MARIE-FRANCE VIGNÉRAS

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BANACH $\ell$-ADIC REPRESENTATIONS OF $p$-ADIC GROUPS

by

Marie-France Vigneras

Abstract. — Let $p \neq \ell$ be two distinct prime numbers, let $F$ be a $p$-adic field and let $E$ be an $\ell$-adic field. We prove that the smooth part and the completion are inverse equivalences of categories between the category of admissible Banach unitary $E$-representations of $GL(n,F)$ and the category of admissible smooth $E$-representations of $GL(n,F)$ equipped with a commensurability class of lattices. We formulate the $\ell$-adic local Langlands correspondence as a canonical bijection between the $n$-dimensional $\ell$-adic representations of the absolute Galois group $\text{Gal}_F$ and the topologically irreducible admissible Banach unitary $\ell$-adic representations of $GL(n,F)$.

1. Introduction

Let $p$ be a prime number, let $F$ be a finite extension of $\mathbb{Q}_p$ or a field of Laurent series $k((T))$ over a finite field $k$ of characteristic $p$, let $\overline{F}$ be an algebraic closure of $F$ and let $n$ be an integer $\geq 1$.

For any topological field $C$, the continuous representations of $GL(n,F)$ on topological vector spaces over $C$ are interesting for their applications in arithmetic, geometry or physics, via the theory of $L$-functions associated to automorphic representations. When $C$ varies, the theories of $C$-representations of $GL(n,F)$ present simultaneously

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strong similarities and strong different features but the Langlands insight, when \( C \) is the complex field, to use the smooth complex representations of \( \text{Gal}_F = \text{Gal}(\overline{F}/F) \) as a classifying scheme, seems to extend to other fields.

Why moving the coefficient field \( C \)? There are many reasons.

1) The representations of \( \text{Gal}_F \) appearing naturally are not smooth complex. In the étale cohomology of proper smooth algebraic varieties, they are continuous \( \ell \)-adic representations on finite dimensional vector spaces \( V \) over finite extensions \( E/\mathbb{Q}_\ell \), for a prime number \( \ell \). By a reduction of a stable \( O_E \)-lattice of \( V \), they give smooth mod \( \ell \)-representations over the residual field of \( E \).

2) The local Langlands correspondence for \( GL(n,F) \), over any algebraically closed field \( R \) of characteristic different from \( p \), is a bijection

\[
\pi \leftrightarrow (\rho, N)
\]

between the equivalence classes of the smooth irreducible \( R \)-representations \( \pi \) of \( GL(n,F) \) and of the pairs \((\rho, N)\) where \( \rho \) is a \( n \)-dimensional smooth semi-simple \( R \)-representation of the Weil group \( W_F \) and \( N \) a nilpotent endomorphism of the space of \( \rho \) such that \( \rho(w)N = N|w|\rho(w) \) where \(|?|\) is the unramified \( R \)-character of \( W_F \) sending a geometric Frobenius to \( q \), the order of the residual field of \( F \).

Our purpose is to obtain a local Langlands correspondence for continuous \( \ell \)-adic representations.

**Theorem 1.** — Let \( \ell \) be a prime number different from \( p \). The \( \ell \)-adic local Langlands correspondence for \( GL(n,F) \) is a canonical bijection between the equivalence classes of

a) \( n \)-dimensional continuous \( \ell \)-adic representations of \( \text{Gal}_F \) with a semi-simple action of the Frobenius,

b) topologically irreducible admissible Banach unitary \( \ell \)-adic representations of \( GL(n,F) \).

This theorem\(^{(1)}\) is motivated by the fascinating work and conjectures of Christophe Breuil on the \( p \)-adic local Langlands correspondence, where topologically irreducible admissible Banach unitary \( p \)-adic representations of \( GL(2,\mathbb{Q}_p) \) appear naturally.

With the existing literature, one translates the local Langlands complex correspondence for \( GL(n,F) \) into a canonical bijection between the isomorphism classes of a) and of

\[ c) \text{Irreducible smooth } \overline{\mathbb{Q}}_p \text{-representations of } GL(n,F) \text{ with a stable lattice.} \]

Indeed, as is well known,

(i) The smooth complex local Langlands correspondence \( LL(\rho,N) \) twisted by a suitable unramified character,

\[
(\rho,N) \leftrightarrow LL(\rho,N) \otimes |\det?|^{-(n-1)/2},
\]

called the smooth complex local Hecke correspondence, is Aut $\mathbf{C}$-equivariant [H prop.6].

