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EXTENSIONS FOR SUPERSINGULAR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbb{Q}_p)$

by

Vytautas Paškūnas

Abstract. — Let $p > 2$ be a prime number. Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and π, τ smooth irreducible representations of G on $\overline{\mathbb{F}}_p$ -vector spaces with a central character. We show if π is supersingular then $\mathrm{Ext}_G^1(\tau, \pi) \neq 0$ implies $\tau \cong \pi$ and compute the dimension of $\mathrm{Ext}_G^1(\pi, \pi)$. This answers affirmatively for $p > 2$ a question of Colmez. We also determine $\mathrm{Ext}_G^1(\tau, \pi)$, when π is the Steinberg representation. As a consequence of our results combined with those already in the literature one knows the extensions between all the irreducible representations of G .

Résumé (Extensions aux représentations supersingulières de $\mathrm{GL}_2(\mathbb{Q}_p)$). — Soit $p > 2$ un nombre premier. Soient $G := \mathrm{GL}_2(\mathbb{Q}_p)$ et π, τ des représentations lisses irréductibles de G sur des $\overline{\mathbb{F}}_p$ -espaces vectoriels avec caractère central. Nous montrons que si π est supersingulière alors $\mathrm{Ext}_G^1(\tau, \pi) \neq 0$ implique $\tau \cong \pi$ et nous calculons la dimension de $\mathrm{Ext}_G^1(\pi, \pi)$. Cela répond par l'affirmative pour $p > 2$ à une question de Colmez. Nous déterminons aussi $\mathrm{Ext}_G^1(\tau, \pi)$, quand π est la représentation de Steinberg. En conséquence de nos résultats, combinés avec ceux de la littérature, nous connaissons maintenant les extensions entre toutes les représentations irréductibles de G .

1. Introduction

In this paper we study the category Rep_G of smooth representations of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ on $\overline{\mathbb{F}}_p$ -vector spaces. Smooth irreducible $\overline{\mathbb{F}}_p$ -representations of G with a central character have been classified by Barthel-Livne [1] and Breuil [4]. A smooth irreducible representation π of G is supersingular, if it is not a subquotient of any principal series representation. Roughly speaking a supersingular representation is an $\overline{\mathbb{F}}_p$ -analog of a supercuspidal representation.

Theorem 1.1. — Assume that $p > 2$ and let τ and π be irreducible smooth representations of G admitting a central character. If π is supersingular and $\mathrm{Ext}_G^1(\tau, \pi) \neq 0$ then $\tau \cong \pi$. Moreover, if $p \geq 5$ then $\dim \mathrm{Ext}_G^1(\pi, \pi) = 5$.

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This answers affirmatively for $p > 2$ a question of Colmez, see the introduction of [7]. When $p = 3$ there are two cases and we can show that in one of them $\dim \mathrm{Ext}_G^1(\pi, \pi) = 5$, in the other $\dim \mathrm{Ext}_G^1(\pi, \pi) \leq 6$, which is the expected dimension. We note that if τ is a twist of Steinberg representation by a character or irreducible principal series then Colmez [7, VII.5.3] and Emerton [8, Prop. 4.3.14] prove by different methods that $\mathrm{Ext}_G^1(\tau, \pi) = 0$. Our result is new when τ is supersingular or a character.

We now explain the strategy of the proof. We first get rid of the extensions coming from the centre Z of G . Let $\zeta : Z \rightarrow \overline{\mathbb{F}}_p^\times$ be the central character of π , and let $\mathrm{Rep}_{G,\zeta}$ be the full subcategory of Rep_G consisting of representations with the central character ζ . We show in Theorem 8.1 that if $\mathrm{Ext}_G^1(\tau, \pi) \neq 0$ then τ also admits a central character ζ . Let $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi)$ parameterise all the isomorphism classes of extensions between π and τ admitting a central character ζ . We show that if $\tau \not\cong \pi$ then $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \cong \mathrm{Ext}_G^1(\tau, \pi)$ and there exists an exact sequence:

$$(1) \quad 0 \rightarrow \mathrm{Ext}_{G,\zeta}^1(\pi, \pi) \rightarrow \mathrm{Ext}_G^1(\pi, \pi) \rightarrow \mathrm{Hom}(Z, \overline{\mathbb{F}}_p) \rightarrow 0,$$

where Hom denotes continuous group homomorphisms. Let I be the ‘standard’ Iwahori subgroup of G , (see §2), and I_1 the maximal pro- p subgroup of I . Since ζ is smooth, it is trivial on the pro- p subgroup $I_1 \cap Z$, hence we may consider ζ as a character of ZI_1 . Let $\mathcal{H} := \mathrm{End}_G(\mathrm{c}\text{-}\mathrm{Ind}_{ZI_1}^G \zeta)$ be the Hecke algebra, and $\mathrm{Mod}_{\mathcal{H}}$ the category of right $\mathcal{H}$ -modules. Let $\mathcal{I} : \mathrm{Rep}_{G,\zeta} \rightarrow \mathrm{Mod}_{\mathcal{H}}$ be the functor

$$\mathcal{I}(\kappa) := \kappa^{I_1} \cong \mathrm{Hom}_G(\mathrm{c}\text{-}\mathrm{Ind}_{ZI_1}^G \zeta, \kappa).$$

Vignéras shows in [18] that $\mathcal{I}$ induces a bijection between irreducible representations of G with the central character ζ and irreducible $\mathcal{H}$ -modules. Using results of Ollivier [13] we show that there exists an E_2 -spectral sequence:

$$(2) \quad \mathrm{Ext}_{\mathcal{H}}^i(\mathcal{I}(\tau), \mathbb{R}^j \mathcal{I}(\pi)) \implies \mathrm{Ext}_{G,\zeta}^{i+j}(\tau, \pi).$$

The 5-term sequence associated to (2) gives an exact sequence:

$$(3) \quad 0 \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(\mathcal{I}(\tau), \mathcal{I}(\pi)) \rightarrow \mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \rightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1 \mathcal{I}(\pi)).$$

Now $\mathrm{Ext}_{\mathcal{H}}^1(\mathcal{I}(\tau), \mathcal{I}(\pi))$ has been determined in [6] and in fact is zero if $\tau \not\cong \pi$. The problem is to understand $\mathbb{R}^1 \mathcal{I}(\pi)$ as an $\mathcal{H}$ -module.

We have two approaches to this. Results of Kisin [10] imply that the dimension of $\mathrm{Ext}_G^1(\pi, \pi)$ is bounded below by the dimension of $\mathrm{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\rho, \rho)$, where ρ is the 2-dimensional irreducible $\overline{\mathbb{F}}_p$ -representation of $\mathcal{G}_{\mathbb{Q}_p}$, the absolute Galois group of $\mathbb{Q}_p$, corresponding to π under the mod p Langlands, see [5], [7]. (Excluding one case when $p = 3$.) Let $\mathfrak{I}$ be the image of $\mathrm{Ext}_{G,\zeta}^1(\pi, \pi) \rightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\pi), \mathbb{R}^1 \mathcal{I}(\pi))$. Using (1) and (3) we obtain a lower bound on the dimension of $\mathfrak{I}$. By forgetting the $\mathcal{H}$ -module structure we obtain an isomorphism of vector spaces:

$$\mathbb{R}^1 \mathcal{I}(\pi) \cong H^1(I_1/Z_1, \pi),$$

where Z_1 is the maximal pro- p subgroup of Z . The key idea is to bound the dimension of $H^1(I_1/Z_1, \pi)$ from above and use this to show if $\mathcal{J}(\tau)$ was a submodule of $\mathbb{R}^1 \mathcal{J}(\pi)$ for some $\tau \not\cong \pi$ then this would force the dimension of $\mathcal{J}$ to be smaller than calculated before.

At the time of writing (an n-th draft of) this, [10] was not written up and there were some technical issues with the outline of the argument in the introductions of [7] and [9], caused by the fact that all the representations in [7] are assumed to have a central character. Since we only need a lower bound on the dimension of $\mathrm{Ext}_G^1(\pi, \pi)$ and only in the supersingular case, we have written up the proof of a weaker statement in the appendix. The proof given there is a variation on Colmez-Kisin argument.

In order to bound the dimension of $H^1(I_1/Z_1, \pi)$ we prove a new result about the structure of supersingular representations of G . Let M be the subspace of π generated by π^{I_1} and the semi-group $\begin{pmatrix} p^{\mathbb{N}} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$. One may show that M is a representation of I .

Theorem 1.2. — *The map $(v, w) \mapsto v - w$ induces an exact sequence of I -representations:*

$$0 \rightarrow \pi^{I_1} \rightarrow M \oplus \Pi \cdot M \rightarrow \pi \rightarrow 0,$$

$$\text{where } \Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}.$$

We show that the restrictions of M and M/π^{I_1} to $I \cap U$, where U is the unipotent upper triangular matrices, are injective objects in $\mathrm{Rep}_{I \cap U}$. If $\psi : I \rightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character and $p > 2$, using this, we work out $\mathrm{Ext}_{I/Z_1}^1(\psi, M)$ and $\mathrm{Ext}_{I/Z_1}^1(\psi, M/\pi^{I_1})$. Theorem 1.2 enables us to determine $H^1(I_1/Z_1, \pi)$ as a representation of I , see Theorem 7.9 and Corollary 7.10. Once one has this it is quite easy to work out $\mathbb{R}^1 \mathcal{J}(\pi)$ as an $\mathcal{H}$ -module in the regular case, see Proposition 10.5, without using Colmez's work. It is also possible to work out directly the $\mathcal{H}$ -module structure of $\mathbb{R}^1 \mathcal{J}(\pi)$ in the Iwahori case. However, the proof relies on heavy calculations of $\mathrm{Ext}_K^1(\mathbf{1}, \pi)$ and $\mathrm{Ext}_K^1(St, \pi)$, where $K := \mathrm{GL}_2(\mathbb{Z}_p)$ and St is the Steinberg representation of $K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_p)$. So we decided to exclude it and use “stratégie de Kisin” instead.

The primes $p = 2$, $p = 3$ require some special attention. Theorem 1.2 holds when $p = 2$, but our calculation of $H^1(I_1/Z_1, \pi)$ breaks down for the technical reason that the trivial character is the only smooth character of I , when $p = 2$. However, if $p = 2$ and we fix a central character ζ then there exists only one supersingular representation (up to isomorphism) with central character ζ . Hence, it is enough to show that $\mathrm{Ext}_G^1(\tau, \pi) = 0$ when τ is a character, since all the other cases are handled in [7, VII.5.3], [8, §4]. It might be easier to do this directly.

Let Sp be the Steinberg representation of G . After the first draft of this paper, it was pointed out to me by Emerton that it was not known (although expected) that $\mathrm{Ext}_G^1(\eta, \mathrm{Sp}) = 0$, when $\eta : G \rightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character of order 2 (all the other cases have been worked out in [8, §4], see also [7, §VII.4, §VII.5]). A slight modification of our proof for supersingular representations also works for the Steinberg

representation. In the last section we work out $\mathrm{Ext}_G^1(\tau, \mathrm{Sp})$ for all irreducible τ , when $p > 2$. As a result of this and the results already in the literature ([6], [7], [8]), one knows $\mathrm{Ext}_G^1(\tau, \pi)$ for all irreducible τ and π , when $p > 2$. We record this in the last section.

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2. Notation

Let $G := \mathrm{GL}_2(\mathbb{Q}_p)$, let P be the subgroup of upper-triangular matrices, T the subgroup of diagonal matrices, U be the unipotent upper triangular matrices and $K := \mathrm{GL}_2(\mathbb{Z}_p)$. Let $\mathfrak{p} := p\mathbb{Z}_p$ and

$$I := \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ \mathfrak{p} & \mathbb{Z}_p^\times \end{pmatrix}, \quad I_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathbb{Z}_p \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}, \quad K_1 := \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix}.$$

For $\lambda \in \mathbb{F}_p$ we denote the Teichmüller lift of λ to $\mathbb{Z}_p$ by $[\lambda]$. Set

$$H := \left\{ \begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix} : \lambda, \mu \in \mathbb{F}_p^\times \right\}.$$

Let $\alpha : H \rightarrow \overline{\mathbb{F}}_p^\times$ be the character

$$\alpha\left(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}\right) := \lambda\mu^{-1}.$$

Further, define

$$\Pi := \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \quad s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t := \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

For $\lambda \in \overline{\mathbb{F}}_p^\times$ we define an unramified character $\mu_\lambda : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$, by $x \mapsto \lambda^{\mathrm{val}(x)}$.

Let Z be the centre of G , and set $Z_1 := Z \cap I_1$. Let $G^0 := \{g \in G : \det g \in \mathbb{Z}_p^\times\}$ and set $G^+ := ZG^0$.

Let $\mathcal{G}$ be a topological group. We denote by $\mathrm{Hom}(\mathcal{G}, \overline{\mathbb{F}}_p)$ the continuous group homomorphism, where the additive group $\overline{\mathbb{F}}_p$ is given the discrete topology. If $\mathcal{V}$ is a representation of $\mathcal{G}$ and S is a subset of $\mathcal{V}$ we denote by $\langle \mathcal{G}, S \rangle$ the smallest subspace of $\mathcal{V}$ stable under the action of $\mathcal{G}$ and containing S . Let $\mathrm{Rep}_{\mathcal{G}}$ be the category of smooth representations of $\mathcal{G}$ on $\overline{\mathbb{F}}_p$ -vector spaces. If $\mathcal{Z}$ is the centre of $\mathcal{G}$ and $\zeta : \mathcal{Z} \rightarrow \overline{\mathbb{F}}_p^\times$ is

a smooth character then we denote by $\mathrm{Rep}_{\mathcal{G}, \zeta}$ the full subcategory of $\mathrm{Rep}_{\mathcal{G}}$ consisting of representations with central character ζ .

All the representations in this paper are on $\overline{\mathbb{F}}_p$ -vector spaces.

3. Irreducible representations of K

We recall some facts about the irreducible representations of K and introduce some notation. Let σ be an irreducible smooth representation of K . Since K_1 is an open pro- p subgroup of K , the space of K_1 -invariants σ^{K_1} is non-zero, and since K_1 is normal in K , σ^{K_1} is a non-zero K -subrepresentation of σ , and since σ is irreducible we obtain $\sigma^{K_1} = \sigma$. Hence the smooth irreducible representations of K coincide with the irreducible representations of $K/K_1 \cong \mathrm{GL}_2(\mathbb{F}_p)$, and so there exists a uniquely determined pair of integers (r, a) with $0 \leq r \leq p-1$, $0 \leq a < p-1$, such that

$$\sigma \cong \mathrm{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det^a.$$

Note that $r = \dim \sigma - 1$ and throughout the paper given σ , r will always mean $\dim \sigma - 1$. The space of I_1 -invariants σ^{I_1} is 1-dimensional and so H acts on σ^{I_1} by a character $\chi_\sigma = \chi$. Explicitly,

$$\chi\left(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}\right) = \lambda^r (\lambda\mu)^a.$$

We define an involution $\sigma \mapsto \tilde{\sigma}$ on the set of isomorphism classes of smooth irreducible representations of K by setting

$$\tilde{\sigma} := \mathrm{Sym}^{p-r-1} \overline{\mathbb{F}}_p^2 \otimes \det^{r+a}.$$

Note that $\chi_{\tilde{\sigma}} = \chi_\sigma^s$. For the computational purposes it is convenient to identify $\mathrm{Sym}^r \overline{\mathbb{F}}_p^2$ with the space of homogeneous polynomials in $\overline{\mathbb{F}}_p[x, y]$ of degree r . The action of $\mathrm{GL}_2(\mathbb{F}_p)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(x, y) := P(ax + cy, bx + dy).$$

With this identification σ^{I_1} is spanned by x^r .

Lemma 3.1. — *Let $0 \leq j \leq r$ be an integer and define $f_j \in \mathrm{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det^a$ by*

$$f_j := \sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1-j} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} s x^r.$$

If $r = p-1$ and $j = 0$ then $f_0 = (-1)^{a+1}(x^r + y^r)$, otherwise $f_j = (-1)^{a+1} \binom{r}{j} x^j y^{r-j}$.

Proof. — It is enough to prove the statement when $a = 0$, since twisting the action by $\det^a$ multiplies f_j by $(\det s)^a = (-1)^a$. We have

$$(4) \quad f_j = \sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1-j} (\lambda x + y)^r = \sum_{i=0}^r \binom{r}{i} \left(\sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1+i-j} \right) x^i y^{r-i}.$$

If $a \geq 0$ is an integer then $\Lambda_a := \sum_{\lambda \in \mathbb{F}_p} \lambda^a$ is zero, unless $a > 0$ and $p-1$ divides a , in which case $\Lambda_a = -1$. Note that $0^0 = 1$. If $a = p-1+i-j$ then $\Lambda_a \neq 0$ if and only if $i = j$ or $i-j = p-1$, which is equivalent to $r = i = p-1$ and $j = 0$. This implies the assertion. $\square$

Let $\overline{\mathbb{F}}_p[[I \cap U]]$ denote the completed group algebra of $I \cap U$. Since $I \cap U \cong \mathbb{Z}_p$ mapping X to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$ induces an isomorphism between the ring of formal power series in one variable $\overline{\mathbb{F}}_p[[X]]$ and $\overline{\mathbb{F}}_p[[I \cap U]]$. Every smooth representation τ of $I \cap U$ is naturally a module over $\overline{\mathbb{F}}_p[[I \cap U]]$, and we will also view τ as a module over $\overline{\mathbb{F}}_p[[X]]$ via the above isomorphism.

Lemma 3.2. — *Let $x^r \in \text{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det^a$ then $X^r s x^r = (-1)^a r! x^r$.*

Proof. — We have $s x^r = (-1)^a y^r$. If $0 \leq i \leq r$ then $X \cdot x^{r-i} y^i = x^{r-i} (y+1)^i - x^{r-i} y^i = i x^{r-i+1} y^{i-1} + Q$, where Q is a homogeneous polynomial of degree r , such that the degree of Q in y is less than $i-1$. Applying this observation r times we obtain that $X^r \cdot y^r = r! x^r$. $\square$

4. Irreducible representations of G

We recall some facts about the irreducible representations of G . We fix an integer r with $0 \leq r \leq p-1$. We consider $\text{Sym}^r \overline{\mathbb{F}}_p^2$ as a representation of KZ by making p act trivially. It is shown in [1, Prop. 8] that there exists an isomorphism of algebras:

$$\text{End}_G(\text{c-Ind}_{KZ}^G \text{Sym}^r \overline{\mathbb{F}}_p^2) \cong \overline{\mathbb{F}}_p[T]$$

for a certain $T \in \text{End}_G(\text{c-Ind}_{KZ}^G \text{Sym}^r \overline{\mathbb{F}}_p^2)$ defined in [1, §3]. One may describe T as follows. Let $\varphi \in \text{c-Ind}_{KZ}^G \text{Sym}^r \overline{\mathbb{F}}_p^2$ be such that $\text{Supp } \varphi = ZK$ and $\varphi(1) = x^r$. Since φ generates $\text{c-Ind}_{KZ}^G \text{Sym}^r \overline{\mathbb{F}}_p^2$ as a G -representation T is determined by $T\varphi$.

Lemma 4.1. — (i) *If $r = 0$ then*

$$T\varphi = \Pi\varphi + \sum_{\lambda \in \mathbb{F}_p} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t\varphi.$$

(ii) *Otherwise,*

$$T\varphi = \sum_{\lambda \in \mathbb{F}_p} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t\varphi.$$

Proof. — In the notation of [1] this is a calculation of $T([1, e_{\bar{0}}])$. The claim follows from the formula (19) in the proof of [1] Theorem 19. $\square$

It follows from [1, Thm. 19] that the map $T - \lambda$ is injective, for all $\lambda \in \overline{\mathbb{F}}_p$.

Definition 4.2. — Let $\pi(r, \lambda)$ be a representation of G defined by the exact sequence:

$$0 \longrightarrow \mathrm{c}\text{-Ind}_{ZK}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2 \xrightarrow{T-\lambda} \mathrm{c}\text{-Ind}_{ZK}^G \mathrm{Sym}^r \overline{\mathbb{F}}_p^2 \longrightarrow \pi(r, \lambda) \longrightarrow 0.$$

If $\eta : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character then set $\pi(r, \lambda, \eta) := \pi(r, \lambda) \otimes \eta \circ \det$.

