Astérisque

JOCHEN BRÜNING

The signature operator on manifolds with a conical singular stratum

Astérisque, tome 328 (2009), p. 1-44

<http://www.numdam.org/item?id=AST_2009__328__1_0>

Astérisque
328, 2009, p. 1–44

THE SIGNATURE OPERATOR ON MANIFOLDS WITH A CONICAL SINGULAR STRATUM

by

Jochen Brüning

Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — We consider a Riemannian manifold, M, which can be compactified by adjoining a smooth compact oriented Riemannian manifold such that a neighbourhood of the singular stratum B, of codimension at least two, is given by a family of metric cones. Under the assumption that the middle cohomology of the cross-section vanishes, we show that there is a natural self-adjoint extension for the Dirac operator on forms with discrete spectrum, and we determine the condition of essential self-adjointness. We describe the boundary conditions analytically and construct a good parametrix which leads to the asymptotic expansion of a suitable resolvent trace as in our previous work. We also give a new proof of the local formula for the L^2-signature.

***Résumé* (Opérateur de signature sur les variétés avec une strate singulière conique)**
Nous considérons une variété riemannienne M, qui peut être compactifiée en lui adjoignant une variété riemannienne C^∞ compacte orientée, telle qu'un voisinage de la strate singulière B, de codimension au moins deux, est donné par une famille de cônes métriques. Sous une hypothèse d'annulation de la cohomologie de la section du cône en dimension moitié, nous montrons qu'il existe une extension auto-adjointe naturelle de l'opérateur de Dirac agissant sur les formes qui est de spectre discret, et nous déterminons la condition sous laquelle l'opérateur de Dirac est essentiellement auto-adjoint. Nous décrivons les conditions de bord, et nous construisons une parametrix qui donne le développement asymptotique de la trace de la résolvante, comme dans un travail antérieur. Nous donnons aussi une preuve nouvelle de la formule locale pour la signature L^2.

Introduction

In this article, we analyze the signature operator on an oriented Riemannian manifold (M, g), of dimension $m = 4k$, with one compact singular stratum B of dimension

2000 Mathematics Subject Classification. — 53C20.
Key words and phrases. — Signature operator, closed extension, parametrix, Witt space.

This work was partially supported by SFB 288 and SFB 647 (HU Berlin).

h (the "horizontal dimension"), such that $m-h \geq 2$. A neighbourhood of the singular set is given by

$$U := U_{\varepsilon_0} := (0, \varepsilon_0) \times N, \; \varepsilon_0 \in (0, 1/2), \tag{0.1}$$

with an oriented compact Riemannian manifold N of dimension $4k-1$ and metric g^{TN}, and M decomposes as

$$M =: U_{\varepsilon_0} \cup M_{\varepsilon_0} \tag{0.2}$$

into points of distance at most and at least ε_0 of the singular set, respectively. For $\varepsilon \in (0, \varepsilon_0]$, we use analogous notation and write $U_\varepsilon, M_\varepsilon$, with

$$M = U_\varepsilon \cup M_\varepsilon.$$

We assume that the orientation on M and N induce the boundary orientation on U, such that $\{-\frac{\partial}{\partial t}, e_1, \ldots, e_{m-1}\}$ is oriented on U if $t \in (0, \varepsilon_0)$ and $\{e_1, \ldots, e_{m-1}\}$ is oriented on N. We assume in addition that the singularity is of the following special type. There is a fibration of oriented compact Riemannian manifolds,

$$\pi : Y \hookrightarrow N \to B, \tag{0.3}$$

with fibers $Y_b = \pi^{-1}(b)$, $b \in B$, of dimension $v := 4k - 1 - h \geq 1$ (the "vertical dimension"); in particular, B carries a metric g^{TB} such that π becomes a Riemannian submersion. Then the tangent bundle TN of N splits under g^{TN} into the *vertical* and the *horizontal* tangent bundle, consisting of the tangent vectors to the fibers and their orthogonal complement,

$$TN_p \;=:\; T_H N_p \oplus T_V N_p, \tag{0.4}$$

with induced metrics $g^{T_H N}$ and $g^{T_V N}$; the corresponding orthogonal projections in TN will be denoted by P_H and P_V, respectively. Next we assume that the metric $g^{TU} := g^{TM}|U$ takes the form

$$g^{TU} := dt^2 \oplus g^{T_H N} \oplus t^2 g^{T_V N}, \tag{0.5}$$

which we will call a metric of *conic type*. Thus, $M \cup B$ is a Riemannian pseudomanifold with one singular stratum of conic type.

The boundary of U is the Riemannian manifold

$$N_{\varepsilon_0} := (N, g^{TN}_{\varepsilon_0} := g^{T_H N} \oplus \varepsilon_0^2 g^{T_V N}). \tag{0.6}$$

The splitting of TN induces a splitting of the cotangent bundle,

$$T^*N =: T_H^*N \oplus T_V^*N,$$

into cotangent vectors annihilating $T_V N$ and $T_H N$, respectively. This splitting induces a bigrading of the exterior algebra ΛT^*N which will be important for our analysis; we write

$$\begin{aligned} \Lambda T^*N &= \Lambda T_H^*N \otimes \Lambda T_V^*N \\ &= \oplus_{j=p+q} \Lambda^p T_H^*N \otimes \Lambda^q T_V^*N =: \oplus_{p,q} \Lambda^{p,q} T^*N. \end{aligned} \tag{0.7}$$

The smooth sections of ΛT^*N and $\Lambda T^*_{H/V}N$ will be denoted $\lambda(N)$ and $\lambda_{H/V}(N)$, respectively, with degree or bidegree noted with superscripts.

Our main object will be the canonical Dirac operator associated with ΛT^*M,

$$D^\Lambda := D^\Lambda_M := d_M + d^\dagger_M, \tag{0.8}$$

with $d_M =: d$ the exterior derivative on M and $d^\dagger$ its formal adjoint with respect to the metric g^{TM}.

D defined on forms with compact support, denoted by $\lambda_c(M)$, is symmetric in $L^2(M, \Lambda T^*M) =: \lambda_{(2)}(M)$ but may not be essentially self-adjoint; we refer to the closure of this operator as $D^\Lambda_{\min} =: D_{\min}$, and $d_{\min}, d^\dagger_{\min}$ are defined analogously.

