these maps are sectionwise weak equivalences of simplicial sets.

Since the B.G. property (see definition 2.1.6) is preserved under filtered colimits and the fibrant objects for \mathcal{M}_* in particular satisfy the B.G. property (see Theorem 2.1.7), we have that the pointed simplicial presheaves $(Q_T JX)^n$ satisfy the B.G. property. Therefore Theorem 2.1.7 implies that the maps $(Q_T JX)^n \rightarrow (IQ_T JX)^n$ are sectionwise weak equivalences of simplicial sets.

Remark 2.3.9(1) implies that the pointed simplicial presheaves $(Q_T JX)^n$ are all motivic flasque, and since the simplicial presheaves $(IQ_T JX)^n$ are fibrant in \mathcal{M}_* , we have that $(IQ_T JX)^n$ are also motivic flasque. Now since T is compact in the sense of Jardine (see Proposition 2.4.1), we have that the maps $\Omega_T(Q_T JX)^{n+1} \rightarrow \Omega_T(IQ_T JX)^{n+1}$ are sectionwise weak equivalences of simplicial sets. This finishes the proof. \square

Corollary 2.4.21. — *Let A in \mathcal{M}_* be an arbitrary pointed simplicial presheaf, and let X be a stably fibrant T -spectrum. Then $\Omega_A X$ is also stably fibrant.*

Proof. — Using Proposition 2.4.11 we have that Ω_A is a right Quillen functor for the projective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$, therefore in particular $\Omega_A X$ is level

fibrant. Lemma 2.4.19(2) implies that $\sigma_*^n : X^n \rightarrow \Omega_T X^{n+1}$ are motivic weak equivalences between motivic fibrant objects. \mathcal{M}_* is a symmetric monoidal model category, then Ken Brown's Lemma 1.1.5 implies that $\Omega_A(\sigma_*^n) : \Omega_A X^n \rightarrow \Omega_A \Omega_T X^{n+1}$ is a motivic weak equivalence. Let $\theta^n : \Omega_A X^n \rightarrow \Omega_T \Omega_A X^{n+1}$ be the adjoint to the bonding map $T \wedge \Omega_A X^n \rightarrow \Omega_A X^{n+1}$ for the spectrum $\Omega_A X$, then we have the following commutative diagram:

$$\begin{array}{ccc} \Omega_A X^n & \xrightarrow{\theta^n} & \Omega_T \Omega_A X^{n+1} \\ & \searrow \Omega_A(\sigma_*^n) & \cong \downarrow t \\ & & \Omega_A \Omega_T X^{n+1} \end{array}$$

where t is the isomorphism which flips loop factors. Then the two out of three property for weak equivalences in \mathcal{M}_* implies that the maps $\theta^n : \Omega_A X^n \rightarrow \Omega_T \Omega_A X^{n+1}$ are motivic weak equivalences. Finally, lemma 2.4.19(2) implies that $\Omega_A X$ is stably fibrant as we wanted. \square

Lemma 2.4.22 (Jardine). — *Let $f : A \rightarrow B$ be a map of T -spectra. The following are equivalent:*

1. f is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.
2. For every stably fibrant injective object X , f induces a bijection

$$f^* : [B, X]_{\text{Spt}} \longrightarrow [A, X]_{\text{Spt}}$$

in the homotopy category associated to $\text{Spt}_T \mathcal{M}_*$.

3. For every stably fibrant injective object X , f induces a bijection

$$f^* : [B, X] \longrightarrow [A, X]$$

in the projective homotopy category for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.

4. For every stably fibrant injective object X , f induces a weak equivalence of simplicial sets

$$f^* : \text{Map}(B, X) \longrightarrow \text{Map}(A, X)$$

Proof. — We refer the reader to [14, Lemma 2.11 and Corollary 2.12]. \square

Proposition 2.4.23. — *Let A in \mathcal{M}_* be an arbitrary presheaf of pointed simplicial sets. Then the adjunction*

$$(- \wedge A, \Omega_A, \varphi_A) : \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a Quillen adjunction.

Proof. — Since the cofibrations in the stable and projective model structures for T -spectra coincide, we have that $- \wedge A$ preserves stable cofibrations (since $- \wedge A$ is a left Quillen functor for the projective model structure). So it only remains to show that

if $j : B \rightarrow C$ is a trivial cofibration in $\text{Spt}_T \mathcal{M}_*$, then $j \wedge \text{id} : B \wedge A \rightarrow C \wedge A$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Let X be an arbitrary stably fibrant injective T -spectrum, corollary 2.4.21 implies that $\Omega_A X$ is also stably fibrant, and since Ω_A is a right Quillen functor for the injective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ (see Proposition 2.4.11), we have that $\Omega_A X$ is also fibrant in the injective model structure. Thus $\Omega_A X$ is stably fibrant injective, then Lemma 2.4.22 implies that $j^* : \text{Map}(C, \Omega_A X) \rightarrow \text{Map}(B, \Omega_A X)$ is a weak equivalence of simplicial sets. Using the enriched adjunction of Proposition 2.4.12, j^* becomes $(j \wedge \text{id})^* : \text{Map}(C \wedge A, X) \rightarrow \text{Map}(B \wedge A, X)$. Finally since $(j \wedge \text{id})^*$ is a weak equivalence for every stably fibrant injective spectrum X , we get that $j \wedge \text{id} : C \wedge A \rightarrow B \wedge A$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. \square

Proposition 2.4.24. — $\text{Spt}_T \mathcal{M}_*$ is a \mathcal{M}_* -model category (see definition 1.7.12).

Proof. — Condition (2) in definition 1.7.12 follows automatically since the unit in \mathcal{M}_* is cofibrant. It only remains to prove that if $i : A \rightarrow B$ is a cofibration in \mathcal{M}_* and $j : C \rightarrow D$ is a cofibration in $\text{Spt}_T \mathcal{M}_*$ then $i \square j : D \wedge A \amalg_{C \wedge A} C \wedge B \rightarrow D \wedge B$ is a cofibration in $\text{Spt}_T \mathcal{M}_*$ which is trivial if either i or j is a weak equivalence. Since the cofibrations in the projective and the motivic stable model structure for $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ coincide, and Proposition 2.4.10 implies in particular that the category of T -spectra equipped with the projective model structure is a \mathcal{M}_* -model category, we have that $i \square j$ is a cofibration in the motivic stable structure. It only remains to show that $i \square j$ is a stable weak equivalence when either i or j is a weak equivalence. If i is a weak equivalence (i.e. a trivial cofibration) then using Proposition 2.4.10 again we have that $i \square j$ is a level weak equivalence, therefore $i \square j$ is also a stable equivalence (see Remark 2.4.15). Finally if j is a stable equivalence (i.e. a trivial cofibration in the motivic stable structure) then we consider the following commutative diagram

$$\begin{array}{ccc}
 C \wedge A & \xrightarrow{\text{id}_C \wedge i} & C \wedge B \\
 \downarrow j \wedge \text{id}_A & & \downarrow f \\
 D \wedge A & \xrightarrow{\quad} & D \wedge A \amalg_{C \wedge A} C \wedge B \\
 & \searrow i \square j & \downarrow \\
 & & D \wedge B
 \end{array}$$

$\begin{array}{l} \text{curved arrow } C \wedge A \rightarrow D \wedge B \\ \text{curved arrow } C \wedge B \rightarrow D \wedge B \end{array}$

Proposition 2.4.23 implies that $j \wedge \text{id}_A$ and $j \wedge \text{id}_B$ are both trivial cofibrations in $\text{Spt}_T \mathcal{M}_*$. Thus f is also a trivial cofibration (since it is the pushout of $j \wedge \text{id}_A$ along $\text{id}_C \wedge i$), and therefore the two out of three property for stable weak equivalences implies that $i \square j$ is a stable equivalence. This finishes the proof. \square

In order to prove that the motivic stable model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ is in fact “stable”, i.e. that the T -suspension functor Σ_T is indeed a Quillen equivalence, Jardine

introduces bigraded stable homotopy groups which allow to give another criterion to detect motivic stable weak equivalences.

Definition 2.4.25. — *Let X be an arbitrary T -spectrum. The weighted stable homotopy groups of X are presheaves of abelian groups $\pi_{t,s}X$ (where $t, s \in \mathbb{Z}$) on $\mathcal{S}m|_S$. For $U \in (\mathcal{S}m|_S)$ the sections $\pi_{t,s}X(U)$ are defined as the colimit of the inductive system:*

$$[S^{t+n} \wedge \mathbb{G}_m^{s+n}, X^n|_U] \longrightarrow [S^{t+n+1} \wedge \mathbb{G}_m^{s+n+1}, X^{n+1}|_U] \longrightarrow \dots$$

where $[-, X^i|_U]$ denotes the set of maps in the homotopy category associated to the motivic model structure on the category $\Delta^{\text{op}}\text{Pre}_*(\mathcal{S}m|_U)_{\text{Nis}}$ of pointed simplicial presheaves on the smooth Nisnevich site over the base scheme U , and the transition maps are given by taking suspension with T and composing with the bonding maps of X . The index t is called the degree and the index s is called the weight of $\pi_{t,s}X$.

Proposition 2.4.26. — *Consider $t, s \in \mathbb{Z}$ and $U \in (\mathcal{S}m|_S)$. Then the following functor:*

$$\begin{array}{ccc} \text{Spt}_T\mathcal{M}_* & \longrightarrow & \text{Abelian Groups} \\ X & \longmapsto & \pi_{t,s}X(U) \end{array}$$

is representable in the homotopy category associated to $\text{Spt}_T\mathcal{M}_*$. To represent it we can choose any spectrum of the form (see definition 2.4.4)

$$F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+)$$

where $n, p, q \geq 0$, $p - n = t$ and $q - n = s$.

Proof. — Since every pointed simplicial presheaf on $\mathcal{S}m|_S$ is cofibrant in \mathcal{M}_* , Proposition 2.4.17 and Corollary 2.4.20 imply that

$$[F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), X]_{\text{Spt}} \cong [S^p \wedge \mathbb{G}_m^q \wedge U_+, (IQ_T JX)^n]$$

where $[-, -]_{\text{Spt}}$ denotes the set of maps between two objects in the homotopy category associated to $\text{Spt}_T\mathcal{M}_*$, and $[-, -]$ denotes the set of maps in the homotopy category associated to \mathcal{M}_* . Since $Q_T JX \rightarrow IQ_T JX$ is in particular a level motivic trivial fibration we have the natural isomorphism

$$[S^p \wedge \mathbb{G}_m^q \wedge U_+, (IQ_T JX)^n] \cong [S^p \wedge \mathbb{G}_m^q \wedge U_+, (Q_T JX)^n]$$

Now since $S^p \wedge \mathbb{G}_m^q \wedge U_+$ is compact in the sense of Jardine (see Proposition 2.4.1), using Proposition 2.3.11 we have that

$$[S^p \wedge \mathbb{G}_m^q \wedge U_+, (Q_T JX)^n] \cong \varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j} \wedge U_+, (JX)^{n+j}]$$

Since \mathcal{M}_* is in particular a symmetric monoidal model category (see Proposition 2.3.7) and $U_+ \in \Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_S)_{\text{Nis}}$ is cofibrant, we have that

$$\varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j} \wedge U_+, (JX)^{n+j}] \cong \varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j}, \mathbf{Hom}_{\mathcal{M}_*}(U_+, (JX)^{n+j})]$$

Proposition 2.1.10 implies that

$$\varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j}, \mathbf{Hom}_{\mathcal{M}_*}(U_+, (JX)^{n+j})] \cong \varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_* \phi^{-1}(JX)^{n+j}]$$

where $\phi : U \rightarrow S$ is the structure map defining U as an object in $\mathcal{M}|_S$.

Now since \mathcal{M}_* is in particular a simplicial model category, and $\phi_* \phi^{-1}(JX)^{n+j} \cong \mathbf{Hom}_{\mathcal{M}_*}(U_+, (JX)^{n+j})$ is a fibrant object, we have that

$$\pi_0(\text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_* \phi^{-1}(JX)^{n+j}))$$

computes $[S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_* \phi^{-1}(JX)^{n+j}]$. The enriched adjunctions of proposition 2.1.11 imply that

$$\begin{aligned} \text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_* \phi^{-1}(JX)^{n+j}) &\cong \text{Map}(\phi^{-1}(S^{p+j} \wedge \mathbb{G}_m^{q+j}), \phi^{-1}(JX)^{n+j}) \\ &= \text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi^{-1}(JX)^{n+j}) \end{aligned}$$

Let $r_U : \phi^{-1}(JX)^{n+j} \rightarrow R_U \phi^{-1}(JX)^{n+j}$ be a functorial fibrant replacement for $\phi^{-1}(JX)^{n+j}$ in the category of pointed simplicial presheaves $\Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_U)_{\text{Nis}}$ on the smooth Nisnevich site over U equipped with the motivic model structure. It is clear that $(JX)^{n+j}$ is motivic flasque (see definition 2.3.8) and satisfies the B.G. property (see definition 2.1.6) on $\Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_S)_{\text{Nis}}$, and since ϕ^{-1} preserves both properties we have that $\phi^{-1}(JX)^{n+j}$ is motivic flasque and satisfies the B.G. property on $\Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_U)_{\text{Nis}}$. Thus r_U is a sectionwise weak equivalence, and since $S^{p+j} \wedge \mathbb{G}_m^{q+j}$ is compact in the sense of Jardine in $\Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_U)_{\text{Nis}}$ (see Proposition 2.4.1) we have that

$$(17) \quad \begin{array}{c} \mathbf{Hom}_{\mathcal{M}_*}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi^{-1}(JX)^{n+j}) \\ \downarrow r_{U*} \\ \mathbf{Hom}_{\mathcal{M}_*}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, R_U \phi^{-1}(JX)^{n+j}) \end{array}$$

is also a sectionwise weak equivalence. Taking global sections at U we get a weak equivalence of simplicial sets:

$$(18) \quad \begin{array}{c} \text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi^{-1}(JX)^{n+j}) \\ \downarrow r_{U*} \\ \text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, R_U \phi^{-1}(JX)^{n+j}) \end{array}$$

Thus $\text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_*\phi^{-1}(JX)^{n+j})$ and $\text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, R_U\phi^{-1}(JX)^{n+j})$ are naturally weakly equivalent simplicial sets. Since $\Delta^{\text{op}}\text{Pre}_*(\mathcal{J}m|_U)_{\text{Nis}}$ is a simplicial model category we have that

$$\pi_0\text{Map}(S^{p+j} \wedge \mathbb{G}_m^{q+j}, R_U\phi^{-1}(JX)^{n+j})$$

computes $[S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi^{-1}(JX)^{n+j}]_U = [S^{p+j} \wedge \mathbb{G}_m^{q+j}, (JX)^{n+j}|_U]$, where $[-, -]_U$ denotes the set of maps in the homotopy category associated to the motivic model structure on $\Delta^{\text{op}}\text{Pre}_*(\mathcal{J}m|_U)_{\text{Nis}}$. Thus $[S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_*\phi^{-1}(JX)^{n+j}]$ is naturally isomorphic to $[S^{p+j} \wedge \mathbb{G}_m^{q+j}, (JX)^{n+j}|_U]$. This implies that

$$\begin{aligned} (19) \quad [F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), X]_{\text{Spt}} &\cong \varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j}, \phi_*\phi^{-1}(JX)^{n+j}] \\ &\cong \varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j}, (JX)^{n+j}|_U] \\ &\cong \varinjlim_{j \geq 0} [S^{p+j} \wedge \mathbb{G}_m^{q+j}, X^{n+j}|_U] \\ &\cong \pi_{p-n, q-n}X(U) = \pi_{t,s}X(U) \end{aligned}$$

Therefore the functors $[F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), -]_{\text{Spt}}$ and $\pi_{t,s}(-)(U)$ have canonically isomorphic image for every T -spectrum X . To finish the proof we will give an element $\alpha \in \pi_{t,s}(F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+))(U)$ which induces an isomorphism of functors

$$[F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), -]_{\text{Spt}} \xrightarrow{\alpha_*} \pi_{t,s}(-)(U)$$

Let β^j be the unit of the adjunction between $- \wedge U_+$ and $\mathbf{Hom}_{\mathcal{M}_*}(U_+, -)$ evaluated in $S^{t+j} \wedge \mathbb{G}_m^{s+j}$:

$$S^{t+j} \wedge \mathbb{G}_m^{s+j} \xrightarrow{\beta^j} \mathbf{Hom}_{\mathcal{M}_*}(U_+, S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+) \cong \phi_*\phi^{-1}(S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+)$$

Now let γ^j be the adjoint to β^j corresponding to the adjunction between ϕ^{-1} and ϕ_* :

$$\phi^{-1}(S^{t+j} \wedge \mathbb{G}_m^{s+j}) \xrightarrow{\gamma^j} \phi^{-1}(S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+)$$

Let $[\gamma^j] \in [\phi^{-1}(S^{t+j} \wedge \mathbb{G}_m^{s+j}), \phi^{-1}(S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+)]_U = [S^{t+j} \wedge \mathbb{G}_m^{s+j}, (S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+)|_U]$ denote the map induced by γ^j in the homotopy category associated to $\Delta^{\text{op}}\text{Pre}_*(\mathcal{J}m|_U)_{\text{Nis}}$ equipped with the motivic model structure. It is clear that the maps $[\gamma^j]$ define an element

$$\alpha \in \varinjlim_{j \geq 0} [S^{t+j} \wedge \mathbb{G}_m^{s+j}, (S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+)|_U]$$

But

$$\begin{aligned} [S^{t+j} \wedge \mathbb{G}_m^{s+j}, (S^{t+j} \wedge \mathbb{G}_m^{s+j} \wedge U_+)|_U] &= [S^{t+j} \wedge \mathbb{G}_m^{s+j}, (S^{p+j-n} \wedge \mathbb{G}_m^{q+j-n} \wedge U_+)|_U] \\ &= [S^{t+j} \wedge \mathbb{G}_m^{s+j}, (F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+))^j|_U] \end{aligned}$$

Thus

$$\alpha \in \varinjlim_{j \geq 0} [S^{t+j} \wedge \mathbb{G}_m^{s+j}, (F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+))^j]_U = \pi_{t,s}(F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+))$$

Finally it is clear that α induces the required isomorphism of functors

$$[F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), -]_{\text{Spt}} \xrightarrow{\alpha_*} \pi_{t,s}(-)(U)$$

since by construction α is compatible with the isomorphisms in (19) which are induced by $r_U : \phi^{-1}(JX)^{n+j} \rightarrow R_U \phi^{-1}(JX)^{n+j}$ via the natural maps r_U^* in the diagrams (17) and (18), where r_U denotes a functorial fibrant replacement in the category $\Delta^{\text{op}}\text{Pre}_*(\mathcal{S}m|_U)_{\text{Nis}}$ equipped with the motivic model structure. \square

Proposition 2.4.27 (Jardine). — *Let $f : X \rightarrow Y$ be a map of T -spectra. The following are equivalent:*

1. f is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.
2. For every $t, s \in \mathbb{Z}$, f induces an isomorphism

$$\pi_{t,s}(f) : \pi_{t,s}X \longrightarrow \pi_{t,s}Y$$

of presheaves of abelian groups on $\mathcal{S}m|_S$.

Proof. — We refer the reader to [14, Lemma 3.7]. \square

Corollary 2.4.28. — *Let $f : X \rightarrow Y$ be a map of T -spectra. The following are equivalent:*

1. f is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.
2. For every $n, p, q \geq 0$ and every $U \in \mathcal{S}m|_S$, f induces an isomorphism

$$[F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), X]_{\text{Spt}} \xrightarrow{f_*} [F_n(S^p \wedge \mathbb{G}_m^q \wedge U_+), Y]_{\text{Spt}}$$

in the homotopy category associated to $\text{Spt}_T \mathcal{M}_*$.

Proof. — It follows immediately from Propositions 2.4.27 and 2.4.26. \square

Theorem 2.4.29 (Jardine). — *The Quillen adjunction:*

$$(\Sigma_T, \Omega_T, \varphi) : \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — We refer the reader to [14, Theorem 3.11 and Corollary 3.17]. \square

Proposition 2.4.30 (Jardine). — *The natural map $\Sigma_T^\ell X \rightarrow X[1]$ from the fake suspension functor to the shift functor is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Therefore the fake suspension functor and the shift functor are naturally equivalent in the homotopy category associated to $\text{Spt}_T \mathcal{M}_*$.*

Proof. — We refer the reader to [14, Lemma 3.19]. \square

Proposition 2.4.31 (Jardine). — *The fake suspension functor Σ_T^ℓ and the suspension functor Σ_T are naturally equivalent in the homotopy category associated to $\text{Spt}_T\mathcal{M}_*$.*

Proof. — We refer the reader to [14, Lemma 3.20]. \square

Corollary 2.4.32 (Jardine). — *The T -loops functor Ω_T , fake T -loops functor Ω_T^ℓ , and shift functor s_{-1} ($s_{-1}X = X[-1]$) are all naturally equivalent in the homotopy category associated to $\text{Spt}_T\mathcal{M}_*$.*

Proof. — It follows immediately from Propositions 2.4.30 and 2.4.31. \square

Proposition 2.4.33. — *Let $X \in \Delta^{\text{op}}\text{Pre}_*(\mathcal{M}|_S)_{\text{Nis}}$ be a pointed simplicial presheaf which is compact in the sense of Jardine (see definition 2.3.10), and let $F_n(X)$ be the T -spectrum constructed in definition 2.4.4. Consider an inductive system of T -spectra:*

$$Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \cdots$$

Then

$$[F_n(X), \varinjlim Z_i]_{\text{Spt}} \cong \varinjlim [F_n(X), Z_i]_{\text{Spt}}$$

where $[-, -]_{\text{Spt}}$ denotes the set of maps in the homotopy category associated to $\text{Spt}_T\mathcal{M}_*$.

Proof. — Since X is cofibrant in \mathcal{M}_* , Proposition 2.4.17 and corollary 2.4.20 imply that

$$\begin{aligned} [F_n(X), \varinjlim Z_i]_{\text{Spt}} &\cong [X, (IQ_T J \varinjlim Z_i)^n] \\ &\cong [X, (Q_T J \varinjlim Z_i)^n] \end{aligned}$$

where $[-, -]$ denotes the set of maps in the homotopy category associated to \mathcal{M}_* . Since X is compact in the sense of Jardine, we have that Proposition 2.3.11 implies the following:

$$\begin{aligned} [X, (Q_T J \varinjlim Z_i)^n] &\cong \varinjlim_{j \geq 0} [S^j \wedge \mathbb{G}_m^j \wedge X, (J \varinjlim Z_i)^{n+j}] \\ &\cong \varinjlim_{j \geq 0} [S^j \wedge \mathbb{G}_m^j \wedge X, (\varinjlim Z_i)^{n+j}] \end{aligned}$$

Now Lemma 2.2(4) in [14] implies that $S^j \wedge \mathbb{G}_m^j \wedge X$ are all compact in the sense of Jardine, therefore using proposition 2.3.11 again, we have:

$$\begin{aligned}
\varinjlim_{j \geq 0} [S^j \wedge \mathbb{G}_m^j \wedge X, (\varinjlim Z_i)^{n+j}] &\cong \varinjlim_{j \geq 0} \varinjlim_{i \geq 0} [S^j \wedge \mathbb{G}_m^j \wedge X, (Z_i)^{n+j}] \\
&\cong \varinjlim_{i \geq 0} \varinjlim_{j \geq 0} [S^j \wedge \mathbb{G}_m^j \wedge X, (Z_i)^{n+j}] \\
&\cong \varinjlim_{i \geq 0} [X, (Q_T J Z_i)^n] \\
&\cong \varinjlim_{i \geq 0} [X, (I Q_T J Z_i)^n] \\
&\cong \varinjlim_{i \geq 0} [F_n(X), Z_i]_{\text{Spt}}
\end{aligned}$$

and this finishes the proof. \square

2.5. Cellularity of the Motivic Stable Model Structure

In this section we will show that $\text{Spt}_T \mathcal{M}_*$ is a cellular model category. For this we will use the cellularity of \mathcal{M}_* (see Proposition 2.3.7) together with some results of Hovey [11].

The cellularity for the motivic stable model structure is also proved in [8, Corollary 1.6]. However, our proof is different since we use the characterization for weak equivalences given in corollary 1.6.11(2) (which holds in any simplicial model category) whereas the argument given in [8, Corollary 1.6] relies on [11, Theorem 4.12] which does not apply to the model category \mathcal{M}_* described in Proposition 2.3.7 (see [11, p. 83]).

Theorem 2.5.1 (Hovey). — *Let $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ be the category of T -spectra equipped with the projective model structure (see Theorem 2.4.5). Then the category $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ is a cellular model category where the sets of generating cofibrations and trivial cofibrations are the ones described in Theorem 2.4.5.*

Proof. — Proposition 2.3.7 implies that the model category \mathcal{M}_* is in particular cellular and left proper. Therefore we can apply Theorem A.9 in [11], which says that the category of T -spectra equipped with the projective model structure is also cellular under our conditions. \square

Theorem 2.4.5 together with Theorem 2.5.1 imply that the projective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ is cellular, proper and simplicial. Therefore we can apply Hirschhorn's localization technology to it. If we are able to find a suitable set of

maps such that the left Bousfield localization with respect to this set recovers the motivic stable model structure, then an immediate corollary of this will be the cellularity of the motivic stable model structure for $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$.

Definition 2.5.2 (Hovey, cf. [11]). — Let $I_{\mathcal{M}_*} = \{Y_+ \hookrightarrow (\Delta_U^n)_+\}$ be the set of generating cofibrations for \mathcal{M}_* (see Proposition 2.3.7). Notice that Y_+ may be equal to $(\Delta_U^n)_+$. We consider the following set of maps of T -spectra

$$S = \{F_{k+1}(T \wedge Y_+) \xrightarrow{\zeta_k^Y} F_k Y_+\}$$

where ζ_k^Y is the adjoint to the identity map (in $\Delta^{\mathrm{op}}\mathrm{Pre}_*(\mathcal{M}|_S)_{\mathrm{Nis}}$)

$$\mathrm{id} : T \wedge Y_+ \rightarrow \mathrm{Ev}_{k+1}(F_k Y_+) = T \wedge Y_+$$

coming from the adjunction between F_{k+1} and Ev_{k+1} (see definition 2.4.4).

Proposition 2.5.3 (Hovey). — Let X be a T -spectrum. The following conditions are equivalent:

1. X is stably fibrant, i.e. X is a fibrant object in $\mathrm{Spt}_T \mathcal{M}_*$.
2. X is S -local.

Proof. — It follows from [11, Theorem 3.4] and lemma 2.4.19. □

Now it is very easy to show that the motivic stable model structure for T -spectra is in fact cellular.

Theorem 2.5.4. — $\mathrm{Spt}_T \mathcal{M}_*$ is a cellular model category with the following sets $I_{\mathcal{M}_*}^T$, $J_{\mathcal{M}_*}^T$ of generating cofibrations and trivial cofibrations respectively:

$$I_{\mathcal{M}_*}^T = \bigcup_{k \geq 0} \{F_k(Y_+ \hookrightarrow (\Delta_U^n)_+) \mid U \in (\mathcal{M}|_S), n \geq 0\}$$

$$J_{\mathcal{M}_*}^T = \{j : A \rightarrow B\}$$

where j satisfies the following conditions:

1. j is an inclusion of $I_{\mathcal{M}_*}^T$ -complexes.
2. j is a stable weak equivalence.
3. the size of B as an $I_{\mathcal{M}_*}^T$ -complex is less than κ , where κ is the regular cardinal described by Hirschhorn in [6, definition 4.5.3].

Proof. — By Theorem 2.5.1 we know that $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$ is cellular when it is equipped with the projective model structure. Therefore we can apply Hirschhorn's localization techniques to construct the left Bousfield localization with respect to the set S of definition 2.5.2. We claim that this localization coincides with $\mathrm{Spt}_T \mathcal{M}_*$. In effect, using Proposition 2.5.3, we have that the fibrant objects in the left Bousfield localization with respect to S coincide with the fibrant objects in $\mathrm{Spt}_T \mathcal{M}_*$. Therefore

a map $f : X \rightarrow Y$ of T -spectra is a weak equivalence in the left Bousfield localization with respect to S if and only if $Qf^* : \text{Map}(QY, Z) \rightarrow \text{Map}(QX, Z)$ is a weak equivalence of simplicial sets for every stably fibrant object Z (here Q denotes the cofibrant replacement functor in $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the projective model structure). But since $\text{Spt}_T\mathcal{M}_*$ is a simplicial model category and the cofibrations coincide with the projective cofibrations, using corollary 1.6.11(2) we get exactly the same characterization for the stable equivalences. Hence the weak equivalences in both the motivic stable structure and the left Bousfield localization with respect to S coincide. This implies that the motivic stable model structure and the left Bousfield localization with respect to S are identical, since the cofibrations in both cases are just the cofibrations for the projective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.

Therefore using Theorem 4.1.1 in [6] we have that $\text{Spt}_T\mathcal{M}_*$ is cellular, since it is constructed applying Hirschhorn technology with respect to the set S .

The claim with respect to the sets of generating cofibrations and trivial cofibrations also follows from [6, Theorem 4.1.1] and the fact that $I_{M_*}^T$ is just the set of generating cofibrations for the projective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$. \square

Theorem 2.5.4 will be one of the main technical ingredients for the construction of new model structures on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ which lift Voevodsky’s slice filtration to the model category level.

2.6. The Motivic Symmetric Stable Model Structure

One of the technical disadvantages of the category of T -spectra $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ (see definition 2.4.2) is that it does not inherit a closed symmetric monoidal structure from the category of pointed simplicial presheaves \mathcal{M}_* . Symmetric spectra were introduced by Hovey, Shipley and Smith in [12] to solve this problem in the context of simplicial sets.

Their construction was lifted to the motivic setting by Jardine in [14], where he constructs a closed symmetric monoidal category of T -spectra together with a suitable model structure which is Quillen equivalent to the category $\text{Spt}_T\mathcal{M}_*$ (see Theorem 2.4.16). In this section we describe some of his constructions and results that will be necessary for our study of the multiplicative properties of the slice filtration.

Definition 2.6.1. — For $n \geq 0$, let Σ_n denote the symmetric group on n letters where Σ_0 is by definition the group with only one element.

The (q, p) -shuffle $c_{q,p} \in \Sigma_{p+q}$ is given by the following formula:

$$c_{q,p}(i) = \begin{cases} i + p & \text{if } 1 \leq i \leq q. \\ i - q & \text{if } q + 1 \leq i \leq p + q. \end{cases}$$

Definition 2.6.2 (Jardine, cf. [14]). — 1. A symmetric T -spectrum X is a collection of pointed simplicial presheaves $(X^n)_{n \geq 0}$ on the smooth Nisnevich site $\mathcal{S}m|_S$, together with:

(a) Left actions

$$\Sigma_n \times X^n \longrightarrow X^n$$

(b) Bonding maps

$$T \wedge X^n \xrightarrow{\sigma^n} X^{n+1}$$

such that the iterated composition

$$T^r \wedge X^n \longrightarrow X^{n+r}$$

is $\Sigma_r \times \Sigma_n$ -equivariant for $r \geq 1$ and $n \geq 0$.

2. A map $f : X \rightarrow Y$ of symmetric T -spectra is a collection of maps

$$X^n \xrightarrow{f^n} Y^n$$

in \mathcal{M}_* satisfying the following conditions:

(a) Compatibility with the bonding maps, i.e. the following diagram:

$$\begin{array}{ccc} T \wedge X^n & \xrightarrow{\text{id} \wedge f^n} & T \wedge Y^n \\ \sigma^n \downarrow & & \downarrow \sigma^n \\ X^{n+1} & \xrightarrow{f^{n+1}} & Y^{n+1} \end{array}$$

commutes for all $n \geq 0$

(b) f^n is Σ_n -equivariant.

3. With the previous definitions we get a category, called the category of symmetric T -spectra which will be denoted by $\text{Spt}_T^\Sigma(\mathcal{S}m|_S)_{\text{Nis}}$.

Example 2.6.3. — Given any pointed simplicial presheaf X in \mathcal{M}_* , the T -spectrum $F_0(X)$ has the structure of a symmetric T -spectrum; where the left action of Σ_n on $F_0(X)^n = T^n \wedge X$ is given by the permutation of the T factors.

In particular if we take $X = S^0$, we get the sphere T -spectrum; which will be denoted by $\mathbf{1}$.

The category of symmetric T -spectra has a simplicial structure similar to the one that exists for T -spectra, which is induced from the one on pointed simplicial presheaves.

Given a symmetric T -spectrum X , the tensor objects are defined as follows:

$$\begin{array}{ccc} X \wedge - : \mathbf{S}\text{Sets} & \longrightarrow & \text{Spt}_T^\Sigma(\mathcal{S}m|_S)_{\text{Nis}} \\ K & \longmapsto & X \wedge K \end{array}$$

where $(X \wedge K)^n = X^n \wedge K_+$ which has an action of Σ_n induced by the one in X^n and the functor $- \wedge K_+$, and with bonding maps

$$T \wedge (X^n \wedge K_+) \xrightarrow{\cong} (T \wedge X^n) \wedge K_+ \xrightarrow{\sigma^n \wedge \text{id}_{K_+}} X^{n+1} \wedge K_+$$

The simplicial functor in two variables is:

$$\begin{aligned} \text{Map}_\Sigma(-, -) : (\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \times \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \mathbf{S}\text{Sets} \\ (X, Y) &\longmapsto \text{Map}_\Sigma(X, Y) \end{aligned}$$

where $\text{Map}_\Sigma(X, Y)_n = \text{Hom}_{\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}}(X \wedge \Delta_+^n, Y)$, and finally for any symmetric T -spectrum Y we have the following functor

$$\begin{aligned} Y^- : \mathbf{S}\text{Sets} &\longrightarrow (\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \\ K &\longmapsto Y^K \end{aligned}$$

where $(Y^K)^n = (Y^n)^{K_+}$ which has an action of Σ_n induced by the one in Y^n and the K_+ -loops functor, and with bonding maps

$$T \wedge (Y^n)^{K_+} \xrightarrow{\alpha} (T \wedge Y^n)^{K_+} \xrightarrow{(\sigma^n)_*} (Y^{n+1})^{K_+}$$

where for $U \in (\mathcal{M}|_S)$, $\alpha(U)$ is adjoint to

$$T(U) \wedge (Y^n(U))^{K_+} \wedge K_+ \xrightarrow{\text{id}_{T(U)} \wedge \text{ev}_{K_+}} T(U) \wedge Y^n(U)$$

In a similar way, it is possible to promote the action of \mathcal{M}_* on the category of T -spectra to the category of symmetric T -spectra, i.e. the category of symmetric T -spectra $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ has the structure of a closed \mathcal{M}_* -module, which is obtained by extending the symmetric monoidal structure for \mathcal{M}_* levelwise.

The bifunctor giving the adjunction of two variables is defined as follows:

$$\begin{aligned} - \wedge - : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \times \mathcal{M}_* &\longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \\ (X, A) &\longmapsto X \wedge A \end{aligned}$$

with $(X \wedge A)^n = X^n \wedge A$ which has an action of Σ_n induced by the one in X^n and the functor $- \wedge A$, and with bonding maps

$$T \wedge (X^n \wedge A) \xrightarrow{\cong} (T \wedge X^n) \wedge A \xrightarrow{\sigma^n \wedge \text{id}_A} X^{n+1} \wedge A$$

The adjoints are given by:

$$\begin{aligned} \Omega_- - : \mathcal{M}_*^{\text{op}} \times \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \\ (A, X) &\longmapsto \Omega_A X \end{aligned}$$

$$\begin{aligned} \mathbf{hom}_r^\Sigma(-, -) : (\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \times \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \mathcal{M}_* \\ (X, Y) &\longmapsto \mathbf{hom}_r^\Sigma(X, Y) \end{aligned}$$

where $(\Omega_A X)^n = \Omega_A X^n$ which has an action of Σ_n induced by the one in X^n and the A -loops functor, with bonding maps $T \wedge (\Omega_A X^n) \rightarrow \Omega_A X^{n+1}$ adjoint to

$$T \wedge (\Omega_A X^n) \wedge A \xrightarrow{\text{id} \wedge \text{ev}_A} T \wedge X^n \xrightarrow{\sigma^n} X^{n+1}$$

and $\mathbf{hom}_r^\Sigma(X, Y)$ is the following pointed simplicial presheaf on $\mathcal{M}|_S$:

$$\begin{aligned} \mathbf{hom}_r^\Sigma(X, Y) : (\mathcal{M}|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto \text{Hom}_{\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}}(X \wedge (\Delta_U^n)_+, Y) \end{aligned}$$

The main difference between the categories of T -spectra and symmetric T -spectra is that the latter has a closed symmetric monoidal structure, i.e. it is possible to construct the smash product of two symmetric T -spectra.

Definition 2.6.4 (cf. [14]). — 1. A symmetric sequence X is a collection of pointed simplicial presheaves $(X^n)_{n \geq 0}$ on the smooth Nisnevich site $\mathcal{M}|_S$, together with left actions

$$\Sigma_n \times X^n \longrightarrow X^n$$

2. A map $f : X \rightarrow Y$ of symmetric sequences consists of a collection of Σ_n -equivariant maps

$$f^n : X^n \longrightarrow Y^n$$

in \mathcal{M}_* .

3. With these definitions we get a category, called the category of symmetric sequences which will be denoted by $(\mathcal{M}_*)^\Sigma$.

Definition 2.6.5. — Let X and Y be two symmetric sequences. Then the product $X \otimes Y$ is given by the following symmetric sequence:

$$(X \otimes Y)^n = \bigvee_{p+q=n} \Sigma_n \otimes_{\Sigma_p \times \Sigma_q} X^p \wedge Y^q$$

Remark 2.6.6. — A symmetric T -spectrum X can be identified with a symmetric sequence X equipped with a module structure over the sphere spectrum, i.e. with a map of symmetric sequences:

$$\mathbf{1} \otimes X \xrightarrow{\sigma_X} X$$

satisfying the usual associativity conditions.

Definition 2.6.7 (cf. [12]). — For every $n \geq 0$, we have the following adjunction:

$$(G_n, Ev_n, \varphi) : \mathcal{M}_* \longrightarrow (\mathcal{M}_*)^\Sigma$$

where Ev_n is the n -evaluation functor

$$\begin{aligned} Ev_n : (\mathcal{M}_*)^\Sigma &\longrightarrow \mathcal{M}_* \\ X &\longmapsto X^n \end{aligned}$$

and G_n is the n -free symmetric sequence functor:

$$\begin{aligned} G_n : \mathcal{M}_* &\longrightarrow (\mathcal{M}_*)^\Sigma \\ X &\longmapsto G_n(X) \end{aligned}$$

where

$$G_n(X)^m = \begin{cases} * & \text{if } m \neq n. \\ \bigvee_{\sigma \in \Sigma_n} X & \text{if } m = n. \end{cases}$$

Definition 2.6.8 (cf. [14]). — For every $n \geq 0$, we have the following adjunction:

$$(F_n^\Sigma, Ev_n, \varphi) : \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$$

where Ev_n is the n -evaluation functor

$$\begin{aligned} Ev_n : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \mathcal{M}_* \\ X &\longmapsto X^n \end{aligned}$$

and F_n^Σ is the n -free symmetric T -spectrum functor:

$$\begin{aligned} F_n^\Sigma : \mathcal{M}_* &\longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \\ X &\longmapsto \mathbf{1} \otimes G_n(X) \end{aligned}$$

Definition 2.6.9 (cf. [14]). — Let X and Y be two symmetric T -spectra. Then the smash product $X \wedge Y$ is given by the colimit of the following diagram

$$\mathbf{1} \otimes X \otimes Y \begin{array}{c} \xrightarrow{\sigma_X \otimes \text{id}} \\ \xrightarrow{\quad \quad \quad} \end{array} X \otimes Y$$

where the bottom arrow is the following composition

$$\mathbf{1} \otimes X \otimes Y \xrightarrow{t} X \otimes \mathbf{1} \otimes Y \xrightarrow{\text{id} \otimes \sigma_Y} X \otimes Y$$

Proposition 2.6.10 (Jardine). — The category of symmetric T -spectra $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ has a closed symmetric monoidal structure where the product is given by the smash product described in definition 2.6.9, and the functor that gives the adjunction of two variables is the following:

$$\begin{aligned} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(-, -) : (\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}})^{\text{op}} \times \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} &\longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \\ (X, Y) &\longmapsto \mathbf{Hom}_{\text{Spt}_T^\Sigma}(X, Y) \end{aligned}$$

where $\mathbf{Hom}_{Spt_T^\Sigma}(X, Y)^n = \mathbf{hom}_r^\Sigma(F_n^\Sigma(S^0) \wedge X, Y)$, and the adjoints σ_*^n to the bonding maps are given as follows: Let $\zeta : F_{n+1}^\Sigma(T) \cong F_{n+1}^\Sigma(S^0) \wedge T \rightarrow F_n^\Sigma(S^0)$ be the adjoint corresponding to the inclusion determined by the identity in Σ_{n+1} :

$$\iota_e : T \hookrightarrow Ev_{n+1}(F_n^\Sigma(S^0)) = \Sigma_{n+1} \otimes_{\Sigma_1 \times \Sigma_n} (T \wedge \bigvee_{\sigma \in \Sigma_n} S^0) = \bigvee_{\sigma \in \Sigma_{n+1}} T$$

then σ_*^n is the following map induced by $\zeta \wedge \text{id}$:

$$\mathbf{hom}_r^\Sigma(F_n^\Sigma(S^0) \wedge X, Y) \xrightarrow{(\zeta \wedge \text{id})^*} \mathbf{hom}_r^\Sigma(F_{n+1}^\Sigma(S^0) \wedge T \wedge X, Y)$$

The twist isomorphism $\tau : X \wedge Y \rightarrow Y \wedge X$ is induced levelwise by:

$$\begin{array}{ccc} X^p \wedge Y^q & \xrightarrow{t} & Y^q \wedge X^p \\ \alpha \downarrow & & \downarrow \alpha_{c_{q,p}} \\ (X \otimes Y)^{p+q} & & (Y \otimes X)^{p+q} \end{array}$$

Finally, the unit is given by the sphere T -spectrum $F_0^\Sigma(S^0) = \mathbf{1}$.

Proof. — We refer the reader to [14, Section 4.3]. □

Proposition 2.6.11. — Let X, Y be two arbitrary symmetric T -spectra and let A in \mathcal{M}_* be an arbitrary pointed simplicial presheaf. Then we have the following enriched adjunctions:

$$(20) \quad \text{Map}(A, \mathbf{hom}_r^\Sigma(X, Y)) \xrightarrow[\cong]{\alpha} \text{Map}_\Sigma(X \wedge A, Y) \xrightarrow[\cong]{\beta} \text{Map}_\Sigma(X, \Omega_A Y)$$

$$(21) \quad \mathbf{Hom}_{\mathcal{M}_*}(A, \mathbf{hom}_r^\Sigma(X, Y)) \xrightarrow[\cong]{\delta} \mathbf{hom}_r^\Sigma(X \wedge A, Y) \xrightarrow[\cong]{\epsilon} \mathbf{hom}_r^\Sigma(X, \Omega_A Y)$$

$$(22) \quad \mathbf{Hom}_{Spt_T^\Sigma}(X \wedge A, Y) \xrightarrow[\cong]{\gamma} \mathbf{Hom}_{Spt_T^\Sigma}(X, \Omega_A Y)$$

where the maps in (20) are isomorphisms of simplicial sets, the maps in (21) are isomorphisms of simplicial presheaves, and the map in (22) is an isomorphism of symmetric T -spectra.

Proof. — We consider first the simplicial adjunctions: To any n -simplex t in $\text{Map}(A, \mathbf{hom}_r^\Sigma(X, Y))$

$$A \wedge \Delta^n \xrightarrow{t} \mathbf{hom}_r^\Sigma(X, Y)$$

associate the following n -simplex in $\text{Map}_\Sigma(X \wedge A, Y)$:

$$X \wedge A \wedge \Delta^n \xrightarrow{\alpha(t)} Y$$

corresponding to the adjunction between $X \wedge -$ and $\mathbf{hom}_r^\Sigma(X, -)$.

To any n -simplex t in $\text{Map}_\Sigma(X \wedge A, Y)$

$$\Delta^n \wedge X \wedge A \xrightarrow{\cong} X \wedge A \wedge \Delta^n \xrightarrow{t} Y$$

associate the following n -simplex in $\text{Map}_\Sigma(X, \Omega_A Y)$:

$$X \wedge \Delta^n \xrightarrow{\cong} \Delta^n \wedge X \xrightarrow{\beta(t)} \Omega_A Y$$

corresponding to the adjunction between $- \wedge A$ and Ω_A .

We consider now the isomorphisms of simplicial presheaves: To any simplex s in $\mathbf{Hom}_{\mathcal{M}_*}(A, \mathbf{hom}_r^\Sigma(X, Y))$

$$A \wedge \Delta_U^n \xrightarrow{s} \mathbf{hom}_r^\Sigma(X, Y)$$

we associate the following simplex in $\mathbf{hom}_r^\Sigma(X \wedge A, Y)$

$$X \wedge A \wedge \Delta_U^n \xrightarrow{\delta(s)} Y$$

corresponding to the adjunction between $X \wedge -$ and $\mathbf{hom}_r^\Sigma(X, -)$.

To any simplex s in $\mathbf{hom}_r^\Sigma(X \wedge A, Y)$

$$X \wedge \Delta_U^n \wedge A \xrightarrow{\cong} X \wedge A \wedge \Delta_U^n \xrightarrow{s} Y$$

we associate the following simplex in $\mathbf{hom}_r^\Sigma(X, \Omega_A Y)$

$$X \wedge \Delta_U^n \xrightarrow{\epsilon(s)} \Omega_A Y$$

corresponding to the adjunction between $- \wedge A$ and Ω_A .

Finally, we consider the isomorphism of symmetric T -spectra: Using the adjunction given by ϵ in (21), we get for every $n \geq 0$ the following commutative diagram, where the vertical maps are isomorphisms of simplicial presheaves:

$$\begin{array}{ccc} \mathbf{hom}_r^\Sigma(F_n^\Sigma(S^0) \wedge X \wedge A, Y) & \xrightarrow{(\zeta \wedge \text{id}_{X \wedge A})^*} & \mathbf{hom}_r^\Sigma(F_{n+1}^\Sigma(T) \wedge X \wedge A, Y) \\ \cong \downarrow \epsilon & & \cong \downarrow \epsilon \\ \mathbf{hom}_r^\Sigma(F_n^\Sigma(S^0) \wedge X, \Omega_A Y) & \xrightarrow{(\zeta \wedge \text{id}_X)^*} & \mathbf{hom}_r^\Sigma(F_{n+1}^\Sigma(T) \wedge X, \Omega_A Y) \end{array}$$

By definition (see Proposition 2.6.10) the diagram above is equal to:

$$\begin{array}{ccc} \mathbf{Hom}_{Spt_T^\Sigma}(X \wedge A, Y)^n & \xrightarrow{\sigma_*^n} & \Omega_T \mathbf{Hom}_{Spt_T^\Sigma}(X \wedge A, Y)^{n+1} \\ \cong \downarrow \epsilon & & \cong \downarrow \epsilon \\ \mathbf{Hom}_{Spt_T^\Sigma}(X, \Omega_A Y)^n & \xrightarrow{\sigma_*^n} & \Omega_T \mathbf{Hom}_{Spt_T^\Sigma}(X, \Omega_A Y)^{n+1} \end{array}$$

This induces the isomorphism γ . □

Proposition 2.6.12. — *Let X, Y, Z be three arbitrary symmetric T -spectra. Then we have the following enriched adjunctions:*

$$(23) \quad \text{Map}_\Sigma(X \wedge Y, Z) \xrightarrow[\cong]{\lambda} \text{Map}_\Sigma(X, \mathbf{Hom}_{Spt_T^\Sigma}(Y, X))$$

$$(24) \quad \mathbf{hom}_r^\Sigma(X \wedge Y, Z) \xrightarrow[\cong]{\kappa} \mathbf{hom}_r^\Sigma(X, \mathbf{Hom}_{Spt_T^\Sigma}(Y, Z))$$

$$(25) \quad \mathbf{Hom}_{Spt_T^\Sigma}(X \wedge Y, Z) \xrightarrow[\cong]{\mu} \mathbf{Hom}_{Spt_T^\Sigma}(X, \mathbf{Hom}_{Spt_T^\Sigma}(Y, Z))$$

where the map in (23) is an isomorphism of simplicial sets, the map in (24) is an isomorphism of simplicial presheaves, and the map in (25) is an isomorphism of symmetric T -spectra.

Proof. — We consider first the simplicial adjunctions: To any n -simplex t in $\text{Map}_\Sigma(X \wedge Y, Z)$

$$\Delta^n \wedge X \wedge Y \xrightarrow{\cong} X \wedge Y \wedge \Delta^n \xrightarrow{t} Z$$

associate the following n -simplex in $\text{Map}_\Sigma(X, \mathbf{Hom}_{Spt_T^\Sigma}(Y, Z))$:

$$X \wedge \Delta^n \xrightarrow{\cong} \Delta^n \wedge X \xrightarrow{\lambda(t)} \mathbf{Hom}_{Spt_T^\Sigma}(Y, Z)$$

corresponding to the adjunction between $- \wedge Y$ and $\mathbf{Hom}_{Spt_T^\Sigma}(Y, -)$.

We consider now the isomorphisms of simplicial presheaves: To any simplex s in $\mathbf{hom}_r^\Sigma(X \wedge Y, Z)$

$$\Delta_U^n \wedge X \wedge Y \xrightarrow{\cong} X \wedge Y \wedge \Delta_U^n \xrightarrow{s} Z$$

we associate the following simplex in $\mathbf{hom}_r^\Sigma(X, \mathbf{Hom}_{Spt_T^\Sigma}(Y, Z))$

$$X \wedge \Delta_U^n \xrightarrow{\cong} \Delta_U^n \wedge X \xrightarrow{\kappa(s)} \mathbf{Hom}_{Spt_T^\Sigma}(Y, Z)$$

corresponding to the adjunction between $- \wedge Y$ and $\mathbf{Hom}_{Spt_T^\Sigma}(Y, -)$.

Finally, we consider the isomorphism of symmetric T -spectra: Using the adjunction given by κ in (24), we get for every $n \geq 0$ the following commutative diagram, where

the vertical maps are isomorphisms of simplicial presheaves:

$$\begin{array}{ccc}
 \mathbf{hom}_r^\Sigma(F_n^\Sigma(S^0) \wedge X \wedge Y, Z) & \xrightarrow{(\zeta \wedge \text{id}_{X \wedge Y})^*} & \mathbf{hom}_r^\Sigma(F_{n+1}^\Sigma(T) \wedge X \wedge Y, Z) \\
 \cong \downarrow \kappa & & \downarrow \cong \kappa \\
 \mathbf{hom}_r^\Sigma(F_n^\Sigma(S^0) \wedge X, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(Y, Z)) & \xrightarrow{(\zeta \wedge \text{id}_X)^*} & \mathbf{hom}_r^\Sigma(F_{n+1}^\Sigma(T) \wedge X, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(Y, Z))
 \end{array}$$

By definition (see Proposition 2.6.10) the diagram above is equal to:

$$\begin{array}{ccc}
 \mathbf{Hom}_{\text{Spt}_T^\Sigma}(X \wedge Y, Z)^n & \xrightarrow{\sigma_*^n} & \Omega_T \mathbf{Hom}_{\text{Spt}_T^\Sigma}(X \wedge Y, Z)^{n+1} \\
 \cong \downarrow \kappa & & \cong \downarrow \kappa \\
 \mathbf{Hom}_{\text{Spt}_T^\Sigma}(X, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(Y, Z))^n & \xrightarrow{\sigma_*^n} & \Omega_T \mathbf{Hom}_{\text{Spt}_T^\Sigma}(X, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(Y, Z))^{n+1}
 \end{array}$$

This induces the isomorphism μ . □

The following proposition will have remarkable consequences in our study of the multiplicative properties for Voevodsky’s slice filtration.

Proposition 2.6.13 (Jardine). — *Let A, B be two arbitrary pointed simplicial presheaves in \mathcal{M}_* . Then we have an isomorphism:*

$$F_n^\Sigma(A) \wedge F_m^\Sigma(B) \xrightarrow{\cong} F_{m+n}^\Sigma(A \wedge B)$$

which is natural in A and B .

Proof. — We refer the reader to [14, Corollary 4.18]. □

For the construction of the motivic stable model structure on the category of T -spectra, it was necessary to introduce the projective and injective model structures (see Theorem 2.4.16). In [14], Jardine considers an injective model structure for symmetric T -spectra as a preliminary step in the construction of a model structure which turns out to be Quillen equivalent to $\text{Spt}_T \mathcal{M}_*$. We will also need to consider a projective model structure for symmetric T -spectra, in order to show that this stable model structure for symmetric T -spectra is cellular.

Definition 2.6.14. — Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We say that f is a level cofibration (respectively level fibration, level weak equivalence), if for every $n \geq 0$, the map $f^n : X^n \rightarrow Y^n$ is a cofibration (respectively a fibration, a weak equivalence) in \mathcal{M}_* .

In Proposition 2.3.7 we used I_{M_*} and J_{M_*} to denote the sets of generating cofibrations and trivial cofibrations for \mathcal{M}_* .

Theorem 2.6.15 (Hovey). — There exists a cofibrantly generated model structure for the category $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ of symmetric T -spectra with the following choices:

1. The weak equivalences are the level weak equivalences.
2. The set I of generating cofibrations is

$$I = \bigcup_{n \geq 0} F_n^\Sigma(I_{M_*})$$

3. The set J of generating trivial cofibrations is

$$J = \bigcup_{n \geq 0} F_n^\Sigma(J_{M_*})$$

This model structure will be called the projective model structure for symmetric T -spectra. Furthermore, the projective model structure is left proper and simplicial.

Proof. — Proposition 2.3.7 implies that the model category \mathcal{M}_* is in particular pointed, proper, simplicial and symmetric monoidal. We also have that every pointed simplicial presheaf in \mathcal{M}_* is cofibrant. Then the result follows immediately from Theorems 8.2 and 8.3 in [11]. \square

Remark 2.6.16. — Let $f : X \rightarrow Y$ be a map of symmetric T -spectra.

1. f is a fibration in $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ equipped with the projective model structure if and only if f is a level fibration.
2. f is a trivial fibration in $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ equipped with the projective model structure if and only if f is both a level fibration and a level weak equivalence.

It follows directly from the definition that every symmetric T -spectrum after forgetting the Σ_n -actions becomes a T -spectrum in $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$. Therefore we get a functor:

$$U : \mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}} \longrightarrow \mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$$

It turns out that this forgetful functor has a left adjoint.

Definition 2.6.17 (Jardine, cf. [14]). — Let X be an arbitrary T -spectrum in $\mathrm{Spt}_T(\mathcal{M}|_S)_{\mathrm{Nis}}$. Then X has a natural filtration $\{L_n X\}_{n \geq 0}$ called the layer filtration, where $L_n X$ is defined as

$$X^0, X^1, \dots, X^n, T \wedge X^n, T^2 \wedge X^n, \dots$$

and furthermore

$$X \cong \varinjlim L_n X$$

It is also possible to give an inductive definition for the layers $L_n X$ using the following pushout diagrams (see definition 2.4.4):

$$\begin{array}{ccc} F_{n+1}(T \wedge X^n) & \longrightarrow & L_n X \\ \downarrow & & \downarrow \\ F_{n+1}(X^{n+1}) & \longrightarrow & L_{n+1} X \end{array}$$

Proposition 2.6.18 (Jardine). — We have the following adjunction

$$(V, U, \varphi) : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$$

The functor V is called the symmetrization functor and is defined as follows:

1. For every pointed simplicial presheaf X on the smooth Nisnevich site $(\mathcal{M}|_S)_{\text{Nis}}$ we have

$$V(F_n(X)) = F_n^\Sigma(X)$$

2. V is constructed inductively using the layer filtration (see definition 2.6.17) together with the following pushout diagrams (see definition 2.6.8):

$$\begin{array}{ccc} F_{n+1}^\Sigma(T \wedge X^n) & \longrightarrow & V(L_n X) \\ \downarrow & & \downarrow \\ F_{n+1}^\Sigma(X^{n+1}) & \longrightarrow & V(L_{n+1} X) \end{array}$$

3. Finally, $V(X) = \varinjlim V(L_n X)$

Proof. — We refer the reader to [14, p. 507] □

Proposition 2.6.19. — The adjunction

$$(V, U, \varphi) : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$$

is enriched in the categories of simplicial sets and pointed simplicial presheaves on $(\mathcal{M}|_S)_{\text{Nis}}$, i.e. for every T -spectrum X and for every symmetric T -spectrum Y we have the following natural isomorphisms:

$$\text{Map}_\Sigma(VX, Y) \xrightarrow[\cong]{\epsilon} \text{Map}(X, UY)$$

$$\mathbf{hom}_r^\Sigma(VX, Y) \xrightarrow[\cong]{\eta} \mathbf{hom}_r(X, UY)$$

Proof. — We consider first the simplicial isomorphism: Given any n -simplex t in $\text{Map}_\Sigma(VX, Y)$

$$VX \wedge \Delta^n \xrightarrow{t} Y$$

consider the map corresponding to the adjunction between $- \wedge \Delta^n$ and $-\Delta^n$ in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$

$$VX \xrightarrow{t'} Y^{\Delta^n}$$

Now use the adjunction between V and U to get the map:

$$X \xrightarrow{t''} U(Y^{\Delta^n}) = (UY)^{\Delta^n}$$

and finally use the adjunction between $- \wedge \Delta^n$ and $-\Delta^n$ in $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ to get the associated n -simplex $\epsilon(t)$ in $\text{Map}(X, UY)$:

$$X \wedge \Delta^n \xrightarrow{\epsilon(t)} UY$$

We consider now the isomorphism of simplicial presheaves: Given any simplex s in $\mathbf{hom}_r^\Sigma(VX, Y)$

$$VX \wedge \Delta_W^n \xrightarrow{s} Y$$

consider the map corresponding to the adjunction between $- \wedge \Delta_W^n$ and $\Omega_{\Delta_W^n} -$ in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$

$$VX \xrightarrow{s'} \Omega_{\Delta_W^n} Y$$

Now use the adjunction between V and U to get the map:

$$X \xrightarrow{s''} U(\Omega_{\Delta_W^n} Y) = \Omega_{\Delta_W^n} UY$$

and finally use the adjunction between $- \wedge \Delta_W^n$ and $\Omega_{\Delta_W^n}$ in $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ to get the associated simplex $\eta(s)$ in $\mathbf{hom}_r(X, UY)$:

$$X \wedge \Delta_W^n \xrightarrow{\eta(s)} UY \quad \square$$

We say that a map $f : X \rightarrow Y$ of symmetric T -spectra is an *injective fibration* if it has the right lifting property with respect to the class of maps which are both level cofibrations and level weak equivalences.

Theorem 2.6.20 (Jardine). — *There exists a model structure for the category $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ of symmetric T -spectra with the following choices:*

1. *The weak equivalences are the level weak equivalences.*
2. *The cofibrations are the level cofibrations.*
3. *The fibrations are the injective fibrations.*

This model structure will be called the injective model structure for symmetric T -spectra. Furthermore, the injective model structure is proper, simplicial, and cofibrantly generated with the following sets I , J of generating cofibrations and trivial cofibrations, respectively (see Theorem 2.4.8):

1. The set I of generating cofibrations is

$$I = \{V(i) : VA \rightarrow VB\}$$

where i satisfies the following conditions:

- (a) i is a level cofibration in $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.
- (b) The codomain B of i is κ -bounded.

2. The set J of generating trivial cofibrations is

$$J = \{V(j) : V(A) \rightarrow V(B)\}$$

where j satisfies the following conditions:

- (a) j is a level trivial cofibration in $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.
- (b) The codomain B of j is κ -bounded.

Proof. — We refer the reader to [14, Theorem 4.2]. □

Remark 2.6.21. — The identity functor on $\text{Spt}_T^{\Sigma}(\mathcal{M}|_S)_{\text{Nis}}$ induces a left Quillen functor from the projective model structure to the injective model structure.

Definition 2.6.22. — 1. Let Z be a symmetric T -spectrum. We say that Z is injective stably fibrant if Z satisfies the following conditions:

- (a) Z is fibrant in $\text{Spt}_T^{\Sigma}(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the injective model structure.
- (b) UZ is fibrant in $\text{Spt}_T\mathcal{M}_*$.

2. Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We say that f is a stable weak equivalence if for every injective stably fibrant symmetric T -spectrum Z , the induced map

$$\text{Map}_{\Sigma}(Y, Z) \xrightarrow{f^*} \text{Map}_{\Sigma}(X, Z)$$

is a weak equivalence of simplicial sets.

3. Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We say that f is a stable fibration if Uf is a fibration in $\text{Spt}_T\mathcal{M}_*$ (see theorem 2.4.16).

In Theorem 2.5.4 we used $I_{\mathcal{M}_*}^T$ and $J_{\mathcal{M}_*}^T$ to denote the sets of generating cofibrations and trivial cofibrations for $\text{Spt}_T\mathcal{M}_*$.

Theorem 2.6.23 (Jardine). — There exists a model structure for the category $\text{Spt}_T^{\Sigma}(\mathcal{M}|_S)_{\text{Nis}}$ of symmetric T -spectra with the following choices:

1. The weak equivalences are the stable weak equivalences.

2. The cofibrations are the projective cofibrations (see Theorem 2.6.15), i.e. they are generated by the set

$$\bigcup_{n \geq 0} F_n^\Sigma(I_{M_*}) = V(I_{M_*}^T)$$

3. The fibrations are the stable fibrations.

This model structure will be called motivic symmetric stable, and the category of symmetric T -spectra, equipped with the motivic symmetric stable model structure will be denoted by $\text{Spt}_T^\Sigma \mathcal{M}_*$. Furthermore, $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a proper and simplicial model category.

Proof. — We refer the reader to [14, Proposition 4.4 and Theorem 4.15]. □

Remark 2.6.24. — Let $p : X \rightarrow Y$ be a map of symmetric T -spectra. Then p is a trivial fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if Up is a trivial fibration in $\text{Spt}_T \mathcal{M}_*$.

Proposition 2.6.25. — $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a \mathcal{M}_* -model category (see definition 1.7.12).

Proof. — Condition (2) in definition 1.7.12 follows automatically since the unit in \mathcal{M}_* is cofibrant. It remains to show that

$$- \wedge - : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \times \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$$

is a Quillen bifunctor. By Lemma 1.7.5 it is enough to prove the following claim:

Given a cofibration $i : A \rightarrow B$ in \mathcal{M}_* and a fibration $p : X \rightarrow Y$ in $\text{Spt}_T^\Sigma \mathcal{M}_*$, then the map

$$\Omega_B X \xrightarrow{(i^*, p_*)} \Omega_B Y \times_{\Omega_A Y} \Omega_A X$$

is a fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$ which is trivial if either i or p is a weak equivalence.

But this follows immediately from the following facts:

1. A map of symmetric T -spectra $f : X \rightarrow Y$ is a fibration (respectively a trivial fibration) in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $Uf : UX \rightarrow UY$ is a fibration (respectively a trivial fibration) in $\text{Spt}_T \mathcal{M}_*$.
2. For every symmetric T -spectrum X and for any pointed simplicial presheaf A in \mathcal{M}_* , we have that $U(\Omega_A X) = \Omega_A UX$, where the right hand side denotes the action of \mathcal{M}_* in $\text{Spt}_T \mathcal{M}_*$.
3. $\text{Spt}_T \mathcal{M}_*$ is a \mathcal{M}_* -model category (see Proposition 2.4.24). □

Corollary 2.6.26. — For every pointed simplicial presheaf $A \in \mathcal{M}_*$, the adjunction

$$(- \wedge A, \Omega_A -, \varphi) : \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen adjunction.

Proof. — We have that every pointed simplicial presheaf is cofibrant in \mathcal{M}_* . Then the result follows from Proposition 2.6.25. \square

Theorem 2.6.27 (Jardine). — *Let $T = S^1 \wedge \mathbb{G}_m \in \mathcal{M}_*$. Then the Quillen adjunction:*

$$(- \wedge T, \Omega_T, \varphi) : \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Let η, ϵ denote the unit and counit of the adjunction $(- \wedge T, \Omega_T, \varphi)$. By Proposition 1.3.13 in [10], it suffices to check that the following conditions hold:

1. For every cofibrant symmetric T -spectrum A in $\text{Spt}_T^\Sigma \mathcal{M}_*$, the following composition

$$A \xrightarrow{\eta_A} \Omega_T(T \wedge A) \xrightarrow{\Omega_T(R^{T \wedge A})} \Omega_T R(T \wedge A)$$

is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, where R denotes a fibrant replacement functor in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

2. For every fibrant symmetric T -spectrum X in $\text{Spt}_T^\Sigma \mathcal{M}_*$, the following composition

$$T \wedge Q(\Omega_T X) \xrightarrow{\text{id} \wedge Q^{\Omega_T X}} T \wedge (\Omega_T X) \xrightarrow{\epsilon_X} X$$

is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, where Q denotes a cofibrant replacement functor in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): It follows directly from Corollary 4.26 in [14].

(2): By construction the map $Q^{\Omega_T X} : Q(\Omega_T X) \rightarrow \Omega_T X$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore by Lemma 4.25 in [14], we have that $\text{id} \wedge Q^{\Omega_T X}$ is also a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then by the two out of three property for weak equivalences, it suffices to show that ϵ_X is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Since X is fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, it follows that UX is fibrant in $\text{Spt}_T \mathcal{M}_*$. Therefore by Lemma 2.4.19(2) we have that UX is in particular level fibrant. Then by Corollary 3.16 in [14] it follows that the map:

$$\epsilon_{UX} : T \wedge (\Omega_T UX) \longrightarrow UX$$

is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, but this is just $U(\epsilon_X)$. Hence by Proposition 4.8 in [14], we have that ϵ_X is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, as we wanted. \square

Proposition 2.6.28 (Jardine). — *$\text{Spt}_T^\Sigma \mathcal{M}_*$ is a symmetric monoidal model category (with respect to the smash product of symmetric T -spectra) in the sense of Hovey (see definition 1.7.7).*

Proof. — We refer the reader to [14, Proposition 4.19]. \square

Corollary 2.6.29. — *Let A be a cofibrant symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Then the adjunction:*

$$(- \wedge A, \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, -), \varphi) : \mathrm{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \mathrm{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen adjunction.

Proof. — It follows directly from Proposition 2.6.28. □

Theorem 2.6.30 (Jardine). — *The adjunction:*

$$(V, U, \varphi) : \mathrm{Spt}_T \mathcal{M}_* \longrightarrow \mathrm{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and the forgetful functor is a Quillen equivalence.

Proof. — We refer the reader to [14, Theorem 4.31]. □

2.7. Cellularity of the Motivic Symmetric Stable Model Structure

In this section we will show that the model category $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is cellular. For this we will use the cellularity of \mathcal{M}_* (see Proposition 2.3.7) together with some results of Hovey [11].

Theorem 2.7.1 (Hovey). — *Let $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ be the category of symmetric T -spectra equipped with the projective model structure (see Theorem 2.6.15). Then $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ is a cellular model category where the sets of generating cofibrations and trivial cofibrations are the ones described in Theorem 2.6.15.*

Proof. — Proposition 2.3.7 implies that \mathcal{M}_* is in particular a cellular, left proper and symmetric monoidal model category. We also have that $T = S^1 \wedge \mathbb{G}_m$ is cofibrant in \mathcal{M}_* . Therefore we can apply Theorem A.9 in [11], which says that the category of symmetric T -spectra equipped with the projective model structure is also cellular under our conditions. □

Theorem 2.6.15 together with Theorem 2.7.1 imply that the projective model structure on $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ is cellular, left proper and simplicial. Therefore we can apply Hirschhorn's localization technology to construct left Bousfield localizations. If we are able to find a suitable set of maps such that the left Bousfield localization with respect to this set recovers the motivic stable model structure on $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$, then an immediate corollary of this will be the cellularity of the motivic stable model structure for symmetric T -spectra.

Definition 2.7.2 (Hovey, cf. [11]). — Let $I_{M_*} = \{Y_+ \hookrightarrow (\Delta_U^n)_+\}$ be the set of generating cofibrations for \mathcal{M}_* (see Proposition 2.3.7). Notice that Y_+ may be equal to $(\Delta_U^n)_+$. We consider the following set of maps of symmetric T -spectra

$$S_\Sigma = \{F_{k+1}^\Sigma(T \wedge Y_+) \xrightarrow{\zeta_{\Sigma,k}^Y} F_k^\Sigma(Y_+)\}$$

where $\zeta_{\Sigma,k}^Y$ is the adjoint corresponding to the inclusion determined by the identity in Σ_{k+1}

$$\iota_e : T \wedge Y_+ \hookrightarrow Ev_{k+1}(F_k^\Sigma(Y_+)) = \Sigma_{k+1} \otimes_{\Sigma_1 \times \Sigma_k} (T \wedge \bigvee_{\sigma \in \Sigma_k} Y_+) = \bigvee_{\sigma \in \Sigma_{k+1}} T \wedge Y_+$$

coming from the adjunction between F_{k+1}^Σ and Ev_{k+1} (see definition 2.6.8)

Proposition 2.7.3 (Hovey). — Let X be a symmetric T -spectrum. The following conditions are equivalent:

1. X is stably fibrant, i.e. X is a fibrant object in $\text{Spt}_T^\Sigma \mathcal{M}_*$.
2. X is S_Σ -local.

Proof. — It follows from definition 8.6 and Theorem 8.8 in [11], together with definition 2.6.22(3) and lemma 2.4.19. □

Now it is very easy to show that the motivic symmetric stable model structure for symmetric T -spectra is in fact cellular.

Theorem 2.7.4. — $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a cellular model category with the following sets I_Σ^T, J_Σ^T of generating cofibrations and trivial cofibrations respectively:

$$\begin{aligned} I_\Sigma^T &= V(I_{M_*}^T) = \bigcup_{k \geq 0} F_k^\Sigma(I_{M_*}) \\ &= \bigcup_{k \geq 0} \{F_k^\Sigma(Y_+) \hookrightarrow F_k^\Sigma((\Delta_U^n)_+) \mid U \in (\mathcal{Jm}|_S), n \geq 0\} \\ J_\Sigma^T &= \{j : A \rightarrow B\} \end{aligned}$$

where j satisfies the following conditions:

1. j is an inclusion of I_Σ^T -complexes.
2. j is a stable weak equivalence of symmetric T -spectra.
3. the size of B as an I_Σ^T -complex is less than κ , where κ is the regular cardinal described by Hirschhorn in [6, definition 4.5.3].