It follows from [1, Thm. 30] and [4, Thm. 1.1] that $\pi(r, \lambda)$ is irreducible unless $(r, \lambda) = (0, \pm 1)$ or $(r, \lambda) = (p-1, \pm 1)$. Moreover, one has non-split exact sequences:

$$(5) \quad 0 \rightarrow \mu_{\pm 1} \circ \det \rightarrow \pi(p-1, \pm 1) \rightarrow \mathrm{Sp} \otimes \mu_{\pm 1} \circ \det \rightarrow 0,$$

$$(6) \quad 0 \rightarrow \mathrm{Sp} \otimes \mu_{\pm 1} \circ \det \rightarrow \pi(0, \pm 1) \rightarrow \mu_{\pm 1} \circ \det \rightarrow 0,$$

where Sp is the Steinberg representation of G , (we take (5) as definition) and if $\lambda \in \overline{\mathbb{F}}_p^\times$ then $\mu_\lambda : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$, $x \mapsto \lambda^{\mathrm{val}(x)}$. Further, if $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1)$ then [1, Thm. 30] asserts that

$$(7) \quad \pi(r, \lambda) \cong \mathrm{Ind}_P^G \mu_{\lambda^{-1}} \otimes \mu_\lambda \omega^r.$$

It follows from [1, Thm. 33] and [4, Thm. 1.1] that the irreducible smooth representations of G with the central character fall into 4 disjoint classes:

- (i) characters, $\eta \circ \det$;
- (ii) special series, $\mathrm{Sp} \otimes \eta \circ \det$;
- (iii) (irreducible) principal series $\pi(r, \lambda, \eta)$, $0 < r \leq p-1$, $\lambda \neq 0$, $(r, \lambda) \neq (p-1, \pm 1)$;
- (iv) supersingular $\pi(r, 0, \eta)$, $0 \leq r \leq p-1$.

4.1. Supersingular representations. — We discuss the supersingular representations. Breuil has shown [4, Thm. 1.1] that the representations $\pi(r, 0, \eta)$ are irreducible and using the results [1] classified smooth irreducible representations of G with a central character.

Definition 4.3. — An irreducible representation π with a central character is supersingular if $\pi \cong \pi(r, 0, \eta)$ for some $0 \leq r \leq p-1$ and a smooth character η .

All the isomorphism between supersingular representations corresponding to different r and η are given by

$$(8) \quad \pi(r, 0, \eta) \cong \pi(r, 0, \eta \mu_{-1}) \cong \pi(p-1-r, 0, \eta \omega^r) \cong \pi(p-1-r, 0, \eta \omega^r \mu_{-1})$$

see [4, Thm. 1.3]. It follows from [1, Cor.36] that an irreducible smooth representation of G with a central character is supersingular if and only if it is not a subquotient of any principal series representation.

We fix a supersingular representation π of G and we are interested in $\text{Ext}_G^1(\tau, \pi)$, where τ is an irreducible smooth representation of G . If $\eta : G \rightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character, then twisting by η induces an isomorphism

$$\text{Ext}_G^1(\tau, \pi) \cong \text{Ext}_G^1(\tau \otimes \eta, \pi \otimes \eta).$$

Hence, we may assume that $p \in Z$ acts trivially on π , so that $\pi \cong \pi(r, 0, \omega^a)$, for some $0 \leq r \leq p-1$, and $0 \leq a < p-1$. It follows from [4, Thm. 3.2.4, Cor. 4.1.4] that π^{I_1} is 2-dimensional. Moreover, [4, Cor. 4.1.5] implies that there exists a basis $\{v_\sigma, v_{\tilde{\sigma}}\}$ of π^{I_1} , such that $\Pi v_\sigma = v_{\tilde{\sigma}}$, $\Pi v_{\tilde{\sigma}} = v_\sigma$ and there exists an isomorphism of K -representations:

$$\langle K \cdot v_\sigma \rangle \cong \sigma, \quad \langle K \cdot v_{\tilde{\sigma}} \rangle \cong \tilde{\sigma},$$

where $\sigma := \text{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det^a$. The group H acts on v_σ by a character χ and on $v_{\tilde{\sigma}}$ by a character χ^s . Explicitly,

$$(9) \quad \chi\left(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}\right) = \lambda^r (\lambda\mu)^a, \quad \forall \lambda, \mu \in \mathbb{F}_p^\times.$$

Lemma 4.4. — *The following relations hold:*

$$(10) \quad v_\sigma = (-1)^{a+1} \sum_{\lambda \in \mathbb{F}_p} \lambda^{p-1-r} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v_{\tilde{\sigma}};$$

$$(11) \quad v_{\tilde{\sigma}} = (-1)^{r+a+1} \sum_{\lambda \in \mathbb{F}_p} \lambda^r \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v_\sigma;$$

$$(12) \quad X^r t v_{\tilde{\sigma}} = (-1)^a r! v_\sigma, \quad X^{p-1-r} t v_\sigma = (-1)^{r+a} (p-1-r)! v_{\tilde{\sigma}}.$$

Proof. — Since $t v_{\tilde{\sigma}} = s \Pi v_{\tilde{\sigma}} = s v_\sigma$ this is a calculation in $\text{Sym}^r \overline{\mathbb{F}}_p^2 \otimes \det^a$, which is done in Lemmas 3.1 and 3.2. $\square$

Definition 4.5. — $M := \left\langle \begin{pmatrix} p^N & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \pi^{I_1} \right\rangle$, $M_\sigma := \left\langle \begin{pmatrix} p^{2N} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} v_\sigma \right\rangle$ and $M_{\tilde{\sigma}} := \left\langle \begin{pmatrix} p^{2N} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} v_{\tilde{\sigma}} \right\rangle$.

Lemma 4.6. — *The subspaces M , M_σ , $M_{\tilde{\sigma}}$ are stable under the action of I .*

Proof. — We prove the statement for M , the rest is identical. By definition M is stable under $I \cap U$. Since $I = (I \cap P^s)(I \cap U)$ it is enough to show that M is stable under $I \cap P^s$. Suppose that $g_1 \in I \cap P^s$, $g_2 \in I \cap U$. Since $I = (I \cap U)(I \cap P^s)$ there exists $h_2 \in I \cap U$ and $h_1 \in I \cap P^s$ such that $g_1 g_2 = h_2 h_1$. Moreover, for $n \geq 0$ we have $t^{-n}(I \cap P^s)t^n \subset I$. Hence, if $v \in \pi^{I_1}$ then $(t^{-n} h_1 t^n) v \in \pi^{I_1}$ and so

$$g_1(g_2 t^n v) = h_2 h_1 t^n v = h_2 t^n (t^{-n} h_1 t^n) v \in M, \quad \forall v \in \pi^{I_1}.$$

This implies that M is stable under $I \cap P^s$. $\square$

The isomorphism $\pi(r, 0, \omega^a) \cong \pi(p-r-1, 0, \omega^{r+a})$ allows to exploit the symmetry between M_σ and $M_{\tilde{\sigma}}$. In particular, if we prove a statement about M_σ which holds for all σ , then it also holds for $M_{\tilde{\sigma}}$ (with σ replaced by $\tilde{\sigma}$).

Proposition 4.7. — *The triples $\chi \hookrightarrow M_\sigma$ and $\chi^s \hookrightarrow M_{\tilde{\sigma}}$ are injective envelopes of χ and χ^s in $\mathrm{Rep}_{H(I_1 \cap U)}$. In particular, $M_\sigma^{I_1 \cap U} = \overline{\mathbb{F}}_p v_\sigma$ and $M_{\tilde{\sigma}}^{I_1 \cap U} = \overline{\mathbb{F}}_p v_{\tilde{\sigma}}$.*

Proof. — We will show the claim for M_σ . The relations (12) imply that

$$v_\sigma = (-1)^r r! (p-1-r)! X^{r+p(p-1-r)} t^2 v_\sigma.$$

For $n \geq 0$ define $\lambda_n := ((-1)^r r! (p-1-r)!)^n$, $e_0 := 0$ and $e_n := r+p(p-1-r)+p^2 e_{n-1}$. Further define $M_{\sigma,n} := \langle (I_1 \cap U) t^{2n} v_\sigma \rangle$. Since $t^{2n} v_\sigma = \lambda_1 X^{p^{2n} e_1} t^{2(n+1)} v_\sigma$, $M_{\sigma,n}$ is contained in $M_{\sigma,n+1}$ and hence

$$M_\sigma = \varinjlim_n M_{\sigma,n}.$$

Since $v_\sigma = \lambda_n X^{e_n} t^{2n} v_\sigma$ and $X v_\sigma = 0$ we obtain an isomorphism $M_{\sigma,n} \cong \overline{\mathbb{F}}_p[X]/(X^{e_n+1})$. In particular, for all $n \geq 0$ we have $M_{\sigma,n}^{I_1 \cap U} = \overline{\mathbb{F}}_p v_\sigma$, and so $M_\sigma^{I_1 \cap U} = \overline{\mathbb{F}}_p v_\sigma$. Given $m \geq 0$, set $\mathcal{U}_m := \begin{pmatrix} 1 & p^m \\ 0 & 1 \end{pmatrix}$, choose n such that $e_n > p^m$ and define $M'_{\sigma,m} := \langle (I_1 \cap U) \cdot X^{e_n+1-p^m} t^{2n} v_\sigma \rangle$. Then $M'_{\sigma,m} \cong \overline{\mathbb{F}}_p[X]/(X^{p^m}) \cong M_\sigma^{\mathcal{U}_m}$ is an injective envelope of χ in $\mathrm{Rep}_{H(I_1 \cap U)/\mathcal{U}_m}$. Since $M_\sigma = \varinjlim M'_{\sigma,m}$ we obtain that M_σ is an injective envelope of χ in $\mathrm{Rep}_{H(I_1 \cap U)}$. $\square$

Lemma 4.8. — *Let $n \geq 0$ be an odd integer then $t^n v_\sigma \in M_{\tilde{\sigma}}$ and $t^n v_{\tilde{\sigma}} \in M_\sigma$. Hence, $tM_\sigma \subset M_{\tilde{\sigma}}$ and $tM_{\tilde{\sigma}} \subset M_\sigma$.*

Proof. — It follows from the definition that $t^2 M_{\tilde{\sigma}} \subset M_{\tilde{\sigma}}$. Hence, it is enough to consider $n = 1$. Applying t to (12) we obtain $t v_\sigma = (-1)^a (r!)^{-1} X^{pr} t^2 v_{\tilde{\sigma}} \in M_{\tilde{\sigma}}$. If $k, m \geq 0$ are integers and m even then we have $t(X^k t^m v_\sigma) = X^{pk} t^m (t v_\sigma)$ and since $t v_\sigma \in M_{\tilde{\sigma}}$ and m is even we obtain $t(X^k t^m v_\sigma) \in M_{\tilde{\sigma}}$. The set $\{X^k t^m v_\sigma : k, m \geq 0, 2 \mid m\}$ spans M_σ as an $\overline{\mathbb{F}}_p$ -vector space. Hence, $tM_\sigma \subset M_{\tilde{\sigma}}$. The rest follows by symmetry. $\square$

Lemma 4.9. — *We have $sv_\sigma \in M_\sigma$ and $sv_{\tilde{\sigma}} \in M_{\tilde{\sigma}}$.*

Proof. — Since $sv_\sigma = s\Pi v_{\tilde{\sigma}} = tv_{\tilde{\sigma}}$ this follows from Lemma 4.8. $\square$

Lemma 4.10. — *M is the direct sum of its I -submodules M_σ and $M_{\tilde{\sigma}}$.*

Proof. — Proposition 4.7 implies that $(M_\sigma \cap M_{\tilde{\sigma}})^{I_1} = M_\sigma^{I_1} \cap M_{\tilde{\sigma}}^{I_1} = \overline{\mathbb{F}}_p v_\sigma \cap \overline{\mathbb{F}}_p v_{\tilde{\sigma}} = 0$. Hence $M_\sigma \cap M_{\tilde{\sigma}} = 0$ and so it is enough to show that $M = M_\sigma + M_{\tilde{\sigma}}$. Clearly, $M_\sigma \subset M$ and $M_{\tilde{\sigma}} \subset M$. Lemma 4.8 implies $M \subseteq M_\sigma + M_{\tilde{\sigma}}$. $\square$

Definition 4.11. — *We set $\pi_\sigma := M_\sigma + \Pi \cdot M_{\tilde{\sigma}}$ and $\pi_{\tilde{\sigma}} := M_{\tilde{\sigma}} + \Pi \cdot M_\sigma$.*

Proposition 4.12. — *The subspaces π_σ and $\pi_{\tilde{\sigma}}$ are stable under the action of G^+ .*

Proof. — We claim that $s\pi_\sigma \subseteq \pi_\sigma$. Now $s(\Pi M_{\tilde{\sigma}}) = tM_{\tilde{\sigma}} \subset M_\sigma$ by Lemma 4.8. It is enough to show that $sM_\sigma \subset \pi_\sigma$. By definition of M_σ it is enough to show that $s(ut^n v_\sigma) \in \pi_\sigma$ for all $u \in I_1 \cap U$ and all even non-negative integers n . Lemma 4.9 gives $sv_\sigma \in M_\sigma$ and if $n \geq 2$ is an even integer then $st^n v_\sigma = \Pi t^{n-1} v_\sigma \in \Pi M_{\tilde{\sigma}}$ by Lemma 4.8. Since $s(K_1 \cap U)s = I_1 \cap U^s$ for all $u \in K_1 \cap U$, and $n \geq 0$ even, we get that $sut^n v_\sigma \in \pi_\sigma$. If $u \in (I_1 \cap U) \setminus (K_1 \cap U)$ and $n > 0$ even, then the matrix identity:

$$(13) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\beta^{-1} & 1 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta^{-1} & 1 \end{pmatrix}$$

implies that $sut^n v_\sigma \in M_\sigma$. This settles the claim. By symmetry $\pi_{\tilde{\sigma}}$ is also stable under s , and since $\pi_\sigma = \Pi\pi_{\tilde{\sigma}}$, we obtain that π_σ is stable under $\Pi s \Pi^{-1}$. Lemma 4.6 implies that π_σ is stable under I . Since $s, \Pi s \Pi^{-1}$ and I generate G^0 , we get that π_σ is stable under G^0 . Since Z acts by a central character, π_σ is stable under $G^+ = ZG^0$. The result for $\pi_{\tilde{\sigma}}$ follows by symmetry. $\square$

5. Extensions

In this section we compute extensions of characters for different subgroups of I .

Definition 5.1. — Define functions $\kappa^u, \varepsilon, \kappa^l : I_1 \rightarrow \overline{\mathbb{F}}_p$ as follows, for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in I_1$ we set

$$\kappa^u(A) = \omega(b), \quad \varepsilon(A) = \omega(p^{-1}(a - d)), \quad \kappa^l(A) = \omega(p^{-1}c),$$

where $\omega : \mathbb{Z}_p \rightarrow \overline{\mathbb{F}}_p$ is the reduction map composed with the canonical embedding.

Proposition 5.2. — If $p \neq 2$ then $\text{Hom}(I_1/Z_1, \overline{\mathbb{F}}_p) = \langle \kappa^u, \kappa^l \rangle$. If $p = 2$ then $\dim \text{Hom}(I_1/Z_1, \overline{\mathbb{F}}_p) = 4$.

Proof. — Let $\psi : I_1/Z_1 \rightarrow \overline{\mathbb{F}}_p$ be a continuous group homomorphism. Since $I_1 \cap U \cong I_1 \cap U^s \cong \mathbb{Z}_p$ there exist $\lambda, \mu \in \overline{\mathbb{F}}_p$ such that $\psi|_{I_1 \cap U} = \lambda \kappa^u$ and $\psi|_{I_1 \cap U^s} = \mu \kappa^l$. Then $\psi - \lambda \kappa^u - \mu \kappa^l$ is trivial on $I_1 \cap U$ and $I_1 \cap U^s$. The matrix identity

$$(14) \quad \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha(1 + \alpha\beta)^{-1} & 1 \end{pmatrix} \begin{pmatrix} (1 + \alpha\beta) & \beta \\ 0 & (1 + \alpha\beta)^{-1} \end{pmatrix}$$

implies that $I_1 \cap U$ and $I_1 \cap U^s$ generate $I_1 \cap \text{SL}_2(\mathbb{Q}_p)$. So $\psi - \lambda \kappa^u - \mu \kappa^l$ must factor through $\det$. The image of Z_1 in $1 + \mathfrak{p}$ under $\det$ is $(1 + \mathfrak{p})^2$. If $p > 2$ then $(1 + \mathfrak{p})^2 = 1 + \mathfrak{p}$ and hence $\psi = \lambda \kappa^u + \mu \kappa^l$. If $p = 2$ then $\dim \text{Hom}((1 + \mathfrak{p})/(1 + \mathfrak{p})^2, \overline{\mathbb{F}}_p) = 2$. $\square$

Lemma 5.3. — Assume $p > 2$ then $\text{Hom}((I_1 \cap P)/Z_1, \overline{\mathbb{F}}_p) = \langle \kappa^u, \varepsilon \rangle$ and $\text{Hom}((I_1 \cap P^s)/Z_1, \overline{\mathbb{F}}_p) = \langle \kappa^l, \varepsilon \rangle$.

Proof. — Let $\psi : (I_1 \cap P)/Z_1 \rightarrow \overline{\mathbb{F}}_p$ be a continuous group homomorphism. Since $I_1 \cap U \cong \mathbb{Z}_p$ there exist $\lambda \in \overline{\mathbb{F}}_p$ such that $\psi|_{I_1 \cap U} = \lambda \kappa^u$. Then $\psi - \lambda \kappa^u$ is trivial on $I_1 \cap U$, and hence defines a homomorphism $(I_1 \cap P)/Z_1(I_1 \cap U) \cong (T \cap I_1)/Z_1 \rightarrow \overline{\mathbb{F}}_p$. Since $p > 2$ we have an isomorphism $(T \cap I_1)/Z_1 \cong 1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$. Hence, there exists $\mu \in \overline{\mathbb{F}}_p$ such that $\psi = \mu \varepsilon + \lambda \kappa^u$. Conjugation by Π gives the second assertion. $\square$

Proposition 5.4. — Let $\chi, \psi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters. $\mathrm{Ext}_{I/Z_1}^1(\psi, \chi)$ is non-zero if and only if $\psi = \chi\alpha$ or $\psi = \chi\alpha^{-1}$. Moreover,

- (i) if $p > 3$ then $\dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \chi) = \dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha^{-1}, \chi) = 1$;
- (ii) if $p = 3$ then $\chi\alpha = \chi\alpha^{-1}$ and $\dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \chi) = 2$;
- (iii) if $p = 2$ then $\chi = \chi\alpha = \chi\alpha^{-1} = 1$ and $\dim \mathrm{Ext}_{I/Z_1}^1(1, 1) = 4$.

Proof. — Since the order of H is prime to p and $I = HI_1$ we have

$$\mathrm{Ext}_{I/Z_1}^1(\psi, \chi) \cong \mathrm{Hom}_H(\psi, H^1(I_1/Z_1, \chi)).$$

Now $H^1(I_1/Z_1, \chi) \cong \mathrm{Hom}(I_1/Z_1, \overline{\mathbb{F}}_p)$, where if $\xi \in \mathrm{Hom}(I_1/Z_1, \overline{\mathbb{F}}_p)$ and $h \in H$ then $[h \cdot \xi](u) = \chi(h)\xi(h^{-1}uh)$. The assertion follows from Proposition 5.2. $\square$

Similarly one obtains:

Lemma 5.5. — Let $\chi, \psi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters and let $\mathcal{U} = \begin{pmatrix} 1 & \mathfrak{p}^k \\ 0 & 1 \end{pmatrix}$ for some integer k then $\mathrm{Ext}_{H\mathcal{U}}^1(\psi, \chi) \neq 0$ if and only if $\psi = \chi\alpha^{-1}$. Moreover, $\dim \mathrm{Ext}_{H\mathcal{U}}^1(\chi\alpha^{-1}, \chi) = 1$.

Lemma 5.6. — Let $\chi, \psi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters and let $\mathcal{U} = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^k & 1 \end{pmatrix}$ for some integer k then $\mathrm{Ext}_{H\mathcal{U}}^1(\psi, \chi) \neq 0$ if and only if $\psi = \chi\alpha$. Moreover, $\dim \mathrm{Ext}_{H\mathcal{U}}^1(\chi\alpha, \chi) = 1$.