(ii) Transporting the correspondence (i) with an algebraic isomorphism $j : \mathbf{C} \simeq \overline{\mathbf{Q}}_\ell$, we obtain the smooth local Hecke $\overline{\mathbf{Q}}_\ell$-correspondence, which does not depend on the choice of the isomorphism $j$.

(iii) $N$ disappears when one considers continuous $\overline{\mathbf{Q}}_\ell$-representations of $W_F$ instead of smooth $\overline{\mathbf{Q}}_\ell$-representations. The pairs $(\rho, N)$ are in bijection

$$(\rho, N) \leftrightarrow \sigma$$

with the $n$-dimensional $\ell$-adic representations $\sigma$ of $W_F$ with a semi-simple action of the Frobenius. The reason is that the kernel of the natural morphism $t : I_F \to \mathbf{Z}_\ell$ is a profinite group prime to $\ell$. There is a nilpotent endomorphism $N$ of the space of $\sigma$ such that $\sigma(?) = \exp(t(?)N)$ on a subgroup of finite index of $I_F$ [8].

(iv) The $n$-dimensional $\ell$-adic representation $\sigma$ of $W_F$ in (iii) extends by continuity to an $\ell$-adic representation of $\text{Gal}_F$ if and only if $\rho$ has a bounded image (i.e. the values of determinants of the irreducible components of $\rho$ are units) [8].

(v) $\rho$ has a bounded image if and only if $\pi = LL(\rho, N)$ is integral [10, §1.4]; moreover all stable lattices in $\pi$ are commensurable [11, Theorem 1].

Our task is to show that the completion with respect to a stable lattice gives a bijection between the isomorphism classes of b) and of c).

The beginning of the proof is valid for any locally profinite group $G$, with a countable fundamental system of neighborhoods of the unit, consisting of open profinite groups of pro-order not divisible by $\ell$ (Section 2). We prove (Theorem 2.12) that the completion and the smooth part induce equivalences of categories between the category $\mathcal{M}_\ell(G)_{\text{adm}}$ of admissible smooth $\ell$-adic representations of $G$ equipped with a commensurability class of lattices, and the category $\mathcal{B}_\ell(G)_{\text{adm}}$ of admissible Banach unitary $\ell$-adic representations of $G$.

Then we consider the group of rational points $G_F$ of any reductive connected group over a local non Archimedean field $F$ of residual characteristic $p \neq \ell$ (Section 3). We prove (Theorem 3.6) that the completion and the smooth part induce equivalences of categories between the category $\text{Mod}^{\text{int,fl}}_{\overline{\mathbf{Q}}_\ell}(G_F)$ of integral smooth $\overline{\mathbf{Q}}_\ell$-representations of $G_F$ of finite length and the category $\mathcal{B}_\ell(G_F)_{\text{adm,fl}}$ of admissible Banach unitary $\ell$-adic representations of topological finite length of $G_F$. We deduce the wanted bijection between the isomorphism classes of b) and c) by restricting to irreducible representations and choosing $G_F = \text{GL}(n, F)$.

A natural question was raised by the referee: Is a topologically irreducible Banach unitary $\ell$-adic representation of $G_F$ always admissible? L. Clozel noticed that the examples of B. Diarra [5, th. 4] (van Rooij), give examples of topologically irreducible representations $V \in \mathcal{B}_E(\text{GL}(1, F))$ where any non zero intertwining operator is bijective, which are not admissible.

SOCIÉTÉ MATHÉMATIQUE DE FRANCE 2010
2.

2.1. The two categories. — Let \( \ell \neq p \) be two distinct prime numbers, let \( E/\mathbb{Q}_\ell \) be a finite extension of ring of integers \( O_E \), of uniformizer \( p_E \), and of residual field \( k_E \), and let \( G \) be a topological group admitting a countable fundamental system of neighborhoods of the unit consisting of open pro-\( \ell' \)-subgroups (profinite subgroups of pro-order prime to \( \ell \)).

After having recalled some definitions and properties concerning the representations of the group \( G \) on \( E \)-vector spaces, we will introduce the two categories of representations \( \mathcal{M}_E(G) \) and \( \mathcal{A}_E(G) \) which will be compared in this paper.