Proof. — By Theorem 2.7.1 we know that $\text{Spt}_T^\Sigma(\mathcal{Jm}|_S)_{\text{Nis}}$ is cellular when it is equipped with the projective model structure. Therefore we can apply Hirschhorn’s localization techniques to construct the left Bousfield localization with respect to the set S_Σ of definition 2.7.2. We claim that this localization coincides with $\text{Spt}_T^\Sigma \mathcal{M}_*$. In effect, using Proposition 2.7.3, we have that the fibrant objects in the left Bousfield

localization with respect to S_Σ coincide with the fibrant objects in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore a map $f : X \rightarrow Y$ of symmetric T -spectra is a weak equivalence in the left Bousfield localization with respect to S_Σ if and only if $Qf^* : \text{Map}_\Sigma(QY, Z) \rightarrow \text{Map}_\Sigma(QX, Z)$ is a weak equivalence of simplicial sets for every stably fibrant object Z (here Q denotes the cofibrant replacement functor in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the projective model structure). But since $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category and the cofibrations coincide with the projective cofibrations, using corollary 1.6.11(2) we get exactly the same characterization for the stable equivalences. Hence the weak equivalences in both the motivic symmetric stable structure and the left Bousfield localization with respect to S_Σ coincide. This implies that the motivic symmetric stable model structure and the left Bousfield localization with respect to S_Σ are identical, since the cofibrations in both cases are just the cofibrations for the projective model structure on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$.

Therefore using [6, Theorem 4.1.1] we have that the motivic symmetric stable model structure on $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ is cellular, since it is constructed applying Hirschhorn technology with respect to the set S_Σ .

The claim with respect to the sets of generating cofibrations and trivial cofibrations also follows from [6, Theorem 4.1.1]. \square

Theorem 2.7.4 will be used for the construction of new model structures on $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ which are adequate to study the multiplicative properties of Voevodsky's slice filtration.

2.8. Modules and Algebras of Motivic Symmetric Spectra

In this section A will always denote a ring spectrum with unit in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$, and $A\text{-mod}$ will denote the category of left (or right) A -modules. In case A is a commutative ring spectrum, we will denote the category of A -algebras by $A\text{-alg}$. Our goal is to define the model structures induced by the motivic symmetric stable model structure on the categories of A -modules and A -algebras, and to show some of their properties.

Proposition 2.8.1. — *We have the following adjunction between the categories of symmetric T -spectra and A -modules:*

$$(A \wedge -, U, \varphi) : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow A\text{-mod}$$

where $U(N) = N$ after forgetting the A -module structure, and $A \wedge X$ has a structure of A -module induced by the ring structure on A .

Proof. — The unit η and counit δ of the adjunction are defined as follows:

$$\begin{aligned} \eta_X : X \cong \mathbf{1} \wedge X &\xrightarrow{u_A \wedge \text{id}} U(A \wedge X) = A \wedge X \\ \delta_N : A \wedge U(N) = A \wedge N &\xrightarrow{\mu_N} N \end{aligned}$$

where u_A is the unit of A and μ_N is the map inducing the A -module structure on N . □

The category of A -modules inherits a simplicial structure from the one that exists on symmetric T -spectra (see Section 2.6).

Given an A -module M , the tensor objects are defined as follows:

$$\begin{aligned} M \wedge - : \mathbf{S}\mathbf{Sets} &\longrightarrow A\text{-mod} \\ K &\longmapsto M \wedge K \end{aligned}$$

where $(M \wedge K)^n = M^n \wedge K_+$, i.e. it coincides with the tensor object defined for symmetric T -spectra and has a structure of A -module induced by the one in M .

The simplicial functor in two variables is:

$$\begin{aligned} \text{Map}_{A\text{-mod}}(-, -) : (A\text{-mod})^{\text{op}} \times A\text{-mod} &\longrightarrow \mathbf{S}\mathbf{Sets} \\ (M, N) &\longmapsto \text{Map}_{A\text{-mod}}(M, N) \end{aligned}$$

where $\text{Map}_{A\text{-mod}}(M, N)_n = \text{Hom}_{A\text{-mod}}(M \wedge \Delta_+^n, N)$, and finally for any A -module N we have the following functor

$$\begin{aligned} N^- : \mathbf{S}\mathbf{Sets} &\longrightarrow (A\text{-mod})^{\text{op}} \\ K &\longmapsto N^K \end{aligned}$$

where $(N^K)^n = (N^n)^{K_+}$, i.e. it coincides with the cotensor object defined for symmetric T -spectra and has a structure of A -module $A \wedge (N)^{K_+} \rightarrow N^{K_+}$ adjoint to

$$A \wedge (N)^{K_+} \wedge K_+ \xrightarrow{\text{id} \wedge \text{ev}_{K_+}} A \wedge N \xrightarrow{\mu} N$$

where μ is the map that induces the A -module structure on N .

Similarly, it is possible to promote the action of \mathcal{M}_* on the category of symmetric T -spectra to the category of A -modules, i.e. the category of A -modules $A\text{-mod}$ has the structure of a closed \mathcal{M}_* -module, which is obtained by extending the symmetric monoidal structure for \mathcal{M}_* levelwise.

The bifunctor giving the adjunction of two variables is defined as follows:

$$\begin{aligned} - \wedge - : A\text{-mod} \times \mathcal{M}_* &\longrightarrow A\text{-mod} \\ (M, D) &\longmapsto M \wedge D \end{aligned}$$

with $(M \wedge D)^n = M^n \wedge D$, i.e. it coincides with the tensor object defined for symmetric T -spectra and has a structure of A -module induced by the one in M .

The adjoints are given by:

$$\begin{aligned} \Omega_- : \mathcal{M}_*^{\text{op}} \times A\text{-mod} &\longrightarrow A\text{-mod} \\ (D, N) &\longmapsto \Omega_D N \end{aligned}$$

$$\begin{aligned} \mathbf{hom}_r^{A\text{-mod}}(-, -) : (A\text{-mod})^{\text{op}} \times A\text{-mod} &\longrightarrow \mathcal{M}_* \\ (M, N) &\longmapsto \mathbf{hom}_r^{A\text{-mod}}(M, N) \end{aligned}$$

where $(\Omega_D N)^n = \Omega_D N^n$, i.e. it coincides with the cotensor object defined for symmetric T -spectra and has a structure of A -module $A \wedge (\Omega_D N) \rightarrow \Omega_D N$ adjoint to

$$A \wedge (\Omega_D N) \wedge D \xrightarrow{\text{id} \wedge \text{lev}_D} A \wedge N \xrightarrow{\mu} N$$

and $\mathbf{hom}_r^{A\text{-mod}}(M, N)$ is the following pointed simplicial presheaf on $\mathcal{S}m|_S$:

$$\begin{aligned} \mathbf{hom}_r^{A\text{-mod}}(M, N) : (\mathcal{S}m|_S \times \Delta)^{\text{op}} &\longrightarrow \mathcal{S}ets \\ (U, n) &\longmapsto \mathbf{Hom}_{A\text{-mod}}(M \wedge (\Delta_U^n)_+, N) \end{aligned}$$

Proposition 2.8.2. — *The adjunction (see Proposition 2.8.1)*

$$(A \wedge -, U, \varphi) : \text{Spt}_T^\Sigma(\mathcal{S}m|_S)_{\text{Nis}} \longrightarrow A\text{-mod}$$

is enriched in the categories of simplicial sets and pointed simplicial presheaves on $(\mathcal{S}m|_S)_{\text{Nis}}$, i.e. for every symmetric T -spectrum X and for every A -module N we have the following natural isomorphisms:

$$\text{Map}_{A\text{-mod}}(A \wedge X, N) \xrightarrow[\cong]{\epsilon} \text{Map}_\Sigma(X, UN)$$

$$\mathbf{hom}_r^{A\text{-mod}}(A \wedge X, N) \xrightarrow[\cong]{\eta} \mathbf{hom}_r^\Sigma(X, UN)$$

Proof. — We consider first the simplicial isomorphism: Given any n -simplex t in $\text{Map}_{A\text{-mod}}(A \wedge X, N)$

$$A \wedge X \wedge \Delta_+^n \xrightarrow{t} N$$

use the adjunction between $A \wedge -$ and U to get the associated n -simplex $\epsilon(t)$ in $\text{Map}_\Sigma(X, UN)$:

$$X \wedge \Delta_+^n \xrightarrow{\epsilon(t)} UN$$

We consider now the isomorphism of simplicial presheaves: Given any simplex s in $\mathbf{hom}_r^{A\text{-mod}}(A \wedge X, N)$

$$A \wedge X \wedge (\Delta_W^n)_+ \xrightarrow{s} N$$

use the adjunction between $A \wedge -$ and U to get the associated simplex $\eta(s)$ in $\mathbf{hom}_r^\Sigma(X, UN)$:

$$X \wedge (\Delta_W^n)_+ \xrightarrow{\eta(s)} UN \quad \square$$

If A is a commutative ring spectrum, then $A\text{-mod}$ is a closed symmetric monoidal category, where the monoidal structure is induced by the one existing on $\mathbf{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$. Namely,

$$\begin{aligned} - \wedge_A - : A\text{-mod} \times A\text{-mod} &\longrightarrow A\text{-mod} \\ (M, N) &\longmapsto M \wedge_A N \end{aligned}$$

$$\begin{aligned} \mathbf{Hom}_{A\text{-mod}}(-, -) : (A\text{-mod})^{\text{op}} \times A\text{-mod} &\longrightarrow A\text{-mod} \\ (M, N) &\longmapsto \mathbf{Hom}_{A\text{-mod}}(M, N) \end{aligned}$$

where $M \wedge_A N$ is defined as the colimit of the following diagram

$$A \wedge M \wedge N \xrightarrow{\mu_M \wedge \text{id}} M \wedge N$$

with the bottom arrow given by the following composition

$$A \wedge M \wedge N \xrightarrow{t \wedge \text{id}} M \wedge A \wedge N \xrightarrow{\text{id} \wedge \mu_N} M \wedge N$$

and $\mathbf{Hom}_{A\text{-mod}}(M, N)$ is defined as the limit of the following diagram

$$\mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, N) \begin{array}{c} \xrightarrow{(\mu_M)^*} \\ \xrightarrow{(\mu_N)_*} \end{array} \mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(A \wedge M, N)$$

If A is not commutative, the bifunctor $- \wedge_A -$ defines instead an adjunction of two variables from the categories of right and left A -modules to the category of symmetric T -spectra:

$$\begin{aligned} - \wedge_A - : A\text{-mod}_r \times A\text{-mod}_l &\longrightarrow \mathbf{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \\ (M, N) &\longmapsto M \wedge_A N \end{aligned}$$

given a right A -module M , the right adjoint to

$$M \wedge_A - : A\text{-mod}_l \rightarrow \mathbf{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$$

is given by

$$\mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, -) : \mathbf{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \rightarrow A\text{-mod}_l$$

where $\mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, Z)$ has a structure of left A -module

$$\mu : A \wedge \mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, Z) \rightarrow \mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, Z)$$

defined as the adjoint of the following composition

$$M \wedge A \wedge \mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, Z) \xrightarrow{\mu_M \wedge \text{id}} M \wedge \mathbf{Hom}_{\mathbf{Spt}_T^\Sigma}(M, Z) \xrightarrow{\epsilon_Z} Z$$

where μ_M denotes the map defining the right A -module structure for M and ϵ denotes the counit of the adjunction between $M \wedge -$ and $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(M, -)$. The construction of the remaining adjoint is similar.

Definition 2.8.3. — *We will say that A is a cofibrant ring spectrum if A is a ring spectrum with unit in $\mathrm{Spt}_T^\Sigma(\mathcal{M}|_S)_{\mathrm{Nis}}$ which is cofibrant in the motivic symmetric stable model structure $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 2.6.23).*

Theorem 2.8.4. — *Let A be a cofibrant ring spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Then the adjunction (see Proposition 2.8.1):*

$$(A \wedge -, U, \varphi) : \mathrm{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow A\text{-mod}$$

induces a model structure for the category $A\text{-mod}$ of A -modules, i.e. a map f in $A\text{-mod}$ is a fibration or a weak equivalence if and only if $U(f)$ is a fibration or a weak equivalence in $\mathrm{Spt}_T^\Sigma \mathcal{M}_$ (see Theorem 2.6.23).*

This model structure will be called motivic stable, and the category of A -modules equipped with the motivic stable model structure will be denoted by $A\text{-mod}(\mathcal{M}_)$.*

Proof. — We have that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a cellular model category (see Theorem 2.7.4), i.e. in particular a cofibrantly generated model category, and a monoidal model category in the sense of Hovey (see Proposition 2.6.28). Therefore, since A is cofibrant the result follows from [9, Corollary 2.2]. \square

Remark 2.8.5. — *Notice that Proposition 4.19 in [14] implies that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ satisfies the monoid axiom of Schwede-Shipley [23, definition 3.3] and hence by [23, Theorem 4.1(1)] Theorem 2.8.4 holds even if the ring spectrum A is not cofibrant. However, the reason for restricting our discussion to cofibrant ring spectra in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is that our main results (Theorems 3.6.19 and 3.6.20) only hold under an even stronger condition, namely A being cofibrant in $R_{\mathrm{C}_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.3.50). This turns out to be essential since we need an analogue of Proposition 2.8.9 for $R_{\mathrm{C}_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ (see Proposition 3.5.20 and Theorems 3.5.22, 3.5.70(4)).*

Lemma 2.8.6. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Then the adjunction:*

$$(A' \wedge_A -, U, \varphi) : A\text{-mod}(\mathcal{M}_*) \rightarrow A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction. Furthermore, a map $w : M \rightarrow M'$ in $A'\text{-mod}(\mathcal{M}_)$ is a weak equivalence if and only if Uw is a weak equivalence in $A\text{-mod}(\mathcal{M}_*)$.*

Proof. — It is clear that $U : A'\text{-mod}(\mathcal{M}_*) \rightarrow A\text{-mod}(\mathcal{M}_*)$ is a right Quillen functor, since the fibrations (respectively, trivial fibrations) for both model structures

are detected in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Finally, the claim related to the weak equivalences follows immediately from Theorem 2.8.4. \square

Proposition 2.8.7. — *Let $f : A \rightarrow A'$ be a weak equivalence between cofibrant ring spectra in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Then f induces a Quillen equivalence between the motivic stable model structures of A and A' modules:*

$$(A' \wedge_A -, U, \varphi) : A\text{-mod}(\mathcal{M}_*) \longrightarrow A'\text{-mod}(\mathcal{M}_*)$$

Proof. — It follows immediately from Theorem 2.4 in [9] together with the fact that the domains of the generating cofibrations for $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ are cofibrant (see Theorem 2.7.4). \square

Proposition 2.8.8. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Then the adjunction*

$$(A' \wedge_A -, U, \varphi) : A\text{-mod}(\mathcal{M}_*) \longrightarrow A'\text{-mod}(\mathcal{M}_*)$$

is enriched in the categories of simplicial sets and pointed simplicial presheaves on $(\mathcal{M}|_S)_{\mathrm{Nis}}$, i.e. for every A -module M and for every A' -module N we have the following natural isomorphisms:

$$\mathrm{Map}_{A'\text{-mod}}(A' \wedge_A M, N) \xrightarrow[\cong]{\epsilon} \mathrm{Map}_{A\text{-mod}}(M, UN)$$

$$\mathrm{hom}_r^{A'\text{-mod}}(A' \wedge_A M, N) \xrightarrow[\cong]{\eta} \mathrm{hom}_r^{A\text{-mod}}(M, UN)$$

Proof. — The proof is exactly the same as the one in Proposition 2.8.2. \square

Proposition 2.8.9. — *Let A be a cofibrant ring spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, and let i be a cofibration in $A\text{-mod}(\mathcal{M}_*)$. Then $U(i)$ is also a cofibration in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — Theorem 2.7.4 implies in particular that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a cofibrantly generated model category. Therefore the proposition follows directly from [9, Corollary 2.2]. \square

Proposition 2.8.10. — *$A\text{-mod}(\mathcal{M}_*)$ is a:*

1. *proper model category.*
2. *simplicial model category.*
3. *\mathcal{M}_* -model category (see definition 1.7.12).*

Proof. — (1): It follows directly from the fact that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a proper model category (see Theorem 2.6.23), together with Theorem 2.8.4 and Proposition 2.8.9.

(2): Since the cotensor objects N^K for the simplicial structure are identical in $A\text{-mod}(\mathcal{M}_*)$ and $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, the results follows from theorem 2.8.4 and theorem 2.6.23 which implies in particular that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category.

(3): Since the cotensor objects $\Omega_D N$ for the \mathcal{M}_* -action are identical in $A\text{-mod}(\mathcal{M}_*)$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$, the results follows from the fact that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a \mathcal{M}_* -model category (see Proposition 2.6.25) together with Theorem 2.8.4. \square

Theorem 2.8.11. — *$A\text{-mod}(\mathcal{M}_*)$ is a cellular model category with the following sets $I_{A\text{-mod}}$, $J_{A\text{-mod}}$ of generating cofibrations and trivial cofibrations respectively (see Theorem 2.7.4):*

$$\begin{aligned} I_{A\text{-mod}} &= A \wedge I_\Sigma^T \\ &= \bigcup_{k \geq 0} \{\text{id} \wedge i : A \wedge F_k^\Sigma(Y_+) \rightarrow A \wedge F_k^\Sigma((\Delta_n^U)_+) \mid U \in (\mathcal{A}m|_S), n \geq 0\} \\ J_{A\text{-mod}} &= A \wedge J_\Sigma^T = \{\text{id} \wedge j : A \wedge X \rightarrow A \wedge Y\} \end{aligned}$$

where $j : X \rightarrow Y$ satisfies the following conditions:

1. j is an inclusion of I_Σ^T -complexes.
2. j is a stable weak equivalence of symmetric T -spectra.
3. the size of Y as an I_Σ^T -complex is less than κ , where κ is the regular cardinal described by Hirschhorn in [6, definition 4.5.3].

Proof. — We have to check that the conditions (1)-(4) of definition 1.3.12 hold.

By construction (see Theorem 2.8.4) it is clear that $I_{A\text{-mod}}$ and $J_{A\text{-mod}}$ are generators for the model structure on $A\text{-mod}(\mathcal{M}_*)$. This takes care of (1).

By adjointness, to prove (2) it suffices to show that the domains and codomains of I_Σ^T are compact relative to $I_{A\text{-mod}}$. However, the domains and codomains of I_Σ^T are cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is in particular a cellular model category (see Theorem 2.7.4). Hence [6, Corollary 12.3.4] implies that the domains and codomains of I_Σ^T are compact with respect to the class of cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Finally, Proposition 2.8.9 implies that all the maps in $I_{A\text{-mod}}$ are cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, the domains and codomains of I_Σ^T are compact with respect to $I_{A\text{-mod}}$, as we wanted.

Again by adjointness, to prove (3) it suffices to show that the domains of J_Σ^T are small relative to $I_{A\text{-mod}}$. But Proposition 2.8.9 implies that all the maps in $I_{A\text{-mod}}$ are cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore by [6, Theorem 12.4.4] we have that the domains of J_Σ^T are small relative to $I_{A\text{-mod}}$, since $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a cellular model category (see Theorem 2.7.4).

Finally, Proposition 2.8.9 implies that the cofibrations in $A\text{-mod}(\mathcal{M}_*)$ are in particular cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is a cellular model category (see Theorem 2.7.4). Therefore the cofibrations in $A\text{-mod}(\mathcal{M}_*)$ are effective monomorphisms in $\text{Spt}_T^\Sigma \mathcal{M}_*$. This takes care of (4) since the limits and colimits in $A\text{-mod}(\mathcal{M}_*)$ are computed in $\text{Spt}_T^\Sigma \mathcal{M}_*$. \square

Theorem 2.8.12. — *Let $T = S^1 \wedge \mathbb{G}_m \in \mathcal{M}_*$. Then the adjunction:*

$$(- \wedge T, \Omega_T, \varphi) : A\text{-mod}(\mathcal{M}_*) \longrightarrow A\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence.

Proof. — Every pointed simplicial presheaf in \mathcal{M}_* is cofibrant, therefore proposition 2.8.10(3) implies that $- \wedge T : A\text{-mod}(\mathcal{M}_*) \rightarrow A\text{-mod}(\mathcal{M}_*)$ is a left Quillen functor.

Let η, ϵ denote the unit and counit of the adjunction $(- \wedge T, \Omega_T, \varphi)$. By Proposition 1.3.13 in [10], it suffices to check that the following conditions hold:

1. For every cofibrant A -module M in $A\text{-mod}(\mathcal{M}_*)$, the following composition

$$M \xrightarrow{\eta_M} \Omega_T(T \wedge M) \xrightarrow{\Omega_T(R^{T \wedge M})} \Omega_T R(T \wedge M)$$

is a weak equivalence in $A\text{-mod}(\mathcal{M}_*)$, where R denotes a fibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$.

2. For every fibrant A -module M in $A\text{-mod}(\mathcal{M}_*)$, the following composition

$$T \wedge Q(\Omega_T M) \xrightarrow{\text{id} \wedge Q^{\Omega_T M}} T \wedge (\Omega_T M) \xrightarrow{\epsilon_M} M$$

is a weak equivalence in $A\text{-mod}(\mathcal{M}_*)$, where Q denotes a cofibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$.

(1): By Proposition 2.8.9 we have that M is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Thus the result follows immediately from theorems 2.6.27(1) and 2.8.4.

(2): It follows directly from Theorem 2.8.4, Proposition 2.8.9 and theorem 2.6.27(2). □

Proposition 2.8.13. — *Let A be a cofibrant commutative ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then $A\text{-mod}(\mathcal{M}_*)$ is a symmetric monoidal model category in the sense of Hovey (see definition 1.7.7).*

Proof. — This follows directly from Theorem 2.7.4, Proposition 2.6.28 and [9, Proposition 2.8(2)]. □

If A is not commutative, then we get a weaker version of the previous proposition.

Proposition 2.8.14. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then $- \wedge_A -$ defines a Quillen adjunction of two variables (see definition 1.7.4) from the motivic model structure for right and left A -modules to the motivic symmetric stable model structure:*

$$- \wedge_A - : A\text{-mod}(\mathcal{M}_*)_r \times A\text{-mod}(\mathcal{M}_*)_l \rightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

Proof. — We need to show that given a cofibration $i : M \rightarrow M'$ in $A\text{-mod}(\mathcal{M}_*)_r$ and a cofibration $j : N \rightarrow N'$ in $A\text{-mod}(\mathcal{M}_*)_l$, the induced map

$$i \square_A j : M \wedge_A N' \coprod_{M \wedge_A N} M' \wedge_A N \rightarrow M' \wedge_A N'$$

is a cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is trivial if either i or j are trivial.

Clearly it is enough to do it for the generating cofibrations and trivial cofibrations in $A\text{-mod}(\mathcal{M}_*)$ (see Lemma 3.5 in [23]). Theorem 2.8.11 implies that $A \wedge I_\Sigma^T$, $A \wedge J_\Sigma^T$ ($I_\Sigma^T \wedge A$, $J_\Sigma^T \wedge A$) are the sets of generating cofibrations and trivial cofibrations for $A\text{-mod}(\mathcal{M}_*)_l$ (respectively for $A\text{-mod}(\mathcal{M}_*)_r$), where I_Σ^T , J_Σ^T denote the sets of generating cofibrations and trivial cofibrations for $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Now

$$I_\Sigma^T \wedge A \square_A A \wedge I_\Sigma^T = I_\Sigma^T \square_A A \wedge I_\Sigma^T \subseteq I_\Sigma^T \square I_\Sigma^T\text{-cof} \subseteq I_\Sigma^T\text{-cof}$$

where the equality follows by definition, the first inclusion follows from the fact that A is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a symmetric monoidal model category (see Proposition 2.6.28), and the last inclusion follows from the fact that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a symmetric monoidal model category. A similar argument shows that

$$J_\Sigma^T \wedge A \square_A A \wedge I_\Sigma^T = J_\Sigma^T \square_A A \wedge I_\Sigma^T \subseteq J_\Sigma^T \square I_\Sigma^T\text{-cof} \subseteq J_\Sigma^T\text{-cof}$$

and finally, the remaining case follows from this by symmetry. \square

In the rest of this section, we assume that A is a commutative ring spectrum with unit in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$. The category of A -algebras is a symmetric monoidal category, where the monoidal structure coincides with the one existing on $A\text{-mod}$. Namely,

$$\begin{aligned} - \wedge_A - : A\text{-alg} \times A\text{-alg} &\longrightarrow A\text{-alg} \\ (C, D) &\longmapsto C \wedge_A D \end{aligned}$$

However, the category of A -algebras is not a closed symmetric monoidal category, i.e. the functor $C \wedge_A - : A\text{-alg} \rightarrow A\text{-alg}$ does not have in general a right adjoint.

Proposition 2.8.15. — *We have the following adjunction between the categories of symmetric T -spectra and A -algebras:*

$$(T, U, \varphi) : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \longrightarrow A\text{-alg}$$

where $U(N) = N$ after forgetting the A -algebra structure, and $T(X) = A \wedge \coprod_{n \geq 0} X^{\wedge n}$ has a structure of A -algebra induced by concatenation together with the ring structure on A .

Proof. — The unit η of the adjunction is

$$X \cong \mathbf{1} \wedge X \xrightarrow{u_A \wedge \text{id}} A \wedge X \xleftarrow{i_{A \wedge X}} U(T(X)) = A \wedge \coprod_{n \geq 0} X^{\wedge n}$$

where u_A is the unit of A . On the other hand, the counit δ of the adjunction is induced by iterating the map that induces the A -algebra structure of B

$$A \wedge \coprod B^{\wedge k} \longrightarrow A \wedge \coprod_{n \geq 0} B^{\wedge n} \xrightarrow{\delta_B} B \quad \square$$

Lemma 2.8.16. — *Let I_Σ^T, J_Σ^T be the sets of generating cofibrations and trivial cofibrations for the motivic symmetric stable model structure $\text{Spt}_T^\Sigma \mathcal{M}_*$ of symmetric T -spectra (see Theorem 2.7.4). Then:*

1. *The domains of I_Σ^T are small relative to $X \wedge I_\Sigma^T$ -cell for every symmetric T -spectrum X .*
2. *The domains of J_Σ^T are small relative to $X \wedge J_\Sigma^T$ -cell for every symmetric T -spectrum X .*
3. *The maps of $X \wedge J_\Sigma^T$ -cell are weak equivalences for every symmetric T -spectrum X .*

Proof. — Let I, J denote the sets of generating cofibrations and trivial cofibrations for the category of symmetric T -spectra $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$ equipped with the injective model structure (see Theorem 2.6.20), where the cofibrations and the weak equivalences are defined levelwise. Hence every symmetric T -spectrum is cofibrant in the injective model structure. On the other hand, theorem 2.6.20 implies that the injective model structure is cofibrantly generated and that the codomains of the generating cofibrations I are small relative to I . Thus, applying [6, Corollary 11.2.4] we get that every symmetric T -spectrum is small with respect to the class of level cofibrations.

(1): It suffices to show that every map in $X \wedge I_\Sigma^T$ is a level cofibration. But this follows directly from [14, Proposition 4.19], since every symmetric T -spectrum X is cofibrant in the injective model structure.

(2): It suffices to show that every map in $X \wedge J_\Sigma^T$ is a level cofibration. But this is a consequence of [14, Proposition 4.19], since every symmetric T -spectrum X is cofibrant in the injective model structure.

(3): This follows immediately from [14, Proposition 4.19], since every symmetric T -spectrum is cofibrant in the injective model structure. \square

Theorem 2.8.17. — *Let A be a cofibrant commutative ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then the adjunction (see Proposition 2.8.15):*

$$(T, U, \varphi) : \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow A\text{-alg}$$

induces a model structure for the category $A\text{-alg}$ of A -algebras, i.e. a map f in $A\text{-alg}$ is a fibration or a weak equivalence if and only if $U(f)$ is a fibration or a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_$ (see Theorem 2.6.23).*

This model structure will be called *motivic*, and the category of A -algebras equipped with the motivic model structure will be denoted by $A\text{-alg}(\mathcal{M}_*)$. Furthermore, $A\text{-alg}(\mathcal{M}_*)$ is a cofibrantly generated model category with the following sets $I_{A\text{-alg}}$, $J_{A\text{-alg}}$ of generating cofibrations and trivial cofibrations respectively (see Theorem 2.7.4):

$$\begin{aligned} I_{A\text{-alg}} &= T(I_\Sigma^T) \\ &= \bigcup_{k \geq 0} \{T(i) : T(F_k^\Sigma(Y_+)) \rightarrow T(F_k^\Sigma((\Delta_n^U)_+)) \mid U \in (\mathcal{M}_S), n \geq 0\} \\ J_{A\text{-alg}} &= T(J_\Sigma^T) = \{T(j) : T(X) \rightarrow T(Y)\} \end{aligned}$$

where $j : X \rightarrow Y$ satisfies the following conditions:

1. j is an inclusion of I_Σ^T -complexes.
2. j is a stable weak equivalence of symmetric T -spectra.
3. the size of Y as an I_Σ^T -complex is less than κ , where κ is the regular cardinal described by Hirschhorn in [6, definition 4.5.3].

Proof. — Theorem 2.7.4 implies that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is in particular a cofibrantly generated model category, and by Proposition 2.6.28 we have that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a symmetric monoidal model category. Therefore the result follows immediately from Lemma 2.8.16 and [9, Theorem 3.1]. \square

Proposition 2.8.18. — *Let A be a cofibrant commutative ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and let $f : B \rightarrow B'$ be a map of A -algebras which is a cofibration in the motivic model category $A\text{-alg}(\mathcal{M}_*)$ of A -algebras. If B is cofibrant in $A\text{-mod}(\mathcal{M}_*)$, then Uf is a cofibration in $A\text{-mod}(\mathcal{M}_*)$.*

Proof. — It follows directly from Lemma 6.2 in [23]. \square

CHAPTER 3

MODEL STRUCTURES FOR THE SLICE FILTRATION

This chapter contains our main results. In Section 3.1, we recall Voevodsky's construction of the slice filtration in the context of simplicial presheaves. In Section 3.2, we apply Hirschhorn's localization techniques to the Morel-Voevodsky stable model structure $\mathrm{Spt}_T \mathcal{M}_*$, in order to construct three new families of model structures, namely $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T \mathcal{M}_*$, $L_{<q} \mathrm{Spt}_T \mathcal{M}_*$ and $S^q \mathrm{Spt}_T \mathcal{M}_*$. These model structures will provide a lifting of Voevodsky's slice filtration to the model category setting. Furthermore, we will also get a simple description for the exact functors f_q ($(q-1)$ -connective cover) and s_q (q -slice) defined in Section 3.1, in terms of a suitable composition of cofibrant and fibrant replacement functors.

In Section 3.3, we promote the model structures introduced in section 3.2 to the setting of symmetric T -spectra. These new model structures will be denoted by $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$, $L_{<q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ and $S^q \mathrm{Spt}_T^\Sigma \mathcal{M}_*$. We will prove that the Quillen adjunction given by the symmetrization and the forgetful functors descends to a Quillen equivalence for these three new model structures. As a consequence we will see that the model categories $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$, $L_{<q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ and $S^q \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ provide a lifting for Voevodsky's slice filtration and give an alternative description for the functors f_q and s_q . The great technical advantage of these model structures relies on the fact that the underlying category is symmetric monoidal. Hence, we have a natural framework to describe the multiplicative properties of the slice filtration.

In Section 3.4, we will show that the slice filtration is compatible with the smash product of symmetric T -spectra.

In Section 3.5, we will promote the model structures constructed in Section 3.3 to the category of A -modules, where A is a cofibrant ring spectrum with unit in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. We will denote these new model structures by $R_{C_{\mathrm{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, $L_{<q} A\text{-mod}(\mathcal{M}_*)$ and $S^q A\text{-mod}(\mathcal{M}_*)$. These new model structures will give an analogue of the slice filtration for the motivic stable homotopy category of A -modules. We will see that when one imposes some natural additional conditions on the ring spectrum A , the free

A -module functor $(A \wedge -)$ induces a strict compatibility between the slice filtration in the categories of symmetric T -spectra and A -modules.

In Section 3.6 we will use all our previous results to show that the smash product of symmetric T -spectra induces natural pairings (in the motivic stable homotopy category) for the functors f_q and s_q . We will see that for every symmetric T -spectrum X , and for every $q \in \mathbb{Z}$:

1. $f_q^\Sigma X$ is a module (up to homotopy) over the (-1) -connective cover of the sphere spectrum $f_0^\Sigma \mathbf{1}$.
2. $s_q^\Sigma X$ is a module (up to homotopy) over the zero slice of the sphere spectrum $s_0^\Sigma \mathbf{1}$.

We will verify that the smash product of symmetric T -spectra induces natural external pairings in the motivic Atiyah-Hirzebruch spectral sequence (see definition 3.6.15):

$$\begin{array}{ccc} E_r^{p,q}(Y; X) \otimes E_r^{p',q'}(Y'; X') & \longrightarrow & E_r^{p+p',q+q'}(Y \wedge Y'; X \wedge X') \\ (\alpha, \beta) & \longmapsto & \alpha \smile \beta \end{array}$$

We will also see that for an A -module M , with A a cofibrant ring spectrum with unit in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, which also satisfies the additional hypothesis that are required in section 3.5:

1. $f_q^\Sigma M$ is again an A -module in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ (not just up to homotopy, but in a very strict sense).
2. $s_q^\Sigma X$ is again an A -module in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ (not just up to homotopy, but in a very strict sense).

Then we will prove that if the ring spectrum A and its unit map $u : \mathbf{1} \rightarrow A$ satisfy the conditions that are required in Section 3.5, the free A -module functor $A \wedge -$ induces for every $q \in \mathbb{Z}$ and for every symmetric T -spectrum X , a natural structure of A -module (in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, i.e. not just up to homotopy, but in a very strict sense) on its q -slice $s_q^\Sigma(X)$.

Finally, we will be able to prove a conjecture of Voevodsky (see [16, Corollary 11.1.3], [25]), which says that if the base scheme S is a perfect field, then for every $q \in \mathbb{Z}$ and for every symmetric T -spectrum X , its q -slice $s_q^\Sigma(X)$ is naturally equipped with a module structure in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ over the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$. If we restrict the field even further, considering a field of characteristic zero, then as a consequence we will prove that all the slices $s_q^\Sigma X$ are big motives in the sense of Voevodsky.

3.1. The Slice Filtration

Let $\mathcal{H}(S)$ denote the homotopy category associated to $\text{Spt}_T\mathcal{M}_*$. We call $\mathcal{H}(S)$ the *motivic stable homotopy category*. We will denote by $[-, -]_{\text{Spt}}$ the set of maps between two objects in $\mathcal{H}(S)$, and $q \in \mathbb{Z}$ will be an arbitrary integer. In [25] Voevodsky constructs the *slice filtration* on motivic stable homotopy theory, using sheaves on the Nisnevich site $(\mathcal{M}|_S)_{\text{Nis}}$ instead of simplicial presheaves as the underlying category. In this section we recall his construction in the context of simplicial presheaves.

Definition 3.1.1. — Let Q_s denote a cofibrant replacement functor in $\text{Spt}_T\mathcal{M}_*$; such that for every T -spectrum X , the natural map:

$$Q_s X \xrightarrow{Q_s^X} X$$

is a trivial fibration in $\text{Spt}_T\mathcal{M}_*$.

Proposition 3.1.2. — The motivic stable homotopy category $\mathcal{H}(S)$ has a structure of triangulated category defined as follows:

1. The suspension functor $\Sigma_T^{1,0}$ is given by

$$\begin{aligned} - \wedge S^1 : \mathcal{H}(S) &\longrightarrow \mathcal{H}(S) \\ X &\longmapsto Q_s X \wedge S^1 \end{aligned}$$

2. The distinguished triangles are isomorphic to triangles of the form

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \Sigma_T^{1,0} A$$

where i is a cofibration in $\text{Spt}_T\mathcal{M}_*$, and C is the homotopy cofibre of i .

Proof. — Theorem 2.4.16 implies in particular that $\text{Spt}_T\mathcal{M}_*$ is a pointed simplicial model category, and Theorem 2.4.29 implies that the adjunction:

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : \text{Spt}_T\mathcal{M}_* \longrightarrow \text{Spt}_T\mathcal{M}_*$$

is a Quillen equivalence. The result now follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII] (see [10, Proposition 7.1.6]). \square

Note 3.1.3. — For $n \in \mathbb{Z}$, $\Sigma_T^{n,0}$ will denote the n^{th} iteration of the suspension functor if $n \geq 0$ ($\Sigma_T^{0,0} = \text{id}$) or the $(-n)^{\text{th}}$ iteration of the desuspension functor for $n < 0$.

Lemma 3.1.4. — Let $X \in \mathcal{M}_*$ be a pointed simplicial presheaf which is compact in the sense of Jardine (see definition 2.3.10), and let $F_n(X)$ be the T -spectrum constructed

in definition 2.4.4. Consider an arbitrary collection of T -spectra $\{Z_i\}_{i \in I}$ indexed by a set I . Then

$$[F_n(X), \coprod_{i \in I} Z_i]_{\text{Spt}} \cong \coprod_{i \in I} [F_n(X), Z_i]_{\text{Spt}}$$

Proof. — If the indexing set I is finite then the claim holds trivially since $\mathcal{H}(S)$ is a triangulated category and therefore finite coproducts and finite products are canonically isomorphic. Thus we can assume that the indexing set I is infinite.

Choosing a well ordering for the set I there exists a unique ordinal μ which is isomorphic to the ordered set I (see [6, Proposition 10.2.7]). We will prove the lemma by transfinite induction, so assume that for every ordinal $\lambda < \mu$, $F_n(X)$ commutes in $\mathcal{H}(S)$ with coproducts indexed by λ . If $\mu = \lambda + 1$, i.e. if μ is the successor of λ , then

$$\coprod_{\alpha < \lambda+1} Z_\alpha \cong \left(\coprod_{\alpha < \lambda} Z_\alpha \right) \coprod Z_\lambda$$

Therefore

$$[F_n(X), \coprod_{\alpha < \lambda+1} Z_\alpha]_{\text{Spt}} \xrightarrow{\cong} ([F_n(X), \coprod_{\alpha < \lambda} Z_\alpha]_{\text{Spt}}) \coprod ([F_n(X), Z_\lambda]_{\text{Spt}})$$

but by the induction hypothesis

$$[F_n(X), \coprod_{\alpha < \lambda} Z_\alpha]_{\text{Spt}} \cong \coprod_{\alpha < \lambda} [F_n(X), Z_\alpha]_{\text{Spt}}$$

thus

$$[F_n(X), \coprod_{\alpha < \lambda+1} Z_\alpha]_{\text{Spt}} \cong \coprod_{\alpha < \lambda+1} [F_n(X), Z_\alpha]_{\text{Spt}}$$

as we wanted.

It remains to consider the case when μ is a limit ordinal. In this case Proposition 10.2.7 in [6] implies that we can recover the map $* \rightarrow \coprod_{\alpha < \mu} Z_\alpha$ as the transfinite composition of a μ -sequence:

$$A_0 \rightarrow A_1 \rightarrow \dots \rightarrow A_\beta \rightarrow \dots \quad (\beta < \mu)$$

where $A_0 = *$, $A_\beta = \coprod_{\alpha < \beta} Z_\alpha$, and the maps in the sequence are the obvious ones.

In particular we have that $\coprod_{\alpha < \mu} Z_\alpha \cong \varinjlim_{\beta < \mu} A_\beta$.

Since X is compact, Proposition 2.4.33 implies that:

$$[F_n(X), \varinjlim_{\beta < \mu} A_\beta]_{\text{Spt}} \cong \varinjlim_{\beta < \mu} [F_n(X), A_\beta]_{\text{Spt}}$$

Now using the induction hypothesis we have:

$$[F_n(X), A_\beta]_{\text{Spt}} \cong \coprod_{\alpha < \beta} [F_n(X), Z_\alpha]_{\text{Spt}}$$

and using Proposition 10.2.7 in [6] again, we get:

$$\varinjlim_{\beta < \mu} \coprod_{\alpha < \beta} [F_n(X), Z_\alpha]_{\text{Spt}} \cong \coprod_{\alpha < \mu} [F_n(X), Z_\alpha]_{\text{Spt}}$$

thus

$$[F_n(X), \coprod_{\alpha < \mu} Z_\alpha]_{\text{Spt}} \cong \coprod_{\alpha < \mu} [F_n(X), Z_\alpha]_{\text{Spt}}$$

as we wanted. □

Proposition 3.1.5. — *The motivic stable homotopy category $\mathcal{A}\mathcal{H}(S)$ is a compactly generated triangulated category in the sense of Neeman (see [19, definition 1.7]). The set of compact generators is given by (see definition 2.4.4):*

$$C = \bigcup_{n,r,s \geq 0} \bigcup_{U \in (\mathcal{A}m|_S)} F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

i.e. the smallest triangulated subcategory of $\mathcal{A}\mathcal{H}(S)$ closed under small coproducts and containing all the objects in C coincides with $\mathcal{A}\mathcal{H}(S)$.

Proof. — Since $\mathcal{A}\mathcal{H}(S)$ is closed under small coproducts, we just need to prove the following two claims:

1. For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$; mapping out of $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ commutes with coproducts in $\mathcal{A}\mathcal{H}(S)$, i.e. given a family of T -spectra $\{X_i\}_{i \in I}$ indexed by a set I we have:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \coprod_{i \in I} X_i]_{\text{Spt}} \cong \coprod_{i \in I} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X_i]_{\text{Spt}}$$

2. If a T spectrum X has the following property: $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} = 0$ for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$, then $X \cong *$ in $\mathcal{A}\mathcal{H}(S)$.

(1): It follows immediately from Lemma 3.1.4 since we know by Proposition 2.4.1 that the pointed simplicial presheaves $S^r \wedge \mathbb{G}_m^s \wedge U_+$ are all compact in the sense of Jardine.

(2): Consider the canonical map $X \rightarrow *$ in $\text{Spt}_T \mathcal{M}_*$. Corollary 2.4.28 together with our hypotheses imply that $X \rightarrow *$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, therefore $X \cong *$ in $\mathcal{A}\mathcal{H}(S)$ as we wanted. □

Corollary 3.1.6. — *Let $f : X \rightarrow Y$ be a map in $\mathcal{A}\mathcal{H}(S)$. Then f is an isomorphism if and only if f induces an isomorphism of abelian groups:*

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} \xrightarrow{f_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$.

Proof. — (\Rightarrow): If f is an isomorphism in $\mathcal{A}\mathcal{H}(S)$ it is clear that the induced maps f_* are isomorphisms of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$.

(\Leftarrow): Complete f to a distinguished triangle in $\mathcal{A}\mathcal{H}(S)$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_T^{1,0} X$$

Then f is an isomorphism if and only if $Z \cong *$ in $\mathcal{A}(S)$.

Now since the functor $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), -]_{\text{Spt}}$ is homological, we get the following long exact sequence of abelian groups:

$$\begin{array}{ccc}
 \vdots & & \\
 \downarrow & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} & & \\
 \downarrow f_* & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}} & & \\
 \downarrow g_* & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} & & \\
 \downarrow h_* & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} X]_{\text{Spt}} & \xleftarrow{\cong_{\Sigma_T^{1,0}}} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), X]_{\text{Spt}} \\
 \downarrow \Sigma_T^{1,0} f_* & & \downarrow f_* \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} Y]_{\text{Spt}} & \xleftarrow{\cong_{\Sigma_T^{1,0}}} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Y]_{\text{Spt}} \\
 \downarrow & & \\
 \vdots & &
 \end{array}$$

But by hypothesis all the maps f_* are isomorphisms, therefore $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} = 0$ for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$. Since $\mathcal{A}(S)$ is a compactly generated triangulated category (see Proposition 3.1.5) with set of compact generators C , we have that $Z \cong *$. This implies that f is an isomorphism, as we wanted. \square

Definition 3.1.7 (Voevodsky, cf. [25]). — We define the effective motivic stable homotopy category $\mathcal{A}^{\text{eff}}(S) \subseteq \mathcal{A}(S)$ as the smallest triangulated full subcategory of $\mathcal{A}(S)$ that is closed under small coproducts and contains

$$C_{\text{eff}} = \bigcup_{n,r,s \geq 0; s-n \geq 0} \bigcup_{U \in (\mathcal{A}m|_S)} F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

Definition 3.1.8 (Voevodsky, cf. [25]). — For $q \in \mathbb{Z}$, we define $\Sigma_T^q \mathcal{A}^{\text{eff}}(S) \subseteq \mathcal{A}(S)$ as the smallest triangulated full subcategory of $\mathcal{A}(S)$ that is closed under small coproducts and contains

$$C_{\text{eff}}^q = \bigcup_{n,r,s \geq 0; s-n \geq q} \bigcup_{U \in (\mathcal{A}m|_S)} F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

Definition 3.1.9 (Voevodsky, cf. [25]). — *The collection of triangulated subcategories $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ gives a filtration on $\mathcal{H}(S)$ which is called the slice filtration, i.e. we have an inductive system of full embeddings*

$$\dots \subseteq \Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S) \subseteq \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \subseteq \Sigma_T^{q-1} \mathcal{H}^{\text{eff}}(S) \subseteq \dots$$

and Proposition 3.1.5 implies that the smallest triangulated subcategory of $\mathcal{H}(S)$ containing $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ for all $q \in \mathbb{Z}$ and closed under small coproducts coincides with $\mathcal{H}(S)$.

Proposition 3.1.10. — *The triangulated category $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is compactly generated in the sense of Neeman, where the set of compact generators is*

$$C_{\text{eff}}^q = \bigcup_{n,r,s \geq 0; s-n \geq q} \bigcup_{U \in (\mathcal{M}|_S)} F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

Proof. — By construction $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is closed under small coproducts. Therefore we just need to check the following two properties:

1. For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$; $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ commutes with coproducts in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, i.e. given a family of T -spectra $\{X_i \in \Sigma_T^q \mathcal{H}^{\text{eff}}(S)\}_{i \in I}$ indexed by a set I we have:

$$\begin{array}{c} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \coprod_{i \in I} X_i) \\ \downarrow \cong \\ \coprod_{i \in I} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X_i) \end{array}$$

2. If a T -spectrum $X \in \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ has the following property:

$$\text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X) = 0$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, then $X \cong *$ in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

(1): It follows immediately from proposition 3.1.5 since $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is in particular a full subcategory of $\mathcal{H}(S)$.

(2): The natural map $X \rightarrow *$ is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ if and only if for every $Z \in \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ we get an induced isomorphism of abelian groups

$$\text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(Z, X) \xrightarrow{\cong} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(Z, *) = 0$$

and since $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is a full subcategory of $\mathcal{H}(S)$, this last condition is equivalent to: For every $Z \in \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ we have an induced isomorphism of abelian groups

$$[Z, X]_{\text{Spt}} \xrightarrow{\cong} [Z, *]_{\text{Spt}} = 0$$

Let \mathcal{U}_X be the full subcategory of $\mathcal{H}(S)$ generated by the T -spectra Y satisfying the following property

$$[\Sigma_T^{n,0}Y, X]_{\text{Spt}} \xrightarrow{\cong} [\Sigma_T^{n,0}Y, *]_{\text{Spt}} = 0$$

for all $n \in \mathbb{Z}$. To finish the proof it is enough to show that $\Sigma_T^q \mathcal{H}^{\text{eff}}(S) \subseteq \mathcal{U}_X$, and by construction of $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, it suffices to prove that \mathcal{U}_X is a triangulated subcategory of $\mathcal{H}(S)$ which is closed under small coproducts and contains the objects $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. The claim that \mathcal{U}_X is triangulated follows immediately from the fact that the functor $[-, X]_{\text{Spt}}$ is cohomological. The claim that \mathcal{U}_X is closed under small coproducts follows from the universal property of the coproduct. Finally by hypothesis \mathcal{U}_X contains the generators $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. This finishes the proof. \square

Corollary 3.1.11. — *Let $f : X \rightarrow Y$ be a map in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Then f is an isomorphism if and only if one of the following equivalent conditions holds:*

1. *For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, f induces an isomorphism of abelian groups:*

$$\begin{array}{c} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X) \\ \downarrow f_* \\ \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y) \end{array}$$

2. *For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, f induces an isomorphism of abelian groups:*

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} \xrightarrow{i_q(f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}$$

Proof. — Since by construction $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ contains C_{eff}^q and it is a full subcategory of $\mathcal{H}(S)$, we get immediately that (1) and (2) are equivalent.

We will prove (1). It is clear that if f is an isomorphism then the induced maps f_* considered above are all isomorphisms of abelian groups. Conversely, assume that all the induced maps:

$$\begin{array}{c} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X) \\ \downarrow f_* \\ \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y) \end{array}$$

are isomorphisms for $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Complete the map $f : X \rightarrow Y$ to a distinguished triangle in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_T^{1,0} X$$

then f is an isomorphism if and only if $Z \cong *$ in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Now since the functor $\text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), -)$ is homological, we get the following long exact sequence of abelian groups:

$$\begin{array}{ccc}
 \vdots & & \\
 \downarrow & & \\
 \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X) & & \\
 \downarrow f_* & & \\
 \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y) & & \\
 \downarrow g_* & & \\
 \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) & & \\
 \downarrow h_* & \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), X) & \\
 & \swarrow \Sigma_T^{1,0} & \downarrow f_* \\
 \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} X) & \cong & \\
 \downarrow \Sigma_T^{1,0} f_* & \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Y) & \\
 & \swarrow \Sigma_T^{1,0} & \\
 \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} Y) & \cong & \\
 \downarrow & & \\
 \vdots & &
 \end{array}$$

But by hypothesis all the maps f_* are isomorphisms, therefore

$$\text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) = 0$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Since $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is a compactly generated triangulated category (see Proposition 3.1.10) with set of compact generators C_{eff}^q , we have that $Z \cong *$. This implies that f is an isomorphism, as we wanted. \square

Proposition 3.1.12. — *The inclusion*

$$i_q : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \longrightarrow \mathcal{H}(S)$$

has a right adjoint

$$r_q : \mathcal{H}(S) \longrightarrow \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$$

which is also an exact functor.

Proof. — We have that $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is a compactly generated triangulated category (see Proposition 3.1.10), and it is clear that the inclusion i_q is an exact functor which preserves coproducts. Then the existence of the exact right adjoint r_q follows from Theorem 4.1 in [19]. \square

Remark 3.1.13. — 1. *Since the inclusion $i_q : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \rightarrow \mathcal{H}(S)$ is a full embedding, we have that the unit of the adjunction $\text{id} \xrightarrow{\tau} r_q i_q$ is an isomorphism of functors.*

2. *We define $f_q = i_q r_q$. Then clearly $f_{q+1} f_q = f_{q+1}$ and there exists a canonical natural transformation $f_{q+1} \rightarrow f_q$.*

Proposition 3.1.14. — *Let $g : X \rightarrow Y$ be a map in $\mathcal{H}(S)$. We have that $f_q(g) : f_q X \rightarrow f_q Y$ is an isomorphism in $\mathcal{H}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ the induced map:*

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} \xrightarrow[\cong]{g_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}$$

is an isomorphism of abelian groups.

Proof. — We have that $f_q = i_q r_q$, where $i_q : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \rightarrow \mathcal{H}(S)$ is a full embedding. Therefore, $f_q(g)$ is an isomorphism in $\mathcal{H}(S)$ if and only if $r_q(g)$ is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

Hence, Corollary 3.1.11 implies that $f_q(g)$ is an isomorphism if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ the induced map:

$$\begin{array}{c} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q X) \\ \downarrow r_q(g)_* \\ \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q Y) \end{array}$$

is an isomorphism. Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Finally since i_q, r_q are adjoint functors and $C_{\text{eff}}^q \subseteq \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, we have the following commutative diagram, where

the vertical arrows are all isomorphisms:

$$\begin{array}{ccc}
 \mathrm{Hom}_{\Sigma_T^q \mathcal{H}^{\mathrm{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q X) & & \\
 \downarrow \cong & \searrow r_q(g)_* & \\
 & & \mathrm{Hom}_{\Sigma_T^q \mathcal{H}^{\mathrm{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q Y) \\
 & & \downarrow \cong \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\mathrm{Spt}} & \xrightarrow{g_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\mathrm{Spt}}
 \end{array}$$

Therefore, $f_q(g)$ is an isomorphism if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^q$ the induced map:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\mathrm{Spt}} \xrightarrow[\cong]{g_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\mathrm{Spt}}$$

is an isomorphism, as we wanted. □

Proposition 3.1.15. — *The counit of the adjunction constructed in Proposition 3.1.12, $f_q = i_q r_q \xrightarrow{\theta} \mathrm{id}$, has the following property:*

For any T -spectrum X , and for any compact generator $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^q$, the map $f_q X \xrightarrow{\theta_X} X$ in $\mathcal{H}(S)$ induces an isomorphism of abelian groups:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), f_q X]_{\mathrm{Spt}} \xrightarrow[\cong]{\theta_{X^*}} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\mathrm{Spt}}$$

Proof. — Let $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ be an arbitrary element in C_{eff}^q . Since $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in \Sigma_T^q \mathcal{H}^{\mathrm{eff}}(S)$ for $n, r, s \geq 0$ with $s - n \geq q$, we get the following commutative diagram:

$$\begin{array}{ccc}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), f_q X]_{\mathrm{Spt}} & \xrightarrow{\theta_{X^*}} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\mathrm{Spt}} \\
 \parallel & & \parallel \\
 [i_q(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), i_q r_q X]_{\mathrm{Spt}} & \xrightarrow{\theta_{X^*}} & [i_q(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), X]_{\mathrm{Spt}}
 \end{array}$$

Now using the adjunction between i_q and r_q we have the following commutative diagram:

$$\begin{array}{ccc}
 [i_q(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), i_q r_q X]_{\text{Spt}} & & \\
 \cong \downarrow & \searrow \theta_{X*} & \\
 & & [i_q(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), X]_{\text{Spt}} \\
 \text{Hom}_{\Sigma_T^q, \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q i_q r_q X) & & \cong \downarrow \\
 \tau_{r_q X} \cong \uparrow & \searrow r_q(\theta_X)* & \\
 & & \text{Hom}_{\Sigma_T^q, \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q X) \\
 \text{Hom}_{\Sigma_T^q, \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q X) & & \uparrow \text{id} \\
 & \searrow \text{id} & \\
 & & \text{Hom}_{\Sigma_T^q, \mathcal{H}^{\text{eff}}(S)}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), r_q X)
 \end{array}$$

where τ is the unit of the adjunction between i_q and r_q . This shows that θ_{X*} is an isomorphism, as we wanted. □

Theorem 3.1.16 (Voevodsky, cf. [25]). — *There exist exact functors*

$$s_q : \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

together with natural transformations

$$\begin{array}{ccc}
 \pi_q : f_q & \longrightarrow & s_q \\
 \sigma_q : s_q & \longrightarrow & \Sigma_T^{1,0} f_{q+1}
 \end{array}$$

such that the following conditions hold:

1. Given any T -spectrum X , we get the following distinguished triangle in $\mathcal{H}(S)$

$$(26) \quad f_{q+1} X \longrightarrow f_q X \xrightarrow{\pi_q} s_q X \xrightarrow{\sigma_q} \Sigma_T^{1,0} f_{q+1} X$$

2. For any T -spectrum X , $s_q X$ is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.
3. For any T -spectrum X , and for any T spectrum Y in $\Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S)$, $[Y, s_q X]_{\text{Spt}} = 0$.

Proof. — Since the triangulated categories $\Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S)$ and $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ are both compactly generated (see Proposition 3.1.10), the result follows from Propositions 9.1.19 and 9.1.8 in [20]. □

Definition 3.1.17 (Voevodsky). — Given an arbitrary T -spectrum X , the sequence of distinguished triangles (26) is called the slice tower of X . The T -spectrum $s_q X$ is called the q -slice of X , and the T -spectrum $f_q X$ is called the $(q - 1)$ -connective cover of X .

Theorem 3.1.18. — There exist exact functors

$$s_{<q} : \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

together with natural transformations

$$\begin{aligned} \pi_{<q} : \text{id} &\longrightarrow s_{<q} \\ \sigma_{<q} : s_{<q} &\longrightarrow \Sigma_T^{1,0} f_q \end{aligned}$$

such that the following conditions hold:

1. Given any T -spectrum X , we get the following distinguished triangle in $\mathcal{H}(S)$

$$(27) \quad f_q X \longrightarrow X \xrightarrow{\pi_{<q}} s_{<q} X \xrightarrow{\sigma_{<q}} \Sigma_T^{1,0} f_q X$$

2. For any T -spectrum X , and for any T spectrum Y in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, $[Y, s_{<q} X]_{\text{Spt}} = 0$.

Proof. — The result follows from Propositions 9.1.19 and 9.1.8 in [20], using the fact that the triangulated categories $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ and $\mathcal{H}(S)$ are both compactly generated (see Propositions 3.1.5 and 3.1.10). \square

Proposition 3.1.19. — Let X be an arbitrary T -spectrum. We have the following commutative diagram, where all the rows and columns are distinguished triangles in $\mathcal{H}(S)$:

$$\begin{array}{ccccccc} f_{q+1} X & \longrightarrow & f_q X & \xrightarrow{\pi_q} & s_q X & \xrightarrow{\sigma_q} & \Sigma_T^{1,0} f_{q+1} X \\ \parallel & & \downarrow & & \downarrow j_q & & \parallel \\ f_{q+1} X & \longrightarrow & X & \xrightarrow{\pi_{<q+1}} & s_{<q+1} X & \xrightarrow{\sigma_{<q+1}} & \Sigma_T^{1,0} f_{q+1} X \\ \downarrow & & \downarrow \pi_{<q} & & \downarrow t_q & & \downarrow \\ * & \longrightarrow & s_{<q} X & \xlongequal{\quad} & s_{<q} X & \longrightarrow & * \\ \downarrow & & \downarrow \sigma_{<q} & & \downarrow & & \downarrow \\ \Sigma_T^{1,0} f_{q+1} X & \longrightarrow & \Sigma_T^{1,0} f_q X & \xrightarrow{\Sigma_T^{1,0} \pi_q} & \Sigma_T^{1,0} s_q X & \xrightarrow{\Sigma_T^{1,0} \sigma_q} & \Sigma_T^{2,0} f_{q+1} X \end{array}$$

Proof. — It follows from Theorems 3.1.16 and 3.1.18, together with the octahedral axiom applied to the following commutative diagram:

$$\begin{array}{ccc} f_{q+1}X & \longrightarrow & f_qX \\ & \searrow & \swarrow \\ & X & \end{array} \quad \square$$

Proposition 3.1.20. — *Let $f : X \rightarrow Y$ be a map in $\mathcal{H}(S)$. We have that*

$$s_{<q}f : s_{<q}X \rightarrow s_{<q}Y$$

is an isomorphism in $\mathcal{H}(S)$ if and only if f induces the following isomorphisms of abelian groups:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}X]_{\text{Spt}} \xrightarrow{(s_{<q}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}Y]_{\text{Spt}}$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$.

Proof. — (\Rightarrow): Assume that $s_{<q}f$ is an isomorphism. Then it is clear that $(s_{<q}f)_*$ is also an isomorphism for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$.

(\Leftarrow): Corollary 3.1.6 implies that $s_{<q}f$ is an isomorphism in $\mathcal{H}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$, the induced maps:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}X]_{\text{Spt}} \xrightarrow{(s_{<q}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}Y]_{\text{Spt}}$$

are isomorphisms of abelian groups.

But Theorem 3.1.18(2) implies that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, we have:

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}X]_{\text{Spt}} \xrightarrow[\cong]{(s_{<q}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}Y]_{\text{Spt}} \cong 0$$

thus $(s_{<q}f)_*$ is an isomorphism in this case.

Thus in order to show that $s_{<q}f$ is an isomorphism, we only need to check that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$, the induced maps:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}X]_{\text{Spt}} \xrightarrow{(s_{<q}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q}Y]_{\text{Spt}}$$

are all isomorphisms of abelian groups; but this holds by hypothesis. This finishes the proof. \square

Proposition 3.1.21. — *Let $f : X \rightarrow Y$ be a map in $\mathcal{H}(S)$. We have that*

$$s_qf : s_qX \rightarrow s_qY$$

is an isomorphism in $\mathcal{H}(S)$ if and only if f induces the following isomorphisms of abelian groups:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_qX]_{\text{Spt}} \xrightarrow{(s_qf)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_qY]_{\text{Spt}}$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ where $s - n = q$.

Proof. — (\Rightarrow): Assume that $s_q f$ is an isomorphism. Then it is clear that $(s_q f)_*$ is also an isomorphism for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ with $s - n = q$.

(\Leftarrow): Theorem 3.1.16(2) implies that $s_q X$ and $s_q Y$ are both in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Therefore using Corollary 3.1.11 and the fact that $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is a full subcategory of $\mathcal{H}(S)$, we have that $s_q f$ is an isomorphism if and only if the maps:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q X]_{\text{Spt}} \xrightarrow{(s_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q Y]_{\text{Spt}}$$

are all isomorphisms of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$.

But if $s - n \geq q + 1$, we have that $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is in fact in $\Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S)$; and using Theorem 3.1.16(3) again, we have that in this case:

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q X]_{\text{Spt}} \xrightarrow[\cong]{(s_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q Y]_{\text{Spt}} \cong 0$$

Thus in order to show that $s_q f$ is an isomorphism, we only need to check that the maps:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q X]_{\text{Spt}} \xrightarrow{(s_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q Y]_{\text{Spt}}$$

are all isomorphisms of abelian groups, for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ with $s - n = q$. This finishes the proof. \square

3.2. Model Structures for the Slice Filtration

Our goal in this section is to use the cellularity of $\text{Spt}_T \mathcal{M}_*$ (see Theorem 2.5.4), to construct relying on Hirschhorn’s localization techniques, several families of model structures on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ via left and right Bousfield localization. These new model structures will provide liftings in a suitable sense for the functors

$$f_q, s_{<q}, s_q : \mathcal{H}(S) \rightarrow \mathcal{H}(S)$$

described in Section 3.1.

The first family of model structures on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$ will be constructed via right Bousfield localization. These model structures will have the property that the cofibrant replacement functor coincides in a suitable sense with the functor f_q defined in Remark 3.1.13. This will provide a natural lifting of Voevodsky’s slice filtration to the level of model categories. In the rest of this section $q \in \mathbb{Z}$ will denote an arbitrary integer.

Theorem 3.2.1. — Consider the following set of objects in $\text{Spt}_T \mathcal{M}_*$

$$C_{\text{eff}}^q = \bigcup_{n,r,s \geq 0; s-n \geq q} \bigcup_{U \in (\mathcal{M}|_S)} F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

The right Bousfield localization of $\text{Spt}_T\mathcal{M}_*$ with respect to the class of C_{eff}^q -colocal equivalences exists (see definitions 1.8.6 and 1.9.2). This model structure will be called $(q-1)$ -connected motivic stable, and the category of T -spectra equipped with the $(q-1)$ -connected motivic stable model structure will be denoted by $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$. Furthermore $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$ is a right proper and simplicial model category. The homotopy category associated to $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$ will be denoted by $R_{C_{\text{eff}}^q} \mathcal{H}(S)$.

Proof. — Theorems 2.4.16 and 2.5.4 imply that $\text{Spt}_T\mathcal{M}_*$ is cellular, proper and simplicial. Therefore we can apply Theorem 5.1.1 in [6] to construct the right Bousfield localization of $\text{Spt}_T\mathcal{M}_*$ with respect to the class of C_{eff}^q -colocal equivalences. Using Theorem 5.1.1 in [6] again, we have that this new model structure is right proper and simplicial. \square

Definition 3.2.2. — Let C_q denote a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$; such that for every T -spectrum X , the natural map

$$C_q X \xrightarrow{C_q^X} X$$

is a trivial fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$, and $C_q X$ is always a C_{eff}^q -colocal T -spectrum.

Proposition 3.2.3. — $IQ_T J$ is also a fibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$ (see Corollary 2.4.20).

Proof. — Since $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$ is the right Bousfield localization of $\text{Spt}_T\mathcal{M}_*$ with respect to the C_{eff}^q -colocal equivalences, by construction we have that the fibrations and the trivial cofibrations are identical in $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$ and $\text{Spt}_T\mathcal{M}_*$ respectively. This implies that for every T -spectrum X , $IQ_T JX$ is fibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$, and using [6, Proposition 3.1.5] we have that the natural map:

$$X \xrightarrow{IQ_T J^X} IQ_T JX$$

is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$. Hence $IQ_T J$ is also a fibrant replacement functor for $R_{C_{\text{eff}}^q} \text{Spt}_T\mathcal{M}_*$. \square

Proposition 3.2.4. — Let $f : X \rightarrow Y$ be a map in $\text{Spt}_T\mathcal{M}_*$. We have that f is a C_{eff}^q -colocal equivalence if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, f induces the following isomorphisms of abelian groups:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} \xrightarrow{f_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}$$

Proof. — (\Rightarrow) : Assume that f is a C_{eff}^q -colocal equivalence. Since all the compact generators $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ are cofibrant in $\text{Spt}_T\mathcal{M}_*$, we have that f is a C_{eff}^q -colocal

equivalence if and only if the following maps are weak equivalences of simplicial sets:

$$\begin{array}{c} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) \\ \downarrow (IQ_T Jf)_* \\ \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) \end{array}$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category and $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is cofibrant, we have that $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX)$ and $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY)$ are both Kan complexes, thus we get the following commutative diagram where the top row and the vertical maps are all isomorphisms of abelian groups:

$$\begin{array}{ccc} \pi_0 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) & & \\ \cong \downarrow & \searrow (IQ_T Jf)_* & \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} & \xrightarrow{\cong} & \pi_0 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) \\ & \searrow f_* & \cong \downarrow \\ & & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}} \end{array}$$

Therefore

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} \xrightarrow{f_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}$$

is an isomorphism of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, as we wanted.

(\Leftarrow): Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Let ω_0, η_0 denote the base points corresponding to $\text{Map}_*(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX)$ and $\text{Map}_*(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY)$ respectively. We need to show that the map:

$$\begin{array}{c} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) \\ \downarrow (IQ_T Jf)_* \\ \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) \end{array}$$

is a weak equivalence of simplicial sets.

We know that the map

$$j : F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+) \rightarrow F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

which is adjoint to the identity map

$$\text{id} : S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+ \rightarrow Ev_{n+1}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)) = S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+$$

is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Now since $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ and $F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+)$ are both cofibrant and $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, we can apply Ken Brown's lemma (see Lemma 1.1.4) to conclude that the horizontal maps in the following commutative diagram are weak equivalences of simplicial sets:

$$\begin{array}{ccc}
 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) & & \\
 \downarrow (IQ_T Jf)_* & \searrow j^* & \\
 & \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JX) & \\
 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) & & \downarrow (IQ_T Jf)_* \\
 & \searrow j^* & \\
 & \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JY) &
 \end{array}$$

Hence by the two out of three property for weak equivalences, it is enough to show that the following induced map

$$\begin{array}{c}
 \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JX) \\
 \downarrow (IQ_T Jf)_* \\
 \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JY)
 \end{array}$$

is a weak equivalence of simplicial sets.

On the other hand, since $\text{Spt}_T \mathcal{M}_*$ is a pointed simplicial model category, we have that Lemma 6.1.2 in [10] together with Remark 2.4.3(2) imply that the following

diagram is commutative for $k \geq 0$:

$$\begin{array}{ccc}
 \pi_{k,\omega_0} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) & & \\
 \parallel & \searrow^{(IQ_T Jf)_*} & \\
 \pi_{k,\omega_0} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) & & \\
 \parallel & & \\
 \pi_{k,\omega_0} \text{Map}_*(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) & & \\
 \searrow^{(IQ_T Jf)_*} & & \\
 \pi_{k,\eta_0} \text{Map}_*(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) & & \\
 \parallel & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, IQ_T JX]_{\text{Spt}} & & \\
 \searrow^{(IQ_T Jf)_*} & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, IQ_T JY]_{\text{Spt}} & & \\
 \parallel & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, X]_{\text{Spt}} & & \\
 \searrow^{f_*} & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Y]_{\text{Spt}} & & \\
 \parallel & & \\
 [F_n(S^{k+r} \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} & & \\
 \searrow^{f_*} & & \\
 [F_n(S^{k+r} \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}} & &
 \end{array}$$

but by hypothesis we have that the bottom row is an isomorphism of abelian groups. Therefore all the maps in the top row are also isomorphisms. Then for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, the induced map

$$\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) \xrightarrow{(IQ_T Jf)_*} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY)$$

is a weak equivalence when it is restricted to the path component of $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX)$ containing ω_0 . But $F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+)$ is also in C_{eff}^q , therefore

the following induced map

$$\begin{array}{c} \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JX)) \\ \downarrow (IQ_T Jf)_* \\ \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JY)) \end{array}$$

is a weak equivalence of simplicial sets, since taking S^1 -loops kills the path components that do not contain the base point.

Finally, since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category we have that the rows in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JX)) & & \\ \downarrow (IQ_T Jf)_* & \searrow \cong & \\ \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, IQ_T JX) & & \\ \downarrow (IQ_T Jf)_* & & \downarrow (IQ_T Jf)_* \\ \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JY)) & & \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, IQ_T JY) \\ & \searrow \cong & \end{array}$$

Hence the two out of three property for weak equivalences implies that the right vertical map is a weak equivalence of simplicial sets. But $F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1$ is clearly isomorphic to $F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+)$, therefore the induced map

$$\begin{array}{c} \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JX) \\ \downarrow (IQ_T Jf)_* \\ \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_T JY) \end{array}$$

is a weak equivalence, as we wanted. □

Corollary 3.2.5. — *Let $f : X \rightarrow Y$ be a map in $\text{Spt}_T \mathcal{M}_*$. We have that f is a C_{eff}^q -local equivalence if and only if the following map*

$$r_q X \xrightarrow{r_q(f)} r_q Y$$

is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

Proof. — The result follows immediately from Proposition 3.2.4 and Corollary 3.1.11(2). □

Corollary 3.2.6. — *Let X be an arbitrary T -spectrum X . We have that $X \cong *$ in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ if and only if the following condition holds:*

For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} \cong 0$$

Proof. — We have that X is isomorphic to $*$ in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ if and only if the map $* \rightarrow X$ is a C_{eff}^q -colocal equivalence. But Corollary 3.2.5 implies that $* \rightarrow X$ is a C_{eff}^q -colocal equivalence if and only if

$$* \cong r_q(*) \longrightarrow r_q X$$

becomes an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

Finally by Corollary 3.1.11(2) we have that $* \rightarrow r_q X$ is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ the following induced maps are isomorphisms of abelian groups:

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), *]_{\text{Spt}} \xrightarrow{\cong} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}}$$

as we wanted. □

Lemma 3.2.7. — *Let $f : X \rightarrow Y$ be a map in $\text{Spt}_T \mathcal{M}_*$. We have that f is a C_{eff}^q -colocal equivalence if and only if $\Omega_{S^1} I Q_T J(f)$ is a C_{eff}^q -colocal equivalence.*

Proof. — Assume that f is a C_{eff}^q -colocal equivalence. We need to show that $\Omega_{S^1} I Q_T J(f)$ is a C_{eff}^q -colocal equivalence. Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category and all the compact generators $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ are cofibrant, we have the following commutative diagram:

$$\begin{array}{ccc}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} I Q_T J X]_{\text{Spt}} & \xrightarrow{(\Omega_{S^1} I Q_T J f)_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} I Q_T J Y]_{\text{Spt}} \\
 \cong \downarrow & & \downarrow \cong \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, X]_{\text{Spt}} & \xrightarrow{f_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, Y]_{\text{Spt}} \\
 \cong \downarrow & & \downarrow \cong \\
 [F_n(S^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} & \xrightarrow{f_*} & [F_n(S^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}
 \end{array}$$

but using Proposition 3.2.4 and the fact that f is a C_{eff}^q -colocal equivalence, we have that the bottom row is an isomorphism, therefore the top row is also an isomorphism.