Lemma 5.7. — Assume $p > 2$ and let $\chi, \psi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters then $\mathrm{Ext}_{(I_1 \cap P)/Z_1}^1(\psi, \chi) \neq 0$ if and only if $\psi \in \{\chi, \chi\alpha^{-1}\}$. Moreover,

$$\dim \mathrm{Ext}_{(I \cap P)/Z_1}^1(\chi\alpha^{-1}, \chi) = \dim \mathrm{Ext}_{(I \cap P)/Z_1}^1(\chi, \chi) = 1.$$

Lemma 5.8. — Assume $p > 2$ and let $\chi, \psi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters then $\mathrm{Ext}_{(I_1 \cap P)/Z_1}^1(\psi, \chi) \neq 0$ if and only if $\psi \in \{\chi, \chi\alpha\}$. Moreover,

$$\dim \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, \chi) = \dim \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi, \chi) = 1.$$

Proposition 5.9. — Let $\chi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be a character and let $\chi \hookrightarrow J_\chi$ be an injective envelope of χ in $\mathrm{Rep}_{H(I_1 \cap U)}$, then $(J_\chi/\chi)^{I_1 \cap U}$ is 1-dimensional and H acts on it by $\chi\alpha^{-1}$. Moreover, $\chi\alpha^{-1} \hookrightarrow J_\chi/\chi$ is an injective envelope of $\chi\alpha^{-1}$ in $\mathrm{Rep}_{H(I_1 \cap U)}$.

Proof. — Consider an exact sequence of $H(I \cap U)$ -representations:

$$0 \rightarrow \chi \rightarrow J_\chi \rightarrow J_\chi/\chi \rightarrow 0.$$

Since J_χ is an injective envelope of χ in $\mathrm{Rep}_{I \cap U}$ taking $I_1 \cap U$ invariants induces H -equivariant isomorphism $(J_\chi/\chi)^{I_1 \cap U} \cong H^1(I_1 \cap U, \chi)$. It follows from Lemma 5.5 that $\dim(J_\chi/\chi)^{I_1 \cap U} = 1$ and H acts on $(J_\chi/\chi)^{I_1 \cap U}$ via the character $\chi\alpha^{-1}$. Let $J_{\chi\alpha^{-1}}$ be an injective envelope of $\chi\alpha^{-1}$ in $\mathrm{Rep}_{H(I_1 \cap U)}$, then there exists an exact sequence of $H(I_1 \cap U)$ -representations:

$$0 \rightarrow J_\chi/\chi \rightarrow J_{\chi\alpha^{-1}} \rightarrow Q \rightarrow 0.$$

Since $J_{\chi\alpha^{-1}}$ is an essential extension of $\chi\alpha^{-1}$, we have $J_{\chi\alpha^{-1}}^{I_1 \cap U} \cong \chi\alpha^{-1}$. Hence taking $(I_1 \cap U)$ -invariants induces an isomorphism $Q^{I_1 \cap U} \cong H^1(I_1 \cap U, J_\chi/\chi) \cong H^2(I_1 \cap U, \chi)$. Since $I_1 \cap U \cong \mathbb{Z}_p$ is a free pro- p group we have $H^2(I_1 \cap U, \chi) = 0$, see [17, §3.4]. Hence $Q^{I_1 \cap U} = 0$, which implies $Q = 0$. $\square$

Lemma 5.10. — *Let $\iota : J \hookrightarrow A$ be a monomorphism in an abelian category $\mathcal{A}$. If J is an injective object in $\mathcal{A}$ then there exists $\sigma : A \rightarrow J$ such that $\sigma \circ \iota = \text{id}$.*

Proof. — Since J is injective the map $\text{Hom}_{\mathcal{A}}(A, J) \rightarrow \text{Hom}_{\mathcal{A}}(J, J)$ is surjective. $\square$

6. Exact sequence

Let $\pi := \pi(r, 0, \eta)$ with $0 \leq r \leq p-1$. We use the notation of §4.1, so that $\sigma := \text{Sym}^r \bar{\mathbb{F}}_p^2 \otimes \det^a$, with $\det^a = \eta \circ \det|_K$, and $\chi : H \rightarrow \bar{\mathbb{F}}_p^\times$ a character as in (9). We construct an exact sequence of I -representations which will be used to calculate $H^1(I_1/Z_1, \pi)$.

Lemma 6.1. — *If $r \neq 0$ then set*

$$w_\sigma := \sum_{\lambda \in \mathbb{F}_p} \lambda^{p-r} \begin{pmatrix} 1 & [\lambda] \\ 0 & 1 \end{pmatrix} t v_{\tilde{\sigma}} + \left(\sum_{\mu \in \mathbb{F}_p} \mu \right) v_\sigma.$$

Then w_σ is fixed by $I_1 \cap P^s$ and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_\sigma = w_\sigma - (-1)^a r v_\sigma.$$

If $r = 0$ then set

$$w_\sigma := \sum_{\lambda, \mu \in \mathbb{F}_p} \lambda \begin{pmatrix} 1 & [\mu] + p[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma.$$

Then

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w_\sigma = w_\sigma + v_\sigma, \quad \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} w_\sigma = w_\sigma - \left(\sum_{\mu \in \mathbb{F}_p} \mu^2 \right) v_\sigma.$$

If $\alpha \in [x] + \mathfrak{p}$, $\beta \in [y] + \mathfrak{p}$ then

$$\begin{pmatrix} 1 + p\alpha & 0 \\ 0 & 1 + p\beta \end{pmatrix} w_\sigma = w_\sigma + (x - y) \left(\sum_{\mu \in \mathbb{F}_p} \mu \right) v_\sigma.$$

Proof. — We set $w := w_\sigma$. Suppose that $r \neq 0$. Now $t v_{\tilde{\sigma}} = s \Pi v_{\tilde{\sigma}} = s v_\sigma$. Hence, if we identify v_σ with $x^r \in \text{Sym}^r \bar{\mathbb{F}}_p^2 \otimes \det^a$ then Lemma 3.1 applied to $j = r-1$ gives $w = -(-1)^a r x^{r-1} y$. This implies the assertion.

Suppose that $r = 0$ and let $P(X) := \frac{X^p+1-(X+1)^p}{p} \in \mathbb{Z}[X]$, then [16] implies that

$$(15) \quad \begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w &= \sum_{\lambda, \mu \in \mathbb{F}_p} \lambda \begin{pmatrix} 1 & 1 + [\mu] + p[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma \\ &= \sum_{\lambda, \mu \in \mathbb{F}_p} \lambda \begin{pmatrix} 1 & [\mu + 1] + p[\lambda + P(\mu)] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma. \end{aligned}$$

Hence,

$$(16) \quad \begin{aligned} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w &= \sum_{\lambda, \mu \in \mathbb{F}_p} \lambda \begin{pmatrix} 1 & [\mu] + p[\lambda + P(\mu - 1)] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma \\ &= \sum_{\lambda, \mu \in \mathbb{F}_p} (\lambda - P(\mu - 1)) \begin{pmatrix} 1 & [\mu] + p[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma \\ &= w - \sum_{\lambda, \mu \in \mathbb{F}_p} P(\mu - 1) \begin{pmatrix} 1 & [\mu] + p[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma \\ &= w + (-1)^a \sum_{\mu \in \mathbb{F}_p} P(\mu - 1) \begin{pmatrix} 1 & [\mu] \\ 0 & 1 \end{pmatrix} t v_{\bar{\sigma}} \\ &= w + \left(\sum_{\mu \in \mathbb{F}_p} P(\mu - 1) \right) v_\sigma, \end{aligned}$$

where the last two equalities follow from (10), (11). If $p = 2$ then $P(X - 1) = 1 - X$, otherwise $P(X - 1) = \sum_{i=1}^{p-1} p^{-1} \binom{p}{i} X^i (-1)^{p-i}$. Hence $\sum_{\mu \in \mathbb{F}_p} P(\mu - 1) = -\sum_{\mu \in \mathbb{F}_p^\times} \mu^{p-1} = 1$.

Now $t^2 v_\sigma$ is fixed by $\begin{pmatrix} 1 & \mathfrak{p}^2 \\ 0 & 1 \end{pmatrix}$ and $I_1 \cap P^s$, so the matrix identity

$$(17) \quad \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha(1 + \alpha\beta)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1 + \alpha\beta)^{-1} & 0 \\ \beta & 1 + \alpha\beta \end{pmatrix}$$

implies that

$$(18) \quad \begin{aligned} \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} w &= \sum_{\lambda, \mu \in \mathbb{F}_p} \lambda \begin{pmatrix} 1 & [\mu] + p[\lambda - \mu^2] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma \\ &= \sum_{\lambda, \mu \in \mathbb{F}_p} (\lambda + \mu^2) \begin{pmatrix} 1 & [\mu] + p[\lambda] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma = w - \left(\sum_{\mu \in \mathbb{F}_p} \mu^2 \right) v_\sigma. \end{aligned}$$

If $\alpha \in [x] + \mathfrak{p}$ and $\beta \in [y] + \mathfrak{p}$ then the same argument gives

$$\begin{aligned} \begin{pmatrix} 1 + p\alpha & 0 \\ 0 & 1 + p\beta \end{pmatrix} w &= \sum_{\lambda, \mu \in \mathbb{F}_p} \lambda \begin{pmatrix} 1 & [\mu] + p[\lambda + \mu(x - y)] \\ 0 & 1 \end{pmatrix} t^2 v_\sigma \\ &= w + (x - y) \left(\sum_{\mu \in \mathbb{F}_p} \mu \right) v_\sigma. \end{aligned} \quad \square$$

Proposition 6.2. — We have $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1 \cap U} = (M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1}$. Moreover, let Δ_σ be the image of $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1}$ in $H^1(I_1, \mathbf{1}) \cong \text{Hom}(I_1, \overline{\mathbb{F}}_p)$. Then the following hold:

- (i) if either $r \neq 0$ or $p > 3$ then $\Delta_\sigma = \overline{\mathbb{F}}_p \kappa^u$;
- (ii) if $p = 3$ and $r = 0$ then $\Delta_\sigma = \overline{\mathbb{F}}_p (\kappa^u + \kappa^l)$;
- (iii) if $p = 2$ and $r = 0$ then $\Delta_\sigma = \overline{\mathbb{F}}_p (\kappa^u + \kappa^l + \varepsilon)$.

Proof. — It follows from Proposition 5.9 that $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1 \cap U}$ is 1-dimensional. Since $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1} \neq 0$ the inclusion $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1} \subseteq (M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1 \cap U}$ is an equality. The image of w_σ of Lemma 6.1 spans $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1}$ and the last assertion follows from Lemma 6.1. $\square$

Theorem 6.3. — The map $(v, w) \mapsto v - w$ induces an exact sequence of I -representations:

$$0 \rightarrow \pi^{I_1} \rightarrow M \oplus \Pi \cdot M \rightarrow \pi \rightarrow 0.$$

Proof. — We claim that $M \cap \Pi \cdot M = \pi^{I_1}$. Consider an exact sequence:

$$0 \rightarrow \pi^{I_1} \rightarrow M \cap \Pi \cdot M \rightarrow Q \rightarrow 0.$$

Since $M \cap \Pi \cdot M$ is an I_1 -invariant subspace of π , we have $(M \cap \Pi \cdot M)^{I_1} \subseteq \pi^{I_1}$. Since $M \cap \Pi \cdot M$ contains π^{I_1} the inclusion is an equality. Hence, by taking I_1 -invariants we obtain an injection $\partial : Q^{I_1} \hookrightarrow H^1(I_1, \pi^{I_1}) \cong \text{Hom}(I_1, \overline{\mathbb{F}}_p) \oplus \text{Hom}(I_1, \overline{\mathbb{F}}_p)$. The element Π acts on $H^1(I_1, \pi^{I_1})$ by $\Pi \cdot (\psi_1, \psi_2) = (\psi_2^\Pi, \psi_1^\Pi)$. Let Δ_σ (resp. $\Delta_{\tilde{\sigma}}$) denote the image of $(M_\sigma / \overline{\mathbb{F}}_p v_\sigma)^{I_1}$ (resp. $(M_{\tilde{\sigma}} / \overline{\mathbb{F}}_p v_{\tilde{\sigma}})^{I_1}$) in $\text{Hom}(I_1, \overline{\mathbb{F}}_p)$. Let Δ be the image of $(M / \pi^{I_1})^{I_1}$ in $H^1(I_1, \pi^{I_1})$ so that $\Delta = \Delta_\sigma \oplus \Delta_{\tilde{\sigma}}$. By taking I_1 -invariants of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^{I_1} & \longrightarrow & M \cap \Pi \cdot M & \longrightarrow & Q \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi^{I_1} & \longrightarrow & M & \longrightarrow & M / \pi^{I_1} \longrightarrow 0 \end{array}$$

we obtain a commutative diagram:

$$\begin{array}{ccc} Q^{I_1} \hookrightarrow & \xrightarrow{\partial} & H^1(I_1, \pi^{I_1}) \\ \downarrow & & \downarrow \text{id} \\ (M / \pi^{I_1})^{I_1} \hookrightarrow & \xrightarrow{\partial} & H^1(I_1, \pi^{I_1}). \end{array}$$

and hence an injection $\partial(Q^{I_1}) \hookrightarrow \Delta$. Acting by Π we obtain an injection $\partial(Q^{I_1}) \hookrightarrow \Pi \cdot \Delta$. We claim that $\Delta \cap \Pi \cdot \Delta = 0$. We have

$$\Delta \cap \Pi \cdot \Delta = (\Delta_\sigma \cap \Pi \cdot \Delta_{\bar{\sigma}}) \oplus (\Delta_{\bar{\sigma}} \cap \Pi \cdot \Delta_\sigma).$$

By symmetry we may assume that $r < p - 1$. Proposition 6.2 applied to M_σ and $M_{\bar{\sigma}}$ implies that if $r \neq 0$ then $\Delta = \overline{\mathbb{F}}_p \kappa^u \oplus \overline{\mathbb{F}}_p \kappa^u$, hence $\Pi \cdot \Delta = \overline{\mathbb{F}}_p(\kappa^u)^\Pi \oplus \overline{\mathbb{F}}_p(\kappa^u)^\Pi = \overline{\mathbb{F}}_p \kappa^l \oplus \overline{\mathbb{F}}_p \kappa^l$, so that $\Delta \cap \Pi \cdot \Delta = 0$. If $r = 0$ then Proposition 6.2 implies that $\Delta = \overline{\mathbb{F}}_p(\kappa^u - (\sum_{\mu \in \mathbb{F}_p} \mu^2) \kappa^l + (\sum_{\mu \in \mathbb{F}_p} \mu) \varepsilon) \oplus \overline{\mathbb{F}}_p \kappa^u$, hence $\Pi \cdot \Delta = \overline{\mathbb{F}}_p \kappa^l \oplus \overline{\mathbb{F}}_p(\kappa^l - (\sum_{\mu \in \mathbb{F}_p} \mu^2) \kappa^u - (\sum_{\mu \in \mathbb{F}_p} \mu) \varepsilon)$, again $\Delta \cap \Pi \cdot \Delta = 0$. Note that if $r = 0$ then we have to apply Proposition 6.2 to $M_{\bar{\sigma}}$ with $r = p - 1$, and $p - 1 \neq 0$. This implies that $Q^{I_1} = 0$ and hence $Q = 0$.

Since G^+ and Π generate G , Proposition 4.12 implies that $\pi_\sigma + \pi_{\bar{\sigma}}$ is stable under the action of G . Since π is irreducible we get $\pi = \pi_\sigma + \pi_{\bar{\sigma}}$. This implies surjectivity. $\square$

Corollary 6.4. — We have $M_\sigma \cap \Pi \cdot M_{\bar{\sigma}} = \pi_\sigma^{I_1} = \overline{\mathbb{F}}_p v_\sigma$ and $M_{\bar{\sigma}} \cap \Pi \cdot M_\sigma = \pi_{\bar{\sigma}}^{I_1} = \overline{\mathbb{F}}_p v_{\bar{\sigma}}$.

Proof. — It is enough to show that $\pi_\sigma^{I_1} = \overline{\mathbb{F}}_p v_\sigma$, since by Theorem 6.3 $M_\sigma \cap \Pi \cdot M_{\bar{\sigma}}$ is contained in π^{I_1} . Suppose not. Clearly $v_\sigma \in \pi_\sigma$, so since π^{I_1} is 2-dimensional, we obtain that $v_{\bar{\sigma}} \in \pi_\sigma$. Then there exists $u_1 \in M_\sigma$ and $u_2 \in \Pi \cdot M_{\bar{\sigma}}$ such that $v_{\bar{\sigma}} = u_1 + u_2$. So $u_2 \in \Pi \cdot M_{\bar{\sigma}} \cap (M_\sigma + M_{\bar{\sigma}}) \subset \pi^{I_1}$ by Theorem 6.3. Hence $u_2 = \lambda v_\sigma$ for some $\lambda \in \overline{\mathbb{F}}_p$, and so $u_2 \in M_\sigma$, and so $v_{\bar{\sigma}} \in M_\sigma$. This contradicts $M_\sigma \cap M_{\bar{\sigma}} = 0$. $\square$

Corollary 6.5. — As G^+ -representation π is the direct sum of its subrepresentations π_σ and $\pi_{\bar{\sigma}}$.

Proof. — It follows from Theorem 6.3 that $\pi = \pi_\sigma + \pi_{\bar{\sigma}}$. Now

$$(\pi_\sigma \cap \pi_{\bar{\sigma}})^{I_1} = \pi_\sigma^{I_1} \cap \pi_{\bar{\sigma}}^{I_1} = \overline{\mathbb{F}}_p v_\sigma \cap \overline{\mathbb{F}}_p v_{\bar{\sigma}} = 0.$$

Hence, $\pi_\sigma \cap \pi_{\bar{\sigma}} = 0$. $\square$

Corollary 6.6. — We have $\pi \cong \mathrm{Ind}_{G^+}^G \pi_\sigma \cong \mathrm{Ind}_{G^+}^G \pi_{\bar{\sigma}}$.

7. Computing $H^1(I_1/Z_1, \pi)$

We keep the notation of §6 and compute $H^1(I_1/Z_1, \pi)$ as a representation of H under the assumption $p > 2$.

Lemma 7.1. — Assume that $p > 2$. Let $\psi, \chi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters. Let N be a smooth representation of $(I \cap P)/Z_1$, such that $N|_{H(I_1 \cap U)}$ is an injective envelope of χ in $\mathrm{Rep}_{H(I_1 \cap U)}$. Suppose that $\mathrm{Ext}_{(I \cap P)/Z_1}^1(\psi, N) \neq 0$ then $\psi = \chi$. Moreover, $\mathrm{Ext}_{(I \cap P)/Z_1}^1(\chi, N) \cong \mathrm{Ext}_{(I \cap P)/Z_1}^1(\chi, \chi)$ is 1-dimensional.

Proof. — Suppose that we have a non-split extension $0 \rightarrow N \rightarrow E \rightarrow \psi \rightarrow 0$. Since $N|_{H(I_1 \cap U)}$ is injective Lemma 5.10 implies that the extension splits when restricted to $H(I_1 \cap U)$. Hence, there exists $v \in E^{I_1 \cap U}$ such that H acts on v by ψ and the image of v spans the underlying vector space of ψ . If v is fixed by $I_1 \cap T$, then since $I_1 \cap T$ and

$H(I_1 \cap U)$ generate $I \cap P$ we would obtain a splitting of E as an $I \cap P$ -representation. Hence, there exists some $h \in I_1 \cap T$, such that $(h-1)v \in N$ is non-zero. Since h normalizes $I_1 \cap U$ and v is fixed by $I_1 \cap U$, we obtain that $(h-1)v \in N^{I_1 \cap U}$. Since H acts on v by ψ and T is abelian, we get that H acts on $(h-1)v$ by ψ . Since $N|_{H(I_1 \cap U)}$ is an injective envelope of χ we obtain that $\chi = \psi$.