Let \( \text{Mod}_E \) be the category of \( E \)-vector spaces and let \( M \in \text{Mod}_E \) non-zero. A line in \( M \) is a subspace of dimension 1. A lattice \( L \) in \( M \) is a \( O_E \)-submodule of \( M \) which contains no line and contains a basis of \( M \) over \( E \). When the dimension of \( M \) over \( E \) is countable, a lattice \( L \) in \( M \) is a free \( O_E \)-submodule of \( M \) generated by a basis of \( M \) over \( E \) [9, I Appendice C.5]. Two lattices \( L, L' \) in \( M \) are commensurable when there exists an element \( a \in O_E \) such that \( aL \subset L', aL' \subset L \). We denote by \([L]\) the commensurability class of \( L \).

Remark 2.1. — An \( O_E \)-submodule \( L \) of \( M \in \text{Mod}_E \) is a lattice in \( M \) if and only if any non zero element \( m \in M \) satisfies the two conditions:

a) there exists an integer \( n \in \mathbb{N} \) such that \( \ell^n m \) belongs to \( L \),

b) there exists an integer \( n \in \mathbb{N} \) such that \( \ell^{-n} m \) does not belong to \( L \).

Two lattices \( L, L' \) in \( M \) are commensurable if and only if there exists an integer \( n \in \mathbb{N} \) such that \( \ell^n L \subset L', \ell^n L' \subset L \).

A representation (= a linear action) of \( G \) on \( M \) is called admissible when \( \dim E M^H < \infty \), for any open pro-\( \ell' \)-subgroup \( H \) of \( G \), where \( M^H \in \text{Mod}_E \) is the subspace of \( H \)-invariant vectors of \( M \). The representation \( M \) is called irreducible when \( M \neq 0 \) and \( 0 \) and \( M \) are the only \( G \)-stable subspaces of \( M \), finitely generated when \( M \) is a finitely generated \( EG \)-module, of finite length when there exists a finite \( G \)-stable filtration \( 0 \subset M_1 \subset \cdots \subset M_n = M \) with irreducible quotients. The length of the filtration and the isomorphism classes of the quotients, up to the order, do not depend on the choice of the filtration.

A lattice \( L \) in the representation of \( G \) on \( M \) will always be a \( G \)-stable lattice in \( M \); the lattice will be called finitely generated when it is a finitely generated \( O_E G \)-module. A representation of \( G \) on \( M \) containing a lattice is called integral (we do not suppose that the lattice is \( O_E \)-free as in [9]). There exist finitely generated lattices in a finitely generated integral representation; they form a commensurability class, and any lattice contains a finitely generated lattice.

A continuous \( E \)-representation of \( G \) is a topological Hausdorff \( E \)-vector space \( M \) equipped with a continuous action of \( G \), i.e. such that the map \( (g, v) \to gv : G \times M \to M \) is continuous. It is called topologically irreducible when \( M \neq 0 \) and \( 0 \) and \( M \) are the only closed \( G \)-stable subspaces of \( M \). It is called of finite topological length when
there exists a finite filtration by \(G\)-stable closed subspaces \(0 \subset M_1 \subset \cdots \subset M_n = M\) with topologically irreducible quotients.

The category \(\mathcal{C}_E(G)\) of continuous representations of \(G\) on topological Hausdorff complete \(E\)-vector spaces with continuous \(G\)-equivariant \(E\)-linear morphisms, called intertwining operators, contains the subcategory \(\text{Mod}_E(G)\) of smooth representations and the subcategory \(\mathcal{B}_E(G)\) of Banach unitary representations, defined below. We indicate by the upper index \(\text{adm}\) or \(\text{fl}\) or \(\text{adm, fl}\) or \(\text{int, int, fl}\) the full subcategories representations which are admissible or of finite topological length or admissible and of finite topological length or integral or integral and of finite topological length. Example: \(\mathcal{C}_E(G)^{\text{adm}}, \text{Mod}_E(G)^{\text{adm}}, \mathcal{B}_E(G)^{\text{adm}}\) for admissible representations.

A representation of \(G\) on an \(E\)-vector space \(W\) is smooth when the stabilizer in \(G\) of any vector of \(W\) is open; this is simply a continuous representation of \(G\) on \(W\) when \(W\) is equipped with the discrete topology. The category \(\text{Mod}_E(G)\) of smooth \(E\)-representations of \(G\), with morphisms the \(G\)-equivariant \(E\)-linear maps, is a full subcategory of \(\mathcal{C}_E(G)\).

A Banach unitary \(E\)-representation \(V\) of \(G\) is a Hausdorff complete topological \(E\)-vector space with a topology given by a norm, equipped with a continuous action of \(G\) which respects the norm. A unit ball of \(V\) is \(L = \{v \in V : ||v|| \leq 1\}\) for some norm \(v \mapsto ||v||\) on \(V\) defining the topology [Sch I.3, III]; it is a lattice in \(V\). The unit balls of two norms on \(V\) giving the same topology are commensurable.