A specific self-adjoint extension of this operator can be defined via the Hilbert complex given by the operator $d_{\max}$ which arises from $d^\dagger$ as

$$d_{\max} := (d^\dagger_{\min})^*, \tag{0.9}$$

cf. [**11**, §3]; with a slight abuse of notation we denote this extension again by $D = D^\Lambda = D^\Lambda_M$, with domain $\mathcal{D} = \operatorname{dom} D$. In general, there will be many more self-adjoint extensions but D is of interest since its kernel gives the L^2-cohomology of M. If D is a Fredholm operator we have to break its symmetry to obtain a nontrivial index, e. g. by an anticommuting *supersymmetry* i. e. a self-adjoint involution of ΛT^*M. We will use multiplication by the complex volume element, τ_M, which splits

$$\Lambda T^*M =: \Lambda^+T^*M \oplus \Lambda^-T^*M$$

into ± 1-eigenbundles and analogously

$$\lambda(M) =: \lambda^+(M) \oplus \lambda^-(M),$$

with associated splitting $\sigma = \sigma^+ + \sigma^-$ on the level of forms. If τ_M maps $\mathcal{D}$ to itself than we can define the *Signature Operator* of M, with domain $\mathcal{D}^{\mathrm{sign}} = \mathcal{D}^+ = \frac{1}{2}(I + \tau_M)\mathcal{D}$, by

$$D^{\mathrm{sign}}_M := D^{\mathrm{sign}} := D^\Lambda_M | \mathcal{D}^+ : \mathcal{D}^+ \to \mathcal{D}^-. \tag{0.10}$$

We say that the *case of uniqueness* or the L^2-Stokes Theorem holds on M if

$$d_{\max} = d_{\min}. \tag{0.11}$$

In this case we have $\tau(\mathcal{D}) \subset \mathcal{D}$, and if D is also Fredholm, then so is D^{sign} and its index equals the L^2-signature of M,

$$\operatorname{ind} D^{\mathrm{sign}} = \operatorname{sign}_{(2)} M. \tag{0.12}$$

The above metric data define the crucial object in the analysis of the signature operator: the splitting of T^*N (induced by (0.4)) defines the "vertical de Rham operator" d_V (see (2.5)) and the metric $g_1^{T_VN}$ defines the adjoint $d^\dagger_V$, such that we can form the operator (see (2.31))

$$A_V := (d_V + d^\dagger_V)\alpha + \nu. \tag{0.13}$$

Here α is another supersymmetry on ΛT^*N and ν is an endomorphism (which are defined in (2.19) and (2.13)), and A_V is a first order symmetric differential operator on $C_c^1(N, \Lambda T^*N)$ which is fiberwise elliptic. Now M is called a *Witt space* if

$$H^{v/2}(Y) = 0. \tag{0.14}$$

We will see below (cf. Theorem 3.1) that (0.14) is essentially equivalent to the analytic condition

$$A_V \text{ is invertible,} \tag{0.15}$$

in the sense that the invertibility of A_V implies the Witt condition, whereas the Witt condition does not exclude the existence of zero eigenvalues but only of such which may be called inessential; indeed, they disappear under suitable rescalings of the fiber metric. *We will assume that M is a Witt space.*

Our results can then be summarized in the following theorems. We describe the Signature Operator on M by explicitly constructing its Green kernel which relates it to the symmetric operator $\tilde{D}$ defined as the restriction of $D_{\max}$ to the domain

$$\begin{aligned}(0.16)\quad \{\sigma \in \operatorname{dom} D_{\max} : ||\sigma^+||_{\lambda_{(2)}(N_t)} &= O(t^{1/2-\varepsilon}) \text{ for every } \varepsilon > 0,\\ ||\sigma^-||_{\lambda_{(2)}(N_t)} &= O(t^{-1/2+\eta}) \text{ for some } \eta > 0, t \to 0\};\end{aligned}$$

note that $\tilde{D}$ anticommutes with τ_M by construction.

Theorem 0.1. — *Let the Riemannian manifold (M, g^{TM}), of dimension $m = 4k$, be the top stratum of a Riemannian pseudomanifold, X, which is a Witt space with only one singular stratum B of conic type.*
1. *The operator $\tilde{D}$ defined by* (0.16) *is self-adjoint and discrete and anticommutes with τ_M.*
2. *If $|A_V| \geq \frac{1}{2}$, then $D^{\Lambda}_{M,\min}$ is essentially self-adjoint.*
3. *The case of uniqueness holds for M.*
4. *$D^{\mathrm{sign}} = \tilde{D}^+$.*

This theorem is well known in the case $h = 0$, cf. [**15**], [**12**], and part 2 and part 3 could also be deduced from Cheeger's work [**15**].

It is clear from part 4 of Theorem 0.1 that under the above conditions

$$\operatorname{ind} \tilde{D}^+ = \operatorname{ind} \tilde{D}^{\mathrm{sign}} = \operatorname{sign}_{(2)} M, \tag{0.17}$$

so it is natural to ask for a local formula analogous to Hirzebruch's Signature Theorem in the smooth case. Bismut and Cheeger [**6**, Thm. 5.7] have indicated the adiabatic construction of the homology L-class on the compact singular space associated with M, together with the corresponding L^2-index formula. A crucial role is played by the η-invariant, $\eta(N, g^{TN})$, of the Riemannian manifold (N, g^{TN}), as introduced by Atiyah, Patodi, and Singer in [**1**, Thm. (4.14)], and its adiabatic limit,

$$\tilde{\eta}(N, g^{TN}) := \lim_{\epsilon \to 0} \eta(N, g_\varepsilon^{TN}).$$

The adiabatic limit was first introduced and computed by Witten [**25**], as a gravitational anomaly, in case of a one-dimensional base. Witten's formula was proved rigorously by Cheeger [**16**], and independently by Bismut and Freed [**9**], [**10**]. The computation of the adiabatic limit for arbitrary dimensions and invertible fiber operators was given by Bismut and Cheeger [**6, 7**], who introduced the form $\tilde{\eta} = \tilde{\eta}(\pi, g^{TM}) \in \lambda(B)$ generalizing the η-invariant; the case of the signature operator was treated by Dai [**18**, Thm.0.3] who further introduced the τ-invariant associated to the Leray spectral sequence of the fibration (0.3). There has been done considerable work recently on the computation of L^2-cohomology groups of spaces which can be compactified as pseudomanifolds of the type we consider here, cf. [**19**], [**20**], [**21**], and [**17**]. These calculations lead to topological formulas for $\operatorname{sign}_{(2)} M$, see [**17**, Cor.1.2] for Witt spaces and its extension in [**21**]. Combining these topological formulas with Dai's result quoted above gives the following local signature formula which was stated for even dimensional base spaces in [**8**, Thm.5.7]; in its formulation, we denote by $D_B^{\Lambda\otimes\mathscr{H}(Y)}$ the Dirac operator D_B^{Λ} twisted by the bundle of fiber harmonic forms.