By Proposition 5.9, N/χ is an injective envelope of $\chi\alpha^{-1}$. Since $p > 2$, $\chi \neq \chi\alpha^{-1}$ and so $\text{Hom}_{I \cap P}(\chi, N/\chi) = \text{Ext}_{(I \cap P)/Z_1}^1(\chi, N/\chi) = 0$. So applying $\text{Hom}_{I \cap P}(\chi, \cdot)$ to the short exact sequence of $(I \cap P)/Z_1$ representations $0 \rightarrow \chi \rightarrow N \rightarrow N/\chi \rightarrow 0$ gives us an isomorphism $\text{Ext}_{(I \cap P)/Z_1}^1(\chi, N) \cong \text{Ext}_{(I \cap P)/Z_1}^1(\chi, \chi)$. Lemma 5.7 implies that these spaces are 1-dimensional. $\square$

Proposition 7.2. — Assume that $p > 2$. Let $\psi, \chi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be characters. Let N be a smooth representation of I/Z_1 , such that $N|_{H(I_1 \cap U)}$ is an injective envelope of χ in $\text{Rep}_{H(I_1 \cap U)}$. Suppose that $\text{Ext}_{I/Z_1}^1(\psi, N) \neq 0$ and let $\mathcal{K}$ be the kernel of the restriction map $\text{Ext}_{I/Z_1}^1(\psi, N) \rightarrow \text{Ext}_{(I \cap P)/Z_1}^1(\psi, N)$ then one of the following holds:

- (i) if $\mathcal{K} \neq 0$ then $\psi = \chi\alpha$;
- (ii) if $\mathcal{K} = 0$ then $\psi = \chi$.

Moreover, $\dim \text{Ext}_{I/Z_1}^1(\chi\alpha, N) = 1$, and let R be the submodule of N , fitting in the exact sequence $0 \rightarrow N^{I_1} \rightarrow R \rightarrow (N/N^{I_1})^{I_1} \rightarrow 0$, then there exists an exact sequence:

$$0 \rightarrow \text{Hom}_I(\chi, \chi\alpha^{-2}) \rightarrow \text{Ext}_{I/Z_1}^1(\chi, R) \rightarrow \text{Ext}_{I/Z_1}^1(\chi, N) \rightarrow 0.$$

Proof. — Suppose that $\mathcal{K} \neq 0$ then there exists a non-split extension $0 \rightarrow N \rightarrow E \rightarrow \psi \rightarrow 0$ of I/Z_1 -representations, which splits when restricted to $I \cap P$. Hence, there exists $v \in E^{I_1 \cap P}$ such that H acts on v by ψ and the image of v spans the underlying vector space of ψ . Let k be the smallest integer $k \geq 1$ such that v is fixed by $\begin{pmatrix} 1 & 0 \\ p^k & 1 \end{pmatrix}$. If $k = 1$ then v is fixed by $I \cap U^s$. Since $I \cap U^s$ and $I \cap P$ generate I , we would obtain that I acts on v by ψ and hence the extension splits. Hence, k is at least 2. Set $\mathcal{U} := \begin{pmatrix} 1 & 0 \\ p^{k-1} & 1 \end{pmatrix}$. Our assumption on k implies that $v' := \begin{pmatrix} 1 & 0 \\ p^{k-1} & 1 \end{pmatrix}v - v \in N$ is non-zero. The matrix identity (14) implies that v' is fixed by $I_1 \cap U$. Since $N^{I_1 \cap U}$ is 1-dimensional and H acts on $N^{I_1 \cap U}$ by χ , we obtain a non-zero element in $\text{Ext}_{H\mathcal{U}}^1(\psi, \chi)$. Lemma 5.6 implies that $\psi = \chi\alpha$. Let $\bar{v}$ be the image of v in E/N^{I_1} . Again by Proposition 5.9 $(N/N^{I_1})^{I_1 \cap U}$ is 1-dimensional and H acts on $(N/N^{I_1})^{I_1 \cap U}$ by $\chi\alpha^{-1}$. If the extension $0 \rightarrow N/N^{I_1} \rightarrow E/N^{I_1} \rightarrow \psi \rightarrow 0$ is non-split, then by the same argument we would obtain a non-zero element in $\text{Ext}_{H\mathcal{U}'}^1(\chi\alpha, \chi\alpha^{-1})$, where $\mathcal{U}' := \begin{pmatrix} 1 & 0 \\ p^m & 1 \end{pmatrix}$, for some $m \geq 1$. This contradicts Lemma 5.6, as $p > 2$ and so α is non-trivial. Hence we obtain an exact sequence:

$$(19) \quad 0 \rightarrow \text{Hom}_I(\chi\alpha, \chi\alpha^{-1}) \rightarrow \text{Ext}_{I/Z_1}^1(\chi\alpha, \chi) \rightarrow \text{Ext}_{I/Z_1}^1(\chi\alpha, N) \rightarrow 0.$$

If $p > 3$ then $\dim \text{Hom}_I(\chi\alpha, \chi\alpha^{-1}) = 0$ and $\dim \text{Ext}_{I/Z_1}^1(\chi\alpha, \chi) = 1$. If $p = 3$ then $\dim \text{Hom}_I(\chi\alpha, \chi\alpha^{-1}) = 1$ and $\dim \text{Ext}_{I/Z_1}^1(\chi\alpha, \chi) = 2$. Hence, $\dim \text{Ext}_{I/Z_1}^1(\chi\alpha, N) = 1$.

Assume that $\mathcal{K} = 0$. Since we have assumed that $\text{Ext}_{I/Z_1}^1(\psi, N) \neq 0$ we obtain that $\text{Ext}_{(I \cap P)/Z_1}^1(\psi, N) \neq 0$ and Lemma 7.1 implies that $\psi = \chi$

and $\dim \text{Ext}_{I/Z_1}^1(\chi, N) \leq 1$. Suppose that there exists a non-split extension $0 \rightarrow N \rightarrow E \rightarrow \chi \rightarrow 0$ of I/Z_1 -representations, which remains non-split when restricted to $I \cap P$. Let w_1 be a basis vector of $N^{I_1 \cap U}$. Lemmas 7.1, 5.7 and 5.3 imply that there exists $v \in E$ such that H acts on v by χ and for all $g \in I_1 \cap P$ we have $gv = v + \varepsilon(g)w_1$. In particular, v is fixed by $I \cap U$ and $\begin{pmatrix} 1+p^2 & 0 \\ 0 & 1+p^2 \end{pmatrix}$. As before, let k be the smallest integer $k \geq 1$ such that v is fixed by $\begin{pmatrix} 1 & 0 \\ p^k & 1 \end{pmatrix}$. We claim that $k = 2$. Indeed, if $k > 2$ then let $v' := \begin{pmatrix} 1 & 0 \\ p^{k-1} & 1 \end{pmatrix} v - v$. Then $v' \in N$ is non-zero, and the matrix identity (14) implies that v' is fixed by $I_1 \cap U$. Since $N^{I_1 \cap U}$ is 1-dimensional and H acts on $N^{I_1 \cap U}$ by χ , we obtain a non-zero element in $\text{Ext}_{H\mathcal{U}}^1(\chi, \chi)$, with $\mathcal{U} := \begin{pmatrix} 1 & 0 \\ p^{k-1} & 1 \end{pmatrix}$. Lemma 5.6 implies that $\chi = \chi\alpha$. Since $p > 2$ this cannot happen.

Consider $u := \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} v - v$. Using (14) and the fact that $k \geq 2$ we obtain

$$(20) \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u = \begin{pmatrix} 1 & 0 \\ p(1+p)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1+p & 1 \\ 0 & (1+p)^{-1} \end{pmatrix} v - v \\ = \begin{pmatrix} 1 & 0 \\ p(1+p)^{-1} & 1 \end{pmatrix} (v + 2w_1) - v = u + 2w_1.$$

Since $2w_1 \neq 0$ we get $u \neq 0$ and so $k = 2$. By Proposition 5.9 $(N/\overline{\mathbb{F}}_p w_1)^{I_1 \cap U}$ is 1-dimensional. This implies that $(N/\overline{\mathbb{F}}_p w_1)^{I_1 \cap U} \cong (N/\overline{\mathbb{F}}_p w_1)^{I_1}$ and the image of u in $N/\overline{\mathbb{F}}_p w_1$ spans $(N/\overline{\mathbb{F}}_p w_1)^{I_1 \cap U}$. If we set $R := \langle w_1, u \rangle$ then by construction we obtain that the map $\text{Ext}_{I/Z_1}^1(\chi, N) \rightarrow \text{Ext}_{I/Z_1}^1(\chi, N/R)$ is zero. Proposition 5.9 implies that $(N/R)^{I_1}$ is 1-dimensional and H acts on it by a character $\chi\alpha^{-2}$. This implies the claim. $\square$

Corollary 7.3. — Assume $p > 2$ then the restriction maps

$$\begin{aligned} \text{Ext}_{I/Z_1}^2(\chi, \chi) &\rightarrow \text{Ext}_{(I \cap P^s)/Z_1}^2(\chi, \chi), \\ \text{Ext}_{I/Z_1}^2(\chi, \chi) &\rightarrow \text{Ext}_{(I \cap P)/Z_1}^2(\chi, \chi) \end{aligned}$$

are injective.

Proof. — Consider the exact sequence of I -representations $0 \rightarrow \chi \rightarrow \text{Ind}_{I \cap P^s}^I \chi \rightarrow Q \rightarrow 0$. Iwahori decomposition implies that

$$(\text{Ind}_{I \cap P^s}^I \chi)|_{H(I_1 \cap U)} \cong \text{Ind}_H^{H(I_1 \cap U)} \chi,$$

and hence it is an injective envelope of χ in $\text{Rep}_{H(I_1 \cap U)}$. Proposition 5.9 implies that $Q|_{H(I_1 \cap U)}$ is an injective envelope of $\chi\alpha^{-1}$ in $\text{Rep}_{H(I_1 \cap U)}$. Since $p > 2$ Lemma 5.4 implies that $\text{Ext}_{I/Z_1}^1(\chi, \chi) = 0$, so using Shapiro's lemma we obtain an exact sequence:

$$\begin{aligned} \text{Ext}_{(I \cap P^s)/Z_1}^1(\chi, \chi) &\hookrightarrow \text{Ext}_{I/Z_1}^1(\chi, Q) \rightarrow \text{Ext}_{I/Z_1}^2(\chi, \chi) \\ &\rightarrow \text{Ext}_{(I \cap P^s)/Z_1}^2(\chi, \chi). \end{aligned}$$

Now $\dim \text{Ext}_{(I \cap P^s)/Z_1}^1(\chi, \chi) = 1$ and $\dim \text{Ext}_{I/Z_1}^1(\chi, Q) = 1$ by Proposition 7.2. This implies the result for $I \cap P^s$. By conjugating by Π we obtain the result for $I \cap P$. $\square$

Corollary 7.4. — Assume $p > 2$ and let N be as in Proposition 7.2 then $\dim \operatorname{Ext}_{I/Z_1}^1(\chi, N) = 1$, the natural maps

$$(21) \quad \operatorname{Ext}_{I/Z_1}^2(\chi, \chi) \rightarrow \operatorname{Ext}_{I/Z_1}^2(\chi, N),$$

$$(22) \quad \operatorname{Ext}_{I/Z_1}^1(\chi, N) \rightarrow \operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi, N)$$

are injective and (22) is an isomorphism.

Proof. — We have an exact sequence:

$$\operatorname{Ext}_{I/Z_1}^1(\chi, N) \hookrightarrow \operatorname{Ext}_{I/Z_1}^1(\chi, N/\chi) \rightarrow \operatorname{Ext}_{I/Z_1}^2(\chi, \chi).$$

Proposition 5.9 and Lemma 7.1 imply that $\operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi, N/\chi) = 0$. The commutative diagram:

$$\begin{array}{ccc} \operatorname{Ext}_{I/Z_1}^1(\chi, N/\chi) & \longrightarrow & \operatorname{Ext}_{I/Z_1}^2(\chi, \chi) \\ \downarrow 0 & & \downarrow 7.3 \\ \operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi, N/\chi) & \xrightarrow{0} & \operatorname{Ext}_{(I \cap P)/Z_1}^2(\chi, \chi) \end{array}$$

and Corollary 7.3 implies that $\operatorname{Ext}_{I/Z_1}^1(\chi, N/\chi) \rightarrow \operatorname{Ext}_{I/Z_1}^2(\chi, \chi)$ is the zero map. Hence, (21) is injective and

$$\dim \operatorname{Ext}_{I/Z_1}^1(\chi, N) = \dim \operatorname{Ext}_{I/Z_1}^1(\chi, N/\chi) = 1,$$

where the last equality is given by Propositions 5.9 and 7.2. We know that $\operatorname{Ext}_{I/Z_1}^1(\chi, N) \neq 0$. So if (22) is not injective, then Proposition 7.2 gives $\chi = \chi\alpha$, but this cannot hold, since $p > 2$. Since both sides have dimension 1, (22) is an isomorphism. $\square$

7.1. $p = 3$. — The case $p = 3$ requires some extra arguments. If you are only interested in $p \geq 5$ then please skip this subsection.

Lemma 7.5. — Assume $p = 3$ and let N be as in Proposition 7.2 then the composition:

$$\operatorname{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) \xrightarrow{\partial} \operatorname{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \xrightarrow{\operatorname{Res}} \operatorname{Ext}_{(I \cap P)/Z_1}^2(\chi\alpha, \chi)$$

is injective, where ∂ is induced by a short exact sequence $0 \rightarrow \chi \rightarrow N \rightarrow N/\chi \rightarrow 0$.

Proof. — Since $p = 3$ we have $\alpha = \alpha^{-1}$ and hence it follows from the Corollary 7.4 that $\dim \operatorname{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) = 1$. Corollary 7.2 implies that the restriction map $\operatorname{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) \rightarrow \operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi\alpha, N/\chi)$ is injective. Moreover, Lemma 7.1 gives $\operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi\alpha, N) = 0$, and so the map $\partial : \operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi\alpha, N/\chi) \rightarrow \operatorname{Ext}_{(I \cap P)/Z_1}^2(\chi\alpha, \chi)$ is injective. The assertion follows from the commutative diagram:

$$\begin{array}{ccc} \operatorname{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) & \xrightarrow{\partial} & \operatorname{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \\ \downarrow \text{(22) Res} & & \downarrow \text{Res} \\ \operatorname{Ext}_{(I \cap P)/Z_1}^1(\chi\alpha, N/\chi) & \xhookrightarrow{\partial} & \operatorname{Ext}_{(I \cap P)/Z_1}^2(\chi\alpha, \chi). \end{array} \quad \square$$

Lemma 7.6. — Assume $p = 3$ and let N be as in Proposition 7.2. Assume that $N^{K_1} \cong \mathrm{Ind}_{HK_1}^I \chi$ as a representation of I , then the composition:

$$\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) \xrightarrow{\partial} \mathrm{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \xrightarrow{\mathrm{Res}} \mathrm{Ext}_{(I \cap P^s)/Z_1}^2(\chi\alpha, \chi)$$

is zero, where ∂ is induced by a short exact sequence $0 \rightarrow \chi \rightarrow N \rightarrow N/\chi \rightarrow 0$.

Proof. — Since $p = 3$ we have $\alpha = \alpha^{-1}$ and hence it follows from the Corollary 7.4 that $\dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) = 1$. Let Δ be the image of the restriction map

$$\Delta := \mathrm{Im}(\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) \rightarrow \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N/\chi)).$$

We claim that Δ is contained in the image of the natural map

$$(23) \quad \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N) \rightarrow \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N/\chi).$$

Since $p = 3$ we have $\dim N^{K_1} = 3$ and so the image of N^{K_1} in N/N^{I_1} is a 2-dimensional I -stable subspace. Since it follows from Proposition 5.9 that $(N/N^{I_1})^{I_1}$ and $((N/N^{I_1})/(N/N^{I_1})^{I_1})^{I_1}$ are 1-dimensional we obtain an exact sequence $0 \rightarrow N^{I_1} \rightarrow N^{K_1} \rightarrow R \rightarrow 0$, where R is the subspace of N/χ defined in Proposition 7.2 (with N/χ instead of N). Since $N^{K_1} \cong \mathrm{Ind}_{HK_1}^I \chi$ we get:

$$N^{K_1}|_{I \cap P^s} \cong \chi \oplus \chi\alpha \oplus \chi \cong \chi \oplus R|_{I \cap P^s}.$$

Let ϕ be the composition:

$$\begin{aligned} \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, R) &\rightarrow \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N^{K_1}) \rightarrow \\ &\rightarrow \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N) \rightarrow \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N/\chi). \end{aligned}$$

Then we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, R) & \xrightarrow{7.2} & \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) \\ \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\ \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, R) & \xrightarrow{\phi} & \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N/\chi). \end{array}$$

The top horizontal arrow is surjective by Proposition 7.2. Hence, Δ equals to the image of $\phi \circ \mathrm{Res}$. Since the image of ϕ is contained in the image of (23) we get the claim. The assertion follows from the commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, N/\chi) & \xrightarrow{\partial} & \mathrm{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \\ \downarrow \mathrm{Res} & & \downarrow \mathrm{Res} \\ \mathrm{Ext}_{(I \cap P^s)/Z_1}^1(\chi\alpha, N/\chi) & \xrightarrow{\partial} & \mathrm{Ext}_{(I \cap P^s)/Z_1}^2(\chi\alpha, \chi), \end{array}$$

since the claim implies that the composition $\partial \circ \mathrm{Res}$ is the zero map. $\square$

Lemma 7.7. — Assume $p = 3$ let N_χ and N_{χ^s} be as in Proposition 7.2 with respect to χ and χ^s . Further assume that $N_{\chi^s}^{K_1} \cong \text{Ind}_{HK_1}^I \chi^s$ as a representation of I , then the natural map

$$(24) \quad \text{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \rightarrow \text{Ext}_{I/Z_1}^2(\chi\alpha, N_\chi) \oplus \text{Ext}_{I/Z_1}^2(\chi\alpha, N_{\chi^s}^\Pi)$$

is injective, where $N_{\chi^s}^\Pi$ denotes the twist of action of I on N_{χ^s} by Π .

Proof. — Applying $\text{Hom}_{I/Z_1}(\chi\alpha, \cdot)$ to the short exact sequence $0 \rightarrow \chi \rightarrow N_\chi \rightarrow N_\chi/\chi \rightarrow 0$ gives a long exact sequence. Equation (19) shows that the map $\text{Ext}_{I/Z_1}^1(\chi\alpha, \chi) \rightarrow \text{Ext}_{I/Z_1}^1(\chi\alpha, N_\chi)$ is surjective, which implies that

$$\text{Ker}(\text{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \rightarrow \text{Ext}_{I/Z_1}^2(\chi\alpha, N_\chi)) \cong \text{Ext}_{I/Z_1}^1(\chi\alpha, N_\chi/\chi).$$

If we replace N_χ with N_{χ^s} and χ with χ^s the same isomorphism holds. Twisting by Π gives:

$$\text{Ker}(\text{Ext}_{I/Z_1}^2(\chi\alpha, \chi) \rightarrow \text{Ext}_{I/Z_1}^2(\chi\alpha, N_{\chi^s}^\Pi)) \cong \text{Ext}_{I/Z_1}^1(\chi\alpha, N_{\chi^s}^\Pi/\chi).$$

Lemma 7.5 implies that the composition

$$\text{Res} \circ \partial : \text{Ext}_{I/Z_1}^1(\chi\alpha, N_\chi/\chi) \rightarrow \text{Ext}_{(I \cap P)/Z_1}^2(\chi\alpha, \chi)$$

is an injection. And Lemma 7.6 implies that the composition

$$\text{Res} \circ \partial : \text{Ext}_{I/Z_1}^1(\chi\alpha, N_{\chi^s}^\Pi/\chi) \rightarrow \text{Ext}_{(I \cap P)/Z_1}^2(\chi\alpha, \chi)$$

is zero. Hence, $\partial(\text{Ext}_{I/Z_1}^1(\chi\alpha, N_\chi/\chi)) \cap \partial(\text{Ext}_{I/Z_1}^1(\chi\alpha, N_{\chi^s}^\Pi/\chi)) = 0$ and so the map in (24) is injective. $\square$

Lemma 7.8. — Assume $p = 3$ and $r = 0$ then $M_{\bar{\sigma}}$ satisfies the assumptions of Lemma 7.6.