Using Proposition 3.2.4 again, we have that $\Omega_{S^1}IQ_TJ(f)$ is a C_{eff}^q -colocal equivalence, as we wanted.

Conversely, assume that $\Omega_{S^1}IQ_TJ(f)$ is a C_{eff}^q -colocal equivalence. Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Proposition 3.2.4 implies that the top row in the following commutative diagram is an isomorphism:

$$\begin{array}{ccc}
 [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), \Omega_{S^1}IQ_TJX]_{\text{Spt}} & \xrightarrow[\cong]{IQ_TJ(f)_*} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), \Omega_{S^1}IQ_TJY]_{\text{Spt}} \\
 \cong \downarrow & & \downarrow \cong \\
 [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, X]_{\text{Spt}} & \xrightarrow{f_*} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, Y]_{\text{Spt}} \\
 \cong \uparrow & & \uparrow \cong \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}} & \xrightarrow{f_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}
 \end{array}$$

therefore the bottom row is also an isomorphism. Finally using Proposition 3.2.4 again, we have that f is a C_{eff}^q -colocal equivalence. This finishes the proof. \square

Corollary 3.2.8. — *The adjunction*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Using Corollary 1.3.16 in [10] and Proposition 3.2.3 we have that it suffices to verify the following two conditions:

1. For every cofibrant object X in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$, the following composition

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1}IQ_TJ^{X \wedge S^1}} \Omega_{S^1}IQ_TJ(X \wedge S^1)$$

is a C_{eff}^q -colocal equivalence.

2. Ω_{S^1} reflects C_{eff}^q -colocal equivalences between fibrant objects in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$.

(1): By construction $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$ is a right Bousfield localization of $\text{Spt}_T \mathcal{M}_*$, therefore the identity functor

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a left Quillen functor. Thus X is also cofibrant in $\text{Spt}_T \mathcal{M}_*$. Since the adjunction $(- \wedge S^1, \Omega_{S^1}, \varphi)$ is a Quillen equivalence on $\text{Spt}_T \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies

that the following composition is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$:

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1}IQ_TJ^{X \wedge S^1}} \Omega_{S^1}IQ_TJ(X \wedge S^1)$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a C_{eff}^q -colocal equivalence.

(2): This follows immediately from Proposition 3.2.3 and Lemma 3.2.7. □

Corollary 3.2.9. — $R_{C_{\text{eff}}^q}\mathcal{H}(S)$ has the structure of a triangulated category.

Proof. — Theorem 3.2.1 implies in particular that $R_{C_{\text{eff}}^q}\text{Spt}_T\mathcal{M}_*$ is a pointed simplicial model category, and Corollary 3.2.8 implies that the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : R_{C_{\text{eff}}^q}\text{Spt}_T\mathcal{M}_* \rightarrow R_{C_{\text{eff}}^q}\text{Spt}_T\mathcal{M}_*$$

is a Quillen equivalence. Therefore the result follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII]. □

Remark 3.2.10. — The adjunction $(\Sigma_T, \Omega_T, \varphi)$ is a Quillen equivalence on $\text{Spt}_T\mathcal{M}_*$. However it does not descend even to a Quillen adjunction on the $(q - 1)$ -connected motivic stable model category $R_{C_{\text{eff}}^q}\text{Spt}_T\mathcal{M}_*$.

Proposition 3.2.11. — We have the following adjunction

$$(C_q, IQ_TJ, \varphi) : R_{C_{\text{eff}}^q}\mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

between exact functors of triangulated categories.

Proof. — Since $R_{C_{\text{eff}}^q}\text{Spt}_T\mathcal{M}_*$ is the right Bousfield localization of $\text{Spt}_T\mathcal{M}_*$ with respect to the C_{eff}^q -colocal equivalences, we have that the identity functor $\text{id} : R_{C_{\text{eff}}^q}\text{Spt}_T\mathcal{M}_* \rightarrow \text{Spt}_T\mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(C_q, IQ_TJ, \varphi) : R_{C_{\text{eff}}^q}\mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

Now Proposition 6.4.1 in [10] implies that C_q maps cofibre sequences in $R_{C_{\text{eff}}^q}\mathcal{H}(S)$ to cofibre sequences in $\mathcal{H}(S)$. Therefore using Proposition 7.1.12 in [10] we have that C_q and IQ_TJ are both exact functors between triangulated categories. □

Proposition 3.2.12. — The unit of the adjunction

$$(C_q, IQ_TJ, \varphi) : R_{C_{\text{eff}}^q}\mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

$\sigma_X : X \rightarrow IQ_TJ C_q X$ is an isomorphism in $R_{C_{\text{eff}}^q}\mathcal{H}(S)$ for every T -spectrum X , and the functor:

$$C_q : R_{C_{\text{eff}}^q}\mathcal{H}(S) \rightarrow \mathcal{H}(S)$$

is a full embedding.

Proof. — For any T -spectrum X , we have the following commutative diagram in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$:

$$\begin{array}{ccc} C_q X & \xrightarrow{C_q^X} & X \\ \downarrow IQ_T J^{C_q X} & & \downarrow IQ_T J^X \\ IQ_T J C_q X & \xrightarrow{IQ_T J(C_q^X)} & IQ_T J X \end{array}$$

where $IQ_T J^{C_q X}$ is in particular a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. But since $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$ is the right Bousfield localization of $\text{Spt}_T \mathcal{M}_*$ with respect to the C_{eff}^q -colocal equivalences, Proposition 3.1.5 in [6] implies that $IQ_T J^{C_q X}$ is also a C_{eff}^q -colocal equivalence.

On the other hand, by construction we have that C_q^X is a C_{eff}^q -colocal equivalence. Therefore $IQ_T J^{C_q X}$ and C_q^X both become isomorphisms in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$.

Finally, since σ_X is the following composition in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$:

$$X \xrightarrow[\cong]{(C_q^X)^{-1}} C_q X \xrightarrow[\cong]{IQ_T J^{C_q X}} IQ_T J C_q X$$

it follows that σ_X is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ as we wanted. This also implies that the functor

$$C_q : R_{C_{\text{eff}}^q} \mathcal{H}(S) \rightarrow \mathcal{H}(S)$$

is a full embedding. □

Proposition 3.2.13. — *Let $f : X \rightarrow Y$ be a map in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$. We have that f is an isomorphism if and only if the following condition holds:*

For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, the induced maps

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q X]_{\text{Spt}} \xrightarrow{(C_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q Y]_{\text{Spt}}$$

are all isomorphisms of abelian groups.

Proof. — Complete the map f to a distinguished triangle in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$:

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma_T^{1,0} X$$

We have that

$$(28) \quad C_q X \xrightarrow{C_q f} C_q Y \longrightarrow C_q Z \longrightarrow \Sigma_T^{1,0} C_q X$$

is also a distinguished triangle in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$, therefore $C_q f$ is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ if and only if $C_q Z \cong *$ in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$; and since C_q is a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$ we have that f is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ if and only if $C_q f$ is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$.

Hence, f is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ if and only if $C_q Z \cong *$ in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$. Now Corollary 3.2.6 implies that $C_q Z \cong *$ in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q Z]_{\text{Spt}} \cong 0$$

But Proposition 3.2.11 implies that the diagram (28) is a distinguished triangle in $\mathcal{H}(S)$; and since for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, the functor $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), -]_{\text{Spt}}$ is homological, we get the following long exact sequence of abelian groups

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q X]_{\text{Spt}} & & \\ \downarrow (C_q f)_* & & \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q Y]_{\text{Spt}} & & \\ \downarrow & & \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q Z]_{\text{Spt}} & & \\ \downarrow & & \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} C_q X]_{\text{Spt}} & \xleftarrow[\Sigma_T^{1,0}]{\cong} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), C_q X]_{\text{Spt}} \\ \downarrow (\Sigma_T^{1,0} C_q f)_* & & \downarrow (C_q f)_* \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} C_q Y]_{\text{Spt}} & \xleftarrow[\Sigma_T^{1,0}]{\cong} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), C_q Y]_{\text{Spt}} \\ \downarrow & & \\ \vdots & & \end{array}$$

Therefore $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q Z]_{\text{Spt}} \cong 0$ for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ if and only if the induced map

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q X]_{\text{Spt}} \xrightarrow{(C_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q Y]_{\text{Spt}}$$

is an isomorphism of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. This finishes the proof. \square

Proposition 3.2.14. — *Let A be an arbitrary T -spectrum in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. We have that $(Q_s A) \wedge S^1$ is a C_{eff}^q -colocal T -spectrum in $\text{Spt}_T \mathcal{M}_*$.*

Proof. — Let ω_0, η_0 denote the base points corresponding to $\text{Map}_*(Q_s A, IQ_T JX)$ and $\text{Map}_*(Q_s A, IQ_T JY)$ respectively.

It is clear that $Q_s A \cong A$ in $\mathcal{H}(S)$; then $Q_s A$ is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, since A is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ and $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is a triangulated subcategory of $\mathcal{H}(S)$.

Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, we have that $Q_s A \wedge S^1$ is cofibrant in $\text{Spt}_T \mathcal{M}_*$, hence it suffices to check that for every C_{eff}^q -colocal equivalence $f : X \rightarrow Y$, the induced map

$$\text{Map}(Q_s A \wedge S^1, IQ_T JX) \xrightarrow{(IQ_T Jf)_*} \text{Map}(Q_s A \wedge S^1, IQ_T JY)$$

is a weak equivalence of simplicial sets.

Now Corollary 3.2.8 together with Proposition 3.2.3 imply that for every $n \geq 0$, $\Omega_{S^n} IQ_T J(f)$ is also a C_{eff}^q -colocal equivalence. Hence Corollary 3.2.5 implies that $r_q \Omega_{S^n} IQ_T J(f)$ is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Since $Q_s A \in \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, we have that $i_q Q_s A = Q_s A$, then by Proposition 3.1.12 we get the following commutative diagram where both rows and the left vertical map are isomorphisms of abelian groups:

$$\begin{array}{ccc} \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(Q_s A, r_q \Omega_{S^n} IQ_T JX) & \xrightarrow{\cong} & [Q_s A, \Omega_{S^n} IQ_T JX]_{\text{Spt}} \\ (r_q \Omega_{S^n} IQ_T Jf)_* \downarrow \cong & & \downarrow (\Omega_{S^n} IQ_T Jf)_* \\ \text{Hom}_{\Sigma_T^q \mathcal{H}^{\text{eff}}(S)}(Q_s A, r_q \Omega_{S^n} IQ_T JY) & \xrightarrow{\cong} & [Q_s A, \Omega_{S^n} IQ_T JY]_{\text{Spt}} \end{array}$$

Therefore the right vertical map is also an isomorphism of abelian groups.

Now since $\text{Spt}_T \mathcal{M}_*$ is a pointed simplicial model category, we have that Lemma 6.1.2 in [10] together with Remark 2.4.3(2) imply that the following diagram is commutative for $n \geq 0$, where all the vertical maps together with the bottom row are isomorphisms of abelian groups:

$$\begin{array}{ccc} \pi_{n, \omega_0} \text{Map}(Q_s A, IQ_T JX) & \xrightarrow{(IQ_T Jf)_*} & \pi_{n, \eta_0} \text{Map}(Q_s A, IQ_T JY) \\ \parallel & & \parallel \\ \pi_{n, \omega_0} \text{Map}_*(Q_s A, IQ_T JX) & \xrightarrow{(IQ_T Jf)_*} & \pi_{n, \eta_0} \text{Map}_*(Q_s A, IQ_T JY) \\ \cong \downarrow & & \downarrow \cong \\ [Q_s A, \Omega_{S^n} IQ_T JX]_{\text{Spt}} & \xrightarrow[\cong]{(\Omega_{S^n} IQ_T Jf)_*} & [Q_s A, \Omega_{S^n} IQ_T JY]_{\text{Spt}} \end{array}$$

Therefore all the maps in the top row are also isomorphisms. Thus, the induced map

$$\text{Map}(Q_s A, IQ_T JX) \xrightarrow{(IQ_T Jf)_*} \text{Map}(Q_s A, IQ_T JY)$$

is a weak equivalence when it is restricted to the path component of $\text{Map}(Q_s A, IQ_T JX)$ containing ω_0 . This implies that the following induced map

$$\begin{array}{c} \text{Map}_*(S^1, \text{Map}_*(Q_s A, IQ_T JX)) \\ \downarrow (IQ_T Jf)_* \\ \text{Map}_*(S^1, \text{Map}_*(Q_s A, IQ_T JY)) \end{array}$$

is a weak equivalence since taking S^1 -loops kills the path components that do not contain the base point.

Finally, since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category we have that the rows in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} \text{Map}_*(S^1, \text{Map}_*(Q_s A, IQ_T JX)) & \xrightarrow{\cong} & \text{Map}_*(Q_s A \wedge S^1, IQ_T JX) \\ \downarrow (IQ_T Jf)_* & & \downarrow (IQ_T Jf)_* \\ \text{Map}_*(S^1, \text{Map}_*(Q_s A, IQ_T JY)) & \xrightarrow{\cong} & \text{Map}_*(Q_s A \wedge S^1, IQ_T JY) \end{array}$$

Hence the two out of three property for weak equivalences implies that the right vertical map is a weak equivalence of simplicial sets, as we wanted. \square

Corollary 3.2.15. — *Let X be an arbitrary T -spectrum in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. We have that $Q_s X$ is a C_{eff}^q -colocal T -spectrum in $\text{Spt}_T \mathcal{M}_*$*

Proof. — Let R denote a fibrant replacement functor in $\text{Spt}_T \mathcal{M}_*$ such that for every T -spectrum Y , the natural map

$$Y \xrightarrow{R_Y} RY$$

is a trivial cofibration in $\text{Spt}_T \mathcal{M}_*$. Then $RQ_s X$ is cofibrant in $\text{Spt}_T \mathcal{M}_*$. Now the map

$$Q_s X \xrightarrow{RQ_s X} RQ_s X$$

is in particular a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, therefore using [6, Lemma 3.2.1(2)] we get that $Q_s X$ is C_{eff}^q -colocal if and only if $RQ_s X$ is C_{eff}^q -colocal. We will show that $RQ_s X$ is C_{eff}^q -colocal.

By hypothesis X is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ and it is clear that $Q_s X \cong X$ in $\mathcal{H}(S)$. Hence $Q_s X$ is also in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ since it is a triangulated subcategory of $\mathcal{H}(S)$. Therefore $\Omega_{S^1} RQ_s X$ is also in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ since $\Omega_{S^1} RQ_s X$ computes the desuspension $\Sigma_T^{-1,0} Q_s X$ of $Q_s X$.

Using Proposition 3.2.14 we have that $(Q_s \Omega_{S^1} RQ_s X) \wedge S^1$ is C_{eff}^q -colocal. But since the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a Quillen equivalence, we have the following weak equivalence in $\text{Spt}_T \mathcal{M}_*$:

$$(Q_s \Omega_{S^1} RQ_s X) \wedge S^1 \xrightarrow{\epsilon_{RQ_s X} \circ (Q_s^{\Omega_{S^1} RQ_s X} \wedge \text{id})} RQ_s X$$

where ϵ denotes the counit of the adjunction considered above.

Finally using [6, Lemma 3.2.1(2)] again, we get that $RQ_s X$ is C_{eff}^q -colocal. This finishes the proof. \square

Proposition 3.2.16. — *Let ρ be the counit of the adjunction:*

$$(C_q, IQ_T J, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

For every T -spectrum X , the map

$$r_q(\rho_X) : r_q C_q IQ_T J X \rightarrow r_q X$$

is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$; and this map induces a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \xrightarrow[r_q]{r_q C_q IQ_T J} \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$$

Proof. — The naturality of the counit ρ , implies that $r_q(\rho_-) : r_q C_q IQ_T J \rightarrow r_q$ is a natural transformation. Hence, it is enough to show that for every T -spectrum X , $r_q(\rho_X)$ is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

Consider the following diagram of T -spectra:

$$C_q IQ_T J X \xrightarrow{C_q^{IQ_T J X}} IQ_T J X \xleftarrow{IQ_T J X} X$$

where $IQ_T J X$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$ and $C_q^{IQ_T J X}$ is a C_{eff}^q -colocal equivalence. Then it is clear that $r_q(IQ_T J X)$ is an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. On the other hand, Corollary 3.2.5 implies that $r_q(C_q^{IQ_T J X})$ is also an isomorphism in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

And this proves the result, since ρ_X is just the following composition in $\mathcal{H}(S)$:

$$C_q IQ_T J X \xrightarrow{C_q^{IQ_T J X}} IQ_T J X \xrightarrow{(IQ_T J X)^{-1}} X \quad \square$$

Proposition 3.2.17. — *Let θ be the counit of the adjunction*

$$(i_q, r_q, \varphi) : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \longrightarrow \mathcal{H}^{\text{eff}}(S)$$

For any T -spectrum X in $\text{Spt}_T \mathcal{M}_*$, the map

$$IQ_T J(\theta_X) : IQ_T J(i_q r_q) X \longrightarrow IQ_T J X$$

is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$; and this map induces a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{IQ_T J(i_q r_q)} \\ \xrightarrow{IQ_T J} \end{array} R_{C_{\text{eff}}^q} \mathcal{H}(S)$$

Proof. — The naturality of the counit θ , implies that $IQ_T J(\theta_-) : IQ_T J(i_q r_q) \rightarrow IQ_T J$ is a natural transformation. Hence, it is enough to show that for every T -spectrum X , $IQ_T J(\theta_X)$ is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$.

By Proposition 3.2.13 it is enough to show that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ the induced maps

$$\begin{array}{c} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q IQ_T J(i_q r_q) X]_{\text{Spt}} \\ \downarrow C_q IQ_T J(\theta_X)_* \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q IQ_T JX]_{\text{Spt}} \end{array}$$

are all isomorphisms of abelian groups.

Consider the following commutative diagram in $\mathcal{H}(S)$:

$$\begin{array}{ccc} C_q IQ_T J(i_q r_q) X & \xrightarrow{C_q IQ_T J(\theta_X)} & C_q IQ_T JX \\ C_q^{IQ_T J i_q r_q X} \downarrow & & \downarrow C_q^{IQ_T JX} \\ IQ_T J(i_q r_q) X & \xrightarrow{IQ_T J(\theta_X)} & IQ_T JX \end{array}$$

where $C_q^{IQ_T J i_q r_q X}$ and $C_q^{IQ_T JX}$ are by construction maps of T -spectra and C_{eff}^q -colocal equivalences. Therefore proposition 3.2.4 implies that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ the induced maps

$$\begin{array}{ccc} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q IQ_T J(i_q r_q) X]_{\text{Spt}} & & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q IQ_T JX]_{\text{Spt}} \\ (C_q^{IQ_T J i_q r_q X})_* \downarrow & & \downarrow (C_q^{IQ_T JX})_* \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T J(i_q r_q) X]_{\text{Spt}} & & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX]_{\text{Spt}} \end{array}$$

are both isomorphisms of abelian groups.

On the other hand, Proposition 3.1.15 implies that we have an induced isomorphism of abelian groups:

$$\begin{array}{c} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T J(i_q r_q) X]_{\text{Spt}} \\ \downarrow IQ_T J(\theta_X)_* \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX]_{\text{Spt}} \end{array}$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$.

Finally, this implies that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, we get the following induced isomorphisms of abelian groups

$$\begin{array}{c} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q I Q_T J(i_q r_q) X]_{\text{Spt}} \\ \downarrow C_q I Q_T J(\theta_X)_* \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), C_q I Q_T J X]_{\text{Spt}} \end{array}$$

as we wanted. \square

Proposition 3.2.18. — *Let θ be the counit of the adjunction*

$$(i_q, r_q, \varphi) : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \longrightarrow \mathcal{H}(S)$$

For any T -spectrum X , the map

$$C_q I Q_T J(\theta_X) : C_q I Q_T J(i_q r_q) X \longrightarrow C_q I Q_T J X$$

is an isomorphism in $\mathcal{H}(S)$; and this map induces a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{C_q I Q_T J(i_q r_q)} \\ \xrightarrow{C_q I Q_T J} \end{array} \mathcal{H}(S)$$

Proof. — The naturality of the counit θ , implies that $C_q I Q_T J(\theta_-) : C_q I Q_T J(i_q r_q) \rightarrow C_q I Q_T J$ is a natural transformation. Hence, it is enough to show that for every T -spectrum X , $C_q I Q_T J(\theta_X)$ is an isomorphism in $\mathcal{H}(S)$.

But this follows immediately from Proposition 3.2.17 together with Proposition 3.2.11. \square

Proposition 3.2.19. — *For every T -spectrum X , the natural map*

$$C_q I Q_T J(i_q r_q) X \xrightarrow{C_q^{I Q_T J(i_q r_q) X}} I Q_T J(i_q r_q) X$$

is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Therefore we have a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{C_q I Q_T J(i_q r_q)} \\ \xrightarrow{I Q_T J(i_q r_q)} \end{array} \mathcal{H}(S)$$

Proof. — The naturality of the maps $C_q^X : C_q X \rightarrow X$ implies that we have an induced natural transformation of functors $C_q I Q_T J(i_q r_q) \rightarrow I Q_T J(i_q r_q)$. Hence, it is enough to show that for every T -spectrum X , $C_q^{I Q_T J(i_q r_q) X}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.

Consider the following commutative diagram in $\text{Spt}_T \mathcal{M}_*$:

$$\begin{array}{ccc}
 Q_s C_q I Q_T J(i_q r_q) X & \xrightarrow{Q_s(C_q^{I Q_T J(i_q r_q) X})} & Q_s I Q_T J(i_q r_q) X \\
 Q_s^{C_q I Q_T J(i_q r_q) X} \downarrow & & \downarrow Q_s^{I Q_T J(i_q r_q) X} \\
 C_q I Q_T J(i_q r_q) X & \xrightarrow{C_q^{I Q_T J(i_q r_q) X}} & I Q_T J(i_q r_q) X
 \end{array}$$

Since Q_s is a cofibrant replacement functor in $\text{Spt}_T \mathcal{M}_*$, it follows that the vertical maps are weak equivalences in $\text{Spt}_T \mathcal{M}_*$. Hence by the two out of three property for weak equivalences it suffices to show that $Q_s(C_q^{I Q_T J(i_q r_q) X})$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.

On the other hand we have that by construction $C_q^{I Q_T J(i_q r_q) X}$ is a C_{eff}^q -colocal equivalence, and [6, Proposition 3.1.5] implies that the vertical maps in the diagram above are also C_{eff}^q -colocal equivalences. Then by the two out of three property for C_{eff}^q -colocal equivalences we have that $Q_s(C_q^{I Q_T J(i_q r_q) X})$ is a C_{eff}^q -colocal equivalence.

Now by construction we have that $C_q I Q_T J(i_q r_q) X$ is a C_{eff}^q -colocal T -spectrum, and that $Q_s C_q I Q_T J(i_q r_q) X$ is cofibrant in $\text{Spt}_T \mathcal{M}_*$. Since $Q_s^{C_q I Q_T J(i_q r_q) X}$ is in particular a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, using [6, Lemma 3.2.1(2)] we have that $Q_s C_q I Q_T J(i_q r_q) X$ is also a C_{eff}^q -colocal T -spectrum.

It is clear that $I Q_T J(i_q r_q) X \cong i_q r_q X$ in $\mathcal{H}(S)$, therefore $I Q_T J(i_q r_q) X$ is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ since $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ is a triangulated subcategory of $\mathcal{H}(S)$ and $i_q r_q X$ is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Then using Corollary 3.2.15, we have that $Q_s I Q_T J(i_q r_q) X$ is a C_{eff}^q -colocal T -spectrum.

Finally we have that $Q_s(C_q^{I Q_T J(i_q r_q) X})$ is a C_{eff}^q -colocal equivalence, and that $Q_s C_q I Q_T J(i_q r_q) X$, $Q_s I Q_T J(i_q r_q) X$ are both C_{eff}^q -colocal T -spectra. Then [6, Theorem 3.2.13(2)] implies that $Q_s(C_q^{I Q_T J(i_q r_q) X})$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, as we wanted. \square

Theorem 3.2.20. — For every T -spectrum X , we have the following diagram in $\mathcal{H}(S)$:

$$\begin{array}{ccc}
 f_q X = i_q r_q X & \xrightarrow[\cong]{I Q_T J f_q X} & I Q_T J f_q X \\
 & \uparrow \cong & \\
 & C_q^{I Q_T J f_q X} & \\
 & \uparrow & \\
 C_q I Q_T J f_q X & \xrightarrow[\cong]{C_q I Q_T J(\theta_X)} & C_q I Q_T J X
 \end{array}
 \tag{29}$$

where all the maps are isomorphisms in $\mathcal{H}(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{f_q} \\ \xrightarrow{C_q IQ_T J} \end{array} \mathcal{H}(S)$$

Proof. — Since $IQ_T J$ is a fibrant replacement functor in $\text{Spt}_T \mathcal{M}_*$, it is clear that $IQ_T J f_q X$ becomes an isomorphism in the associated homotopy category $\mathcal{H}(S)$.

The fact that $C_q^{IQ_T J f_q X}$ is an isomorphism in $\mathcal{H}(S)$ follows from Proposition 3.2.19. Finally, Proposition 3.2.18 implies that $C_q IQ_T J(\theta_X)$ is also an isomorphism in $\mathcal{H}(S)$. This shows that all the maps in the diagram (29) are isomorphisms in $\mathcal{H}(S)$, therefore for every T -spectrum X we can define the following composition in $\mathcal{H}(S)$

$$(30) \quad \begin{array}{ccc} f_q X & \xrightarrow[\cong]{IQ_T J f_q X} & IQ_T J f_q X \\ & & \downarrow \cong \\ & & (C_q^{IQ_T J f_q X})^{-1} \\ & & \downarrow \\ & & C_q IQ_T J f_q X \xrightarrow[\cong]{C_q IQ_T J(\theta_X)} C_q IQ_T J X \end{array}$$

which is an isomorphism. The fact that $IQ_T J$ is a functorial fibrant replacement in $\text{Spt}_T \mathcal{M}_*$, propositions 3.2.19 and 3.2.18, imply all together that the isomorphisms defined in diagram (30) induce a natural isomorphism of functors $f_q \xrightarrow{\cong} C_q IQ_T J$. This finishes the proof. \square

Theorem 3.2.20 gives the desired lifting to the model category level for the functor f_q . Now we proceed to show that the homotopy categories $R_{\text{eff}}^{C_q} \mathcal{H}(S)$ are in fact equivalent to the categories $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$ defined in Section 3.1.

Using Propositions 3.1.12 and 3.2.11, we get the following diagram of adjunctions:

$$(31) \quad \begin{array}{ccc} R_{C_{\text{eff}}}^q \mathcal{H}(S) & \xrightarrow{(C_q, IQ_T J, \varphi)} & \mathcal{H}(S) \\ & & \uparrow (i_q, r_q, \varphi) \\ & & \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \end{array}$$

$$\begin{array}{ccc} R_{C_{\text{eff}}}^q \mathcal{H}(S) & \xrightleftharpoons[IQ_T J]{C_q} & \mathcal{H}(S) \\ & & \downarrow r_q \quad \uparrow i_q \\ & & \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \end{array}$$

where all the functors are exact.

Proposition 3.2.21. — *The adjunctions of diagram (31) induce an equivalence of categories:*

$$(32) \quad R_{C_{\text{eff}}}^q \mathcal{H}(S) \xrightleftharpoons[IQ_T J i_q]{r_q C_q} \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$$

between $R_{C_{\text{eff}}}^q \mathcal{H}(S)$ and $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$.

Proof. — It is enough to show the existence of the following natural isomorphisms between functors:

$$(33) \quad \begin{array}{ccc} \text{id} & \xrightarrow[\cong]{\epsilon} & (IQ_T J i_q)(r_q C_q) \\ & & \\ & & (r_q C_q)(IQ_T J i_q) \xrightarrow[\eta]{\cong} \text{id} \end{array}$$

We construct first the natural equivalence ϵ . Let $f : X \rightarrow Y$ be a map in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$. Applying the functor $i_q r_q C_q$, we get the following commutative diagram in $\mathcal{H}(S)$:

$$\begin{array}{ccc} i_q r_q C_q X & \xrightarrow{\theta_{C_q X}} & C_q X \\ \downarrow i_q r_q C_q f & & C_q f \downarrow \\ i_q r_q C_q Y & \xrightarrow{\theta_{C_q Y}} & C_q Y \end{array}$$

where θ denotes the counit of the adjunction between i_q and r_q . Now if we apply the functor $IQ_T J$, we have the following commutative diagram in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$:

$$\begin{array}{ccccc} IQ_T J i_q r_q C_q X & \xrightarrow[\cong]{IQ_T J(\theta_{C_q X})} & IQ_T J C_q X & \xleftarrow[\cong]{\sigma_X} & X \\ IQ_T J i_q r_q C_q f \downarrow & & IQ_T J C_q f \downarrow & & \downarrow f \\ IQ_T J i_q r_q C_q Y & \xrightarrow[\cong]{IQ_T J(\theta_{C_q Y})} & IQ_T J C_q Y & \xleftarrow[\cong]{\sigma_Y} & Y \end{array}$$

where σ denotes the unit of the adjunction between C_q and $IQ_T J$. But Propositions 3.2.17 and 3.2.12 imply that all the horizontal maps are isomorphisms in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$. Now if we define

$$\epsilon_X = (IQ_T J(\theta_{C_q X}))^{-1} \circ (\sigma_X)$$

we get the natural isomorphism of functors $\epsilon : \text{id} \rightarrow (IQ_T J i_q)(r_q C_q)$.

To finish the proof, we proceed to construct the natural equivalence η . Let $f : X \rightarrow Y$ be a map in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Applying the functor $C_q IQ_T J i_q$, we get the following commutative diagram in $\mathcal{H}(S)$:

$$\begin{array}{ccc} C_q IQ_T J i_q X & \xrightarrow{\rho_{i_q X}} & i_q X \\ C_q IQ_T J i_q f \downarrow & & \downarrow i_q f \\ C_q IQ_T J i_q Y & \xrightarrow{\rho_{i_q Y}} & i_q Y \end{array}$$

where ρ denotes the counit of the adjunction between C_q and $IQ_T J$. Now if we apply the functor r_q , we have the following commutative diagram in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$:

$$\begin{array}{ccccc} r_q C_q IQ_T J i_q X & \xrightarrow[\cong]{r_q(\rho_{i_q X})} & r_q i_q X & \xleftarrow[\cong]{\tau_X} & X \\ r_q C_q IQ_T J i_q f \downarrow & & r_q i_q f \downarrow & & \downarrow f \\ r_q C_q IQ_T J i_q Y & \xrightarrow[\cong]{r_q(\rho_{i_q Y})} & r_q i_q Y & \xleftarrow[\cong]{\tau_Y} & Y \end{array}$$

where τ denotes the unit of the adjunction between i_q and r_q . But Proposition 3.2.16 and Remark 3.1.13 imply that all the horizontal maps are isomorphisms in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Now if we define

$$\eta_X = (\tau_X)^{-1} \circ r_q(\rho_{i_q X})$$

we get the natural isomorphism of functors $\eta : (r_q C_q)(IQ_T J i_q) \rightarrow \text{id}$. This finishes the proof. \square

Proposition 3.2.22. — 1. We have the following commutative diagram of left Quillen functors:

$$(34) \quad \begin{array}{ccc} R_{C_{\text{eff}}^{q+1}} \text{Spt}_T \mathcal{M}_* & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* \\ & \searrow \text{id} & \swarrow \text{id} \\ & \text{Spt}_T \mathcal{M}_* & \end{array}$$

2. For every T -spectrum X , the natural map:

$$C_q C_{q+1} X \xrightarrow{C_q^{C_{q+1} X}} C_{q+1} X$$

is a weak equivalence in $\mathcal{H}(S)$, and it induces a natural equivalence $C_q^{C_{q+1}^-} : C_q \circ C_{q+1} \rightarrow C_{q+1}$ between the following functors:

$$\begin{array}{ccc} R_{C_{\text{eff}}^{q+1}} \mathcal{H}(S) & \xrightarrow{C_{q+1}} & R_{C_{\text{eff}}^q} \mathcal{H}(S) \\ & \searrow C_{q+1} & \swarrow C_q \\ & \mathcal{H}(S) & \end{array}$$

3. The natural transformation $f_{q+1} X \rightarrow f_q X$ (see Theorem 3.1.16(1)) gets canonically identified, through the equivalence of categories $r_q C_q, IQ_T J i_q$ constructed in Proposition 3.2.21; with the following composition in $\mathcal{H}(S)$

$$\begin{array}{ccc} & C_q C_{q+1} IQ_T J X & \\ & \uparrow (C_q^{C_{q+1} IQ_T J X})^{-1} & \downarrow C_q (C_{q+1}^{IQ_T J X}) \\ C_{q+1} IQ_T J X & & C_q IQ_T J X \end{array}$$

which is induced by the following commutative diagram in $\text{Spt}_T \mathcal{M}_*$

$$(35) \quad \begin{array}{ccc} C_q C_{q+1} IQ_T J X & \xrightarrow{C_q (C_{q+1}^{IQ_T J X})} & C_q IQ_T J X \\ \downarrow C_q^{C_{q+1} IQ_T J X} & & \downarrow C_q^{IQ_T J X} \\ C_{q+1} IQ_T J X & \xrightarrow{C_{q+1}^{IQ_T J X}} & IQ_T J X \end{array}$$

Proof. — (1): Since $R_{C_{\text{eff}}^{q+1}} \text{Spt}_T \mathcal{M}_*$ and $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$ are both right Bousfield localizations of $\text{Spt}_T \mathcal{M}_*$, by construction the identity functor

$$\begin{array}{ccc} R_{C_{\text{eff}}^{q+1}} \text{Spt}_T \mathcal{M}_* & \xrightarrow{\text{id}} & \text{Spt}_T \mathcal{M}_* \\ R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* & \xrightarrow{\text{id}} & \text{Spt}_T \mathcal{M}_* \end{array}$$

is in both cases a left Quillen functor. To finish the proof, it suffices to show that the identity functor

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* \rightarrow R_{C_{\text{eff}}^{q+1}} \text{Spt}_T \mathcal{M}_*$$

is a right Quillen functor. Using the universal property of right Bousfield localizations (see definition 1.8.2), it is enough to check that if $f : X \rightarrow Y$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$ then $IQ_T J(f)$ is a C_{eff}^{q+1} -colocal equivalence. But since $IQ_T JX$ and $IQ_T JY$ are already fibrant in $\text{Spt}_T \mathcal{M}_*$, we have that $IQ_T J(f)$ is a C_{eff}^{q+1} -colocal equivalence if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q+1}$, the induced map:

$$\begin{array}{c} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JX) \\ \downarrow IQ_T J(f)_* \\ \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T JY) \end{array}$$

is a weak equivalence of simplicial sets. But since $C_{\text{eff}}^{q+1} \subseteq C_{\text{eff}}^q$, and by hypothesis f is a C_{eff}^q -colocal equivalence; we have that all the induced maps $IQ_T J(f)_*$ are weak equivalences of simplicial sets. Thus $IQ_T J(f)$ is a C_{eff}^{q+1} -colocal equivalence, as we wanted.

Finally (2) and (3) follow directly from Proposition 3.2.21, Theorem 3.2.20 together with the commutative diagram (34) of left Quillen functors constructed above and [10, Theorem 1.3.7]. □

Theorem 3.2.23. — *We have the following commutative diagram of left Quillen functors:*

$$(36) \quad \begin{array}{ccc} \vdots & & \\ \text{id} \downarrow & & \\ R_{C_{\text{eff}}^{q+1}} \text{Spt}_T \mathcal{M}_* & \searrow \text{id} & \\ \text{id} \downarrow & & \text{id} \searrow \\ R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_* & \xrightarrow{\text{id}} & \text{Spt}_T \mathcal{M}_* \\ \text{id} \downarrow & & \text{id} \nearrow \\ R_{C_{\text{eff}}^{q-1}} \text{Spt}_T \mathcal{M}_* & & \\ \text{id} \downarrow & & \\ \vdots & & \end{array}$$

and the associated diagram of homotopy categories:

$$(37) \quad \begin{array}{ccccc} & \vdots & & & \\ & \downarrow \uparrow & & & \\ & R_{C_{\text{eff}}^{q+1}} \mathcal{H}(S) & \begin{array}{l} \swarrow C_{q+1} \\ \searrow IQ_T J \end{array} & & \\ & \downarrow C_{q+1} \uparrow IQ_T J & & & \\ & R_{C_{\text{eff}}^q} \mathcal{H}(S) & \begin{array}{l} \xrightarrow{C_q} \\ \xleftarrow{IQ_T J} \end{array} & \mathcal{H}(S) & \\ & \downarrow C_q \uparrow IQ_T J & \begin{array}{l} \swarrow C_{q-1} \\ \searrow IQ_T J \end{array} & & \\ & R_{C_{\text{eff}}^{q-1}} \mathcal{H}(S) & & & \\ & \downarrow \uparrow & & & \\ & \vdots & & & \end{array}$$

gets canonically identified, through the equivalences of categories $r_q C_q, IQ_T J i_q$ constructed in Proposition 3.2.21; with Voevodsky's slice filtration:

$$(38) \quad \begin{array}{ccccc} & \vdots & & & \\ & \downarrow \uparrow & & & \\ & \Sigma_T^{q+1} \Sigma_T^q \mathcal{H}^{\text{eff}}(S) & \begin{array}{l} \swarrow i_{q+1} \\ \searrow r_{q+1} \end{array} & & \\ & \downarrow i_{q+1} \uparrow r_q & & & \\ & \Sigma_T^q \mathcal{H}^{\text{eff}}(S) & \begin{array}{l} \xrightarrow{i_q} \\ \xleftarrow{r_q} \end{array} & \mathcal{H}(S) & \\ & \downarrow i_q \uparrow r_{q-1} & \begin{array}{l} \swarrow i_{q-1} \\ \searrow r_{q-1} \end{array} & & \\ & \Sigma_T^{q-1} \Sigma_T^q \mathcal{H}^{\text{eff}}(S) & & & \\ & \downarrow \uparrow & & & \\ & \vdots & & & \end{array}$$

Proof. — It follows immediately from propositions 3.2.22 and 3.2.21. □

Remark 3.2.24. — The drawback of the model structures on $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$ is that it is not clear if they are cellular again. Therefore in order to recover a lifting for the slice functors s_q , we are forced to take an indirect approach.

The first step in this new approach will be to construct another family of model structures on $\text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}}$, via left Bousfield localization; such that the fibrant replacement functor provides an alternative description of the functors $s_{<q}$ defined in Theorem 3.1.18.

Definition 3.2.25. — For $r \geq 1$, we define D^r using the following pushout diagram of simplicial sets:

$$\begin{array}{ccc} S^{r-1} & \xrightarrow{j_0} & S^{r-1} \times \Delta^1 \\ \downarrow & & \downarrow p \\ * & \longrightarrow & D^r \end{array}$$

where j_0 is the following composition:

$$S^{r-1} \cong S^{r-1} \times \Delta^0 \xrightarrow{\text{id} \times d_1} S^{r-1} \times \Delta^1$$

and let $\iota_1 : S^{r-1} \rightarrow D^r$ be the following composition:

$$S^{r-1} \cong S^{r-1} \times \Delta^0 \xrightarrow{\text{id} \times d_0} S^{r-1} \times \Delta^1 \xrightarrow{p} D^r$$

Remark 3.2.26. — It is clear that the canonical map $* \rightarrow D^r$ is a trivial cofibration in the category of pointed simplicial sets.

Proposition 3.2.27. — For every $r \geq 1, s \geq 0$, and for every scheme $U \in \mathcal{M}|_S$; the pointed simplicial presheaf on the smooth Nisnevich site over S

$$D^r \wedge \mathbb{G}_m^s \wedge U_+$$

has the following properties:

1. it is compact in the sense of Jardine (see definition 2.3.10).
2. the canonical map $* \rightarrow F_n(D^r \wedge \mathbb{G}_m^s \wedge U_+)$ is a trivial cofibration in $\text{Spt}_T \mathcal{M}_*$.
3. the canonical map $F_n(D^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow *$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.

Proof. — (1): It is clear from the construction that D^r has only finitely many non-degenerate simplices. Therefore the result follows from [14, Lemma 2.2].

(2): Proposition 2.3.7 implies that \mathcal{M}_* is a \mathbf{SSets}_* -model category; and since $\mathbb{G}_m^s \wedge U_+$ is cofibrant, we have the following Quillen adjunction:

$$\mathbf{SSets}_* \begin{array}{c} \xrightarrow{- \wedge \mathbb{G}_m^s \wedge U_+} \\ \xleftarrow{\text{Map}_*(\mathbb{G}_m^s \wedge U_+, -)} \end{array} \mathcal{M}_*$$

But $* \rightarrow D^r$ is a trivial cofibration of pointed simplicial sets, therefore the induced map

$$* \cong * \wedge \mathbb{G}_m^s \wedge U_+ \longrightarrow D^r \wedge \mathbb{G}_m^s \wedge U_+$$

is a trivial cofibration in \mathcal{M}_* . Finally using Proposition 2.4.17 we have that

$$(F_n, Ev_n, \varphi) : \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a Quillen adjunction. Hence the canonical map

$$* \cong F_n(*) \longrightarrow F_n(D^r \wedge \mathbb{G}_m^s \wedge U_+)$$

is a trivial cofibration in $\text{Spt}_T \mathcal{M}_*$, as we wanted.

(3): It follows immediately from (2) and the two out of three property for weak equivalences. \square

Proposition 3.2.28. — *For every compact generator $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$ (see Proposition 3.1.5), there exists a natural cofibration:*

$$F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \xrightarrow{\iota_{n,r,s}^U} F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+)$$

in $\text{Spt}_T \mathcal{M}_*$.

Proof. — We define $\iota_{n,r,s}^U$ as $F_n(\iota_1 \wedge \mathbb{G}_m^s \wedge U_+)$, where $\iota_1 : S^r \rightarrow D^{r+1}$ is the map constructed in definition 3.2.25.

It is clear that ι_1 is a cofibration of pointed simplicial sets, therefore the result follows from Propositions 2.4.17 and 2.3.7 which imply that F_n and $-\wedge \mathbb{G}_m^s \wedge U_+$ are both left Quillen functors. \square

Theorem 3.2.29. — *Consider the following set of maps in $\text{Spt}_T \mathcal{M}_*$:*

$$(39) \quad L(< q) = \{ \iota_{n,r,s}^U : F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \mid F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q \}$$

The left Bousfield localization of $\text{Spt}_T \mathcal{M}_*$ with respect to the $L(< q)$ -local equivalences exists. This new model structure will be called *weight^{< q} motivic stable*. $L_{< q} \text{Spt}_T \mathcal{M}_*$ will denote the category of T -spectra equipped with the *weight^{< q} motivic stable* model structure, and $L_{< q} \mathcal{H}(S)$ will denote its associated homotopy category. Furthermore the *weight^{< q} motivic stable* model structure is cellular, left proper and simplicial; with the following sets of generating cofibrations and trivial cofibrations respectively:

$$I_{L(< q)} = I_{M_*}^T = \bigcup_{n \geq 0} \{ F_n(Y_+ \hookrightarrow (\Delta_U^n)_+) \}$$

$$J_{L(< q)} = \{ j : A \rightarrow B \}$$

where j satisfies the following conditions:

1. j is an inclusion of $I_{M_*}^T$ -complexes.
2. j is a $L(< q)$ -local equivalence.
3. the size of B as an $I_{M_*}^T$ -complex is less than κ , where κ is the regular cardinal defined by Hirschhorn in [6, definition 4.5.3].

Proof. — Theorems 2.5.4 and 2.4.16 imply that $\text{Spt}_T\mathcal{M}_*$ is a cellular, proper and simplicial model category. Therefore the existence of the left Bousfield localization follows from [6, Theorem 4.1.1]. Using [6, Theorem 4.1.1] again, we have that $L_{<q}\text{Spt}_T\mathcal{M}_*$ is cellular, left proper and simplicial; where the sets of generating cofibrations and trivial cofibrations are the ones described above. \square

Definition 3.2.30. — Let W_q denote a fibrant replacement functor in $L_{<q}\text{Spt}_T\mathcal{M}_*$; such that for every T -spectrum X , the natural map:

$$X \xrightarrow{W_q^X} W_q X$$

is a trivial cofibration in $L_{<q}\text{Spt}_T\mathcal{M}_*$, and $W_q X$ is $L(<q)$ -local in $\text{Spt}_T\mathcal{M}_*$.

Proposition 3.2.31. — Q_s is also a cofibrant replacement functor in $L_{<q}\text{Spt}_T\mathcal{M}_*$ (see definition 3.1.1), and for every T -spectrum X the natural map

$$Q_s X \xrightarrow{Q_s^X} X$$

is a trivial fibration in $L_{<q}\text{Spt}_T\mathcal{M}_*$.

Proof. — Since $L_{<q}\text{Spt}_T\mathcal{M}_*$ is the left Bousfield localization of $\text{Spt}_T\mathcal{M}_*$ with respect to the $L(<q)$ -local equivalences, by construction we have that the cofibrations and the trivial fibrations are identical in $L_{<q}\text{Spt}_T\mathcal{M}_*$ and $\text{Spt}_T\mathcal{M}_*$ respectively. This implies that for every T -spectrum X , $Q_s X$ is cofibrant in $L_{<q}\text{Spt}_T\mathcal{M}_*$, and we also have that the natural map

$$Q_s X \xrightarrow{Q_s^X} X$$

is a trivial fibration in $L_{<q}\text{Spt}_T\mathcal{M}_*$. Hence Q_s is also a cofibrant replacement functor for $L_{<q}\text{Spt}_T\mathcal{M}_*$. \square

Proposition 3.2.32. — Let Z be an arbitrary T -spectrum. We have that Z is $L(<q)$ -local in $\text{Spt}_T\mathcal{M}_*$ if and only if the following conditions hold:

1. Z is fibrant in $\text{Spt}_T\mathcal{M}_*$.
2. For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$

Proof. — (\Rightarrow): Assume that Z is $L(<q)$ -local. Then by definition we have that Z must be fibrant in $\text{Spt}_T\mathcal{M}_*$. Since all the T -spectra $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ and $F_n(D^r \wedge \mathbb{G}_m^s \wedge U_+)$ are cofibrant, and Z is $L(<q)$ -local; for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ we get the following weak equivalence of simplicial sets:

$$\text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z) \xrightarrow{(\iota_{n,r,s}^U)^*} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)$$

Now we have that $\text{Spt}_T \mathcal{M}_*$ is in particular a simplicial model category, therefore we get the following commutative diagram:

$$\begin{array}{ccc} \pi_0 \text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z) & \xrightarrow[\cong]{(\iota_{n,r,s}^U)^*} & \pi_0 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) \\ \cong \downarrow & & \downarrow \cong \\ [F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} & \xrightarrow[\cong]{(\iota_{n,r,s}^U)^*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \end{array}$$

where the vertical arrows and the top row are isomorphisms. Therefore we get the following isomorphism:

$$[F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \xrightarrow[\cong]{(\iota_{n,r,s}^U)^*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}$$

Finally Proposition 3.2.27(2) implies that $[F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$. Thus, for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ we have that $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$, as we wanted.

(\Leftarrow): Assume that Z satisfies (1) and (2). Let ω_0, η_0 denote the base points corresponding to the pointed simplicial sets $\text{Map}_*(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z)$ and $\text{Map}_*(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)$ respectively. Since $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ and $F_n(D^r \wedge \mathbb{G}_m^s \wedge U_+)$ are always cofibrant, it is enough to show that the induced map:

$$\text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z) \xrightarrow{(\iota_{n,r,s}^U)^*} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)$$

is a weak equivalence of simplicial sets for every map $\iota_{n,r,s}^U \in L(< q)$.

Fix $\iota_{n,r,s}^U \in L(< q)$. By Proposition 3.2.27(3) we know that the map $F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow *$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Then Ken Brown's lemma (see Lemma 1.1.4) together with the fact that $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, imply that the following map is a weak equivalence of simplicial sets:

$$* \cong \text{Map}(*, Z) \longrightarrow \text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z)$$

In particular $\text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z)$ has only one path connected component.

Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, we have the following isomorphism of abelian groups

$$\pi_0 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) \xrightarrow{\cong} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}$$

but our hypothesis implies that $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$, hence $\pi_0 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) \cong 0$, i.e. $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)$ has only one path connected component.

Now Proposition 3.2.27(2) implies that $* \rightarrow F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+)$ is a trivial cofibration in $\text{Spt}_T \mathcal{M}_*$, and since $-\wedge S^1$ is a left Quillen functor, it follows that

$$* \cong * \wedge S^k \longrightarrow F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k$$

is also a trivial cofibration for $k \geq 0$. Therefore $[F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Z]_{\text{Spt}} \cong 0$, and this implies that the induced map $(\iota_{n,r,s}^U)^*$ is an isomorphism of abelian groups:

$$\begin{array}{ccc} 0 \cong [F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Z]_{\text{Spt}} & & \\ & \downarrow (\iota_{n,r,s}^U)^* & \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Z]_{\text{Spt}} & \cong & [F_n(S^{k+r} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \end{array}$$

since by hypothesis $[F_n(S^{k+r} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$.

On the other hand, since $\text{Spt}_T \mathcal{M}_*$ is a pointed simplicial model category, we have that Lemma 6.1.2 in [10] together with Remark 2.4.3(2) imply that the following diagram is commutative for $k \geq 0$ and all the vertical arrows are isomorphisms:

$$\begin{array}{ccc} \pi_{k,\omega_0} \text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z) & & \\ \parallel & \searrow (\iota_{n,r,s}^U)^* & \pi_{k,\eta_0} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) \\ \parallel & & \parallel \\ \pi_{k,\omega_0} \text{Map}_*(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z) & \searrow (\iota_{n,r,s}^U)^* & \pi_{k,\eta_0} \text{Map}_*(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) \\ \cong \downarrow & & \cong \downarrow \\ [F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Z]_{\text{Spt}} & \searrow (\iota_{n,r,s}^U)^* & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Z]_{\text{Spt}} \\ \cong \downarrow & & \cong \downarrow \\ [F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^k, Z]_{\text{Spt}} & \searrow (\iota_{n,r,s}^U)^* & [F_n(S^{k+r} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \end{array}$$

but we know that the bottom row is always an isomorphism of abelian groups, hence the top row is also an isomorphism. This implies that the map

$$\text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z) \xrightarrow{(\iota_{n,r,s}^U)^*} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)$$

is a weak equivalence when it is restricted to the path component of $\text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z)$ containing ω_0 . However we already know that $\text{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z)$ and $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)$ have only one path connected component. This implies that the map defined above is a weak equivalence of simplicial sets, as we wanted. \square

Corollary 3.2.33. — *Let $m, n \in \mathbb{Z}$ with $m > n$. If Z is a $L(< n)$ -local T -spectrum in $\text{Spt}_T \mathcal{M}_*$, then Z is also $L(< m)$ -local in $\text{Spt}_T \mathcal{M}_*$.*

Proof. — We have that $C_{\text{eff}}^m \subseteq C_{\text{eff}}^n$, since $m > n$. The result now follows immediately from the characterization of $L(< q)$ -local objects given in Proposition 3.2.32. \square

Corollary 3.2.34. — *Let Z be a fibrant T -spectrum in $\text{Spt}_T \mathcal{M}_*$. We have that Z is $L(< q)$ -local if and only if $\Omega_{S^1} Z$ is $L(< q)$ -local.*

Proof. — (\Rightarrow): Assume that Z is $L(< q)$ -local. We have that Z is fibrant in $\text{Spt}_T \mathcal{M}_*$; and since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, it follows that $\Omega_{S^1} Z$ is also fibrant.

Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, we have the following natural isomorphisms:

$$\begin{aligned} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} Z]_{\text{Spt}} &\cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, Z]_{\text{Spt}} \\ &\cong [F_n(S^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \end{aligned}$$

but Proposition 3.2.32 implies that $[F_n(S^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$, hence $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} Z]_{\text{Spt}} \cong 0$ for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Finally, using Proposition 3.2.32 again, we have that $\Omega_{S^1} Z$ is $L(< q)$ -local, as we wanted.

(\Leftarrow): Assume that $\Omega_{S^1} Z$ is $L(< q)$ -local. Since by hypothesis Z is fibrant in $\text{Spt}_T \mathcal{M}_*$, Proposition 3.2.32 implies that it is enough to show that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$$

Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, and Z is fibrant by hypothesis; we have the following natural isomorphisms of abelian groups:

$$\begin{aligned} [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), \Omega_{S^1} Z]_{\text{Spt}} &\cong [F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), Z]_{\text{Spt}} \\ &\cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \end{aligned}$$

Now using Proposition 3.2.32 and the fact that $\Omega_{S^1}Z$ is $L(< q)$ -local, it follows that $[F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), \Omega_{S^1}Z]_{\text{Spt}} \cong 0$. Therefore, $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$ for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, as we wanted. \square

Corollary 3.2.35. — *Let Z be a fibrant T -spectrum in $\text{Spt}_T\mathcal{M}_*$. We have that Z is $L(< q)$ -local if and only if $IQ_TJ(Q_s Z \wedge S^1)$ is $L(< q)$ -local.*

Proof. — (\Rightarrow): Assume that Z is $L(< q)$ -local. Since $IQ_TJ(Q_s Z \wedge S^1)$ is fibrant, using proposition 3.2.32 we have that it is enough to check that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_TJ(Q_s Z \wedge S^1)]_{\text{Spt}} \cong 0$. But since $-\wedge S^1$ is a Quillen equivalence, we get the following diagram:

$$\begin{array}{ccc}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_TJ(Q_s Z \wedge S^1)]_{\text{Spt}} & & \\
 \searrow \cong & & \\
 & [F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), IQ_TJ(Q_s Z \wedge S^1)]_{\text{Spt}} & \\
 & \uparrow \cong & \\
 [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Z]_{\text{Spt}} & & \\
 \searrow \Sigma_T^{1,0} \cong & & \\
 & [F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), Q_s Z \wedge S^1]_{\text{Spt}} &
 \end{array}$$

where all the maps are isomorphisms of abelian groups. Since Z is $L(< q)$ -local, Proposition 3.2.32 implies that $[F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Z]_{\text{Spt}} \cong 0$. Therefore

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_TJ(Q_s Z \wedge S^1)]_{\text{Spt}} \cong 0$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, as we wanted.

(\Leftarrow): Assume that $IQ_TJ(Q_s Z \wedge S^1)$ is $L(< q)$ -local. By hypothesis, Z is fibrant; therefore Proposition 3.2.32 implies that it is enough to show that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$. Since $\text{Spt}_T\mathcal{M}_*$ is a simplicial model category and $-\wedge S^1$ is a Quillen equivalence; we have the following diagram:

$$\begin{array}{ccc}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1}IQ_TJ(Q_s Z \wedge S^1)]_{\text{Spt}} & & \\
 \cong \downarrow & & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, Q_s Z \wedge S^1]_{\text{Spt}} & \xleftarrow[\Sigma_T^{1,0}]{\cong} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}
 \end{array}$$

where all the maps are isomorphisms of abelian groups. On the other hand, using Corollary 3.2.34 we have that $\Omega_{S^1}IQ_TJ(Q_s Z \wedge S^1)$ is $L(< q)$ -local. Therefore using Proposition 3.2.32 again, we have that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1}IQ_TJ(Q_s Z \wedge S^1)]_{\text{Spt}} \cong 0$$

and this finishes the proof. □

Corollary 3.2.36. — *Let $f : X \rightarrow Y$ be a map in $\text{Spt}_T \mathcal{M}_*$. We have that f is a $L(< q)$ -local equivalence if and only if for every $L(< q)$ -local T -spectrum Z , f induces the following isomorphism of abelian groups:*

$$[Y, Z]_{\text{Spt}} \xrightarrow{f^*} [X, Z]_{\text{Spt}}$$

Proof. — Suppose that f is a $L(< q)$ -local equivalence, then by definition the induced map:

$$\text{Map}(Q_s Y, Z) \xrightarrow{(Q_s f)^*} \text{Map}(Q_s X, Z)$$

is a weak equivalence of simplicial sets for every $L(< q)$ -local T -spectrum Z . Proposition 3.2.32(1) implies that Z is fibrant in $\text{Spt}_T \mathcal{M}_*$, and since $\text{Spt}_T \mathcal{M}_*$ is in particular a simplicial model category; we get the following commutative diagram, where the top row and all the vertical maps are isomorphisms of abelian groups:

$$\begin{array}{ccc} \pi_0 \text{Map}(Q_s Y, Z) & \xrightarrow[\cong]{(Q_s f)^*} & \pi_0 \text{Map}(Q_s X, Z) \\ \cong \downarrow & & \downarrow \cong \\ [Y, Z]_{\text{Spt}} & \xrightarrow{f^*} & [X, Z]_{\text{Spt}} \end{array}$$

hence f^* is an isomorphism for every $L(< q)$ -local T -spectrum Z , as we wanted.

Conversely, assume that for every $L(< q)$ -local T -spectrum Z , the induced map

$$[Y, Z]_{\text{Spt}} \xrightarrow{f^*} [X, Z]_{\text{Spt}}$$

is an isomorphism of abelian groups.

Since $L_{< q} \text{Spt}_T \mathcal{M}_*$ is the left Bousfield localization of $\text{Spt}_T \mathcal{M}_*$ with respect to the $L(< q)$ -local equivalences, we have that the identity functor $\text{id} : \text{Spt}_T(\mathcal{M}|_S)_{\text{Nis}} \rightarrow L_{< q} \text{Spt}_T \mathcal{M}_*$ is a left Quillen functor. Therefore for every T -spectrum Z , we get the following commutative diagram where all the vertical arrows are isomorphisms:

$$\begin{array}{ccc} \text{Hom}_{L_{< q} \mathcal{H}(S)}(Q_s Y, Z) & \xrightarrow{(Q_s f)^*} & \text{Hom}_{L_{< q} \mathcal{H}(S)}(Q_s X, Z) \\ \cong \downarrow & & \downarrow \cong \\ [Y, W_q Z]_{\text{Spt}} & \xrightarrow[\cong]{f^*} & [X, W_q Z]_{\text{Spt}} \end{array}$$

but $W_q Z$ is by construction $L(< q)$ -local, then by hypothesis the bottom row is an isomorphism of abelian groups. Hence it follows that the induced map:

$$\text{Hom}_{L_{< q} \mathcal{H}(S)}(Q_s Y, Z) \xrightarrow[\cong]{(Q_s f)^*} \text{Hom}_{L_{< q} \mathcal{H}(S)}(Q_s X, Z)$$

is an isomorphism for every T -spectrum Z . This implies that $Q_s f$ is a weak equivalence in $L_{< q} \text{Spt}_T \mathcal{M}_*$, and since Q_s is also a cofibrant replacement functor in $L_{< q} \text{Spt}_T \mathcal{M}_*$, it follows that f is a weak equivalence in $L_{< q} \text{Spt}_T \mathcal{M}_*$. Therefore we have that f is a $L(< q)$ -local equivalence, as we wanted. \square

Lemma 3.2.37. — *Let $f : X \rightarrow Y$ be a map in $\text{Spt}_T \mathcal{M}_*$. We have that f is a $L(< q)$ -local equivalence if and only if*

$$Q_s f \wedge \text{id} : Q_s X \wedge S^1 \rightarrow Q_s Y \wedge S^1$$

is a $L(< q)$ -local equivalence.

Proof. — Assume that f is a $L(< q)$ -local equivalence, and let Z be an arbitrary $L(< q)$ -local T -spectrum. Then Corollary 3.2.34 implies that $\Omega_{S^1} Z$ is also $L(< q)$ -local. Therefore the induced map

$$\text{Map}(Q_s Y, \Omega_{S^1} Z) \xrightarrow{(Q_s f)^*} \text{Map}(Q_s X, \Omega_{S^1} Z)$$

is a weak equivalence of simplicial sets. Now since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Map}(Q_s Y, \Omega_{S^1} Z) & \xrightarrow{(Q_s f)^*} & \text{Map}(Q_s X, \Omega_{S^1} Z) \\ \cong \downarrow & & \downarrow \cong \\ \text{Map}(Q_s Y \wedge S^1, Z) & \xrightarrow{(Q_s f \wedge \text{id})^*} & \text{Map}(Q_s X \wedge S^1, Z) \end{array}$$

and using the two out of three property for weak equivalences of simplicial sets, we have that

$$\text{Map}(Q_s Y \wedge S^1, Z) \xrightarrow{(Q_s f \wedge \text{id})^*} \text{Map}(Q_s X \wedge S^1, Z)$$

is a weak equivalence. Since this holds for every $L(< q)$ -local T -spectrum Z , it follows that

$$Q_s f \wedge \text{id} : Q_s X \wedge S^1 \rightarrow Q_s Y \wedge S^1$$

is a $L(< q)$ -local equivalence, as we wanted.

Conversely, suppose that

$$Q_s f \wedge \text{id} : Q_s X \wedge S^1 \rightarrow Q_s Y \wedge S^1$$

is a $L(< q)$ -local equivalence. Let Z be an arbitrary $L(< q)$ -local T -spectrum. Since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category and $-\wedge S^1$ is a Quillen equivalence, we get

the following commutative diagram:

$$\begin{array}{ccc}
 [Q_s Y \wedge S^1, IQ_T J(Q_s Z \wedge S^1)]_{\text{Spt}} & \xrightarrow{(Q_s f \wedge \text{id})^*} & [Q_s X \wedge S^1, IQ_T J(Q_s Z \wedge S^1)]_{\text{Spt}} \\
 \cong \downarrow & & \downarrow \cong \\
 [Q_s Y \wedge S^1, Q_s Z \wedge S^1]_{\text{Spt}} & \xrightarrow{(Q_s f \wedge \text{id})^*} & [Q_s X \wedge S^1, Q_s Z \wedge S^1]_{\text{Spt}} \\
 \cong \uparrow \Sigma_T^{1,0} & & \Sigma_T^{1,0} \uparrow \cong \\
 [Y, Z]_{\text{Spt}} & \xrightarrow{f^*} & [X, Z]_{\text{Spt}}
 \end{array}$$

Now, Corollary 3.2.35 implies that $IQ_T J(Q_s Z \wedge S^1)$ is also $L(< q)$ -local. Therefore using Corollary 3.2.36 we have that the top row in the diagram above is an isomorphism of abelian groups. This implies that the induced map:

$$[Y, Z]_{\text{Spt}} \xrightarrow{f^*} [X, Z]_{\text{Spt}}$$

is an isomorphism of abelian groups for every $L(< q)$ -local spectrum Z . Finally using Corollary 3.2.36 again, we have that $f : X \rightarrow Y$ is a $L(< q)$ -local equivalence, as we wanted. \square

Corollary 3.2.38. — *The following adjunction:*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : L_{< q} \text{Spt}_T \mathcal{M}_* \longrightarrow L_{< q} \text{Spt}_T \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Using Corollary 1.3.16 in [10] and Proposition 3.2.31 we have that it suffices to verify the following two conditions:

1. For every fibrant object X in $L_{< q} \text{Spt}_T \mathcal{M}_*$, the following composition

$$(Q_s \Omega_{S^1} X) \wedge S^1 \xrightarrow{Q_s^{\Omega_{S^1} X} \wedge \text{id}} (\Omega_{S^1} X) \wedge S^1 \xrightarrow{\epsilon_X} X$$

is a $L(< q)$ -local equivalence.

2. $- \wedge S^1$ reflects $L(< q)$ -local equivalences between cofibrant objects in $L_{< q} \text{Spt}_T \mathcal{M}_*$.

(1): By construction $L_{< q} \text{Spt}_T \mathcal{M}_*$ is a left Bousfield localization of $\text{Spt}_T \mathcal{M}_*$, therefore the identity functor

$$\text{id} : L_{< q} \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a right Quillen functor. Thus X is also fibrant in $\text{Spt}_T \mathcal{M}_*$. Since the adjunction $(- \wedge S^1, \Omega_{S^1}, \varphi)$ is a Quillen equivalence on $\text{Spt}_T \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies

that the following composition is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$:

$$(Q_s \Omega_{S^1} X) \wedge S^1 \xrightarrow{Q_s^\Omega \Omega_{S^1} X \wedge \text{id}} (\Omega_{S^1} X) \wedge S^1 \xrightarrow{\epsilon_X} X$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $L(< q)$ -local equivalence.

(2): This follows immediately from Proposition 3.2.31 and Lemma 3.2.37. \square

Remark 3.2.39. — We have a situation similar to the one described in remark 3.2.10 for the model categories $R_{C_{\text{off}}^q} \text{Spt}_T \mathcal{M}_*$; i.e. although the adjunction $(\Sigma_T, \Omega_T, \varphi)$ is a Quillen equivalence on $\text{Spt}_T \mathcal{M}_*$, it does not descend even to a Quillen adjunction on the weight $^{< q}$ motivic stable model category $L_{< q} \text{Spt}_T \mathcal{M}_*$.

Corollary 3.2.40. — The homotopy category $L_{< q} \mathcal{H}(S)$ associated to $L_{< q} \text{Spt}_T \mathcal{M}_*$ has the structure of a triangulated category.

Proof. — Theorem 3.2.29 implies in particular that $L_{< q} \text{Spt}_T \mathcal{M}_*$ is a pointed simplicial model category, and Corollary 3.2.38 implies that the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : L_{< q} \text{Spt}_T \mathcal{M}_* \rightarrow L_{< q} \text{Spt}_T \mathcal{M}_*$$

is a Quillen equivalence. Therefore the result follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII]. \square

Corollary 3.2.41. — $L_{< q} \text{Spt}_T \mathcal{M}_*$ is a right proper model category.

Proof. — We need to show that the $L(< q)$ -local equivalences are stable under pull-back along fibrations in $L_{< q} \text{Spt}_T \mathcal{M}_*$. Consider the following pullback diagram:

$$\begin{array}{ccc} Z & \xrightarrow{w^*} & X \\ p^* \downarrow & & \downarrow p \\ W & \xrightarrow{w} & Y \end{array}$$

where p is a fibration in $L_{< q} \text{Spt}_T \mathcal{M}_*$, and w is a $L(< q)$ -local equivalence. Let F be the homotopy fibre of p . Then we get the following commutative diagram in $L_{< q} \mathcal{H}(S)$:

$$\begin{array}{ccccccc} \Omega_{S^1} Y & \xrightarrow{q} & F & \xrightarrow{i} & X & \xrightarrow{p} & Y \\ \Omega_{S^1} w \uparrow & & \parallel & & \uparrow w^* & & \uparrow w \\ \Omega_{S^1} W & \xrightarrow{r} & F & \xrightarrow{j} & Z & \xrightarrow{p^*} & W \end{array}$$

Since the rows in the diagram above are both fibre sequences in $L_{< q} \text{Spt}_T \mathcal{M}_*$, it follows that both rows are distinguished triangles in $L_{< q} \mathcal{H}(S)$ (which has the structure of a triangulated category given by Corollary 3.2.40). Now w, id_F are both isomorphisms

in $L_{<q}\mathcal{H}(S)$, hence it follows that w^* is also an isomorphism in $L_{<q}\mathcal{H}(S)$. Therefore w^* is a $L(<q)$ -local equivalence, as we wanted. \square

Proposition 3.2.42. — *We have the following adjunction*

$$(Q_s, W_q, \varphi) : \mathcal{H}(S) \longrightarrow L_{<q}\mathcal{H}(S)$$

of exact functors between triangulated categories.

Proof. — Since $L_{<q}\text{Spt}_T\mathcal{M}_*$ is the left Bousfield localization of $\text{Spt}_T\mathcal{M}_*$ with respect to the $L(<q)$ -local equivalences, we have that the identity functor $\text{id} : \text{Spt}_T\mathcal{M}_* \rightarrow L_{<q}\text{Spt}_T\mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(Q_s, W_q, \varphi) : \mathcal{H}(S) \longrightarrow L_{<q}\mathcal{H}(S)$$

Now Proposition 6.4.1 in [10] implies that Q_s maps cofibre sequences in $\mathcal{H}(S)$ to cofibre sequences in $L_{<q}\mathcal{H}(S)$. Therefore using Proposition 7.1.12 in [10] we have that Q_s and W_q are both exact functors between triangulated categories. \square

Proposition 3.2.43. — *Let $\eta_X : Q_s W_q X \rightarrow X$ denote the counit of the adjunction*

$$(Q_s, W_q, \varphi) : \mathcal{H}(S) \longrightarrow L_{<q}\mathcal{H}(S)$$

Then the following conditions hold:

1. *For every T -spectrum X , we have that η_X is an isomorphism in $L_{<q}\mathcal{H}(S)$.*
2. *The exact functor*

$$W_q : L_{<q}\mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

is a full embedding of triangulated categories.

Proof. — (1): We have that η_X is the following composition in $L_{<q}\mathcal{H}(S)$:

$$Q_s W_q X \xrightarrow{Q_s^{W_q X}} W_q X \xrightarrow{(W_q X)^{-1}} X$$

where $Q_s^{W_q X}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$. Now [6, Proposition 3.1.5] implies that $Q_s^{W_q X}$ is a $L(<q)$ -local equivalence, i.e. a weak equivalence in $L_{<q}\text{Spt}_T\mathcal{M}_*$. Therefore $Q_s^{W_q X}$ becomes an isomorphism in $L_{<q}\mathcal{H}(S)$, and this implies that η_X is an isomorphism in $L_{<q}\mathcal{H}(S)$, as we wanted.

(2): It follows immediately from (1). \square

Proposition 3.2.44. — *For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, the map $*$ $\rightarrow F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is a $L(<q)$ -local equivalence in $\text{Spt}_T\mathcal{M}_*$.*

Proof. — Let Z be an arbitrary $L(< q)$ -local T -spectrum. Then proposition 3.2.32(2) implies that the following induced map

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \longrightarrow [*, Z]_{\text{Spt}} \cong 0$$

is an isomorphism of abelian groups. Therefore using Corollary 3.2.36, it follows that $* \rightarrow F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is a $L(< q)$ -local equivalence. \square

Proposition 3.2.45. — *Let $f : X \rightarrow Y$ be a map in $L_{< q}\mathcal{H}(S)$. We have that f is an isomorphism in $L_{< q}\mathcal{H}(S)$ if and only if one of the following equivalent conditions holds:*

1. *The following map*

$$W_q X \xrightarrow[\cong]{W_q(f)} W_q Y$$

is an isomorphism in $\mathcal{H}(S)$.

2. *For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$, the induced map*

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q X]_{\text{Spt}} \xrightarrow[\cong]{(W_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q Y]_{\text{Spt}}$$

is an isomorphism of abelian groups.