Theorem 0.2. — *We have*

$$\operatorname{ind} D^{\mathrm{sign}} = \lim_{\varepsilon\to 0} \int_M L(\nabla^{TM}) - \int_B L(TB, \nabla^{TB}) \wedge \tilde{\eta} - \frac{1}{2}\eta(D_B^{\Lambda\otimes\mathscr{H}(Y)}).$$

We give here an analytic proof of [**17**, Cor.1.2] in the general case which should be applicable to more general situations; in combination with the results of Atiyah, Patodi, and Singer and Dai's computation, it yields the theorem. The parametrix construction which we give in this paper should, in principle, also lead directly to the local index formula but, so far, we have been unable to overcome the technical difficulties involved.

We also have considered the resolvent trace expansion. We have a proof of the following result, but its presentation would lengthen the paper unduly; we hope to include it in a more general result at some future time.

Theorem 0.3. — 1. *For $\mu \in \mathbb{R}\setminus\{0\}$ and $p > m$, the resolvent $(D - i\mu)^{-1}$ is in the Schatten-von Neumann class of order p in $L^2(M, \Lambda T^*M)$.*
2. *For $z \in \mathbb{R}$ and $l > m/2$, we have the asymptotic expansion*

$$\operatorname{tr}[D^2 + z^2]^{-l} \sim_{z\to\infty} z^{m-2l} \sum_{j\geq 0} a_j z^{-j} + \sum_{j\geq 2l-h} b_j z^{-j} \log z.$$

The plan of the article is as follows. In Section 1, we deal with general Dirac operators and derive some decomposition theorems which are induced by a fibration of the form (0.3) and are needed later on. These results are known for spin Dirac operators, see [**5**, pp. 56, 59].

In Section 2, we represent the signature operator D^{sign} on U in the form

$$D_M^{\mathrm{sign}} \simeq \frac{\partial}{\partial t} + A_H(t) + t^{-1} A_V,$$

acting on $C_c^1\big((0,\varepsilon_0), H^1(N, \Lambda T^*N)\big)$ (see (2.38)). Here $A_H(t)$ and A_V are first order differential operators which can be written as a Dirac operator plus a potential and $A_H(t)$ is linear in t, with derivative a bounded endomorphism, while A_V is given by (0.13). We also show (in Theorem 2.5) that the anticommutator $A_H A_V + A_V A_H$ is a first order vertical differential operator, a crucial fact for our analysis. The guiding principles here are the structure of Dirac systems, as developped in [**3**], and the decomposition results from Sec. 1.

In Section 3, we obtain explicitly the spectral decomposition of the operators $A_V(b) := A_V|Y_b$ (cf. Theorem 3.1). By ellipticity, the spectrum is discrete. It consists of the *harmonic* eigenvalues $\mu = j - v/2, 0 \leq j \leq v$, generated by the harmonic forms on Y_b, and two families $\mu^{\pm}$ generated by the nonzero eigenvalues of the Laplacian on Y_b, with $\mu^+ \subset (-\frac{1}{2}, \infty)$ and $\mu^- \subset (-\infty, \frac{1}{2})$. When the metric on Y_b is scaled down, these eigenvalues tend respectively to $+\infty$ and $-\infty$.

Section 4 introduces appropriate boundary conditions for D^{sign}, based on the spectral analysis of Section 2. For the choice of boundary conditions and hence of a self-adjoint extension, only the small eigenvalues of A_V matter. We treat them by explicitly constructing the resolvent kernel by means of matrix Bessel functions, as introduced in [**13**], and then use this kernel in constructing a good pseudodifferential parametrix for D^{sign} with operator valued symbol, again following the strategy developed in [**13**]. At the end of this section, we give the proof of Theorem 0.1.

In Section 5 we prove Theorem 0.2 by reducing the problem to an APS-type problem on M_ε, for sufficciently small $\varepsilon > 0$. We also prove various related results: a Kato type perturbation result for the APS projection (Theorem 5.9), a vanishing result which is crucial for our approach (Theorem 5.2), and a new identity involving Dai's τ-invariant (Theorem 5.4).

This paper started as a joint project with Bob Seeley to whom it owes a lot. The construction of the Signature Operator was essentially finished several years ago using a less explicit parametrix construction. The publication of the results has been delayed by an attempt to deduce the local signature formula directly from the resolvent expansion in Theorem 0.3. However, this goal has proved elusive so far; we hope that, nevertheless, the results presented here will be of independent value.

We wish to thank Bob Seeley for many years of fruitful exchange and cooperation. We are indebted to Jean-Michel Bismut, Xiaonan Ma, and Henri Moscovici for useful discussions. We are grateful for the support of Deutsche Forschungsgemeinschaft under various grants, especially SFB 288 and SFB 647, and for the generous hospitality of the Ohio State University, the Mittag-Leffler Institute, the University of Bergen, Kyoto University, and MSRI Berkeley. Special thanks are due to an anonymous referee for very helpful remarks based on an unduly preliminary version of this article.

1. Dirac operators on fibrations

In this section, we consider a Riemannian manifold (M, g^{TM}) which we assume to be oriented. For $X, Y \in TM$ we write

$$g^{TM}(X,Y) =: \langle X,Y \rangle_{TM} =: \langle X,Y \rangle,$$

if no confusion may arise, and we use similar notation for vector bundles. Moreover, we consider a second oriented Riemannian manifold (B, g^{TB}) and a Riemannian fibration

$$\pi = \pi_B^M : M \to B \tag{1.1}$$

with generic fiber F; we write

$$F_b := \pi^{-1}(b), \ b \in B. \tag{1.2}$$

We denote the bundle of tangent vectors to the fibers by $T_V M$. Then the fibration induces an orthogonal splitting

$$TM =: T_H M \oplus T_V M, \ g := g^{TM} =: g^{T_H M} \oplus g^{T_V M} =: g_H \oplus g_V,$$

with orthogonal projections $P_{H/V} : TM \to T_{H/V} M$. Note that $T_V M$ and its annihilator $T_H^* M$ are defined independent of the metric.