Proof. — Now $\langle (I \cap U)tv_\sigma \rangle = \langle Isv_{\bar{\sigma}} \rangle \cong St|_I \cong \text{Ind}_{HK_1}^I \chi^s$ as a representation of I , where $St \cong \text{Sym}^2 \bar{\mathbb{F}}_3^2$ is the Steinberg representation of $\text{GL}_2(\mathbb{F}_3)$. Hence we have an injection $\text{Ind}_{HK_1}^I \chi^s \hookrightarrow M_{\bar{\sigma}}$. Since $M_{\bar{\sigma}}|_{H(I \cap U)}$ is an injective envelope of χ^s in $\text{Rep}_{H(I \cap U)}$ we obtain that $M_{\bar{\sigma}}^{K_1 \cap U} \cong \text{Ind}_{H(K_1 \cap U)}^{H(I \cap U)} \chi^s$ as a representation of $H(I \cap U)$. Hence $\dim M_{\bar{\sigma}}^{K_1 \cap U} = 3$ and so we obtain $M_{\bar{\sigma}}^{K_1 \cap U} \cong M_{\bar{\sigma}}^{K_1} \cong \text{Ind}_{HK_1}^I \chi^s$. $\square$

7.2. — Using the lemmas above we prove the main result of this section.

Theorem 7.9. — Assume $p > 2$ and let $\psi : H \rightarrow \bar{\mathbb{F}}_p^\times$ be a character, such that $\text{Ext}_{I/Z_1}^1(\psi, \pi_\sigma) \neq 0$. Then $\psi \in \{\chi\alpha, \chi\}$. Moreover,

- (i) $\dim \text{Ext}_{I/Z_1}^1(\chi, \pi_\sigma) = 2$;
- (ii) if $p > 3$ or $p = 3$ and $r \in \{0, 2\}$ then $\text{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma) = 0$;
- (iii) if $p = 3$ and $r = 1$ then $\dim \text{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma) \leq 1$.

Proof. — Corollary 7.4, (21) gives injections:

$$\begin{aligned}\mathrm{Ext}_{I/Z_1}^2(\chi, \chi) &\hookrightarrow \mathrm{Ext}_{I/Z_1}^2(\chi, M_\sigma), \\ \mathrm{Ext}_{I/Z_1}^2(\chi, \chi) &\hookrightarrow \mathrm{Ext}_{I/Z_1}^2(\chi, \Pi \cdot M_{\bar{\sigma}}).\end{aligned}$$

Moreover, $\mathrm{Ext}_{I/Z_1}^1(\chi, \chi) = 0$. Corollary 6.4 gives a short exact sequence $0 \rightarrow \chi \rightarrow M_\sigma \oplus \Pi \cdot M_{\bar{\sigma}} \rightarrow \pi_\sigma \rightarrow 0$, which induces an isomorphism:

$$\mathrm{Ext}_{I/Z_1}^1(\chi, M_\sigma) \oplus \mathrm{Ext}_{I/Z_1}^1(\chi, \Pi \cdot M_{\bar{\sigma}}) \cong \mathrm{Ext}_{I/Z_1}^1(\chi, \pi_\sigma).$$

Corollary 7.4 implies that $\dim \mathrm{Ext}_{I/Z_1}^1(\chi, \pi_\sigma) = 2$.

Assume that $\psi \neq \chi$. From $0 \rightarrow M_\sigma \rightarrow \pi_\sigma \rightarrow (\Pi \cdot M_{\bar{\sigma}})/\chi \rightarrow 0$ we obtain a long exact sequence:

$$\begin{aligned}\mathrm{Hom}_I(\psi, \chi\alpha) &\hookrightarrow \mathrm{Ext}_{I/Z_1}^1(\psi, M_\sigma) \rightarrow \mathrm{Ext}_{I/Z_1}^1(\psi, \pi_\sigma) \rightarrow \\ &\quad \mathrm{Ext}_{I/Z_1}^1(\psi, (\Pi \cdot M_{\bar{\sigma}})/\chi).\end{aligned}$$

If $\mathrm{Ext}_{I/Z_1}^1(\psi, M_\sigma) \neq 0$ then Proposition 7.2 implies $\psi = \chi\alpha$. Similarly, if $\mathrm{Ext}_{I/Z_1}^1(\psi, (\Pi \cdot M_{\bar{\sigma}})/\chi) \neq 0$ then $\psi = (\chi^s\alpha^{-1})^\Pi = \chi\alpha$. Hence, $\psi = \chi\alpha$ and $\dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma) \leq 1$.

If $p > 3$ then Proposition 7.2 implies that $\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, M_\sigma/\chi) = 0$. Hence the exact sequence $0 \rightarrow \Pi \cdot M_{\bar{\sigma}} \rightarrow \pi_\sigma \rightarrow M_\sigma/\chi \rightarrow 0$ gives an exact sequence:

$$\mathrm{Hom}_I(\chi\alpha, \chi\alpha^{-1}) \hookrightarrow \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \Pi \cdot M_{\bar{\sigma}}) \rightarrow \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma).$$

Since $p > 3$ Proposition 7.2 implies that $\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \Pi \cdot M_{\bar{\sigma}}) = 0$ and hence $\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma) = 0$.

Assume that $p = 3$ and $r = 0$ Lemmas 7.7 and 7.8 give an exact sequence:

$$\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \chi) \hookrightarrow \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, M_\sigma \oplus \Pi \cdot M_{\bar{\sigma}}) \twoheadrightarrow \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma).$$

Since $p = 3$ we have $\dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \chi) = 2$ and Proposition 7.2 gives $\dim \mathrm{Ext}_{I/Z_1}^1(\chi\alpha, M_\sigma \oplus \Pi \cdot M_{\bar{\sigma}}) = 2$. Hence $\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \pi_\sigma) = 0$. Since $p = 3$ and $r = 0$ we have $(\chi\alpha)^\Pi = \chi\alpha$, $\chi = \chi^s$ and since $\pi_{\bar{\sigma}} = \Pi \cdot \pi_\sigma$, we also obtain $\mathrm{Ext}_{I/Z_1}^1(\chi\alpha, \pi_{\bar{\sigma}}) = 0$, which deals with the case $p = 3$ and $r = 2$. $\square$

Corollary 7.10. — Assume $p > 2$ and let $\psi : H \rightarrow \bar{\mathbb{F}}_p^\times$ be a character. Suppose that $\mathrm{Hom}_I(\psi, H^1(I_1/Z_1, \pi)) \neq 0$ then $\psi \in \{\chi, \chi^s\}$. Moreover, the following hold:

- (i) if $p = 3$ and $r = 1$ then $\dim H^1(I_1/Z_1, \pi) \leq 6$;
- (ii) otherwise, $\dim H^1(I_1/Z_1, \pi) = 4$.

Proof. — By Corollary 6.5 $\pi \cong \pi_\sigma \oplus \pi_{\bar{\sigma}}$ as I -representations. The assertion follows from Theorem 7.9. We note that if $p = 3$ and $r = 1$ then $\chi\alpha = \chi^s$ and $\chi^s\alpha = \chi$. $\square$

8. Extensions and central characters

We fix a smooth character $\zeta : Z \rightarrow \overline{\mathbb{F}}_p^\times$ and let $\text{Rep}_{G,\zeta}$ be the full category of Rep_G consisting of representations with central character ζ . Let V be an $\overline{\mathbb{F}}_p$ -vector space with an action of Z , given by $zv = \zeta(z)v$, for all $z \in Z$ and $v \in V$. Then $\text{Ind}_Z^G V$ is an object of $\text{Rep}_{G,\zeta}$, moreover given π in $\text{Rep}_{G,\zeta}$ by Frobenius reciprocity we get

$$(25) \quad \text{Hom}_G(\pi, \text{Ind}_Z^G V) \cong \text{Hom}_Z(\pi, V) \cong \text{Hom}_{\overline{\mathbb{F}}_p}(\pi, V).$$

Hence, the functor $\text{Hom}_G(\cdot, \text{Ind}_Z^G V)$ is exact and so $\text{Ind}_Z^G V$ is an injective object in $\text{Rep}_{G,\zeta}$. Further, if V is the underlying vector space of π then we may embed $\pi \hookrightarrow \text{Ind}_Z^G V$, $v \mapsto [g \mapsto gv]$. Hence, $\text{Rep}_{G,\zeta}$ has enough injectives.

For π_1, π_2 in $\text{Rep}_{G,\zeta}$ we denote $\text{Ext}_{G,\zeta}^1(\pi_1, \pi_2) := \mathbb{R}^1 \text{Hom}(\pi_1, \pi_2)$ computed in the category $\text{Rep}_{G,\zeta}$.

Proposition 8.1. — *Let π_1 and π_2 be irreducible representations of G admitting a central character. Let ζ be the central character of π_2 . If $\text{Ext}_G^1(\pi_1, \pi_2) \neq 0$ then ζ is also the central character of π_1 . If $\pi_1 \not\cong \pi_2$ then $\text{Ext}_{G,\zeta}^1(\pi_1, \pi_2) = \text{Ext}_G^1(\pi_1, \pi_2)$. If $\pi_1 \cong \pi_2$ then there exists an exact sequence:*

$$0 \rightarrow \text{Ext}_{G,\zeta}^1(\pi_1, \pi_2) \rightarrow \text{Ext}_G^1(\pi_1, \pi_2) \rightarrow \text{Hom}(Z, \overline{\mathbb{F}}_p).$$

Moreover, if $p > 2$ then the last arrow is surjective.

Proof. — Suppose that we have a non-split extension $0 \rightarrow \pi_2 \rightarrow E \rightarrow \pi_1 \rightarrow 0$ in Rep_G . For all $z \in Z$ we define $\theta_z : E \rightarrow E$, $v \mapsto zv - \zeta(z)v$. Since z is central in G , θ_z is G -equivariant. If $\theta_z = 0$ for all $z \in Z$ then E admits a central character ζ , and hence ζ is the central character of π_1 and the extension lies in $\text{Ext}_{G,\zeta}^1(\pi_1, \pi_2)$. If $\theta_z \neq 0$ for some $z \in Z$ then it induces an isomorphism $\pi_1 \cong \pi_2$.

We assume that $\pi_1 \cong \pi_2$ and drop the subscript. Then (25) gives $\text{Hom}_G(\pi, \text{Ind}_Z^G \zeta) \cong \pi^*$. Fix a non-zero $\varphi \in \text{Hom}_Z(\pi, \zeta)$. Since π is irreducible we obtain an exact sequence:

$$(26) \quad 0 \rightarrow \pi \xrightarrow{\varphi} \text{Ind}_Z^G \zeta \rightarrow Q \rightarrow 0.$$

Since $\text{Ind}_Z^G \zeta$ is an injective object in $\text{Rep}_{G,\zeta}$, and (26) is in $\text{Rep}_{G,\zeta}$ by applying $\text{Hom}_G(\pi, \cdot)$ to (26) we obtain an exact sequence:

$$(27) \quad \pi^* \rightarrow \text{Hom}_G(\pi, Q) \rightarrow \text{Ext}_{G,\zeta}^1(\pi, \pi) \rightarrow 0.$$

If we consider (26) as an exact sequence in Rep_G then by applying $\text{Hom}_G(\pi, \cdot)$ we get an exact sequence:

$$(28) \quad \pi^* \rightarrow \text{Hom}_G(\pi, Q) \rightarrow \text{Ext}_G^1(\pi, \pi) \rightarrow \text{Ext}_G^1(\pi, \text{Ind}_Z^G \zeta).$$

Putting (27) and (28) together we obtain an exact sequence:

$$0 \rightarrow \text{Ext}_{G,\zeta}^1(\pi, \pi) \rightarrow \text{Ext}_G^1(\pi, \pi) \rightarrow \text{Ext}_G^1(\pi, \text{Ind}_Z^G \zeta).$$

Let $0 \rightarrow \text{Ind}_Z^G \zeta \rightarrow E \rightarrow \pi \rightarrow 0$ be an extension in Rep_G . For all $z \in Z$, $\theta_z : E \rightarrow E$ induces $\theta_z(E) \in \text{Hom}_G(\pi, \text{Ind}_Z^G \zeta)$. Now $\theta_z(E) = 0$ for all $z \in Z$ if and only if E has

a central character ζ , but since $\mathrm{Ind}_Z^G \zeta$ is an injective object in $\mathrm{Rep}_{G,\zeta}$ Lemma 5.10 implies that the sequence is split if and only if E has a central character ζ . Now

$$(29) \quad \begin{aligned} \theta_{z_1 z_2}(v) &= z_1 z_2 v - \zeta(z_1 z_2)v = z_1(z_2 v - \zeta(z_2)v) + z_1 \zeta(z_2)v - \zeta(z_1 z_2)v \\ &= \zeta(z_1)\theta_{z_2}(v) + \zeta(z_2)\theta_{z_1}(v). \end{aligned}$$

Hence, if we set $\psi_E(z) := \zeta(z)^{-1}\theta_z(E)$, then (29) gives $\psi_E(z_1 z_2) = \psi_E(z_1) + \psi_E(z_2)$. Hence, the map $E \mapsto \psi_E$ induces an injection $\mathrm{Ext}_G^1(\pi, \mathrm{Ind}_Z^G \zeta) \hookrightarrow \mathrm{Hom}(Z, \pi^*)$. The image of

$$\mathrm{Ext}_G^1(\pi, \pi) \rightarrow \mathrm{Ext}_G^1(\pi, \mathrm{Ind}_Z^G \zeta) \hookrightarrow \mathrm{Hom}(Z, \pi^*)$$

is contained in $\mathrm{Hom}(Z, \overline{\mathbb{F}}_p \varphi)$, which is isomorphic to $\mathrm{Hom}(Z, \overline{\mathbb{F}}_p)$. To a continuous group homomorphism $\psi : G \rightarrow \overline{\mathbb{F}}_p$ we may associate an extension $\begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} \otimes \pi$. The image of this extension in $\mathrm{Hom}(Z, \overline{\mathbb{F}}_p)$ is equal to the restriction of ψ to Z . If $p > 2$ then the restriction map induces a surjection $\mathrm{Hom}(G, \overline{\mathbb{F}}_p) \twoheadrightarrow \mathrm{Hom}(Z, \overline{\mathbb{F}}_p)$, which yields the last assertion. $\square$

Proposition 8.2. — *Let $\pi := \pi(r, 0, \eta)$ and ζ the central character of π . Assume that $p > 2$ and $(p, r) \neq (3, 1)$ then $\dim \mathrm{Ext}_{G,\zeta}^1(\pi, \pi) \geq 3$.*

Proof. — This follows from [10, 2.3.4]. $\square$

Remark 8.3. — *At the time of writing this note, [10] was not written up and there were some technical issues with the outline of the argument given in the introductions to [7] and [9]. Since we only need a lower bound on the dimension and only in the supersingular case, we have written up another proof of Proposition 8.2 in the appendix. The proof given there is a variation of Colmez-Kisin argument.*

9. Hecke Algebra

Let ζ be the central character of π . Let $\mathcal{H} := \mathrm{End}_G(\mathrm{c}\text{-}\mathrm{Ind}_{Z I_1}^G \zeta)$. Let $\mathcal{I} : \mathrm{Rep}_{G,\zeta} \rightarrow \mathrm{Mod}_{\mathcal{H}}$ be the functor:

$$\mathcal{I}(\pi) := \pi^{I_1} \cong \mathrm{Hom}_G(\mathrm{c}\text{-}\mathrm{Ind}_{Z I_1}^G \zeta, \pi).$$

Let $\mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \rightarrow \mathrm{Rep}_{G,\zeta}$ be the functor:

$$\mathcal{T}(M) := M \otimes_{\mathcal{H}} \mathrm{c}\text{-}\mathrm{Ind}_{Z I_1}^G \zeta.$$

One has $\mathrm{Hom}_{\mathcal{H}}(M, \mathcal{I}(\pi)) \cong \mathrm{Hom}_G(\mathcal{T}(M), \pi)$. Moreover, Vignéras in [18, Thm. 5.4] shows that $\mathcal{I}$ induces a bijection between irreducible objects in $\mathrm{Rep}_{G,\zeta}$ and $\mathrm{Mod}_{\mathcal{H}}$. Let $\mathrm{Rep}_{G,\zeta}^{I_1}$ be the full subcategory of $\mathrm{Rep}_{G,\zeta}$ consisting of representations generated by their I_1 -invariants. Ollivier has shown [13] that

$$(30) \quad \mathcal{I} : \mathrm{Rep}_{G,\zeta}^{I_1} \rightarrow \mathrm{Mod}_{\mathcal{H}}, \quad \mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \rightarrow \mathrm{Rep}_{G,\zeta}^{I_1}$$

are quasi-inverse to each other and so $\text{Mod}_{\mathcal{H}}$ is equivalent to $\text{Rep}_{G,\zeta}^{I_1}$. In particular, suppose that $\tau = \langle G \cdot \tau^{I_1} \rangle$, π in $\text{Rep}_{G,\zeta}$ and let $\pi_1 := \langle G \cdot \pi^{I_1} \rangle \subseteq \pi$ then one has:

$$(31) \quad \begin{aligned} \text{Hom}_G(\tau, \pi) &\cong \text{Hom}_G(\tau, \pi_1) \cong \text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathcal{J}(\pi_1)) \\ &\cong \text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathcal{J}(\pi)) \end{aligned}$$

and the natural map $\mathcal{J}\mathcal{J}(\tau) \rightarrow \tau$ is an isomorphism.

Let J be an injective object in $\text{Rep}_{G,\zeta}$, then the first isomorphism of (31) implies that $J_1 := \langle G \cdot J^{I_1} \rangle$ is an injective object in $\text{Rep}_{G,\zeta}^{I_1}$. Since $\mathcal{T}$ and $\mathcal{J}$ induce an equivalence of categories between $\text{Mod}_{\mathcal{H}}$ and $\text{Rep}_{G,\zeta}^{I_1}$ we obtain that $\mathcal{J}(J_1) = \mathcal{J}(J)$ is an injective object in $\text{Mod}_{\mathcal{H}}$. Hence, (31) gives an E_2 -spectral sequence:

$$(32) \quad \text{Ext}_{\mathcal{H}}^i(\mathcal{J}(\tau), \mathbb{R}^j \mathcal{J}(\pi)) \implies \text{Ext}_{G,\zeta}^{i+j}(\tau, \pi)$$

The 5-term sequence associated to (32) gives us:

Proposition 9.1. — *Let τ and π be in $\text{Rep}_{G,\zeta}$ suppose that τ is generated as a G -representation by τ^{I_1} then there exists an exact sequence:*

$$(33) \quad \begin{aligned} 0 \rightarrow \text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\tau), \mathcal{J}(\pi)) &\rightarrow \text{Ext}_{G,\zeta}^1(\tau, \pi) \rightarrow \text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi)) \\ &\rightarrow \text{Ext}_{\mathcal{H}}^2(\mathcal{J}(\tau), \mathcal{J}(\pi)) \rightarrow \text{Ext}_{G,\zeta}^2(\tau, \pi) \end{aligned}$$

It is easy to write down the first two non-trivial arrows of (33) explicitly. An extension class of $0 \rightarrow \mathcal{J}(\pi) \rightarrow E \rightarrow \mathcal{J}(\tau) \rightarrow 0$ maps to the extension class of $0 \rightarrow \mathcal{T}\mathcal{J}(\pi) \rightarrow \mathcal{T}(E) \rightarrow \mathcal{T}\mathcal{J}(\tau) \rightarrow 0$. Let ϵ be an extension class of $0 \rightarrow \pi \rightarrow \kappa \rightarrow \tau \rightarrow 0$. We may apply $\mathcal{J}$ to get

$$(34) \quad 0 \longrightarrow \mathcal{J}(\pi) \longrightarrow \mathcal{J}(\kappa) \longrightarrow \mathcal{J}(\tau) \xrightarrow{\partial_\epsilon} \mathbb{R}^1 \mathcal{J}(\pi).$$

The second non-trivial arrow in (33) is given by $\epsilon \mapsto \partial_\epsilon$.