3. *For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$, the induced map*

$$\begin{array}{c} \text{Hom}_{L_{< q}\mathcal{H}(S)}(Q_s F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X) \\ \downarrow f_* \\ \text{Hom}_{L_{< q}\mathcal{H}(S)}(Q_s F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y) \end{array}$$

is an isomorphism of abelian groups.

Proof. — Proposition 3.2.43 implies that f is an isomorphism in $L_{< q}\mathcal{H}(S)$ if and only if $W_q f$ becomes an isomorphism in $\mathcal{H}(S)$. Thus it only remains to show that (1), (2) and (3) are all equivalent.

(1) \Leftrightarrow (2) Corollary 3.1.6 implies that $W_q f$ is an isomorphism in $\mathcal{H}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C$ the following induced map

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q X]_{\text{Spt}} \xrightarrow[\cong]{(W_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q Y]_{\text{Spt}}$$

is an isomorphism of abelian groups. But using proposition 3.2.32(2) we have that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$,

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q X]_{\text{Spt}} \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q Y]_{\text{Spt}}$$

since by construction $W_q X$ and $W_q Y$ are both $L(< q)$ -local T -spectra. Hence $W_q f$ is an isomorphism in $\mathcal{H}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$ the

following induced map

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q X]_{\text{Spt}} \xrightarrow[\cong]{(W_q f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q Y]_{\text{Spt}}$$

is an isomorphism of abelian groups.

(2) \Leftrightarrow (3) By Proposition 3.2.42 we have the following adjunction between exact functors of triangulated categories:

$$(Q_s, W_q, \varphi) : \mathcal{H}(S) \longrightarrow L_{<q}\mathcal{H}(S)$$

In particular for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \notin C_{\text{eff}}^q$, we get the following commutative diagram, where all the vertical arrows are isomorphisms of abelian groups:

$$\begin{array}{ccc} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q X]_{\text{Spt}} & & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_q Y]_{\text{Spt}} \\ \cong \downarrow & \searrow^{(W_q f)_*} & \downarrow \cong \\ \text{Hom}_{L_{<q}\mathcal{H}(S)}(Q_s F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), X) & & \text{Hom}_{L_{<q}\mathcal{H}(S)}(Q_s F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y) \\ & \searrow^{f_*} & \end{array}$$

therefore the top row is an isomorphism if and only if the bottom row is an isomorphism of abelian groups, as we wanted. \square

Lemma 3.2.46. — *Let Z be a $L(<q)$ -local T -spectrum. We have that $f_q Z \cong *$ in $\mathcal{H}(S)$ (see Remark 3.1.13).*

Proof. — Let $j : * \rightarrow Z$ denote the canonical map. Proposition 3.1.14 implies that $f_q(j) : * \cong f_q(*) \rightarrow f_q Z$ is an isomorphism in $\mathcal{H}(S)$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$ the induced map

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), *]_{\text{Spt}} \xrightarrow{f_q(j)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}$$

is an isomorphism of abelian groups. Therefore it is enough to show that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, we have $[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}} \cong 0$. But this follows from proposition 3.2.32(2), since Z is $L(<q)$ -local by hypothesis. \square

Corollary 3.2.47. — *For every T -spectrum X , $Q_s f_q X \cong *$ in $L_{<q}\mathcal{H}(S)$.*

Proof. — We will show that the map $* \rightarrow Q_s f_q X$ is an isomorphism in $L_{<q}\mathcal{H}(S)$. By Yoneda’s lemma it suffices to check that for every T -spectrum Z , the induced map

$$\text{Hom}_{L_{<q}\mathcal{H}(S)}(Q_s f_q X, Z) \longrightarrow \text{Hom}_{L_{<q}\mathcal{H}(S)}(*, Z) \cong 0$$

is an isomorphism of abelian groups. Now Propositions 3.2.42 and 3.1.12 imply that we have the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{L_{<q}\mathcal{H}(S)}(Q_s f_q X, Z) &\cong [f_q X, W_q Z]_{\mathrm{Spt}} = [i_q r_q X, W_q Z]_{\mathrm{Spt}} \\ &\cong \mathrm{Hom}_{\Sigma_T^q \mathcal{H}^{\mathrm{eff}}(S)}(r_q X, r_q W_q Z) \end{aligned}$$

Finally since i_q is a full embedding, we have

$$\mathrm{Hom}_{\Sigma_T^q \mathcal{H}^{\mathrm{eff}}(S)}(r_q X, r_q W_q Z) \cong [i_q r_q X, i_q r_q W_q Z]_{\mathrm{Spt}} = [f_q X, f_q W_q Z]_{\mathrm{Spt}}$$

and Lemma 3.2.46 implies that $f_q W_q Z \cong *$ in $\mathcal{H}(S)$. Hence

$$\mathrm{Hom}_{L_{<q}\mathcal{H}(S)}(Q_s f_q X, Z) \cong [f_q X, f_q W_q Z]_{\mathrm{Spt}} \cong [f_q X, *]_{\mathrm{Spt}} \cong 0$$

as we wanted. \square

Proposition 3.2.48. — *For every T -spectrum X , the natural map in $L_{<q}\mathcal{H}(S)$*

$$Q_s X \xrightarrow{Q_s(\pi_{<q} X)} Q_s s_{<q} X$$

is an isomorphism, where $\pi_{<q}$ is the natural transformation defined in Theorem 3.1.18. Furthermore, these maps induce a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{Q_s} \\ \xrightarrow{Q_s s_{<q}} \end{array} L_{<q}\mathcal{H}(S)$$

Proof. — The naturality of $\pi_{<q}$ and the fact that Q_s is a functor imply that the maps $Q_s(\pi_{<q} X)$ induce a natural transformation $Q_s \rightarrow Q_s s_{<q}$. Hence it suffices to show that for every T -spectrum X , the map $Q_s(\pi_{<q} X)$ is an isomorphism in $L_{<q}\mathcal{H}(S)$.

Theorem 3.1.18 implies that we have the following distinguished triangle in $\mathcal{H}(S)$:

$$f_q X \longrightarrow X \xrightarrow{\pi_{<q} X} s_{<q} X \xrightarrow{\sigma_{<q} X} \Sigma_T^{1,0} f_q X$$

and using Proposition 3.2.42, we get the following distinguished triangle in $L_{<q}\mathcal{H}(S)$:

$$Q_s f_q X \longrightarrow Q_s X \xrightarrow{Q_s(\pi_{<q} X)} Q_s s_{<q} X \xrightarrow{Q_s(\sigma_{<q} X)} \Sigma_T^{1,0} Q_s f_q X$$

But Corollary 3.2.47 implies that $Q_s f_q X \cong *$ in $L_{<q}\mathcal{H}(S)$, therefore $Q_s(\pi_{<q} X)$ is an isomorphism in $L_{<q}\mathcal{H}(S)$, as we wanted. \square

Corollary 3.2.49. — *For every T -spectrum X , the natural map in $\mathcal{H}(S)$*

$$W_q Q_s X \xrightarrow{W_q Q_s(\pi_{<q} X)} W_q Q_s s_{<q} X$$

is an isomorphism. Furthermore, these maps induce a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{W_q Q_s} \\ \xrightarrow{W_q Q_{s_{s < q}}} \end{array} \mathcal{H}(S)$$

Proof. — Since Q_s, W_q are both functors and $\pi_{<q} : \text{id} \rightarrow s_{<q}$ is a natural transformation (see Theorem 3.1.18); we have that the maps $W_q Q_s(\pi_{<q} X)$ induce a natural transformation $W_q Q_s \rightarrow W_q Q_{s_{s < q}}$. Therefore it suffices to see that for every T -spectrum X , the map $W_q Q_s(\pi_{<q} X)$ is an isomorphism in $\mathcal{H}(S)$.

But Proposition 3.2.48 implies that the map $Q_s(\pi_{<q} X)$ is an isomorphism in $L_{<q} \mathcal{H}(S)$. Therefore using Proposition 3.2.42, we have that $W_q Q_s(\pi_{<q} X)$ is also an isomorphism in $\mathcal{H}(S)$. \square

Lemma 3.2.50. — *For every T -spectrum X , $IQ_T J(Q_{s_{s < q}} X)$ is $L(<q)$ -local in $\text{Spt}_T \mathcal{M}_*$.*

Proof. — Proposition 3.2.32 implies that it is enough to show that $IQ_T J(Q_{s_{s < q}} X)$ satisfies the following properties:

1. $IQ_T J(Q_{s_{s < q}} X)$ is fibrant in $\text{Spt}_T \mathcal{M}_*$.
2. For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T J(Q_{s_{s < q}} X)]_{\text{Spt}} \cong 0$$

The first condition is obvious since $IQ_T J$ is a fibrant replacement functor in $\text{Spt}_T \mathcal{M}_*$.

Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. Using Theorem 3.1.18(2) and the fact that $C_{\text{eff}}^q \subseteq \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, we have that

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q} X]_{\text{Spt}} \cong 0$$

Therefore

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T J(Q_{s_{s < q}} X)]_{\text{Spt}} \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_{<q} X]_{\text{Spt}} \cong 0$$

for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. This takes care of the second condition and finishes the proof. \square

Proposition 3.2.51. — *For every T -spectrum X the natural map*

$$Q_{s_{s < q}} X \xrightarrow{W_q^{Q_{s_{s < q}} X}} W_q Q_{s_{s < q}} X$$

is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$. Therefore, we have a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{Q_{s^s < q}} \\ \xrightarrow{W_q Q_{s^s < q}} \end{array} \mathcal{H}(S)$$

Proof. — The naturality of the maps $W_q^X : X \rightarrow W_q X$ implies that we have an induced natural transformation of functors $Q_{s^s < q} \rightarrow W_q Q_{s^s < q}$. Hence, it is enough to show that for every T -spectrum X , $W_q^{Q_{s^s < q} X}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

Consider the following commutative diagram in $\text{Spt}_T\mathcal{M}_*$:

$$(40) \quad \begin{array}{ccc} Q_{s^s < q} X & \longrightarrow & IQ_T J(Q_{s^s < q} X) \\ W_q^{Q_{s^s < q} X} \downarrow & & \downarrow IQ_T J(W_q^{Q_{s^s < q} X}) \\ W_q Q_{s^s < q} X & \longrightarrow & IQ_T J(W_q Q_{s^s < q} X) \end{array}$$

where the horizontal maps are weak equivalences in $\text{Spt}_T\mathcal{M}_*$. Hence, the two out of three property for weak equivalences implies that it is enough to show that $IQ_T J(W_q^{Q_{s^s < q} X})$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

By construction the map $W_q^{Q_{s^s < q} X}$ is a $L(< q)$ -local equivalence, and since the horizontal maps in diagram (40) are weak equivalences in $\text{Spt}_T\mathcal{M}_*$, it follows from [6, Proposition 3.1.5] that these horizontal maps are also $L(< q)$ -local equivalences. Therefore, the two out of three property for $L(< q)$ -local equivalences implies that $IQ_T J(W_q^{Q_{s^s < q} X})$ is a $L(< q)$ -local equivalence.

Now Lemma 3.2.50 implies that $IQ_T J(Q_{s^s < q} X)$ is $L(< q)$ -local. On the other hand, since the map

$$W_q Q_{s^s < q} X \longrightarrow IQ_T J(W_q Q_{s^s < q} X)$$

is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$, $W_q Q_{s^s < q} X$ is by construction $L(< q)$ -local, and $IQ_T J(W_q Q_{s^s < q} X)$, $W_q Q_{s^s < q} X$ are both fibrant in $\text{Spt}_T\mathcal{M}_*$; it follows from [6, Lemma 3.2.1] that $IQ_T J(W_q Q_{s^s < q} X)$ is also $L(< q)$ -local.

Finally we have a $L(< q)$ -local equivalence

$$IQ_T J(Q_{s^s < q} X) \xrightarrow{IQ_T J(W_q^{Q_{s^s < q} X})} IQ_T J(W_q Q_{s^s < q} X)$$

where the domain and the codomain are both $L(< q)$ -local. Then Theorem 3.2.13 in [6] implies that $IQ_T J(W_q^{Q_{s^s < q} X})$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$. This finishes the proof. \square

Theorem 3.2.52. — *For every T -spectrum X , we have the following diagram in $\mathcal{H}(S)$:*

$$(41) \quad \begin{array}{ccc} s_{<q}X & \xleftarrow[\cong]{Q_s^{s_{<q}X}} & Q_s s_{<q}X \\ & & \downarrow \cong \\ & & W_q^{Q_s s_{<q}X} \\ & & \downarrow \\ W_q Q_s s_{<q}X & \xleftarrow[\cong]{W_q Q_s(\pi_{<q})} & W_q Q_s X \end{array}$$

where all the maps are isomorphisms in $\mathcal{H}(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{s_{<q}} \\ \xrightarrow{W_q Q_s} \end{array} \mathcal{H}(S)$$

Proof. — Since Q_s is a cofibrant replacement functor in $\text{Spt}_T \mathcal{M}_*$, it is clear that $Q_s^{s_{<q}X}$ becomes an isomorphism in the associated homotopy category $\mathcal{H}(S)$.

The fact that $W_q^{Q_s s_{<q}X}$ is an isomorphism in $\mathcal{H}(S)$ follows from Proposition 3.2.51. Finally, Corollary 3.2.49 implies that $W_q Q_s(\pi_{<q})$ is also an isomorphism in $\mathcal{H}(S)$. This shows that all the maps in the diagram (41) are isomorphisms in $\mathcal{H}(S)$, therefore for every T -spectrum X we can define the following composition in $\mathcal{H}(S)$

$$(42) \quad \begin{array}{ccc} s_{<q}X & \xrightarrow[\cong]{(Q_s^{s_{<q}X})^{-1}} & Q_s s_{<q}X \\ & & \downarrow \cong \\ & & W_q^{Q_s s_{<q}X} \\ & & \downarrow \\ W_q Q_s s_{<q}X & \xrightarrow[\cong]{(W_q Q_s(\pi_{<q}))^{-1}} & W_q Q_s X \end{array}$$

which is an isomorphism. The fact that Q_s is a functorial cofibrant replacement in $\text{Spt}_T \mathcal{M}_*$, proposition 3.2.51 and Corollary 3.2.49, imply all together that the isomorphisms defined in diagram (42) induce a natural isomorphism of functors $s_{<q} \xrightarrow{\cong} W_q Q_s$. This finishes the proof. \square

Remark 3.2.53. — *Theorem 3.2.52 gives the desired lifting to the model category level for the functors $s_{<q}$ defined in Theorem 3.1.18.*

Proposition 3.2.54. — *We have the following commutative diagram of left Quillen functors:*

$$\begin{array}{ccc}
 & \text{Spt}_T \mathcal{M}_* & \\
 \text{id} \swarrow & & \searrow \text{id} \\
 L_{<q+1} \text{Spt}_T \mathcal{M}_* & \xrightarrow{\text{id}} & L_{<q} \text{Spt}_T \mathcal{M}_*
 \end{array}$$

Proof. — Since $L_{<q} \text{Spt}_T \mathcal{M}_*$ and $L_{<q+1} \text{Spt}_T \mathcal{M}_*$ are both left Bousfield localizations for $\text{Spt}_T \mathcal{M}_*$, we have that the identity functors:

$$\begin{aligned}
 \text{id} : \text{Spt}_T \mathcal{M}_* &\longrightarrow L_{<q} \text{Spt}_T \mathcal{M}_* \\
 \text{id} : \text{Spt}_T \mathcal{M}_* &\longrightarrow L_{<q+1} \text{Spt}_T \mathcal{M}_*
 \end{aligned}$$

are both left Quillen functors. Hence, it suffices to show that

$$\text{id} : L_{<q+1} \text{Spt}_T \mathcal{M}_* \longrightarrow L_{<q} \text{Spt}_T \mathcal{M}_*$$

is a left Quillen functor. Using the universal property for left Bousfield localizations (see definition 1.8.1), we have that it is enough to check that if $f : X \rightarrow Y$ is a $L(<q+1)$ -local equivalence then $Q_s(f) : Q_s X \rightarrow Q_s Y$ is a $L(<q)$ -local equivalence.

But Theorem 3.1.6(c) in [6] implies that this last condition is equivalent to the following one: Let Z be an arbitrary $L(<q)$ -local T -spectrum, then Z is also $L(<q+1)$ -local. Finally, this last condition follows immediately from corollary 3.2.33. \square

Corollary 3.2.55. — *We have the following adjunction*

$$(Q_s, W_q, \varphi) : L_{<q+1} \mathcal{H}(S) \longrightarrow L_{<q} \mathcal{H}(S)$$

of exact functors between triangulated categories.

Proof. — Proposition 3.2.54 implies that $\text{id} : L_{<q+1} \text{Spt}_T \mathcal{M}_* \rightarrow L_{<q} \text{Spt}_T \mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories

$$(Q_s, W_q, \varphi) : L_{<q+1} \mathcal{H}(S) \longrightarrow L_{<q} \mathcal{H}(S)$$

Now Proposition 6.4.1 in [10] implies that Q_s maps cofibre sequences in $L_{<q+1} \mathcal{H}(S)$ to cofibre sequences in $L_{<q} \mathcal{H}(S)$. Therefore using Proposition 7.1.12 in [10] we have that Q_s and W_q are both exact functors between triangulated categories. \square

Theorem 3.2.56. — *We have the following tower of left Quillen functors:*

$$(43) \quad \begin{array}{c} \vdots \\ \downarrow \text{id} \\ L_{<q+1}\text{Spt}_T\mathcal{M}_* \\ \begin{array}{ccc} \text{id} \nearrow & & \downarrow \text{id} \\ \text{Spt}_T\mathcal{M}_* & \xrightarrow{\text{id}} & L_{<q}\text{Spt}_T\mathcal{M}_* \\ \text{id} \searrow & & \downarrow \text{id} \\ & & L_{<q-1}\text{Spt}_T\mathcal{M}_* \\ & & \downarrow \text{id} \\ & & \vdots \end{array} \end{array}$$

together with the corresponding tower of associated homotopy categories:

$$(44) \quad \begin{array}{c} \vdots \\ \begin{array}{ccc} Q_s \downarrow & \uparrow W_{q+1} & \\ & L_{<q+1}\mathcal{H}(S) & \\ Q_s \downarrow & \uparrow W_q & \\ \mathcal{H}(S) & \begin{array}{ccc} Q_s \nearrow & & \downarrow Q_s \\ \leftarrow W_{q+1} & & L_{<q}\mathcal{H}(S) \\ \leftarrow W_q & & \\ \leftarrow W_{q-1} & & \downarrow Q_s \\ & & L_{<q-1}\mathcal{H}(S) \\ & & \downarrow Q_s \\ & & \uparrow W_{q-2} \\ & & \vdots \end{array} & \\ & & \vdots \end{array} \end{array}$$

Furthermore, the tower (44) satisfies the following properties:

1. All the categories are triangulated.
2. All the functors are exact.
3. Q_s is a left adjoint for all the functors W_q .

Proof. — It follows immediately from propositions 3.2.42, 3.2.54 and Corollary 3.2.55. □

Remark 3.2.57. — *The great technical advantage of the categories $L_{<q}\mathrm{Spt}_T\mathcal{M}_*$ over the categories $R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T\mathcal{M}_*$ is the fact that $L_{<q}\mathrm{Spt}_T\mathcal{M}_*$ are always cellular, whereas it is not clear if $R_{C_{\mathrm{eff}}^q}\mathrm{Spt}_T\mathcal{M}_*$ satisfies the cellularity property. Therefore we still can apply Hirschhorn’s localization technology to the categories $L_{<q}\mathrm{Spt}_T\mathcal{M}_*$. This will be the final step in our approach to get the desired lifting for the functors s_q (see Theorem 3.1.16) to the model category level.*

Definition 3.2.58. — *We consider the following set of T -spectra*

$$S(q) = \{F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C \mid s - n = q\} \subseteq C_{\mathrm{eff}}^q$$

(see Proposition 3.1.5 and definition 3.1.8).

Theorem 3.2.59. — *The right Bousfield localization of the model category $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ with respect to the $S(q)$ -colocal equivalences exists. This new model structure will be called q -slice motivic stable. $S^q\mathrm{Spt}_T\mathcal{M}_*$ will denote the category of T -spectra equipped with the q -slice motivic stable model structure, and $S^q\mathcal{H}(S)$ will denote its associated homotopy category. Furthermore the q -slice motivic stable model structure is right proper and simplicial.*

Proof. — Theorem 3.2.29 implies that $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ is a cellular and simplicial model category. On the other hand, corollary 3.2.41 implies that $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ is right proper. Therefore we can apply Theorem 5.1.1 in [6] to construct the right Bousfield localization of $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ with respect to the $S(q)$ -colocal equivalences. Using [6, Theorem 5.1.1] again, we have that $S^q\mathrm{Spt}_T\mathcal{M}_*$ is a right proper and simplicial model category. □

Definition 3.2.60. — *Let P_q denote a functorial cofibrant replacement functor in $S^q\mathrm{Spt}_T\mathcal{M}_*$; such that for every T -spectrum X , the natural map*

$$P_q X \xrightarrow{P_q^X} X$$

is a trivial fibration in $S^q\mathrm{Spt}_T\mathcal{M}_$, and $P_q X$ is a $S(q)$ -colocal T -spectrum in $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$.*

Proposition 3.2.61. — *We have that W_{q+1} is also a fibrant replacement functor in $S^q\mathrm{Spt}_T\mathcal{M}_*$ (see definition 3.2.30), and for every T -spectrum X the natural map*

$$X \xrightarrow{W_{q+1}^X} W_{q+1} X$$

is a trivial cofibration in $S^q\mathrm{Spt}_T\mathcal{M}_$.*

Proof. — Since $S^q\mathrm{Spt}_T\mathcal{M}_*$ is the right Bousfield localization of $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ with respect to the $S(q)$ -colocal equivalences, by construction we have that the fibrations and the trivial cofibrations are identical in $S^q\mathrm{Spt}_T\mathcal{M}_*$ and $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ respectively.

This implies that for every T -spectrum X , $W_{q+1}X$ is fibrant in $S^q\text{Spt}_T\mathcal{M}_*$, and we also have that the natural map:

$$X \xrightarrow{W_{q+1}^X} W_{q+1}X$$

is a trivial cofibration in $S^q\text{Spt}_T\mathcal{M}_*$. Hence W_{q+1} is also a fibrant replacement functor for $S^q\text{Spt}_T\mathcal{M}_*$. \square

Proposition 3.2.62. — *Let $f : X \rightarrow Y$ be a map of T -spectra. We have that f is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$ if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$ the induced map*

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X]_{\text{Spt}} \xrightarrow[\cong]{(W_{q+1}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y]_{\text{Spt}}$$

is an isomorphism of abelian groups.

Proof. — (\Rightarrow): Assume that f is a $S(q)$ -colocal equivalence. All the compact generators $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ are cofibrant in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$, since they are cofibrant in $\text{Spt}_T\mathcal{M}_*$, and the cofibrations are exactly the same in both model structures.

Therefore we have that f is a $S(q)$ -colocal equivalence if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$ the following maps are weak equivalences of simplicial sets:

$$\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X) \xrightarrow{(W_{q+1}f)_*} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y)$$

Since $L_{<q+1}\text{Spt}_T\mathcal{M}_*$ is a simplicial model category, we have that $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X)$ and $\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y)$ are both Kan complexes. Now Proposition 3.2.32(1) implies that $W_{q+1}X, W_{q+1}Y$ are both fibrant in $\text{Spt}_T\mathcal{M}_*$, therefore since $\text{Spt}_T\mathcal{M}_*$ is a simplicial model category we get the following commutative diagram where the top row and all the vertical maps are isomorphisms of abelian groups:

$$\begin{array}{ccc} \pi_0\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X) & & \\ \cong \downarrow & \searrow^{(W_{q+1}f)_*} & \\ & \cong & \pi_0\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y) \\ & & \cong \downarrow \\ [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X]_{\text{Spt}} & \xrightarrow{(W_{q+1}f)_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y]_{\text{Spt}} \end{array}$$

Therefore

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X]_{\text{Spt}} \xrightarrow[\cong]{(W_{q+1}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y]_{\text{Spt}}$$

is an isomorphism of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$, as we wanted.

(\Leftarrow): Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$. Let ω_0, η_0 be the base points corresponding to $\text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X)$ and $\text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y)$ respectively. We need to show that the map:

$$\text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X) \xrightarrow{(W_{q+1}f)_*} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y)$$

is a weak equivalence of simplicial sets. Let

$$j : F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+) \rightarrow F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

be the adjoint to the identity map

$$\text{id} : S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+ \rightarrow \text{Ev}_{n+1}F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) = S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+$$

We know that j is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$, therefore [6, Proposition 3.1.5] implies that j is a $L(< q + 1)$ -local equivalence, i.e. a weak equivalence in $L_{< q+1}\text{Spt}_T\mathcal{M}_*$. Now since $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ and $F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+)$ are both cofibrant in $L_{< q+1}\text{Spt}_T\mathcal{M}_*$, and $L_{< q+1}\text{Spt}_T\mathcal{M}_*$ is a simplicial model category, we can apply Ken Brown's lemma (see Lemma 1.1.4) to conclude that the horizontal maps in the following commutative diagram are weak equivalences of simplicial sets:

$$\begin{array}{ccc} \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X) & \xrightarrow{j^*} & \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X) \\ \downarrow (W_{q+1}f)_* & & \downarrow (W_{q+1}f)_* \\ \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y) & \xrightarrow{j^*} & \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y) \end{array}$$

Hence by the two out of three property for weak equivalences, it is enough to show that the following induced map

$$\begin{array}{c} \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X) \\ \downarrow (W_{q+1}f)_* \\ \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y) \end{array}$$

is a weak equivalence of simplicial sets.

On the other hand, since $\text{Spt}_T\mathcal{M}_*$ is a pointed simplicial model category and $W_{q+1}X, W_{q+1}Y$ are both fibrant in $\text{Spt}_T\mathcal{M}_*$ by proposition 3.2.32(1); we have that Lemma 6.1.2 in [10] together with Remark 2.4.3(2) imply that the following diagram

is commutative for $k \geq 0$:

$$\begin{array}{ccc}
 \pi_{k,\omega_0} \text{Map}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X) & & \\
 \parallel & \searrow (W_{q+1}f)_* & \\
 & \pi_{k,\eta_0} \text{Map}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y) & \\
 \parallel & & \parallel \\
 \pi_{k,\omega_0} \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X) & & \pi_{k,\eta_0} \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y) \\
 \parallel & \searrow (W_{q+1}f)_* & \parallel \\
 \mathbb{R} \downarrow & & \mathbb{R} \downarrow \\
 [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^k, W_{q+1}X]_{\text{Spt}} & & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^k, W_{q+1}Y]_{\text{Spt}} \\
 \parallel & \searrow (W_{q+1}f)_* & \parallel \\
 \mathbb{R} \downarrow & & \mathbb{R} \downarrow \\
 [F_{n+1}(S^{k+r} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X]_{\text{Spt}} & & [F_{n+1}(S^{k+r} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y]_{\text{Spt}} \\
 \parallel & \searrow (W_{q+1}f)_* & \parallel \\
 \mathbb{R} \downarrow & & \mathbb{R} \downarrow \\
 [F_{n+1}(S^{k+r} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X]_{\text{Spt}} & & [F_{n+1}(S^{k+r} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y]_{\text{Spt}}
 \end{array}$$

but by hypothesis we have that the bottom row is an isomorphism of abelian groups, since $F_{n+1}(S^{k+r} \wedge \mathbb{G}_m^{s+1} \wedge U_+)$ is also in $S(q)$. Therefore all the maps in the top row are also isomorphisms. Hence, the induced map

$$\begin{array}{c}
 \text{Map}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X) \\
 \downarrow (W_{q+1}f)_* \\
 \text{Map}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y)
 \end{array}$$

is a weak equivalence when it is restricted to the path component of $\text{Map}(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X)$ containing ω_0 . This implies that the following induced map

$$\begin{array}{c}
 \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X)) \\
 \downarrow (W_{q+1}f)_* \\
 \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y))
 \end{array}$$

is a weak equivalence since taking S^1 -loops kills the path components that do not contain the base point.

Finally, since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category we have that the rows in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc}
 \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X)) & & \\
 \downarrow (W_{q+1}f)_* & \searrow \cong & \\
 & \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, W_{q+1}X) & \\
 & \downarrow (W_{q+1}f)_* & \\
 \text{Map}_*(S^1, \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y)) & & \\
 & \searrow \cong & \\
 & \text{Map}_*(F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, W_{q+1}Y) &
 \end{array}$$

Hence the two out of three property for weak equivalences implies that the right vertical map is a weak equivalence of simplicial sets. But $F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1$ is clearly isomorphic to $F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+)$, therefore the induced map

$$\begin{array}{c}
 \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}X) \\
 \downarrow (W_{q+1}f)_* \\
 \text{Map}(F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1}Y)
 \end{array}$$

is a weak equivalence, as we wanted. □

Corollary 3.2.63. — *Let $f : X \rightarrow Y$ be a map of T -spectra. We have that f is a $S(q)$ -colocal equivalence in $L_{<q+1} \text{Spt}_T \mathcal{M}_*$ if and only if*

$$W_{q+1}X \xrightarrow{W_{q+1}f} W_{q+1}Y$$

is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_$.*

Proof. — (\Rightarrow): Assume that f is a $S(q)$ -colocal equivalence, and fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$. By Proposition 3.2.4 it suffices to show that the induced map

$$(45) \quad \begin{array}{c}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X]_{\text{Spt}} \\
 \downarrow (W_{q+1}f)_* \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y]_{\text{Spt}}
 \end{array}$$

is an isomorphism of abelian groups.

Since $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^q$, we have two possibilities:

1. $s - n = q$, i.e. $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$.
2. $s - n \geq q + 1$, i.e. $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q+1}$

In case (1), Proposition 3.2.62 implies that the induced map in diagram (45) is an isomorphism of abelian groups.

On the other hand, in case (2), we have by Proposition 3.2.32(2) that

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X]_{\text{Spt}} \cong 0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y]_{\text{Spt}}$$

since by construction $W_{q+1}X$ and $W_{q+1}Y$ are both $L(< q + 1)$ -local T -spectra. Hence the induced map in diagram (45) is also an isomorphism of abelian groups in this case, as we wanted.

(\Leftarrow): Assume that $W_{q+1}f$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T\mathcal{M}_*$, and fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$.

Since $S(q) \subseteq C_{\text{eff}}^q$, it follows from Proposition 3.2.4 that the induced map

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}X]_{\text{Spt}} \xrightarrow{(W_{q+1}f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}Y]_{\text{Spt}}$$

is an isomorphism of abelian groups. Therefore, Proposition 3.2.62 implies that f is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$. This finishes the proof. \square

Lemma 3.2.64. — *Let $f : X \rightarrow Y$ be a map of T -spectra. We have that f is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$ if and only if*

$$\Omega_{S^1}W_{q+1}(f) : \Omega_{S^1}W_{q+1}X \longrightarrow \Omega_{S^1}W_{q+1}Y$$

is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_$.*

Proof. — Assume that f is a $S(q)$ -colocal equivalence. We need to show that $\Omega_{S^1}W_{q+1}(f)$ is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$.

Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$. Corollary 3.2.34 implies that $\Omega_{S^1}W_{q+1}X$ and $\Omega_{S^1}W_{q+1}Y$ are both $L(< q + 1)$ -local; and proposition 3.2.32(1) implies that $\Omega_{S^1}W_{q+1}X$ and $\Omega_{S^1}W_{q+1}Y$ are both fibrant in $\text{Spt}_T\mathcal{M}_*$. Therefore using the fact that $\text{Spt}_T\mathcal{M}_*$ is a simplicial model category, we get the following commutative

diagram:

$$\begin{array}{ccc}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} W_{q+1} X]_{\text{Spt}} & & \\
 \downarrow \cong & \searrow (\Omega_{S^1} W_{q+1} f)_* & \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} W_{q+1} Y]_{\text{Spt}} & & \\
 \downarrow \cong & & \downarrow \cong \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, W_{q+1} X]_{\text{Spt}} & \xrightarrow{(W_{q+1} f)_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, W_{q+1} Y]_{\text{Spt}} \\
 \downarrow \cong & & \downarrow \cong \\
 [F_n(S^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1} X]_{\text{Spt}} & \xrightarrow[\cong]{(W_{q+1} f)_*} & [F_n(S^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1} Y]_{\text{Spt}}
 \end{array}$$

but using Proposition 3.2.62 and the fact that f is a $S(q)$ -colocal equivalence, we have that the bottom row is an isomorphism, therefore the top row is also an isomorphism. Hence, the induced map:

$$\begin{array}{c}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} W_{q+1} X]_{\text{Spt}} \\
 \cong \downarrow (\Omega_{S^1} W_{q+1} f)_* \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} W_{q+1} Y]_{\text{Spt}}
 \end{array}$$

is an isomorphism of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$. Finally, using Proposition 3.2.62 again, together with the fact that $\Omega_{S^1} W_{q+1} X$ and $\Omega_{S^1} W_{q+1} Y$ are both $L(< q + 1)$ -local T -spectra; we have that $\Omega_{S^1} W_{q+1}(f)$ is a $S(q)$ -colocal equivalence in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, as we wanted.

Conversely, assume that $\Omega_{S^1} W_{q+1}(f)$ is a $S(q)$ -colocal equivalence in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, and fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$. Corollary 3.2.34 implies that $\Omega_{S^1} W_{q+1} X$ and $\Omega_{S^1} W_{q+1} Y$ are both $L(< q + 1)$ -local; and proposition 3.2.32(1) implies that $\Omega_{S^1} W_{q+1} X$ and $\Omega_{S^1} W_{q+1} Y$ are both fibrant in $\text{Spt}_T \mathcal{M}_*$. Therefore using the fact that

$\text{Spt}_T \mathcal{M}_*$ is a simplicial model category, we get the following commutative diagram:

$$\begin{array}{ccc}
 [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), \Omega_{S^1} W_{q+1} X]_{\text{Spt}} & & \\
 \cong \downarrow & \xrightarrow[\cong]{(\Omega_{S^1} W_{q+1} f)_*} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), \Omega_{S^1} W_{q+1} Y]_{\text{Spt}} \\
 [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, W_{q+1} X]_{\text{Spt}} & & \\
 \cong \downarrow & \xrightarrow[(W_{q+1} f)_*]{\cong} & [F_{n+1}(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+) \wedge S^1, W_{q+1} Y]_{\text{Spt}} \\
 [F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1} X]_{\text{Spt}} & & \\
 \cong \downarrow & \xrightarrow[(W_{q+1} f)_*]{\cong} & [F_{n+1}(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), W_{q+1} Y]_{\text{Spt}} \\
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1} X]_{\text{Spt}} & & \\
 \cong \downarrow & \xrightarrow[(W_{q+1} f)_*]{\cong} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1} Y]_{\text{Spt}}
 \end{array}$$

Since $\Omega_{S^1} W_{q+1} f$ is a $S(q)$ -colocal equivalence, we have that Proposition 3.2.62 together with the fact that $\Omega_{S^1} W_{q+1} X$ and $\Omega_{S^1} W_{q+1} Y$ are both $L(< q + 1)$ -local imply that the top row in the diagram above is an isomorphism; therefore the bottom row is also an isomorphism. Thus, the induced map:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1} X]_{\text{Spt}} \xrightarrow[\cong]{(W_{q+1} f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1} Y]_{\text{Spt}}$$

is an isomorphism of abelian groups for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$. Now using Proposition 3.2.62 again, we have that f is a $S(q)$ -colocal equivalence. This finishes the proof. \square

Corollary 3.2.65. — *The adjunction*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : S^q \text{Spt}_T \mathcal{M}_* \longrightarrow S^q \text{Spt}_T \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Using Corollary 1.3.16 in [10] and Proposition 3.2.61 we have that it suffices to verify the following two conditions:

1. For every cofibrant object X in $S^q\text{Spt}_T\mathcal{M}_*$, the following composition

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1}W_{q+1}^{X \wedge S^1}} \Omega_{S^1}W_{q+1}(X \wedge S^1)$$

is a $S(q)$ -colocal equivalence.

2. Ω_{S^1} reflects $S(q)$ -colocal equivalences between fibrant objects in $S^q\text{Spt}_T\mathcal{M}_*$.

(1): By construction $S^q\text{Spt}_T\mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1}\text{Spt}_T\mathcal{M}_*$, therefore the identity functor

$$\text{id} : S^q\text{Spt}_T\mathcal{M}_* \longrightarrow L_{<q+1}\text{Spt}_T\mathcal{M}_*$$

is a left Quillen functor. Thus X is also cofibrant in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$. Since the adjunction $(- \wedge S^1, \Omega_{S^1}, \varphi)$ is a Quillen equivalence on $L_{<q+1}\text{Spt}_T\mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies that the following composition is a weak equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$:

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1}W_{q+1}^{X \wedge S^1}} \Omega_{S^1}W_{q+1}(X \wedge S^1)$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $S(q)$ -colocal equivalence.

- (2): This follows immediately from Proposition 3.2.61 and Lemma 3.2.64. □

Remark 3.2.66. — *We have a situation similar to the one described in remarks 3.2.10 and 3.2.39 for the model categories $R_{C_{\text{eff}}}^q\text{Spt}_T\mathcal{M}_*$ and $L_{<q}\text{Spt}_T\mathcal{M}_*$; i.e. although the adjunction $(\Sigma_T, \Omega_T, \varphi)$ is a Quillen equivalence on $\text{Spt}_T\mathcal{M}_*$, it does not descend even to a Quillen adjunction on the q -slice motivic stable model category $S^q\text{Spt}_T\mathcal{M}_*$.*

Corollary 3.2.67. — *$S^q\mathcal{A}\mathcal{H}(S)$ has the structure of a triangulated category.*

Proof. — Theorem 3.2.59 implies in particular that $S^q\text{Spt}_T\mathcal{M}_*$ is a pointed simplicial model category, and Corollary 3.2.65 implies that the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : S^q\text{Spt}_T\mathcal{M}_* \rightarrow S^q\text{Spt}_T\mathcal{M}_*$$

is a Quillen equivalence. Therefore the result follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII]. □

Proposition 3.2.68. — *We have the following adjunction*

$$(P_q, W_{q+1}, \varphi) : S^q\mathcal{A}\mathcal{H}(S) \longrightarrow L_{<q+1}\mathcal{A}\mathcal{H}(S)$$

of exact functors between triangulated categories.

Proof. — Since $S^q\mathrm{Spt}_T\mathcal{M}_*$ is the right Bousfield localization of $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ with respect to the $S(q)$ -colocal equivalences, we have that the identity functor $\mathrm{id} : S^q\mathrm{Spt}_T\mathcal{M}_* \rightarrow L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(P_q, W_{q+1}, \varphi) : S^q\mathcal{H}(S) \longrightarrow L_{<q+1}\mathcal{H}(S)$$

Now Proposition 6.4.1 in [10] implies that P_q maps cofibre sequences in $S^q\mathcal{H}(S)$ to cofibre sequences in $L_{<q+1}\mathcal{H}(S)$. Therefore using Proposition 7.1.12 in [10] we have that P_q and W_{q+1} are both exact functors between triangulated categories. \square

Proposition 3.2.69. — *The identity functor*

$$\mathrm{id} : S^q\mathrm{Spt}_T\mathcal{M}_* \longrightarrow R_{\mathrm{eff}}^{C_q}\mathrm{Spt}_T\mathcal{M}_*$$

is a right Quillen functor.

Proof. — Consider the following diagram of right Quillen functors

$$\begin{array}{ccccc} L_{<q+1}\mathrm{Spt}_T\mathcal{M}_* & \xrightarrow{\mathrm{id}} & \mathrm{Spt}_T\mathcal{M}_* & \xrightarrow{\mathrm{id}} & R_{\mathrm{eff}}^{C_q}\mathrm{Spt}_T\mathcal{M}_* \\ \mathrm{id} \downarrow & & & \nearrow \mathrm{id} & \\ S^q\mathrm{Spt}_T\mathcal{M}_* & & & & \end{array}$$

By the universal property of right Bousfield localizations (see definition 1.8.2) it suffices to check that if $f : X \rightarrow Y$ is a $S(q)$ -colocal equivalence in $L_{<q+1}\mathrm{Spt}_T\mathcal{M}_*$, then $W_{q+1}f : W_{q+1}X \rightarrow W_{q+1}Y$ is a $R_{\mathrm{eff}}^{C_q}$ -colocal equivalence in $\mathrm{Spt}_T\mathcal{M}_*$. But this follows immediately from Corollary 3.2.63. \square

Corollary 3.2.70. — *We have the following adjunction*

$$(C_q, W_{q+1}, \varphi) : R_{\mathrm{eff}}^{C_q}\mathcal{H}(S) \longrightarrow S^q\mathcal{H}(S)$$

of exact functors between triangulated categories.

Proof. — By Proposition 3.2.69 the identity functor $\mathrm{id} : R_{\mathrm{eff}}^{C_q}\mathrm{Spt}_T\mathcal{M}_* \rightarrow S^q\mathrm{Spt}_T\mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(C_q, W_{q+1}, \varphi) : R_{\mathrm{eff}}^{C_q}\mathcal{H}(S) \longrightarrow S^q\mathcal{H}(S)$$

Now Proposition 6.4.1 in [10] implies that C_q maps cofibre sequences in $R_{\mathrm{eff}}^{C_q}\mathcal{H}(S)$ to cofibre sequences in $S^q\mathcal{H}(S)$. Therefore using Proposition 7.1.12 in [10] we have that C_q and W_{q+1} are both exact functors between triangulated categories. \square

Lemma 3.2.71. — *If A is a cofibrant T -spectrum in $S^q\mathrm{Spt}_T\mathcal{M}_*$, then the map $*$ $\rightarrow A$ is a trivial cofibration in $L_{<q}\mathrm{Spt}_T\mathcal{M}_*$.*

Proof. — Let Z be an arbitrary $L(< q)$ -local T -spectrum in $\text{Spt}_T \mathcal{M}_*$. We claim that the map $Z \rightarrow *$ is a trivial fibration in $S^q \text{Spt}_T \mathcal{M}_*$. In effect, using Corollary 3.2.33 we have that Z is $L(< q + 1)$ -local in $\text{Spt}_T \mathcal{M}_*$, i.e. a fibrant object in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$. By construction $S^q \text{Spt}_T \mathcal{M}_*$ is a right Bousfield localization of $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, hence Z is also fibrant in $S^q \text{Spt}_T \mathcal{M}_*$. Then by Proposition 3.2.62 it suffices to show that for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$ (i.e. $s - n = q$):

$$0 \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}$$

But this follows immediately from Proposition 3.2.32, since Z is $L(< q)$ -local.

Now since $S^q \text{Spt}_T \mathcal{M}_*$ is a simplicial model category and A is cofibrant in $S^q \text{Spt}_T \mathcal{M}_*$, we have that the following map is a trivial fibration of simplicial sets:

$$\text{Map}(A, Z) \longrightarrow \text{Map}(A, *) = *$$

The identity functor

$$\text{id} : S^q \text{Spt}_T \mathcal{M}_* \longrightarrow L_{< q+1} \text{Spt}_T \mathcal{M}_*$$

is a left Quillen functor, since $S^q \text{Spt}_T \mathcal{M}_*$ is a right Bousfield localization of $L_{< q+1} \text{Spt}_T \mathcal{M}_*$. Therefore A is also cofibrant in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, and since $L_{< q+1} \text{Spt}_T \mathcal{M}_*$ is a left Bousfield localization of $\text{Spt}_T \mathcal{M}_*$; it follows that A is also cofibrant in $\text{Spt}_T \mathcal{M}_*$. On the other hand, we have that Z is in particular fibrant in $\text{Spt}_T \mathcal{M}_*$. Hence $\pi_0 \text{Map}(A, Z)$ computes $[A, Z]_{\text{Spt}}$, since $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category. But $\text{Map}(A, Z) \rightarrow *$ is in particular a weak equivalence of simplicial sets, then

$$[A, Z]_{\text{Spt}} \cong 0$$

for every $L(< q)$ -local T -spectrum Z . Finally, Corollary 3.2.36 implies that $* \rightarrow A$ is a weak equivalence in $L_{< q} \text{Spt}_T \mathcal{M}_*$. This finishes the proof, since we already know that A is cofibrant in $L_{< q} \text{Spt}_T \mathcal{M}_*$. \square

Lemma 3.2.72. — *The natural map*

$$C_q s_q X \xrightarrow{C_q^{s_q X}} s_q X$$

is a weak equivalence in $\text{Spt}_T \mathcal{M}_$.*

Proof. — Consider the following commutative diagram in $\text{Spt}_T \mathcal{M}_*$:

$$\begin{array}{ccc} s_q X & \xleftarrow{Q_s^{s_q X}} & Q_s s_q X \\ C_q^{s_q X} \uparrow & & \uparrow C_q^{Q_s s_q X} \\ C_q s_q X & \xleftarrow{C_q(Q_s^{s_q X})} & C_q Q_s s_q X \end{array}$$

By construction $C_q^{s_q X}$, $C_q^{Q_s s_q X}$ are both weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$; and [6, Proposition 3.1.5] implies that $Q_s^{s_q X}$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$, i.e. a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$. Then the two out of three property for weak equivalences implies that $C_q(Q_s^{s_q X})$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$.

Now [6, Theorem 3.2.13(2)] implies that $C_q(Q_s^{s_q X})$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, since $C_q s_q X$ and $C_q Q_s s_q X$ are by construction C_{eff}^q -colocal T -spectra in $\text{Spt}_T \mathcal{M}_*$. It is clear that $Q_s^{s_q X}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, then by the two out of three property for weak equivalences, it suffices to show that $C_q^{Q_s s_q X}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.

By Theorem 3.1.16(2) we have that $s_q X$ is in $\Sigma_T^q \mathcal{H}^{\text{eff}}(S)$, then corollary 3.2.15 implies that $Q_s s_q X$ is C_{eff}^q -colocal in $\text{Spt}_T \mathcal{M}_*$. We already know that $C_q^{Q_s s_q X}$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$; then [6, Theorem 3.2.13(2)] implies that $C_q^{Q_s s_q X}$ is also a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, since by construction $C_q Q_s s_q X$ is a C_{eff}^q -colocal T -spectrum. This finishes the proof. \square

Lemma 3.2.73. — *For every T -spectrum X , we have that $IQ_T J s_q X$ (see Theorem 3.1.16) is $L(< q + 1)$ -local.*

Proof. — Proposition 3.2.32 implies that it suffices to check that $IQ_T J s_q X$ satisfies the following conditions:

1. $IQ_T J s_q X$ is fibrant in $\text{Spt}_T \mathcal{M}_*$.
2. For every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q+1}$,

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T J s_q X]_{\text{Spt}} \cong 0$$

Condition (1) holds trivially, since $IQ_T J$ is a fibrant replacement functor in $\text{Spt}_T \mathcal{M}_*$.

Fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q+1}$. Since $C_{\text{eff}}^{q+1} \subseteq \Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S)$, it follows from Theorem 3.1.16(3) that:

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), IQ_T J s_q X]_{\text{Spt}} \cong [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), s_q X]_{\text{Spt}} \cong 0$$

and this takes care of condition (2). \square

Lemma 3.2.74. — *For every T -spectrum X , $C_q IQ_T J f_{q+1} X \cong *$ in $S^q \mathcal{H}(S)$.*

Proof. — Consider the following commutative diagram in $\text{Spt}_T \mathcal{M}_*$:

$$(46) \quad \begin{array}{ccccc} Q_s f_{q+1} X & \xrightarrow{Q_s^{f_{q+1} X}} & f_{q+1} X & \xrightarrow{IQ_T J f_{q+1} X} & IQ_T J f_{q+1} X \\ C_q^{Q_s f_{q+1} X} \uparrow & & \uparrow C_q^{f_{q+1} X} & & \uparrow C_q^{IQ_T J f_{q+1} X} \\ C_q Q_s f_{q+1} X & \xrightarrow{C_q(Q_s^{f_{q+1} X})} & C_q f_{q+1} X & \xrightarrow{C_q(IQ_T J f_{q+1} X)} & C_q IQ_T J f_{q+1} X \end{array}$$

We claim that all the maps in the diagram (46) above are weak equivalences in $\text{Spt}_T \mathcal{M}_*$. In effect, it is clear that all the maps in the top row are weak equivalences in $\text{Spt}_T \mathcal{M}_*$. Hence, by the two out of three property for weak equivalences it suffices to show that $C_q^{Q_s f_{q+1} X}$, $C_q(Q_s^{f_{q+1} X})$ and $C_q(IQ_T J f_{q+1} X)$ are all weak equivalences in $\text{Spt}_T \mathcal{M}_*$.

On the other hand, [6, Proposition 3.1.5] implies that all the maps in the top row are weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$, and it is clear that all the vertical maps are also weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$. Thus, by the two out of three property for weak equivalences we have that all the maps in the diagram (46) above are weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$.

By construction we have that $C_q Q_s f_{q+1} X$, $C_q f_{q+1} X$ and $C_q IQ_T J f_{q+1} X$ are all C_{eff}^q -colocal T -spectra in $\text{Spt}_T \mathcal{M}_*$. Then [6, Theorem 3.2.13(2)] implies that $C_q(Q_s^{f_{q+1} X})$ and $C_q(IQ_T J f_{q+1} X)$ are both weak equivalences in $\text{Spt}_T \mathcal{M}_*$.

Now, by Proposition 3.1.12 we have that $f_{q+1} X \in \Sigma_T^{q+1} \mathcal{H}^{\text{eff}}(S) \subseteq \Sigma_T^q \mathcal{H}^{\text{eff}}(S)$. Thus, Corollary 3.2.15 implies that $Q_s f_{q+1} X$ is a C_{eff}^q -colocal T -spectrum in $\text{Spt}_T \mathcal{M}_*$. Then using [6, Theorem 3.2.13(2)] again, we have that $C_q^{Q_s f_{q+1} X}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$ since by construction $C_q Q_s f_{q+1} X$ is a C_{eff}^q -colocal T -spectrum and $C_q^{Q_s f_{q+1} X}$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$.

This proves the claim, i.e. all the maps in the diagram (46) above are weak equivalences in $\text{Spt}_T \mathcal{M}_*$. Then using [6, Proposition 3.1.5] again, we have that all the maps in the diagram (46) above are also weak equivalences in $S^q \text{Spt}_T \mathcal{M}_*$. Therefore, to finish the proof it is enough to check that $* \rightarrow Q_s f_{q+1} X$ is a weak equivalence in $S^q \text{Spt}_T \mathcal{M}_*$.

But Corollary 3.2.47 implies that $* \rightarrow Q_s f_{q+1} X$ is a weak equivalence in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$. Therefore, using [6, Proposition 3.1.5], we have that $* \rightarrow Q_s f_{q+1} X$ is a $S(q)$ -colocal equivalence in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, i.e. a weak equivalence in $S^q \text{Spt}_T \mathcal{M}_*$. This finishes the proof. \square

Proposition 3.2.75. — *For every T -spectrum X , the following maps of T -spectra:*

$$(47) \quad s_q X \xrightarrow{IQ_T J s_q X} IQ_T J s_q X \xleftarrow{C_q^{IQ_T J s_q X}} C_q IQ_T J s_q X$$

are both weak equivalences in $\text{Spt}_T \mathcal{M}_$.*

Furthermore, these weak equivalences induce natural isomorphisms between the following exact functors

$$\begin{array}{ccc} \mathcal{H}(S) & \xrightarrow{s_q} & \mathcal{H}(S) \\ & \xrightarrow{IQ_T J s_q} & \\ \mathcal{H}(S) & \xrightarrow{IQ_T J s_q} & \mathcal{H}(S) \\ & \xrightarrow{C_q IQ_T J s_q} & \end{array}$$

Proof. — The naturality of the maps $IQ_T J^X : X \rightarrow IQ_T JX$ and $C_q^X : C_q X \rightarrow X$ implies that we have induced natural transformations of functors $s_q \rightarrow IQ_T J s_q$ and $C_q IQ_T J s_q \rightarrow IQ_T J s_q$. Hence, it is enough to show that for every T -spectrum X , $IQ_T J^{s_q X}$ and $C_q^{IQ_T J^{s_q X}}$ are weak equivalences in $\text{Spt}_T \mathcal{M}_*$.

It is clear that $IQ_T J^{s_q X}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, since $IQ_T J$ is a fibrant replacement functor for $\text{Spt}_T \mathcal{M}_*$.

We now proceed to show that $C_q^{IQ_T J^{s_q X}}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Consider the following commutative diagram in $\text{Spt}_T \mathcal{M}_*$:

$$\begin{array}{ccc}
 s_q X & \xleftarrow{C_q^{s_q X}} & C_q s_q X \\
 \downarrow IQ_T J^{s_q X} & & \downarrow C_q(IQ_T J^{s_q X}) \\
 IQ_T J s_q X & \xleftarrow{C_q^{IQ_T J^{s_q X}}} & C_q IQ_T J s_q X
 \end{array}$$

Lemma 3.2.72 implies that $C_q^{s_q X}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$. Since we know that $IQ_T J^{s_q X}$ is always a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, the two out of three property for weak equivalences implies that it suffices to check that $C_q(IQ_T J^{s_q X})$ is also a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.

Using [6, Proposition 3.1.5], we have that $IQ_T J^{s_q X}$ is a C_{eff}^q -colocal equivalence. Then the two out of three property for C_{eff}^q -colocal equivalences implies that $C_q(IQ_T J^{s_q X})$ is a C_{eff}^q -colocal equivalence, since by construction $C_q^{s_q X}$ and $C_q^{IQ_T J^{s_q X}}$ are both C_{eff}^q -colocal equivalences.

Finally, by construction $C_q s_q X$ and $C_q IQ_T J s_q X$ are both C_{eff}^q -colocal, therefore [6, Theorem 3.2.13(2)] implies that $C_q(IQ_T J^{s_q X})$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$, as we wanted. \square

Proposition 3.2.76. — *For every T -spectrum X , the natural map:*

$$C_q IQ_T J s_q X \xrightarrow{W_{q+1}^{C_q IQ_T J^{s_q X}}} W_{q+1} C_q IQ_T J s_q X$$

is a weak equivalence in $\text{Spt}_T \mathcal{M}_$.*

Furthermore, this weak equivalence induces a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{C_q IQ_T J s_q} \\ \xrightarrow{W_{q+1} C_q IQ_T J s_q} \end{array} \mathcal{H}(S)$$

Proof. — The naturality of the maps $W_{q+1}^X : X \rightarrow W_{q+1} X$ implies that we have an induced natural transformation of functors $C_q IQ_T J s_q \rightarrow W_{q+1} C_q IQ_T J s_q$. Hence, it is enough to show that for every T -spectrum X , $W_{q+1}^{C_q IQ_T J^{s_q X}}$ is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$.

Consider the following commutative diagram in $\text{Spt}_T\mathcal{M}_*$:

$$\begin{array}{ccc}
 IQ_TJs_qX & \xrightarrow{W_{q+1}^{IQ_TJs_qX}} & W_{q+1}IQ_TJs_qX \\
 C_q^{IQ_TJs_qX} \uparrow & & \uparrow W_{q+1}(C_q^{IQ_TJs_qX}) \\
 C_qIQ_TJs_qX & \xrightarrow{W_{q+1}^{C_qIQ_TJs_qX}} & W_{q+1}C_qIQ_TJs_qX
 \end{array}$$

By construction, $W_{q+1}^{IQ_TJs_qX}$ is a $L(< q + 1)$ -local equivalence, and $W_{q+1}IQ_TJs_qX$ is $L(< q + 1)$ -local in $\text{Spt}_T\mathcal{M}_*$. By Lemma 3.2.73 we have that IQ_TJs_qX is also $L(< q + 1)$ -local. Therefore, [6, Theorem 3.2.13(1)] implies that $W_{q+1}^{IQ_TJs_qX}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

Now, it follows directly from Proposition 3.2.75 that $C_q^{IQ_TJs_qX}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$. Hence by the two out of three property for weak equivalences, it suffices to show that $W_{q+1}(C_q^{IQ_TJs_qX})$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

We already know that $C_q^{IQ_TJs_qX}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$, then using [6, Proposition 3.1.5] we have that $C_q^{IQ_TJs_qX}$ is a $L(< q + 1)$ -local equivalence. Then the two out of three property for $L(< q + 1)$ -local equivalences implies that $W_{q+1}(C_q^{IQ_TJs_qX})$ is also a $L(< q + 1)$ -local equivalence, since by construction $W_{q+1}^{IQ_TJs_qX}$ and $W_{q+1}^{C_qIQ_TJs_qX}$ are both $L(< q + 1)$ -local equivalences.

Finally, by construction $W_{q+1}IQ_TJs_qX$ and $W_{q+1}C_qIQ_TJs_qX$ are $L(< q + 1)$ -local in $\text{Spt}_T\mathcal{M}_*$, then [6, Theorem 3.2.13(1)] implies that $W_{q+1}(C_q^{IQ_TJs_qX})$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$, as we wanted. \square

Proposition 3.2.77. — *For every T -spectrum X , the natural map:*

$$W_{q+1}C_qIQ_TJs_qX \xleftarrow{C_q^{W_{q+1}C_qIQ_TJs_qX}} C_qW_{q+1}C_qIQ_TJs_qX$$

is a weak equivalence in $\text{Spt}_T\mathcal{M}_$.*

Furthermore, this weak equivalence induces a natural isomorphism between the following exact functors

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{W_{q+1}C_qIQ_TJs_q} \\ \xrightarrow{C_qW_{q+1}C_qIQ_TJs_q} \end{array} \mathcal{H}(S)$$

Proof. — The naturality of the maps $C_q^X : C_qX \rightarrow X$ implies that we have an induced natural transformation of functors $C_qW_{q+1}C_qIQ_TJs_q \rightarrow W_{q+1}C_qIQ_TJs_q$. Hence, it is enough to show that for every T -spectrum X , $C_q^{W_{q+1}C_qIQ_TJs_qX}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

Consider the following commutative diagram in $\text{Spt}_T\mathcal{M}_*$:

$$\begin{array}{ccc}
 C_q I Q_T J s_q X & \xleftarrow{C_q^{C_q I Q_T J s_q X}} & C_q C_q I Q_T J s_q X \\
 W_{q+1}^{C_q I Q_T J s_q X} \downarrow & & \downarrow C_q(W_{q+1}^{C_q I Q_T J s_q X}) \\
 W_{q+1} C_q I Q_T J s_q X & \xleftarrow{C_q^{W_{q+1} C_q I Q_T J s_q X}} & C_q W_{q+1} C_q I Q_T J s_q X
 \end{array}$$

By construction $C_q^{C_q I Q_T J s_q X}$ is a C_{eff}^q -colocal equivalence, and $C_q I Q_T J s_q X$, $C_q C_q I Q_T J s_q X$ are both C_{eff}^q -colocal in $\text{Spt}_T\mathcal{M}_*$. Therefore, [6, Theorem 3.2.13(2)] implies that $C_q^{C_q I Q_T J s_q X}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

Now, it follows directly from Proposition 3.2.76 that $W_{q+1}^{C_q I Q_T J s_q X}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$. Hence by the two out of three property for weak equivalences, it suffices to show that $C_q(W_{q+1}^{C_q I Q_T J s_q X})$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$.

We already know that $W_{q+1}^{C_q I Q_T J s_q X}$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$, then using [6, Proposition 3.1.5] we have that $W_{q+1}^{C_q I Q_T J s_q X}$ is a C_{eff}^q -colocal equivalence. Then the two out of three property for C_{eff}^q -colocal equivalences implies that $C_q(W_{q+1}^{C_q I Q_T J s_q X})$ is also a C_{eff}^q -colocal equivalence, since by construction $C_q^{C_q I Q_T J s_q X}$ and $C_q^{W_{q+1} C_q I Q_T J s_q X}$ are both C_{eff}^q -colocal equivalences.

Finally, by construction $C_q C_q I Q_T J s_q X$ and $C_q W_{q+1} C_q I Q_T J s_q X$ are C_{eff}^q -colocal in $\text{Spt}_T\mathcal{M}_*$, then [6, Theorem 3.2.13(2)] implies that $C_q(W_{q+1}^{C_q I Q_T J s_q X})$ is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$, as we wanted. \square

Proposition 3.2.78. — For every T -spectrum X , the following natural maps in $R_{C_{\text{eff}}^q} \mathcal{A}\mathcal{H}(S)$ (see Proposition 3.1.15 and Theorem 3.1.16):

$$I Q_T J X \xleftarrow{I Q_T J(\theta_X)} I Q_T J f_q X \xrightarrow{I Q_T J(\pi_q^X)} I Q_T J s_q X$$

become isomorphisms in $S^q \mathcal{A}\mathcal{H}(S)$ after applying the functor C_q :

$$C_q I Q_T J X \xleftarrow[\cong]{C_q I Q_T J(\theta_X)} C_q I Q_T J f_q X \xrightarrow[\cong]{C_q I Q_T J(\pi_q^X)} C_q I Q_T J s_q X$$

Proof. — Proposition 3.2.17 implies that the map

$$I Q_T J f_q X \xrightarrow[\cong]{I Q_T J(\theta_X)} I Q_T J X$$

is an isomorphism in $R_{C_{\text{eff}}^q} \mathcal{A}\mathcal{H}(S)$. Hence using Corollary 3.2.70 we have that

$$C_q I Q_T J f_q X \xrightarrow[\cong]{C_q I Q_T J(\theta_X)} C_q I Q_T J X$$

is an isomorphism in $S^q \mathcal{A}\mathcal{H}(S)$.

On the other hand, Theorem 3.1.16(1) implies that we have the following distinguished triangle in $\mathcal{H}(S)$:

$$f_{q+1}X \longrightarrow f_qX \xrightarrow{\pi_q^X} s_qX \longrightarrow \Sigma_T^{1,0} f_{q+1}X$$

Proposition 3.2.11 implies that after applying $IQ_T J$ we get the following distinguished triangle in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$

$$IQ_T J f_{q+1}X \longrightarrow IQ_T J f_qX \xrightarrow{IQ_T J(\pi_q^X)} IQ_T J s_qX \longrightarrow \Sigma_T^{1,0} IQ_T J f_{q+1}X$$

Now Corollary 3.2.70 implies that after applying C_q we get the following distinguished triangle in $S^q \mathcal{H}(S)$

$$\begin{array}{ccc} C_q IQ_T J f_{q+1}X & \longrightarrow & C_q IQ_T J f_qX \xrightarrow{C_q IQ_T J(\pi_q^X)} C_q IQ_T J s_qX \\ & & \downarrow \\ & & \Sigma_T^{1,0} C_q IQ_T J f_{q+1}X \end{array}$$

Therefore it is enough to check that $C_q IQ_T J f_{q+1}X \cong *$ in $S^q \mathcal{H}(S)$. But this follows directly from Lemma 3.2.74. \square

Corollary 3.2.79. — *For every T -spectrum X , the following natural maps in $\mathcal{H}(S)$ (see Proposition 3.1.15 and Theorem 3.1.16):*

$$X \xleftarrow{\theta_X} f_qX \xrightarrow{\pi_q^X} s_qX$$

become isomorphisms in $\mathcal{H}(S)$ after applying the functor $C_q W_{q+1} C_q IQ_T J$:

$$\begin{array}{ccc} C_q W_{q+1} C_q IQ_T J X & \xleftarrow[\cong]{C_q W_{q+1} C_q IQ_T J(\theta_X)} & C_q W_{q+1} C_q IQ_T J f_qX \\ & & \cong \downarrow C_q W_{q+1} C_q IQ_T J(\pi_q^X) \\ & & C_q W_{q+1} C_q IQ_T J s_qX \end{array}$$

Furthermore, these maps induce natural isomorphisms between the following exact functors

$$\begin{array}{ccc} \mathcal{H}(S) & \xrightarrow{C_q W_{q+1} C_q IQ_T J s_q} & \mathcal{H}(S) \\ & \xrightarrow{C_q W_{q+1} C_q IQ_T J f_q} & \\ \mathcal{H}(S) & \xrightarrow{C_q W_{q+1} C_q IQ_T J f_q} & \mathcal{H}(S) \\ & \xrightarrow{C_q W_{q+1} C_q IQ_T J} & \end{array}$$

Proof. — The naturality of the maps $\pi_q^X : f_q X \rightarrow s_q X$ and $\theta_X : f_q X \rightarrow X$ implies that we have induced natural transformations of functors $C_q W_{q+1} C_q I Q_T J f_q \rightarrow C_q W_{q+1} C_q I Q_T J s_q$ and $C_q W_{q+1} C_q I Q_T J f_q \rightarrow C_q W_{q+1} C_q I Q_T J$. Hence, it is enough to show that for every T -spectrum X , $C_q W_{q+1} C_q I Q_T J(\pi_q^X)$ and $C_q W_{q+1} C_q I Q_T J(\theta_X)$ are weak equivalences in $\text{Spt}_T \mathcal{M}_*$.

Proposition 3.2.78 implies that the following natural maps

$$C_q I Q_T J X \xleftarrow[\cong]{C_q I Q_T J(\theta_X)} C_q I Q_T J f_q X \xrightarrow[\cong]{C_q I Q_T J(\pi_q^X)} C_q I Q_T J s_q X$$

are isomorphisms in $S^q \mathcal{H}(S)$. Then the result follows immediately from Corollary 3.2.70 and Proposition 3.2.11. \square

Theorem 3.2.80. — *For every T -spectrum X , we have the following diagram in $\mathcal{H}(S)$:*

$$(48) \quad \begin{array}{ccc} & W_{q+1} C_q I Q_T J s_q X & \\ & \swarrow \cong & \nwarrow \cong \\ C_q I Q_T J s_q X & & C_q W_{q+1} C_q I Q_T J s_q X \\ \downarrow \cong & & \uparrow \cong \\ C_q I Q_T J s_q X & & C_q W_{q+1} C_q I Q_T J(\pi_q^X) \\ \uparrow \cong & & \downarrow \cong \\ I Q_T J s_q X & & C_q W_{q+1} C_q I Q_T J f_q X \\ & & \downarrow \cong \\ & & C_q W_{q+1} C_q I Q_T J(\theta_X) \\ & & \downarrow \cong \\ & & C_q W_{q+1} C_q I Q_T J X \\ & & \downarrow \cong \\ & & s_q X \end{array}$$

where all the maps are isomorphisms in $\mathcal{H}(S)$. Furthermore, this diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}(S) \begin{array}{c} \xrightarrow{s_q} \\ \xrightarrow{C_q W_{q+1} C_q I Q_T J} \end{array} \mathcal{H}(S)$$

Proof. — The fact that $I Q_T J s_q X$ and $C_q I Q_T J s_q X$ are isomorphisms in $\mathcal{H}(S)$ follows from Proposition 3.2.75. Now Proposition 3.2.76 implies that $W_{q+1} C_q I Q_T J s_q X$ is an isomorphism in $\mathcal{H}(S)$, and proposition 3.2.77 implies that $C_q W_{q+1} C_q I Q_T J s_q X$ is also an isomorphism in $\mathcal{H}(S)$. Finally, Corollary 3.2.79 implies that $C_q W_{q+1} C_q I Q_T J(\pi_q^X)$ and $C_q W_{q+1} C_q I Q_T J(\theta_X)$ are both isomorphisms in $\mathcal{H}(S)$.

This shows that all the maps in the diagram (48) are isomorphisms in $\mathcal{H}(S)$, therefore for every T -spectrum X we can define the following composition in $\mathcal{H}(S)$

$$(49) \quad \begin{array}{ccc} & W_{q+1}C_qIQ_TJs_qX & \\ & \cong \nearrow & \searrow \cong \\ C_qIQ_TJs_qX & & C_qW_{q+1}C_qIQ_TJs_qX \\ \uparrow \cong & & \downarrow \cong \\ (C_q^{IQ_TJs_qX})^{-1} & & (C_qW_{q+1}C_qIQ_TJ(\pi_q^X))^{-1} \\ \uparrow \cong & & \downarrow \cong \\ IQ_TJs_qX & & C_qW_{q+1}C_qIQ_TJf_qX \\ \uparrow \cong & & \downarrow \cong \\ IQ_TJs_qX & & C_qW_{q+1}C_qIQ_TJ(\theta_X) \\ \uparrow \cong & & \downarrow \cong \\ s_qX & & C_qW_{q+1}C_qIQ_TJX \end{array}$$

which is an isomorphism.

On the other hand, Propositions 3.2.75, 3.2.76 and 3.2.77, and Corollary 3.2.79 imply all together that the isomorphisms defined in diagram (49) induce a natural isomorphism of functors $s_q \xrightarrow{\cong} C_qW_{q+1}C_qIQ_TJ$. This finishes the proof. \square

Proposition 3.2.81. — *Let η denote the unit of the adjunction $(C_q, W_{q+1}, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}(S) \rightarrow S^q \mathcal{H}(S)$ constructed in Corollary 3.2.70. We have that the natural transformation $\pi_q : f_q \rightarrow s_q$ (see Theorem 3.1.16) gets canonically identified, through the equivalence of categories $r_q C_q, IQ_T J i_q$ constructed in Proposition 3.2.21 with the following map in $\mathcal{H}(S)$:*

$$C_qIQ_TJX \xrightarrow{C_q(\eta_{IQ_TJX})} C_qW_{q+1}C_qIQ_TJX$$

Proof. — It follows directly from Theorem 3.1.16, Corollary 3.2.63 together with [20, Proposition 9.1.8]. \square

Remark 3.2.82. — *Theorem 3.2.80 gives the desired lifting to the model category level for the functors s_q defined in Theorem 3.1.16; and it completes the program that we started at the beginning of this section, where the goal was to get a lifting for the slice functors s_q .*

3.3. The Symmetric Model Structure for the Slice Filtration

Our goal now is to lift the model structures constructed in section 3.2 to the category of symmetric T -spectra, in order to have a natural framework for the study of the multiplicative properties of Voevodsky’s slice filtration.

Let $\mathcal{H}^\Sigma(S)$ denote the homotopy category associated to $\text{Spt}_T^\Sigma \mathcal{M}_*$. We call $\mathcal{H}^\Sigma(S)$ the *motivic symmetric stable homotopy category*. We will denote by $[-, -]_{\text{Spt}}^\Sigma$ the set of maps between two objects in $\mathcal{H}^\Sigma(S)$, and $q \in \mathbb{Z}$ will be an arbitrary integer.

Definition 3.3.1. — Let Q_Σ denote a cofibrant replacement functor in $\text{Spt}_T^\Sigma \mathcal{M}_*$; such that for every symmetric T -spectrum X , the natural map

$$Q_\Sigma X \xrightarrow{Q_\Sigma^X} X$$

is a trivial fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Definition 3.3.2. — Let R_Σ denote a fibrant replacement functor in $\text{Spt}_T^\Sigma \mathcal{M}_*$; such that for every symmetric T -spectrum X , the natural map

$$X \xrightarrow{R_\Sigma^X} R_\Sigma X$$

is a trivial cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proposition 3.3.3. — The motivic symmetric stable homotopy category $\mathcal{H}^\Sigma(S)$ has a structure of triangulated category defined as follows:

1. The suspension $\Sigma_T^{1,0}$ functor is given by

$$\begin{aligned} - \wedge S^1 : \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ X &\longmapsto Q_\Sigma X \wedge S^1 \end{aligned}$$

2. The distinguished triangles are isomorphic to triangles of the form

$$A \xrightarrow{i} B \xrightarrow{j} C \xrightarrow{k} \Sigma_T^{1,0} A$$

where i is a cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and C is the homotopy cofibre of i .