An \(E\)-linear map \(f : V_1 \to V_2\) between two Banach \(E\)-vector spaces \(V_1, V_2\) is continuous if and only if there exists some non zero \(a \in E\) such that \(f(L_1) \subset af(L_2)\) for some unit balls \(L_1, L_2\) of \(V_1, V_2\) [Sch I.3.1]. The topology quotient topology on the image of \(f\) is the topology induced by \(V_2\) if and only if \(f(L_1)\) and \(L_2 \cap f(V_1)\) are commensurable (this does not depend on the choice of the unit balls \(L_1, L_2\)). When \(f\) is continuous and bijective, the inverse of \(f\) is continuous [Sch I.8.7].

We will compare \(\mathcal{B}_E(G)\) with the category \(\mathcal{M}_E(G)\) of smooth \(E\)-representations \(W\) of \(G\) equipped with a commensurability class \([L]\) of lattices; a morphism \((W, [L]) \to (W', [L'])\) is a morphism \(f : W \to W'\) in \(\text{Mod}_E(G)\) such that \(f(L) \subset aL'\) for some \(a \in E\). The pair \((W, [L])\) is called admissible or of finite length when \(W\) is admissible or of finite length, and \(\mathcal{M}_E(G)^{\text{adm}}\) or \(\mathcal{M}_E(G)^{\text{fl}}\) is the full subcategory of admissible or of finite length pairs in \(\mathcal{M}_E(G)\).

2.2. The two functors. — We introduce two natural functors in opposite directions between the categories \(\mathcal{M}_E(G)\) and \(\mathcal{B}_E(G)\).

There is the natural functor \(\mathcal{C}_E(G) \to \text{Mod}_E(G)\) sending \(M \in \mathcal{C}_E(G)\) to its smooth part

\[ M^{\infty} := \bigcup_H M^H, \]

for all open pro-\(\ell\)-subgroups \(H\) of \(G\). When \(V \in \mathcal{B}_E(G)\) is a Banach unitary representation of \(G\), the smooth part \(L^{\infty} = V^{\infty} \cap L\) of a unit ball \(L\) of \(V\) is a lattice of \(V^{\infty}\). Two unit balls of \(V\) are commensurable and their smooth parts are commensurable, hence \((V^{\infty}, [L^{\infty}]) \in \mathcal{M}_E(G)\) is well defined. A continuous morphism \(f : V_1 \to V_2\) of Banach unitary \(E\)-representations of \(G\) with unit balls \(L_1, L_2\), restricts to a morphism
We get a functor
\[ \mathcal{B}_E(G) \to \mathcal{M}_E(G). \]
In the opposite direction there is the natural functor
\[ \mathcal{M}_E(G) \to \mathcal{B}_E(G) \]
sending \((W, [L])\) to the completion of \(W\) for the \(L\)-adic topology [Sch 7.5]:
\[ \hat{W}_L := \lim_{n} W/\ell^n L \cong E \otimes_{O_E} \hat{L}, \quad \hat{L} := \lim_{n} L/\ell^n L. \]
Any element \(v \in \hat{W}_L\) is written
\[ v = (w_n + \ell^n L)_n, \quad w_n \in W, \quad w_{n+1} \in w_n + \ell^n L, \]
for all \(n \in \mathbb{N}\). The lattice \(\hat{L}\) is a unit ball of \(\hat{W}_L\) for the gauge norm \(||v|| = \inf_{a \in E, v \in aL} |a|\). The completions of \(W\) defined by two commensurable lattices of \(W\) are the same. The group \(G\) acts naturally on \(\hat{W}_L\), for \(g \in G\) and \(v\) as above,
\[ gv = (gw_n + \ell^n L)_{n \in \mathbb{N}}, \]
and \(\hat{W}_L\) is a Banach unitary \(E\)-representation of \(G\) of unit ball \(\hat{L}\), well defined by \((W, [L])\). A morphism \(f : (W, [L]) \to (W', [L'])\) in \(\mathcal{M}_E(G)\) extends by continuity to an intertwinning operator \(\hat{f} : \hat{W}_L \to \hat{W}'_L\).

**Remark 2.2.** — The map \(W \mapsto \hat{W}_L\) sending \(w\) to \((w + \ell^n L)_{n \in \mathbb{N}}\) is injective, because \(L\) contains no line. We will identify \(W\) with its image in \(\hat{W}_L\).