We are interested in (32) when both π and τ are irreducible. We recall some facts about the structure of $\mathcal{H}$ and its irreducible modules, for proofs see [18] or [14, §1]. As an $\overline{\mathbb{F}}_p$ -vector space $\mathcal{H}$ has a basis indexed by double cosets $I_1 \backslash G / ZI_1$, we write T_g for the element corresponding to a double coset $I_1 g ZI_1$. Given π in $\text{Rep}_{G,\zeta}$, and $v \in \pi^{I_1}$, the action of T_g is given by:

$$(35) \quad vT_g = \sum_{u \in I_1 / (I_1 \cap g^{-1} I_1 g)} u g^{-1} v.$$

Let $\chi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be a character then we define $e_\chi \in \mathcal{H}$ by

$$e_\chi := \frac{1}{|H|} \sum_{h \in H} \chi(h) T_h.$$

Then $e_\chi e_\psi = e_\chi$ if $\chi = \psi$ and 0 otherwise and it follows from (35) that $\pi^{I_1} e_\chi$ is the χ -isotypical subspace of π^{I_1} as a representation of H . The elements T_{n_s}, T_{Π} and

e_χ , for all χ generate $\mathcal{H}$ as an algebra, and are subject to the following relations:
 $T_\Pi^2 = \zeta(p)^{-1}$,

$$(36) \quad e_\chi T_{n_s} = T_{n_s} e_{\chi^s}, \quad e_\chi T_\Pi = T_\Pi e_{\chi^s}, \quad e_\chi T_{n_s}^2 = -e_\chi e_{\chi^s} T_{n_s}.$$

Note that $e_\chi e_{\chi^s} = e_\chi$ if $\chi = \chi^s$ and $e_\chi e_{\chi^s} = 0$, otherwise. We let $\mathcal{H}^+$ be the subalgebra of $\mathcal{H}$ generated by T_{n_s} , $T_\Pi T_{n_s} T_\Pi^{-1}$ and e_χ for all characters χ . One may naturally identify $\mathcal{H}^+ \cong \mathrm{End}_{G^+}(\mathrm{c}\text{-Ind}_{Z_{I_1}}^{G^+} \zeta)$.

Definition 9.2. — Let $0 \leq r \leq p-1$ be an integer, $\lambda \in \overline{\mathbb{F}}_p$ and $\eta : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$ a smooth character, and let ζ be the central character of $\pi(r, \lambda, \eta)$ then we define $\mathcal{H}$ -modules $M(r, \lambda) := \pi(r, \lambda)^{I_1}$, $M(r, \lambda, \eta) := \pi(r, \lambda, \eta)^{I_1}$.

Assume for simplicity that $\zeta(p) = 1$ then it is shown in [6, Cor. 6.4] that $M(r, \lambda, \eta)$ has an $\overline{\mathbb{F}}_p$ -basis $\{v_1, v_2\}$ such that

- (i) $v_1 e_\chi = v_1$, $v_1 T_\Pi = v_2$, $v_2 e_{\chi^s} = v_2$, $v_2 T_\Pi = v_1$ and such that $v_1 T_{n_s} = -v_1$ if $r = p-1$ and $v_1 T_{n_s} = 0$ otherwise.
- (ii) $v_2(1 + T_{n_s}) = \eta(-p^{-1})\lambda v_1$ if $r = 0$ and $v_2 T_{n_s} = \eta(-p^{-1})\lambda v_1$ otherwise,

where $\chi : H \rightarrow \overline{\mathbb{F}}_p^\times$ is the character $\chi\left(\begin{pmatrix} [\lambda] & 0 \\ 0 & [\mu] \end{pmatrix}\right) = \lambda^r \eta([\lambda\mu])$. If $\lambda = 0$ so that $\pi(r, \lambda, \eta)$ is supersingular, then $v_1 = v_\sigma$ and $v_2 = v_{\bar{\sigma}}$.

Lemma 9.3. — Let π be a supersingular representation of G then

- (i) if $r \in \{0, p-1\}$ then
 - (a) $\dim \mathrm{Ext}_{\mathcal{H}}^1(\pi^{I_1}, \pi^{I_1}) = 1$;
 - (b) $\mathrm{Ext}_{\mathcal{H}}^i(\pi^{I_1}, *) = 0$ for $i > 1$;
- (ii) otherwise, $\dim \mathrm{Ext}_{\mathcal{H}}^1(\pi^{I_1}, \pi^{I_1}) = 2$.

Proof. — [6, Cor. 6.7, 6.6]. □

We look more closely at the regular case. Let π be supersingular with $0 < r < p-1$ and assume for simplicity that $p \in Z$ acts trivially on π . For $(\lambda_1, \lambda_2) \in \overline{\mathbb{F}}_p^2$ we define an $\mathcal{H}$ -module E_{λ_1, λ_2} to be a 4-dimensional vector space with basis $\{v_\chi, v_{\chi^s}, w_\chi, w_{\chi^s}\}$ with the action of $\mathcal{H}$ given on the generators

$$(37) \quad w_\chi T_{n_s} = \lambda_1 v_{\chi^s}, \quad w_{\chi^s} T_{n_s} = \lambda_2 v_\chi, \quad v_\chi T_{n_s} = v_{\chi^s} T_{n_s} = 0$$

and $w_\psi T_\Pi = w_{\psi^s}$, $v_\psi T_\Pi = v_{\psi^s}$, $w_\psi e_\psi = w_\psi$, $v_\psi e_{\psi^s} = v_\psi$, for $\psi \in \{\chi, \chi^s\}$. Then $\langle v_\chi, v_{\chi^s} \rangle$ is stable under the action of $\mathcal{H}$ and we have an exact sequence:

$$(38) \quad 0 \rightarrow \mathcal{I}(\pi) \rightarrow E_{\lambda_1, \lambda_2} \rightarrow \mathcal{I}(\pi) \rightarrow 0$$

The extension (38) is split if and only if $(\lambda_1, \lambda_2) = (0, 0)$. It is immediate that the map $\overline{\mathbb{F}}_p^2 \rightarrow \mathrm{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi), \mathcal{I}(\pi))$ sending (λ_1, λ_2) to the equivalence class of (38) is an isomorphism of $\overline{\mathbb{F}}_p$ -vector spaces.

Lemma 9.4. — Let $\lambda \in \overline{\mathbb{F}}_p^\times$ then

$$(39) \quad \mathcal{I}(E_{0,\lambda}) \cong \frac{\text{c-Ind}_{KZ}^G \sigma}{(T_\sigma^2)}, \quad \mathcal{I}(E_{\lambda,0}) \cong \frac{\text{c-Ind}_{KZ}^G \tilde{\sigma}}{(T_{\tilde{\sigma}}^2)}$$

where $T_\sigma \in \text{End}_G(\text{c-Ind}_{KZ}^G \sigma)$ is given by Lemma 4.1.

Proof. — Let $\varphi \in \text{c-Ind}_{KZ}^G \sigma$ such that $\text{Supp } \varphi = KZ$ and $\varphi(1)$ spans σ^{I_1} . Let $\tau := \frac{\text{c-Ind}_{KZ}^G \sigma}{(T_\sigma^2)}$ and v the image of φ in τ . Then $\tau = \langle G \cdot v \rangle = \langle G \cdot \tau^{I_1} \rangle$. And so it is enough to show that $\mathcal{I}(\tau) \cong E_{0,\lambda}$. Since $T_\sigma : \text{c-Ind}_{KZ}^G \sigma \rightarrow \text{c-Ind}_{KZ}^G \sigma$ is injective and $\pi \cong \frac{\text{c-Ind}_{KZ}^G \sigma}{(T_\sigma)}$, we have an exact sequence

$$(40) \quad 0 \rightarrow \pi \rightarrow \tau \rightarrow \pi \rightarrow 0$$

and we may identify the subobject with $T_\sigma(\tau)$. Now, $v, \Pi v, T_\sigma(v)$ and $T_\sigma(\Pi v)$ are linearly independent and I_1 -invariant. Thus $\dim \tau^{I_1} \geq 4$ and since $\dim \pi^{I_1} = 2$ we obtain an exact sequence of $\mathcal{H}$ -modules

$$(41) \quad 0 \rightarrow \mathcal{I}(\pi) \rightarrow \mathcal{I}(\tau) \rightarrow \mathcal{I}(\pi) \rightarrow 0$$

Hence, $\mathcal{I}(\tau) \cong E_{\lambda_1, \lambda_2}$ for some $\lambda_1, \lambda_2 \in \overline{\mathbb{F}}_p$. Since $\sigma \cong \langle K \cdot \varphi \rangle \cong \langle K \cdot v \rangle$ and $\langle K \cdot T_\sigma(v) \rangle \cong T_\sigma(\langle K \cdot v \rangle) \cong \sigma$, [14, 3.1.3] gives

$$(42) \quad v e_\chi = v, \quad (T_\sigma(v)) e_\chi = T_\sigma(v), \quad v T_{n_s} = (T_\sigma(v)) T_{n_s} = 0.$$

Hence, $\lambda_1 = 0$. If $\lambda_2 = 0$ then (41) would split and so would (40). Hence, $\lambda_2 \neq 0$. We leave it to the reader to check that for any $\lambda \in \overline{\mathbb{F}}_p^\times$, $E_{0,\lambda} \cong E_{0,1}$. $\square$

Lemma 9.5. — If $E = E_{\lambda_1, \lambda_2}$, $\lambda_1 \lambda_2 \neq 0$ then $\dim \text{Ext}_{\mathcal{H}}^1(E, \mathcal{I}(\pi)) = 1$.

Proof. — Applying $\text{Hom}_{\mathcal{H}}(*, \mathcal{I}(\pi))$ to (38) gives an exact sequence

$$(43) \quad \begin{aligned} \text{Hom}_{\mathcal{H}}(\mathcal{I}(\pi), \mathcal{I}(\pi)) &\hookrightarrow \text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi), \mathcal{I}(\pi)) \\ &\rightarrow \text{Ext}_{\mathcal{H}}^1(E, \mathcal{I}(\pi)) \rightarrow \text{Ext}_{\mathcal{H}}^1(\mathcal{I}(\pi), \mathcal{I}(\pi)) \end{aligned}$$

Hence, $\dim \text{Ext}_{\mathcal{H}}^1(E, \mathcal{I}(\pi)) = 1 + \dim \Upsilon$, where Υ is the image of the last arrow in (43). Yoneda's interpretation of Ext says that $\Upsilon \neq 0$ is equivalent to the following commutative diagram of $\mathcal{H}$ -modules:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}(\pi) & \longrightarrow & A & \longrightarrow & \mathcal{I}(\pi) \longrightarrow 0 \\ & & \downarrow = & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}(\pi) & \longrightarrow & B & \longrightarrow & E \longrightarrow 0 \end{array}$$

with A non-split. Then $A \cong E_{\mu_1, \mu_2}$ for some $\mu_1, \mu_2 \in \overline{\mathbb{F}}_p$. The condition $v T_{n_s}^2 = 0$ for all $v \in B$ is equivalent to $\mu_1 \lambda_2 = 0$ and $\mu_2 \lambda_1 = 0$. Since $\lambda_1 \lambda_2 \neq 0$ we obtain $\mu_1 = \mu_2 = 0$ and hence a contradiction to a non-split A . $\square$

10. Main result

Let π an irreducible representation with a central character ζ . A construction of [14, §6], [6, §9] gives an injection $\pi \hookrightarrow \Omega$, where Ω is in $\mathrm{Rep}_{G,\zeta}$ and $\Omega|_K$ is an injective envelope of $\mathrm{soc}_K \pi$ in $\mathrm{Rep}_{K,\zeta}$.

Lemma 10.1. — *If $\pi \cong \pi(r, 0, \eta)$ with $0 < r < p-1$ then $\Omega^{I_1} \cong E_{\lambda_1, \lambda_2}$ with $\lambda_1 \lambda_2 \neq 0$. Otherwise, $\Omega^{I_1} \cong \pi^{I_1}$.*

Proof. — Let σ be an irreducible smooth representation of K and $\mathrm{Inj} \sigma$ injective envelope of σ in $\mathrm{Rep}_{K,\zeta}$. If $\sigma = \chi \circ \det$ or $\sigma \cong St \otimes \chi \circ \det$ then $\dim(\mathrm{Inj} \sigma)^{I_1} = \dim \sigma^{I_1} = 1$ and $\dim(\mathrm{Inj} \sigma)^{I_1} = 2$ otherwise, [14, 6.4.1, §4.1]. If π is either a character, special series, a twist of unramified series or $\pi \cong \pi(0, 0, \eta)$ then $\mathrm{soc}_K \pi$ is a direct summand of $(1 \oplus St) \otimes \chi \circ \det$. Hence,

$$\Omega^{I_1} = (\mathrm{soc}_K \Omega)^{I_1} = (\mathrm{soc}_K \pi)^{I_1} \subseteq \pi^{I_1} \subseteq \Omega^{I_1}$$

and so $\pi^{I_1} \cong \Omega^{I_1}$. If π is a tamely ramified principal series, which is not a twist of unramified principal series, then $\dim \pi^{I_1} = 2$ and $\mathrm{soc}_K \pi$ is irreducible, so $\dim \Omega^{I_1} = 2$. Finally, if $\pi \cong \pi(r, 0, \eta)$ with $0 < r < p-1$ then it follows from [14, 6.4.5] that $\Omega^{I_1} \cong E_{\lambda_1, \lambda_2}$ with $\lambda_1 \lambda_2 \neq 0$. $\square$

Proposition 10.2. — *Let π, τ be irreducible representations of G with a central character, and let ζ be the central character of π . Suppose that $\mathrm{Ext}_G^1(\tau, \pi) \neq 0$. If*

$$(44) \quad \mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \rightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi))$$

is not surjective then $\tau \cong \pi \cong \pi(r, 0, \eta)$ with $0 < r < p-1$.

Proof. — We note that Proposition 8.1 implies that ζ is the central character of τ . Since $\Omega|_K$ is an injective object in $\mathrm{Rep}_{K,\zeta}$, $\Omega|_{I_1}$ is an injective object in $\mathrm{Rep}_{I_1,\zeta}$. Hence, $\mathbb{R}^1 \mathcal{J}(\Omega) = 0$ and we have an exact sequence:

$$(45) \quad 0 \rightarrow \mathcal{J}(\pi) \rightarrow \mathcal{J}(\Omega) \rightarrow \mathcal{J}(\Omega/\pi) \rightarrow \mathbb{R}^1 \mathcal{J}(\pi) \rightarrow 0.$$

Assume $\pi \cong \pi(r, 0, \eta)$, $0 < r < p-1$. Let $\partial \in \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi))$ be non-zero. Suppose that $\tau \not\cong \pi$ then $\mathrm{Ext}_{\mathcal{H}}^1(\mathcal{J}(\tau), \mathcal{J}(\pi)) = 0$, [6, 6.5], Lemma 10.1 implies $\mathcal{J}(\Omega)/\mathcal{J}(\pi) \cong \mathcal{J}(\pi)$. So we have a surjection

$$(46) \quad \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathcal{J}(\Omega/\pi)) \twoheadrightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi)).$$

Further, we have an isomorphism

$$(47) \quad \mathrm{Hom}_G(\tau, \Omega/\pi) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathcal{J}(\Omega/\pi)).$$

Choose $\psi \in \mathrm{Hom}_G(\tau, \Omega/\pi)$ mapping to ∂ under the composition of (47) and (46). Since τ is irreducible, by pulling back the image of ψ we obtain an extension $0 \rightarrow \pi \rightarrow E_\psi \rightarrow \tau \rightarrow 0$ inside of Ω . By construction, (44) maps the class of this extension to ∂ .

If $\pi \not\cong \pi(r, 0, \eta)$ with $0 < r < p-1$ then Lemma 10.1 says that $\mathcal{J}(\Omega/\pi) \cong \mathbb{R}^1 \mathcal{J}(\pi)$ and arguing as above we get that (44) is surjective. $\square$

Corollary 10.3. — *Let π, τ be irreducible representations of G with a central character, and suppose that π is supersingular with a central character ζ . If $\text{Ext}_G^1(\tau, \pi) \neq 0$ and $\tau \not\cong \pi$ then*

$$\text{Ext}_G^1(\tau, \pi) \cong \text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi)).$$

Proof. — Proposition 8.1 implies that the central character of τ is ζ and $\text{Ext}_G^1(\tau, \pi) \cong \text{Ext}_{G, \zeta}^1(\tau, \pi)$. By [6, Cor.6.5], $\text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\tau), \mathcal{J}(\pi)) = 0$. The assertion follows from Propositions 9.1, 10.2. $\square$

Lemma 10.4. — *Let π and τ be supersingular representations of G with the same central character. Suppose that $\pi^{I_1} \cong \tau^{I_1}$ as H -representations then $\pi \cong \tau$.*

Proof. — It follows from the explicit description of supersingular modules $M(r, 0, \eta)$ of $\mathcal{H}$ in §9 or [14, Def.2.1.2] that $\mathcal{J}(\tau) \cong \mathcal{J}(\pi)$ as $\mathcal{H}$ -modules. Hence, $\tau \cong \mathcal{T} \mathcal{J}(\tau) \cong \mathcal{T} \mathcal{J}(\pi) \cong \pi$. $\square$

Proposition 10.5. — *Let $\pi = \pi(r, 0, \eta)$ with $0 < r < p - 1$, and let ζ be the central character of π . Assume that $p \geq 5$ then $\mathbb{R}^1 \mathcal{J}^1(\pi) \cong \mathcal{J}(\pi) \oplus \mathcal{J}(\pi)$.*

Proof. — Corollary 6.6 implies that we have an isomorphism of $\mathcal{H}^+$ -modules $\mathbb{R}^1 \mathcal{J}(\pi) \cong \mathbb{R}^1 \mathcal{J}(\pi_\sigma) \oplus \mathbb{R}^1 \mathcal{J}(\pi_{\bar{\sigma}})$. Let $v \in \mathbb{R}^1 \mathcal{J}(\pi_\sigma)$ it follows from Theorem 7.9 that $ve_\chi = v$. Since $0 < r < p - 1$ we have $\chi \neq \chi^s$ and so $ve_{\chi^s} = 0$. Since $T_{n_s} \in \mathcal{H}^+$, $vT_{n_s} \in \mathbb{R}^1 \mathcal{J}(\pi_\sigma)$ and hence $vT_{n_s} = vT_{n_s}e_\chi = ve_{\chi^s}T_{n_s} = 0$. So T_{n_s} kills $\mathbb{R}^1 \mathcal{J}(\pi_\sigma)$ and by symmetry it also kills $\mathbb{R}^1 \mathcal{J}(\pi_{\bar{\sigma}})$. Theorem 7.9 gives $\dim \mathbb{R}^1 \mathcal{J}(\pi_\sigma) = 2$. If we chose a basis $\{v, w\}$ of $\mathbb{R}^1 \mathcal{J}(\pi_\sigma)$ then $\{vT_\Pi, wT_\Pi\}$ is a basis of $\mathbb{R}^1 \mathcal{J}(\pi_{\bar{\sigma}})$. And it follows from the explicit description of $M(r, 0, \eta)$ in §9 that $\langle v, vT_\Pi \rangle$ is stable under the action of $\mathcal{H}$ and is isomorphic to $M(r, 0, \eta)$. $\square$

Proposition 10.6. — *Let π and ζ be as in Proposition 10.5 and let τ be an irreducible representation of G with a central character ζ . Assume $p > 2$ and $\tau \not\cong \pi$ then $\text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi)) = 0$.*

Proof. — Assume that $\text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi)) \neq 0$ if $p \geq 5$ then Proposition 10.5 implies that $\mathcal{J}(\tau) \cong \mathcal{J}(\pi)$, and hence $\tau \cong \pi$. Assume that $p = 3$ then the assumption $0 < r < p - 1$ forces $r = 1$. Corollary 7.10 implies that $\tau^{I_1} \cong \chi \oplus \chi^s$ as an H -representation, where χ is as in (9). It follows from Lemma 10.4 that τ cannot be supersingular. Since $\chi \neq \chi^s$ we get that τ is a principal series representation. Corollary 10.3 implies that $\text{Ext}_G^1(\tau, \pi) \neq 0$. Let η be one of the characters $\omega \circ \det$, $\mu_{-1} \circ \det$, $\omega\mu_{-1} \circ \det$. Since $p = 3$ and $r = 1$, (8) gives $\pi \cong \pi \otimes \eta$. Twisting by η gives $\text{Ext}_G^1(\tau \otimes \eta, \pi) \neq 0$, and hence $\text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau \otimes \eta), \mathbb{R}^1 \mathcal{J}(\pi)) \neq 0$. Since $p > 2$ [1, Thm. 34, Cor. 36] imply that $\tau \not\cong \tau \otimes \eta$ and so $\mathcal{J}(\tau) \not\cong \mathcal{J}(\tau \otimes \eta)$ as $\mathcal{H}$ -modules. This implies that $\dim \mathbb{R}^1 \mathcal{J}(\pi)$ is at least $4 \times 2 = 8$, which contradicts Corollary 7.10. $\square$

Theorem 10.7. — *Assume that $p > 2$ and let τ and π be irreducible smooth representations of G admitting a central character. Suppose that π is supersingular and $\text{Ext}_G^1(\tau, \pi) \neq 0$ then $\tau \cong \pi$.*

Proof. — If $0 < r < p - 1$ the assertion follows from Corollary 10.3 and Proposition 10.6. Suppose that $r \in \{0, p - 1\}$. Let $\mathfrak{J}$ be the image of $\mathrm{Ext}_{G, \zeta}^1(\pi, \pi) \rightarrow \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\pi), \mathbb{R}^1 \mathcal{J}(\pi))$. Then it follows from Propositions 8.2, 9.1 and Lemma 9.3 that $\dim \mathfrak{J} \geq 3 - 1 = 2$. Hence, $\mathcal{J}(\pi) \oplus \mathcal{J}(\pi)$ is a submodule of $\mathbb{R}^1 \mathcal{J}(\pi)$. By forgetting the action of $\mathcal{H}$ we obtain an isomorphism of vector spaces $\mathbb{R}^1 \mathcal{J}(\pi) \cong H^1(I_1/Z_1, \pi)$. Corollary 7.10 implies that $\dim \mathbb{R}^1 \mathcal{J}(\pi) = 4$. Since $\dim \mathcal{J}(\pi) = 2$ we obtain

$$(48) \quad \mathbb{R}^1 \mathcal{J}(\pi) \cong \mathcal{J}(\pi) \oplus \mathcal{J}(\pi).$$

Corollary 10.3 implies the result. $\square$

Remark 10.8. — We note that the proof in the regular case $0 < r < p - 1$ is purely representation theoretic and makes no use of Colmez's functor. The Iwahori case $r \in \{0, p - 1\}$ could also be done representation theoretically. One needs to work out the action of $\mathcal{H}$ on $H^1(I_1/Z_1, \pi)$. This can be done, but it is not so pleasant, in particular $p = 3$ requires extra arguments.