Proof. — Theorem 2.6.23 implies in particular that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a pointed simplicial model category, and Theorem 2.6.27 implies that the adjunction:

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence. The result now follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII] (see [10, Proposition 7.1.6]). □

Theorem 3.3.4. — The adjunction

$$(V, U, \varphi) : \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and the forgetful functors, induces an adjunction

$$(VQ_s, UR_\Sigma, \varphi) : \mathcal{H}(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

of exact functors between triangulated categories. Furthermore, VQ_s and UR_Σ are both equivalences of categories.

Proof. — Theorem 2.6.30 implies that the adjunction (V, U, φ) is a Quillen equivalence. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(VQ_s, UR_\Sigma, \varphi) : \mathcal{H}(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

Now [10, Proposition 1.3.13] implies that VQ_s, UR_Σ are both equivalences of categories. Finally, Proposition 2.6.19 together with [10, Proposition 6.4.1] imply that VQ_s maps cofibre sequences in $\mathcal{H}(S)$ to cofibre sequences in $\mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that VQ_s and UR_Σ are both exact functors between triangulated categories. \square

Corollary 3.3.5. — 1. The exact functor (see Remark 3.1.13)

$$f_q : \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

gets canonically identified with the following exact functor:

$$\begin{aligned} \tilde{f}_q : \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ X &\longmapsto VQ_s(f_q(UR_\Sigma X)) \end{aligned}$$

i.e. $\tilde{f}_q = VQ_s \circ f_q \circ UR_\Sigma$.

2. The exact functor (see Theorem 3.1.18)

$$s_{<q} : \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

gets canonically identified with the following exact functor:

$$\begin{aligned} \tilde{s}_{<q} : \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ X &\longmapsto VQ_s(s_{<q}(UR_\Sigma X)) \end{aligned}$$

i.e. $\tilde{s}_{<q} = VQ_s \circ s_{<q} \circ UR_\Sigma$.

3. The exact functor (see Theorem 3.1.16)

$$s_q : \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$

gets canonically identified with the following exact functor:

$$\begin{aligned} \tilde{s}_q : \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ X &\longmapsto VQ_s(s_q(UR_\Sigma X)) \end{aligned}$$

i.e. $\tilde{s}_q = VQ_s \circ s_q \circ UR_\Sigma$.

Proof. — It follows immediately from Theorem 3.3.4. \square

Lemma 3.3.6. — *Let $X \in \mathcal{M}_*$ be a pointed simplicial presheaf which is compact in the sense of Jardine (see definition 2.3.10), and let $F_n^\Sigma(X)$ be the symmetric T -spectrum constructed in definition 2.6.8. Consider an arbitrary collection of symmetric T -spectra $\{Z_i\}_{i \in I}$ indexed by a set I . Then*

$$[F_n^\Sigma(X), \coprod_{i \in I} Z_i]_{\text{Spt}}^\Sigma \cong \prod_{i \in I} [F_n^\Sigma(X), Z_i]_{\text{Spt}}^\Sigma$$

Proof. — We have that $F_n^\Sigma = V \circ F_n$ (see Proposition 2.6.18). Therefore, the result follows directly from Lemma 3.1.4 together with Theorems 2.6.30 and 3.3.4. □

Proposition 3.3.7. — *The motivic symmetric stable homotopy category $\mathcal{H}^\Sigma(S)$ is a compactly generated triangulated category in the sense of Neeman (see [19, definition 1.7]). The set of compact generators is given by (see definition 2.6.8):*

$$C^\Sigma = \bigcup_{n,r,s \geq 0} \bigcup_{U \in (\mathcal{A}m|_S)} F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

i.e. the smallest triangulated subcategory of $\mathcal{H}^\Sigma(S)$ closed under small coproducts and containing all the objects in C^Σ coincides with $\mathcal{H}^\Sigma(S)$.

Proof. — It follows directly from Proposition 3.1.5, Lemma 3.3.6 and theorems 2.6.30, 3.3.4. □

Corollary 3.3.8. — *Let $f : X \rightarrow Y$ be a map in $\mathcal{H}^\Sigma(S)$. Then f is an isomorphism if and only if f induces an isomorphism of abelian groups:*

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}}^\Sigma \xrightarrow{f_*} [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}^\Sigma$$

for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^\Sigma$.

Proof. — (\Rightarrow): If f is an isomorphism in $\mathcal{H}^\Sigma(S)$ it is clear that the induced maps f_* are isomorphisms of abelian groups for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^\Sigma$.

(\Leftarrow): Complete f to a distinguished triangle in $\mathcal{H}^\Sigma(S)$:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_T^{1,0} X$$

Then f is an isomorphism if and only if $Z \cong *$ in $\mathcal{H}^\Sigma(S)$.

Now since the functor $[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), -]_{\text{Spt}}^\Sigma$ is homological, we get the following long exact sequence of abelian groups:

$$\begin{array}{ccc}
 \vdots & & \\
 \downarrow & & \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}}^\Sigma & & \\
 \downarrow f_* & & \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}^\Sigma & & \\
 \downarrow g_* & & \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}^\Sigma & & \\
 \downarrow h_* & & \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} X]_{\text{Spt}}^\Sigma & \xleftarrow[\Sigma_T^{1,0}]{\cong} & [F_{n+1}^\Sigma(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), X]_{\text{Spt}}^\Sigma \\
 \downarrow \Sigma_T^{1,0} f_* & & \downarrow f_* \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Sigma_T^{1,0} Y]_{\text{Spt}}^\Sigma & \xleftarrow[\Sigma_T^{1,0}]{\cong} & [F_{n+1}^\Sigma(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Y]_{\text{Spt}}^\Sigma \\
 \downarrow & & \\
 \vdots & &
 \end{array}$$

But by hypothesis all the maps f_* are isomorphisms, therefore $[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}^\Sigma = 0$ for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^\Sigma$. Since $\mathcal{H}^\Sigma(S)$ is a compactly generated triangulated category (see Proposition 3.3.7) with set of compact generators C^Σ , we have that $Z \cong *$. This implies that f is an isomorphism, as we wanted. \square

Theorem 3.3.9. — Consider the following set of objects in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.2.1):

$$C_{\text{eff}}^{q,\Sigma} = \bigcup_{n,r,s \geq 0; s-n \geq q} \bigcup_{U \in (\mathcal{A}m|_S)} F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

The right Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the class of $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalences exists (see definitions 1.8.6 and 1.9.2). This model structure will be called $(q-1)$ -connected motivic symmetric stable, and the category of symmetric T -spectra equipped with the $(q-1)$ -connected motivic symmetric stable model structure will be denoted by $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Furthermore $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right proper and simplicial model category. The homotopy category associated to $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ will be denoted by $R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$.

Proof. — Theorems 2.6.23 and 2.7.4 imply that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a cellular, proper and simplicial model category. Therefore we can apply Theorem 5.1.1 in [6] to construct the right Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the class of $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalences. Using Theorem 5.1.1 in [6] again, we have that this new model structure is right proper and simplicial. \square

Definition 3.3.10. — Let C_q^Σ denote a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$; such that for every symmetric T -spectrum X , the natural map

$$C_q^\Sigma X \xrightarrow{C_q^{\Sigma,X}} X$$

is a trivial fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, and $C_q^\Sigma X$ is always $C_{\text{eff}}^{q,\Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proposition 3.3.11. — R_Σ is also a fibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.2), and for every symmetric T -spectrum X the natural map

$$X \xrightarrow{R_\Sigma^X} R_\Sigma X$$

is a trivial cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — Since $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is the right Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalences, by construction we have that the fibrations and the trivial cofibrations are identical in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$ respectively. This implies that for every symmetric T -spectrum X , $R_\Sigma X$ is fibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, and we also have that the natural map

$$X \xrightarrow{R_\Sigma^X} R_\Sigma X$$

is a trivial cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Hence R_Σ is also a fibrant replacement functor for $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. \square

Proposition 3.3.12. — A map of symmetric T -spectra $f : X \rightarrow Y$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if the underlying map $UR_\Sigma(f) : UR_\Sigma X \rightarrow UR_\Sigma Y$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$.

Proof. — Consider $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q,\Sigma}$. Using the enriched adjunctions of proposition 2.6.19, we get the following commutative diagram where all the vertical

arrows are isomorphisms:

$$\begin{array}{ccc}
 \mathrm{Map}_{\Sigma}(F_n^{\Sigma}(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_{\Sigma}X) & \xrightarrow{R_{\Sigma}f_*} & \mathrm{Map}_{\Sigma}(F_n^{\Sigma}(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_{\Sigma}Y) \\
 \parallel & & \parallel \\
 \mathrm{Map}_{\Sigma}(V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), R_{\Sigma}X) & \xrightarrow{R_{\Sigma}f_*} & \mathrm{Map}_{\Sigma}(V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), R_{\Sigma}Y) \\
 \cong \downarrow & & \downarrow \cong \\
 \mathrm{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_{\Sigma}X) & \xrightarrow{UR_{\Sigma}f_*} & \mathrm{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_{\Sigma}Y)
 \end{array}$$

Since $UR_{\Sigma}X$ and $UR_{\Sigma}Y$ are both fibrant in $\mathrm{Spt}_T\mathcal{M}_*$, we have that $UR_{\Sigma}(f)$ is a C_{eff}^q -colocal equivalence in $\mathrm{Spt}_T\mathcal{M}_*$ if and only if the bottom row in the diagram above is a weak equivalence of simplicial sets for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^q$. By the two out of three property for weak equivalences we have that this happens if and only if the top row in the diagram above is a weak equivalence for every $F_n^{\Sigma}(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^{q,\Sigma}$. But this last condition holds if and only if f is a $C_{\mathrm{eff}}^{q,\Sigma}$ -colocal equivalence in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$. This finishes the proof. \square

Proposition 3.3.13. — *Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We have that f is a $C_{\mathrm{eff}}^{q,\Sigma}$ -colocal equivalence in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$ if and only if for every $F_n^{\Sigma}(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^{q,\Sigma}$, the induced map:*

$$[F_n^{\Sigma}(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\mathrm{Spt}}^{\Sigma} \xrightarrow{f_*} [F_n^{\Sigma}(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\mathrm{Spt}}^{\Sigma}$$

is an isomorphism of abelian groups.

Proof. — By Proposition 3.3.12, f is a $C_{\mathrm{eff}}^{q,\Sigma}$ -colocal equivalence in $\mathrm{Spt}_T^{\Sigma}\mathcal{M}_*$ if and only if $UR_{\Sigma}(f)$ is a C_{eff}^q -colocal equivalence in $\mathrm{Spt}_T\mathcal{M}_*$. Using Proposition 3.2.4 we have that $UR_{\Sigma}(f)$ is a C_{eff}^q -colocal equivalence if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^q$, the induced map

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_{\Sigma}X]_{\mathrm{Spt}} \xrightarrow{UR_{\Sigma}(f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_{\Sigma}Y]_{\mathrm{Spt}}$$

is an isomorphism of abelian groups.

Now Theorem 2.6.30 implies that we have the following commutative diagram, where all the vertical arrows are isomorphisms:

$$\begin{array}{ccc}
 [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_\Sigma X]_{\text{Spt}} & \xrightarrow{UR_\Sigma(f)_*} & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_\Sigma Y]_{\text{Spt}} \\
 \cong \downarrow & & \downarrow \cong \\
 [V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), X]_{\text{Spt}}^\Sigma & \xrightarrow{f_*} & [V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), Y]_{\text{Spt}}^\Sigma \\
 \parallel & & \parallel \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), X]_{\text{Spt}}^\Sigma & \xrightarrow{f_*} & [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Y]_{\text{Spt}}^\Sigma
 \end{array}$$

Therefore f is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence if and only if for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q,\Sigma}$, the bottom row is an isomorphism of abelian groups. This finishes the proof. \square

Lemma 3.3.14. — *Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We have that f is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $\Omega_{S^1} R_\Sigma f$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — It follows from Proposition 3.3.12 that f is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $UR_\Sigma f$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$. Since $UR_\Sigma X, UR_\Sigma Y$ are both fibrant in $\text{Spt}_T \mathcal{M}_*$, using Lemma 3.2.7 we have that $UR_\Sigma f$ is a C_{eff}^q -colocal equivalence if and only if $\Omega_{S^1} UR_\Sigma f = U(\Omega_{S^1} R_\Sigma f)$ is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$.

Finally, since $\Omega_{S^1} R_\Sigma X, \Omega_{S^1} R_\Sigma Y$ are both fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, we have by Proposition 3.3.12 that $U(\Omega_{S^1} R_\Sigma f)$ is a C_{eff}^q -colocal equivalence if and only if $\Omega_{S^1} R_\Sigma f$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence. This finishes the proof. \square

Corollary 3.3.15. — *The adjunction*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Using Corollary 1.3.16 in [10] and Proposition 3.3.11 we have that it suffices to verify the following two conditions:

1. For every cofibrant object X in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, the following composition

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1} R_\Sigma^{X \wedge S^1}} \Omega_{S^1} R_\Sigma(X \wedge S^1)$$

is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence.

2. Ω_{S^1} reflects $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalences between fibrant objects in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): By construction $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore the identity functor

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Thus X is also cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Since the adjunction $(-\wedge S^1, \Omega_{S^1}, \varphi)$ is a Quillen equivalence on $\text{Spt}_T^\Sigma \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies that the following composition is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$:

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1} R_\Sigma^{X \wedge S^1}} \Omega_{S^1} R_\Sigma(X \wedge S^1)$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $C_{\text{eff}}^{q, \Sigma}$ -colocal equivalence.

(2): This follows immediately from Proposition 3.3.11 and Lemma 3.3.14. \square

Remark 3.3.16. — *The adjunction $(\Sigma_T, \Omega_T, \varphi)$ is a Quillen equivalence on $\text{Spt}_T^\Sigma \mathcal{M}_*$. However it does not descend even to a Quillen adjunction on the $(q-1)$ -connected motivic symmetric stable model category $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*

Corollary 3.3.17. — *$R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$ has the structure of a triangulated category.*

Proof. — Theorem 3.3.9 implies in particular that $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a pointed simplicial model category, and Corollary 3.3.15 implies that the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence. Therefore the result follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII]. \square

Proposition 3.3.18. — *We have the following adjunction*

$$(C_q^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

between exact functors of triangulated categories.

Proof. — Since $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is the right Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $C_{\text{eff}}^{q, \Sigma}$ -colocal equivalences, we have that the identity functor $\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(C_q^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

Now Proposition 6.4.1 in [10] implies that C_q^Σ maps cofibre sequences in $R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$ to cofibre sequences in $\mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that C_q^Σ and R_Σ are both exact functors between triangulated categories. \square

Theorem 3.3.19. — *The adjunction*

$$(V, U, \varphi) : R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and the forgetful functors is a Quillen equivalence.

Proof. — Proposition 3.3.12 together with the universal property for right Bousfield localizations (see definition 1.8.2) imply that

$$U : R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$$

is a right Quillen functor. Using Corollary 1.3.16 in [10] and Proposition 3.3.11 we have that it suffices to verify the following two conditions:

1. For every cofibrant object X in $R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$, the following composition

$$X \xrightarrow{\eta_X} UV(X) \xrightarrow{UR_\Sigma^{V,X}} UR_\Sigma V(X)$$

is a weak equivalence in $R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$.

2. U reflects weak equivalences between fibrant objects in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): By construction $R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$ is a right Bousfield localization of $\text{Spt}_T \mathcal{M}_*$, therefore the identity functor

$$\text{id} : R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* \longrightarrow \text{Spt}_T \mathcal{M}_*$$

is a left Quillen functor. Thus X is also cofibrant in $\text{Spt}_T \mathcal{M}_*$. Since the adjunction (V, U, φ) is a Quillen equivalence between $\text{Spt}_T \mathcal{M}_*$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies that the following composition is a weak equivalence in $\text{Spt}_T \mathcal{M}_*$:

$$X \xrightarrow{\eta_X} UV(X) \xrightarrow{UR_\Sigma^{V,X}} UR_\Sigma V(X)$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a C_{eff}^q -colocal equivalence in $\text{Spt}_T \mathcal{M}_*$, i.e. a weak equivalence in $R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$.

- (2): This follows immediately from Propositions 3.3.11 and 3.3.12. □

Corollary 3.3.20. — *The adjunction*

$$(V, U, \varphi) : R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and the forgetful functors, induces an adjunction

$$(VC_q, UR_\Sigma, \varphi) : R_{C_{\text{eff}}}^q \mathcal{H}(S) \longrightarrow R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S)$$

of exact functors between triangulated categories. Furthermore, VC_q and UR_Σ are both equivalences of categories.

Proof. — Theorem 3.3.19 implies that the adjunction (V, U, φ) is a Quillen equivalence. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(VC_q, UR_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}(S) \longrightarrow R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$$

Now [10, Proposition 1.3.13] implies that VC_q, UR_Σ are both equivalences of categories. Finally, Proposition 2.6.19 together with [10, Proposition 6.4.1] imply that VC_q maps cofibre sequences in $R_{C_{\text{eff}}^q} \mathcal{H}(S)$ to cofibre sequences in $R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that VC_q and UR_Σ are both exact functors between triangulated categories. \square

Now it is very easy to find the desired lifting for the functor $\tilde{f}_q : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$ (see Corollary 3.3.5(1)) to the model category level.

Lemma 3.3.21. — 1. *Let X be an arbitrary T -spectrum in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$. Then the following maps in $\text{Spt}_T^\Sigma \mathcal{M}_*$*

$$VQ_s(C_q X) \xrightarrow{V(Q_s^{C_q X})} VC_q X \xleftarrow{C_q^{\Sigma, VC_q X}} C_q^\Sigma(VC_q X)$$

induce natural isomorphisms between the functors:

$$C_q^\Sigma \circ VC_q, VC_q, VQ_s \circ C_q : R_{C_{\text{eff}}^q} \mathcal{H}(S) \rightarrow \mathcal{H}^\Sigma(S)$$

$$\begin{array}{ccc}
 & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \\
 VC_q \nearrow & & \searrow C_q^\Sigma \\
 R_{C_{\text{eff}}^q} \mathcal{H}(S) & \xrightarrow{VC_q} & \mathcal{H}^\Sigma(S) \\
 C_q \searrow & & \nearrow VQ_s \\
 & \mathcal{H}(S) &
 \end{array}$$

Given a T -spectrum X

$$\alpha_X : VQ_s(C_q X) \xrightarrow{\cong} C_q^\Sigma(VC_q X)$$

will denote the isomorphism in $\mathcal{H}^\Sigma(S)$ corresponding to the natural isomorphism between $VQ_s \circ C_q$ and $C_q^\Sigma \circ VC_q$.

2. *Let X be an arbitrary symmetric T -spectrum. Then the following maps in $R_{C_{\text{eff}}^q} \text{Spt}_T \mathcal{M}_*$*

$$IQ_T J(UR_\Sigma X) \xleftarrow{IQ_T J^{UR_\Sigma X}} UR_\Sigma X \xrightarrow{U(R_\Sigma^{R_\Sigma X})} UR_\Sigma(R_\Sigma X)$$

induce natural isomorphisms between the functors:

$$IQ_T J \circ UR_\Sigma, UR_\Sigma, UR_\Sigma \circ R_\Sigma : \mathcal{H}^\Sigma(S) \rightarrow R_{C_{\text{eff}}}^q \mathcal{H}(S)$$

$$\begin{array}{ccc}
 & \mathcal{H}(S) & \\
 UR_\Sigma \nearrow & & \searrow IQ_T J \\
 \mathcal{H}^\Sigma(S) & \xrightarrow{UR_\Sigma} & R_{C_{\text{eff}}}^q \mathcal{H}(S) \\
 R_\Sigma \searrow & & \nearrow UR_\Sigma \\
 & R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) &
 \end{array}$$

Given a symmetric T -spectrum X

$$\beta_X : IQ_T J(UR_\Sigma X) \xrightarrow{\cong} UR_\Sigma(R_\Sigma X)$$

will denote the isomorphism in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$ corresponding to the natural isomorphism between $IQ_T J \circ UR_\Sigma$ and $UR_\Sigma \circ R_\Sigma$.

Proof. — (1): It follows immediately from Theorem 1.3.7 in [10] and the following commutative diagram of left Quillen functors:

$$\begin{array}{ccc}
 R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \text{id} \downarrow & & \downarrow \text{id} \\
 \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & \text{Spt}_T^\Sigma \mathcal{M}_*
 \end{array}$$

(2): It follows immediately from the dual of Theorem 1.3.7 in [10] and the following commutative diagram of right Quillen functors:

$$\begin{array}{ccc}
 R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* & \xleftarrow{U} & R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \text{id} \uparrow & & \uparrow \text{id} \\
 \text{Spt}_T \mathcal{M}_* & \xleftarrow{U} & \text{Spt}_T^\Sigma \mathcal{M}_*
 \end{array}
 \quad \square$$

Theorem 3.3.22. — *Let X be an arbitrary symmetric T -spectrum.*

1. The diagram (29) in Theorem 3.2.20 induces the following diagram in $\mathcal{H}^\Sigma(S)$:

$$(50) \quad \begin{array}{ccc} VQ_s(IQ_T J f_q(UR_\Sigma X)) & \xleftarrow[VQ_s(C_q^{IQ_T J f_q(UR_\Sigma X)})]{\cong} & VQ_s(C_q IQ_T J f_q(UR_\Sigma X)) \\ \uparrow \cong & & \downarrow \cong \\ VQ_s(IQ_T J^f_q UR_\Sigma X) & & VQ_s(C_q IQ_T J(\theta_{UR_\Sigma X})) \\ \tilde{f}_q X = VQ_s(f_q(UR_\Sigma X)) & & VQ_s(C_q IQ_T J(UR_\Sigma X)) \end{array}$$

where all the maps are isomorphisms in $\mathcal{H}^\Sigma(S)$. Furthermore, this diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow{\tilde{f}_q} \\ \xrightarrow[VQ_s \circ C_q IQ_T J \circ UR_\Sigma]{} \end{array} \mathcal{H}^\Sigma(S)$$

2. Let ϵ be the counit of the adjunction (see Corollary 3.3.20):

$$(VC_q, UR_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}(S) \longrightarrow R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$$

Then we have the following diagram in $\mathcal{H}^\Sigma(S)$ (see Lemma 3.3.21):

$$(51) \quad \begin{array}{ccc} C_q^\Sigma(VC_q(IQ_T J(UR_\Sigma X))) & \xrightarrow[C_q^\Sigma(VC_q(\beta_X))]{\cong} & C_q^\Sigma(VC_q(UR_\Sigma(R_\Sigma X))) \\ \uparrow \cong & & \downarrow \cong \\ \alpha_{IQ_T J(UR_\Sigma X)} & & C_q^\Sigma(\epsilon_{R_\Sigma X}) \\ VQ_s(C_q IQ_T J(UR_\Sigma X)) & & C_q^\Sigma R_\Sigma X = f_q^\Sigma X \end{array}$$

where all the maps are isomorphisms in $\mathcal{H}^\Sigma(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow[VQ_s \circ C_q IQ_T J \circ UR_\Sigma]{} \\ \xrightarrow[C_q^\Sigma R_\Sigma = f_q^\Sigma]{} \end{array} \mathcal{H}^\Sigma(S)$$

3. Combining the diagrams (50) and (51) above we get a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow{\tilde{f}_q} \\ \xrightarrow{f_q^\Sigma} \end{array} \mathcal{H}^\Sigma(S)$$

Proof. — It is clear that it suffices to prove only the first two claims.

(1): It follows immediately from theorems 3.2.20 and 3.3.4.

(2): It follows immediately from Lemma 3.3.21 and Corollary 3.3.20. □

Proposition 3.3.23. — *Let ϵ denote the counit of the adjunction $(C_q^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$ constructed in Proposition 3.3.18. Then the natural transformation $\theta_q : f_q \rightarrow \text{id}$ (see Proposition 3.1.15) gets canonically identified, through the equivalence of categories $r_q C_q, IQ_T J_i, VC_q, UR_\Sigma, VQ_s$ and UR_Σ constructed in Proposition 3.2.21, corollary 3.3.20 and Theorem 3.3.4; with $\theta_q^\Sigma = \epsilon$.*

Proof. — By construction θ_q is the counit of the adjunction

$$(i_q, r_q, \varphi) : \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \rightarrow \mathcal{H}(S)$$

(see Proposition 3.1.15). The result follows immediately from Proposition 3.2.21, corollary 3.3.20 and theorem 3.3.4. □

The functor f_q^Σ gives the desired lifting for the functor \tilde{f}_q to the model category level, and it will be used in the study of the multiplicative properties of Voevodsky’s slice filtration.

Proposition 3.3.24. — 1. *We have the following commutative diagram of left Quillen functors:*

$$(52) \quad \begin{array}{ccc} R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \\ & \searrow \text{id} & \swarrow \text{id} \\ & \text{Spt}_T^\Sigma \mathcal{M}_* & \end{array}$$

2. *For every symmetric T-spectrum X, the natural map:*

$$C_q^\Sigma C_{q+1}^\Sigma X \xrightarrow{C_q^{\Sigma, C_{q+1}^\Sigma X}} C_{q+1}^\Sigma X$$

is a weak equivalence in $\mathcal{H}^\Sigma(S)$, and it induces a natural equivalence $C_q^{\Sigma, C_{q+1}^{\Sigma-}}$: $C_q^\Sigma \circ C_{q+1}^\Sigma \rightarrow C_{q+1}^\Sigma$ between the following functors:

$$\begin{array}{ccc} R_{C_{\text{eff}}^{q+1}} \mathcal{H}^\Sigma(S) & \xrightarrow{C_{q+1}^\Sigma} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \\ & \searrow C_{q+1}^\Sigma & \swarrow C_q^\Sigma \\ & \mathcal{H}^\Sigma(S) & \end{array}$$

3. The natural transformation $f_{q+1}X \rightarrow f_qX$ (see Theorem 3.1.16(1)) gets canonically identified, through the equivalence of categories r_qC_q, IQ_TJi_q, VC_q and UR_Σ constructed in Proposition 3.2.21 and corollary 3.3.20; with the following composition $\rho_q^X : f_{q+1}^\Sigma X \rightarrow f_q^\Sigma X$ in $\mathcal{A}^\Sigma(S)$

$$\begin{array}{ccc}
 & C_q^\Sigma C_{q+1}^\Sigma R_\Sigma X & \\
 (C_q^{\Sigma, C_{q+1}^\Sigma R_\Sigma X})^{-1} \nearrow & & \searrow C_q^\Sigma (C_{q+1}^{\Sigma, R_\Sigma X}) \\
 C_{q+1}^\Sigma R_\Sigma X & & C_q^\Sigma R_\Sigma X
 \end{array}$$

which is induced by the following commutative diagram in $\text{Spt}_T^\Sigma \mathcal{M}_*$

$$(53) \quad \begin{array}{ccc}
 C_q^\Sigma C_{q+1}^\Sigma R_\Sigma X & \xrightarrow{C_q^\Sigma (C_{q+1}^{\Sigma, R_\Sigma X})} & C_q^\Sigma R_\Sigma X \\
 C_q^{\Sigma, C_{q+1}^\Sigma R_\Sigma X} \downarrow & & \downarrow C_q^{\Sigma, R_\Sigma X} \\
 C_{q+1}^\Sigma R_\Sigma X & \xrightarrow{C_{q+1}^{\Sigma, R_\Sigma X}} & R_\Sigma X
 \end{array}$$

Proof. — (1): Since $R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ are both right Bousfield localizations of $\text{Spt}_T^\Sigma \mathcal{M}_*$, by construction the identity functor

$$\text{id} : R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is in both cases a left Quillen functor. To finish the proof, it suffices to show that the identity functor

$$\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a right Quillen functor. Using the universal property of right Bousfield localizations (see definition 1.8.2), it is enough to check that if $f : X \rightarrow Y$ is a $C_{\text{eff}}^{q, \Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ then $R_\Sigma f$ is a $C_{\text{eff}}^{q+1, \Sigma}$ -colocal equivalence. But since $R_\Sigma X$ and $R_\Sigma Y$ are already fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, we have that $R_\Sigma(f)$ is a $C_{\text{eff}}^{q+1, \Sigma}$ -colocal equivalence if and only if for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q+1, \Sigma}$, the induced map:

$$\begin{array}{c}
 \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_\Sigma X) \\
 \downarrow (R_\Sigma f)_* \\
 \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_\Sigma Y)
 \end{array}$$

is a weak equivalence of simplicial sets. But since $C_{\text{eff}}^{q+1, \Sigma} \subseteq C_{\text{eff}}^{q, \Sigma}$, and by hypothesis f is a $C_{\text{eff}}^{q, \Sigma}$ -colocal equivalence; we have that all the induced maps $(R_\Sigma f)_*$ are weak equivalences of simplicial sets. Thus $R_\Sigma f$ is a $C_{\text{eff}}^{q+1, \Sigma}$ -colocal equivalence, as we wanted.

Finally (2) and (3) follow directly from Proposition 3.2.21, Corollary 3.3.20, Theorems 3.2.20, 3.3.22 together with the commutative diagram (52) of left Quillen functors constructed above and [10, Theorem 1.3.7]. □

Theorem 3.3.25. — *We have the following commutative diagram of left Quillen functors:*

$$\begin{array}{c}
 \vdots \\
 \text{id} \downarrow \\
 R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \text{id} \downarrow \quad \searrow \text{id} \\
 R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{\text{id}} \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \text{id} \downarrow \quad \nearrow \text{id} \\
 R_{C_{\text{eff}}^{q-1}} \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \text{id} \downarrow \\
 \vdots
 \end{array}
 \tag{54}$$

and the associated diagram of homotopy categories:

$$\begin{array}{c}
 \vdots \\
 \downarrow \uparrow \\
 R_{C_{\text{eff}}^{q+1}} \mathcal{H}^\Sigma(S) \begin{array}{l} \searrow C_{q+1}^\Sigma \\ \swarrow R_\Sigma \end{array} \\
 \downarrow C_{q+1}^\Sigma \uparrow R_\Sigma \\
 R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \begin{array}{l} \searrow C_q^\Sigma \\ \swarrow R_\Sigma \end{array} \mathcal{H}^\Sigma(S) \\
 \downarrow C_q^\Sigma \uparrow R_\Sigma \\
 R_{C_{\text{eff}}^{q-1}} \mathcal{H}^\Sigma(S) \begin{array}{l} \searrow C_{q-1}^\Sigma \\ \swarrow R_\Sigma \end{array} \\
 \downarrow \uparrow \\
 \vdots
 \end{array}
 \tag{55}$$

gets canonically identified, through the equivalences of categories $r_q C_q$, $IQ_T J_{i_q}$, VC_q and UR_Σ constructed in Proposition 3.2.21 and Corollary 3.3.20; with Voevodsky's

slice filtration:

$$\begin{array}{c}
 \vdots \\
 \downarrow \uparrow \\
 \Sigma_T^{q+1} \Sigma_T^q \mathcal{H}^{\text{eff}}(S) \\
 \begin{array}{ccc}
 & \swarrow^{i_{q+1}} & \\
 i_{q+1} \downarrow & & \searrow^{r_{q+1}} \\
 \Sigma_T^q \mathcal{H}^{\text{eff}}(S) & \xrightarrow{i_q} & \mathcal{H}(S) \\
 \uparrow^{i_q} & \xleftarrow{r_q} & \\
 \Sigma_T^{q-1} \Sigma_T^q \mathcal{H}^{\text{eff}}(S) & \swarrow^{i_{q-1}} & \\
 & \searrow^{r_{q-1}} & \\
 \vdots & &
 \end{array}
 \end{array}
 \tag{56}$$

Proof. — It follows immediately from proposition 3.3.24, Corollary 3.3.20 and Theorem 3.2.23. □

Theorem 3.3.26. — Consider the following set of maps in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.2.29):

$$\begin{aligned}
 (57) \quad L^\Sigma(< q) = \{ & V(\iota_{n,r,s}^U : F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow F_n^\Sigma(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \mid \\
 & F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q,\Sigma} \}
 \end{aligned}$$

The left Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $L^\Sigma(< q)$ -local equivalences exists. This new model structure will be called $\text{weight}^{< q}$ motivic symmetric stable. $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$ will denote the category of symmetric T -spectra equipped with the $\text{weight}^{< q}$ motivic symmetric stable model structure, and $L_{< q} \mathcal{H}^\Sigma(S)$ will denote its associated homotopy category. Furthermore the $\text{weight}^{< q}$ motivic symmetric stable model structure is cellular, left proper and simplicial; with the following sets of generating cofibrations and trivial cofibrations respectively:

$$I_{L^\Sigma(< q)} = I_\Sigma^T = \bigcup_{n \geq 0} \{ F_n^\Sigma(Y_+ \hookrightarrow (\Delta_U^n)_+) \}$$

$$J_{L^\Sigma(< q)} = \{ j : A \rightarrow B \}$$

where j satisfies the following conditions:

1. j is an inclusion of I_Σ^T -complexes.
2. j is a $L^\Sigma(< q)$ -local equivalence.

3. the size of B as an I_Σ^T -complex is less than κ , where κ is the regular cardinal defined by Hirschhorn in [6, definition 4.5.3].

Proof. — Theorems 2.7.4 and 2.6.23 imply that $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a cellular, proper and simplicial model category. Therefore the existence of the left Bousfield localization follows from [6, Theorem 4.1.1]. Using [6, Theorem 4.1.1] again, we have that $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is cellular, left proper and simplicial; where the sets of generating cofibrations and trivial cofibrations are the ones described above. \square

Definition 3.3.27. — Let W_q^Σ denote a fibrant replacement functor in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$; such that for every symmetric T -spectrum X , the natural map:

$$X \xrightarrow{W_q^{\Sigma, X}} W_q^\Sigma X$$

is a trivial cofibration in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$, and $W_q^\Sigma X$ is $L^\Sigma(<q)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proposition 3.3.28. — Q_Σ is also a cofibrant replacement functor in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.1), and for every symmetric T -spectrum X the natural map

$$Q_\Sigma X \xrightarrow{Q_\Sigma^X} X$$

is a trivial fibration in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — Since $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is the left Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $L^\Sigma(<q)$ -local equivalences, by construction we have that the cofibrations and the trivial fibrations are identical in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$ respectively. This implies that for every symmetric T -spectrum X , $Q_\Sigma X$ is cofibrant in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$, and we also have that the natural map

$$Q_\Sigma X \xrightarrow{Q_\Sigma^X} X$$

is a trivial fibration in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Hence Q_Σ is also a cofibrant replacement functor for $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$. \square

Proposition 3.3.29. — Let Z be a symmetric T -spectrum. We have that Z is $L^\Sigma(<q)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if UZ is $L(<q)$ -local in $\text{Spt}_T \mathcal{M}_*$.

Proof. — We have that Z is $L^\Sigma(<q)$ -local if and only if Z is fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and for every

$$\iota_{n,r,s}^U : F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \in L(<q)$$

the induced map

$$\begin{array}{c} \mathrm{Map}_\Sigma(V(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+)), Z) \\ \downarrow V(\iota_{n,r,s}^U)^* \\ \mathrm{Map}_\Sigma(V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), Z) \end{array}$$

is a weak equivalence of simplicial sets.

On the other hand, we have that UZ is $L(< q)$ -local in $\mathrm{Spt}_T\mathcal{M}_*$ if and only if UZ is fibrant in $\mathrm{Spt}_T\mathcal{M}_*$ and for every $\iota_{n,r,s}^U : F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \in L(< q)$, the induced map

$$\mathrm{Map}(F_n(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+), UZ) \xrightarrow{(\iota_{n,r,s}^U)^*} \mathrm{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UZ)$$

is a weak equivalence of simplicial sets.

Then the result follows from the following facts:

1. By definition, Z is fibrant in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ if UZ is fibrant in $\mathrm{Spt}_T\mathcal{M}_*$.
2. Proposition 2.6.19, which implies that the adjunction

$$(V, U, \varphi) : \mathrm{Spt}_T\mathcal{M}_* \longrightarrow \mathrm{Spt}_T^\Sigma\mathcal{M}_*$$

is enriched in the category of simplicial sets. □

Proposition 3.3.30. — *Let Z be a symmetric T -spectrum. We have that Z is $L^\Sigma(< q)$ -local in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ if and only if the following conditions hold:*

1. Z is fibrant in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$.
2. For every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^{q,\Sigma}$, $[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\mathrm{Spt}}^\Sigma \cong 0$

Proof. — It follows directly from Propositions 3.3.29 and 3.2.32, together with the fact that $(V, U, \varphi) : \mathrm{Spt}_T\mathcal{M}_* \rightarrow \mathrm{Spt}_T^\Sigma\mathcal{M}_*$ is a Quillen adjunction. □

Corollary 3.3.31. — *Let Z be a fibrant symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$. We have that Z is $L^\Sigma(< q)$ -local in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ if and only if $\Omega_{S^1}Z$ is $L^\Sigma(< q)$ -local in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$.*

Proof. — By Proposition 3.3.29 we have that Z is $L^\Sigma(< q)$ -local if and only if UZ is $L(< q)$ -local in $\mathrm{Spt}_T\mathcal{M}_*$. Now Corollary 3.2.34 implies that UZ is $L(< q)$ -local if and only if $\Omega_{S^1}UZ = U(\Omega_{S^1}Z)$ is $L(< q)$ -local.

Therefore using Proposition 3.3.29 again, we get that Z is $L^\Sigma(< q)$ -local if and only if $\Omega_{S^1}Z$ is $L^\Sigma(< q)$ -local. □

Corollary 3.3.32. — *Let Z be a fibrant symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$. We have that Z is $L^\Sigma(< q)$ -local in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ if and only if $R_\Sigma(Q_\Sigma Z \wedge S^1)$ is $L^\Sigma(< q)$ -local in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$.*

Proof. — (\Rightarrow): Assume that Z is $L^\Sigma(< q)$ -local. Since $R_\Sigma(Q_\Sigma Z \wedge S^1)$ is fibrant, using proposition 3.3.30 we have that it is enough to check that for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q, \Sigma}$, $[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma \cong 0$. But since $-\wedge S^1$ is a Quillen equivalence, we get the following diagram:

$$\begin{array}{ccc}
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma & & \\
 \searrow \cong & & \\
 & [F_{n+1}^\Sigma(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma & \\
 & \uparrow \cong & \\
 [F_{n+1}^\Sigma(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Z]_{\text{Spt}}^\Sigma & & \\
 \searrow \cong & & \\
 & [F_{n+1}^\Sigma(S^{r+1} \wedge \mathbb{G}_m^{s+1} \wedge U_+), Q_\Sigma Z \wedge S^1]_{\text{Spt}}^\Sigma & \\
 \swarrow \Sigma_T^{1,0} & &
 \end{array}$$

where all the maps are isomorphisms of abelian groups. Since Z is $L^\Sigma(< q)$ -local, Proposition 3.3.30 implies that $[F_{n+1}^\Sigma(S^r \wedge \mathbb{G}_m^{s+1} \wedge U_+), Z]_{\text{Spt}}^\Sigma \cong 0$. Therefore

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma \cong 0$$

for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q, \Sigma}$, as we wanted.

(\Leftarrow): Assume that $R_\Sigma(Q_\Sigma Z \wedge S^1)$ is $L^\Sigma(< q)$ -local. By hypothesis, Z is fibrant; therefore Proposition 3.3.30 implies that it is enough to show that for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q, \Sigma}$, $[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}^\Sigma \cong 0$. Since $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category and $-\wedge S^1$ is a Quillen equivalence; we have the following diagram:

$$\begin{array}{ccc}
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma & & \\
 \downarrow \cong & & \\
 [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \wedge S^1, Q_\Sigma Z \wedge S^1]_{\text{Spt}}^\Sigma & \xleftarrow[\cong]{\Sigma_T^{1,0}} & [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}^\Sigma
 \end{array}$$

where all the maps are isomorphisms of abelian groups. On the other hand, using Corollary 3.3.31 we have that $\Omega_{S^1} R_\Sigma(Q_\Sigma Z \wedge S^1)$ is $L^\Sigma(< q)$ -local. Therefore using Proposition 3.3.30 again, we have that for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q, \Sigma}$:

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}^\Sigma \cong [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \Omega_{S^1} R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma \cong 0$$

and this finishes the proof. □

Corollary 3.3.33. — *Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We have that f is a $L^\Sigma(< q)$ -local equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if for every $L^\Sigma(< q)$ -local*

symmetric T -spectrum Z , f induces the following isomorphism of abelian groups:

$$[Y, Z]_{\text{Spt}}^{\Sigma} \xrightarrow{f^*} [X, Z]_{\text{Spt}}^{\Sigma}$$

Proof. — Suppose that f is a $L^{\Sigma}(< q)$ -local equivalence, then by definition the induced map:

$$\text{Map}_{\Sigma}(Q_{\Sigma}Y, Z) \xrightarrow{(Q_{\Sigma}f)^*} \text{Map}_{\Sigma}(Q_{\Sigma}X, Z)$$

is a weak equivalence of simplicial sets for every $L^{\Sigma}(< q)$ -local symmetric T -spectrum Z . Proposition 3.3.30(1) implies that Z is fibrant in $\text{Spt}_T^{\Sigma}\mathcal{M}_{*}$, and since $\text{Spt}_T^{\Sigma}\mathcal{M}_{*}$ is in particular a simplicial model category; we get the following commutative diagram, where the top row and all the vertical maps are isomorphisms of abelian groups:

$$\begin{array}{ccc} \pi_0 \text{Map}_{\Sigma}(Q_{\Sigma}Y, Z) & \xrightarrow[\cong]{(Q_{\Sigma}f)^*} & \pi_0 \text{Map}_{\Sigma}(Q_{\Sigma}X, Z) \\ \cong \downarrow & & \downarrow \cong \\ [Y, Z]_{\text{Spt}}^{\Sigma} & \xrightarrow{f^*} & [X, Z]_{\text{Spt}}^{\Sigma} \end{array}$$

hence f^* is an isomorphism for every $L^{\Sigma}(< q)$ -local symmetric T -spectrum Z , as we wanted.

Conversely, assume that for every $L^{\Sigma}(< q)$ -local symmetric T -spectrum Z , the induced map

$$[Y, Z]_{\text{Spt}}^{\Sigma} \xrightarrow{f^*} [X, Z]_{\text{Spt}}^{\Sigma}$$

is an isomorphism of abelian groups.

Since $L_{< q}\text{Spt}_T^{\Sigma}\mathcal{M}_{*}$ is the left Bousfield localization of $\text{Spt}_T^{\Sigma}\mathcal{M}_{*}$ with respect to the $L^{\Sigma}(< q)$ -local equivalences, we have that the identity functor $\text{id} : \text{Spt}_T^{\Sigma}\mathcal{M}_{*} \rightarrow L_{< q}\text{Spt}_T^{\Sigma}\mathcal{M}_{*}$ is a left Quillen functor. Therefore for every symmetric T -spectrum Z , we get the following commutative diagram where all the vertical arrows are isomorphisms:

$$\begin{array}{ccc} \text{Hom}_{L_{< q}\mathcal{H}^{\Sigma}(S)}(Q_{\Sigma}Y, Z) & \xrightarrow{(Q_{\Sigma}f)^*} & \text{Hom}_{L_{< q}\mathcal{H}^{\Sigma}(S)}(Q_{\Sigma}X, Z) \\ \cong \downarrow & & \downarrow \cong \\ [Y, W_q^{\Sigma}Z]_{\text{Spt}}^{\Sigma} & \xrightarrow[\cong]{f^*} & [X, W_q^{\Sigma}Z]_{\text{Spt}}^{\Sigma} \end{array}$$

but $W_q^{\Sigma}Z$ is by construction $L^{\Sigma}(< q)$ -local, then by hypothesis the bottom row is an isomorphism of abelian groups. Hence it follows that the induced map:

$$\text{Hom}_{L_{< q}\mathcal{H}^{\Sigma}(S)}(Q_{\Sigma}Y, Z) \xrightarrow[\cong]{(Q_{\Sigma}f)^*} \text{Hom}_{L_{< q}\mathcal{H}^{\Sigma}(S)}(Q_{\Sigma}X, Z)$$

is an isomorphism for every symmetric T -spectrum Z . This implies that $Q_{\Sigma}f$ is a weak equivalence in $L_{< q}\text{Spt}_T^{\Sigma}\mathcal{M}_{*}$, and since Q_{Σ} is also a cofibrant replacement functor in

$L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$, it follows that f is a weak equivalence in $L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore we have that f is a $L^\Sigma(<q)$ -local equivalence, as we wanted. \square

Lemma 3.3.34. — *Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We have that f is a $L^\Sigma(<q)$ -local equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if*

$$Q_\Sigma f \wedge \text{id} : Q_\Sigma X \wedge S^1 \rightarrow Q_\Sigma Y \wedge S^1$$

is a $L^\Sigma(<q)$ -local equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — Assume that f is a $L^\Sigma(<q)$ -local equivalence, and let Z be an arbitrary $L^\Sigma(<q)$ -local symmetric T -spectrum. Then Corollary 3.3.31 implies that $\Omega_{S^1} Z$ is also $L^\Sigma(<q)$ -local. Therefore the induced map

$$\text{Map}_\Sigma(Q_\Sigma Y, \Omega_{S^1} Z) \xrightarrow{(Q_\Sigma f)^*} \text{Map}_\Sigma(Q_\Sigma X, \Omega_{S^1} Z)$$

is a weak equivalence of simplicial sets. Now since $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category, we have the following commutative diagram:

$$\begin{array}{ccc} \text{Map}_\Sigma(Q_\Sigma Y, \Omega_{S^1} Z) & \xrightarrow{(Q_\Sigma f)^*} & \text{Map}_\Sigma(Q_\Sigma X, \Omega_{S^1} Z) \\ \cong \downarrow & & \downarrow \cong \\ \text{Map}_\Sigma(Q_\Sigma Y \wedge S^1, Z) & \xrightarrow{(Q_\Sigma f \wedge \text{id})^*} & \text{Map}_\Sigma(Q_\Sigma X \wedge S^1, Z) \end{array}$$

and using the two out of three property for weak equivalences of simplicial sets, we have that

$$\text{Map}_\Sigma(Q_\Sigma Y \wedge S^1, Z) \xrightarrow{(Q_\Sigma f \wedge \text{id})^*} \text{Map}_\Sigma(Q_\Sigma X \wedge S^1, Z)$$

is a weak equivalence. Since this holds for every $L^\Sigma(<q)$ -local symmetric T -spectrum Z , it follows that

$$Q_\Sigma f \wedge \text{id} : Q_\Sigma X \wedge S^1 \rightarrow Q_\Sigma Y \wedge S^1$$

is a $L^\Sigma(<q)$ -local equivalence, as we wanted.

Conversely, suppose that

$$Q_\Sigma f \wedge \text{id} : Q_\Sigma X \wedge S^1 \rightarrow Q_\Sigma Y \wedge S^1$$

is a $L^\Sigma(<q)$ -local equivalence. Let Z be an arbitrary $L^\Sigma(<q)$ -local symmetric T -spectrum. Since $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category and $-\wedge S^1$ is a Quillen equivalence,

we get the following commutative diagram:

$$\begin{array}{ccc}
 [Q_\Sigma Y \wedge S^1, R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma & \xrightarrow{(Q_\Sigma f \wedge \text{id})^*} & [Q_\Sigma X \wedge S^1, R_\Sigma(Q_\Sigma Z \wedge S^1)]_{\text{Spt}}^\Sigma \\
 \cong \downarrow & & \downarrow \cong \\
 [Q_\Sigma Y \wedge S^1, Q_\Sigma Z \wedge S^1]_{\text{Spt}}^\Sigma & \xrightarrow{(Q_\Sigma f \wedge \text{id})^*} & [Q_\Sigma X \wedge S^1, Q_\Sigma Z \wedge S^1]_{\text{Spt}}^\Sigma \\
 \cong \uparrow \Sigma_T^{1,0} & & \Sigma_T^{1,0} \uparrow \cong \\
 [Y, Z]_{\text{Spt}}^\Sigma & \xrightarrow{f^*} & [X, Z]_{\text{Spt}}^\Sigma
 \end{array}$$

Now, Corollary 3.3.32 implies that $R_\Sigma(Q_\Sigma Z \wedge S^1)$ is also $L^\Sigma(< q)$ -local. Therefore using Corollary 3.3.33 we have that the top row in the diagram above is an isomorphism of abelian groups. This implies that the induced map:

$$[Y, Z]_{\text{Spt}}^\Sigma \xrightarrow{f^*} [X, Z]_{\text{Spt}}^\Sigma$$

is an isomorphism of abelian groups for every $L^\Sigma(< q)$ -local symmetric spectrum Z . Finally using Corollary 3.3.33 again, we have that $f : X \rightarrow Y$ is a $L^\Sigma(< q)$ -local equivalence, as we wanted. \square

Corollary 3.3.35. — *The following adjunction:*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Using Corollary 1.3.16 in [10] and Proposition 3.3.28 we have that it suffices to verify the following two conditions:

1. For every fibrant object X in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$, the following composition

$$(Q_\Sigma \Omega_{S^1} X) \wedge S^1 \xrightarrow{Q_\Sigma^{\Omega_{S^1} X} \wedge \text{id}} (\Omega_{S^1} X) \wedge S^1 \xrightarrow{\epsilon_X} X$$

is a $L^\Sigma(< q)$ -local equivalence.

2. $- \wedge S^1$ reflects $L^\Sigma(< q)$ -local equivalences between cofibrant objects in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): By construction $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore the identity functor

$$\text{id} : L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a right Quillen functor. Thus X is also fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Since the adjunction $(- \wedge S^1, \Omega_{S^1}, \varphi)$ is a Quillen equivalence on $\text{Spt}_T^\Sigma \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies

that the following composition is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$:

$$(Q_\Sigma \Omega_{S^1} X) \wedge S^1 \xrightarrow{Q_\Sigma^{\Omega_{S^1} X} \wedge \text{id}} (\Omega_{S^1} X) \wedge S^1 \xrightarrow{\epsilon_X} X$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $L^\Sigma(< q)$ -local equivalence.

(2): This follows immediately from Proposition 3.3.28 and Lemma 3.3.34. □

Remark 3.3.36. — We have a situation similar to the one described in remark 3.3.16 for the model categories $R_{C^q_{\text{eff}}} \text{Spt}_T^\Sigma \mathcal{M}_*$; i.e. although the adjunction $(\Sigma_T, \Omega_T, \varphi)$ is a Quillen equivalence on $\text{Spt}_T^\Sigma \mathcal{M}_*$, it does not descend even to a Quillen adjunction on the $\text{weight}^{< q}$ motivic symmetric stable model category $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Corollary 3.3.37. — The homotopy category $L_{< q} \mathcal{H}^\Sigma(S)$ associated to $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$ has the structure of a triangulated category.

Proof. — Theorem 3.3.26 implies in particular that $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a pointed simplicial model category, and Corollary 3.3.35 implies that the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence. Therefore the result follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII]. □

Corollary 3.3.38. — $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right proper model category.

Proof. — We need to show that the $L^\Sigma(< q)$ -local equivalences are stable under pull-back along fibrations in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Consider the following pullback diagram:

$$\begin{array}{ccc} Z & \xrightarrow{w^*} & X \\ p^* \downarrow & & \downarrow p \\ W & \xrightarrow{w} & Y \end{array}$$

where p is a fibration in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$, and w is a $L^\Sigma(< q)$ -local equivalence. Let F be the homotopy fibre of p . Then we get the following commutative diagram in $L_{< q} \mathcal{H}^\Sigma(S)$:

$$\begin{array}{ccccccc} \Omega_{S^1} Y & \xrightarrow{q} & F & \xrightarrow{i} & X & \xrightarrow{p} & Y \\ \Omega_{S^1} w \uparrow & & \parallel & & \uparrow w^* & & \uparrow w \\ \Omega_{S^1} W & \xrightarrow{r} & F & \xrightarrow{j} & Z & \xrightarrow{p^*} & W \end{array}$$

Since the rows in the diagram above are both fibre sequences in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$, it follows that both rows are distinguished triangles in $L_{< q} \mathcal{H}^\Sigma(S)$ (which has the structure of a triangulated category given by Corollary 3.3.37). Now w, id_F are both isomorphisms in

$L_{<q}\mathcal{H}^\Sigma(S)$, hence it follows that w^* is also an isomorphism in $L_{<q}\mathcal{H}^\Sigma(S)$. Therefore w^* is a $L^\Sigma(<q)$ -local equivalence, as we wanted. \square

Proposition 3.3.39. — *We have the following adjunction*

$$(Q_\Sigma, W_q^\Sigma, \varphi) : \mathcal{H}^\Sigma(S) \longrightarrow L_{<q}\mathcal{H}^\Sigma(S)$$

of exact functors between triangulated categories.

Proof. — Since $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$ is the left Bousfield localization of $\text{Spt}_T^\Sigma\mathcal{M}_*$ with respect to the $L^\Sigma(<q)$ -local equivalences, we have that the identity functor $\text{id} : \text{Spt}_T^\Sigma\mathcal{M}_* \rightarrow L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(Q_\Sigma, W_q^\Sigma, \varphi) : \mathcal{H}^\Sigma(S) \longrightarrow L_{<q}\mathcal{H}^\Sigma(S)$$

Now Proposition 6.4.1 in [10] implies that Q_Σ maps cofibre sequences in $\mathcal{H}^\Sigma(S)$ to cofibre sequences in $L_{<q}\mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that Q_Σ and W_q^Σ are both exact functors between triangulated categories. \square

Lemma 3.3.40. — *Let X be a $L(<q)$ -local spectrum in $\text{Spt}_T\mathcal{M}_*$. We have that Q_sX and $UR_\Sigma VQ_sX$ are also $L(<q)$ -local in $\text{Spt}_T\mathcal{M}_*$.*

Proof. — Since X is $L(<q)$ -local, it follows that X is fibrant in $\text{Spt}_T\mathcal{M}_*$. By definition we have that the natural map

$$Q_sX \xrightarrow{Q_s^X} X$$

is a trivial fibration in $\text{Spt}_T\mathcal{M}_*$, therefore Q_sX is also fibrant in $\text{Spt}_T\mathcal{M}_*$. Hence [6, Lemma 3.2.1(a)] implies that Q_sX is $L(<q)$ -local.

Since the adjunction (V, U, φ) is a Quillen equivalence between $\text{Spt}_T\mathcal{M}_*$ and $\text{Spt}_T^\Sigma\mathcal{M}_*$, we have that $UR_\Sigma VQ_sX$ is fibrant in $\text{Spt}_T\mathcal{M}_*$, and [10, Proposition 1.3.13(b)] implies that the composition

$$Q_sX \xrightarrow{\eta_{Q_sX}} UV(Q_sX) \xrightarrow{U(R_\Sigma^{VQ_sX})} UR_\Sigma VQ_sX$$

is a weak equivalence in $\text{Spt}_T\mathcal{M}_*$. Since we already know that Q_sX is $L(<q)$ -local, using [6, Lemma 3.2.1(a)] again we get that $UR_\Sigma VQ_sX$ is also $L(<q)$ -local in $\text{Spt}_T\mathcal{M}_*$. This finishes the proof. \square

Proposition 3.3.41. — *Let $f : X \rightarrow Y$ be a map in $\text{Spt}_T\mathcal{M}_*$. We have that f is a $L(<q)$ -local equivalence in $\text{Spt}_T\mathcal{M}_*$ if and only if $VQ_s f$ is a $L^\Sigma(<q)$ -local equivalence in $\text{Spt}_T^\Sigma\mathcal{M}_*$.*

Proof. — (\Rightarrow): Assume that f is a $L(< q)$ -local equivalence, and let Z be an arbitrary $L^\Sigma(< q)$ -local symmetric T -spectrum. Then Z is fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and using theorem 2.6.30 we get the following commutative diagram where all the vertical arrows are isomorphisms:

$$\begin{array}{ccc} [VQ_s Y, Z]_{\text{Spt}}^\Sigma & \xrightarrow{(VQ_s f)^*} & [VQ_s X, Z]_{\text{Spt}}^\Sigma \\ \cong \downarrow & & \downarrow \cong \\ [Y, UZ]_{\text{Spt}} & \xrightarrow{f^*} & [X, UZ]_{\text{Spt}} \end{array}$$

By Proposition 3.3.29 we have that UZ is $L(< q)$ -local in $\text{Spt}_T \mathcal{M}_*$, hence corollary 3.2.36 implies that the bottom row in the diagram above is always an isomorphism. Therefore the top row in the diagram above is an isomorphism for every $L^\Sigma(< q)$ -local symmetric T -spectrum Z , then by Corollary 3.3.33 it follows that $VQ_s f$ is a $L^\Sigma(< q)$ -local equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

(\Leftarrow): Assume that $VQ_s f$ is a $L^\Sigma(< q)$ -local equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and let Z be an arbitrary $L(< q)$ -local T -spectrum in $\text{Spt}_T \mathcal{M}_*$. We need to show that the induced map:

$$\text{Map}(Q_s Y, Z) \xrightarrow{(Q_s f)^*} \text{Map}(Q_s X, Z)$$

is a weak equivalence of simplicial sets.

But Theorem 2.6.30 implies that the adjunction (V, U, φ) is a Quillen equivalence between $\text{Spt}_T \mathcal{M}_*$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore using [10, Proposition 1.3.13(b)] we have that all the maps in the following diagram are weak equivalences in $\text{Spt}_T \mathcal{M}_*$:

$$Z \xleftarrow{Q_s^Z} Q_s Z \xrightarrow{U(R_\Sigma^{VQ_s Z}) \circ \eta_{Q_s Z}} UR_\Sigma VQ_s Z$$

Lemma 3.3.40 implies in particular that $Z, Q_s Z, UR_\Sigma VQ_s Z$ are all fibrant in $\text{Spt}_T \mathcal{M}_*$. Now using the fact that $\text{Spt}_T \mathcal{M}_*$ is a simplicial model category together with Ken Brown’s lemma (see Lemma 1.1.5) and the two out of three property for weak equivalences, we have that it suffices to prove that the induced map:

$$\text{Map}(Q_s Y, UR_\Sigma VQ_s Z) \xrightarrow{(Q_s f)^*} \text{Map}(Q_s X, UR_\Sigma VQ_s Z)$$

is a weak equivalence of simplicial sets. Using the enriched adjunctions of Proposition 2.6.19, we get the following commutative diagram where all the vertical arrows

are isomorphisms:

$$\begin{array}{ccc} \mathrm{Map}(Q_s Y, UR_\Sigma VQ_s Z) & \xrightarrow{(Q_s f)^*} & \mathrm{Map}(Q_s X, UR_\Sigma VQ_s Z) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{Map}_\Sigma(VQ_s Y, R_\Sigma VQ_s Z) & \xrightarrow{(VQ_s f)^*} & \mathrm{Map}_\Sigma(VQ_s X, R_\Sigma VQ_s Z) \end{array}$$

Finally, Lemma 3.3.40 implies that $UR_\Sigma VQ_s Z$ is $L(< q)$ -local in $\mathrm{Spt}_T \mathcal{M}_*$, therefore by Proposition 3.3.29 we have that $R_\Sigma VQ_s Z$ is $L^\Sigma(< q)$ -local in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Since $VQ_s f$ is a $L^\Sigma(< q)$ -local equivalence and $VQ_s X, VQ_s Y$ are both cofibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, it follows that the bottom row in the diagram above is a weak equivalence of simplicial sets. This implies that the top row is also a weak equivalence of simplicial sets, as we wanted. \square

Theorem 3.3.42. — *The adjunction*

$$(V, U, \varphi) : L_{< q} \mathrm{Spt}_T \mathcal{M}_* \longrightarrow L_{< q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and the forgetful functor is a Quillen equivalence.

Proof. — Proposition 3.3.41 together with the universal property for left Bousfield localizations (see definition 1.8.1) imply that

$$V : L_{< q} \mathrm{Spt}_T \mathcal{M}_* \longrightarrow L_{< q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Using Corollary 1.3.16 in [10] and Proposition 3.2.31 we have that it suffices to verify the following two conditions:

1. For every fibrant object X in $L_{< q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$, the following composition

$$VQ_s U(X) \xrightarrow{V(Q_s^{U^X})} VU(X) \xrightarrow{\epsilon_X} X$$

is a weak equivalence in $L_{< q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$.

2. V reflects weak equivalences between cofibrant objects in $L_{< q} \mathrm{Spt}_T \mathcal{M}_*$.

(1): By construction $L_{< q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization of $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, therefore the identity functor

$$\mathrm{id} : L_{< q} \mathrm{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \mathrm{Spt}_T^\Sigma \mathcal{M}_*$$

is a right Quillen functor. Thus X is also fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Since the adjunction (V, U, φ) is a Quillen equivalence between $\mathrm{Spt}_T \mathcal{M}_*$ and $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies that the following composition is a weak equivalence in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$:

$$VQ_s U(X) \xrightarrow{V(Q_s^{U^X})} VU(X) \xrightarrow{\epsilon_X} X$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $L^\Sigma(< q)$ -local equivalence.

(2): This follows immediately from Propositions 3.2.31 and 3.3.41. □

Corollary 3.3.43. — *The adjunction*

$$(V, U, \varphi) : L_{< q} \text{Spt}_T \mathcal{M}_* \longrightarrow L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

given by the symmetrization and the forgetful functors, induces an adjunction

$$(VQ_s, UW_q^\Sigma, \varphi) : L_{< q} \mathcal{H}(S) \longrightarrow L_{< q} \mathcal{H}^\Sigma(S)$$

of exact functors between triangulated categories. Furthermore, VQ_s and UW_q^Σ are both equivalences of categories.

Proof. — Theorem 3.3.42 implies that the adjunction (V, U, φ) is a Quillen equivalence. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(VQ_s, UW_q^\Sigma, \varphi) : L_{< q} \mathcal{H}(S) \longrightarrow L_{< q} \mathcal{H}^\Sigma(S)$$

Now [10, Proposition 1.3.13] implies that VQ_s, UW_q^Σ are both equivalences of categories. Finally, Proposition 2.6.19 together with [10, Proposition 6.4.1] imply that VQ_s maps cofibre sequences in $L_{< q} \mathcal{H}(S)$ to cofibre sequences in $L_{< q} \mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that VQ_s and UW_q^Σ are both exact functors between triangulated categories. □

Now it is very easy to find the desired lifting for the functor $\tilde{s}_{< q} : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$ (see Corollary 3.3.5(2)) to the model category level.

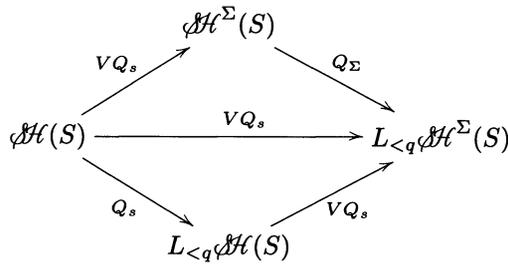
Lemma 3.3.44. — *Let X be an arbitrary symmetric T -spectrum.*

1. *The following maps in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$*

$$Q_\Sigma(VQ_s X) \xrightarrow{Q_\Sigma^{VQ_s X}} VQ_s X \xleftarrow{V(Q_s^{Q_s X})} VQ_s(Q_s X)$$

induce natural isomorphisms between the functors:

$$Q_\Sigma \circ VQ_s, VQ_s, VQ_s \circ Q_s : \mathcal{H}(S) \rightarrow L_{< q} \mathcal{H}^\Sigma(S)$$



Given a T -spectrum X

$$\kappa_X : Q_\Sigma(VQ_s X) \xrightarrow{\cong} VQ_s(Q_s X)$$

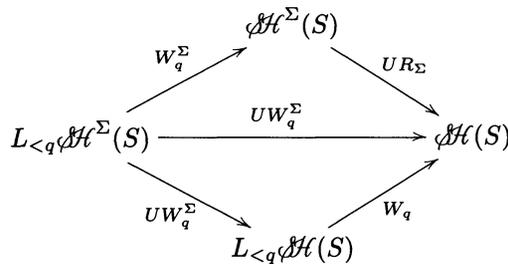
will denote the isomorphism in $L_{<q}\mathcal{H}^\Sigma(S)$ corresponding to the natural isomorphism between $Q_\Sigma \circ VQ_s$ and $VQ_s \circ Q_s$.

2. The following maps in $\text{Spt}_T \mathcal{M}_*$

$$UR_\Sigma(W_q^\Sigma X) \xleftarrow{U(R_\Sigma^{W_q^\Sigma X})} UW_q^\Sigma X \xrightarrow{W_q^{UW_q^\Sigma X}} W_q(UW_q^\Sigma X)$$

induce natural isomorphisms between the functors:

$$UR_\Sigma \circ W_q^\Sigma, UW_q^\Sigma, W_q \circ UW_q^\Sigma : L_{<q}\mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}(S)$$



Given a symmetric T -spectrum X

$$\mu_X : UR_\Sigma(W_q^\Sigma X) \xrightarrow{\cong} W_q(UW_q^\Sigma X)$$

will denote the isomorphism in $\mathcal{H}(S)$ corresponding to the natural isomorphism between $UR_\Sigma \circ W_q^\Sigma$ and $W_q \circ UW_q^\Sigma$.

Proof. — (1): It follows immediately from Theorem 1.3.7 in [10] and the following commutative diagram of left Quillen functors:

$$\begin{array}{ccc}
 \mathrm{Spt}_T \mathcal{M}_* & \xrightarrow{V} & \mathrm{Spt}_T^\Sigma \mathcal{M}_* \\
 \mathrm{id} \downarrow & & \downarrow \mathrm{id} \\
 L_{<q} \mathrm{Spt}_T \mathcal{M}_* & \xrightarrow{V} & L_{<q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*
 \end{array}$$

(2): It follows immediately from the dual of Theorem 1.3.7 in [10] and the following commutative diagram of right Quillen functors:

$$\begin{array}{ccc}
 \mathrm{Spt}_T \mathcal{M}_* & \xleftarrow{U} & \mathrm{Spt}_T^\Sigma \mathcal{M}_* & \square \\
 \mathrm{id} \uparrow & & \uparrow \mathrm{id} & \\
 L_{<q} \mathrm{Spt}_T \mathcal{M}_* & \xleftarrow{U} & L_{<q} \mathrm{Spt}_T^\Sigma \mathcal{M}_* &
 \end{array}$$

Theorem 3.3.45. — *Let X be an arbitrary symmetric T -spectrum.*

1. *The diagram (41) in Theorem 3.2.52 induces the following diagram in $\mathcal{H}^\Sigma(S)$:*

$$(58) \quad \begin{array}{ccc}
 VQ_s(Q_{s <q}(UR_\Sigma X)) & \xrightarrow[\cong]{VQ_s(W_q^{Q_{s <q}(UR_\Sigma X)})} & VQ_s(W_q Q_{s <q}(UR_\Sigma X)) \\
 \downarrow \cong & & \uparrow \cong \\
 VQ_s(Q_{s <q}^{UR_\Sigma X}) & & VQ_s(W_q Q_s(\pi_{<q}^{UR_\Sigma X})) \\
 \downarrow & & \uparrow \\
 \tilde{s}_{<q} X = VQ_s(s_{<q}(UR_\Sigma X)) & & VQ_s(W_q Q_s(UR_\Sigma X))
 \end{array}$$

where all the maps are isomorphisms in $\mathcal{H}^\Sigma(S)$. Furthermore, this diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow{\tilde{s}_{<q}} \\ \xrightarrow{VQ_s \circ W_q Q_s \circ UR_\Sigma} \end{array} \mathcal{H}^\Sigma(S)$$

2. *Let η be the unit of the adjunction (see Corollary 3.3.43):*

$$(VQ_s, UW_q^\Sigma, \varphi) : L_{<q} \mathcal{H}(S) \longrightarrow L_{<q} \mathcal{H}^\Sigma(S)$$

Then we have the following diagram in $\mathcal{A}\mathcal{H}(S)$ (see Lemma 3.3.44):

$$(59) \quad \begin{array}{ccc} W_q(UW_q^\Sigma(Q_\Sigma(VQ_sX))) & \xrightarrow[\cong]{W_qUW_q^\Sigma(\kappa_X)} & W_q(UW_q^\Sigma(VQ_s(Q_sX))) \\ \mu_{Q_\Sigma(VQ_sX)} \uparrow \cong & & \cong \downarrow W_q(\eta_{Q_sX})^{-1} \\ UR_\Sigma(W_q^\Sigma Q_\Sigma(VQ_sX)) & & W_qQ_sX \end{array}$$

where all the maps are isomorphisms in $\mathcal{A}\mathcal{H}^\Sigma(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{A}\mathcal{H}(S) \begin{array}{c} \xrightarrow{UR_\Sigma \circ W_q^\Sigma Q_\Sigma \circ VQ_s} \\ \xrightarrow{W_qQ_s} \end{array} \mathcal{A}\mathcal{H}(S)$$

3. Let ϵ denote the counit of the adjunction (see Theorem 3.3.4):

$$(VQ_s, UR_\Sigma, \varphi) : \mathcal{A}\mathcal{H}(S) \longrightarrow \mathcal{A}\mathcal{H}^\Sigma(S)$$

and let γ denote the natural isomorphism constructed above in (2). Then we have the following diagram in $\mathcal{A}\mathcal{H}^\Sigma(S)$:

$$(60) \quad \begin{array}{ccc} VQ_s(UR_\Sigma W_q^\Sigma Q_\Sigma VQ_s(UR_\Sigma X)) & \xrightarrow[\cong]{\epsilon_{W_q^\Sigma Q_\Sigma VQ_s(UR_\Sigma X)}} & W_q^\Sigma Q_\Sigma VQ_s(UR_\Sigma X) \\ VQ_s(\gamma_{UR_\Sigma X})^{-1} \uparrow \cong & & \downarrow W_q^\Sigma Q_\Sigma(\epsilon_X) \\ VQ_s(W_qQ_s(UR_\Sigma X)) & & W_q^\Sigma Q_\Sigma X = s_{<q}^\Sigma X \end{array}$$

where all the maps are isomorphisms in $\mathcal{A}\mathcal{H}^\Sigma(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{A}\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow{VQ_s \circ W_qQ_s \circ UR_\Sigma} \\ \xrightarrow{W_q^\Sigma Q_\Sigma = s_{<q}^\Sigma} \end{array} \mathcal{A}\mathcal{H}^\Sigma(S)$$

4. Combining the diagrams (58) and (60) above we get a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow{\tilde{s}_{<q}} \\ \xrightarrow{s_{<q}^\Sigma} \end{array} \mathcal{H}^\Sigma(S)$$

Proof. — It is clear that it suffices to prove only the first three claims.

- (1): It follows immediately from theorems 3.2.52 and 3.3.4.
- (2): It follows immediately from Lemma 3.3.44 and Corollary 3.3.43.
- (3): It follows immediately from (2) above, and Theorem 3.3.4. □

The functor $s_{<q}^\Sigma$ gives the desired lifting for the functor $\tilde{s}_{<q}$ to the model category level.