The bundle (TM, g^{TM}) has a distinguished metric connection, the Levi-Civita connection ∇^{TM}; all bundles associated to the principal bundle of orthonormal frames in TM inherit a metric and a metric connection from (TM, g^{TM}). This holds in particular for the exterior algebra of the cotangent bundle, $\Lambda T^* M$, and for the bundle of Clifford algebras, $Cl(TM)$, and its complexification, $\mathbb{C}l(TM) = Cl(TM) \otimes_{\mathbb{R}} \mathbb{C}$.

We are interested in the class of *Dirac bundles* as defined in [**23**, p. 114], i.e. the smooth hermitian bundles (E, h^E) over M equipped with hermitian connections ∇^E such that the following conditions are satisfied: There is a smooth bundle map cl from the tangent bundle, TM, to the skew-hermitian endomorphisms, $\mathrm{End}_{\mathrm{as}}\, E$, of E such that

$$\mathrm{cl}(X) \circ \mathrm{cl}(X) = -g(X,X) I_E, \ X \in TM, \tag{1.3}$$

which implies that cl extends to an algebra homomorphism

$$\mathrm{cl} : \mathbb{C}l(TM) \to \mathrm{End}\, E, \tag{1.4}$$

turning E into a left Clifford module. Moreover, ∇^E is required to be compatible with the Levi-Civita connection in the sense that

$$\nabla^E_X \mathrm{cl}(Y)\sigma = \mathrm{cl}(\nabla^{TM}_X Y)\sigma + \mathrm{cl}(Y)\nabla^E_X \sigma, \tag{1.5}$$

for $X, Y \in TM, \sigma \in C^1(M, E)$. A prototypical Dirac bundle is, of course, $\mathbb{C}l(TM)$ itself with the metric structure induced from g^{TM}. This bundle is canonically isomorphic to the exterior algebra bundle $\Lambda T^* M$, with Clifford action

$$\mathrm{cl}(X)\omega = \mathrm{w}(X^\flat)\omega - \mathrm{i}(X)\omega, \ X \in TM, \ \omega \in \Lambda T^* M,$$

where "w" and "i" refer to wedge and interior multiplication, respectively, while $\flat : TM \to T^*M$ denotes the "musical" isomorphism induced by g^{TM} with inverse $\sharp$. Note that these definitions extend naturally to Hilbert bundles over M.

The notion of Dirac bundle was introduced to define the *Dirac operator* naturally associated with it, i.e. the operator

$$D := D_M^E := \sum_{i=1}^m \mathrm{cl}(e_i)\nabla_{e_i}^E, \tag{1.6}$$

which we will regard as an unbounded operator in $L^2(M,E)$ with domain $C_c^1(M,E)$ if not stated otherwise. Then D is symmetric in $L^2(M,E)$ and essentially self-adjoint e.g. if M is complete.

To obtain a nontrivial index, the symmetry of D must be broken. This is achieved by a *supersymmetry* or grading, α, on E, i.e. by a self-adjoint involution $\alpha \in \operatorname{End} E$ which is parallel with respect to ∇^E and anticommutes with Clifford multiplication, and hence with D. Then the bundle E splits as

$$E = E^+ \oplus E^-,\ E^\pm = \frac{1}{2}(I \pm \alpha)E.$$

$\mathbb{C}l(TM)$ has a natural grading obtained by lifting the map $X \mapsto -X$ from TM to $\mathbb{C}l(TM)$, with the property that

$$\mathbb{C}l(TM)^\pm E^+ \subset E^\pm,\ \mathbb{C}l(TM)^\pm E^- \subset E^\mp,$$

for any graded Dirac bundle E.

We are now interested in splitting the Dirac operator $D = D_M^E$ along the fibration $\pi : M \to B$ into a "horizontal" and a "vertical" part. The notion of horizontality we use will be introduced below, while we will call a differential operator Q on $C_c^1(M,E)$ *vertical* if Q commutes with multiplication by functions pulled back from the base, i.e. if Q differentiates only in fiber directions; if Q is of first order this is also equivalent to saying that

$$\hat{Q}(\xi) = 0,\ \xi \in T_H^*M, \tag{1.7}$$

with $\hat{Q}$ the principal symbol of Q. The desired splitting of D will reflect the geometry of the fibration π, through the second fundamental form, which is defined for $X, Y \in T_VM$ and $Z \in T_HM$ by

$$\begin{aligned} \langle II(X,Y), Z\rangle &= \langle \nabla_Z X - P_V[Z,X], Y\rangle \\ &= \langle \nabla_X Z, Y\rangle \\ &= -\langle \nabla_X Y, Z\rangle; \end{aligned} \tag{1.8}$$

and through the curvature of π, which is for $Z_1, Z_2 \in T_HM$ defined as

$$R_{Z_1,Z_2} := -P_V[Z_1, Z_2].$$

Before we state the results on the splitting of D we need to introduce some notation concerning local orthonormal frames. We will always denote by $(e_i)_{i=1}^h$ and $(f_j)_{j=1}^v$ an oriented local orthonormal frame for T_HM and T_VM, respectively, where $h = \dim B$

and $v := \dim F$ denote the "horizontal" and "vertical" dimensions, with $h + v = m := \dim M$, and we assume that $\{e_1, \dots, f_v\}$ is oriented in TM. More specifically, we may assume that $(e_i)_{i=1}^h$ consists of the horizontal lifts of an oriented local orthonormal frame $(\underline{e}_i)_{i=1}^h$ for TB; if this frame is defined in some open set U then $(e_i)_{i=1}^h$ is defined in $\pi^{-1}(U)$.

There are two operators generated by D which naturally belong to the horizontal and the vertical space, respectively, to wit

$$\tilde{D}_H := \sum_{i=1}^h \mathrm{cl}(e_i)\nabla_{e_i}^E, \tag{1.9}$$

$$\tilde{D}_V := \sum_{j=1}^v \mathrm{cl}(f_j)\nabla_{f_j}^E, \tag{1.10}$$

such that $D = \tilde{D}_H + \tilde{D}_V$. However, these operators are not easy to interpret and in spite of having a symmetric principal symbol, they are not symmetric in general. This defect is easily cured as follows. Since D is symmetric on $C_c^1(M, E)$, i. e. $D = D^\dagger$, its formal adjoint, we obtain

$$\begin{aligned} D &= \frac{1}{2}(\tilde{D}_H + \tilde{D}_H^\dagger) + \frac{1}{2}(\tilde{D}_V + \tilde{D}_V^\dagger) \\ &=: D_H + D_V, \end{aligned} \tag{1.11}$$

with $D_{H/V}$ symmetric. But since $\tilde{D}_V$ has symmetric principal symbol, we see that

$$\tilde{D}_V^\dagger = \tilde{D}_V + \beta_1, \tag{1.12}$$

with some endomorphism $\beta_1 \in C^\infty(M, \mathrm{End}\, E)$ such that

$$D_H = \tilde{D}_H - \frac{1}{2}\beta_1, \tag{1.13}$$

$$D_V = \tilde{D}_V + \frac{1}{2}\beta_1; \tag{1.14}$$

note that β_1 is necessarily skew-symmetric.