**2.3.** — To study the two functors, smooth part and completion, between \(\mathcal{M}_E(G)\) and \(\mathcal{B}_E(G)\), the key point is the exactness of the \(H\)-invariants functor.

**Proposition 2.3.** — Let \(H\) be any open pro-\(\ell\)-subgroup of \(G\). The \(H\)-invariants functor
\[ M \mapsto M^H : \mathcal{C}_E(G) \to \text{Mod}_E \]
is exact.

**Proof.** — This is well known for the subcategory \(\text{Mod}_E(G)\) of smooth representations in \(\mathcal{C}_E(G)\). The exactness results from the existence of a Haar \(O_E\)-measure \(dg\) on \(G\) such that the volume \(\text{vol}(H, dg)\) of \(H\) is a unit in \(O_E\). The function \(e_H\) equal to \(\text{vol}(H, dg)^{-1}\) on \(H\) and 0 on \(G - H\), is an idempotent in the convolution algebra \(C^\infty_c(G; O_E)\) of locally constant compactly supported functions \(G \to O_E\), for the Haar measure \(dg\). The idempotent \(e_H\) acts on \(M \in \mathcal{C}_E(G)\), as follows. One chooses a decreasing sequence of normal subgroups \(H_n\) of \(H\) of finite index such that \(\cap_{n \in \mathbb{N}} H_n\) is trivial, and a system of representatives \(X_n\) in \(H\) of \(H/H_n\). The continuity of the action of \(G\) on \(M\) implies that the sequence
\[ v_n = [H : H_n]^{-1} \sum_{g \in X_n} gv \]
converges to a unique element $e_H \ast v$ in the Hausdorff complete space $M$. This element $e_H \ast v$ does not depend on the choice of $(H_n, X_n)_{n \in \mathbb{N}}$ and clearly $v \mapsto e_H \ast v$ is a linear projector $M \rightarrow M^H$ of its $H$-invariants. \hfill \square

**Corollary 2.4.** — The smooth part functor $M \mapsto M^\infty : \mathcal{E}_E(G) \rightarrow \text{Mod}_E(G)$ is exact.

**Proposition 2.5.** — A Banach unitary $E$-representation $V$ of $G$ is equal to the closure of its smooth part $V^\infty$.

**Proof.** — Let $v$ be an arbitrary element of $V$ and let $L$ be a unit ball of $V$. For any integer $n \geq 1$, there is an open pro-$\ell'$-subgroup $H_n$ of $G$ such that $H_n v \subset v + \ell^n L$, by the continuity of the action of $G$ on $V$. The element $e_H \ast v$ is fixed by $H_n$ and belongs to $v + \ell^n L$. The element $(e_H \ast v + \ell^n L)_{n \in \mathbb{N}}$ belongs to the closure of $V^\infty$ and is equal to $v$. \hfill \square

**Corollary 2.6.** — The smooth part functor $\mathcal{B}_E(G) \rightarrow \mathcal{M}_E(G)$ is fully faithful.

**Proof.** — For $i = 1, 2$, let $V_i \in \mathcal{B}_E(G)$ with unit ball $L_i$. The embedding $V_i^\infty \rightarrow (V_i^\infty)_{L_i}^\infty$ extends by continuity to an isomorphism $\tau_i : V_i \rightarrow (V_i^\infty)_{L_i}^\infty$ in $\mathcal{B}_E(G)$ by the Proposition 2.5 and its proof. We deduce that arbitrary intertwinning operators $\phi : (V_1^\infty, [L_1^\infty]) \rightarrow (V_2^\infty, [L_2^\infty])$ and $f : V_1 \rightarrow V_2$ satisfy $\phi = (\tau_2^{-1} \circ \phi \circ \tau_1)^\infty$, $f = \tau_2^{-1} (f^\infty) \circ \tau_1$. \hfill \square

We show that the completion commutes with the $H$-invariants.

**Proposition 2.7.** — Let $V$ be the completion of an integral smooth $E$-representation $W$ of $G$ with respect to a lattice $L$, and let $H$ be an open pro-$\ell'$-subgroup of $G$. The $H$-invariants $V^H$ of $V$ is equal to the closure of $W^H$ in $V$,

$$V^H = \overline{W^H}.$$  

**Proof.** — For $X = W, L$ or $V$, we have $e_H \ast X = X^H$. Let $v = (w_n + \ell^n L)_{n \in \mathbb{N}}$ be an element of $V$ as in (1). Then $e_H \ast w_{n+1} \in e_H \ast w_n + \ell^n L$, and $e_H \ast v = (e_H \ast w_n + \ell^n L)_{n \in \mathbb{N}}$. \hfill \square

**Corollary 2.8.** — An admissible smooth $E$-representation of $G$ with a commensurability class of lattices is equal to the smooth part of its completion.