Lemma 10.9. — Let $\pi \cong \pi(r, 0, \eta)$ with $0 < r < p - 1$, then

$$\dim \mathcal{J}(\Omega/\pi) \cdot e_{\chi} T_{n_s} \geq 1, \quad \dim \mathcal{J}(\Omega/\pi) \cdot e_{\chi^s} T_{n_s} \geq 1.$$

Proof. — We have an exact sequence of K -representations:

$$(49) \quad 0 \rightarrow \pi^{K_1} \rightarrow \Omega^{K_1} \rightarrow (\Omega/\pi)^{K_1}$$

Since $\Omega|_K \cong \mathrm{Inj} \sigma \oplus \mathrm{Inj} \tilde{\sigma}$, we have $\Omega^{K_1} \cong \mathrm{inj} \sigma \oplus \mathrm{inj} \tilde{\sigma}$, where inj denotes an injective envelope in the category Rep_{K/K_1} , [14, 6.2.4]. In [6, 20.1, §16] we have determined the K -representation $\pi^{K_1} \cong \pi_{\sigma}^{K_1} \oplus \pi_{\tilde{\sigma}}^{K_1}$. It follows from the description and [6, 3.4, 3.5] that $\pi_{\sigma}^{K_1}$ is isomorphic to the kernel of $\mathrm{inj} \sigma \rightarrow \mathrm{Ind}_I^K \chi$. Hence, $(\Omega/\pi)^{K_1}$ contains $\mathrm{Ind}_I^K \chi \oplus \mathrm{Ind}_I^K \chi^s$ as a subobject and so $(\Omega/\pi)^{I_1}$ contains $V := (\mathrm{Ind}_I^K \chi \oplus \mathrm{Ind}_I^K \chi^s)^{I_1}$. Moreover, V is stable under the action of T_{n_s} , and $\dim V e_{\chi} T_{n_s} = \dim V e_{\chi^s} T_{n_s} = 1$, [14, 3.1.11]. This yields the claim. $\square$

Proposition 10.10. — Let $\pi \cong \pi(r, 0, \eta)$ with $0 < r < p - 1$. If $p \geq 5$ then

$$(50) \quad \dim \mathrm{Ext}_{K/Z_1}^1(\sigma, \pi) \leq 2, \quad \dim \mathrm{Ext}_{K/Z_1}^1(\tilde{\sigma}, \pi) \leq 2.$$

If $p = 3$ then

$$(51) \quad \dim \mathrm{Ext}_{K/Z_1}^1(\sigma, \pi) \leq 3, \quad \dim \mathrm{Ext}_{K/Z_1}^1(\tilde{\sigma}, \pi) \leq 3.$$

Proof. — We have $\mathrm{Hom}_{K/Z_1}(\sigma, \pi) \cong \mathrm{Hom}_{K/Z_1}(\sigma, \Omega)$, since by construction $\mathrm{soc}_K \Omega \cong \mathrm{soc}_K \pi$. Moreover, since $\Omega|_K$ is injective in $\mathrm{Rep}_{K, \zeta}$ we have $\mathrm{Ext}_{K/Z_1}^1(\sigma, \Omega) = 0$. Hence,

$$(52) \quad \mathrm{Hom}_{K/Z_1}(\sigma, \Omega/\pi) \cong \mathrm{Ext}_{K/Z_1}^1(\sigma, \pi).$$

It follows from [14, 4.1.5] that if κ is any smooth K -representation then one has

$$(53) \quad \mathrm{Hom}_{K/Z_1}(\sigma, \kappa) \cong \mathrm{Ker}(\mathcal{J}(\kappa) e_{\chi} \xrightarrow{T_{n_s}} \mathcal{J}(\kappa) e_{\chi^s}).$$

Now Lemma 10.9, (52) and (53) imply that

$$(54) \quad \dim \mathrm{Ext}_{K/Z_1}^1(\sigma, \pi) \leq \dim \mathcal{J}(\Omega/\pi) e_{\chi} - 1 = \dim \mathbb{R}^1 \mathcal{J}(\pi) e_{\chi}.$$

It follows from Theorem 7.9 that if $p \geq 5$ then $\dim \mathbb{R}^1 \mathcal{J}(\pi)e_\chi = 2$ and if $p = 3$ then $\dim \mathbb{R}^1 \mathcal{J}(\pi)e_\chi \leq 3$. The same proof also works for $\tilde{\sigma}$. $\square$

Proposition 10.11. — *Let $\pi \cong \pi(r, 0, \eta)$ with $0 < r < p - 1$. If $p \geq 5$ then*

$$(55) \quad \dim \operatorname{Ext}_{G, \zeta}^1(\pi, \pi) \leq 3.$$

If $p = 3$ then

$$(56) \quad \dim \operatorname{Ext}_{G, \zeta}^1(\pi, \pi) \leq 4.$$

Proof. — Recall that we have an exact sequence:

$$(57) \quad 0 \rightarrow \operatorname{c-Ind}_{KZ}^G \sigma \xrightarrow{T} \operatorname{c-Ind}_{KZ}^G \sigma \rightarrow \pi \rightarrow 0.$$

Applying $\operatorname{Hom}_G(*, \pi)$ to (57) gives an exact sequence

$$(58) \quad \operatorname{Hom}_G(\operatorname{c-Ind}_{KZ}^G \sigma, \pi) \hookrightarrow \operatorname{Ext}_{G, \zeta}^1(\pi, \pi) \rightarrow \operatorname{Ext}_{G, \zeta}^1(\operatorname{c-Ind}_{KZ}^G \sigma, \pi).$$

We may think of this exact sequence first as Yoneda Exts in $\operatorname{Rep}_{G, \zeta}$, but since $\operatorname{Rep}_{G, \zeta}$ has enough injectives Yoneda's Ext^n is isomorphic to $\mathbb{R}^n \operatorname{Hom} \cong \operatorname{Ext}_{G, \zeta}^n$. For any A in $\operatorname{Rep}_{G, \zeta}$ we have

$$\operatorname{Hom}_G(\operatorname{c-Ind}_{KZ}^G \sigma, A) \cong \operatorname{Hom}_{K/Z_1}(\sigma, FA),$$

where $F : \operatorname{Rep}_{G, \zeta} \rightarrow \operatorname{Rep}_{K, \zeta}$ is the restriction. The functor F is exact and maps injectives to injectives, hence

$$(59) \quad \operatorname{Ext}_{G, \zeta}^1(\operatorname{c-Ind}_{KZ}^G \sigma, A) \cong \operatorname{Ext}_{K/Z_1}^1(\sigma, FA).$$

Now (58), (59) and Proposition 10.10 give the assertion. $\square$

The same proof gives:

Corollary 10.12. — *Let $n \geq 1$ and $\tau = \frac{\operatorname{c-Ind}_{KZ}^G \sigma}{(T^n)}$ or $\tau = \frac{\operatorname{c-Ind}_{KZ}^G \tilde{\sigma}}{(T^n)}$. If $p \geq 5$ then $\dim \operatorname{Ext}_{G, \zeta}^1(\tau, \pi) \leq 3$; if $p = 3$ then $\dim \operatorname{Ext}_{G, \zeta}^1(\tau, \pi) \leq 4$.*

Theorem 10.13. — *Assume $p > 2$ and $\pi \cong \pi(r, 0, \eta)$ supersingular. If $(p, r) \neq (3, 1)$ then $\dim \operatorname{Ext}_{G, \zeta}^1(\pi, \pi) = 3$.*

Proof. — Proposition 8.2 or §A gives $\dim \operatorname{Ext}_{G, \zeta}^1(\pi, \pi) \geq 3$. If $0 < r < p - 1$ then equality follows from Proposition 10.11. If $r = 0$ or $r = p - 1$ then $\operatorname{Ext}_{\mathcal{H}}^2(\mathcal{J}(\pi), \mathcal{J}(\pi)) = 0$ and $\operatorname{Ext}_{\mathcal{H}}^1(\mathcal{J}(\pi), \mathcal{J}(\pi))$ is 1-dimensional by Lemma 9.3. Hence, (48) and (33) give $\dim \operatorname{Ext}_{G, \zeta}^1(\pi, \pi) = 3$. $\square$

For future use we record the following:

Proposition 10.14. — *Assume $p > 2$ and $\pi \cong \pi(r, 0, \eta)$ supersingular. Let $0 \rightarrow \mathcal{J}(\pi) \rightarrow E \rightarrow \mathcal{J}(\pi) \rightarrow 0$ be a non-split extension of $\mathcal{H}$ -modules. If $(p, r) \neq (3, 1)$ then $\dim \operatorname{Ext}_{G, \zeta}^1(\mathcal{J}(E), \pi) \leq 3$.*

Proof. — If $(p, r) \neq (3, 1)$ then we have $\mathbb{R}^1 \mathcal{J}(\pi) \cong \mathcal{J}(\pi) \oplus \mathcal{J}(\pi)$ and so $\dim \mathrm{Hom}_{\mathcal{H}}(E, \mathbb{R}^1 \mathcal{J}(\pi)) = 2$. So if $\dim \mathrm{Ext}_{\mathcal{H}}^1(E, \mathcal{J}(\pi)) \leq 1$ then (33) allows us to conclude. If $r = 0$ or $r = p - 1$ the latter may be deduced from Lemma 9.3. If $0 < r < p - 1$ and $E \cong E_{\lambda_1, \lambda_2}$ with $\lambda_1 \lambda_2 \neq 0$ then this is given by Lemma 9.5. If $\lambda_1 \lambda_2 = 0$ then $\mathcal{J}(E) \cong \frac{c\text{-Ind}_{KZ}^G \sigma}{(T^2)}$ or $\frac{c\text{-Ind}_{KZ}^G \tilde{\sigma}}{(T^2)}$ and the assertion is given by Corollary 10.12. $\square$

11. Non-supersingular representations

We compute $\mathrm{Ext}_{G, \zeta}^1(\tau, \pi)$, when π is the Steinberg representation of G or a character and τ is an irreducible representation of G under the assumption $p > 2$. The results of this paper combined with [6] give all the extensions between irreducible representations of G , when $p > 2$. We record this below. A lot of cases have been worked out by different methods by Colmez [7] and Emerton [8]. The new results of this section are determination of $\mathbb{R}^1 \mathcal{J}(\mathrm{Sp})$, where Sp is the Steinberg representation, and showing that if $\eta : G \rightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character of order 2 then $\mathrm{Ext}_G^1(\eta, \mathrm{Sp}) = 0$.

Proposition 11.1. — Assume $p > 2$ and let $\psi : H \rightarrow \overline{\mathbb{F}}_p^\times$ be a character. Suppose that $\mathrm{Ext}_{I/Z_1}^1(\psi, \mathrm{Sp}) \neq 0$ then $\psi = \mathbf{1}$ the trivial character. Moreover, $\dim \mathrm{Ext}_{I/Z_1}^1(\mathbf{1}, \mathrm{Sp}) = 2$.

Proof. — It follows from (7) that $\pi(p-1, 1) \cong \mathrm{Ind}_P^G \mathbf{1}$. By restricting (5) to I we obtain an exact sequence of I -representations:

$$(60) \quad 0 \rightarrow \mathbf{1} \rightarrow \mathrm{Ind}_{I \cap P^s}^I \mathbf{1} \oplus \mathrm{Ind}_{I \cap P}^I \mathbf{1} \rightarrow \mathrm{Sp} \rightarrow 0.$$

If we set $M := \mathrm{Ind}_{I \cap P^s}^I \mathbf{1}$ then $\mathrm{Ind}_{I \cap P}^I \mathbf{1} \cong M^\Pi$, and $M|_{H(I \cap U)} \cong \mathrm{Ind}_H^{H(I \cap U)} \mathbf{1}$ is an injective envelope of $\mathbf{1}$ in $\mathrm{Rep}_{H(I \cap U)}$. So (60) is an analog of Theorem 6.3. The proof of Theorem 7.9 goes through without any changes. For $p = 3$ we note that $M^{K_1} \cong \mathrm{Ind}_{HK_1}^I \mathbf{1}$ and hence M satisfies the assumptions of Lemma 7.7. $\square$

Let $\omega : \mathbb{Q}_p^\times \rightarrow \overline{\mathbb{F}}_p^\times$ be a character, such that $\omega(p) = 1$ and $\omega|_{\mathbb{Z}_p^\times}$ is the reduction mod p composed with the canonical embedding.

Proposition 11.2. — Assume $p > 2$ then $\mathbb{R}^1 \mathcal{J}(\mathbf{1}) \cong M(p-3, 1, \omega)$ and $\mathbb{R}^1 \mathcal{J}(\mathrm{Sp}) \cong M(p-1, 1)$,

Proof. — Recall (5) gives an exact sequence

$$(61) \quad 0 \rightarrow \mathbf{1} \rightarrow \pi(p-1, 1) \rightarrow \mathrm{Sp} \rightarrow 0.$$

Applying $\mathcal{J}$ to (61) we get an exact sequence:

$$0 \rightarrow \mathbb{R}^1 \mathcal{J}(\mathbf{1}) \rightarrow \mathbb{R}^1 \mathcal{J}(\pi(p-1, 1)) \rightarrow \mathbb{R}^1 \mathcal{J}(\mathrm{Sp}).$$

Now [6, Thm. 7.16] asserts that $\mathbb{R}^1 \mathcal{J}(\pi(p-1, 1)) \cong M(p-3, 1, \omega) \oplus M(p-1, 1)$. Now H acts on $\mathbb{R}^1 \mathcal{J}(\mathbf{1})$ and $\mathbb{R}^1 \mathcal{J}(\pi(p-1, 1))$ via $h \mapsto T_{h-1}$. It follows from Definition 9.2 that

$$M(p-1, 1) \cong \mathbf{1} \oplus \mathbf{1}, \quad M(p-3, 1, \omega) \cong \alpha \oplus \alpha^{-1}$$

as H -representations. Propositions 5.2, 5.4 imply that

$$\mathbb{R}^1 \mathcal{J}(\mathbf{1}) \cong H^1(I_1/Z_1, \mathbf{1}) \cong \text{Hom}(I_1/Z_1, \overline{\mathbb{F}}_p) = \langle \kappa^u, \kappa^l \rangle \cong \alpha \oplus \alpha^{-1}$$

as H -representations. Since $p > 2$ we get $\mathbb{R}^1 \mathcal{J}(\mathbf{1}) \cong M(p-3, 1, \omega)$. Then $M(p-1, 1)$ is a 2-dimensional submodule of $\mathbb{R}^1 \mathcal{J}(\text{Sp})$. However, Proposition 11.1 implies that $\mathbb{R}^1 \mathcal{J}(\text{Sp})$ is 2-dimensional, so the injection is an isomorphism. $\square$

Lemma 11.3. — *Let M be an irreducible $\mathcal{H}$ -module. If $\text{Ext}_{\mathcal{H}}^1(M, \mathcal{J}(\mathbf{1}))$ or $\text{Ext}_{\mathcal{H}}^1(M, \mathcal{J}(\text{Sp}))$ is non-zero then $M \in \{\mathcal{J}(\mathbf{1}), \mathcal{J}(\text{Sp})\}$. Moreover,*

$$\dim \text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\mathbf{1}), \mathcal{J}(\text{Sp})) = \dim \text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\text{Sp}), \mathcal{J}(\mathbf{1})) = 1.$$

If $p > 2$ then $\text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\mathbf{1}), \mathcal{J}(\mathbf{1}))$ and $\text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\text{Sp}), \mathcal{J}(\text{Sp}))$ are zero, and if $p = 2$ then both spaces are 1-dimensional.

Proof. — Recall that (6) gives an exact sequence:

$$(62) \quad 0 \rightarrow \text{Sp} \rightarrow \pi(0, 1) \rightarrow \mathbf{1} \rightarrow 0.$$

Applying $\mathcal{J}$ we obtain an exact sequence:

$$(63) \quad 0 \rightarrow \mathcal{J}(\text{Sp}) \rightarrow M(0, 1) \rightarrow \mathcal{J}(\mathbf{1}) \rightarrow 0.$$

If $\text{Ext}_{\mathcal{H}}^1(M, \mathcal{J}(\text{Sp})) \neq 0$ and $M \not\cong \mathcal{J}(\mathbf{1})$ then from (63) we obtain that $\text{Ext}_{\mathcal{H}}^1(N, M(0, 1)) \neq 0$, and [6, Cor.6.5] implies that M is either a subquotient of $M(0, 1)$ or a subquotient of $M(p-1, 1)$. Hence $M \cong \mathcal{J}(\text{Sp})$. Using (61) one can deal in the same way with $\text{Ext}_{\mathcal{H}}^1(N, \mathcal{J}(\mathbf{1}))$. Since $\mathcal{J}(\mathbf{1})$ and $\mathcal{J}(\text{Sp})$ are one dimensional, one can verify the rest by hand using the description of $\mathcal{H}$ in terms of generators and relations given in (36). $\square$

Let π and τ be irreducible representations of G admitting the same central character ζ . Assume that π is not supersingular. When $p > 2$ for given π we are going to list all τ such that $\text{Ext}_{G, \zeta}^1(\tau, \pi) \neq 0$. If one is interested in $\text{Ext}_G^1(\tau, \pi)$ then this can be deduced from Proposition 8.1. If $\eta : G \rightarrow \overline{\mathbb{F}}_p^\times$ is a smooth character then $\text{Ext}_{G, \zeta}^1(\tau \otimes \eta, \pi \otimes \eta) \cong \text{Ext}_{G, \zeta}^1(\tau, \pi)$. Hence, we may assume that π is $\mathbf{1}$, Sp or $\pi(r, \lambda)$ with $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1)$, $(r, \lambda) \neq (p-1, \pm 1)$. Recall if $\lambda \neq 0$ and $(r, \lambda) \neq (0, \pm 1)$ then [1, Thm. 30] asserts that

$$(64) \quad \pi(r, \lambda) \cong \text{Ind}_P^G \mu_{\lambda^{-1}} \otimes \mu_{\lambda} \omega^r.$$

It follows from (64) that if $\lambda \neq \pm 1$ then $\pi(0, \lambda) \cong \pi(p-1, \lambda)$. Hence, we may assume that $1 \leq r \leq p-1$. Propositions 9.1 and 10.2 gives us an exact sequence:

$$(65) \quad \text{Ext}_{\mathcal{H}}^1(\mathcal{J}(\tau), \mathcal{J}(\pi)) \hookrightarrow \text{Ext}_{G, \zeta}^1(\tau, \pi) \twoheadrightarrow \text{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi)).$$

Theorem 11.4. — *Let π, τ and ζ be as above. Assume that $p > 2$ and $\text{Ext}_{G, \zeta}^1(\tau, \pi) \neq 0$. Let d be the dimension of $\text{Ext}_{G, \zeta}^1(\tau, \pi)$.*

(i) *if $\pi \cong \mathbf{1}$ then one of the following holds:*

(a) $\tau \cong \text{Sp}$, and $d = 1$;

(b) $p \geq 5$, $\tau \cong \pi(p-3, 1, \omega) \cong \text{Ind}_P^G \omega \otimes \omega^{-1}$ and $d = 1$;

- (c) $p = 3$, $\tau \cong \mathrm{Sp} \otimes \omega \circ \det$ and $d = 1$;
- (ii) if $\pi \cong \mathrm{Sp}$ then $\tau \cong \mathbf{1}$ and $d = 2$;

Proof. — This follows from (65), Lemma 11.3 and Proposition 11.2. We note that if $p = 3$ then $\pi(p-3, 1, \omega)$ is reducible, but has a unique irreducible subobject isomorphic to $\mathrm{Sp} \otimes \omega \circ \det$. $\square$

For the sake of completeness we also deal with $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi)$ when π is irreducible principal series. We deduce the results from [6, §8], but they are also contained in [7] and [8].