Lemma 1.1. — 1. *In* (1.12), *we have*

$$\beta_1 = -v\,\mathrm{cl}(H_F), \tag{1.15}$$

where

$$H_F := -\frac{1}{v}\sum_{j=1}^v P_H \nabla_{f_j}^{TM} f_j$$

is the mean curvature vector field of the fibers of π.
2. *For any horizontal vector field* Z *on* M *we have*

$$\mathrm{cl}(Z)D_V + D_V\,\mathrm{cl}(Z) = 0. \tag{1.16}$$

Proof. — 1. We compute $\tilde{D}_V^\dagger$ by calculating for $\sigma_k \in C_c^1(M,E)$, $k=1,2$, the expression

$$(\tilde{D}_V\sigma_1,\sigma_2)_{L^2(M,E)} - (\sigma_1,\tilde{D}_V\sigma_2)_{L^2(M,E)}$$
$$= \sum_{j=1}^{v}\int_M \big(\langle \mathrm{cl}(f_j)\nabla^E_{f_j}\sigma_1,\sigma_2\rangle_E - \langle\sigma_1,\mathrm{cl}(f_j)\nabla^E_{f_j}\sigma_2\rangle_E\big)$$
$$= \sum_{j=1}^{v}\int_M \big(-f_j\langle\sigma_1,\mathrm{cl}(f_j)\sigma_2\rangle_E + \langle\sigma_1,\mathrm{cl}(\nabla^{TM}_{f_j}f_j)\sigma_2\rangle_E\big)$$
$$= \sum_{j=1}^{v}\int_M \big(-f_j\langle\sigma_1,\mathrm{cl}(f_j)\sigma_2\rangle_E + \langle\sigma_1,\mathrm{cl}(\nabla^{T_VM}_{f_j}f_j)\sigma_2\rangle_E\big) - (\sigma_1, v\,\mathrm{cl}(H_F)\sigma_2)_{L^2(M,E)}, \tag{1.17}$$

where we have used the properties (1.3) through (1.5). Now we introduce a vertical vector field, X, by setting

$$\langle X,Y\rangle_{T_VM} := \langle\sigma_1,\mathrm{cl}(Y)\sigma_2\rangle_E,\ Y\in C(M,T_VM).$$

Then it is easy to see that the integrand in (1.17) equals the divergence of $X|F_b$ and hence vanishes upon integration over F_b, for any $b\in B$. It follows that

$$\tilde{D}_V^\dagger - \tilde{D}_V = -v\,\mathrm{cl}(H_F),$$

as claimed.

2. We compute, using again the basic relations (1.3) through (1.5),

$$\begin{aligned}
\mathrm{cl}(X)D_V + D_V\,\mathrm{cl}(X) &= \mathrm{cl}(X)\tilde{D}_V + \tilde{D}_V\,\mathrm{cl}(X) + v\langle X,H_F\rangle_{TM}\\
&= \sum_j \big(\mathrm{cl}(X)\,\mathrm{cl}(f_j)\nabla^E_{f_j} + \mathrm{cl}(f_j)\nabla^E_{f_j}\,\mathrm{cl}(X)\big) + v\langle X,H_F\rangle_{TM}\\
&= \sum_j \mathrm{cl}(f_j)\,\mathrm{cl}(\nabla^{TM}_{f_j}X) + v\langle X,H_F\rangle_{TM}\\
&= \Big(\sum_{j,l}\mathrm{cl}(f_j)\,\mathrm{cl}(f_l)\langle\nabla^{TM}_{f_j}X,f_l\rangle_{TM} + v\langle X,H_F\rangle_{TM}\Big)\\
&= \sum_{j\neq l}\mathrm{cl}(f_j)\,\mathrm{cl}(f_l)\langle X,\nabla^{TM}_{f_j}f_l\rangle_{TM}\\
&= 0. \qquad \square
\end{aligned}$$

We will use below a stronger property of this decomposition, namely that (in the case of D^Λ)

$$\tilde{D}_{HV} := D_HD_V + D_VD_H \tag{1.18}$$

is a first order vertical differential operator. Note that while $\tilde{D}_{HV}$ is always of first order, it need not be vertical in general. But this can be achieved if we further modify the decomposition (1.11) by bringing in the curvature of π.

Theorem 1.2. — *Define a symmetric endomorphism field of E by*

$$\beta_2 := \frac{1}{4}\sum_{j;i,k}\langle\nabla^{TM}_{e_i}e_k, f_j\rangle \operatorname{cl}(f_j)\operatorname{cl}(e_k)\operatorname{cl}(e_i). \tag{1.19}$$

Then the operator

$$\tilde{D}_{HV} := (D_H+\beta_2)(D_V-\beta_2)+(D_V-\beta_2)(D_H+\beta_2) \tag{1.20}$$

is first order vertical.

Proof. — β_2 is clearly well defined. We compute

$$\tilde{D}_{HV} = (D_HD_V + D_VD_H) - (D_H\beta_2+\beta_2D_H) + (D_V\beta_2+\beta_2D_V) \tag{1.21}$$

$$=: I + II + III. \tag{1.22}$$

Since III is first order vertical, we compute the coefficient, γ_m, of $\nabla^E_{e_m}$ from I and II:

$$\begin{aligned}\gamma_m &= -\frac{v}{2}\big(\operatorname{cl}(H_F)\operatorname{cl}(e_m)+\operatorname{cl}(e_m)\operatorname{cl}(H_F)\big)+\sum_j \operatorname{cl}(f_j)\operatorname{cl}(\nabla^{TM}_{f_j}e_m)\\ &\quad+\big(\beta_2\operatorname{cl}(e_m)+\operatorname{cl}(e_m)\beta_2\big)\\ &= v\langle H_F, e_m\rangle + \sum_{j,k}\operatorname{cl}(f_j)\operatorname{cl}(f_k)\langle\nabla^{TM}_{f_j}e_m, f_k\rangle\\ &\quad+\sum_{j;i}\operatorname{cl}(f_j)\operatorname{cl}(e_i)\langle\nabla^{TM}_{f_j}e_m, e_i\rangle + \big(\beta_2\operatorname{cl}(e_m)+\operatorname{cl}(e_m)\beta_2\big)\\ &= \sum_{j;i}\operatorname{cl}(f_j)\operatorname{cl}(e_i)\langle\nabla^{TM}_{f_j}e_m, e_i\rangle + \big(\beta_2\operatorname{cl}(e_m)+\operatorname{cl}(e_m)\beta_2\big)\\ &= 0,\end{aligned}$$