**Proof.** — When the representation $W$ is admissible, the $E$-vector space $W^H$ is finite dimensional and already complete, hence $V^H = \overline{W^H}$ in the Proposition 2.7. \hfill \square

It is clear that the functor smooth part respects admissible representations, the corollary shows that the completion respects also admissible representations.

**Theorem 2.9.** — The smooth part and completion are inverse equivalences of categories between $\mathcal{M}_E^{\text{adm}}(G)$ and $\mathcal{B}_E^{\text{adm}}(G)$.

**Proof.** — Proposition 2.5, Corollaries 2.6, 2.8. \hfill \square
In particular, the smooth part and the completion induce inverse equivalences of categories between admissible and of finite topological length representations $\mathcal{M}_E^{adm,fl}(G)$ and $\mathcal{B}_E^{adm,fl}(G)$.

We consider now $\ell$-adic representations of $G$. For any finite extensions $E'/E/\mathbb{Q}_\ell$ contained in a fixed algebraic closure $\overline{\mathbb{Q}}$, the scalar extension $s_{E'/E}$ from $E$ to $E'$

$$\mathcal{C}_E(G) \rightarrow \mathcal{C}_{E'}(G)$$

sends $M \in \mathcal{C}_E(G)$ to $M_{E'} := E' \otimes_E M = \oplus (e_i \otimes M)$, for a finite basis $(e_i)$ of the $E$-vector space $E'$, with the topology induced by $M$ (independent of the choice of the basis) and a morphism $f : M \rightarrow M'$ in $\mathcal{C}_E(G)$ to $\text{id}_{E'} \otimes f$. The inductive limit

$$\mathcal{C}_\ell(G) := \lim_{E'/E} \mathcal{C}_{E'}(G)$$

is the category of $\ell$-adic representations of $G$. The scalar extension respects smooth representations, and the inductive limit

$$\text{Mod}_\ell(G) := \lim_{E'/E} \text{Mod}_{E'}(G)$$

is the category of smooth $\ell$-adic representations of $G$, which is a (not full) subcategory of the classical category $\text{Mod}_{\overline{\mathbb{Q}}_\ell}(G)$ of smooth $\overline{\mathbb{Q}}_\ell$-representations of $G$.

Let $R/E$ be any extension contained in $\overline{\mathbb{Q}}_\ell$ and let $O_R$ be the ring of integers in $R$. As the dimension of $R/E$ is countable, $O_R$ is an $O_E$-free module [9, Appendice C, C.4]. We denote by $L_{O_R} := O_R \otimes_{O_E} L$, the scalar extension from $O_E$ to $O_R$ of an $O_E$-module $L$.

**Lemma 2.10.** — Let $M \in \mathcal{C}_E(G)$ equipped with a lattice $L$.

(i) The $H$-invariants commute with the scalar extension, $e_HM_R = (e_HM)_R$, $e_HL_{O_R} = (e_HL)_{O_R}$ for any pro-$\ell$-subgroup $H$ of $G$ (this is true for any extension $R/E$ of fields of characteristic different from $p$). In particular, $M$ is admissible if and only if $M_R$ is admissible.

(ii) The intersection $L = M \cap L'$ of a lattice $L'$ of $M_R$ with $M$ is a lattice in $M$. In particular, if $M_R$ is integral then $M$ is integral.

(iii) The scalar extension $L_{O_R}$ of a lattice $L$ in $M$ is a lattice in $M_R$. In particular, if $M$ is integral then $M_R$ is integral.

(iv) Two lattices $L, L'$ of $M$ are commensurable if and only if their scalar extensions $L_{O_R}, L'_{O_R}$ are commensurable.

**Proof.** — (i) is clear. The other properties are clear using the Remark 2.1 and $L_{O_R} = \oplus_i (e_i \otimes L)$ for a basis $(e_i)$ of the free $O_E$-module $O_R$. 

**Lemma 2.11.** — The scalar extension $s_{E'/E}$ from $E$ to a finite extension $E'$ commutes with the smooth part functor $\mathcal{C}_E(G) \rightarrow \text{Mod}_E(G)$ and with the smooth part and completion functors between $\mathcal{B}_E(G)$ and $\mathcal{M}_E(G)$.