Theorem 11.5. — *Let π , τ and ζ be as above. Assume that $p > 2$, $\pi \cong \pi(r, \lambda)$ with $1 \leq r \leq p-1$, $\lambda \in \overline{\mathbb{F}}_p^\times$ and $(r, \lambda) \neq (p-1, \pm 1)$. Then*

$$\mathrm{Ext}_{G,\zeta}^1(\pi(r, \lambda), \pi(r, \lambda)) \cong \mathrm{Hom}(\mathbb{Q}_p^\times, \overline{\mathbb{F}}_p).$$

In particular, $\dim \mathrm{Ext}_{G,\zeta}^1(\pi(r, \lambda), \pi(r, \lambda)) = 2$. Moreover, suppose that $\tau \not\cong \pi$ and $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \neq 0$. Let d be the dimension of $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi)$ then one of the following holds:

- (i) if $(r, \lambda) = (p-2, \pm 1)$ then such τ does not exist;
- (ii) if $(r, \lambda) = (p-3, \pm 1)$ (hence $p \geq 5$) then $\tau \cong \mathrm{Sp} \otimes \omega^{-1} \mu_{\pm 1} \circ \det$ and $d = 1$;
- (iii) otherwise, $\tau \cong \pi(s, \lambda^{-1}, \omega^{r+1})$, where $0 \leq s \leq p-2$ and $s \equiv p-3-r \pmod{p-1}$, and $d = 1$.

Remark 11.6. — *Note that if $\pi = \pi(r, \lambda)$ is as in (iii) and we write $\pi \cong \mathrm{Ind}_P^G \psi_1 \otimes \psi_2 \omega^{-1}$, then it follows from (64) that $\pi(s, \lambda^{-1}, \omega^{r+1}) \cong \mathrm{Ind}_P^G \psi_2 \otimes \psi_1 \omega^{-1}$.*

Proof. — The first assertion follows from [6, Cor.8.2]. Assume that $\tau \not\cong \pi$ then it follows from [6, Cor.6.5, 6.6, 6.7] that $\mathrm{Ext}_{\mathcal{H}}^1(\mathcal{J}(\tau), \mathcal{J}(\pi)) = 0$. Hence, (65) implies that $\mathrm{Ext}_{G,\zeta}^1(\tau, \pi) \cong \mathrm{Hom}_{\mathcal{H}}(\mathcal{J}(\tau), \mathbb{R}^1 \mathcal{J}(\pi))$. The assertions (i), (ii) and (iii) follow from [6, Thm. 7.16], where $\mathbb{R}^1 \mathcal{J}(\pi)$ is determined. The difference between (ii) and (iii) is accounted for by the fact that if $r = p-3$ then $s = 0$ and if $\lambda = \pm 1$ then $\pi(s, \lambda^{-1}, \omega^{r+1}) \cong \pi(0, \pm 1, \omega^{p-2})$, which is reducible, but has a unique irreducible submodule isomorphic to $\mathrm{Sp} \otimes \omega^{-1} \mu_{\pm 1} \circ \det$. $\square$

Appendix A

Lower bound on $\dim \mathrm{Ext}_G^1(\pi, \pi)$

Let $\mathbb{F}$ be a finite field of characteristic $p > 2$ and $W(\mathbb{F})$ the ring of Witt vectors. Let $0 \leq r \leq p-1$ be an integer and set

$$\pi(r) := \frac{\mathrm{c}\text{-Ind}_{KZ}^G \mathrm{Sym}^r \mathbb{F}^2}{(T)}.$$

We note that the endomorphism T is defined over $\mathbb{F}$, see [1, Prop 1]. In this section, we bound the dimension of $\mathrm{Ext}_G^1(\pi(r), \pi(r))$ from below, using the ideas of Colmez and Kisin. Let L be a finite extension of $W(\mathbb{F})[1/p]$ and $\mathcal{O}$ the ring of integers in L .

Let $\mathcal{G}_{\mathbb{Q}_p}$ be the absolute Galois group of $\mathbb{Q}_p$. Let $\text{Rep}_{\mathcal{O}} G$ be the category of $\mathcal{O}[G]$ -modules of finite length, with the central character, and such that the action of G is continuous for the discrete topology. Let $\text{Rep}_{\mathcal{O}} \mathcal{G}_{\mathbb{Q}_p}$ be the category of $\mathcal{O}[\mathcal{G}_{\mathbb{Q}_p}]$ -modules of finite length, such that the action of $\mathcal{G}_{\mathbb{Q}_p}$ is continuous for the discrete topology. Colmez in [7] has defined an exact functor

$$\mathbf{V} : \text{Rep}_{\mathcal{O}} G \rightarrow \text{Rep}_{\mathcal{O}} \mathcal{G}_{\mathbb{Q}_p}.$$

Set $\rho(r) := \mathbf{V}(\pi(r))$, then $\rho(r)$ is an absolutely irreducible 2-dimensional $\mathbb{F}$ -representation of $\mathcal{G}_{\mathbb{Q}_p}$, uniquely determined by the following: $\det \rho = \omega^{r+1}$; the restriction of ρ to inertia is isomorphic to $\omega_2^{r+1} \oplus \omega_2^{p(r+1)}$, where ω_2 is the fundamental character of Serre of niveau 2. In the notation of [5], $\rho(r) = \text{ind } \omega_2^{r+1}$. We note that since, $\pi(r)$ and $\rho(r)$ are absolutely irreducible, the functor $\mathbf{V}$ induces an isomorphism:

$$(66) \quad \text{Hom}_G(\pi(r), \pi(r)) \cong \text{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\rho(r), \rho(r)) \cong \mathbb{F}.$$

Let $\eta : \mathcal{G}_{\mathbb{Q}_p} \rightarrow \mathcal{O}^\times$ be a crystalline character lifting $\zeta := \omega^r$ the central character of $\pi(r)$. We consider η as a character of the centre of G , $Z(G) \cong \mathbb{Q}_p^\times$ via the class field theory. To simplify the notation we set $\pi := \pi(r)$ and $\rho := \rho(r)$. Let $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$ be the full subcategory of $\text{Rep}_{\mathcal{O}} G$, such that τ is an object in $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$ if and only if the central character of τ is equal to (the image of) η , and the irreducible subquotients of τ are isomorphic to π . We note that $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$ is abelian.

For τ and κ in $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$ we let $\text{Ext}_G^1(\kappa, \tau)$ be the Yoneda Ext^1 in $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$, so an element of $\text{Ext}_G^1(\kappa, \tau)$ can be viewed as an equivalence class of an exact sequence

$$(67) \quad 0 \rightarrow \tau \rightarrow E \rightarrow \kappa \rightarrow 0,$$

where E lies in $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$. Applying $\mathbf{V}$ to (67) we get an exact sequence $0 \rightarrow \mathbf{V}(\tau) \rightarrow \mathbf{V}(E) \rightarrow \mathbf{V}(\kappa) \rightarrow 0$. Hence, a map

$$(68) \quad \text{Ext}_G^1(\kappa, \tau) \rightarrow \text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\mathbf{V}(\kappa), \mathbf{V}(\tau)).$$

A theorem of Colmez [7, VII.5.3] asserts that (68) is injective, when $\tau = \kappa = \pi$.

Lemma A.1. — *Let τ and κ be in $\text{Rep}_{\mathcal{O}}^{\pi, \eta} G$ then $\mathbf{V}$ induces an isomorphism, and an injection respectively:*

$$\begin{aligned} \text{Hom}_G(\kappa, \tau) &\cong \text{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\mathbf{V}(\kappa), \mathbf{V}(\tau)), \\ \text{Ext}_G^1(\kappa, \tau) &\hookrightarrow \text{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\mathbf{V}(\kappa), \mathbf{V}(\tau)). \end{aligned}$$

Proof. — We may assume that $\tau \neq 0$ and $\kappa \neq 0$. We argue by induction on $\ell(\tau) + \ell(\kappa)$, where ℓ is the length as an $\mathcal{O}[G]$ -module. If $\ell(\tau) + \ell(\kappa) = 2$ then $\tau \cong \kappa \cong \pi$ and the assertion about Ext^1 is a Theorem of Colmez cited above, the assertion about Hom follows from (66). Assume that $\ell(\tau) > 1$ then we have an exact sequence:

$$(69) \quad 0 \rightarrow \tau' \rightarrow \tau \rightarrow \pi \rightarrow 0.$$

Since $\mathbf{V}$ is exact we get an exact sequence:

$$(70) \quad 0 \rightarrow \mathbf{V}(\tau') \rightarrow \mathbf{V}(\tau) \rightarrow \mathbf{V}(\pi) \rightarrow 0.$$

Applying $\mathrm{Hom}_G(\kappa, \cdot)$ to (69) and $\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\mathbf{V}(\kappa), \cdot)$ to (70) we obtain two long exact sequences, and a map between them induced by $\mathbf{V}$. With the obvious notation we get a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^0 & \longrightarrow & B^0 & \longrightarrow & C^0 & \longrightarrow & A^1 & \longrightarrow & B^1 & \longrightarrow & C^1 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{B}^0 & \longrightarrow & \mathcal{C}^0 & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \mathcal{B}^1 & \longrightarrow & \mathcal{C}^1. \end{array}$$

The first and third vertical arrows are isomorphisms, fourth and sixth injections by induction hypothesis. This implies that the second arrow is an isomorphism, and the fifth is an injection. Hence,

$$\mathrm{Hom}_G(\kappa, \tau) \cong \mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\mathbf{V}(\kappa), \mathbf{V}(\tau)),$$

$$\mathrm{Ext}_G^1(\kappa, \tau) \hookrightarrow \mathrm{Ext}_{\mathcal{G}_{\mathbb{Q}_p}}^1(\mathbf{V}(\kappa), \mathbf{V}(\tau)).$$

If $\ell(\tau) = 1$ and $\ell(\kappa) > 1$ then one may argue similarly with $\mathrm{Hom}_G(\cdot, \tau)$ and $\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\cdot, \mathbf{V}(\tau))$. $\square$

From now on we assume that $(p, r) \neq (3, 1)$. Let $\mathrm{Rep}_{\mathcal{O}}^{\pi, \eta} \mathcal{G}_{\mathbb{Q}_p}$ be the full subcategory of $\mathrm{Rep}_{\mathcal{O}} \mathcal{G}_{\mathbb{Q}_p}$, with objects ρ' , such that there exists π' in $\mathrm{Rep}_{\mathcal{O}}^{\pi, \eta} G$ with $\rho' \cong \mathbf{V}(\pi')$. Lemma A.1 implies that $\mathbf{V}$ induces an equivalence of categories between $\mathrm{Rep}_{\mathcal{O}}^{\pi, \eta} G$ and $\mathrm{Rep}_{\mathcal{O}}^{\pi, \eta} \mathcal{G}_{\mathbb{Q}_p}$. In particular, $\mathrm{Rep}_{\mathcal{O}}^{\pi, \eta} \mathcal{G}_{\mathbb{Q}_p}$ is abelian. We define three deformation problems for ρ , closely following Mazur [12]. Let D^u be the universal deformation problem; $D^{\omega, \eta}$ the deformation problem with the determinant condition, so that we consider the deformations with determinant equal to $\omega\eta$, [12, §24]; $D^{\pi, \eta}$ a deformation problem with the categorical condition, so that we consider those deformations, which as representations of $\mathcal{O}[\mathcal{G}_{\mathbb{Q}_p}]$ lie in $\mathrm{Rep}_{\mathcal{O}}^{\pi, \eta} \mathcal{G}_{\mathbb{Q}_p}$, [12, §25], [15]. Since ρ is absolutely irreducible, the functors D^u , $D^{\omega, \eta}$, $D^{\pi, \eta}$ are (pro-)representable by complete local noetherian $\mathcal{O}$ -algebras R^u , $R^{\omega, \eta}$, $R^{\pi, \eta}$ respectively. By the universality of R^u we have surjections $R^u \twoheadrightarrow R^{\omega, \eta}$ and $R^u \twoheadrightarrow R^{\pi, \eta}$.

For ρ' in $\mathrm{Rep}_{\mathbb{F}} \mathcal{G}_{\mathbb{Q}_p}$ we set $h^i(\rho') := \dim_{\mathbb{F}} H^i(\mathcal{G}_{\mathbb{Q}_p}, \rho')$. Let V be the underlying vector space of ρ , the $\mathcal{G}_{\mathbb{Q}_p}$ acts by conjugation on $\mathrm{End}_{\mathbb{F}} V$. We denote this representation by $\mathrm{Ad}(\rho)$, in particular $\mathrm{Ad}(\rho) \cong \rho \otimes \rho^*$. Local Tate duality gives

$$h^2(\rho \otimes \rho^*) = h^0(\rho \otimes \rho^* \otimes \omega) = \dim \mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\rho, \rho \otimes \omega).$$

Now [4, Lem. 4.2.2] implies that $\rho \cong \rho \otimes \omega$ if and only if $p = 2$ or $(p, r) = (3, 1)$. Since both cases are excluded here, we have $h^2(\mathrm{Ad}(\rho)) = 0$. Since ρ is absolutely irreducible $h^0(\rho \otimes \rho^*) = 1$. The local Euler characteristic gives:

$$4 = \dim \rho \otimes \rho^* = -h^0(\rho \otimes \rho^*) + h^1(\rho \otimes \rho^*) - h^2(\rho \otimes \rho^*)$$

and so $h^1(\mathrm{Ad}(\rho)) = 5$. Since $p > 2$ the exact sequence of $\mathcal{G}_{\mathbb{Q}_p}$ -representations:

$$0 \rightarrow \mathrm{Ad}^0(\rho) \rightarrow \mathrm{Ad}(\rho) \xrightarrow{\mathrm{trace}} \mathbb{F} \rightarrow 0$$

splits. Hence $h^1(\mathrm{Ad}^0(\rho)) = 3$ and $h^2(\mathrm{Ad}^0(\rho)) = 0$. It follows from [11] that $R^u \cong \mathcal{O}[[t_1, \dots, t_5]]$ and $R^{\omega, \eta} \cong \mathcal{O}[[t_1, t_2, t_3]]$.

Inverting p we get surjections $R^u[1/p] \twoheadrightarrow R^{\omega\eta}[1/p]$ and $R^u[1/p] \twoheadrightarrow R^{\pi,\eta}[1/p]$, and hence closed embeddings

$$\mathrm{Spec} R^{\omega\eta}[1/p] \hookrightarrow \mathrm{Spec} R^u[1/p], \quad \mathrm{Spec} R^{\pi,\eta}[1/p] \hookrightarrow \mathrm{Spec} R^u[1/p].$$

Let $x \in \mathrm{Spec} R^{\omega\eta}[1/p]$ be a closed point with residue field E . Specializing at x we obtain a continuous 2-dimensional E -representation V_x of $\mathcal{G}_{\mathbb{Q}_p}$. Suppose that V_x is crystalline, and if λ_1, λ_2 are eigenvalues of φ on $D_{\mathrm{crys}}(V_x^*)$ then $\lambda_1 \neq \lambda_2$ and $\lambda_1 \neq \lambda_2 p^{\pm 1}$ then Berger-Breuil in [3] associate to V_x a unitary E -Banach space representation B_x of G . Choose a G -invariant norm $\|\cdot\|$ on B_x defining the topology and such that $\|B_x\| \subseteq |E|$ and let B_x^0 be the unit ball with respect to $\|\cdot\|$. Berger has shown in [2] that $B_x^0 \otimes_{\mathcal{O}_E} \mathbb{F} \cong \pi$ as G -representations. The constructions in [3] and [7] are mutually inverse to one another. This means

$$V_x \cong E \otimes_{\mathcal{O}_E} \varprojlim \mathbf{V}(B_x^0 / \varpi_E^n B_x^0).$$

Hence, every such x also lies in $\mathrm{Spec} R^{\pi,\eta}[1/p]$. A Theorem of Kisin [10, 1.3.4] asserts that the set of crystalline points, satisfying the conditions above, is Zariski dense in $\mathrm{Spec} R^{\omega\eta}[1/p]$. Since $\mathrm{Spec} R^{\omega\eta}[1/p]$ and $\mathrm{Spec} R^{\pi,\eta}[1/p]$ are closed subsets of $\mathrm{Spec} R^u[1/p]$, we get that $\mathrm{Spec} R^{\omega\eta}[1/p]$ is contained in $\mathrm{Spec} R^{\pi,\eta}[1/p]$. Since $R^{\omega\eta}[1/p]$ is reduced we get a surjective homomorphism $R^{\pi,\eta}[1/p] \twoheadrightarrow R^{\omega\eta}[1/p]$. Let I be the kernel of $R^u \twoheadrightarrow R^{\pi,\eta}$ and let $a \in I$. The image of a in $R^{\pi,\eta}[1/p]$ is zero, hence a maps to 0 in $R^{\omega\eta}[1/p]$. Since $R^{\omega\eta}$ is p -torsion free, the map $R^{\omega\eta} \rightarrow R^{\omega\eta}[1/p]$ is injective, and hence the image of a in $R^{\omega\eta}$ is zero. So the surjection $R^u \twoheadrightarrow R^{\omega\eta}$ factors through $R^{\pi,\eta} \twoheadrightarrow R^{\omega\eta}$. Let $\mathfrak{m}_{\pi,\eta}$ and $\mathfrak{m}_{\omega\eta}$ be the maximal ideals in $R^{\pi,\eta}$ and $R^{\omega\eta}$ respectively. Then we obtain a surjection:

$$(71) \quad D^{\pi,\eta}(\mathbb{F}[\varepsilon])^* \cong \frac{\mathfrak{m}_{\pi,\eta}}{\varpi_L R^{\pi,\eta} + \mathfrak{m}_{\pi,\eta}^2} \twoheadrightarrow \frac{\mathfrak{m}_{\omega\eta}}{\varpi_L R^{\omega\eta} + \mathfrak{m}_{\omega\eta}^2} \cong D^{\omega\eta}(\mathbb{F}[\varepsilon])^*,$$

where $\mathbb{F}[\varepsilon]$ is the dual numbers, $\varepsilon^2 = 0$, and star denotes $\mathbb{F}$ -linear dual. It follows from (71) that $\dim_{\mathbb{F}} D^{\pi,\eta}(\mathbb{F}[\varepsilon]) \geq \dim_{\mathbb{F}} D^{\omega\eta}(\mathbb{F}[\varepsilon]) = 3$. Now $D^u(\mathbb{F}[\varepsilon]) \cong \mathrm{Ext}_{\mathbb{F}[\mathcal{G}_{\mathbb{Q}_p}]}^1(\rho, \rho)$, [12, §22] and so $D^{\pi,\eta}(\mathbb{F}[\varepsilon])$ is isomorphic to the image of $\mathrm{Ext}_{G,\zeta}^1(\pi, \pi)$ in $\mathrm{Ext}_{\mathbb{F}[\mathcal{G}_{\mathbb{Q}_p}]}^1(\rho, \rho)$ via (68), where $\mathrm{Ext}_{G,\zeta}^1(\pi, \pi)$ is Yoneda Ext in the category of smooth $\mathbb{F}$ -representations of G with central character ζ . Now, [7, VII.5.3] implies that the map $\mathrm{Ext}_{G,\zeta}^1(\pi, \pi) \rightarrow \mathrm{Ext}_{\mathbb{F}[\mathcal{G}_{\mathbb{Q}_p}]}^1(\rho, \rho)$ is an injection. We obtain:

Theorem A.2. — *Let π be as above and assume that $(p, r) \neq (3, 1)$ then $\dim_{\mathbb{F}} \mathrm{Ext}_{G,\zeta}^1(\pi, \pi) \geq 3$.*

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