if we plug in the definition of β_2 in the penultimate line. □

Our next goal is to interpret the new operators D_H and D_V as Dirac operators in a natural way. This is more obvious for D_V since the fibers F_b, $b \in B$, inherit a lot of structure from M and E. Indeed, denoting by $j_b : F_b \to M$ the inclusion map, we obtain a hermitian bundle with hermitian connection over F_b by defining

$$E_b := j_b^*E,\ h^{E_b} := j_b^*h^E,\ \nabla^{E_b} := j_b^*\nabla^E.$$

Clearly, the relations (1.3) and (1.4) remain valid, so what remains to be checked is the compatibility condition (1.5) which now needs to involve ∇^{TF_b}. To achieve this we are going to modify ∇^{E_b} as follows. For $X, Y \in TF_b \subset T_VM, Z \in TF_b^\perp \subset T_HM$, we introduce the shape operator $S = S_b$ of F_b by

$$S_ZX := -P_V(\nabla^{TM}_X Z),$$

such that

$$\begin{aligned}\langle S_ZX, Y\rangle_{TF_b} &:= \langle\nabla^{TM}_X Y, Z\rangle_{TM}\\ &= -\langle \mathrm{II}_{F_b}(X,Y), Z\rangle.\end{aligned}$$

Then we define a new connection on TF by

$$\nabla_X^{E,b} := \nabla_X^{E_b} - \frac{1}{2}\sum_i \mathrm{cl}(S_{e_i}X)\,\mathrm{cl}(e_i), \tag{1.23}$$

which is clearly invariantly defined.

Theorem 1.3. — *The data $(E_b, h^{E_b}, \nabla^{E,b})$ define a Dirac bundle over F_b, for all $b \in B$, with Dirac operator*

$$D_V(b) := D_V|F_b. \tag{1.24}$$

Proof. — We compute with the notation used above:

$$\begin{aligned}
&\nabla_X^{E,b}\,\mathrm{cl}(Y) - \mathrm{cl}(Y)\nabla_X^{E,b}\\
&\quad= \mathrm{cl}(\nabla_X^{TM}Y) - \frac{1}{2}\sum_i \big(\mathrm{cl}(S_{e_i}X)\,\mathrm{cl}(e_i)\,\mathrm{cl}(Y) - \mathrm{cl}(Y)\,\mathrm{cl}(S_{e_i}X)\,\mathrm{cl}(e_i)\big)\\
&\quad= \mathrm{cl}(\nabla_X^{TM}Y) + \frac{1}{2}\sum_i \big(\mathrm{cl}(S_{e_i}X)\,\mathrm{cl}(Y) + \mathrm{cl}(Y)\,\mathrm{cl}(S_{e_i}X)\big)\,\mathrm{cl}(e_i)\\
&\quad= \mathrm{cl}(\nabla_X^{TM}Y) - \sum_i \langle S_{e_i}X, Y\rangle_{TM}\,\mathrm{cl}(e_i)\\
&\quad= \mathrm{cl}(\nabla_X^{TM}Y) - \sum_i \langle \nabla_X^{TM}Y, e_i\rangle_{TM}\,\mathrm{cl}(e_i)\\
&\quad= \mathrm{cl}(\nabla_X^{TF_b}Y).
\end{aligned}$$

Next we compute the Dirac operator, $\bar{D}_V$, associated to $(E_b, h^{E_b}, \nabla^{E,b})$:

$$\begin{aligned}
\bar{D}_V &= \sum_j \mathrm{cl}(f_j)\nabla_{f_j}^{E,b}\\
&= \sum_j \mathrm{cl}(f_j)\nabla_{f_j}^{E} - \frac{1}{2}\sum_{i,j,k}\langle S_{e_i}f_j, f_k\rangle_{TM}\,\mathrm{cl}(f_j)\,\mathrm{cl}(f_k)\,\mathrm{cl}(e_i)\\
&= \sum_j \mathrm{cl}(f_j)\nabla_{f_j}^{E} - \frac{1}{2}\sum_{i,j,k}\langle \nabla_{f_j}^{TM}f_k, e_i\rangle_{TM}\,\mathrm{cl}(f_j)\,\mathrm{cl}(f_k)\,\mathrm{cl}(e_i)\\
&= \sum_j \mathrm{cl}(f_j)\nabla_{f_j}^{E} + \frac{1}{2}\sum_{i,j}\langle \nabla_{f_j}^{TM}f_j, e_i\rangle_{TM}\,\mathrm{cl}(e_i)\\
&= \sum_j \mathrm{cl}(f_j)\nabla_{f_j}^{E} - \frac{v}{2}\,\mathrm{cl}(H_F)\\
&= \tilde{D}_V - \frac{v}{2}\,\mathrm{cl}(H_F)\\
&= D_V.
\end{aligned}$$

□

To exhibit D_H as a Dirac operator, too, we have to extend our setting to smooth Dirac-Hilbert bundles. This does not require new definitions but only natural extensions, as indicated above. If we introduce the family of Hilbert spaces over B,

$$\mathscr{E}_b := L^2(F_b, E_b),\ b \in B, \tag{1.25}$$

and put $\mathscr{E} := \bigcup_{b\in B} \mathscr{E}_b$ then the restriction map

$$R : C^1_c(M,E) \to \Gamma(B,\mathscr{E}),\ R\sigma(b) := R_b\sigma := \sigma|F_b,\ b \in B, \tag{1.26}$$

is an isometry by the Fubini Theorem,

$$||\sigma||^2_{L^2(M,E)} = \int_B ||R\sigma(b)||^2_{\mathscr{E}_b}\,\mathrm{vol}_B(b). \tag{1.27}$$

We define a metric on $\mathscr{E}$ by setting

$$h^{\mathscr{E}}(b)\big(R\sigma_1, R\sigma_2\big) := \int_{F_b} h^E(b)\big(\sigma_1,\sigma_2\big)\,\mathrm{vol}_{F_b},\ \sigma_j \in C_c(M,E), j = 1,2, \tag{1.28}$$

and the Clifford action by

$$\mathrm{cl}_B(\underline{X})R\sigma := R\,\mathrm{cl}(X)\sigma,\ \sigma \in C_c(M,E), \tag{1.29}$$

where $\underline{X} \in TB$ with horizontal lift $X \in T_HM$. The connection requires again some modification: we put

$$\nabla^{\mathscr{E}}_{\underline{X}}R\sigma := R\nabla^E_X\sigma - \frac{1}{2}\sum_j R\,\mathrm{cl}(\nabla^{TM}_{f_j}X)\,\mathrm{cl}(f_j)\sigma, \tag{1.30}$$

where again $\underline{X} \in TB$ with horizontal lift $X \in TM$, and $\sigma \in C^1_c(M,E)$. Then we have the following pleasant interpretation of D_H.