**Proof.** — We choose a basis $(e_i)$ of the free $O_E$-module $O_{E'}$. The scalar extension $V_{E'} = \oplus_i (e_i \otimes V)$ of the completion $V$ of $(W, [L]) \in \mathcal{M}_E(G)$ is clearly the completion of the scalar extension $(W_{E'} = \oplus_i (e_i \otimes W), [L_{O_{E'}} = \oplus_i (e_i \otimes L)])$ of $(W, [L])$. The
scalar extension of the $H$-invariants of $V \in \mathcal{B}_E(G)$ is the $H$-invariants of the scalar extension $V_{E'}$ (Lemma 2.10).

As the scalar extension $s_{E'/E}$ from $E$ to a finite extension $E'$ respects admissibility, lattices, commensurability of lattices, Banach spaces (Lemma 2.10), the inductive limit over $s_{E'/E}$ for all finite extensions $E'/E/\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}_\ell}$, defines the categories

a) $\mathcal{C}_\ell(G)^{\text{adm}}$ of admissible $\ell$-adic representations of $G$,

b) $\mathcal{M}_\ell(G)^{\text{int}}$ of integral smooth $\ell$-adic representations,

c) $\mathcal{M}_\ell(G)$ of smooth $\ell$-adic representations of $G$ equipped with a commensurability class of lattices,

d) $\mathcal{B}_\ell(G)$ of Banach unitary $\ell$-adic representations of $G$.

We define the completion and smooth part functors between $\mathcal{M}_\ell(G)^{\text{adm}}$ and $\mathcal{B}_\ell(G)^{\text{adm}}$ using the Lemma 2.11.

**Theorem 2.12.** — The completion and smooth part functors induce equivalence of categories between the categories $\mathcal{M}_\ell(G)^{\text{adm}}$ and $\mathcal{B}_\ell(G)^{\text{adm}}$.

**Proof.** — Theorem 2.9.

**3.**

Let $G_F$ be the group of rational points of a connected reductive group over a local non Archimedean field $F$ of residual characteristic $p$. The group $G_F$ is a locally pro-$p$-group. As before $E/\mathbb{Q}_\ell$ is a finite extension contained in $\overline{\mathbb{Q}_\ell}$ and $\ell \neq p$.

**Proposition 3.1.** — Let $R/R_o$ be any extension of fields of characteristic different from $p$. Then $W \in \text{Mod}_{R_o}(G_F)$ has finite length if and only if $W_R \in \text{Mod}_R(G_F)$ has finite length.

**Proof.** — [9, II.4.3.c].

**Proposition 3.2.** — Any finite length smooth representation $W$ of $G_F$ over a field of characteristic different from $p$ is admissible.

**Proof.** — This is proved in [9, II.2.8] when the field is algebraically closed. The scalar extension is not sensitive to admissibility and finite length (Lemma 2.10, Proposition 3.1) for any extension of fields of characteristic different from $p$.

**Proposition 3.3.** — The lattices in an integral finite length representation $W \in \text{Mod}_E(G_F)$ are commensurable (hence finitely generated).

**Proof.** — This is proved [11, th.1] when the field is $\overline{\mathbb{Q}_p}$. The scalar extension is not sensitive to integrality, commensurability of lattices, and finite length, (Proposition 2.10, Proposition 3.1).

**Remark 3.4.** — One cannot replace “finite length” by “admissible” in the Proposition 3.3.
Lemma 3.5. — The category of smooth $\overline{Q}_\ell$-representation of $G_F$ of finite length is equal to the category of smooth $\ell$-adic representations of $G_F$ of finite length,