Proposition 3.3.46. — *We have the following commutative diagram of left Quillen functors:*

$$\begin{array}{ccc} & \text{Spt}_T^\Sigma \mathcal{M}_* & \\ \text{id} \swarrow & & \searrow \text{id} \\ L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_* \end{array}$$

Proof. — Since $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ are both left Bousfield localizations for $\text{Spt}_T^\Sigma \mathcal{M}_*$, we have that the identity functors:

$$\begin{aligned} \text{id} : \text{Spt}_T^\Sigma \mathcal{M}_* &\longrightarrow L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_* \\ \text{id} : \text{Spt}_T^\Sigma \mathcal{M}_* &\longrightarrow L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_* \end{aligned}$$

are both left Quillen functors. Hence, it suffices to show that

$$\text{id} : L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Using the universal property for left Bousfield localizations (see definition 1.8.1), we have that it is enough to check that if $f : X \rightarrow Y$ is a $L^\Sigma(< q+1)$ -local equivalence then $Q_\Sigma f : Q_\Sigma X \rightarrow Q_\Sigma Y$ is a $L^\Sigma(< q)$ -local equivalence.

But Theorem 3.1.6(c) in [6] implies that this last condition is equivalent to the following one: Let Z be an arbitrary $L^\Sigma(< q)$ -local symmetric T -spectrum, then Z is also $L^\Sigma(< q+1)$ -local. Finally, this last condition follows immediately from proposition 3.3.29 and corollary 3.2.33. □

Corollary 3.3.47. — *We have the following adjunction*

$$(Q_\Sigma, W_q^\Sigma, \varphi) : L_{<q+1} \mathcal{H}^\Sigma(S) \longrightarrow L_{<q} \mathcal{H}^\Sigma(S)$$

of exact functors between triangulated categories.

Proof. — Proposition 3.3.46 implies that $\text{id} : L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories

$$(Q_\Sigma, W_q^\Sigma, \varphi) : L_{<q+1}\mathcal{H}^\Sigma(S) \longrightarrow L_{<q}\mathcal{H}^\Sigma(S)$$

Now Proposition 6.4.1 in [10] implies that Q_Σ maps cofibre sequences in $L_{<q+1}\mathcal{H}^\Sigma(S)$ to cofibre sequences in $L_{<q}\mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that Q_Σ and W_q^Σ are both exact functors between triangulated categories. \square

Theorem 3.3.48. — *We have the following tower of left Quillen functors:*

$$(61) \quad \begin{array}{c} \vdots \\ \downarrow \text{id} \\ L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_* \\ \begin{array}{ccc} \text{id} \nearrow & & \downarrow \text{id} \\ \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_* \\ \text{id} \searrow & & \downarrow \text{id} \\ & & L_{<q-1}\text{Spt}_T^\Sigma \mathcal{M}_* \\ & & \downarrow \text{id} \\ & & \vdots \end{array} \end{array}$$

together with the corresponding tower of associated homotopy categories:

$$(62) \quad \begin{array}{c} \vdots \\ \begin{array}{ccc} Q_\Sigma \downarrow & \uparrow W_{q+1}^\Sigma & \\ & L_{<q+1}\mathcal{H}^\Sigma(S) & \\ Q_\Sigma \downarrow & \uparrow W_q^\Sigma & \\ \mathcal{H}^\Sigma(S) & \begin{array}{ccc} \xrightarrow{Q_\Sigma} & & \xrightarrow{Q_\Sigma} \\ \xleftarrow{W_{q+1}^\Sigma} & & \xleftarrow{W_q^\Sigma} \\ \xrightarrow{Q_\Sigma} & & \xrightarrow{Q_\Sigma} \\ \xleftarrow{W_q^\Sigma} & & \xleftarrow{W_{q-1}^\Sigma} \end{array} & L_{<q}\mathcal{H}^\Sigma(S) \\ Q_\Sigma \downarrow & \uparrow W_{q-1}^\Sigma & \\ & L_{<q-1}\mathcal{H}^\Sigma(S) & \\ Q_\Sigma \downarrow & \uparrow W_{q-2}^\Sigma & \\ & \vdots & \end{array} \end{array}$$

The tower (62) gets canonically identified, through the equivalences of categories VQ_s , UR_Σ and UW_q^Σ constructed in Theorem 3.3.4 and Corollary 3.3.43; with the tower (44) defined in theorem 3.2.56. Moreover, this tower also satisfies the following properties:

1. All the categories are triangulated.
2. All the functors are exact.
3. Q_Σ is a left adjoint for all the functors W_q^Σ .

Proof. — It follows immediately from propositions 3.3.39, 3.3.46, Corollary 3.3.47 together with Theorem 3.3.4 and Corollary 3.3.43. □

Definition 3.3.49. — We consider the following set of symmetric T -spectra

$$S^\Sigma(q) = \{F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^\Sigma | s - n = q\} \subseteq C_{\text{eff}}^{q, \Sigma}$$

(see Proposition 3.1.5 and definition 3.1.8).

Theorem 3.3.50. — The right Bousfield localization of the model category $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $S^\Sigma(q)$ -colocal equivalences exists. This new model structure will be called q -slice motivic symmetric stable. $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ will denote the category of symmetric T -spectra equipped with the q -slice motivic symmetric stable model structure, and $S^q \mathcal{A}^\Sigma(S)$ will denote its associated homotopy category. Furthermore, the q -slice motivic symmetric stable model structure is right proper and simplicial.

Proof. — Theorem 3.3.26 implies that $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a cellular and simplicial model category. On the other hand, corollary 3.3.38 implies that $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ is right proper. Therefore we can apply Theorem 5.1.1 in [6] to construct the right Bousfield localization of $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ with respect to the $S^\Sigma(q)$ -colocal equivalences. Using [6, Theorem 5.1.1] again, we have that $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right proper and simplicial model category. □

Definition 3.3.51. — Let P_q^Σ denote a cofibrant replacement functor in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$; such that for every symmetric T -spectrum X , the natural map

$$P_q^\Sigma X \xrightarrow{P_q^{\Sigma, X}} X$$

is a trivial fibration in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$, and $P_q^\Sigma X$ is always a $S^\Sigma(q)$ -colocal symmetric T -spectrum in $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Proposition 3.3.52. — W_{q+1}^Σ is also a fibrant replacement functor in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.27), and for every symmetric T -spectrum X the natural map

$$X \xrightarrow{W_{q+1}^{\Sigma, X}} W_{q+1}^\Sigma X$$

is a trivial cofibration in $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$.

Proof. — Since $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$ is the right Bousfield localization of $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ with respect to the $S^\Sigma(q)$ -colocal equivalences, by construction we have that the fibrations and the trivial cofibrations are identical in $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$ and $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ respectively. This implies that for every symmetric T -spectrum X , $W_{q+1}^\Sigma X$ is fibrant in $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$, and we also have that the natural map

$$X \xrightarrow{W_{q+1}^{\Sigma, X}} W_{q+1}^\Sigma X$$

is a trivial cofibration in $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$. Hence W_{q+1}^Σ is also a fibrant replacement functor for $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$. \square

Proposition 3.3.53. — *Let $f : X \rightarrow Y$ be a map in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$. We have that f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ if and only if the underlying map $UW_{q+1}^\Sigma(f) : UW_{q+1}^\Sigma X \rightarrow UW_{q+1}^\Sigma Y$ is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$.*

Proof. — Consider $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$. Using the enriched adjunctions of proposition 2.6.19, we get the following commutative diagram where the vertical arrows are all isomorphisms:

$$\begin{array}{ccc}
 \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma X) & & \\
 \parallel & \searrow^{W_{q+1}^\Sigma f_*} & \\
 & & \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma Y) \\
 \parallel & & \parallel \\
 \text{Map}_\Sigma(V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), W_{q+1}^\Sigma X) & & \text{Map}_\Sigma(V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), W_{q+1}^\Sigma Y) \\
 \cong \downarrow & \searrow^{W_{q+1}^\Sigma f_*} & \cong \downarrow \\
 \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma X) & & \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma Y) \\
 & \searrow^{UW_{q+1}^\Sigma f_*} & \\
 & & \text{Map}(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma Y)
 \end{array}$$

Since $UW_{q+1}^\Sigma X$ and $UW_{q+1}^\Sigma Y$ are both fibrant in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$, we have that $UW_{q+1}^\Sigma(f)$ is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$ if and only if the bottom row in the diagram above is a weak equivalence of simplicial sets for every

$F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$. By the two out of three property for weak equivalences we have that this happens if and only if the top row in the diagram above is a weak equivalence for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$. But this last condition holds if and only if f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$. This finishes the proof. \square

Proposition 3.3.54. — *Let $f : X \rightarrow Y$ be a map in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$. We have that f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$, the induced map:*

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma X]_{\text{Spt}}^\Sigma \xrightarrow{(W_{q+1}^\Sigma f)_*} [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma Y]_{\text{Spt}}^\Sigma$$

is an isomorphism of abelian groups.

Proof. — By Proposition 3.3.53, f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $UW_{q+1}^\Sigma(f)$ is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T \mathcal{M}_*$. Since $UW_{q+1}^\Sigma X, UW_{q+1}^\Sigma Y$ are both fibrant in $L_{<q+1}\text{Spt}_T \mathcal{M}_*$, using Proposition 3.2.62 we have that $UW_{q+1}^\Sigma(f)$ is a $S(q)$ -colocal equivalence if and only if for every $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S(q)$, the induced map

$$[F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma X]_{\text{Spt}} \xrightarrow{UW_{q+1}^\Sigma(f)_*} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma Y]_{\text{Spt}}$$

is an isomorphism of abelian groups.

Now since $W_{q+1}^\Sigma X, W_{q+1}^\Sigma Y$ are also fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, Theorem 2.6.30 implies that we have the following commutative diagram, where all the vertical arrows are isomorphisms:

$$\begin{array}{ccc} [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma X]_{\text{Spt}} & & [F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UW_{q+1}^\Sigma Y]_{\text{Spt}} \\ \cong \searrow & \xrightarrow{(UW_{q+1}^\Sigma f)_*} & \downarrow \cong \\ [V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), W_{q+1}^\Sigma X]_{\text{Spt}}^\Sigma & & [V(F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)), W_{q+1}^\Sigma Y]_{\text{Spt}}^\Sigma \\ \parallel \searrow & \xrightarrow{(W_{q+1}^\Sigma f)_*} & \parallel \\ [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma X]_{\text{Spt}}^\Sigma & & [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma Y]_{\text{Spt}}^\Sigma \\ & \xrightarrow{(W_{q+1}^\Sigma f)_*} & \end{array}$$

Therefore f is a $S^\Sigma(q)$ -colocal equivalence if and only if for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$, the bottom row is an isomorphism of abelian groups. This finishes the proof. \square

Corollary 3.3.55. — *Let $f : X \rightarrow Y$ be a map of symmetric T -spectra. We have that f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ if and only if*

$$W_{q+1}^\Sigma X \xrightarrow{W_{q+1}^\Sigma f} W_{q+1}^\Sigma Y$$

is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma\mathcal{M}_$.*

Proof. — (\Rightarrow): Assume that f is a $S^\Sigma(q)$ -colocal equivalence, and fix $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q,\Sigma}$. By Proposition 3.3.13 it suffices to show that the induced map

$$(63) \quad \begin{array}{c} [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma X]_{\text{Spt}}^\Sigma \\ (W_{q+1}^\Sigma f)_* \downarrow \\ [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma Y]_{\text{Spt}}^\Sigma \end{array}$$

is an isomorphism of abelian groups.

Since $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q,\Sigma}$, we have two possibilities:

1. $s - n = q$, i.e. $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$.
2. $s - n \geq q + 1$, i.e. $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q+1,\Sigma}$.

In case (1), Proposition 3.3.54 implies that the induced map in diagram (63) is an isomorphism of abelian groups.

On the other hand, in case (2), we have by Proposition 3.3.30(2) that

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma X]_{\text{Spt}}^\Sigma \cong 0 \cong [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma Y]_{\text{Spt}}^\Sigma$$

since by construction $W_{q+1}^\Sigma X$ and $W_{q+1}^\Sigma Y$ are both $L^\Sigma(< q + 1)$ -local symmetric T -spectra. Hence the induced map in diagram (63) is also an isomorphism of abelian groups in this case, as we wanted.

(\Leftarrow): Assume that $W_{q+1}^\Sigma f$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma\mathcal{M}_*$, and fix $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$.

Since $S^\Sigma(q) \subseteq C_{\text{eff}}^{q,\Sigma}$, it follows from Proposition 3.3.13 that the induced map

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma X]_{\text{Spt}}^\Sigma \xrightarrow{(W_{q+1}^\Sigma f)_*} [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), W_{q+1}^\Sigma Y]_{\text{Spt}}^\Sigma$$

is an isomorphism of abelian groups. Therefore, Proposition 3.3.54 implies that f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$. This finishes the proof. \square

Lemma 3.3.56. — *Let $f : X \rightarrow Y$ be a map in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$. We have that f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ if and only if $\Omega_{S^1} W_{q+1}^\Sigma f$ is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$.*

Proof. — It follows from Proposition 3.3.53 that f is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ if and only if $UW_{q+1}^\Sigma f$ is a $S(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$. Since $UW_{q+1}^\Sigma X, UW_{q+1}^\Sigma Y$ are both fibrant in $L_{<q+1}\text{Spt}_T\mathcal{M}_*$, using Lemma 3.2.64

we have that $UW_{q+1}^\Sigma f$ is a $S(q)$ -colocal equivalence if and only if $\Omega_{S^1}UW_{q+1}^\Sigma f = U(\Omega_{S^1}W_{q+1}^\Sigma f)$ is a $S(q)$ -colocal equivalence.

Finally, since $\Omega_{S^1}W_{q+1}^\Sigma X, \Omega_{S^1}W_{q+1}^\Sigma Y$ are both fibrant in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, we have by Proposition 3.3.53 that $U(\Omega_{S^1}W_{q+1}^\Sigma f)$ is a $S(q)$ -colocal equivalence if and only if $\Omega_{S^1}W_{q+1}^\Sigma f$ is a $S^\Sigma(q)$ -colocal equivalence. This finishes the proof. \square

Corollary 3.3.57. — *The adjunction*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : S^q\text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow S^q\text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen equivalence.

Proof. — Using Corollary 1.3.16 in [10] and Proposition 3.3.52 we have that it suffices to verify the following two conditions:

1. For every cofibrant object X in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$, the following composition

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1}W_{q+1}^{\Sigma, X \wedge S^1}} \Omega_{S^1}W_{q+1}^\Sigma(X \wedge S^1)$$

is a $S^\Sigma(q)$ -colocal equivalence.

2. Ω_{S^1} reflects $S^\Sigma(q)$ -colocal equivalences between fibrant objects in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): By construction $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore the identity functor

$$\text{id} : S^q\text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Thus X is also cofibrant in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$. Since the adjunction $(- \wedge S^1, \Omega_{S^1}, \varphi)$ is a Quillen equivalence on $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies that the following composition is a weak equivalence in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$:

$$X \xrightarrow{\eta_X} \Omega_{S^1}(X \wedge S^1) \xrightarrow{\Omega_{S^1}W_{q+1}^{\Sigma, X \wedge S^1}} \Omega_{S^1}W_{q+1}^\Sigma(X \wedge S^1)$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $S^\Sigma(q)$ -colocal equivalence.

- (2): This follows immediately from Proposition 3.3.52 and Lemma 3.3.56. \square

Remark 3.3.58. — *The adjunction $(\Sigma_T, \Omega_T, \varphi)$ is a Quillen equivalence on $\text{Spt}_T^\Sigma \mathcal{M}_*$. However it does not descend even to a Quillen adjunction on the q -slice motivic symmetric stable model category $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.*

Corollary 3.3.59. — *$S^q\mathcal{H}^\Sigma(S)$ has the structure of a triangulated category.*

Proof. — Theorem 3.3.50 implies in particular that $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$ is a pointed simplicial model category, and Corollary 3.3.57 implies that the adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : S^q\text{Spt}_T^\Sigma\mathcal{M}_* \rightarrow S^q\text{Spt}_T^\Sigma\mathcal{M}_*$$

is a Quillen equivalence. Therefore the result follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII]. \square

Proposition 3.3.60. — *We have the following adjunction*

$$(P_q^\Sigma, W_{q+1}^\Sigma, \varphi) : S^q\mathcal{H}^\Sigma(S) \longrightarrow L_{<q+1}\mathcal{H}^\Sigma(S)$$

between exact functors of triangulated categories.

Proof. — Since $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$ is the right Bousfield localization of $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ with respect to the $S^\Sigma(q)$ -colocal equivalences, we have that the identity functor $\text{id} : S^q\text{Spt}_T^\Sigma\mathcal{M}_* \rightarrow L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(P_q^\Sigma, W_{q+1}^\Sigma, \varphi) : S^q\mathcal{H}^\Sigma(S) \longrightarrow L_{<q+1}\mathcal{H}^\Sigma(S)$$

Now Proposition 6.4.1 in [10] implies that P_q^Σ maps cofibre sequences in $S^q\mathcal{H}^\Sigma(S)$ to cofibre sequences in $L_{<q+1}\mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that P_q^Σ and W_{q+1}^Σ are both exact functors between triangulated categories. \square

Proposition 3.3.61. — *The identity*

$$\text{id} : S^q\text{Spt}_T^\Sigma\mathcal{M}_* \longrightarrow R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma\mathcal{M}_*$$

is a right Quillen functor.

Proof. — Consider the following diagram of right Quillen functors

$$\begin{array}{ccccc} L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_* & \xrightarrow{\text{id}} & \text{Spt}_T^\Sigma\mathcal{M}_* & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma\mathcal{M}_* \\ \text{id} \downarrow & & \text{---} \text{---} \text{---} \text{---} \text{---} & \text{---} \text{---} \text{---} \text{---} \text{---} & \\ & & S^q\text{Spt}_T^\Sigma\mathcal{M}_* & \xrightarrow{\text{id}} & \end{array}$$

By the universal property of right Bousfield localizations (see definition 1.8.2) it suffices to check that if $f : X \rightarrow Y$ is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma\mathcal{M}_*$, then $W_{q+1}^\Sigma f : W_{q+1}^\Sigma X \rightarrow W_{q+1}^\Sigma Y$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma\mathcal{M}_*$. But this follows immediately from Corollary 3.3.55. \square

Corollary 3.3.62. — *We have the following adjunction*

$$(C_{\text{eff}}^\Sigma, W_{q+1}^\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow S^q\mathcal{H}^\Sigma(S)$$

of exact functors between triangulated categories.

Proof. — By Proposition 3.3.61 the identity functor $\text{id} : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Quillen functor. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(C_q^\Sigma, W_{q+1}^\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow S^q \mathcal{H}^\Sigma(S)$$

Now Proposition 6.4.1 in [10] implies that C_q^Σ maps cofibre sequences in $R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$ to cofibre sequences in $S^q \mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that C_q^Σ and W_{q+1}^Σ are both exact functors between triangulated categories. □

Lemma 3.3.63. — *If A is a cofibrant symmetric T -spectrum in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$, then the map $* \rightarrow A$ is a trivial cofibration in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — Let Z be an arbitrary $L^\Sigma(< q)$ -local symmetric T -spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$. We claim that the map $Z \rightarrow *$ is a trivial fibration in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. In effect, using Proposition 3.3.29 and Corollary 3.2.33 we have that Z is $L^\Sigma(< q + 1)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$, i.e. a fibrant object in $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$. By construction $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$, hence Z is also fibrant in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. Then by Proposition 3.3.54 it suffices to show that for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$ (i.e. $s - n = q$):

$$0 \cong [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z]_{\text{Spt}}^\Sigma$$

But this follows immediately from Proposition 3.3.30, since Z is $L^\Sigma(< q)$ -local.

Now since $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category and A is cofibrant in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$, we have that the following map is a trivial fibration of simplicial sets:

$$\text{Map}_\Sigma(A, Z) \longrightarrow \text{Map}_\Sigma(A, *) = *$$

The identity functor

$$\text{id} : S^q \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor, since $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore A is also cofibrant in $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$, and since $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$; it follows that A is also cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. On the other hand, we have that Z is in particular fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Hence $\pi_0 \text{Map}_\Sigma(A, Z)$ computes $[A, Z]_{\text{Spt}}^\Sigma$, since $\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category. But $\text{Map}_\Sigma(A, Z) \rightarrow *$ is in particular a weak equivalence of simplicial sets, then

$$[A, Z]_{\text{Spt}}^\Sigma \cong 0$$

for every $L^{\Sigma}(< q)$ -local symmetric T -spectrum Z . Finally, Corollary 3.3.33 implies that $* \rightarrow A$ is a weak equivalence in $L_{< q} \text{Spt}_T^{\Sigma} \mathcal{M}_*$. This finishes the proof, since we already know that A is cofibrant in $L_{< q} \text{Spt}_T^{\Sigma} \mathcal{M}_*$. \square

Theorem 3.3.64. — *The adjunction*

$$(V, U, \varphi) : S^q \text{Spt}_T \mathcal{M}_* \longrightarrow S^q \text{Spt}_T^{\Sigma} \mathcal{M}_*$$

given by the symmetrization and the forgetful functors is a Quillen equivalence.

Proof. — Proposition 3.3.53 together with the universal property for right Bousfield localizations (see definition 1.8.2) imply that

$$U : S^q \text{Spt}_T^{\Sigma} \mathcal{M}_* \longrightarrow S^q \text{Spt}_T \mathcal{M}_*$$

is a right Quillen functor. Using Corollary 1.3.16 in [10] and Proposition 3.3.52 we have that it suffices to verify the following two conditions:

1. For every cofibrant object X in $S^q \text{Spt}_T \mathcal{M}_*$, the following composition

$$X \xrightarrow{\eta_X} UV(X) \xrightarrow{UW_{q+1}^{\Sigma, V^X}} UW_{q+1}^{\Sigma} V(X)$$

is a weak equivalence in $S^q \text{Spt}_T \mathcal{M}_*$.

2. U reflects weak equivalences between fibrant objects in $S^q \text{Spt}_T^{\Sigma} \mathcal{M}_*$.

(1): By construction $S^q \text{Spt}_T \mathcal{M}_*$ is a right Bousfield localization of $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, therefore the identity functor

$$\text{id} : S^q \text{Spt}_T \mathcal{M}_* \longrightarrow L_{< q+1} \text{Spt}_T \mathcal{M}_*$$

is a left Quillen functor. Thus X is also cofibrant in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$. Since the adjunction (V, U, φ) is a Quillen equivalence between $L_{< q+1} \text{Spt}_T \mathcal{M}_*$ and $L_{< q+1} \text{Spt}_T^{\Sigma} \mathcal{M}_*$, [10, Proposition 1.3.13(b)] implies that the following composition is a weak equivalence in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$:

$$X \xrightarrow{\eta_X} UV(X) \xrightarrow{UW_{q+1}^{\Sigma, V^X}} UW_{q+1}^{\Sigma} V(X)$$

Hence using [6, Proposition 3.1.5] it follows that the composition above is a $S(q)$ -local equivalence in $L_{< q+1} \text{Spt}_T \mathcal{M}_*$, i.e. a weak equivalence in $S^q \text{Spt}_T \mathcal{M}_*$.

- (2): This follows immediately from Propositions 3.3.52 and 3.3.53. \square

Corollary 3.3.65. — *The adjunction*

$$(V, U, \varphi) : S^q \text{Spt}_T \mathcal{M}_* \longrightarrow S^q \text{Spt}_T^{\Sigma} \mathcal{M}_*$$

given by the symmetrization and the forgetful functors, induces an adjunction

$$(VP_q, UW_{q+1}^{\Sigma}, \varphi) : S^q \mathcal{H}(S) \longrightarrow S^q \mathcal{H}^{\Sigma}(S)$$

of exact functors between triangulated categories. Furthermore, VP_q and UW_{q+1}^Σ are both equivalences of categories.

Proof. — Theorem 3.3.64 implies that the adjunction (V, U, φ) is a Quillen equivalence. Therefore we get the following adjunction at the level of the associated homotopy categories:

$$(VP_q, UW_{q+1}^\Sigma, \varphi) : S^q \mathcal{H}(S) \longrightarrow S^q \mathcal{H}^\Sigma(S)$$

Now [10, Proposition 1.3.13] implies that VP_q, UW_{q+1}^Σ are both equivalences of categories. Finally, Proposition 2.6.19 together with [10, Proposition 6.4.1] imply that VP_q maps cofibre sequences in $S^q \mathcal{H}(S)$ to cofibre sequences in $S^q \mathcal{H}^\Sigma(S)$. Therefore using Proposition 7.1.12 in [10] we have that VP_q and UW_{q+1}^Σ are both exact functors between triangulated categories. □

Now it is very easy to find the desired lifting for the functor $\tilde{s}_q : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$ (see Corollary 3.3.5(3)) to the model category level.

Lemma 3.3.66. — 1. *Let X be an arbitrary T -spectrum in $R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$. Then the following maps in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$*

$$VP_q(C_q X) \xrightarrow{V(P_q^{C_q X})} VC_q X \xleftarrow{C_q^{\Sigma, VC_q X}} C_q^\Sigma(VC_q X)$$

induce natural isomorphisms between the functors:

$$C_q^\Sigma \circ VC_q, VC_q, VP_q \circ C_q : R_{C_{\text{eff}}}^q \mathcal{H}(S) \rightarrow S^q \mathcal{H}^\Sigma(S)$$

$$\begin{array}{ccc}
 & R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) & \\
 VC_q \nearrow & & \searrow C_q^\Sigma \\
 R_{C_{\text{eff}}}^q \mathcal{H}(S) & \xrightarrow{VC_q} & S^q \mathcal{H}^\Sigma(S) \\
 C_q \searrow & & \nearrow VP_q \\
 & S^q \mathcal{H}(S) &
 \end{array}$$

Given a T -spectrum X

$$\sigma_X : VP_q(C_q X) \xrightarrow{\cong} C_q^\Sigma(VC_q X)$$

will denote the isomorphism in $S^q \mathcal{H}^\Sigma(S)$ corresponding to the natural isomorphism between $VP_q \circ C_q$ and $C_q^\Sigma \circ VC_q$.

2. Let X be an arbitrary symmetric T -spectrum in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. Then the following maps in $R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_*$

$$W_{q+1}(UW_{q+1}^\Sigma X) \xleftarrow{W_{q+1}^{UW_{q+1}^\Sigma X}} UW_{q+1}^\Sigma X \xrightarrow{U(R_\Sigma^{W_{q+1}^\Sigma X})} UR_\Sigma(W_{q+1}^\Sigma X)$$

induce natural isomorphisms between the functors:

$$W_{q+1} \circ UW_{q+1}^\Sigma, UW_{q+1}^\Sigma, UR_\Sigma \circ W_{q+1}^\Sigma : S^q \mathcal{H}^\Sigma(S) \rightarrow R_{C_{\text{eff}}}^q \mathcal{H}(S)$$

$$\begin{array}{ccc}
 & S^q \mathcal{H}(S) & \\
 UW_{q+1}^\Sigma \nearrow & & \searrow W_{q+1} \\
 S^q \mathcal{H}^\Sigma(S) & \xrightarrow{UW_{q+1}^\Sigma} & R_{C_{\text{eff}}}^q \mathcal{H}(S) \\
 W_{q+1}^\Sigma \searrow & & \nearrow UR_\Sigma \\
 & R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) &
 \end{array}$$

Given a symmetric T -spectrum X

$$\tau_X : W_{q+1}(UW_{q+1}^\Sigma X) \xrightarrow{\cong} UR_\Sigma(W_{q+1}^\Sigma X)$$

will denote the isomorphism in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$ corresponding to the natural isomorphism between $W_{q+1} \circ UW_{q+1}^\Sigma$ and $UR_\Sigma \circ W_{q+1}^\Sigma$.

Proof. — (1): It follows immediately from Theorem 1.3.7 in [10] and the following commutative diagram of left Quillen functors:

$$\begin{array}{ccc}
 R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \text{id} \downarrow & & \downarrow \text{id} \\
 S^q \text{Spt}_T \mathcal{M}_* & \xrightarrow{V} & S^q \text{Spt}_T^\Sigma \mathcal{M}_*
 \end{array}$$

(2): It follows immediately from the dual of Theorem 1.3.7 in [10] and the following commutative diagram of right Quillen functors:

$$\begin{array}{ccc}
 R_{C_{\text{eff}}}^q \text{Spt}_T \mathcal{M}_* & \xleftarrow{U} & R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_* & \square \\
 \text{id} \uparrow & & \uparrow \text{id} & \\
 S^q \text{Spt}_T \mathcal{M}_* & \xleftarrow{U} & S^q \text{Spt}_T^\Sigma \mathcal{M}_* &
 \end{array}$$

Lemma 3.3.67. — *Let X be an arbitrary T -spectrum, and let η be the unit of the adjunction (see Corollary 3.3.65):*

$$(VP_q, UW_{q+1}^\Sigma, \varphi) : S^q \mathcal{H}(S) \longrightarrow S^q \mathcal{H}^\Sigma(S)$$

We have the following diagram in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$ (see Lemma 3.3.66):

$$\begin{array}{ccc}
 W_{q+1}UW_{q+1}^\Sigma VP_q C_q X & \xrightarrow[\cong]{W_{q+1}UW_{q+1}^\Sigma(\sigma_X)} & W_{q+1}UW_{q+1}^\Sigma C_q^\Sigma VC_q X \\
 \uparrow \cong & & \downarrow \cong \\
 W_{q+1}C_q X & & UR_\Sigma W_{q+1}^\Sigma C_q^\Sigma VC_q X
 \end{array}$$

$\tau_{C_q^\Sigma VC_q X}$

where all the maps are isomorphisms in $R_{C_{\text{eff}}}^q \mathcal{H}(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\begin{array}{ccc}
 R_{C_{\text{eff}}}^q \mathcal{H}(S) & \xrightarrow{W_{q+1}C_q} & R_{C_{\text{eff}}}^q \mathcal{H}(S) \\
 & \xrightarrow{UR_\Sigma W_{q+1}^\Sigma C_q^\Sigma VC_q} &
 \end{array}$$

Proof. — It follows immediately from lemma 3.3.66 and Corollary 3.3.65. □

Theorem 3.3.68. — *Let X be an arbitrary symmetric T -spectrum.*

1. The diagram (48) in Theorem 3.2.80 induces the following diagram in $\mathcal{H}^\Sigma(S)$:

$$\begin{array}{c}
 \tilde{s}_q X = VQ_s(s_q UR_\Sigma X) \\
 \downarrow \cong \\
 VQ_s(IQ_T J s_q UR_\Sigma X) \\
 \uparrow \cong \\
 VQ_s(C_q^{IQ_T J s_q UR_\Sigma X}) \\
 \downarrow \cong \\
 VQ_s(C_q IQ_T J s_q UR_\Sigma X) \\
 \downarrow \cong \\
 VQ_s(W_{q+1}^{C_q IQ_T J s_q UR_\Sigma X}) \\
 \uparrow \cong \\
 VQ_s(W_{q+1} C_q IQ_T J s_q UR_\Sigma X) \\
 \downarrow \cong \\
 VQ_s(C_q W_{q+1} C_q IQ_T J s_q UR_\Sigma X) \\
 \uparrow \cong \\
 VQ_s(C_q W_{q+1} C_q IQ_T J(\pi_q^{UR_\Sigma X})) \\
 \downarrow \cong \\
 VQ_s(C_q W_{q+1} C_q IQ_T J f_q UR_\Sigma X) \\
 \downarrow \cong \\
 VQ_s(C_q W_{q+1} C_q IQ_T J(\theta_{UR_\Sigma X})) \\
 \downarrow \cong \\
 VQ_s(C_q W_{q+1} C_q IQ_T J UR_\Sigma X)
 \end{array}
 \tag{64}$$

where all the maps are isomorphisms in $\mathcal{H}^\Sigma(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \begin{array}{c} \xrightarrow{\tilde{s}_q} \\ \xrightarrow{VQ_s \circ C_q W_{q+1} C_q IQ_T J \circ UR_\Sigma} \end{array} \mathcal{H}^\Sigma(S)$$

2. Let ϵ denote the counit of the adjunction (see Corollary 3.3.20):

$$(VC_q, UR_\Sigma, \varphi) : R_{C_{\text{eff}}}^q \mathcal{H}(S) \longrightarrow R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S)$$

and let δ denote the natural isomorphism constructed in Lemma 3.3.67. Then we have the following diagram in $\mathcal{H}^\Sigma(S)$ (see Lemmas 3.3.67 and 3.3.21):

$$\begin{array}{c}
 C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma X = s_q^\Sigma X \\
 \uparrow \cong \\
 C_q^\Sigma(\epsilon_{W_{q+1}^\Sigma, C_q^\Sigma R_\Sigma X}) \\
 C_q^\Sigma(VC_qUR_\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma X) \\
 \uparrow \cong \\
 C_q^\Sigma VC_qUR_\Sigma W_{q+1}^\Sigma C_q^\Sigma(\epsilon_{R_\Sigma X}) \\
 C_q^\Sigma VC_qUR_\Sigma W_{q+1}^\Sigma C_q^\Sigma(VC_qUR_\Sigma R_\Sigma X) \\
 \uparrow \cong \\
 C_q^\Sigma VC_q(\delta_{UR_\Sigma R_\Sigma X}) \\
 C_q^\Sigma VC_q(W_{q+1}C_q)UR_\Sigma R_\Sigma X \\
 \uparrow \cong \\
 C_q^\Sigma VC_q W_{q+1} C_q(\beta_X) \\
 C_q^\Sigma VC_q W_{q+1} C_q(IQ_T J U R_\Sigma X) \\
 \uparrow \cong \\
 \alpha_{W_{q+1} C_q I Q_T J U R_\Sigma X} \\
 VQ_s C_q(W_{q+1} C_q I Q_T J U R_\Sigma X)
 \end{array}
 \tag{65}$$

where all the maps are isomorphisms in $\mathcal{H}^\Sigma(S)$. This diagram induces a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \xrightarrow[\begin{array}{c} C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma = s_q^\Sigma \end{array}]{VQ_s \circ C_q W_{q+1} C_q I Q_T J \circ U R_\Sigma} \mathcal{H}^\Sigma(S)$$

3. Combining the diagrams (64) and (65) above we get a natural isomorphism between the following exact functors:

$$\mathcal{H}^\Sigma(S) \xrightarrow[\begin{array}{c} s_q^\Sigma \end{array}]{\tilde{s}_q} \mathcal{H}^\Sigma(S)$$

Proof. — It is clear that it suffices to prove only the first two claims.

(1): This follows immediately from theorems 3.2.80 and 3.3.4.

(2): This follows immediately from lemmas 3.3.21 and 3.3.67 together with Corollary 3.3.20. □

Proposition 3.3.69. — Let η denote the unit of the adjunction $(C_q^\Sigma, W_{q+1}^\Sigma, \varphi) : R_{C_q^\Sigma} \mathcal{H}^\Sigma(S) \rightarrow S^q \mathcal{H}^\Sigma(S)$ constructed in Corollary 3.3.62. The natural transformation $\pi_q : f_q \rightarrow s_q$ (see Theorem 3.1.16) gets canonically identified, through

the equivalence of categories $r_q C_q$, $IQ_T J i_q$, VC_q and UR_Σ constructed in Proposition 3.2.21 and corollary 3.3.20; with the following map $\pi_q^\Sigma : f_q^\Sigma \rightarrow s_q^\Sigma$ in $\mathcal{H}^\Sigma(S)$:

$$C_q^\Sigma R_\Sigma X \xrightarrow{C_q^\Sigma(\eta_{R_\Sigma X})} C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma X$$

Proof. — The result follows easily from Proposition 3.2.81, Corollaries 3.3.20, 3.3.65 and Theorem 3.3.68. \square

Remark 3.3.70. — The functor s_q^Σ gives the desired lifting for the functor \tilde{s}_q to the model category level, and it will be the main ingredient for the study of the multiplicative properties of Voevodsky's slice filtration. This completes the program that we started at the beginning of this section.

3.4. Multiplicative Properties of the Slice Filtration

Our goal in this section is to show that the smash product of symmetric spectra is compatible in a suitable sense with the slice filtration. To establish this compatibility in a formal way, we will use the model structures constructed in Section 3.3. In the rest of this section $p, q \in \mathbb{Z}$ will denote arbitrary integers.

Lemma 3.4.1. — The sphere spectrum $\mathbf{1}$ is cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$, $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$ and $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — By proposition 3.3.61 and Theorem 3.3.9 we have that it is enough to show that $\mathbf{1}$ is cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Now, Corollary 3.2.15 implies that $F_0(S^0)$ is a C_{eff}^0 -colocal T -spectrum in $\text{Spt}_T \mathcal{M}_*$, since $F_0(S^0) \in \mathcal{H}^{\text{eff}}(S)$. Then using [6, Theorem 5.1.1(2)] we have that $F_0(S^0)$ is a cofibrant object in $R_{C_{\text{eff}}^0} \text{Spt}_T \mathcal{M}_*$, and this implies that $\mathbf{1} = V(F_0(S^0))$ is also cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$, since the symmetrization functor

$$V : R_{C_{\text{eff}}^0} \text{Spt}_T \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. \square

Lemma 3.4.2. — Let A be a symmetric T -spectrum.

1. If A is cofibrant in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$, then the functor $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, -)$ maps fibrations in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$ to fibrations in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.
2. If A is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, then the functor $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, -)$ maps fibrations in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$ to fibrations in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — Since $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$, $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$ are all right Bousfield localizations of $\text{Spt}_T^\Sigma \mathcal{M}_*$, we have that the fibrations in all these model structures coincide and also the identity functor

$$\text{id} : R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Therefore if A is cofibrant in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$, then A is also cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Hence it suffices to prove (2).

So assume that A is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and let $f : X \rightarrow Y$ be an arbitrary fibration in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$. Then using Corollary 2.6.29 together with the fact that A is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, we get that

$$\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, X) \xrightarrow{f_*} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, Y)$$

is a fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$, or equivalently a fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. □

Lemma 3.4.3. — *If $A = F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is an arbitrary element in $S^\Sigma(p)$, i.e. $s - n = p$, and F is a symmetric T -spectrum such that the map $F \rightarrow *$ is a trivial fibration in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$; then $\pi : \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F) \rightarrow *$ is a trivial fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — Since A is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, it follows directly from Lemma 3.4.2 that π is a fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, it only remains to show that π is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Fix $F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \in C_{\text{eff}}^{q,\Sigma}$. By construction $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore F is also fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Since A is cofibrant and F is fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, corollary 2.6.29 implies that we have the following natural isomorphism of abelian groups:

$$[F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)]_{\text{Spt}}^\Sigma \cong [F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge A, F]_{\text{Spt}}^\Sigma$$

and Proposition 2.6.13 implies that:

$$\begin{aligned} F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge A &= F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \\ &\cong F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+) \end{aligned}$$

But clearly $F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+) \in C_{\text{eff}}^{p+q,\Sigma}$, and since $F \rightarrow *$ is a weak equivalence in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$, we have by Proposition 3.3.13:

$$\begin{aligned} 0 &\cong [F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+), F]_{\text{Spt}}^\Sigma \\ &\cong [F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)]_{\text{Spt}}^\Sigma \end{aligned}$$

Finally, using Proposition 3.3.13 again, we get that π is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, as we wanted. □

Lemma 3.4.4. — *If A is a cofibrant symmetric T -spectrum in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$, and F is a symmetric T -spectrum such that the map $F \rightarrow *$ is a trivial fibration in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$; then $\pi : \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F) \rightarrow *$ is a trivial fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — Since A is cofibrant in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$, it follows from Lemma 3.4.2(1) that π is a fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, it only remains to show that π is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Fix $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q, \Sigma}$. Then Lemma 3.4.3 together with Proposition 3.3.24(1) imply that

$$\mathbf{Hom}_{\text{Spt}_T^\Sigma}(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), F) \longrightarrow *$$

is a trivial fibration in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$. Now, since A is cofibrant in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$ which is in particular a simplicial model category, we have that the induced map:

$$\text{Map}_\Sigma(A, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), F)) \longrightarrow \text{Map}_\Sigma(A, *) = *$$

is a trivial fibration of simplicial sets. Finally using the enriched adjunctions of Proposition 2.6.12, this last map gets canonically identified with

$$\begin{array}{c} \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)) \\ \downarrow \\ \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), *) = * \end{array}$$

Since $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)$ is in particular fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, by definition we have that π is a $C_{\text{eff}}^{q, \Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, i.e. a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. This finishes the proof. \square

Theorem 3.4.5. — *The smash product of symmetric T -spectra*

$$- \wedge - : R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen bifunctor in the sense of Hovey (see definition 1.7.4).

Proof. — By Lemma 1.7.5, it is enough to prove the following claim:

Given a cofibration $i : A \rightarrow B$ in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$ and a fibration $f : X \rightarrow Y$ in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$, the induced map

$$(i^*, f_*) : \mathbf{Hom}_{\text{Spt}_T^\Sigma}(B, X) \longrightarrow \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, X) \times_{\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, Y)} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(B, Y)$$

is a fibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ which is trivial if either i or f is a weak equivalence.

Since $R_{C_{\text{eff}}}^p \text{Spt}_T^\Sigma \mathcal{M}_*$, $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$ and $R_{C_{\text{eff}}}^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$ are all right Bousfield localizations of $\text{Spt}_T^\Sigma \mathcal{M}_*$, we have that the fibrations in all these model structures coincide and also the identity functor

$$(66) \quad \text{id} : R_{C_{\text{eff}}}^p \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow \text{Spt}_T^\Sigma \mathcal{M}_*$$

is a left Quillen functor. Hence it follows that i is cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and f is fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then Proposition 2.6.28 implies that (i^*, f_*) is a fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$, or equivalently a fibration in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

Now assume that i is a trivial cofibration in $R_{C_{\text{eff}}}^p \text{Spt}_T^\Sigma \mathcal{M}_*$. Since the identity functor considered in (66) above is a left Quillen functor, we have that i is also a trivial cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Hence using proposition 2.6.28 again, we have that (i^*, f_*) is in particular a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then [6, Proposition 3.1.5] implies that (i^*, f_*) is also a weak equivalence in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

Finally, assume that f is a trivial fibration in $R_{C_{\text{eff}}}^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Consider the following commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{\kappa} & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ * & \xrightarrow{\iota} & B/A \end{array}$$

where the diagram on the left is a pullback in $R_{C_{\text{eff}}}^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and the diagram on the right is a pushout in $R_{C_{\text{eff}}}^p \text{Spt}_T^\Sigma \mathcal{M}_*$. We already know that the map (i^*, f_*) is a fibration in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$, therefore it is clear that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(B/A, F)$ is the homotopy fibre of (i^*, f_*) in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$. On the other hand, it is clear that κ is a trivial fibration in $R_{C_{\text{eff}}}^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and ι is a cofibration in $R_{C_{\text{eff}}}^p \text{Spt}_T^\Sigma \mathcal{M}_*$.

By Corollary 3.3.17 we have that the homotopy category associated to $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is triangulated, hence to check that (i^*, f_*) is a weak equivalence in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$ it is enough to show that the map

$$\pi : \mathbf{Hom}_{\text{Spt}_T^\Sigma}(B/A, F) \longrightarrow *$$

is a weak equivalence in $R_{C_{\text{eff}}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$. But this follows immediately from lemma 3.4.4. □

Lemma 3.4.6. — *If $i : A \rightarrow B$ is a cofibration in $L_{<p} \text{Spt}_T^\Sigma \mathcal{M}_*$, and $j : C \rightarrow D$ is a cofibration in $L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_*$; then for every $r \in \mathbb{Z}$, the map*

$$B \wedge C \coprod_{A \wedge C} A \wedge D \xrightarrow{i \square j} B \wedge D$$

is a cofibration in $L_{<r} \text{Spt}_T^\Sigma \mathcal{M}_$.*

Proof. — Since $L_{<p}\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, $L_{<q}\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ and $L_{<r}\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ are all left Bousfield localizations of $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, we have that the cofibrations in these four model categories coincide.

Then the result follows immediately from Proposition 2.6.28. \square

Lemma 3.4.7. — *If $A = F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is an arbitrary element in $S^\Sigma(p)$, i.e. $s - n = p$, and Z is an arbitrary $L^\Sigma(< p + q)$ -local symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$; then $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is a $L^\Sigma(< q)$ -local symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — By Proposition 3.3.30 it is enough to check that the following two conditions hold:

1. $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$.
2. For every $F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \in C_{\mathrm{eff}}^{q, \Sigma}$

$$[F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)]_{\mathrm{Spt}}^\Sigma \cong 0$$

Since Z is $L^\Sigma(< p + q)$ -local in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, we have that Z is in particular fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Now Corollary 2.6.29 together with the fact that A is cofibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ imply that $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. This takes care of the first condition.

Fix $F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \in C_{\mathrm{eff}}^{q, \Sigma}$. Since A is cofibrant and Z is fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, it follows from Corollary 2.6.29 that we have the following natural isomorphism of abelian groups:

$$[F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)]_{\mathrm{Spt}}^\Sigma \cong [F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge A, Z]_{\mathrm{Spt}}^\Sigma$$

Using Proposition 2.6.13 we have the following isomorphisms of symmetric T -spectra:

$$\begin{aligned} F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge A &= F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \\ &\cong F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+) \end{aligned}$$

But clearly $F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+) \in C_{\mathrm{eff}}^{p+q, \Sigma}$. Since Z is $L^\Sigma(< p + q)$ -local in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, Proposition 3.3.30 implies:

$$\begin{aligned} 0 &\cong [F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+), Z]_{\mathrm{Spt}}^\Sigma \\ &\cong [F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)]_{\mathrm{Spt}}^\Sigma \end{aligned}$$

This finishes the proof. \square

Lemma 3.4.8. — *If A is a symmetric T -spectrum such that the map $* \rightarrow A$ is a trivial cofibration in $L_{<p}\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, and Z is an arbitrary $L^\Sigma(< p + q)$ -local symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$; then $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is a $L^\Sigma(< q)$ -local symmetric T -spectrum in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — By Proposition 3.3.30 it is enough to check that the following two conditions hold:

1. $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$.
2. For every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^{q, \Sigma}$

$$[F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)]_{\mathrm{Spt}}^\Sigma \cong 0$$

Since Z is $L^\Sigma(< p + q)$ -local in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, we have that Z is in particular fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. By construction $L_{<p} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization of $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, then it follows that A is cofibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Therefore, Corollary 2.6.29 implies that $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. This takes care of the first condition.

Fix $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^{q, \Sigma}$. Using Lemma 3.4.7 together with Proposition 3.3.46, we get that the induced map

$$\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z) \longrightarrow *$$

is a fibration in $L_{<p} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Since $L_{<p} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category and $* \rightarrow A$ is a trivial cofibration in $L_{<p} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$, it follows that the following map is a trivial fibration of simplicial sets:

$$\mathrm{Map}_\Sigma(A, \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), Z)) \longrightarrow \mathrm{Map}_\Sigma(A, *) = *$$

Finally using the enriched adjunctions of Proposition 2.6.12, the map above becomes:

$$\begin{array}{c} \mathrm{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)) \\ \downarrow \\ \mathrm{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), *) = * \end{array}$$

which is in particular a weak equivalence of simplicial sets. We already know that $\mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)$ is fibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, and we have that $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is cofibrant in $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Since $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category, we have that

$$\begin{aligned} 0 &\cong \pi_0 \mathrm{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)) \\ &\cong [F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \mathbf{Hom}_{\mathrm{Spt}_T^\Sigma}(A, Z)]_{\mathrm{Spt}}^\Sigma \end{aligned}$$

for every $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\mathrm{eff}}^{q, \Sigma}$. This finishes the proof. □

Lemma 3.4.9. — *If A, C are symmetric T -spectra such that $* \rightarrow A$ is a trivial cofibration in $L_{<p} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$, and $* \rightarrow C$ is a trivial cofibration in $L_{<q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$; then $\iota : * \rightarrow C \wedge A$ is a trivial cofibration in $L_{<p+q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — Since A is in particular cofibrant in $L_{<p} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ (respectively C is cofibrant in $L_{<q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$), it follows directly from Lemma 3.4.6 that ι is a cofibration in $L_{<p+q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$. Thus, it only remains to show that ι is a weak equivalence in $L_{<p+q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$.

Let Z be an arbitrary $L^\Sigma(< p + q)$ -local symmetric T -spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then by Lemma 3.4.8, we have that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, Z)$ is $L^\Sigma(< q)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Now Corollary 3.3.33 implies that

$$[C, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, Z)]_{\text{Spt}}^\Sigma \cong 0$$

But A, C are in particular cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and Z is in particular fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, then using Corollary 2.6.29 we get the following isomorphism:

$$[C \wedge A, Z]_{\text{Spt}}^\Sigma \cong [C, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, Z)]_{\text{Spt}}^\Sigma \cong 0$$

Hence the induced map

$$0 \cong [C \wedge A, Z]_{\text{Spt}}^\Sigma \xrightarrow{\iota^*} [*, Z]_{\text{Spt}}^\Sigma \cong 0$$

is an isomorphism for every $L^\Sigma(< p + q)$ -local T -spectrum Z . Thus, using Corollary 3.3.33 again, we have that ι is a $L^\Sigma(< p + q)$ -local equivalence. This finishes the proof. \square

Lemma 3.4.10. — *If $A = F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ is an arbitrary element in $S^\Sigma(p)$, i.e. $s - n = p$, and F is a symmetric T -spectrum such that the map $F \rightarrow *$ is a trivial fibration in $S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$; then $\pi : \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F) \rightarrow *$ is a trivial fibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — F is fibrant in $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, since by construction $S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$. Applying Lemma 3.4.7, we get that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)$ is fibrant in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$; and since $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, it follows that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)$ is fibrant in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.

By Proposition 3.3.54 it only remains to check that for every $F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \in S^\Sigma(q)$, i.e. $l - j = q$,

$$[F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)]_{\text{Spt}}^\Sigma \cong 0$$

Since A is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and F is in particular fibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$, corollary 2.6.29 implies that we have the following natural isomorphism of abelian groups:

$$[F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)]_{\text{Spt}}^\Sigma \cong [F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge A, F]_{\text{Spt}}^\Sigma$$

But using Proposition 2.6.13 we get:

$$\begin{aligned} F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge A &= F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \\ &\cong F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+) \end{aligned}$$

and it is clear that $F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+) \in S^\Sigma(p + q)$.

Finally, since $F \rightarrow *$ is a trivial fibration in $S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$, using Proposition 3.3.54 we get that for every $F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+) \in S^\Sigma(q)$:

$$\begin{aligned} 0 &\cong [F_{j+n}^\Sigma(S^{k+r} \wedge \mathbb{G}_m^{l+s} \wedge U \times_S V_+), F]_{\text{Spt}}^\Sigma \\ &\cong [F_j^\Sigma(S^k \wedge \mathbb{G}_m^l \wedge V_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)]_{\text{Spt}}^\Sigma \end{aligned}$$

as we wanted. □

Lemma 3.4.11. — *If A is a cofibrant symmetric T -spectrum in $S^p\text{Spt}_T^\Sigma \mathcal{M}_*$, and F is a symmetric T -spectrum such that the map $F \rightarrow *$ is a trivial fibration in $S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$; then $\pi : \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F) \rightarrow *$ is a trivial fibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — F is fibrant in $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, since by construction $S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$. Now, Lemma 3.3.63 implies that $* \rightarrow A$ is a trivial cofibration in $L_{<p}\text{Spt}_T^\Sigma \mathcal{M}_*$. Applying Lemma 3.4.8, we get that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)$ is fibrant in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$; and since $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, it follows that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)$ is fibrant in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.

Fix $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$, i.e $s - n = q$. Applying Lemma 3.4.10 we have that

$$\mathbf{Hom}_{\text{Spt}_T^\Sigma}(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), F) \longrightarrow *$$

is a trivial fibration in $S^p\text{Spt}_T^\Sigma \mathcal{M}_*$ which is in particular a simplicial model category. Therefore the induced map

$$\text{Map}_\Sigma(A, \mathbf{Hom}_{\text{Spt}_T^\Sigma}(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), F)) \longrightarrow \text{Map}_\Sigma(A, *) = *$$

is a trivial fibration of simplicial sets. Finally using the enriched adjunctions of Proposition 2.6.12, the map above becomes:

$$\begin{array}{c} \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)) \\ \downarrow \\ \text{Map}_\Sigma(F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), *) = * \end{array}$$

which is in particular a weak equivalence of simplicial sets. We already know that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, F)$ is fibrant in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, then by definition it follows that π is a $S^\Sigma(q)$ -colocal equivalence in $L_{<q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, i.e. a weak equivalence in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$. This finishes the proof. □

Theorem 3.4.12. — *The smash product of symmetric T -spectra*

$$- \wedge - : S^p\text{Spt}_T^\Sigma \mathcal{M}_* \times S^q\text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$$

is a Quillen bifunctor in the sense of Hovey (see definition 1.7.4).

Proof. — Since

$$- \wedge - : S^p \text{Spt}_T^\Sigma \mathcal{M}_* \times S^q \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$$

is an adjunction of two variables (see Lemma 1.7.5), it follows that it is enough to prove the following two claims:

1. Let $i : A \rightarrow B$ be a cofibration in $S^p \text{Spt}_T^\Sigma \mathcal{M}_*$, and let $j : C \rightarrow D$ be a cofibration in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. Assume that either i or j is trivial. Then

$$B \wedge C \coprod_{A \wedge C} A \wedge D \xrightarrow{i \square j} B \wedge D$$

is a trivial cofibration in $S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

2. Let $i : A \rightarrow B$ be a cofibration in $S^p \text{Spt}_T^\Sigma \mathcal{M}_*$, and let $p : X \rightarrow Y$ be a trivial fibration in $S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Then

$$\mathbf{Hom}_{\text{Spt}_T^\Sigma}(B, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(B, Y) \times \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, Y) \mathbf{Hom}_{\text{Spt}_T^\Sigma}(A, X)$$

is a trivial fibration in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): By symmetry, it is enough to consider the case where i is a cofibration in $S^p \text{Spt}_T^\Sigma \mathcal{M}_*$, and j is a trivial cofibration in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. Since $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ and $S^p \text{Spt}_T^\Sigma \mathcal{M}_*$ are right Bousfield localizations of $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $L_{<p+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ respectively, we have that the identity functor

$$\begin{aligned} \text{id} : S^q \text{Spt}_T^\Sigma \mathcal{M}_* &\longrightarrow L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_* \\ \text{id} : S^p \text{Spt}_T^\Sigma \mathcal{M}_* &\longrightarrow L_{<p+1} \text{Spt}_T^\Sigma \mathcal{M}_* \end{aligned}$$

is in both cases a left Quillen functor. This implies in particular that i is a cofibration in $L_{<p+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ and j is a cofibration in $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$. Then by Lemma 3.4.6 we have that $i \square j$ is a cofibration in $L_{<p+q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$.

By construction $S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<p+q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$, hence the trivial cofibrations in both model structures are exactly the same. Thus, it only remains to show that $i \square j$ is a weak equivalence in $L_{<p+q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Consider the following pushout diagrams in $\text{Spt}_T \mathcal{M}_*$:

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ * & \xrightarrow{\iota} & B/A \end{array} \quad \begin{array}{ccc} C & \xrightarrow{j} & D \\ \downarrow & & \downarrow \\ * & \xrightarrow{\kappa} & D/C \end{array}$$

By construction $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization of $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$; therefore the trivial cofibrations coincide in both model structures. This implies that j and κ are both trivial cofibrations in $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$. On the other hand, it is clear that ι is a cofibration in $S^p \text{Spt}_T^\Sigma \mathcal{M}_*$. Then Lemma 3.3.63 implies that ι is a trivial cofibration

in $L_{<p}\text{Spt}_T^\Sigma \mathcal{M}_*$. Using Lemma 3.4.9, we get that the map $* \rightarrow (B/A) \wedge (D/C)$ is a trivial cofibration in $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$.

Finally, since $i \square j$ is a cofibration in $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, it follows that $(B/A) \wedge (D/C)$ is the homotopy cofibre of $i \square j$ in $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$. But Corollary 3.3.37 implies that the homotopy category associated to $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$ is triangulated. Therefore $i \square j$ is a trivial cofibration in $L_{<p+q+1}\text{Spt}_T^\Sigma \mathcal{M}_*$, since its homotopy cofibre is contractible.

(2): Using (1) above together with the fact that

$$- \wedge - : S^p\text{Spt}_T^\Sigma \mathcal{M}_* \times S^q\text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$$

is an adjunction of two variables, we have that (i^*, p_*) is a fibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, it only remains to show that (i^*, p_*) is a weak equivalence in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.

Consider the following diagrams in $\text{Spt}_T \mathcal{M}_*$:

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ * & \xrightarrow{\iota} & B/A \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\kappa} & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{p} & Y \end{array}$$

where the diagram on the left is a pushout square and the diagram on the right is a pullback square. It is clear that ι is a cofibration in $S^p\text{Spt}_T^\Sigma \mathcal{M}_*$ and that κ is a trivial fibration in $S^{p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$. Then Lemma 3.4.11 implies that $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(B/A, F) \rightarrow *$ is a trivial fibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.

We already know that (i^*, p_*) is a fibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore $\mathbf{Hom}_{\text{Spt}_T^\Sigma}(B/A, F)$ is the homotopy fibre of (i^*, p_*) in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$. Finally, by Corollary 3.3.59 we have that the homotopy category associated to $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ is triangulated. Therefore it follows that (i^*, p_*) is a trivial fibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$, since its homotopy fibre is contractible. \square

Remark 3.4.13. — Notice that the weight^{<q} motivic symmetric stable model structures are not compatible with the smash product of symmetric T-spectra, since the bifunctor

$$- \wedge - : L_{<p}\text{Spt}_T^\Sigma \mathcal{M}_* \times L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow L_{<p+q}\text{Spt}_T^\Sigma \mathcal{M}_*$$

is not a Quillen bifunctor, contrary to the case of the q-connected motivic symmetric stable model structures $R_{\text{eff}}^q \text{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.4.5) and the q-slice motivic symmetric stable model structures $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.4.12).

In effect, fix $q \in \mathbb{Z}$ and assume that the bifunctor defined above is a Quillen bifunctor. It follows from Lemmas 3.3.63 and 3.4.1 that the map $* \rightarrow \mathbf{1}$ is a trivial cofibration in $L_{<0}\text{Spt}_T^\Sigma \mathcal{M}_*$. Then, for every fibrant symmetric T-spectrum F in $L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$ we have that the map

$$\mathbf{Hom}_{\text{Spt}_T^\Sigma}(\mathbf{1}, F) \cong F \rightarrow *$$

is a trivial fibration in $L_{<q}\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ (contrast with Lemma 3.4.8). Therefore the homotopy category $L_{<q}\mathcal{H}^\Sigma(S)$ associated to $L_{<q}\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ is trivial. By Theorem 3.3.45 this implies that the functors

$$s_{<q}^\Sigma : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$$

are all trivial, i.e. $s_{<q}^\Sigma X = 0$ for every symmetric T -spectrum X . Then, it follows from Proposition 3.1.19, Corollary 3.3.5 and Theorem 3.3.68 that all the slices are also trivial, i.e. $s_q^\Sigma X = 0$ for every symmetric T -spectrum X .

Finally, if our base scheme S is a perfect field then the zero slice of the sphere spectrum $s_0^\Sigma(\mathbf{1})$ is given by the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$ (see [26] and [16, Theorem 10.5.1]), which is not trivial. So we have a contradiction.

Remark 3.4.14. — Theorems 3.4.5 and 3.4.12 establish formally the compatibility between the smash product of symmetric T -spectra and Voevodsky’s slice filtration. In the remaining sections we will see that they have remarkable consequences; for instance if A is a cofibrant ring spectrum with unit in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$, then under mild conditions the slice filtration preserves A -module structures in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$.

3.5. Further Multiplicative Properties of the Slice Filtration

In this section A will always denote a cofibrant ring spectrum with unit in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$. Our goal is to use the motivic model structure $A\text{-mod}(\mathcal{M}_*)$ for the category of A -modules (see Section 2.8) together with the model structures for the category of symmetric T -spectra constructed in Section 3.3 (which provide a lifting of the slice filtration to the model category level), in order to get an analogue of the slice filtration for the category of A -modules. The main results of this section guarantee that under suitable conditions, the $(q - 1)$ -connective cover $f_q^\Sigma(M)$, $s_{<q}^\Sigma(M)$ and the q -slice $s_q^\Sigma(M)$ of an arbitrary A -module M in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$, inherit a natural structure of A -module in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$; and that the unit map $u : \mathbf{1} \rightarrow A$ satisfying some natural additional conditions, induces for every symmetric T -spectrum X a natural structure of A -module in $\mathrm{Spt}_T^\Sigma\mathcal{M}_*$ on its q -slice $s_q^\Sigma(X)$.

Let $\mathcal{H}(A\text{-mod})$ denote the homotopy category associated to $A\text{-mod}(\mathcal{M}_*)$. We call $\mathcal{H}(A\text{-mod})$ the *motivic stable homotopy category of A -modules*. We will denote by $[-, -]_m$ the set of maps between two objects in $\mathcal{H}(A\text{-mod})$, and $p, q \in \mathbb{Z}$ will be arbitrary integers.

Definition 3.5.1. — Let Q_m denote a cofibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$; such that for every A -module M , the natural map

$$Q_m M \xrightarrow{Q_m^M} M$$

is a trivial fibration in $A\text{-mod}(\mathcal{M}_*)$.

Definition 3.5.2. — Let R_m denote a fibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$; such that for every A -module M , the natural map

$$M \xrightarrow{R_m^M} R_m M$$

is a trivial cofibration in $A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.3. — The motivic stable homotopy category of A -modules $\mathcal{H}(A\text{-mod})$ has a structure of triangulated category defined as follows:

1. The suspension $\Sigma_T^{1,0}$ functor is given by

$$\begin{aligned} - \wedge S^1 : \mathcal{H}(A\text{-mod}) &\longrightarrow \mathcal{H}(A\text{-mod}) \\ M &\longmapsto Q_m M \wedge S^1 \end{aligned}$$

2. The distinguished triangles are isomorphic to triangles of the form

$$M \xrightarrow{i} N \xrightarrow{j} O \xrightarrow{k} \Sigma_T^{1,0} M$$

where i is a cofibration in $A\text{-mod}(\mathcal{M}_*)$, and O is the homotopy cofibre of i .

Proof. — By Proposition 2.8.10(2) we have that $A\text{-mod}(\mathcal{M}_*)$ is a pointed simplicial model category, and Theorem 2.8.12 implies that the adjunction:

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : A\text{-mod}(\mathcal{M}_*) \longrightarrow A\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence. The result now follows from the work of Quillen in [21, sections I.2 and I.3] and the work of Hovey in [10, chapters VI and VII] (see [10, Proposition 7.1.6]). □

Theorem 3.5.4. — The adjunction

$$(A \wedge -, U, \varphi) : \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow A\text{-mod}(\mathcal{M}_*)$$

defined in Proposition 2.8.1, induces an adjunction

$$(A \wedge Q_{\Sigma^-,} UR_m, \varphi) : \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}(A\text{-mod})$$

of exact functors between triangulated categories.

Proof. — The proof is exactly the same as in Theorem 3.3.4. We leave the details to the reader. □

Lemma 3.5.5. — Let $X \in \mathcal{M}_*$ be a pointed simplicial presheaf which is compact in the sense of Jardine (see definition 2.3.10), and let $F_n^\Sigma(X)$ be the symmetric T -spectrum

constructed in definition 2.6.8. Consider an arbitrary collection of A -modules $\{M_i\}_{i \in I}$ indexed by a set I . Then

$$[A \wedge F_n^\Sigma(X), \coprod_{i \in I} M_i]_m \cong \prod_{i \in I} [A \wedge F_n^\Sigma(X), M_i]_m$$

Proof. — The proof is exactly the same as in Lemma 3.3.6. We leave the details to the reader. \square

Proposition 3.5.6. — *The motivic stable homotopy category of A -modules $\mathcal{H}(A\text{-mod})$ is a compactly generated triangulated category in the sense of Neeman (see [19, definition 1.7]). The set of compact generators is given by (see definition 2.6.8):*

$$C^m = \bigcup_{n,r,s \geq 0} \bigcup_{U \in (\mathcal{M}|_S)} A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

i.e. the smallest triangulated subcategory of $\mathcal{H}(A\text{-mod})$ closed under small coproducts and containing all the objects in C^m coincides with $\mathcal{H}(A\text{-mod})$.

Proof. — The proof is exactly the same as in Proposition 3.3.7. We leave the details to the reader. \square

Corollary 3.5.7. — *Let $f : M \rightarrow N$ be a map in $\mathcal{H}(A\text{-mod})$. Then f is an isomorphism if and only if f induces an isomorphism of abelian groups:*

$$[A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), M]_m \xrightarrow{f_*} [A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), N]_m$$

for every $A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^m$.

Proof. — The proof is exactly the same as in Corollary 3.3.8. We leave the details to the reader. \square

In the rest of this section some results will be just stated without proof. In every case, the proof is exactly the same as the one given in Section 3.3, taking into consideration all that has been proved so far in this section together with proposition 2.8.2, the cellularity for the motivic model category of A -modules (see Theorem 2.8.11), and the fact that the generators $A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C^m$ are all cofibrant in $A\text{-mod}(\mathcal{M}_*)$ (this follows immediately from Theorem 2.8.4).

Theorem 3.5.8. — *Consider the following set of objects in $A\text{-mod}(\mathcal{M}_*)$ (see Theorem 3.2.1):*

$$C_{\text{eff}}^{q,m} = \bigcup_{n,r,s \geq 0; s-n \geq q} \bigcup_{U \in (\mathcal{M}|_S)} A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$$

The right Bousfield localization of $A\text{-mod}(\mathcal{M}_)$ with respect to the class of $C_{\text{eff}}^{q,m}$ -colocal equivalences exists (see definitions 1.8.6 and 1.9.2). This model structure will be called $(q-1)$ -connected motivic stable, and the category of A -modules equipped with the*

$(q - 1)$ -connected motivic stable model structure will be denoted by $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$. Furthermore $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$ is a right proper and simplicial model category. The homotopy category associated to $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$ will be denoted by $R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod})$.

Remark 3.5.9. — Notice that we can not use the adjunction

$$(A \wedge -, U, \varphi) : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$$

for the construction of $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, since we do not know if the model structure for $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is cofibrantly generated.

Definition 3.5.10. — Let C_q^m denote a cofibrant replacement functor in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$; such that for every A -module M , the natural map

$$C_q^m M \xrightarrow{C_q^{m,M}} M$$

is a trivial fibration in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, and $C_q^m M$ is always $C_{\text{eff}}^{q,m}$ -colocal in $A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.11. — R_m is also a fibrant replacement functor in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$ (see definition 3.5.2), and for every A -module M the natural map

$$M \xrightarrow{R_m^M} R_m M$$

is a trivial cofibration in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.12. — The adjunction

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \longrightarrow R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence, and $R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod})$ has the structure of a triangulated category.

Proposition 3.5.13. — We have the following Quillen adjunction

$$(\text{id}, \text{id}, \varphi) : R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \longrightarrow A\text{-mod}(\mathcal{M}_*)$$

which induces an adjunction

$$(C_q^m, R_m, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) \longrightarrow \mathcal{H}(A\text{-mod})$$

between exact functors of triangulated categories.

Theorem 3.5.14. — The adjunction (see Theorem 3.2.1)

$$(A \wedge -, U, \varphi) : R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$$

given by the free A -module and the forgetful functors is a Quillen adjunction, and it induces an adjunction

$$(A \wedge C_q^\Sigma -, UR_m, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod})$$

of exact functors between triangulated categories.

Proposition 3.5.15. — *We have the following commutative diagram of left Quillen functors:*

$$(67) \quad \begin{array}{ccccc} R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & & \\ \downarrow A \wedge - & \searrow \text{id} & \swarrow \text{id} & & \downarrow A \wedge - \\ & \text{Spt}_T^\Sigma \mathcal{M}_* & & & \\ R_{C_{\text{eff}}^{q+1}} A\text{-mod}(\mathcal{M}_*) & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) & & \\ & \searrow \text{id} & \swarrow \text{id} & & \\ & A\text{-mod}(\mathcal{M}_*) & & & \end{array}$$

and the following associated commutative diagrams of homotopy categories:

$$(68) \quad \begin{array}{ccccc} & & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & & \\ & \nearrow A \wedge C_q^\Sigma - & & \searrow C_q^m & \\ R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge C_q^\Sigma -} & \mathcal{H}(A\text{-mod}) & & \\ & \searrow C_q^\Sigma & \nearrow A \wedge Q_\Sigma - & & \\ & & \mathcal{H}^\Sigma(S) & & \end{array}$$

$$(69) \quad \begin{array}{ccccc} & & \mathcal{H}^\Sigma(S) & & \\ & \nearrow UR_m & & \searrow R_\Sigma & \\ \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & & \\ & \searrow R_m & \nearrow UR_m & & \\ & & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & & \end{array}$$

Theorem 3.5.16. — *We have the following commutative diagram of left Quillen functors:*

$$(70) \quad \begin{array}{ccccc} & \vdots & & \vdots & \\ & \text{id} \downarrow & & \text{id} \downarrow & \\ & R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{A \wedge -} & R_{C_{\text{eff}}^{q+1}} A\text{-mod}(\mathcal{M}_*) & \\ \text{id} \swarrow & \text{id} \downarrow & & \text{id} \downarrow & \text{id} \searrow \\ \text{Spt}_T^\Sigma \mathcal{M}_* & \xleftarrow{\text{id}} R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{A \wedge -} & R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) & \xrightarrow{\text{id}} A\text{-mod}(\mathcal{M}_*) \\ \text{id} \swarrow & \text{id} \downarrow & & \text{id} \downarrow & \text{id} \searrow \\ & R_{C_{\text{eff}}^{q-1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{A \wedge -} & R_{C_{\text{eff}}^{q-1}} A\text{-mod}(\mathcal{M}_*) & \\ & \text{id} \downarrow & & \text{id} \downarrow & \\ & \vdots & & \vdots & \end{array}$$

and the following associated commutative diagrams of homotopy categories:

$$(71) \quad \begin{array}{ccccc} & \vdots & & \vdots & \\ & \downarrow & & \downarrow & \\ & R_{C_{\text{eff}}^{q+1}} \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge C_{q+1}^\Sigma -} & R_{C_{\text{eff}}^{q+1}} \mathcal{H}(A\text{-mod}) & \\ C_{q+1}^\Sigma \swarrow & C_{q+1}^\Sigma \downarrow & & C_{q+1}^m \downarrow & C_{q+1}^m \searrow \\ \mathcal{H}^\Sigma(S) & \xleftarrow{C_q^\Sigma -} R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge C_q^\Sigma -} & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & \xrightarrow{C_q^m} \mathcal{H}(A\text{-mod}) \\ C_{q-1}^\Sigma \swarrow & C_q^\Sigma \downarrow & & C_q^m \downarrow & C_{q-1}^m \searrow \\ & R_{C_{\text{eff}}^{q-1}} \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge C_{q-1}^\Sigma -} & R_{C_{\text{eff}}^{q-1}} \mathcal{H}(A\text{-mod}) & \\ & \downarrow & & \downarrow & \\ & \vdots & & \vdots & \end{array}$$

$$(72) \quad \begin{array}{ccccc} & & \vdots & & \vdots \\ & & \uparrow & & \uparrow \\ & & R_{C_{\text{eff}}^{q+1}} \mathcal{H}^\Sigma(S) & \xleftarrow{UR_m} & R_{C_{\text{eff}}^{q+1}} \mathcal{H}(A\text{-mod}) \\ & \nearrow R_\Sigma & \uparrow R_\Sigma & & \uparrow R_m \\ \mathcal{H}^\Sigma(S) & \xrightarrow{-R_\Sigma} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \xleftarrow{UR_m} & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) \xleftarrow{R_m} \mathcal{H}(A\text{-mod}) \\ & \searrow R_\Sigma & \uparrow R_\Sigma & & \uparrow R_m \\ & & R_{C_{\text{eff}}^{q-1}} \mathcal{H}^\Sigma(S) & \xleftarrow{UR_m} & R_{C_{\text{eff}}^{q-1}} \mathcal{H}(A\text{-mod}) \\ & & \uparrow & & \uparrow \\ & & \vdots & & \vdots \end{array}$$

Theorem 3.5.17. — *If A is a commutative cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, then the symmetric monoidal structure for the category of A -modules, induces the following Quillen bifunctor in the sense of Hovey (see definition 1.7.4):*

$$- \wedge_A - : R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*) \times R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \longrightarrow R_{C_{\text{eff}}^{p+q}} A\text{-mod}(\mathcal{M}_*)$$

Proof. — The proof is similar to the one given for Theorem 3.4.5. We leave the details to the reader. □

If the ring A is not commutative, then we need to impose some additional conditions in order to get a weaker version of the previous result (see Theorem 3.5.21).