Theorem 1.4. — *The data $(\mathscr{E}, h^{\mathscr{E}}, \nabla^{\mathscr{E}})$ define a (Hilbert-) Dirac bundle over B such that its Dirac operator, $D^{\mathscr{E}}$, is given by*

$$D^{\mathscr{E}}R\sigma := D^{\mathscr{E}}_B R\sigma := RD_H\sigma,$$

for $\sigma \in C^1_c(M,E)$.

2. Representation of the signature operator near the singularity

We now restrict the general considerations of the previous section to a manageable and important special case, namely the Dirac operator on differential forms on a manifold with a conic singular stratum. Hence we will assume in the remainder of this work that we deal with the geometric situation explained in the Introduction. Thus, we consider a Riemannian manifold (M, g^{TM}), of dimension $m = 4k$, such that for $\varepsilon \in (0, \varepsilon_0]$ we have decompositions

$$M := U_\varepsilon \cup M_\varepsilon, \tag{2.1}$$

where $(M_{\varepsilon_0}, g^{TM_{\varepsilon_0}})$ is a compact Riemannian manifold with boundary $\partial M_{\varepsilon_0} = N_{\varepsilon_0}$. We further assume that the singular part, U_{ε_0}, is a bundle of metric cones over another compact Riemannian manifold, (B, g^{TB}), as explained above.

In order to construct a self-adjoint Fredholm extension of the operator

$$D^{\Lambda}_{M,\min} := D_{\min} := (d_M + d_M^{\dagger})_{\min}, \tag{2.2}$$

we need to construct a good representation of D on U_{ε_0}. To obtain a nontrivial index, we use the supersymmetry leading to the signature operator which is defined, on any oriented Riemannian manifold (M, g^{TM}) and for any local orthonormal and oriented frame $(\tilde{e}_i)_{i=1}^m$ of tangent vectors, by

$$\begin{aligned}\tau_M := \tau_{M,g^{TM}} &:= \sqrt{-1}^{[(m+1)/2]}\,\mathrm{cl}(\tilde{e}_1)\ldots\mathrm{cl}(\tilde{e}_m)\\ &= (-1)^k\,\mathrm{cl}(\tilde{e}_1)\ldots\mathrm{cl}(\tilde{e}_m);\end{aligned} \tag{2.3}$$

note that τ anticommutes with any Dirac operator on sections with compact support if m is even. If the signature operator can be defined then it is derived from the maximal de Rham complex. Thus, we state next the decomposition of d_N under the Riemannian fibration (0.3), as described somewhat more generally in [**4**, Prop. 10.1]. For this, a few further preparations are needed.

In the decomposition (0.7),

$$\Lambda T^*N = \oplus_{p,q}\Lambda^{p,q}T^*_N,$$

we count the degree of forms by operators hd and vd of horizontal and vertical degree, respectively, that is,

$$\mathrm{hd}|\Lambda^{p,q}T^*N = p,\ \ \mathrm{vd}|\Lambda^{p,q}T^*N = q.$$

Furthermore, we note the natural isometry of hermitian bundles

$$\psi : \pi^*\Lambda T^*B \otimes \Lambda T^*_V M \to \Lambda T^*_H N \otimes \Lambda T^*_V N, \tag{2.4}$$

such that the smooth sections of ΛT^*N are generated over $C^\infty(N)$ by sections of the form $\pi^*\omega_1 \otimes \omega_2$, with $\omega_1 \in \lambda(B)$ and $\omega_2 \in \lambda_V(N)$. Thus we can define the first order vertical operator d_V figuring in (0.13) by

$$d_V(\pi^*\omega_1 \otimes \omega_2) := \pi^*\varepsilon_H\omega_1 \otimes d_F\omega_2, \tag{2.5}$$

where

$$\varepsilon_H := (-1)^{\mathrm{hd}}. \tag{2.6}$$

Finally, we note the following decomposition of the Levi-Civita connection on N,

$$\begin{aligned}\nabla^{TN} &:= (P_H\nabla^{TN}P_H + P_V\nabla^{TN}P_V) + P_H\nabla^{TN}P_V + P_V\nabla^{TN}P_H\\ &=: \nabla^{TN,\delta} + \nabla^{HV} + \nabla^{VH},\end{aligned} \tag{2.7}$$

where $\nabla^{TN,\delta}$ is a connection while the other two terms are endomorphisms; observe that all operators in (2.7) act as derivations on tensors.

The decomposition of d_N for fibrations $\pi : N \to B$ then reads as follows.

Lemma 2.1. — *In local oriented orthonormal frames $(e_i)_{i=1}^h$ and $(f_j)_{j=1}^v$ for T_HN and T_VN, respectively, we have*

$$d_N = \Big(\sum_i^h \mathrm{w}(e_i^\flat)\nabla_{e_i}^{TN,\delta} - \sum_{i;j,l}\langle\nabla_{f_j}^{TN} f_l, e_i\rangle_{TN}\, \mathrm{w}(e_i^\flat)\otimes \mathrm{w}(f_j^\flat)\mathrm{i}(f_l)\Big) \tag{2.8}$$

$$+\frac{1}{2}\sum_{i,k;j}\langle[e_k,e_i],f_j\rangle_{TN}\,\mathrm{w}(e_i^\flat)\,\mathrm{w}(e_k^\flat)\otimes \mathrm{i}(f_j) \tag{2.9}$$

$$+ d_V$$

$$=: d_H^{(1,0)} + d_H^{(2,-1)} + d_V^{(0,1)} \tag{2.10}$$

$$=: d_H^1 + d_H^2 + d_V. \tag{2.11}$$

In (2.8) *and* (2.9), *the indices i,k run from 1 to h and indices j,l from 1 to v, while the upper indices in* (2.10) *indicate the change in bidegree effected by the respective operators; and $d_V = d_V^{(0,1)}$ is defined in* (2.5).