$$\text{Mod}_{\overline{Q}_\ell}(G_F)^{fl} \simeq \text{Mod}_{\ell}(G_F)^{fl}.$$ 

Proof. — Let $W \in \text{Mod}_{Q_{\ell}}(G_F)^{fl}$. There exists a finite extension $E/Q_{\ell}$ and $W_E \in \text{Mod}_E(G_F)^{fl}$ such that $W$ is the scalar extension of $W_E$. When $W$ is irreducible, this is proved in [9, II.4.7]. In general, let $H$ be an open pro-$p$-subgroup of $G_F$ such that the length of $W$ is equal to the length of the module $e_HW$ over the Hecke algebra $\text{End}_{Q_{\ell},G_F} \overline{Q}_\ell[G_F/H]$. Let $(w_i)$ be a finite $Q_{\ell}$-basis of $e_HW$. The convolution algebra $\text{End}_{Z_{\ell},G_F} Z_{\ell}[G_F/H]$ is finitely generated [9, II.2.13], and the dimension of $e_HW$ over $Q_{\ell}$ is finite. Hence there exists a finite extension $E/Q_{\ell}$ such that the $E$-vector space $\oplus_i Ew_i$ in $e_HW$ is stable by the Hecke algebra $\text{End}_{\overline{Q}_\ell,G_F} E[G_F/H]$. The $E$-representation $U$ of $G_F$ generated by $(w_i)$ in $W$ satisfies $e_HU = \oplus_i Ew_i$. The scalar extension $Q_{\ell} \otimes E U$ is equal to $W$ because it is a subrepresentation of $W$ with the same $H$-invariants. By the Proposition 3.1, $W_E$ has finite length.

Let $E, E'$ be two finite extensions. Let $W_E \in \text{Mod}_E(G_F)^{fl}, W_{E'} \in \text{Mod}_{E'}(G_F)^{fl}$, let $f : \overline{Q}_\ell \otimes E W_E \to \overline{Q}_\ell \otimes E' W_{E'}$ be a $\overline{Q}_\ell$-morphism between their scalar extensions to $\overline{Q}_\ell$. There exists a finite extension $E''$ containing $E, E'$ such that $f$ is defined on $E''$, i.e. induces a $E''G_F$-morphism $f_{E''} : E'' \otimes E W_E \to E'' \otimes E' W_{E'}$ between their scalar extensions to $E''$ [9, proof of II.4.7].

The scalar extension $s_{E'/E}$ for smooth representations of $G_F$ respects finite length (Proposition 3.1) and the category $\text{Mod}_{\ell}(G_F)^{fl}$ of smooth $\ell$-adic representations of $G_F$ of finite length is well defined, contained in the category $\text{Mod}_{\ell}(G_F)^{\text{adm}}$ of admissible smooth $\ell$-adic representations of $G_F$ (Proposition 3.2). The category $\text{Mod}_{\ell}(G_F)^{\text{int},fl}$ of integral smooth $\ell$-adic representations of $G_F$ of finite length is equivalent by the forgetful functor composite with the completion and the smooth part to the category $\mathcal{B}_{\ell}(G_F)^{\text{adm},fl}$ of Banach unitary $\ell$-adic representations which are admissible and of finite length.

Theorem 3.6. — The completion and the smooth part define equivalence of categories between $\text{Mod}^\text{int,fl}_{Q_{\ell}}(G_F)$ and $\mathcal{B}_{\ell}(G_F)^{\text{adm,fl}}$.

In particular, they give bijections between the irreducible integral smooth $\overline{Q}_\ell$-representations of $G_F$ and the topologically irreducible admissible Banach unitary $\ell$-adic representations of $G_F$.

For $G_F = GL(n, F)$, we deduce the $\ell$-adic local Langlands correspondence for $GL(n, F)$, given in the introduction.

A very natural question (asked by the referee) for a Banach unitary $\ell$-adic representation $V$ of $G_F$ (notations of the Sections 2 and 3) is: does $V$ topologically irreducible imply $V$ admissible? The answer is no.
A necessary condition for a positive answer is that any non zero intertwining operator $f$ of $V$ is surjective (it is clearly injective).

**Proposition 3.7.** — When $V \in \mathcal{B}_E(G_F)$ is admissible and topologically irreducible, any non zero intertwining operator $f$ of $V$ is bijective.

**Proof.** — The restriction $f^\infty$ of $f$ to the smooth part $V^\infty$ is non zero (Theorem 2.9), bijective because $V^\infty$ is irreducible, hence $f$ is bijective.

L. Clozel noticed that the examples of B. Diarra [5, th. 4] (van Rooij), give examples of topologically irreducible representations $V \in \mathcal{B}_E(GL(1,F))$ where any non zero intertwining operator is bijective, which are not admissible.

When $F$ is replaced by $\mathbb{R}$, Fokko Du Cloux has proved that a topologically irreducible Banach unitary complex representation $V$ of $G_{\mathbb{R}}$ is admissible if and only if any intertwining operator of $V$ is a scalar [3, 3.2.9, 3.2.10]. Soergel has proved that there exists topologically irreducible Banach unitary complex representations of $SL(2, \mathbb{R})$ which are not admissible [7].

**References**


M.-F. Vigneras, Institut de Mathémathiques de Jussieu, 175 rue du Chevaleret, 75013 Paris

E-mail : vigneras@math.jussieu.fr

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