Lemma 3.5.18. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures.*

1. *The adjunction:*

$$(A' \wedge_A -, U, \varphi) : R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \rightarrow R_{C_{\text{eff}}^q} A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction.

2. *Furthermore, a map $w : M \rightarrow M'$ in $R_{C_{\text{eff}}^q} A'\text{-mod}(\mathcal{M}_*)$ is a weak equivalence if and only if Uw is a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$.*

Proof. — (1): Lemma 2.8.6 implies that $U : A'\text{-mod}(\mathcal{M}_*) \rightarrow A\text{-mod}(\mathcal{M}_*)$ is a right Quillen functor. Consider the following commutative diagram of right Quillen functors:

$$\begin{array}{ccc}
 A'\text{-mod}(\mathcal{M}_*) & \xrightarrow{U} & A\text{-mod}(\mathcal{M}_*) \\
 \text{id} \downarrow & & \downarrow \text{id} \\
 R_{\text{eff}}^{C^q} A'\text{-mod}(\mathcal{M}_*) & \xrightarrow{U} & R_{\text{eff}}^{C^q} A\text{-mod}(\mathcal{M}_*)
 \end{array}$$

then the universal property of right Bousfield localizations together with proposition 2.8.8 imply that the dotted arrow U is a right Quillen functor.

(2): Let $R_m, R_{m'}$ denote fibrant replacement functors in $A\text{-mod}(\mathcal{M}_*)$ and $A'\text{-mod}(\mathcal{M}_*)$ respectively, and let N be an arbitrary A' -module. We have the following commutative diagram in $A\text{-mod}(\mathcal{M}_*)$:

$$\begin{array}{ccc}
 N & \xrightarrow{R_m^N} & R_m N \\
 R_{m'}^N \downarrow & & \downarrow R_m(R_{m'}^N) \\
 R_{m'} N & \xrightarrow[R_{m'}^{R_{m'} N}]{} & R_m R_{m'} N
 \end{array}$$

Lemma 2.8.6 implies that all the maps in the diagram above are weak equivalences in $A\text{-mod}(\mathcal{M}_*)$.

Now fix $F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q, \Sigma}$. Using the naturality of the diagram above together with Proposition 2.8.8, we get the following commutative diagram of simplicial

sets:

$$\begin{array}{ccc}
 \text{Map}_{A'\text{-mod}}(A' \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_{m'}M) & & \\
 \cong \downarrow & \searrow^{(R_{m'}w)_*} & \\
 \text{Map}_{A'\text{-mod}}(A' \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_{m'}M') & & \\
 \cong \downarrow & & \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_{m'}M) & & \\
 \downarrow^{(R_m^{UR_{m'}M})_*} & \searrow^{(UR_{m'}w)_*} & \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_{m'}M') & & \\
 \downarrow^{(R_m UR_{m'}w)_*} & & \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_m UR_{m'}M) & & \\
 \downarrow^{(UR_m(R_{m'}^M))_*} & \searrow^{(R_m UR_{m'}w)_*} & \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), R_m UR_{m'}M') & & \\
 \downarrow^{(UR_m w)_*} & & \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_m M) & & \\
 \downarrow^{(UR_m(R_{m'}^M))_*} & & \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+), UR_m M') & &
 \end{array}$$

where the top vertical arrows are isomorphisms of simplicial sets. But $A\text{-mod}(\mathcal{M}_*)$, $A'\text{-mod}(\mathcal{M}_*)$ are simplicial model categories (see proposition 2.8.10(2)) and the natural maps $R_m^{UR_{m'}M}$, $UR_m(R_{m'}^M)$, $R_m^{UR_{m'}M'}$ and $UR_m(R_{m'}^M')$ are all weak equivalences between fibrant objects, thus by Ken Brown’s lemma (see Lemma 1.1.5) all the vertical arrows are weak equivalences of simplicial sets.

Therefore, the top row is a weak equivalence of simplicial sets if and only if the bottom row is a weak equivalence of simplicial sets. This proves the claim. \square

Proposition 3.5.19. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$, which is compatible with the ring structures. Assume that f is a weak equivalence in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$. Then the adjunction*

$$(A' \wedge_A -, U, \varphi) : R_{\text{off}}^q A\text{-mod}(\mathcal{M}_*) \rightarrow R_{\text{off}}^q A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence.

Proof. — We have shown in lemma 3.5.18(1) that

$$(A' \wedge_A -, U, \varphi) : R_{\text{off}}^q A\text{-mod}(\mathcal{M}_*) \rightarrow R_{\text{off}}^q A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction.

Now let η, ϵ denote the unit and counit of the adjunction $(A' \wedge_A -, U, \varphi)$. By Corollary 1.3.16(c) in [10], it suffices to check that the following conditions hold:

1. For every cofibrant A -module M in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, the following composition

$$M \cong A \wedge_A M \xrightarrow{\eta_M = f \wedge \text{id}} A' \wedge_A M \xrightarrow{R_{m'}^{A' \wedge_A M}} R_{m'}(A' \wedge_A M)$$

is a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, where $R_{m'}$ denotes a fibrant replacement functor in $A'\text{-mod}(\mathcal{M}_*)$ (see Proposition 3.5.11).

2. U reflects weak equivalences between fibrant objects in $R_{C_{\text{eff}}^q} A'\text{-mod}(\mathcal{M}_*)$.

(1): Since $\text{id} : R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \rightarrow A\text{-mod}(\mathcal{M}_*)$ is a left Quillen functor, we have that M is also cofibrant in $A\text{-mod}(\mathcal{M}_*)$. Hence, proposition 2.8.7 implies that $R_{m'}^{A' \wedge_A M} \circ \eta_M$ is a weak equivalence in $A\text{-mod}(\mathcal{M}_*)$. Finally, by [6, Proposition 3.1.5] we have that $R_{m'}^{A' \wedge_A M} \circ \eta_M$ is a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, as we wanted.

(2): This follows immediately from Lemma 3.5.18(2). □

Proposition 3.5.20. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also $C_{\text{eff}}^{0, \Sigma}$ -colocal (equivalently cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$). For every cofibration $f : M \rightarrow N$ in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, we have that f is also a cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — Let (see Theorem 2.8.11)

$$\begin{aligned} \overline{\Lambda(K)} &= J_{A\text{-mod}} \cup \{A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \partial \Delta^k \rightarrow \\ &A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \Delta^k \mid s - n \geq q, k \geq 0\} \end{aligned}$$

Since $A\text{-mod}(\mathcal{M}_*)$ is in particular a simplicial model category (see Proposition 2.8.10(2)), using definitions 5.2.1, 16.3.1 and Propositions 5.3.6, 16.1.3 in [6], we have that f is a retract of a cofibration $g : M \rightarrow O$ in $A\text{-mod}(\mathcal{M}_*)$ for which there is a weak equivalence $h : O \rightarrow P$ in $A\text{-mod}(\mathcal{M}_*)$ such that the composition $h \circ g$ is a relative $\overline{\Lambda(K)}$ -cell complex.

It is clear that it is enough to check that g is a cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Now, using Lemma 5.3.4 in [6], we have that this follows from:

1. g is a cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$.
2. h is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.
3. $h \circ g$ is a cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): This follows directly from Proposition 2.8.9.

(2): This follows directly from Theorem 2.8.4.

(3): Let \mathcal{C} denote the class of cofibrations in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Theorem 2.8.11 implies that $J_{A\text{-mod}}$ is a set of generating trivial cofibrations for $A\text{-mod}(\mathcal{M}_*)$, therefore

Proposition 2.8.9 and Theorem 2.8.4 imply that all the maps in $J_{A\text{-mod}}$ are trivial cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$. But $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization with respect to $\text{Spt}_T^\Sigma \mathcal{M}_*$, hence all the maps in $J_{A\text{-mod}}$ are also trivial cofibrations in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. We have that in particular

$$J_{A\text{-mod}} = A \wedge J_\Sigma^T = \{\text{id} \wedge j : A \wedge X \rightarrow A \wedge Y\}$$

is contained in \mathcal{C} . On the other hand, by hypothesis the map $* \rightarrow A$ is a cofibration in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$ and by construction $* \rightarrow F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ are cofibrations in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ for $s - n \geq q$. Then Theorem 3.4.5 together with the fact that $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category (see Theorem 3.3.9) imply that

$$\{A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \partial \Delta^k \rightarrow A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \Delta^k \mid s - n \geq q, k \geq 0\}$$

is also contained in \mathcal{C} . Therefore, we have that all the maps in $\overline{\Lambda(K)}$ are contained in \mathcal{C} .

Finally since limits and colimits in $A\text{-mod}$ are computed in $\text{Spt}_T^\Sigma(\mathcal{J}m|_S)_{\text{Nis}}$, we have that $h \circ g$ is a relative \mathcal{C} -cell complex in $\text{Spt}_T^\Sigma(\mathcal{J}m|_S)_{\text{Nis}}$, and since \mathcal{C} is clearly closed under coproducts, pushouts and filtered colimits, we have that $h \circ g$ is a cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. \square

Theorem 3.5.21. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$. Then $-\wedge_A-$ defines a Quillen adjunction of two variables (see definition 1.7.4) from the $(p-1)$ -connected motivic model structure for right A -modules and the $(q-1)$ -connected motivic model structure for left A -modules to the $(p+q-1)$ -connected motivic symmetric stable model structure:*

$$-\wedge_A- : R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)_r \times R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)_l \longrightarrow R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$$

Proof. — By Lemma 1.7.5, it is enough to prove the following claim:

Given a cofibration $i : N \rightarrow N'$ in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)_l$ and a fibration $f : X \rightarrow Y$ in $R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_*$, the induced map

$$\begin{array}{c} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N', X) \\ \downarrow (i^*, f_*) \\ \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N, X) \times \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N, Y) \times \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N', Y) \end{array}$$

is a fibration in $R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)_r$ which is trivial if either i or f is a weak equivalence.

However, Proposition 3.5.20 and Lemma 3.5.18(2) imply that i is also a cofibration in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, which is trivial if i is a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)_l$. Now, it follows from Theorem 3.4.5 that (i^*, f_*) is a fibration in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$ which is

trivial if either i or f is a weak equivalence. By Lemma 3.5.18(2) we have that it suffices to check that (i^*, f_*) is a fibration in $R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)_r$.

Since $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization with respect to $\text{Spt}_T^\Sigma \mathcal{M}_*$, we have that (i^*, f_*) is a fibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$; hence it follows from Theorem 2.8.4 that (i^*, f_*) is also a fibration in $A\text{-mod}(\mathcal{M}_*)_r$. However, by construction $R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)_r$ is a right Bousfield localization with respect to $A\text{-mod}(\mathcal{M}_*)_r$, therefore the classes of fibrations in both model structures are identical. Thus (i^*, f_*) is a fibration in $R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)_r$, as we wanted. \square

Theorem 3.5.22. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also $C_{\text{eff}}^{0,\Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$), and let M be an arbitrary A -module. Then the solid arrows in the following commutative diagram:*

$$(73) \quad \begin{array}{ccccccc} C_q^\Sigma R_m M & \xrightarrow{C_q^\Sigma(R_m^M)} & C_q^\Sigma M & \xleftarrow{C_q^\Sigma(C_q^m, M)} & C_q^\Sigma C_q^m M & \xrightarrow{C_q^\Sigma(R_m^{C_q^m M})} & C_q^\Sigma R_m C_q^m M \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_q^{\Sigma, R_m M} & & C_q^{\Sigma, M} & & C_q^{\Sigma, C_q^m M} & & C_q^{\Sigma, R_m C_q^m M} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R_m M & \dashrightarrow^{R_m^M} & M & \dashleftarrow^{C_q^m, M} & C_q^m M & \dashrightarrow^{R_m^{C_q^m M}} & R_m C_q^m M \end{array}$$

induce a natural equivalence between the functors:

$$(74) \quad \begin{array}{ccc} & \mathcal{H}(A\text{-mod}) & \\ & \nearrow^{C_q^m} & \searrow^{UR_m} \\ R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & & \mathcal{H}^\Sigma(S) \\ & \searrow^{UR_m} & \nearrow^{C_q^\Sigma} \\ & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \end{array}$$

Proof. — Clearly it suffices to show that $C_q^\Sigma(R_m^M)$, $C_q^\Sigma(C_q^m, M)$, $C_q^\Sigma(R_m^{C_q^m M})$ and $C_q^{\Sigma, R_m C_q^m M}$ are all weak equivalences in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proposition 3.5.11 implies that R_m^M is a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, then applying lemma 3.5.18(2) to the unit map $\mathbf{1} \rightarrow A$ we have that R_m^M is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. By construction $C_q^{\Sigma, R_m M}$ and $C_q^{\Sigma, M}$ are both weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Hence, the two out of three property for weak equivalences implies that $C_q^\Sigma(R_m^M)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. However, $C_q^\Sigma R_m M$ and $C_q^\Sigma M$ are both $C_{\text{eff}}^{q,\Sigma}$ -colocal; therefore [6, Theorem 3.2.13(2)] implies that $C_q^\Sigma(R_m^M)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Using Lemma 3.5.18(2) again, we have that $C_q^{m,M}$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. But $C_q^{\Sigma,M}$ and $C_q^{\Sigma,C_q^m M}$ are both weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, thus the two out of three property for weak equivalences implies that $C_q^{\Sigma}(C_q^{m,M})$ is also a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. However, by construction $C_q^{\Sigma} M$ and $C_q^{\Sigma} C_q^m M$ are both $C_{\text{eff}}^{q,\Sigma}$ -colocal; hence by [6, Theorem 3.2.13(2)] we have that $C_q^{\Sigma}(C_q^{m,M})$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

By Proposition 3.5.11 we have that $R_m^{C_q^m M}$ is a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$, then lemma 3.5.18(2) implies that $R_m^{C_q^m M}$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Now, $C_q^{\Sigma,C_q^m M}$ and $C_q^{\Sigma,R_m C_q^m M}$ are both weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, the two out of three property for weak equivalences implies that $C_q^{\Sigma}(R_m^{C_q^m M})$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. However, by construction, $C_q^{\Sigma} C_q^m M$ and $C_q^{\Sigma} R_m C_q^m M$ are both $C_{\text{eff}}^{q,\Sigma}$ -colocal; therefore [6, Theorem 3.2.13(2)] implies that $C_q^{\Sigma}(R_m^{C_q^m M})$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

We already know that $C_q^{\Sigma}(R_m^{C_q^m M})$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and definition 3.5.2 together with Theorem 2.8.4 imply that $R_m^{C_q^m M}$ is also a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore, to show that $C_q^{\Sigma,R_m C_q^m M}$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, it suffices to check that $C_q^{\Sigma,C_q^m M}$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Now, by construction we have that $C_q^{\Sigma,C_q^m M}$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and that $C_q^{\Sigma} C_q^m M$ is a $C_{\text{eff}}^{q,\Sigma}$ -colocal spectrum, thus by [6, Theorem 3.2.13(2)] it only remains to show that $C_q^m M$ is $C_{\text{eff}}^{q,\Sigma}$ -colocal. But this follows from our hypothesis which says that A is $C_{\text{eff}}^{0,\Sigma}$ -colocal together with Proposition 3.5.20. This finishes the proof. \square

Remark 3.5.23. — Notice that Theorem 3.5.22 does not follow from the general nonsense (see Proposition 3.5.15), since it gives a compatibility between compositions of left and right adjoints.

Theorem 3.5.24. — Fix $q \in \mathbb{Z}$. Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Assume that there exists $p \in \mathbb{Z}$ such that A, A' are both $C_{\text{eff}}^{p,\Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and f is a weak equivalence in $R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently in $R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)$). Then f induces a Quillen equivalence between the $(q-1)$ -connected motivic stable model structures of A and A' modules:

$$(A' \wedge_A -, U, \varphi) : R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \longrightarrow R_{C_{\text{eff}}^q} A'\text{-mod}(\mathcal{M}_*)$$

Proof. — Since A and A' are $C_{\text{eff}}^{p,\Sigma}$ -colocal, [6, Theorem 3.2.13(2)] implies that f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore, the result follows directly from proposition 3.5.19. \square

Theorem 3.5.25. — Consider the following set of maps in $A\text{-mod}(\mathcal{M}_*)$ (see Theorem 3.3.26):

$$(75) \quad \begin{aligned} L^m(< q) = \{ \text{id} \wedge V(\iota_{n,r,s}^U) : A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \rightarrow \\ A \wedge F_n^\Sigma(D^{r+1} \wedge \mathbb{G}_m^s \wedge U_+) \mid F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in C_{\text{eff}}^{q,\Sigma} \} \end{aligned}$$

The left Bousfield localization of $A\text{-mod}(\mathcal{M}_*)$ with respect to the $L^m(< q)$ -local equivalences exists. This new model structure will be called $\text{weight}^{< q}$ motivic stable. $L_{< q}A\text{-mod}(\mathcal{M}_*)$ will denote the category of A -modules equipped with the $\text{weight}^{< q}$ motivic stable model structure, and $L_{< q}\mathcal{H}(A\text{-mod})$ will denote its associated homotopy category. Furthermore the $\text{weight}^{< q}$ motivic stable model structure is cellular, left proper and simplicial; with the following sets of generating cofibrations and trivial cofibrations respectively:

$$I_{L^m(< q)} = I_{A\text{-mod}} = \bigcup_{n \geq 0} \{ A \wedge F_n^\Sigma(Y_+ \hookrightarrow (\Delta_U^n)_+) \}$$

$$J_{L^m(< q)} = \{ j : A \rightarrow B \}$$

where j satisfies the following conditions:

1. j is an inclusion of $I_{A\text{-mod}}$ -complexes.
2. j is a $L^m(< q)$ -local equivalence.
3. the size of B as an $I_{A\text{-mod}}$ -complex is less than κ , where κ is the regular cardinal defined by Hirschhorn in [6, definition 4.5.3].

Remark 3.5.26. — Notice that the model category $L_{< q}\text{Spt}_T^\Sigma \mathcal{M}_*$ is not a symmetric monoidal model category (see Remark 3.4.13), i.e. the smash product and the model structure are not compatible, therefore in general it is not possible to use the adjunction

$$(A \wedge -, U, \varphi) : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \rightarrow A\text{-mod}$$

for the construction of a model structure on the category of A -modules. However, if A satisfies additional conditions (see Proposition 3.5.41), then the adjunction above induces a model structure on the category of A -modules which coincides with $L_{< q}A\text{-mod}(\mathcal{M}_*)$ (see Proposition 3.5.41 and Theorem 3.5.44).

Definition 3.5.27. — Let W_q^m denote a fibrant replacement functor in $L_{< q}A\text{-mod}(\mathcal{M}_*)$; such that for every A -module M , the natural map:

$$M \xrightarrow{W_q^{m,M}} W_q^m M$$

is a trivial cofibration in $L_{< q}A\text{-mod}(\mathcal{M}_*)$, and $W_q^m M$ is $L^m(< q)$ -local in $A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.28. — Q_m is also a cofibrant replacement functor in $L_{<q}A\text{-mod}(\mathcal{M}_*)$ (see definition 3.5.1), and for every A -module M the natural map

$$Q_m M \xrightarrow{Q_m^M} M$$

is a trivial fibration in $L_{<q}A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.29. — Let M be an A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$. We have that M is $L^m(<q)$ -local in $A\text{-mod}(\mathcal{M}_*)$ if and only if UM is $L^\Sigma(<q)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proposition 3.5.30. — The adjunction:

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : L_{<q}A\text{-mod}(\mathcal{M}_*) \longrightarrow L_{<q}A\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence, and the homotopy category $L_{<q}\mathcal{H}(A\text{-mod})$ associated to $L_{<q}A\text{-mod}(\mathcal{M}_*)$ has the structure of a triangulated category.

Corollary 3.5.31. — $L_{<q}A\text{-mod}(\mathcal{M}_*)$ is a right proper model category.

Proposition 3.5.32. — We have the following Quillen adjunction:

$$(\text{id}, \text{id}, \varphi) : A\text{-mod}(\mathcal{M}_*) \longrightarrow L_{<q}A\text{-mod}(\mathcal{M}_*)$$

which induces an adjunction:

$$(Q_m, W_q^m, \varphi) : \mathcal{H}(A\text{-mod}) \longrightarrow L_{<q}\mathcal{H}(A\text{-mod})$$

of exact functors between triangulated categories.

Theorem 3.5.33. — The adjunction:

$$(A \wedge -, U, \varphi) : L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow L_{<q}A\text{-mod}(\mathcal{M}_*)$$

given by the free A -module and the forgetful functor is a Quillen adjunction, and it induces an adjunction:

$$(A \wedge Q_s -, UW_q^m, \varphi) : L_{<q}\mathcal{H}^\Sigma(S) \longrightarrow L_{<q}\mathcal{H}(A\text{-mod})$$

of exact functors between triangulated categories.

Proposition 3.5.34. — *We have the following commutative diagram of left Quillen functors:*

$$(76) \quad \begin{array}{ccccc} & & \text{Spt}_T^\Sigma \mathcal{M}_* & & \\ & \swarrow \text{id} & \downarrow A \wedge - & \searrow \text{id} & \\ L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & & \xrightarrow{\text{id}} & L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_* \\ & \downarrow A \wedge - & & \downarrow A \wedge - & \\ & & A\text{-mod}(\mathcal{M}_*) & & \\ & \swarrow \text{id} & & \searrow \text{id} & \\ L_{<q+1} A\text{-mod}(\mathcal{M}_*) & \xrightarrow{\text{id}} & & \xrightarrow{\text{id}} & L_{<q} A\text{-mod}(\mathcal{M}_*) \end{array}$$

and the following associated commutative diagrams of homotopy categories:

$$(77) \quad \begin{array}{ccccc} & & \mathcal{H}(A\text{-mod}) & & \\ & \swarrow A \wedge Q_{\Sigma-} & & \searrow Q_m & \\ \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge Q_{\Sigma-}} & & \xrightarrow{} & L_{<q} \mathcal{H}(A\text{-mod}) \\ & \searrow Q_\Sigma & & \swarrow A \wedge Q_{\Sigma-} & \\ & & L_{<q} \mathcal{H}^\Sigma(S) & & \end{array}$$

$$(78) \quad \begin{array}{ccccc} & & \mathcal{H}(A\text{-mod}) & & \\ & \swarrow W_q^m & & \searrow UR_m & \\ L_{<q} \mathcal{H}(A\text{-mod}) & \xrightarrow{UW_q^m} & & \xrightarrow{} & \mathcal{H}^\Sigma(S) \\ & \searrow UW_q^m & & \swarrow W_q^\Sigma & \\ & & L_{<q} \mathcal{H}^\Sigma(S) & & \end{array}$$

Theorem 3.5.35. — *We have the following commutative diagram of left Quillen functors:*

$$(79) \quad \begin{array}{ccccc} & \vdots & & \vdots & \\ & \text{id} \downarrow & & \text{id} \downarrow & \\ & L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{A \wedge -} L_{<q+1} A\text{-mod}(\mathcal{M}_*) & & & \\ \text{id} \nearrow & \text{id} \downarrow & & \text{id} \downarrow & \text{id} \nwarrow \\ \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{\text{id}} & L_{<q} \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{A \wedge -} L_{<q} A\text{-mod}(\mathcal{M}_*) & \xleftarrow{\text{id}} & A\text{-mod}(\mathcal{M}_*) & \\ \text{id} \searrow & \text{id} \downarrow & & \text{id} \downarrow & \text{id} \swarrow \\ & L_{<q-1} \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{A \wedge -} L_{<q-1} A\text{-mod}(\mathcal{M}_*) & & & \\ & \text{id} \downarrow & & \text{id} \downarrow & \\ & \vdots & & \vdots & \end{array}$$

and the following associated commutative diagrams of homotopy categories:

$$(80) \quad \begin{array}{ccccc} & \vdots & & \vdots & \\ & \downarrow & & \downarrow & \\ & L_{<q+1} \mathcal{H}^\Sigma(S) \xrightarrow{A \wedge_{Q_\Sigma} -} L_{<q+1} \mathcal{H}(A\text{-mod}) & & & \\ Q_\Sigma \nearrow & Q_\Sigma \downarrow & & Q_m \downarrow & Q_m \nwarrow \\ \mathcal{H}^\Sigma(S) \xrightarrow{Q_\Sigma} & L_{<q} \mathcal{H}^\Sigma(S) \xrightarrow{A \wedge_{Q_\Sigma} -} L_{<q} \mathcal{H}(A\text{-mod}) & \xleftarrow{Q_m} & \mathcal{H}(A\text{-mod}) & \\ Q_\Sigma \searrow & Q_\Sigma \downarrow & & Q_m \downarrow & Q_m \swarrow \\ & L_{<q-1} \mathcal{H}^\Sigma(S) \xrightarrow{A \wedge_{Q_\Sigma} -} L_{<q-1} \mathcal{H}(A\text{-mod}) & & & \\ & \downarrow & & \downarrow & \\ & \vdots & & \vdots & \end{array}$$

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow \\
 & & L_{<q+1}\mathcal{H}^\Sigma(S) & \xleftarrow{UW_{q+1}^m} & L_{<q+1}\mathcal{H}(A\text{-mod}) \\
 & \swarrow W_{q+1}^\Sigma & \uparrow W_q^\Sigma & & \uparrow W_q^m & \searrow W_{q+1}^m \\
 (81) \quad & \mathcal{H}^\Sigma(S) & \xleftarrow{W_q^\Sigma} & L_{<q}\mathcal{H}^\Sigma(S) & \xleftarrow{UW_q^m} & L_{<q}\mathcal{H}(A\text{-mod}) & \xrightarrow{-W_q^m} & \mathcal{H}(A\text{-mod}) \\
 & \swarrow W_{q-1}^\Sigma & \uparrow W_{q-1}^\Sigma & & \uparrow W_{q-1}^m & \searrow W_{q-1}^m \\
 & & L_{<q-1}\mathcal{H}^\Sigma(S) & \xleftarrow{UW_{q-1}^m} & L_{<q-1}\mathcal{H}(A\text{-mod}) \\
 & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots
 \end{array}$$

Lemma 3.5.36. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. If $g : M \rightarrow N$ is a $L^m(< q)$ -local equivalence in $A\text{-mod}(\mathcal{M}_*)$, then $\text{id} \wedge_A Q_m g : A' \wedge_A Q_m M \rightarrow A' \wedge_A Q_m N$ is a $L^{m'}(< q)$ -local equivalence in $A'\text{-mod}(\mathcal{M}_*)$, where Q_m denotes a cofibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$.*

Proof. — Let Z be an arbitrary $L^{m'}(< q)$ -local A' -module in $A'\text{-mod}(\mathcal{M}_*)$. Lemma 2.8.6 implies that $A' \wedge_A Q_m M, A' \wedge_A Q_m N$ are both cofibrant in $A'\text{-mod}(\mathcal{M}_*)$. Therefore it suffices to show that the induced map

$$\text{Map}_{A'\text{-mod}}(A' \wedge_A Q_m N, Z) \xrightarrow{(\text{id} \wedge_A Q_m g)^*} \text{Map}_{A'\text{-mod}}(A' \wedge_A Q_m M, Z)$$

is a weak equivalence of simplicial sets. However, using Proposition 2.8.8 we get the following commutative diagram, where the vertical maps are isomorphisms of simplicial sets

$$\begin{array}{ccc}
 \text{Map}_{A'\text{-mod}}(A' \wedge_A Q_m N, Z) & \xrightarrow{(\text{id} \wedge_A Q_m g)^*} & \text{Map}_{A'\text{-mod}}(A' \wedge_A Q_m M, Z) \\
 \cong \downarrow & & \downarrow \cong \\
 \text{Map}_{A\text{-mod}}(Q_m N, UZ) & \xrightarrow{(Q_m g)^*} & \text{Map}_{A\text{-mod}}(Q_m M, UZ)
 \end{array}$$

Finally, Proposition 3.5.29 implies that UZ is $L^m(< q)$ -local in $A\text{-mod}(\mathcal{M}_*)$, therefore the bottom row is a weak equivalence of simplicial sets, since by hypothesis g is a $L^m(< q)$ -local equivalence in $A\text{-mod}(\mathcal{M}_*)$. Hence, the two out of three property for

weak equivalences implies that the top row is also a weak equivalence of simplicial sets, as we wanted. \square

Proposition 3.5.37. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Then the adjunction:*

$$(A' \wedge_A -, U, \varphi) : L_{<q} A\text{-mod}(\mathcal{M}_*) \rightarrow L_{<q} A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction.

Proof. — Lemma 2.8.6 implies that $A' \wedge_A - : A\text{-mod}(\mathcal{M}_*) \rightarrow A'\text{-mod}(\mathcal{M}_*)$ is a left Quillen functor. Consider the following commutative diagram of left Quillen functors:

$$\begin{array}{ccc} A\text{-mod}(\mathcal{M}_*) & \xrightarrow{A' \wedge_A -} & A'\text{-mod}(\mathcal{M}_*) \\ \text{id} \downarrow & & \downarrow \text{id} \\ L_{<q} A\text{-mod}(\mathcal{M}_*) & \xrightarrow{A' \wedge_A -} & L_{<q} A'\text{-mod}(\mathcal{M}_*) \end{array}$$

then the universal property of left Bousfield localizations together with lemma 3.5.36 imply that the dotted arrow $A' \wedge_A -$ is a left Quillen functor. \square

Lemma 3.5.38. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. If f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently in $A\text{-mod}(\mathcal{M}_*)$), then for every $L^m(<q)$ -local A -module M in $A\text{-mod}(\mathcal{M}_*)$, we have that $Q_m M$ and $UR_{m'}(A' \wedge_A Q_m M)$ are also $L^m(<q)$ -local in $A\text{-mod}(\mathcal{M}_*)$, where Q_m denotes a cofibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$ and $R_{m'}$ denotes a fibrant replacement functor in $A'\text{-mod}(\mathcal{M}_*)$.*

Proof. — Since M is $L^m(<q)$ -local, it follows that M is fibrant in $A\text{-mod}(\mathcal{M}_*)$. By definition we have that the natural map

$$Q_m M \xrightarrow{Q_m^M} M$$

is a trivial fibration in $A\text{-mod}(\mathcal{M}_*)$, therefore $Q_m M$ is also fibrant in $A\text{-mod}(\mathcal{M}_*)$. Hence [6, Lemma 3.2.1(a)] implies that $Q_m M$ is $L^m(<q)$ -local. Proposition 2.8.7 implies that the adjunction $(A' \wedge_A -, U, \varphi)$ is a Quillen equivalence between $A\text{-mod}(\mathcal{M}_*)$ and $A'\text{-mod}(\mathcal{M}_*)$, therefore we have that $UR_{m'}(A' \wedge_A Q_m M)$ is fibrant in $A\text{-mod}(\mathcal{M}_*)$, and [10, Proposition 1.3.13(b)] implies that the composition

$$Q_m M \xrightarrow{\eta_{Q_m M}} U(A' \wedge_A Q_m M) \xrightarrow{U(R_{m'}^{A' \wedge_A Q_m M})} UR_{m'}(A' \wedge_A Q_m M)$$

is a weak equivalence in $A\text{-mod}(\mathcal{M}_*)$. Since we already know that $Q_m M$ is $L^m(<q)$ -local, using [6, Lemma 3.2.1(a)] again we get that $UR_{m'}(A' \wedge_A Q_m M)$ is also $L^m(<q)$ -local in $A\text{-mod}(\mathcal{M}_*)$. This finishes the proof. \square

Lemma 3.5.39. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. If f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently in $A\text{-mod}(\mathcal{M}_*)$); then $g : M \rightarrow N$ is a $L^m(< q)$ -local equivalence in $A\text{-mod}(\mathcal{M}_*)$ if and only if $\text{id} \wedge_A Q_m g : A' \wedge_A Q_m M \rightarrow A' \wedge_A Q_m N$ is a $L^{m'}(< q)$ -local equivalence in $A'\text{-mod}(\mathcal{M}_*)$, where Q_m denotes a cofibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$.*

Proof. — (\Rightarrow): It follows directly from Lemma 3.5.36.

(\Leftarrow): Assume that $\text{id} \wedge_A Q_m g$ is a $L^{m'}(< q)$ -local equivalence in $A'\text{-mod}(\mathcal{M}_*)$, and let Z be an arbitrary $L^m(< q)$ -local A -module in $A\text{-mod}(\mathcal{M}_*)$. We need to show that the induced map:

$$\text{Map}_{A\text{-mod}}(Q_m N, Z) \xrightarrow{(Q_m g)^*} \text{Map}_{A\text{-mod}}(Q_m M, Z)$$

is a weak equivalence of simplicial sets.

But Proposition 2.8.7 implies that the adjunction $(A' \wedge_A -, U, \varphi)$ is a Quillen equivalence between $A\text{-mod}(\mathcal{M}_*)$ and $A'\text{-mod}(\mathcal{M}_*)$, therefore using [10, Proposition 1.3.13(b)] we have that all the maps in the following diagram are weak equivalences in $A\text{-mod}(\mathcal{M}_*)$:

$$Z \xleftarrow{Q_m^Z} Q_m Z \xrightarrow{U(R_{m'}^{A' \wedge_A Q_m Z}) \circ \eta_{Q_m Z}} UR_{m'}(A' \wedge_A Q_m Z)$$

where $R_{m'}$ denotes a fibrant replacement functor in $A'\text{-mod}(\mathcal{M}_*)$. Lemma 3.5.38 implies in particular that $Z, Q_m Z, UR_{m'}(A' \wedge_A Q_m Z)$ are all fibrant in $A\text{-mod}(\mathcal{M}_*)$. Now using the fact that $A\text{-mod}(\mathcal{M}_*)$ is a simplicial model category together with Ken Brown’s lemma (see Lemma 1.1.5) and the two out of three property for weak equivalences, we have that it suffices to prove that the induced map:

$$\begin{array}{c} \text{Map}_{A\text{-mod}}(Q_m N, UR_{m'}(A' \wedge_A Q_m Z)) \\ \downarrow (Q_m g)^* \\ \text{Map}_{A\text{-mod}}(Q_m M, UR_{m'}(A' \wedge_A Q_m Z)) \end{array}$$

is a weak equivalence of simplicial sets. Using the enriched adjunctions of proposition 2.8.8, we get the following commutative diagram where all the vertical arrows are

isomorphisms:

$$\begin{array}{ccc}
 \text{Map}_{A\text{-mod}}(Q_m N, UR_{m'}(A' \wedge_A Q_m Z)) & & \\
 \downarrow \cong & \searrow^{(Q_m g)^*} & \\
 \text{Map}_{A\text{-mod}}(Q_m M, UR_{m'}(A' \wedge_A Q_m Z)) & & \\
 \downarrow \cong & & \\
 \text{Map}_{A'\text{-mod}}(A' \wedge_A Q_m N, R_{m'}(A' \wedge_A Q_m Z)) & \searrow^{(\text{id} \wedge_A Q_m g)^*} & \\
 & & \text{Map}_{A'\text{-mod}}(A' \wedge_A Q_m M, R_{m'}(A' \wedge_A Q_m Z))
 \end{array}$$

Finally, Lemma 3.5.38 implies that $UR_{m'}(A' \wedge_A Q_m Z)$ is $L^{m'}(< q)$ -local in $A\text{-mod}(\mathcal{M}_*)$, therefore by Proposition 3.5.29 we have that $R_{m'}(A' \wedge_A Q_m Z)$ is $L^{m'}(< q)$ -local in $A'\text{-mod}(\mathcal{M}_*)$. Since $\text{id} \wedge_A Q_m g$ is a $L^{m'}(< q)$ -local equivalence and $A' \wedge_A Q_m M, A' \wedge_A Q_m N$ are both cofibrant in $A'\text{-mod}(\mathcal{M}_*)$, it follows that the bottom row in the diagram above is a weak equivalence of simplicial sets. This implies that the top row is also a weak equivalence of simplicial sets, as we wanted. \square

Proposition 3.5.40. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. If f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, then the adjunction:*

$$(A' \wedge_A -, U, \varphi) : L_{< q} A\text{-mod}(\mathcal{M}_*) \rightarrow L_{< q} A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence.

Proof. — Proposition 3.5.37 implies that the adjunction $(A' \wedge_A -, U, \varphi)$ is a Quillen adjunction. Using Corollary 1.3.16 in [10] and Proposition 3.5.28 we have that it suffices to verify the following two conditions:

1. For every fibrant A' -module M in $L_{< q} A'\text{-mod}(\mathcal{M}_*)$, the following composition

$$A' \wedge_A Q_m U M \xrightarrow{\text{id} \wedge_A (Q_m^{UM})} A' \wedge_A U M \xrightarrow{\epsilon_M} M$$

is a weak equivalence in $L_{< q} A'\text{-mod}(\mathcal{M}_*)$, where Q_m denotes a cofibrant replacement functor in $A\text{-mod}(\mathcal{M}_*)$ (see proposition 3.5.28).

2. $A' \wedge_A -$ reflects weak equivalences between cofibrant A -modules in $L_{< q} A\text{-mod}(\mathcal{M}_*)$.

(1): By construction $L_{< q} A'\text{-mod}(\mathcal{M}_*)$ is a left Bousfield localization of $A'\text{-mod}(\mathcal{M}_*)$, therefore the identity functor

$$\text{id} : L_{< q} A'\text{-mod}(\mathcal{M}_*) \rightarrow A'\text{-mod}(\mathcal{M}_*)$$

is a right Quillen functor. Thus M is also fibrant in A' -mod(\mathcal{M}_*). Proposition 2.8.7 implies that the adjunction $(A' \wedge_A -, U, \varphi)$ is a Quillen equivalence between A -mod(\mathcal{M}_*) and A' -mod(\mathcal{M}_*), hence using [10, Proposition 1.3.13(b)] we have that the following composition is a weak equivalence in A' -mod(\mathcal{M}_*):

$$A' \wedge_A Q_m U M \xrightarrow{\text{id} \wedge_A (Q_m^{UM})} A' \wedge_A U M \xrightarrow{\epsilon_M} M$$

Therefore [6, Proposition 3.1.5] implies that the composition above is a $L^{m'}(< q)$ -local equivalence.

(2): This follows immediately from Proposition 3.5.28 and Lemma 3.5.39. □

Proposition 3.5.41. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$. The adjunction:*

$$(A \wedge -, U, \varphi) : \text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}} \rightarrow A\text{-mod}$$

between symmetric T -spectra and A -modules, together with the model structure $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see Theorem 3.3.26), induces a model structure on A -mod, which we will denote by $\widetilde{L}_{< q} A\text{-mod}(\mathcal{M}_*)$; i.e. a map $f : M \rightarrow N$ of A -modules is a fibration or a weak equivalence in $\widetilde{L}_{< q} A\text{-mod}(\mathcal{M}_*)$ if and only if Uf is a fibration or a weak equivalence in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Furthermore, the model category $\widetilde{L}_{< q} A\text{-mod}(\mathcal{M}_*)$ is cofibrantly generated, with the following sets of generating cofibrations and trivial cofibrations respectively:

$$\begin{aligned} I_{L^{m}(< q)} &= I_{A\text{-mod}} = A \wedge I_\Sigma^T \\ &= \bigcup_{k \geq 0} \{ \text{id} \wedge i : A \wedge F_k^\Sigma(Y_+) \rightarrow A \wedge F_k^\Sigma((\Delta_n^U)_+) \mid U \in (\mathcal{M}|_S), n \geq 0 \} \\ \widetilde{J}_{L^{m}(< q)} &= \{ \text{id} \wedge j : A \wedge X \rightarrow A \wedge Y \} \end{aligned}$$

where $j : X \rightarrow Y$ satisfies the following conditions:

1. j is an inclusion of I_Σ^T -complexes in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$.
2. j is a $L^\Sigma(< q)$ -local equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.
3. the size of Y as an I_Σ^T -complex is less than κ , where κ is the regular cardinal defined by Hirschhorn in [6, definition 4.5.3].

Proof. — Using a result of D. Kan (see Theorem 11.3.2 in [6]), we have that it is enough to prove that the following conditions hold:

1. The domains of $I_{L^{m}(< q)}$ (respectively $\widetilde{J}_{L^{m}(< q)}$) are small relative to the $I_{L^{m}(< q)}$ -cells (respectively $\widetilde{J}_{L^{m}(< q)}$ -cells) in the category of A -modules.
2. U maps relative $\widetilde{J}_{L^{m}(< q)}$ -cell complexes to weak equivalences in $L_{< q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): By adjointness it suffices to check that the domains of $I_{L^\Sigma(<q)}$ (respectively $J_{L^\Sigma(<q)}$) are small relative to the $I_{L^m(<q)}$ -cells (respectively $\widetilde{J_{L^m(<q)}}$ -cells) in $\text{Spt}_T^\Sigma(\mathcal{A}|_S)_{\text{Nis}}$. Theorem 3.3.26 implies that $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$ is in particular a cofibrantly generated model category with the sets $I_{L^\Sigma(<q)}$ and $J_{L^\Sigma(<q)}$ as generating cofibrations and trivial cofibrations, therefore by [10, Proposition 2.1.16] it only remains to show that all the maps in $I_{L^m(<q)}$ -cells (respectively $\widetilde{J_{L^m(<q)}}$ -cells) are cofibrations (respectively trivial cofibrations) in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$.

Since A is in particular cofibrant in $\text{Spt}_T^\Sigma\mathcal{M}_*$ and the cofibrations in $\text{Spt}_T^\Sigma\mathcal{M}_*$ and $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$ are identical, Proposition 2.6.28 implies that all the maps in $I_{L^m(<q)}$ are cofibrations in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$. However, the class of cofibrations is closed under coproducts and filtered colimits, and the limits and colimits in the category of A -modules are computed in $\text{Spt}_T^\Sigma(\mathcal{A}|_S)_{\text{Nis}}$, hence all the maps in $I_{L^m(<q)}$ -cells are cofibrations in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$.

By hypothesis A is cofibrant in $S^0\text{Spt}_T^\Sigma\mathcal{M}_*$, and every map j in $J_{L^\Sigma(<q)}$ is clearly a trivial cofibration in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$. Since $S^{q-1}\text{Spt}_T^\Sigma\mathcal{M}_*$ is a right Bousfield localization with respect to $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$, we have that every map j in $J_{L^\Sigma(<q)}$ is also a trivial cofibration in $S^{q-1}\text{Spt}_T^\Sigma\mathcal{M}_*$. Therefore, Theorem 3.4.12 implies that all the maps in $\widetilde{J_{L^m(<q)}}$ are trivial cofibrations in $S^{q-1}\text{Spt}_T^\Sigma\mathcal{M}_*$, and since $S^{q-1}\text{Spt}_T^\Sigma\mathcal{M}_*$ is a right Bousfield localization with respect to $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$; we get that all the maps in $\widetilde{J_{L^m(<q)}}$ are also trivial cofibrations in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$. Finally, since the class of trivial cofibrations is closed under coproducts and filtered colimits, and the limits and colimits in the category of A -modules are computed in $\text{Spt}_T^\Sigma(\mathcal{A}|_S)_{\text{Nis}}$, we have that all the maps in $\widetilde{J_{L^m(<q)}}$ -cells are also trivial cofibrations in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$.

(2): We have shown that every map in $\widetilde{J_{L^m(<q)}}$ -cells is a trivial cofibration in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$. In particular, every relative $\widetilde{J_{L^m(<q)}}$ -cell complex is a weak equivalence in $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$, as we wanted. \square

Remark 3.5.42. — Notice that we can not use the same argument as in Theorem 2.8.4 to construct the model structure $\widetilde{L_{<q}A\text{-mod}(\mathcal{M}_*)}$, since the model category $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$ is not a symmetric monoidal model category (see Remark 3.4.13), i.e. the monoidal structure on symmetric T -spectra is not compatible with the model structure on $L_{<q}\text{Spt}_T^\Sigma\mathcal{M}_*$. Therefore, the hypothesis of A being cofibrant in $S^0\text{Spt}_T^\Sigma\mathcal{M}_*$ is really necessary.

Lemma 3.5.43. — Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma\mathcal{M}_*$, which is also cofibrant in $S^0\text{Spt}_T^\Sigma\mathcal{M}_*$. Then the model category $\widetilde{L_{<q}A\text{-mod}(\mathcal{M}_*)}$ described in Proposition 3.5.41 is simplicial.

Proof. — Since the cotensor objects N^K for the simplicial structure are identical in $\widetilde{L}_{<q}A\text{-mod}(\mathcal{M}_*)$ and $L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$, the results follows from proposition 3.5.41 and theorem 3.3.26 which implies in particular that $L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category. □

Theorem 3.5.44. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$. We have that the model structures $L_{<q}A\text{-mod}(\mathcal{M}_*)$ (see Theorem 3.5.25) and $\widetilde{L}_{<q}A\text{-mod}(\mathcal{M}_*)$ (see Proposition 3.5.41) on the category of A -modules are identical.*

Proof. — Theorem 3.5.25 and Proposition 3.5.41 imply that both $L_{<q}A\text{-mod}(\mathcal{M}_*)$ and $\widetilde{L}_{<q}A\text{-mod}(\mathcal{M}_*)$ have

$$\bigcup_{k \geq 0} \{ \text{id} \wedge i : A \wedge F_k^\Sigma(Y_+) \rightarrow A \wedge F_k^\Sigma((\Delta_n^U)_+) \mid U \in (\mathcal{J}m|_S), n \geq 0 \}$$

as set of generating cofibrations. Hence the cofibrations in $L_{<q}A\text{-mod}(\mathcal{M}_*)$ and $\widetilde{L}_{<q}A\text{-mod}(\mathcal{M}_*)$ are exactly the same. It suffices to check that the weak equivalences in both model structures are identical.

However, Theorem 3.3.26 and Lemma 3.5.43 imply that $L_{<q}A\text{-mod}(\mathcal{M}_*)$ and $\widetilde{L}_{<q}A\text{-mod}(\mathcal{M}_*)$ are both simplicial model categories. Therefore, corollary 1.6.11(2) implies that it is enough to show that the fibrant objects in $L_{<q}A\text{-mod}(\mathcal{M}_*)$ and $\widetilde{L}_{<q}A\text{-mod}(\mathcal{M}_*)$ coincide. But this follows directly from Propositions 3.5.29 and 3.5.41. □

Theorem 3.5.45. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, and let M be an arbitrary A -module. Then the solid arrows in the following commutative diagram:*

$$(82) \quad \begin{array}{ccccccc} Q_\Sigma R_m M & \xleftarrow{Q_\Sigma(R_m^M)} & Q_\Sigma M & \xleftarrow{Q_\Sigma(Q_m^M)} & Q_\Sigma Q_m M & \xleftarrow{Q_\Sigma(W_q^m, Q_m M)} & Q_\Sigma W_q^m Q_m M \\ \downarrow Q_\Sigma^{R_m M} & & \downarrow Q_\Sigma^M & & \downarrow Q_\Sigma^{Q_m M} & & \downarrow Q_\Sigma^{W_q^m, Q_m M} \\ R_m M & \xleftarrow{R_m^M} & M & \xleftarrow{Q_m^M} & Q_m M & \xrightarrow{W_q^m, Q_m M} & W_q^m Q_m M \end{array}$$

induce a natural equivalence between the functors:

$$(83) \quad \begin{array}{ccc} & L_{<q}\mathcal{H}(A\text{-mod}) & \\ \mathcal{H}(A\text{-mod}) \swarrow^{Q_m} & & \searrow^{UW_q^m} \\ & & L_{<q}\mathcal{H}^\Sigma(S) \\ \mathcal{H}(A\text{-mod}) \searrow_{UR_m} & & \swarrow_{Q_\Sigma} \\ & \mathcal{H}^\Sigma(S) & \end{array}$$

Proof. — It suffices to show that all the maps $W_q^{m,Q_m M}$, Q_m^M , R_m^M and $Q_\Sigma^{R_m M}$ are weak equivalences in $L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$. Proposition 3.3.28 implies that $Q_\Sigma^{R_m M}$ is a weak equivalence in $L_{<q}\text{Spt}_T^\Sigma \mathcal{M}_*$.

Since A is cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, Theorem 3.5.44 and Proposition 3.5.41 imply that it is enough to show that $W_q^{m,Q_m M}$, Q_m^M and R_m^M are weak equivalences in $L_{<q}A\text{-mod}(\mathcal{M}_*)$. By construction (see definition 3.5.27) $W_q^{m,Q_m M}$ is a weak equivalence in $L_{<q}A\text{-mod}(\mathcal{M}_*)$, and Proposition 3.5.28 implies that Q_m^M is a weak equivalence in $L_{<q}A\text{-mod}(\mathcal{M}_*)$. Finally, by construction (see definition 3.5.2) R_m^M is a weak equivalence in $A\text{-mod}(\mathcal{M}_*)$, and [6, Proposition 3.1.5] implies that R_m^M is also a weak equivalence in $L_{<q}A\text{-mod}(\mathcal{M}_*)$. This finishes the proof. \square

Remark 3.5.46. — Notice that Theorem 3.5.45 does not follow from the general nonsense (see Proposition 3.5.34), since it gives a compatibility between compositions of left and right adjoints.

Theorem 3.5.47. — Fix $q \in \mathbb{Z}$. Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Assume that one of the following conditions holds:

1. f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently in $A\text{-mod}(\mathcal{M}_*)$).
2. There exists $p \in \mathbb{Z}$ such that A, A' are both $L^\Sigma(<p)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and f is a weak equivalence in $L_{<p}\text{Spt}_T^\Sigma \mathcal{M}_*$.
3. There exists $p \in \mathbb{Z}$ such that A, A' are both $C_{\text{eff}}^{p,\Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and f is a weak equivalence in $R_{C_{\text{eff}}^p}\text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently in $R_{C_{\text{eff}}^p}A\text{-mod}(\mathcal{M}_*)$).

Then f induces a Quillen equivalence between the weight $<q$ motivic stable model structures of A and A' modules:

$$(A' \wedge_A -, U, \varphi) : L_{<q}A\text{-mod}(\mathcal{M}_*) \longrightarrow L_{<q}A'\text{-mod}(\mathcal{M}_*)$$

Proof. — (1): This is just proposition 3.5.40.

(2): Since A and A' are $L^\Sigma(< p)$ -local in $\text{Spt}_T^\Sigma \mathcal{M}_*$, [6, Theorem 3.2.13(1)] implies that f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore the result follows from Proposition 3.5.40.

(3): Since A and A' are $C_{\text{eff}}^{p, \Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$, using [6, Theorem 3.2.13(2)] we have that f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, the result follows from Proposition 3.5.40. □

Definition 3.5.48. — We consider the following set of A -modules

$$S^m(q) = \{A \wedge F_n^\Sigma(S^r \wedge G_m^s \wedge U_+) \in C^m \mid s - n = q\} \subseteq C_{\text{eff}}^{q, m}$$

(see definition 3.3.49).

Theorem 3.5.49. — The right Bousfield localization of the model category $L_{< q+1} A\text{-mod}(\mathcal{M}_*)$ with respect to the $S^m(q)$ -colocal equivalences exists. This new model structure will be called q -slice motivic stable. $S^q A\text{-mod}(\mathcal{M}_*)$ will denote the category of A -modules equipped with the q -slice motivic stable model structure, and $S^q \mathcal{H}(A\text{-mod})$ will denote its associated homotopy category. Furthermore, the q -slice motivic stable model structure is right proper and simplicial.

Remark 3.5.50. — Notice that we can not use the adjunction $(A \wedge -, U, \varphi) : S^q \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow S^q A\text{-mod}(\mathcal{M}_*)$ for the construction of $S^q A\text{-mod}(\mathcal{M}_*)$, since we do not know if the model structure for $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is cofibrantly generated.

Definition 3.5.51. — Let P_q^m denote a cofibrant replacement functor in $S^q A\text{-mod}(\mathcal{M}_*)$; such that for every A -module M , the natural map

$$P_q^m M \xrightarrow{P_q^{m, M}} M$$

is a trivial fibration in $S^q A\text{-mod}(\mathcal{M}_*)$, and $P_q^m M$ is always a $S^m(q)$ -colocal A -module in $L_{< q+1} A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.52. — W_{q+1}^m is also a fibrant replacement functor in $S^q A\text{-mod}(\mathcal{M}_*)$ (see definition 3.5.27), and for every A -module M the natural map

$$M \xrightarrow{W_{q+1}^{m, M}} W_{q+1}^m M$$

is a trivial cofibration in $S^q A\text{-mod}(\mathcal{M}_*)$.

Corollary 3.5.53. — Let $f : M \rightarrow N$ be a map of A -modules. Then f is a $S^m(q)$ -colocal equivalence in $L_{< q+1} A\text{-mod}(\mathcal{M}_*)$ if and only if

$$W_{q+1}^m M \xrightarrow{W_{q+1}^m f} W_{q+1}^m N$$

is a $C_{\text{eff}}^{q, m}$ -colocal equivalence in $A\text{-mod}(\mathcal{M}_*)$.

Proposition 3.5.54. — *The adjunction*

$$(- \wedge S^1, \Omega_{S^1}, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \longrightarrow S^q A\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence, and $S^q \mathcal{H}(A\text{-mod})$ has the structure of a triangulated category.

Proposition 3.5.55. — *We have the following adjunction*

$$(P_q^m, W_{q+1}^m, \varphi) : S^q \mathcal{H}(A\text{-mod}) \longrightarrow L_{<q+1} \mathcal{H}(A\text{-mod})$$

between exact functors of triangulated categories.

Proposition 3.5.56. — *The identity functor*

$$\text{id} : S^q A\text{-mod}(\mathcal{M}_*) \longrightarrow R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$$

is a right Quillen functor, and it induces the following adjunction

$$(C_q^m, W_{q+1}^m, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) \longrightarrow S^q \mathcal{H}(A\text{-mod})$$

of exact functors between triangulated categories.

Lemma 3.5.57. — *If M is a cofibrant A -module in $S^q A\text{-mod}(\mathcal{M}_*)$, then the map $* \rightarrow M$ is a trivial cofibration in $L_{<q} A\text{-mod}(\mathcal{M}_*)$.*

Theorem 3.5.58. — *The adjunction*

$$(A \wedge -, U, \varphi) : S^q \text{Spt}_T^\Sigma \mathcal{M}_* \longrightarrow S^q A\text{-mod}(\mathcal{M}_*)$$

given by the free A -module and the forgetful functors is a Quillen adjunction, and it induces an adjunction

$$(A \wedge P_q^\Sigma -, UW_{q+1}^m, \varphi) : S^q \mathcal{H}^\Sigma(S) \longrightarrow S^q \mathcal{H}(A\text{-mod})$$

of exact functors between triangulated categories.

Proposition 3.5.59. — *We have the following commutative diagram of left Quillen functors:*

$$(84) \quad \begin{array}{ccc} R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{A \wedge -} & R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*) \\ \text{id} \downarrow & & \downarrow \text{id} \\ S^q \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{A \wedge -} & S^q A\text{-mod}(\mathcal{M}_*) \end{array}$$

and the following associated commutative diagrams of homotopy categories:

$$(85) \quad \begin{array}{ccc} & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & \\ A \wedge C_q^\Sigma - \nearrow & & \searrow C_q^m \\ R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \xrightarrow{A \wedge C_q^\Sigma -} & S^q \mathcal{H}(A\text{-mod}) \\ C_q^\Sigma \searrow & & \nearrow A \wedge P_q^\Sigma - \\ & S^q \mathcal{H}^\Sigma(S) & \end{array}$$

$$(86) \quad \begin{array}{ccc} & S^q \mathcal{H}^\Sigma(S) & \\ UW_{q+1}^m \nearrow & & \searrow W_{q+1}^\Sigma \\ S^q \mathcal{H}(A\text{-mod}) & \xrightarrow{UW_{q+1}^m} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \\ W_{q+1}^m \searrow & & \nearrow UR_m \\ & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & \end{array}$$

Theorem 3.5.60. — Let A be a commutative cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Then the symmetric monoidal structure for the category of A -modules, induces the following Quillen bifunctor in the sense of Hovey (see definition 1.7.4):

$$- \wedge_A - : S^p A\text{-mod}(\mathcal{M}_*) \times S^q A\text{-mod}(\mathcal{M}_*) \longrightarrow S^{p+q} A\text{-mod}(\mathcal{M}_*)$$

Proof. — The proof is similar to the one given for Theorem 3.4.12. We leave the details to the reader. □

If the ring A is not commutative, then we need to impose some additional conditions in order to get a weaker version of the previous result (see Theorem 3.5.65).

Lemma 3.5.61. — Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Then the adjunction:

$$(A' \wedge_A -, U, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \rightarrow S^q A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction.

Proof. — Proposition 3.5.37 implies that $U : L_{<q+1} A'\text{-mod}(\mathcal{M}_*) \rightarrow L_{<q+1} A\text{-mod}(\mathcal{M}_*)$ is a right Quillen functor. Consider the following commutative diagram of right Quillen

functors:

$$\begin{array}{ccc}
 L_{<q+1}A'\text{-mod}(\mathcal{M}_*) & \xrightarrow{U} & L_{<q+1}A\text{-mod}(\mathcal{M}_*) \\
 \text{id} \downarrow & & \downarrow \text{id} \\
 S^q A'\text{-mod}(\mathcal{M}_*) & \dashrightarrow^U & S^q A\text{-mod}(\mathcal{M}_*)
 \end{array}$$

then the universal property of right Bousfield localizations together with proposition 2.8.8 imply that the dotted arrow U is a right Quillen functor. \square

Proposition 3.5.62. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. If f is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, then the adjunction:*

$$(A' \wedge_A -, U, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \rightarrow S^q A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence.

Proof. — We have shown in lemma 3.5.61 that

$$(A' \wedge_A -, U, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \rightarrow S^q A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction.

Now let η, ϵ denote the unit and counit of the adjunction $(A' \wedge_A -, U, \varphi)$. By Corollary 1.3.16(c) in [10], it suffices to check that the following conditions hold:

1. For every cofibrant A -module M in $S^q A\text{-mod}(\mathcal{M}_*)$, the following composition

$$M \cong A \wedge_A M \xrightarrow{\eta_M = f \wedge_A \text{id}} A' \wedge_A M \xrightarrow{W_{q+1}^{m', A' \wedge_A M}} W_{q+1}^{m'}(A' \wedge_A M)$$

is a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$, where $W_{q+1}^{m'}$ denotes a fibrant replacement functor in $S^q A'\text{-mod}(\mathcal{M}_*)$ (see Proposition 3.5.52).

2. U reflects weak equivalences between fibrant objects in $S^q A'\text{-mod}(\mathcal{M}_*)$.

(1): Since $\text{id} : S^q A\text{-mod}(\mathcal{M}_*) \rightarrow L_{<q+1}A\text{-mod}(\mathcal{M}_*)$ is a left Quillen functor, we have that M is also cofibrant in $L_{<q+1}A\text{-mod}(\mathcal{M}_*)$. Hence, theorem 3.5.47(1) implies that $W_{q+1}^{m', A' \wedge_A M} \circ \eta_M$ is a weak equivalence in $L_{<q+1}A\text{-mod}(\mathcal{M}_*)$. Finally, by [6, Proposition 3.1.5] we have that $W_{q+1}^{m', A' \wedge_A M} \circ \eta_M$ is a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$, as we wanted.

(2): Let $g : M \rightarrow N$ be a map between fibrant A' -modules in $S^q A'\text{-mod}(\mathcal{M}_*)$, such that Ug is a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$.

Fix $F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^\Sigma(q)$ (see definition 3.3.49). Using the enriched adjunctions of Proposition 2.8.8, we get the following commutative diagram of simplicial sets

where the vertical arrows are isomorphisms

$$\begin{array}{ccc}
 \text{Map}_{A'\text{-mod}}(A' \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), M) & & \\
 \cong \downarrow & \searrow^{g_*} & \\
 & \text{Map}_{A'\text{-mod}}(A' \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), N) & \\
 \downarrow & & \cong \downarrow \\
 \text{Map}_{A\text{-mod}}(A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), UM) & & \\
 & \searrow^{(Ug)_*} & \\
 & \text{Map}_{A\text{-mod}}(A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+), UN) &
 \end{array}$$

Now M and N are both fibrant in $L_{<q+1}A'\text{-mod}(\mathcal{M}_*)$ (this follows from Proposition 3.5.52), hence Proposition 3.5.37 implies that UM and UN are also fibrant in $L_{<q+1}A\text{-mod}(\mathcal{M}_*)$. Therefore, the bottom row in the diagram above is a weak equivalence of simplicial sets, since by hypothesis Ug is a weak equivalence in $S^qA\text{-mod}(\mathcal{M}_*)$. Finally, by the two out of three property for weak equivalences we get that the top row is also a weak equivalence of simplicial sets, and this implies that g is a weak equivalence in $S^qA'\text{-mod}(\mathcal{M}_*)$, since M and N are both fibrant in $L_{<q+1}A'\text{-mod}(\mathcal{M}_*)$. \square

Lemma 3.5.63. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Assume that A and A' are cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$. Then $w : M \rightarrow M'$ is a weak equivalence in $S^qA'\text{-mod}(\mathcal{M}_*)$ if and only if Uw is a weak equivalence in $S^qA\text{-mod}(\mathcal{M}_*)$.*

Proof. — Let $W_{q+1}^m, W_{q+1}^{m'}$ denote fibrant replacement functors in $L_{<q+1}A\text{-mod}(\mathcal{M}_*)$ and $L_{<q+1}A'\text{-mod}(\mathcal{M}_*)$ respectively, and let N be an arbitrary A' -module. We have the following commutative diagram in $L_{<q+1}A\text{-mod}(\mathcal{M}_*)$:

$$\begin{array}{ccc}
 N & \xrightarrow{W_{q+1}^{m,N}} & W_{q+1}^m N \\
 W_{q+1}^{m',N} \downarrow & & \downarrow W_{q+1}^m(W_{q+1}^{m',N}) \\
 W_{q+1}^{m'} N & \xrightarrow{W_{q+1}^{m,W_{q+1}^{m'} N}} & W_{q+1}^m W_{q+1}^{m'} N
 \end{array}$$

since A, A' are both cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, Theorem 3.5.44 and Proposition 3.5.41 imply that all the maps in the diagram above are weak equivalences in $L_{<q+1}A\text{-mod}(\mathcal{M}_*)$.

Now fix $A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \in S^m(q)$ (see definition 3.5.48). Using the naturality of the diagram above together with Proposition 2.8.8, we get the following

Proof. — Let (see Theorem 3.5.25)

$$\overline{\Lambda(K)} = J_{L^m(\langle q+1 \rangle)} \cup \{A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \partial \Delta^k \rightarrow A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \Delta^k \mid s - n = q, k \geq 0\}$$

Since $L_{\langle q+1 \rangle}A\text{-mod}(\mathcal{M}_*)$ is in particular a simplicial model category (see Theorem 3.5.25), using definitions 5.2.1, 16.3.1 and Propositions 5.3.6, 16.1.3 in [6], we have that f is a retract of a cofibration $g : M \rightarrow O$ in $L_{\langle q+1 \rangle}A\text{-mod}(\mathcal{M}_*)$ for which there is a weak equivalence $h : O \rightarrow P$ in $L_{\langle q+1 \rangle}A\text{-mod}(\mathcal{M}_*)$ such that the composition $h \circ g$ is a relative $\overline{\Lambda(K)}$ -cell complex.

It is clear that it is enough to check that g is a cofibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$. Now, using Lemma 5.3.4 in [6], we have that this follows from:

1. g is a cofibration in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$.
2. h is a weak equivalence in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$.
3. $h \circ g$ is a cofibration in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$.

(1): Since $L_{\langle q+1 \rangle}A\text{-mod}(\mathcal{M}_*)$ is a left Bousfield localization with respect to $A\text{-mod}(\mathcal{M}_*)$, we have that the cofibrations are exactly the same in both model structures. Hence g is a cofibration in $A\text{-mod}(\mathcal{M}_*)$, and Proposition 2.8.9 implies that g is also a cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$. But $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization with respect to $\text{Spt}_T^\Sigma \mathcal{M}_*$, therefore g is a cofibration in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$.

(2): Since A is cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, Theorem 3.5.44 and Proposition 3.5.41 imply that h is a weak equivalence in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$.

(3): Let \mathcal{C} denote the class of cofibrations in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$. Theorem 3.5.25 implies that $J_{L^m(\langle q+1 \rangle)}$ is a set of generating trivial cofibrations for $L_{\langle q+1 \rangle}A\text{-mod}(\mathcal{M}_*)$, and since A is cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, theorem 3.5.44 together with Proposition 3.5.41 imply that all the maps in $J_{L^m(\langle q+1 \rangle)}$ are weak equivalences in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$.

Now, $L_{\langle q+1 \rangle}A\text{-mod}(\mathcal{M}_*)$ is a left Bousfield localization with respect to $A\text{-mod}(\mathcal{M}_*)$, thus all the maps in $J_{L^m(\langle q+1 \rangle)}$ are cofibrations in $A\text{-mod}(\mathcal{M}_*)$, and Proposition 2.8.9 implies that the maps in $J_{L^m(\langle q+1 \rangle)}$ are also cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$. However, $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization with respect to $\text{Spt}_T^\Sigma \mathcal{M}_*$, hence all the maps in $J_{L^m(\langle q+1 \rangle)}$ are cofibrations in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$.

Therefore, all the maps in $J_{L^m(\langle q+1 \rangle)}$ are trivial cofibrations in $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$. But $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization with respect to $L_{\langle q+1 \rangle}\text{Spt}_T^\Sigma \mathcal{M}_*$, hence all the maps in $J_{L^m(\langle q+1 \rangle)}$ are also trivial cofibrations in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$. We have that in particular $J_{L^m(\langle q+1 \rangle)}$ is contained in \mathcal{C} . On the other hand, by construction $* \rightarrow F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ are cofibrations in $R_{C^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ for $s - n = q$, thus, Proposition 3.3.61 implies that $* \rightarrow F_n(S^r \wedge \mathbb{G}_m^s \wedge U_+)$ are also cofibrations in $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ for $s - n = q$. By hypothesis the map $* \rightarrow A$ is a cofibration in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, then Theorem 3.4.12 together with the fact that $S^q\text{Spt}_T^\Sigma \mathcal{M}_*$ is a simplicial model category

(see Theorem 3.3.50) imply that

$$\{A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \partial \Delta^k \rightarrow A \wedge F_n^\Sigma(S^r \wedge \mathbb{G}_m^s \wedge U_+) \otimes \Delta^k \mid s - n = q, k \geq 0\}$$

is also contained in \mathcal{C} . Therefore, we have that all the maps in $\overline{\Lambda(K)}$ are contained in \mathcal{C} .

Finally since limits and colimits in $A\text{-mod}$ are computed in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$, we have that $h \circ g$ is a relative \mathcal{C} -cell complex in $\text{Spt}_T^\Sigma(\mathcal{M}|_S)_{\text{Nis}}$, and since \mathcal{C} is clearly closed under coproducts, pushouts and filtered colimits, we have that $h \circ g$ is a cofibration in $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$. \square

Theorem 3.5.65. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma\mathcal{M}_*$, which is also cofibrant in $S^0\text{Spt}_T^\Sigma\mathcal{M}_*$. Then $-\wedge_A-$ defines a Quillen adjunction of two variables (see definition 1.7.4) from the p -slice motivic model structure for right A -modules and the q -slice motivic model structure for left A -modules to the $(p+q)$ -slice motivic symmetric stable model structure:*

$$-\wedge_A- : S^pA\text{-mod}(\mathcal{M}_*)_r \times S^qA\text{-mod}(\mathcal{M}_*)_l \longrightarrow S^{p+q}\text{Spt}_T^\Sigma\mathcal{M}_*$$

Proof. — By Lemma 1.7.5, it is enough to prove the following claim:

Given a cofibration $i : N \rightarrow N'$ in $S^qA\text{-mod}(\mathcal{M}_*)_l$ and a fibration $f : X \rightarrow Y$ in $S^{p+q}\text{Spt}_T^\Sigma\mathcal{M}_*$, the induced map

$$\begin{array}{c} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N', X) \\ \downarrow (i^*, f_*) \\ \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N, X) \times_{\mathbf{Hom}_{\text{Spt}_T^\Sigma}(N, Y)} \mathbf{Hom}_{\text{Spt}_T^\Sigma}(N', Y) \end{array}$$

is a fibration in $S^pA\text{-mod}(\mathcal{M}_*)_r$ which is trivial if either i or f is a weak equivalence.

However, Proposition 3.5.64 and Lemma 3.5.63 (1 is cofibrant in $S^0\text{Spt}_T^\Sigma\mathcal{M}_*$ by Lemma 3.4.1) imply that i is also a cofibration in $S^q\text{Spt}_T^\Sigma\mathcal{M}_*$, which is trivial if i is a weak equivalence in $S^qA\text{-mod}(\mathcal{M}_*)_l$. Now, it follows from Theorem 3.4.12 that (i^*, f_*) is a fibration in $S^p\text{Spt}_T^\Sigma\mathcal{M}_*$ which is trivial if either i or f is a weak equivalence. By Lemma 3.5.63 we have that it suffices to check that (i^*, f_*) is a fibration in $S^pA\text{-mod}(\mathcal{M}_*)_r$.

By definition $S^p\text{Spt}_T^\Sigma\mathcal{M}_*$ is a right Bousfield localization with respect to $L_{<p+1}\text{Spt}_T^\Sigma\mathcal{M}_*$, hence the fibrations in both model structures coincide. This implies that (i^*, f_*) is a fibration in $L_{<p+1}\text{Spt}_T^\Sigma\mathcal{M}_*$. Now, proposition 3.5.41 and Theorem 3.5.44 imply that (i^*, f_*) is also a fibration in $L_{<p+1}A\text{-mod}(\mathcal{M}_*)_r$ since we are assuming that A is cofibrant in $S^0\text{Spt}_T^\Sigma\mathcal{M}_*$. However, by construction $S^pA\text{-mod}(\mathcal{M}_*)_r$ is a right Bousfield localization with respect to $L_{<p+1}A\text{-mod}(\mathcal{M}_*)_r$,

therefore the classes of fibrations in both model structures are identical. Thus (i^*, f_*) is a fibration in $S^p A\text{-mod}(\mathcal{M}_*)_r$, as we wanted. □

Lemma 3.5.66. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Assume that A and A' are cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$. Furthermore, assume that A' is also cofibrant in $A\text{-mod}(\mathcal{M}_*)$. If f is a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, then for every cofibrant A -module M in $S^q A\text{-mod}(\mathcal{M}_*)$, the induced map*

$$M \cong A \wedge_A M \xrightarrow{f \wedge_A \text{id}} A' \wedge_A M$$

is a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$.

Proof. — Lemma 3.4.1 implies that $\mathbf{1}$ is cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$ and A is by hypothesis cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, thus by Lemma 3.5.63 it suffices to check that $f \wedge_A \text{id}$ is a weak equivalence in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

Using Lemma 3.5.63 again, we get that f is a weak equivalence in $S^0 A\text{-mod}(\mathcal{M}_*)$. Now, M is cofibrant in $S^q A\text{-mod}(\mathcal{M}_*)$ and $f : A \rightarrow A'$ may be considered as a map of right A -modules; therefore theorem 3.5.65 together with Ken Brown’s lemma (see Lemma 1.1.4) imply that it suffices to show that A and A' are both cofibrant in $S^0 A\text{-mod}(\mathcal{M}_*)$.

We have that $\mathbf{1}$ is cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$ by Lemma 3.4.1, therefore Theorem 3.5.58 implies that A is cofibrant in $S^0 A\text{-mod}(\mathcal{M}_*)$.

Now, since $S^0 A\text{-mod}(\mathcal{M}_*)$ is a right Bousfield localization with respect to $L_{<1} A\text{-mod}(\mathcal{M}_*)$, [6, Proposition 3.2.2(2)] implies that to show that A' is cofibrant in $S^0 A\text{-mod}(\mathcal{M}_*)$ it suffices to check that A' is cofibrant in $L_{<1} A\text{-mod}(\mathcal{M}_*)$ and that f is a weak equivalence in $L_{<1} A\text{-mod}(\mathcal{M}_*)$.

On the other hand, $L_{<1} A\text{-mod}(\mathcal{M}_*)$ is a left Bousfield localization with respect to $A\text{-mod}(\mathcal{M}_*)$, hence A' is cofibrant in $L_{<1} A\text{-mod}(\mathcal{M}_*)$ since by hypothesis A' is cofibrant in $A\text{-mod}(\mathcal{M}_*)$.

Finally, we are assuming that f is a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$ and that A, A' are both cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$; therefore, [6, Theorem 3.2.13(2)] implies that f is also a weak equivalence in $L_{<1} \text{Spt}_T^\Sigma \mathcal{M}_*$. But since A is cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, we can apply Proposition 3.5.41 and Theorem 3.5.44 to conclude that f is a weak equivalence in $L_{<1} A\text{-mod}(\mathcal{M}_*)$, as we wanted. □

Proposition 3.5.67. — *Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is compatible with the ring structures. Assume that A and A' are cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$. Furthermore, assume that A' is also cofibrant in $A\text{-mod}(\mathcal{M}_*)$. If f is a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, then it induces a Quillen equivalence between*

the q -slice motivic stable model structures of A and A' modules:

$$(A' \wedge_A -, U, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \longrightarrow S^q A'\text{-mod}(\mathcal{M}_*)$$

Proof. — We have shown in lemma 3.5.61 that

$$(A' \wedge_A -, U, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \rightarrow S^q A'\text{-mod}(\mathcal{M}_*)$$

is a Quillen adjunction.

Now let η, ϵ denote the unit and counit of the adjunction $(A' \wedge_A -, U, \varphi)$. By Corollary 1.3.16(c) in [10], it suffices to check that the following conditions hold:

1. For every cofibrant A -module M in $S^q A\text{-mod}(\mathcal{M}_*)$, the following composition

$$M \cong A \wedge_A M \xrightarrow{\eta_M = f \wedge_A \text{id}} A' \wedge_A M \xrightarrow{W_{q+1}^{m', A' \wedge_A M}} W_{q+1}^{m'}(A' \wedge_A M)$$

is a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$, where $W_{q+1}^{m'}$ denotes a fibrant replacement functor in $S^q A'\text{-mod}(\mathcal{M}_*)$ (see Proposition 3.5.52).

2. U reflects weak equivalences between fibrant objects in $S^q A'\text{-mod}(\mathcal{M}_*)$.

(1): Lemma 3.5.66 implies that $f \wedge_A \text{id}$ is a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$, and Lemma 3.5.63 implies that $W_{q+1}^{m', A' \wedge_A M}$ is also a weak equivalence in $S^q A\text{-mod}(\mathcal{M}_*)$. Therefore, the result follows from the two out of three property for weak equivalences.

- (2): This follows immediately from Lemma 3.5.63. □

Theorem 3.5.68. — *Let A be a cofibrant ring spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$, which is also cofibrant in $R_{\text{eff}} C_0 \text{Spt}_T^\Sigma \mathcal{M}_*$, and let M be an arbitrary A -module. Then the solid arrows in the following commutative diagram:*

$$(87) \quad \begin{array}{ccccccc} C_q^\Sigma W_{q+1}^m C_q^m M & \xleftarrow{C_q^\Sigma(W_{q+1}^{m, C_q^m M})} & C_q^\Sigma C_q^m M & \xrightarrow{C_q^\Sigma(C_q^m, M)} & C_q^\Sigma M & \xrightarrow{C_q^\Sigma(R_m^M)} & C_q^\Sigma R_m M \\ \downarrow C_q^{\Sigma, W_{q+1}^m, C_q^m M} & & \downarrow C_q^{\Sigma, C_q^m M} & & \downarrow C_q^{\Sigma, M} & & \downarrow C_q^{\Sigma, R_m M} \\ W_{q+1}^m C_q^m M & \xleftarrow{W_{q+1}^{m, C_q^m M}} & C_q^m M & \xrightarrow{C_q^{m, M}} & M & \xrightarrow{R_m^M} & R_m M \end{array}$$

induce a natural equivalence between the functors:

$$(88) \quad \begin{array}{ccc} & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \\ UR_m \nearrow & & \searrow C_q^\Sigma \\ R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & & S^q \mathcal{H}^\Sigma(S) \\ C_q^\Sigma \searrow & & \nearrow UW_{q+1}^m \\ & S^q \mathcal{H}(A\text{-mod}) & \end{array}$$

Proof. — Clearly, it is enough to prove that the maps $W_{q+1}^{m, C_q^m M}$, $C_q^{\Sigma, C_q^m M}$, $C_q^\Sigma(C_q^{m, M})$ and $C_q^\Sigma(R_m^M)$ are all weak equivalences in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

Lemma 3.4.1 implies that $\mathbf{1}$ is cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, and Proposition 3.3.61 implies that A is also cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$.

Now, Proposition 3.5.52 implies that W_{q+1}^m is a fibrant replacement functor in $S^q A\text{-mod}(\mathcal{M}_*)$, then using Lemma 3.5.63 we get that $W_{q+1}^{m, C_q^m M}$ is a weak equivalence in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$.

By construction $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization with respect to $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$, and on the other hand, $L_{< q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization with respect to $\text{Spt}_T^\Sigma \mathcal{M}_*$. Hence, [6, Proposition 3.1.5] implies that it suffices to show that the remaining maps $C_q^{\Sigma, C_q^m M}$, $C_q^\Sigma(C_q^{m, M})$ and $C_q^\Sigma(R_m^M)$ are weak equivalences in $\text{Spt}_T^\Sigma \mathcal{M}_*$. We will show that this is the case.

Since A is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, Proposition 3.5.20 implies that $C_q^m M$ is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, and $C_q^{\Sigma, C_q^m M}$ is by definition a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$; therefore [6, Theorem 3.2.13(2)] implies that $C_q^{\Sigma, C_q^m M}$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, since $C_q^\Sigma C_q^m M$ is also cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Since $C_q^\Sigma C_q^m M$ and $C_q^\Sigma M$ are both cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ by construction, using Theorem 3.2.13(2) in [6] we get that if $C_q^\Sigma(C_q^{m, M})$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, then it is also a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. But it is clear that $C_q^{\Sigma, C_q^m M}$ and $C_q^{\Sigma, M}$ are both weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, then by the two out of three property of weak equivalences, it is enough to check that the map $C_q^{m, M}$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Applying lemma 3.5.18 we get that $C_q^{m, M}$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, since $C_q^{m, M}$ is by construction a weak equivalence in $R_{C_{\text{eff}}^q} A\text{-mod}(\mathcal{M}_*)$.

Since $C_q^\Sigma M$ and $C_q^\Sigma R_m M$ are both cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ by construction, using Theorem 3.2.13(2) in [6] again, we get that if $C_q^\Sigma(R_m^M)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, then it is also a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. But it is clear that

$C_q^{\Sigma, M}$ and $C_q^{\Sigma, R_m M}$ are both weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T^{\Sigma} \mathcal{M}_*$, then by the two out of three property of weak equivalences, it is enough to check that the map R_m^M is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^{\Sigma} \mathcal{M}_*$. However, Theorem 2.8.4 and definition 3.5.2 imply that R_m^M is a weak equivalence in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$, and by [6, Proposition 3.1.5] we have that R_m^M is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^{\Sigma} \mathcal{M}_*$. This finishes the proof. \square

Remark 3.5.69. — Notice that Theorem 3.5.68 does not follow from the general nonsense (see Proposition 3.5.59), since it gives a compatibility between compositions of left and right adjoints.

Theorem 3.5.70. — Fix $q \in \mathbb{Z}$. Let $f : A \rightarrow A'$ be a map between cofibrant ring spectra in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$, which is compatible with the ring structures. Assume that one of the following conditions holds:

1. f is a weak equivalence in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$.
2. There exists $p \in \mathbb{Z}$ such that A, A' are both $L^{\Sigma}(\langle p \rangle)$ -local in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$ and f is a weak equivalence in $L_{<p} \text{Spt}_T^{\Sigma} \mathcal{M}_*$.
3. There exists $p \in \mathbb{Z}$ such that A, A' are both $C_{\text{eff}}^{p, \Sigma}$ -colocal in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$ and f is a weak equivalence in $R_{C_{\text{eff}}^p} \text{Spt}_T^{\Sigma} \mathcal{M}_*$ (equivalently in $R_{C_{\text{eff}}^p} A\text{-mod}(\mathcal{M}_*)$).
4. A, A' are both cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^{\Sigma} \mathcal{M}_*$, A' is also cofibrant in $A\text{-mod}(\mathcal{M}_*)$ and f is a weak equivalence in $S^0 \text{Spt}_T^{\Sigma} \mathcal{M}_*$.
5. A, A' are both cofibrant in $S^0 \text{Spt}_T^{\Sigma} \mathcal{M}_*$, A' is also cofibrant in $A\text{-mod}(\mathcal{M}_*)$ and f is a weak equivalence in $S^0 \text{Spt}_T^{\Sigma} \mathcal{M}_*$.

Then f induces a Quillen equivalence between the q -slice motivic stable model structures of A and A' modules:

$$(A' \wedge_A -, U, \varphi) : S^q A\text{-mod}(\mathcal{M}_*) \longrightarrow S^q A'\text{-mod}(\mathcal{M}_*)$$

Proof. — (1): This is just Proposition 3.5.62.

(2): Since A and A' are $L^{\Sigma}(\langle p \rangle)$ -local in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$, [6, Theorem 3.2.13(1)] implies that f is a weak equivalence in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$. Therefore the result follows from Proposition 3.5.62.

(3): Since A and A' are $C_{\text{eff}}^{p, \Sigma}$ -colocal in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$, using [6, Theorem 3.2.13(2)] we have that f is a weak equivalence in $\text{Spt}_T^{\Sigma} \mathcal{M}_*$. Thus, the result follows from Proposition 3.5.62.

(4): Proposition 3.3.61 implies that A and A' are both cofibrant in $S^0 \text{Spt}_T^{\Sigma} \mathcal{M}_*$, therefore the result follows from Proposition 3.5.67.

(5): This is just proposition 3.5.67. \square

3.6. Applications

In this section we will describe some of the consequences that follow from the compatibility of the slice filtration with the smash product of symmetric T -spectra in the sense of Theorems 3.4.5 and 3.4.12, as well as those that follow from the compatibility between the slice filtration on the categories of symmetric T -spectra and A -modules in the sense of Propositions 3.5.15, 3.5.34, 3.5.59 and Theorems 3.5.22, 3.5.45, 3.5.68, 3.5.70. In the rest of this section $p, q \in \mathbb{Z}$ will denote arbitrary integers.

Proposition 3.6.1. — *The model categories $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$ are both symmetric monoidal model categories (with respect to the smash product of symmetric T -spectra) in the sense of Hovey (see definition 1.7.7).*

Proof. — It follows directly from Lemma 3.4.1, together with theorems 3.4.5 and 3.4.12 □

Theorem 3.6.2. — *The triangulated categories $\mathcal{H}^\Sigma(S)$, $R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S)$ and $S^0 \mathcal{H}^\Sigma(S)$ inherit a natural symmetric monoidal structure from the smash product of symmetric T -spectra. The symmetric monoidal structure is defined as follows:*

1.

$$\begin{aligned} - \wedge^{\mathbf{L}} - : \mathcal{H}^\Sigma(S) \times \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ (X, Y) &\longmapsto Q_\Sigma X \wedge Q_\Sigma Y \end{aligned}$$

2.

$$\begin{aligned} - \wedge^{\mathbf{L}} - : R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) \times R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) &\longrightarrow R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) \\ (X, Y) &\longmapsto C_0^\Sigma X \wedge C_0^\Sigma Y \end{aligned}$$

3.

$$\begin{aligned} - \wedge^{\mathbf{L}} - : S^0 \mathcal{H}^\Sigma(S) \times S^0 \mathcal{H}^\Sigma(S) &\longrightarrow S^0 \mathcal{H}^\Sigma(S) \\ (X, Y) &\longmapsto P_0^\Sigma X \wedge P_0^\Sigma Y \end{aligned}$$

Proof. — It follows directly from Propositions 2.6.28 and 3.6.1, together with Theorem 1.7.15. □

Proposition 3.6.3. — *The following exact functors between triangulated categories are both strong symmetric monoidal:*

$$\begin{aligned} C_0^\Sigma : R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ C_0^\Sigma : R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) &\longrightarrow S^0 \mathcal{H}^\Sigma(S) \end{aligned}$$

Proof. — Propositions 2.6.28 and 3.6.1 imply that $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$, $R_{C_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ and $S^0 \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ are all symmetric monoidal model categories in the sense of Hovey. Now, using Theorem 3.3.9 and Proposition 3.3.61 we have that the following adjunctions

$$\begin{aligned} (\mathrm{id}, \mathrm{id}, \varphi) : R_{C_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_* &\longrightarrow \mathrm{Spt}_T^\Sigma \mathcal{M}_* \\ (\mathrm{id}, \mathrm{id}, \varphi) : R_{C_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_* &\longrightarrow S^0 \mathrm{Spt}_T^\Sigma \mathcal{M}_* \end{aligned}$$

are both symmetric monoidal Quillen adjunctions (see definition 1.7.11). The result then follows immediately from Theorem 4.3.3 in [10]. \square

Corollary 3.6.4. — *The following exact functors between triangulated categories are both lax symmetric monoidal:*

$$\begin{aligned} R_\Sigma : \mathcal{H}^\Sigma(S) &\longrightarrow R_{C_{\mathrm{eff}}^0} \mathcal{H}^\Sigma(S) \\ W_1^\Sigma : S^0 \mathcal{H}^\Sigma(S) &\longrightarrow R_{C_{\mathrm{eff}}^0} \mathcal{H}^\Sigma(S) \end{aligned}$$

Proof. — By Proposition 3.3.18 and Corollary 3.3.62 we have the following adjunctions

$$\begin{aligned} (C_0^\Sigma, R_\Sigma, \varphi) : R_{C_{\mathrm{eff}}^0} \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\ (C_0^\Sigma, W_1^\Sigma, \varphi) : R_{C_{\mathrm{eff}}^0} \mathcal{H}^\Sigma(S) &\longrightarrow S^0 \mathcal{H}^\Sigma(S) \end{aligned}$$

Using Proposition 3.6.3 we have that the left adjoints for R_Σ and W_1^Σ are both strong symmetric monoidal. Finally by standard results in category theory we get that the right adjoints R_Σ and W_1^Σ are both lax symmetric monoidal (see [15, Theorem 1.5]). \square

Proposition 3.6.5. — *The smash product of symmetric T -spectra induces the following Quillen adjunctions of two variables (see definition 1.7.12):*

1. $R_{C_{\mathrm{eff}}^q} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a $R_{C_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ -model category in the sense of Hovey.
2. $S^q \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a $S^0 \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ -model category in the sense of Hovey.
3. $\mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a $R_{C_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ -model category in the sense of Hovey.
4. $S^q \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ is a $R_{C_{\mathrm{eff}}^0} \mathrm{Spt}_T^\Sigma \mathcal{M}_*$ -model category in the sense of Hovey.

Proof. — (1): This follows immediately from Lemma 3.4.1 and Theorem 3.4.5.

(2): This follows immediately from Lemma 3.4.1 and Theorem 3.4.12.

(3): It follows from proposition 2.6.28 and Theorem 3.3.9 which imply that the following composition is a Quillen adjunction of two variables:

$$R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_* \times \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{(\text{id}, \text{id})} \text{Spt}_T^\Sigma \mathcal{M}_* \times \text{Spt}_T^\Sigma \mathcal{M}_* \downarrow -\wedge- \\ \text{Spt}_T^\Sigma \mathcal{M}_*$$

(4): It follows from proposition 3.3.61 and Theorem 3.4.12 which imply that the following composition is a Quillen adjunction of two variables:

$$R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_* \times S^q \text{Spt}_T^\Sigma \mathcal{M}_* \xrightarrow{(\text{id}, \text{id})} S^0 \text{Spt}_T^\Sigma \mathcal{M}_* \times S^q \text{Spt}_T^\Sigma \mathcal{M}_* \quad \square \\ \downarrow -\wedge- \\ S^q \text{Spt}_T^\Sigma \mathcal{M}_*$$

Theorem 3.6.6. — *The smash product of symmetric T -spectra induces the following natural module structures (see definition 1.7.1):*

1. *The triangulated category $R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S)$ has a natural structure of $R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S)$ -module, defined as follows:*

$$-\wedge^{\mathbf{L}}- : R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) \times R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \\ (X, Y) \longmapsto C_0^\Sigma X \wedge C_q^\Sigma Y$$

2. *The triangulated category $S^q \mathcal{H}^\Sigma(S)$ has a natural structure of $S^0 \mathcal{H}^\Sigma(S)$ -module, defined as follows:*

$$-\wedge^{\mathbf{L}}- : S^0 \mathcal{H}^\Sigma(S) \times S^q \mathcal{H}^\Sigma(S) \longrightarrow S^q \mathcal{H}^\Sigma(S) \\ (X, Y) \longmapsto P_0^\Sigma X \wedge P_q^\Sigma Y$$

3. *The triangulated category $\mathcal{H}^\Sigma(S)$ has a natural structure of $R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S)$ -module, defined as follows:*

$$-\wedge^{\mathbf{L}}- : R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) \times \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}^\Sigma(S) \\ (X, Y) \longmapsto C_0^\Sigma X \wedge Q_\Sigma Y$$

4. *The triangulated category $S^q \mathcal{H}^\Sigma(S)$ has a natural structure of $R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S)$ -module, defined as follows:*

$$-\wedge^{\mathbf{L}}- : R_{C_{\text{eff}}^0} \mathcal{H}^\Sigma(S) \times S^q \mathcal{H}^\Sigma(S) \longrightarrow S^q \mathcal{H}^\Sigma(S) \\ (X, Y) \longmapsto C_0^\Sigma X \wedge P_q^\Sigma Y$$

Proof. — This follows directly from Lemma 3.4.1, Proposition 3.6.5 and [10, Theorem 4.3.4]. \square

Theorem 3.6.7. — *The smash product of symmetric T -spectra induces the following adjunctions of two variables (see definition 1.7.2) which are also bilinear pairings:*

1.

$$\begin{aligned} - \wedge^{\mathbf{L}} - : R_{C_{\text{eff}}^p} \mathcal{H}^{\Sigma}(S) \times R_{C_{\text{eff}}^q} \mathcal{H}^{\Sigma}(S) &\longrightarrow R_{C_{\text{eff}}^{p+q}} \mathcal{H}^{\Sigma}(S) \\ (X, Y) &\longmapsto C_p^{\Sigma} X \wedge C_q^{\Sigma} Y \end{aligned}$$

2.

$$\begin{aligned} - \wedge^{\mathbf{L}} - : S^p \mathcal{H}^{\Sigma}(S) \times S^q \mathcal{H}^{\Sigma}(S) &\longrightarrow S^{p+q} \mathcal{H}^{\Sigma}(S) \\ (X, Y) &\longmapsto P_p^{\Sigma} X \wedge P_q^{\Sigma} Y \end{aligned}$$

Proof. — (1): By Theorem 3.4.5 we have that

$$- \wedge - : R_{C_{\text{eff}}^p} \text{Spt}_T^{\Sigma} \mathcal{M}_{*} \times R_{C_{\text{eff}}^q} \text{Spt}_T^{\Sigma} \mathcal{M}_{*} \longrightarrow R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^{\Sigma} \mathcal{M}_{*}$$

is a Quillen bifunctor. Then Proposition 1.7.14 implies that

$$\begin{aligned} - \wedge^{\mathbf{L}} - : R_{C_{\text{eff}}^p} \mathcal{H}^{\Sigma}(S) \times R_{C_{\text{eff}}^q} \mathcal{H}^{\Sigma}(S) &\longrightarrow R_{C_{\text{eff}}^{p+q}} \mathcal{H}^{\Sigma}(S) \\ (X, Y) &\longmapsto C_p^{\Sigma} X \wedge C_q^{\Sigma} Y \end{aligned}$$

is an adjunction of two variables. Finally, since the coproduct of two cofibrant objects is always cofibrant, and $X \wedge (Y \amalg Z)$ is canonically isomorphic in $\text{Spt}_T^{\Sigma}(\mathcal{M}|_S)_{\text{Nis}}$ to $(X \wedge Y) \amalg (X \wedge Z)$, we get that the pairing $- \wedge^{\mathbf{L}} -$ is bilinear.

(2): By Theorem 3.4.12 we have that

$$- \wedge - : S^p \text{Spt}_T^{\Sigma} \mathcal{M}_{*} \times S^q \text{Spt}_T^{\Sigma} \mathcal{M}_{*} \longrightarrow S^{p+q} \text{Spt}_T^{\Sigma} \mathcal{M}_{*}$$

is a Quillen bifunctor. Then Proposition 1.7.14 implies that

$$\begin{aligned} - \wedge^{\mathbf{L}} - : S^p \mathcal{H}^{\Sigma}(S) \times S^q \mathcal{H}^{\Sigma}(S) &\longrightarrow S^{p+q} \mathcal{H}^{\Sigma}(S) \\ (X, Y) &\longmapsto P_p^{\Sigma} X \wedge P_q^{\Sigma} Y \end{aligned}$$

is an adjunction of two variables. Finally, since the coproduct of two cofibrant objects is always cofibrant and $X \wedge (Y \amalg Z)$ is canonically isomorphic in $\text{Spt}_T^{\Sigma}(\mathcal{M}|_S)_{\text{Nis}}$ to $(X \wedge Y) \amalg (X \wedge Z)$, we get that the pairing $- \wedge^{\mathbf{L}} -$ is bilinear. \square

Proposition 3.6.8. — *Let X, Y be two arbitrary symmetric T -spectra.*

1. *There exists a natural bilinear isomorphism in $\mathcal{H}^{\Sigma}(S)$:*

$$C_p^{\Sigma} X \wedge C_q^{\Sigma} Y \xrightarrow{m_1^{X,Y}} C_{p+q}^{\Sigma} (C_p^{\Sigma} X \wedge C_q^{\Sigma} Y)$$

2. There exists a natural bilinear map in $R_{C_{\text{eff}}^{p+q}} \mathcal{H}^\Sigma(S)$:

$$C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y \xrightarrow{m_2^{X,Y}} R_\Sigma(X \wedge Y)$$

3. There exists a natural bilinear isomorphism in $S^{p+q} \mathcal{H}^\Sigma(S)$:

$$C_p^\Sigma X \wedge C_q^\Sigma Y \xrightarrow{m_3^{X,Y}} C_{p+q}^\Sigma(C_p^\Sigma X \wedge C_q^\Sigma Y)$$

4. There exists a natural bilinear map in $R_{C_{\text{eff}}^{p+q}} \mathcal{H}^\Sigma(S)$:

$$C_p^\Sigma W_{p+1}^\Sigma X \wedge C_q^\Sigma W_{q+1}^\Sigma Y \xrightarrow{m_4^{X,Y}} W_{p+q+1}^\Sigma(X \wedge Y)$$

Proof. — (1): Theorems 3.4.5 and 3.3.9 imply that we have the following commutative diagram of Quillen bifunctors:

$$\begin{array}{ccc} R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & & \\ \begin{array}{c} \text{---} \wedge \text{---} \\ \downarrow \end{array} & \searrow \text{---} \wedge \text{---} & \\ R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & \text{Spt}_T^\Sigma \mathcal{M}_* \end{array}$$

Using [10, Theorem 1.3.7] we get the natural isomorphism m_1 , which is bilinear since the functors C_p^Σ , C_q^Σ , C_{p+q}^Σ are all exact and the smash product is bilinear.

(2): By proposition 3.3.18 we have the following adjunctions:

$$(C_p^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^p} \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

$$(C_q^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

$$(C_{p+q}^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^{p+q}} \mathcal{H}^\Sigma(S) \longrightarrow \mathcal{H}^\Sigma(S)$$

Let ϵ_p , ϵ_q denote the respective counits, and let $\tilde{m}_2^{X,Y}$ be the following composition in $\mathcal{H}^\Sigma(S)$:

$$\begin{array}{ccc} C_{p+q}^\Sigma(C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y) & \xrightarrow{(m_1^{R_\Sigma X, R_\Sigma Y})^{-1}} & C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y \\ \tilde{m}_2^{X,Y} \downarrow & & \swarrow \epsilon_p^X \wedge \epsilon_q^Y \\ X \wedge Y & & \end{array}$$

Then using the adjunction between C_{p+q}^Σ and R_Σ considered above, we define $m_2^{X,Y}$ as the adjoint of $\tilde{m}_2^{X,Y}$. The naturality of m_2 follows from:

1. the naturality of m_1
2. the naturality of the fibrant replacement functor, and
3. the naturality of the counits ϵ_p and ϵ_q .

Finally we have that m_2 is bilinear since:

1. m_1 is bilinear
2. the functors $C_p^\Sigma, C_q^\Sigma, C_{p+q}^\Sigma$ and R_Σ are all exact, and
3. the smash product is bilinear.

(3): Theorem 3.4.5 and proposition 3.3.61 imply that we have the following commutative diagram of Quillen bifunctors:

$$\begin{array}{ccc}
 R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & & \\
 \begin{array}{c} \text{\scriptsize } -\wedge - \\ \downarrow \end{array} & \begin{array}{c} \text{\scriptsize } -\wedge - \\ \searrow \end{array} & \\
 R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_*
 \end{array}$$

Using [10, Theorem 1.3.7] we get the natural isomorphism m_3 , which is bilinear since the functors $C_p^\Sigma, C_q^\Sigma, C_{p+q}^\Sigma$ are all exact and the smash product is bilinear.

(4): By corollary 3.3.62 we have the following adjunctions:

$$\begin{aligned}
 (C_p^\Sigma, W_{p+1}^\Sigma, \varphi) : R_{C_{\text{eff}}^p} \mathcal{H}^\Sigma(S) &\longrightarrow S^p \mathcal{H}^\Sigma(S) \\
 (C_q^\Sigma, W_{q+1}^\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) &\longrightarrow S^q \mathcal{H}^\Sigma(S) \\
 (C_{p+q}^\Sigma, W_{p+q+1}^\Sigma, \varphi) : R_{C_{\text{eff}}^{p+q}} \mathcal{H}^\Sigma(S) &\longrightarrow S^{p+q} \mathcal{H}^\Sigma(S)
 \end{aligned}$$

Let ϵ_p, ϵ_q denote the respective counits, and let $\tilde{m}_4^{X,Y}$ be the following composition in $S^{p+q} \mathcal{H}^\Sigma(S)$:

$$\begin{array}{ccc}
 C_{p+q}^\Sigma (C_p^\Sigma W_{p+1}^\Sigma X \wedge C_q^\Sigma W_{q+1}^\Sigma Y) & \xrightarrow{(m_3^{W_{p+1}^\Sigma X, W_{q+1}^\Sigma Y})^{-1}} & C_p^\Sigma W_{p+1}^\Sigma X \wedge C_q^\Sigma W_{q+1}^\Sigma Y \\
 \begin{array}{c} \downarrow \\ \tilde{m}_4^{X,Y} \\ \downarrow \\ X \wedge Y \end{array} & & \begin{array}{c} \swarrow \\ \epsilon_p^X \wedge \epsilon_q^Y \\ \swarrow \end{array}
 \end{array}$$

Then using the adjunction between C_{p+q}^Σ and W_{p+q+1}^Σ considered above, we define $m_4^{X,Y}$ as the adjoint of $\tilde{m}_4^{X,Y}$. The naturality of m_4 follows from:

1. the naturality of m_3
2. the naturality of the fibrant replacement functors, and
3. the naturality of the counits ϵ_p and ϵ_q .

Finally we have that m_4 is bilinear since:

1. m_3 is bilinear
2. the functors $C_p^\Sigma, C_q^\Sigma, C_{p+q}^\Sigma, W_{p+1}^\Sigma, W_{q+1}^\Sigma$ and W_{p+q+1}^Σ are all exact, and
3. the smash product is bilinear.

□

Theorem 3.6.9. — *The smash product of symmetric T-spectra induces the following natural pairings (external products):*

1. *For every couple of symmetric T-spectra X, Y we have the following natural map in $\mathcal{H}^\Sigma(S)$ (see Proposition 3.6.8):*

$$\begin{array}{ccc}
 f_p^\Sigma X \wedge f_q^\Sigma Y & \xrightarrow{\cup_{p,q}^c} & f_{p+q}^\Sigma(X \wedge Y) \\
 \parallel & & \parallel \\
 C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y & & C_{p+q}^\Sigma R_\Sigma(X \wedge Y) \\
 \searrow^{m_1^{R_\Sigma X, R_\Sigma Y}} & & \nearrow^{C_{p+q}^\Sigma(m_2^{X,Y})} \\
 \cong & & \\
 C_{p+q}^\Sigma(C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y) & &
 \end{array}$$

which induces a bilinear natural transformation between the functors:

$$\begin{array}{ccc}
 \mathcal{H}^\Sigma(S) \times \mathcal{H}^\Sigma(S) & \longrightarrow & \mathcal{H}^\Sigma(S) \\
 (X, Y) & \longmapsto & f_p^\Sigma X \wedge f_q^\Sigma Y
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{H}^\Sigma(S) \times \mathcal{H}^\Sigma(S) & \longrightarrow & \mathcal{H}^\Sigma(S) \\
 (X, Y) & \longmapsto & f_{p+q}^\Sigma(X \wedge Y)
 \end{array}$$

2. For every couple of symmetric T -spectra X, Y we have the following natural map in $\mathcal{H}^\Sigma(S)$ (see Proposition 3.6.8):

$$\begin{array}{ccc}
 s_p^\Sigma X \wedge s_q^\Sigma Y & & \\
 \parallel & \searrow \cup_{p,q}^s & \\
 C_p^\Sigma W_{p+1}^\Sigma C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma Y & & s_{p+q}^\Sigma (X \wedge Y) \\
 \downarrow \cong & & \parallel \\
 m_1^{W_{p+1}^\Sigma C_p^\Sigma R_\Sigma X, W_{q+1}^\Sigma C_q^\Sigma R_\Sigma Y} & \cong & C_{p+q}^\Sigma W_{p+q+1}^\Sigma C_{p+q}^\Sigma R_\Sigma (X \wedge Y) \\
 \downarrow & & \uparrow \\
 C_{p+q}^\Sigma (C_p^\Sigma W_{p+1}^\Sigma C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma Y) & & \\
 \downarrow & & \\
 C_{p+q}^\Sigma (m_4^{C_p^\Sigma R_\Sigma X, C_q^\Sigma R_\Sigma Y}) & & \\
 \downarrow & & \\
 C_{p+q}^\Sigma W_{p+q+1}^\Sigma (C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y) & & \\
 \searrow \cong & & \uparrow \\
 C_{p+q}^\Sigma W_{p+q+1}^\Sigma (m_3^{R_\Sigma X, R_\Sigma Y}) & & C_{p+q}^\Sigma W_{p+q+1}^\Sigma C_{p+q}^\Sigma (m_2^{X, Y}) \\
 & & \uparrow \\
 & & C_{p+q}^\Sigma W_{p+q+1}^\Sigma C_{p+q}^\Sigma (C_p^\Sigma R_\Sigma X \wedge C_q^\Sigma R_\Sigma Y)
 \end{array}$$

which induces a bilinear natural transformation between the following functors:

$$\begin{array}{ccc}
 \mathcal{H}^\Sigma(S) \times \mathcal{H}^\Sigma(S) & \longrightarrow & \mathcal{H}^\Sigma(S) \\
 (X, Y) & \longmapsto & s_p^\Sigma X \wedge s_q^\Sigma Y
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{H}^\Sigma(S) \times \mathcal{H}^\Sigma(S) & \longrightarrow & \mathcal{H}^\Sigma(S) \\
 (X, Y) & \longmapsto & s_{p+q}^\Sigma (X \wedge Y)
 \end{array}$$

Proof. — (1): This follows immediately from (1) and (2) in Proposition 3.6.8.

(2): This follows immediately from (1), (4), (3) and (2) in Proposition 3.6.8. \square

Theorem 3.6.10. — The pairings $\cup_{p,q}^c$ and $\cup_{p,q}^s$ constructed in theorem 3.6.9 are compatible with the natural transformations ρ and π^Σ (see propositions 3.3.24(3) and 3.3.69) in the following sense:

1. For every couple of symmetric T -spectra X, Y ; the following diagram is commutative in $\mathcal{H}^\Sigma(S)$:

$$\begin{array}{ccc} f_{p+1}^\Sigma X \wedge f_q^\Sigma Y & \xrightarrow{\rho_p^X \wedge \text{id}} & f_p^\Sigma X \wedge f_q^\Sigma Y \\ \cup_{p+1,q}^c \downarrow & & \downarrow \cup_{p,q}^c \\ f_{p+q+1}^\Sigma(X \wedge Y) & \xrightarrow{\rho_{p+q+1}^{X \wedge Y}} & f_{p+q}^\Sigma(X \wedge Y) \end{array}$$

2. For every couple of symmetric T -spectra X, Y ; the following diagram is commutative in $\mathcal{H}^\Sigma(S)$:

$$\begin{array}{ccc} f_p^\Sigma X \wedge f_{q+1}^\Sigma Y & \xrightarrow{\text{id} \wedge \rho_q^Y} & f_p^\Sigma X \wedge f_q^\Sigma Y \\ \cup_{p,q+1}^c \downarrow & & \downarrow \cup_{p,q}^c \\ f_{p+q+1}^\Sigma(X \wedge Y) & \xrightarrow{\rho_{p+q+1}^{X \wedge Y}} & f_{p+q}^\Sigma(X \wedge Y) \end{array}$$

3. For every couple of symmetric T -spectra X, Y ; the following diagram is commutative in $\mathcal{H}^\Sigma(S)$:

$$\begin{array}{ccc} f_p^\Sigma X \wedge f_q^\Sigma Y & \xrightarrow{\pi_p^{\Sigma,X} \wedge \pi_q^{\Sigma,Y}} & s_p^\Sigma X \wedge s_q^\Sigma Y \\ \cup_{p,q}^c \downarrow & & \downarrow \cup_{p,q}^s \\ f_{p+q}^\Sigma(X \wedge Y) & \xrightarrow{\pi_{p+q}^{\Sigma,X \wedge Y}} & s_{p+q}^\Sigma(X \wedge Y) \end{array}$$

Proof. — (1): This follows from the following commutative diagram of left Quillen (bi)functors (see Theorems 3.4.5 and 3.3.25), together with the construction of the external pairing \cup^c given in Theorem 3.6.9(1) and the construction of the natural transformation ρ given in Proposition 3.3.24(3):

$$\begin{array}{ccccc} R_{C_{\text{eff}}^{p+1}} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id} \times \text{id}} & R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & & \\ \downarrow -\wedge- & \searrow -\wedge- & \swarrow -\wedge- & & \downarrow -\wedge- \\ & \text{Spt}_T^\Sigma \mathcal{M}_* & & & \\ \downarrow -\wedge- & \nearrow \text{id} & \nwarrow \text{id} & & \downarrow -\wedge- \\ R_{C_{\text{eff}}^{p+q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_* & & \end{array}$$

(2): This follows from the following commutative diagram of left Quillen (bi)functors (see Theorems 3.4.5 and 3.3.25), together with the construction of

the external pairing \cup^c given in theorem 3.6.9(1) and the construction of the natural transformation ρ given in Proposition 3.3.24(3):

$$\begin{array}{ccc}
 R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^{q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id} \times \text{id}} & R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* \\
 \downarrow -\wedge- & \searrow -\wedge- & \swarrow -\wedge- \\
 & \text{Spt}_T^\Sigma \mathcal{M}_* & \\
 \swarrow \text{id} & & \searrow \text{id} \\
 R_{C_{\text{eff}}^{p+q+1}} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{\text{id}} & R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_* \\
 & \downarrow -\wedge- & \\
 & &
 \end{array}$$

(3): This follows from the following commutative diagram of left Quillen (bi)functors (see Theorems 3.3.25, 3.4.5, 3.4.12 and Propositions 2.6.28, 3.3.61), together with the construction of the external pairings \cup^c , \cup^s given in theorem 3.6.9(1)-(2) and the construction of the natural transformation π^Σ given in Proposition 3.3.69:

$$\begin{array}{ccc}
 \text{Spt}_T^\Sigma \mathcal{M}_* \times \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{-\wedge-} & \text{Spt}_T^\Sigma \mathcal{M}_* & \square \\
 \uparrow \text{id} \times \text{id} & & \uparrow \text{id} & \\
 R_{C_{\text{eff}}^p} \text{Spt}_T^\Sigma \mathcal{M}_* \times R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{-\wedge-} & R_{C_{\text{eff}}^{p+q}} \text{Spt}_T^\Sigma \mathcal{M}_* & \\
 \downarrow \text{id} \times \text{id} & & \downarrow \text{id} & \\
 S^p \text{Spt}_T^\Sigma \mathcal{M}_* \times S^q \text{Spt}_T^\Sigma \mathcal{M}_* & \xrightarrow{-\wedge-} & S^{p+q} \text{Spt}_T^\Sigma \mathcal{M}_* &
 \end{array}$$

Definition 3.6.11. — Consider the following functors:

$$\begin{aligned}
 f^\Sigma : \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\
 X &\longmapsto \bigoplus_{q \in \mathbb{Z}} f_q^\Sigma X \\
 s^\Sigma : \mathcal{H}^\Sigma(S) &\longrightarrow \mathcal{H}^\Sigma(S) \\
 X &\longmapsto \bigoplus_{q \in \mathbb{Z}} s_q^\Sigma X
 \end{aligned}$$

Proposition 3.6.12. — The functors:

1. $f^\Sigma : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$
2. $s^\Sigma : \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$

are both exact.

Proof. — (1): Theorem 3.3.22(3) implies that all the functors f_q^Σ are exact. Therefore $f^\Sigma = \bigoplus_{q \in \mathbb{Z}} f_q^\Sigma$ is also an exact functor, since the coproduct of a collection of distinguished triangles is a distinguished triangle.

(2): Theorem 3.3.68(3) implies that all the functors s_q^Σ are exact. Therefore $s^\Sigma = \bigoplus_{q \in \mathbb{Z}} s_q^\Sigma$ is also an exact functor, since the coproduct of a collection of distinguished triangles is a distinguished triangle. \square

Theorem 3.6.13. — *Let X be a ring spectrum in $\mathcal{H}^\Sigma(S)$ and let M be an X -module.*

1. *The (-1) -connective cover of X , $f_0^\Sigma X$ (see Theorem 3.3.22(3)) also has the structure of a ring spectrum in $\mathcal{H}^\Sigma(S)$.*
2. *The $(q - 1)$ -connective cover of M , $f_q^\Sigma M$ is a module in $\mathcal{H}^\Sigma(S)$ over the (-1) -connective cover of X , $f_0^\Sigma X$.*
3. *The coproduct of all the connective covers of X , $f^\Sigma X$ has the structure of a graded ring spectrum in $\mathcal{H}^\Sigma(S)$.*
4. *The coproduct of all the connective covers of M , $f^\Sigma M$ is a graded module in $\mathcal{H}^\Sigma(S)$ over the graded ring $f^\Sigma X$.*
5. *The zero slice of X , $s_0^\Sigma X$ (see Theorem 3.3.68(3)) also has the structure of a ring spectrum in $\mathcal{H}^\Sigma(S)$.*
6. *The q -slice of M , $s_q^\Sigma M$ is a module in $\mathcal{H}^\Sigma(S)$ over the zero slice of X , $s_0^\Sigma X$.*
7. *The coproduct of all the slices of X , $s^\Sigma X$ has the structure of a graded ring spectrum in $\mathcal{H}^\Sigma(S)$.*
8. *The coproduct of all the slices of M , $s^\Sigma M$ is a graded module in $\mathcal{H}^\Sigma(S)$ over the graded ring $s^\Sigma X$.*

Proof. — We have that (1) and (5) follow immediately from proposition 3.6.3 and corollary 3.6.4. On the other hand, (2), (3) and (4) follow directly from Theorem 3.6.9(1). Finally, (6), (7) and (8) follow directly from Theorem 3.6.9(2). \square

Theorem 3.6.14. — *Let X be an arbitrary symmetric T -spectrum.*

1. *The (-1) -connective cover of the sphere spectrum, $f_0^\Sigma \mathbf{1}$ has the structure of a ring spectrum in $\mathcal{H}^\Sigma(S)$.*
2. *The $(q - 1)$ -connective cover of X , $f_q^\Sigma X$ is a module in $\mathcal{H}^\Sigma(S)$ over the (-1) -connective cover of the sphere spectrum, $f_0^\Sigma \mathbf{1}$.*
3. *The coproduct of all the connective covers of the sphere spectrum, $f^\Sigma \mathbf{1}$ has the structure of a graded ring spectrum in $\mathcal{H}^\Sigma(S)$.*
4. *The coproduct of all the connective covers of X , $f^\Sigma X$ is a graded module in $\mathcal{H}^\Sigma(S)$ over the graded ring $f^\Sigma \mathbf{1}$.*
5. *The zero slice of the sphere spectrum, $s_0^\Sigma \mathbf{1}$ has the structure of a ring spectrum in $\mathcal{H}^\Sigma(S)$.*
6. *The q -slice of X , $s_q^\Sigma X$ is a module in $\mathcal{H}^\Sigma(S)$ over the zero slice of the sphere spectrum, $s_0^\Sigma \mathbf{1}$.*
7. *The coproduct of all the slices of the sphere spectrum, $s^\Sigma \mathbf{1}$ has the structure of a graded ring spectrum in $\mathcal{H}^\Sigma(S)$.*

8. The coproduct of all the slices of X , $s^\Sigma X$ is a graded module in $\mathcal{H}^\Sigma(S)$ over the graded ring $s^\Sigma \mathbf{1}$.

Proof. — It is clear that the sphere spectrum $\mathbf{1}$ is a ring spectrum in $\mathcal{H}^\Sigma(S)$, and by construction we have that every symmetric T -spectrum X is a module in $\mathcal{H}^\Sigma(S)$ over the sphere spectrum.

The result then follows immediately from Theorem 3.6.13. □

Using the slice filtration, it is possible to construct a spectral sequence which is an analogue of the classical Atiyah-Hirzebruch spectral sequence in algebraic topology.

Definition 3.6.15 (Motivic Atiyah-Hirzebruch Spectral Sequence). — *Let X, Y be a pair of symmetric T -spectra. Then the collection of distinguished triangles in $\mathcal{H}^\Sigma(S)$ (see Theorem 3.1.16 and Propositions 3.3.24(3), 3.3.69):*

$$f_{q+1}^\Sigma X \xrightarrow{\rho_q^X} f_q^\Sigma X \xrightarrow{\pi_q^{\Sigma, X}} s_q^\Sigma X \xrightarrow{\sigma_q^{\Sigma, X}} \Sigma_T^{1,0} f_{q+1}^\Sigma X$$

generates an exact couple $(D_1^{p,q}(Y; X), E_1^{p,q}(Y; X))$, where:

1. $D_1^{p,q}(Y; X) = [Y, \Sigma_T^{p+q,0} f_p^\Sigma X]_{\text{Spt}}^\Sigma$, and
2. $E_1^{p,q}(Y; X) = [Y, \Sigma_T^{p+q,0} s_p^\Sigma X]_{\text{Spt}}^\Sigma$.

The compatibility of the slice filtration with the smash product of symmetric T -spectra implies that the smash product of symmetric T -spectra induces a pairing of spectral sequences:

Theorem 3.6.16. — *Let X, X', Y, Y' be in $\text{Spt}_T^\Sigma \mathcal{M}_*$. The smash product of symmetric T -spectra induces natural external pairings in the motivic Atiyah-Hirzebruch spectral sequence:*

$$E_r^{p,q}(Y; X) \otimes E_r^{p',q'}(Y'; X') \longrightarrow E_r^{p+p',q+q'}(Y \wedge Y'; X \wedge X')$$

$$(\alpha, \beta) \longmapsto \alpha \smile \beta$$

Furthermore, this pairing has the following explicit description on the E_1 terms:

- Given $\alpha : Y \rightarrow \Sigma_T^{p+q,0} s_p^\Sigma X$ in $E_1^{p,q}(Y; X)$ and $\beta : Y' \rightarrow \Sigma_T^{p'+q',0} s_{p'}^\Sigma X'$ in $E_1^{p',q'}(Y'; X')$, $\alpha \smile \beta$ is the following composition (see Theorem 3.6.9(2)) in $E_1^{p+p',q+q'}(Y \wedge Y'; X \wedge X')$:

$$\begin{array}{ccc}
 Y \wedge Y' & & \Sigma_T^{p+p'+q+q',0} s_{p+p'}^\Sigma (X \wedge X') \\
 \alpha \wedge \beta \downarrow & & \uparrow \Sigma_T^{p+p'+q+q',0} \circ \cup_{p,p'}^s \\
 (\Sigma_T^{p+q,0} s_p^\Sigma X) \wedge (\Sigma_T^{p'+q',0} s_{p'}^\Sigma X') & \xrightarrow{\cong} & \Sigma_T^{p+p'+q+q',0} (s_p^\Sigma X) \wedge (s_{p'}^\Sigma X')
 \end{array}$$

Proof. — Using the naturality of the external pairings $\cup_{p,q}^c, \cup_{p,q}^s$ (see Theorem 3.6.9) and Theorem 3.6.10, the result follows immediately from the work of Massey [17] together with [2, Proposition 14.3]. \square

Definition 3.6.17. — Consider a cofibrant ring spectrum A with unit in $\text{Spt}_{\mathbb{T}}^{\Sigma} \mathcal{M}_{*}$.

1. Let f_q^m denote the following composition of exact functors between triangulated categories (see Proposition 3.5.13)

$$\begin{array}{ccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{f_q^m} & \mathcal{H}(A\text{-mod}) \\ & \searrow R_m & \nearrow C_q^m \\ & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & \end{array}$$

2. Let $s_{<q}^m$ denote the following composition of exact functors between triangulated categories (see Proposition 3.5.32)

$$\begin{array}{ccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{s_{<q}^m} & \mathcal{H}(A\text{-mod}) \\ & \searrow Q_m & \nearrow W_q^m \\ & L_{<q} \mathcal{H}(A\text{-mod}) & \end{array}$$

3. Let s_q^m denote the following composition of exact functors between triangulated categories (see Propositions 3.5.13 and 3.5.56)

$$\begin{array}{ccccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{s_q^m} & \mathcal{H}(A\text{-mod}) & & \\ R_m \downarrow & & \uparrow C_q^m & & \\ R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & \xrightarrow{C_q^m} & S^q \mathcal{H}(A\text{-mod}) & \xrightarrow{W_{q+1}^m} & R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) \end{array}$$

Remark 3.6.18. — Notice that the next two theorems are much stronger than Theorem 3.6.13; since the module structures in the latter are defined just up to homotopy (i.e. they make sense in $\mathcal{H}^{\Sigma}(S)$), whereas the module structures in Theorems 3.6.19 and 3.6.20 are strict (i.e. they are defined in the model category $\text{Spt}_{\mathbb{T}}^{\Sigma} \mathcal{M}_{*}$).

Theorem 3.6.19. — Let A be a cofibrant ring spectrum with unit in $\text{Spt}_{\mathbb{T}}^{\Sigma} \mathcal{M}_{*}$.

1. If A is cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_{\mathbb{T}}^{\Sigma} \mathcal{M}_{*}$, then the functor $f_q^{\Sigma} \circ UR_m$ (see Theorems 3.3.22 and 3.5.4)

$$\mathcal{H}(A\text{-mod}) \xrightarrow{UR_m} \mathcal{H}^{\Sigma}(S) \xrightarrow{f_q^{\Sigma}} \mathcal{H}^{\Sigma}(S)$$

factors up to a canonical isomorphism through $\mathcal{H}(A\text{-mod})$ (see definition 3.6.17(1)) as follows:

$$\begin{array}{ccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \\ f_q^m \downarrow & & \downarrow f_q^\Sigma \\ \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \end{array}$$

i.e. for every A -module M , its $(q-1)$ -connective cover $f_q^\Sigma(M)$ inherits a natural strict structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

2. If A is cofibrant in $S^0\text{Spt}_T^\Sigma \mathcal{M}_*$, then the functor $s_{<q}^\Sigma \circ UR_m$ (see Theorems 3.3.45 and 3.5.4)

$$\mathcal{H}(A\text{-mod}) \xrightarrow{UR_m} \mathcal{H}^\Sigma(S) \xrightarrow{s_{<q}^\Sigma} \mathcal{H}^\Sigma(S)$$

factors up to a canonical isomorphism through $\mathcal{H}(A\text{-mod})$ (see definition 3.6.17(2)) as follows:

$$\begin{array}{ccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \\ s_{<q}^m \downarrow & & \downarrow s_{<q}^\Sigma \\ \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \end{array}$$

i.e. for every A -module M , $s_{<q}^\Sigma(M)$ inherits a natural strict structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

3. If A is cofibrant in $R_{\text{eff}} C^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, then the functor $s_q^\Sigma \circ UR_m$ (see Theorems 3.3.68 and 3.5.4)

$$\mathcal{H}(A\text{-mod}) \xrightarrow{UR_m} \mathcal{H}^\Sigma(S) \xrightarrow{s_q^\Sigma} \mathcal{H}^\Sigma(S)$$

factors up to a canonical isomorphism through $\mathcal{H}(A\text{-mod})$ (see definition 3.6.17(3)) as follows:

$$\begin{array}{ccc} \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \\ s_q^m \downarrow & & \downarrow s_q^\Sigma \\ \mathcal{H}(A\text{-mod}) & \xrightarrow{UR_m} & \mathcal{H}^\Sigma(S) \end{array}$$

i.e. for every A -module M , its q -slice $s_q^\Sigma(M)$ inherits a natural strict structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — (1): By construction (see Theorem 3.3.22) the functor f_q^Σ is defined as the following composition

$$\begin{array}{ccc} \mathcal{H}^\Sigma(S) & \xrightarrow{f_q^\Sigma} & \mathcal{H}^\Sigma(S) \\ & \searrow R_\Sigma & \nearrow C_q^\Sigma \\ & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \end{array}$$

Since we are assuming that A is cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently $C_{\text{eff}}^{0,\Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$), the result follows directly from diagram (69) in Proposition 3.5.15 and Theorem 3.5.22.

(2): By construction (see Theorem 3.3.45) the functor $s_{<q}^\Sigma$ is defined as the following composition

$$\begin{array}{ccc} \mathcal{H}^\Sigma(S) & \xrightarrow{s_{<q}^\Sigma} & \mathcal{H}^\Sigma(S) \\ & \searrow Q_\Sigma & \nearrow W_q^\Sigma \\ & L_{<q} \mathcal{H}^\Sigma(S) & \end{array}$$

Since we are assuming that A is cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, the result follows directly from Theorem 3.5.45 and diagram (78) in Proposition 3.5.34.

(3): By construction (see Theorem 3.3.68) the functor s_q^Σ is defined as the following composition

$$\begin{array}{ccc} \mathcal{H}^\Sigma(S) & \xrightarrow{s_q^\Sigma} & \mathcal{H}^\Sigma(S) \\ R_\Sigma \downarrow & & \uparrow C_q^\Sigma \\ R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \xrightarrow{C_q^\Sigma} S^q \mathcal{H}^\Sigma(S) \xrightarrow{W_{q+1}^\Sigma} R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & \end{array}$$

Since we are assuming that A is cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$ (equivalently $C_{\text{eff}}^{0,\Sigma}$ -colocal in $\text{Spt}_T^\Sigma \mathcal{M}_*$), the result is a consequence of diagram (69) in Proposition 3.5.15, Theorem 3.5.68, diagram (86) in Proposition 3.5.59 and Theorem 3.5.22. \square

Theorem 3.6.20. — *Let A be a cofibrant ring spectrum in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$ such that the unit map $u : \mathbf{1} \rightarrow A$ is a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$. Consider the following composition of exact functors between triangulated categories (see Proposition 3.3.18,*

Corollary 3.3.62, Theorem 3.5.58, propositions 3.5.56, 3.5.13 and Theorem 3.5.4)

$$(89) \quad \begin{array}{ccccc} \mathcal{H}^\Sigma(S) & & & & \mathcal{H}^\Sigma(S) \\ R_\Sigma \downarrow & & & & \uparrow UR_m \\ R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) & & & & \mathcal{H}(A\text{-mod}) \\ C_q^\Sigma \downarrow & & & & \uparrow C_q^m \\ S^q \mathcal{H}^\Sigma(S) & \xrightarrow[A \wedge P_q^\Sigma -]{} & S^q \mathcal{H}(A\text{-mod}) & \xrightarrow[W_{q+1}^m]{} & R_{C_{\text{eff}}}^q \mathcal{H}(A\text{-mod}) \end{array}$$

Then u induces a natural isomorphism between s_q^Σ (see Theorem 3.3.68) and the functor defined above in diagram (89), i.e. for every symmetric T -spectrum X , its q -slice $s_q^\Sigma(X)$ is equipped with a natural strict structure of A -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Proof. — The functor s_q^Σ (see Theorem 3.3.68) is defined as the following composition

$$\begin{array}{ccc} \mathcal{H}^\Sigma(S) & \xrightarrow{s_q^\Sigma} & \mathcal{H}^\Sigma(S) \\ R_\Sigma \downarrow & & \uparrow C_q^\Sigma \\ R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) & \xrightarrow[C_q^\Sigma]{} S^q \mathcal{H}^\Sigma(S) \xrightarrow[W_{q+1}^\Sigma]{} R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) \end{array}$$

By hypothesis A is cofibrant in $R_{C_{\text{eff}}}^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, and Lemma 3.4.1 implies that $\mathbf{1}$ is also cofibrant in $R_{C_{\text{eff}}}^0 \text{Spt}_T^\Sigma \mathcal{M}_*$. Since the unit map $u : \mathbf{1} \rightarrow A$ is assumed to be a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, it follows from theorem 3.5.70(4) that the adjunction

$$(A \wedge -, U, \varphi) : S^q \text{Spt}_T^\Sigma \mathcal{M}_* \rightarrow S^q A\text{-mod}(\mathcal{M}_*)$$

is a Quillen equivalence. Therefore the functor s_q^Σ is naturally isomorphic to the following composition

$$\begin{array}{ccccc} \mathcal{H}^\Sigma(S) & & & & \mathcal{H}^\Sigma(S) \\ R_\Sigma \downarrow & & & & \uparrow C_q^\Sigma \\ R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) & & & & R_{C_{\text{eff}}}^q \mathcal{H}^\Sigma(S) \\ C_q^\Sigma \downarrow & & & & \uparrow W_{q+1}^\Sigma \\ S^q \mathcal{H}^\Sigma(S) & \xrightarrow[A \wedge P_q^\Sigma -]{} & S^q \mathcal{H}(A\text{-mod}) & \xrightarrow[UW_{q+1}^m]{} & S^q \mathcal{H}^\Sigma(S) \end{array}$$

Now, Proposition 3.3.61 implies that A is cofibrant in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$, therefore using diagram (86) in Proposition 3.5.59, we get that the functor s_q^Σ becomes naturally

isomorphic to the following composition

$$\begin{array}{ccccc}
 \mathcal{H}^\Sigma(S) & & & & \mathcal{H}^\Sigma(S) \\
 R_\Sigma \downarrow & & & & \uparrow C_q^\Sigma \\
 R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & & & & R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \\
 C_q^\Sigma \downarrow & & & & \uparrow UR_m \\
 S^q \mathcal{H}^\Sigma(S) \xrightarrow[A \wedge P_q^\Sigma -]{} S^q \mathcal{H}(A\text{-mod}) \xrightarrow[W_{q+1}^m]{} R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & & & &
 \end{array}$$

Finally, since A is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ we can apply Theorem 3.5.22 and we get that s_q^Σ is naturally isomorphic to the following composition

$$\begin{array}{ccccc}
 \mathcal{H}^\Sigma(S) & & & & \mathcal{H}^\Sigma(S) \\
 R_\Sigma \downarrow & & & & \uparrow UR_m \\
 R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) & & & & \mathcal{H}(A\text{-mod}) \\
 C_q^\Sigma \downarrow & & & & \uparrow C_q^m \\
 S^q \mathcal{H}^\Sigma(S) \xrightarrow[A \wedge P_q^\Sigma -]{} S^q \mathcal{H}(A\text{-mod}) \xrightarrow[W_{q+1}^m]{} R_{C_{\text{eff}}^q} \mathcal{H}(A\text{-mod}) & & & &
 \end{array}$$

This finishes the proof. □

Lemma 3.6.21. — *Let $g : X \rightarrow Y$ be a map between cofibrant spectra in $\text{Spt}_T^\Sigma \mathcal{M}_*$.*

1. *The natural map $\theta_q^{\Sigma, X} : f_q^\Sigma(X) \rightarrow X$ (see Proposition 3.3.23) is an isomorphism in $\mathcal{H}^\Sigma(S)$ if and only if X is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*
2. *The induced map $f_q^\Sigma(g) : f_q^\Sigma(X) \rightarrow f_q^\Sigma(Y)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if g is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.*
3. *If $X \cong s_q^\Sigma(X)$ in $\mathcal{H}^\Sigma(S)$, then X is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ and $X \cong f_q^\Sigma(X)$ in $\mathcal{H}^\Sigma(S)$.*
4. *Furthermore, assume that X, Y are both cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Then g is a weak equivalence in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if the induced map $s_q^\Sigma(g) : s_q^\Sigma(X) \rightarrow s_q^\Sigma(Y)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.*

Proof. — (1): Consider the following diagram in $\text{Spt}_T^\Sigma \mathcal{M}_*$

$$X \xrightarrow{R_\Sigma^X} R_\Sigma X \xleftarrow{C_q^{\Sigma, R_\Sigma X}} C_q^\Sigma R_\Sigma X$$

We claim that $\theta_q^{\Sigma, X}$ is an isomorphism in $\mathcal{H}^\Sigma(S)$ if and only if $C_q^{\Sigma, R_\Sigma X}$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. In effect, Proposition 3.3.23 implies that the natural map θ_q^Σ

is just the counit of the adjunction (see Proposition 3.3.18)

$$(C_q^\Sigma, R_\Sigma, \varphi) : R_{C_{\text{eff}}^q} \mathcal{H}^\Sigma(S) \rightarrow \mathcal{H}^\Sigma(S)$$

and by construction (see definition 3.3.2) R_Σ^X is always a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$; therefore, $\theta_q^{\Sigma, X}$ is an isomorphism in $\mathcal{H}^\Sigma(S)$ if and only if $C_q^{\Sigma, R_\Sigma X}$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

We have that X is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and by construction R_Σ^X is a trivial cofibration in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.2), thus $R_\Sigma X$ is also cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. On the other hand, C_q^Σ is a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.10). Hence, [6, Theorem 3.2.13(2)] and [6, Proposition 3.2.2] imply that $C_q^{\Sigma, R_\Sigma X}$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $R_\Sigma X$ is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Finally, [6, Proposition 3.2.2] implies that X is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $R_\Sigma X$ is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

(2): By construction, we have that $f_q^\Sigma = C_q^\Sigma \circ R_\Sigma$ (see Theorem 3.3.22). Consider the following commutative diagram in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ R_\Sigma^X \downarrow & & \downarrow R_\Sigma^Y \\ R_\Sigma X & \xrightarrow{R_\Sigma(g)} & R_\Sigma Y \\ C_q^{\Sigma, R_\Sigma X} \uparrow & & \uparrow C_q^{\Sigma, R_\Sigma Y} \\ C_q^\Sigma R_\Sigma X & \xrightarrow{C_q^\Sigma R_\Sigma(g)} & C_q^\Sigma R_\Sigma Y \end{array}$$

Proposition 3.3.11 and definition 3.3.10 imply that all the vertical arrows are weak equivalences in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Hence, using the two out of three property for weak equivalences we get that the top row is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if the bottom row is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

On the other hand, by construction $C_q^\Sigma R_\Sigma X$, $C_q^\Sigma R_\Sigma Y$ are both cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.10); thus, [6, Theorem 3.2.13(2)] and [6, Proposition 3.1.5] imply that $C_q^\Sigma R_\Sigma(g)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $C_q^\Sigma R_\Sigma(g)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

(3): By (1) above, it suffices to show that X is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Since we are assuming that X is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$ and $X \cong s_q^\Sigma(X)$ in $\mathcal{H}^\Sigma(S)$, [6, Proposition 3.2.2] implies that it is enough to check that $s_q^\Sigma(X)$ is cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$.

However, by definition $s_q^\Sigma = C_q^\Sigma \circ W_{q+1}^\Sigma \circ C_q^\Sigma \circ R_\Sigma$ (see Theorem 3.3.68), and by construction C_q^Σ is a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.10). Therefore, $s_q^\Sigma(X)$ is always cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, as we wanted.

(4): By construction, we have that $s_q^\Sigma = C_q^\Sigma \circ W_{q+1}^\Sigma \circ C_q^\Sigma \circ R_\Sigma$ (see Theorem 3.3.68). Consider the following commutative diagram in $\text{Spt}_T^\Sigma \mathcal{M}_*$

$$\begin{array}{ccc}
 X & \xrightarrow{g} & Y \\
 R_\Sigma^X \downarrow & & \downarrow R_\Sigma^Y \\
 R_\Sigma X & \xrightarrow{R_\Sigma(g)} & R_\Sigma Y \\
 C_q^{\Sigma, R_\Sigma X} \uparrow & & \uparrow C_q^{\Sigma, R_\Sigma Y} \\
 C_q^\Sigma R_\Sigma X & \xrightarrow{C_q^\Sigma R_\Sigma(g)} & C_q^\Sigma R_\Sigma Y \\
 W_{q+1}^{\Sigma, C_q^\Sigma R_\Sigma X} \downarrow & & \downarrow W_{q+1}^{\Sigma, C_q^\Sigma R_\Sigma Y} \\
 W_{q+1}^\Sigma C_q^\Sigma R_\Sigma X & \xrightarrow{W_{q+1}^\Sigma C_q^\Sigma R_\Sigma(g)} & W_{q+1}^\Sigma C_q^\Sigma R_\Sigma Y \\
 C_q^{\Sigma, W_{q+1}^\Sigma C_q^\Sigma R_\Sigma X} \uparrow & & \uparrow C_q^{\Sigma, W_{q+1}^\Sigma C_q^\Sigma R_\Sigma Y} \\
 C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma X & \xrightarrow{s_q^\Sigma(g)} & C_q^\Sigma W_{q+1}^\Sigma C_q^\Sigma R_\Sigma Y
 \end{array}$$

We claim that $C_q^\Sigma R_\Sigma(g)$ is a weak equivalence in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $s_q^\Sigma(g)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$. In effect, Corollary 3.3.55 implies that $C_q^\Sigma R_\Sigma(g)$ is a weak equivalence in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $W_{q+1}^\Sigma C_q^\Sigma R_\Sigma(g)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. But C_q^Σ is by construction a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.10); thus, $W_{q+1}^\Sigma C_q^\Sigma R_\Sigma(g)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $s_q^\Sigma(g)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Finally, since $s_q^\Sigma(X)$, $s_q^\Sigma(Y)$ are always cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, we have that [6, Theorem 3.2.13(2)] and [6, Proposition 3.1.5] imply that $s_q^\Sigma(g)$ is a weak equivalence in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ if and only if $s_q^\Sigma(g)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

Now, the two out of three property for weak equivalences implies that it is enough to show that the maps R_Σ^X , R_Σ^Y , $C_q^{\Sigma, R_\Sigma X}$ and $C_q^{\Sigma, R_\Sigma Y}$ are all weak equivalences in $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$. But $S^q \text{Spt}_T^\Sigma \mathcal{M}_*$ is a right Bousfield localization with respect to $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$, and similarly $L_{<q+1} \text{Spt}_T^\Sigma \mathcal{M}_*$ is a left Bousfield localization with respect to $\text{Spt}_T^\Sigma \mathcal{M}_*$; thus, [6, Proposition 3.1.5] implies that it is enough to check that R_Σ^X , R_Σ^Y , $C_q^{\Sigma, R_\Sigma X}$ and $C_q^{\Sigma, R_\Sigma Y}$ are weak equivalences in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

By construction the maps R_Σ^X , R_Σ^Y are trivial cofibrations in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.2); hence, they are in particular weak equivalences in $\text{Spt}_T^\Sigma \mathcal{M}_*$, and $R_\Sigma X$, $R_\Sigma Y$ are both cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$ since we are assuming that X and Y are cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. Now [6, Proposition 3.2.2] implies that $R_\Sigma X$ and $R_\Sigma Y$ are also cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, since by hypothesis X , Y are both cofibrant in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$. Therefore, [6, Theorem 3.2.13(2)] and [6, Proposition 3.1.5] imply that $C_q^{\Sigma, R_\Sigma X}$ and $C_q^{\Sigma, R_\Sigma Y}$ are weak equivalences in $\text{Spt}_T^\Sigma \mathcal{M}_*$, if and only if they are weak equivalences

in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$, but this is clear since C_q^Σ is a cofibrant replacement functor in $R_{C_{\text{eff}}^q} \text{Spt}_T^\Sigma \mathcal{M}_*$ (see definition 3.3.10). \square

The next theorem proves a conjecture of Voevodsky (see [16, Corollary 11.1.3], [25]).

Theorem 3.6.22. — *Let $H\mathbb{Z}$ denote the motivic Eilenberg-MacLane spectrum in $\text{Spt}_T^\Sigma \mathcal{M}_*$ (see [16, Example 8.2.2(2)]), and assume that the base scheme is a perfect field k . Then for every symmetric T -spectrum X in $\mathcal{H}^\Sigma(k)$:*

- *The q -slice of X , $s_q^\Sigma X$ has a natural structure of $H\mathbb{Z}$ -module in $\text{Spt}_T^\Sigma \mathcal{M}_*$, i.e. $s_q^\Sigma X$ is in a natural way an object in the motivic stable homotopy category of $H\mathbb{Z}$ -modules $\mathcal{H}(H\mathbb{Z}\text{-mod})$.*

Proof. — The work of Voevodsky [26] in the case of a field of characteristic zero, and Levine [16, Theorem 10.5.1] in general, shows that $s_0^\Sigma(u)$ is a weak equivalence in $\text{Spt}_T^\Sigma \mathcal{M}_*$, where u denotes the unit map $u : \mathbf{1} \rightarrow H\mathbb{Z}$ for the commutative ring spectrum $H\mathbb{Z}$ in $\text{Spt}_T^\Sigma \mathcal{M}_*$.

By Theorem 2.8.17, Proposition 2.8.18 and Lemma 3.4.1, we can assume that $H\mathbb{Z}$ is cofibrant in $\text{Spt}_T^\Sigma \mathcal{M}_*$. On the other hand, Lemma 10.4.1 in [16] shows that $s_0^\Sigma(H\mathbb{Z})$ is isomorphic to $H\mathbb{Z}$ in $\mathcal{H}^\Sigma(S)$; hence by Lemma 3.6.21(3) we get that $H\mathbb{Z}$ is cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$. Furthermore, Lemma 3.4.1 implies that $\mathbf{1}$ is also cofibrant in $R_{C_{\text{eff}}^0} \text{Spt}_T^\Sigma \mathcal{M}_*$.

Therefore we can apply lemma 3.6.21(4) to conclude that $u : \mathbf{1} \rightarrow H\mathbb{Z}$ is a weak equivalence in $S^0 \text{Spt}_T^\Sigma \mathcal{M}_*$. Thus, the result follows from Theorem 3.6.20. \square

The motivic stable model category of $H\mathbb{Z}$ -modules has been studied in detail by Røndigs and Østvær [22], as a consequence of their work we get that the slices may be interpreted as motives in the sense of Voevodsky.

Theorem 3.6.23. — *Let k be a field of characteristic zero. Then for every symmetric T -spectrum X in $\mathcal{H}^\Sigma(k)$:*

- *The q -slice of X , $s_q^\Sigma X$ is a big motive (see [24], [22, Section 2.3]) in the sense of Voevodsky .*

Proof. — The work of Røndigs and Østvær [22] shows in particular that over a field of characteristic zero, the motivic stable homotopy category $\mathcal{H}(H\mathbb{Z}\text{-mod})$ of modules over the motivic Eilenberg-MacLane spectrum $H\mathbb{Z}$ is equivalent to Voevodsky’s big category of motives DM_k , where the equivalence preserves the monoidal and triangulated structures (see [22, Theorem 1]).

Therefore, the result is an immediate consequence of Theorem 3.6.22 \square

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