Proof. — The proof follows straightforwardly from the well known representation

$$d_M = \sum_i \mathrm{w}(e_i^\flat)\nabla_{e_i}^{TM} + \sum_j \mathrm{w}(f_j^\flat)\nabla_{f_j}^{TM},$$

and the decomposition (2.7). □

We will use this result to determine the decomposition (1.11) for the fibration

$$\pi_{(0,\infty)} : U_\infty \to (0,\infty),$$

where we now allow ε to be any number with $0 < \varepsilon \le \infty$, by an obvious extension. This gives the *boundary representation* needed in the approach of Atiyah, Patodi, and Singer (APS) which will be applied here to reduce the index problem to an APS-type problem, cf. [**1**]. The geometry is, however, not cylindrical near the boundary as assumed in loc. cit. which will cause additional difficulties later.

We will base our analysis on the unitary transformation

$$\begin{aligned} &\Psi_1 : L^2(\mathbb{R}_+, \mathbb{C}^2\otimes\lambda(N)) \to \lambda_{(2)}(U_\infty),\\ &\Psi_1(\sigma_1,\sigma_2)(t) := \pi_N^* t^\nu \sigma_1(t) + dt\wedge \pi_N^* t^\nu\sigma_2(t), \end{aligned} \tag{2.12}$$

where π_N denotes the canonical projection $U_\infty \to N$ and

$$\nu := \mathrm{vd} - \frac{v}{2}. \tag{2.13}$$

Ψ_1 generalizes the unitary transformation used in [**12**] for simple cones; note that it arises as the parallel transport along normal geodesics with respect to the metric connection defined by the fibration $\pi_{(0,\infty)} : U_\infty \to (0,\infty)$ according to Theorem 1.4. Then a straightforward calculation gives

Lemma 2.2. — *We have*

$$\Psi_1^{-1} d_{U_\infty} \Psi_1 = \begin{pmatrix} d_H^1 + t d_H^2 + t^{-1} d_V & 0 \\ \frac{\partial}{\partial t} + t^{-1}\nu & -d_H^1 - t d_H^2 - t^{-1} d_V \end{pmatrix}.$$

Taking adjoints and adding we obtain the transformation of $D_{U_\infty}^\Lambda$.

Corollary 2.3. — *With the notation*

$$\tilde{A}_H(t) := (d_H^1 + t d_H^2) + (d_H^1 + t d_H^2)^\dagger, \tag{2.14}$$

$$\tilde{A}_{0V} := d_V + d_V^\dagger, \tag{2.15}$$

$$\tilde{A}_0(t) := \tilde{A}_H(t) + t^{-1}\tilde{A}_{0V}, \tag{2.16}$$

and

$$\gamma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.17}$$

we have

$$\begin{aligned} &\Psi_1^{-1} D_{U_\infty}^\Lambda \Psi_1 =: \tilde{D}_{U_\infty}^\Lambda = \\ &\gamma\Big(\frac{\partial}{\partial t} + \begin{pmatrix} 0 & -\tilde{A}_H(t) \\ -\tilde{A}_H(t) & 0 \end{pmatrix} + t^{-1}\begin{pmatrix} \nu & -\tilde{A}_{0V} \\ -\tilde{A}_{0V} & -\nu \end{pmatrix}\Big) \end{aligned} \tag{2.18}$$

To transform the signature operator we need to incorporate the self-adjoint involution τ_{U_∞} which defines it. From (2.3) it is easy to derive its transformation law:

Lemma 2.4. — *We have*

$$\begin{aligned} \tilde{\tau} := \Psi_1^{-1}\tau_M\Psi_1 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes \varepsilon_H^v \tau_H \otimes \tau_V \\ &=: \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes (-\alpha), \end{aligned} \tag{2.19}$$

where with oriented frames $\{e_1, \dots, e_h\}$, $\{f_1, \dots, f_v\}$ for $T_H N$ and $T_V N$, respectively, we have

$$\tau_H := \sqrt{-1}^{[(h+1)/2]}\,\mathrm{cl}(e_1)\dots\mathrm{cl}(e_h), \tag{2.20}$$

$$\tau_V := \sqrt{-1}^{[(v+1)/2]}\,\mathrm{cl}(f_1)\dots\mathrm{cl}(f_v); \tag{2.21}$$

note that ε_H^v and τ_H commute.

The signature operator transforms to the positive part of $\tilde{D}_{U_\infty}$ with respect to $\tilde{\tau}$:

$$\tilde{D}_{U_\infty}^{\mathrm{sign}} := \frac{1}{2}(I + \tilde{\tau})\tilde{D}_{U_\infty}\frac{1}{2}(I + \tilde{\tau}). \tag{2.22}$$

To further transform $\tilde{D}^{\mathrm{sign}}_{U_\infty}$, we observe that the orthogonal projection onto the $+1$-eigenspace of $\tilde{\tau}$,

$$P^+(\tilde{\tau}) := \frac{1}{2}\begin{pmatrix} I & -\alpha \\ -\alpha & I \end{pmatrix}, \tag{2.23}$$

is conjugate to the standard projection

$$P := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{2.24}$$

under the unitary transformation

$$U := \frac{1}{\sqrt{2}}\begin{pmatrix} I & \alpha \\ -\alpha & I \end{pmatrix}, \tag{2.25}$$

i. e.

$$P = U^{-1}P^+(\tilde{\tau})U, \tag{2.26}$$

or equivalently,

$$U^{-1}\tilde{\tau}U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{2.27}$$

Now we obtain the final representation of $D^{\mathrm{sign}}_{U_\infty}$ by transforming all terms in (2.18) under U, observing the commutation relations

$$\nu\alpha = -\alpha\nu, \tag{2.28}$$

$$\tilde{A}(t)\alpha = \alpha\tilde{A}(t), \tag{2.29}$$

and using the notation

$$A_H(t) := \tilde{A}_H(t)\alpha,\ A_{0V}(t) := \tilde{A}_{0V}\alpha, \tag{2.30}$$

$$A_V := A_{0V} + \nu, \tag{2.31}$$

$$A_{(0)}(t) := A_H(t) + t^{-1}A_{(0)V}, \tag{2.32}$$

$$\Psi := \Psi_1 U, \tag{2.33}$$

where all operators are acting on $\lambda(N)$. We will call A_V the *cone coefficient* and

$$D^{\Lambda}_{\mathrm{cone}} := \gamma\Big(\frac{\partial}{\partial t} + t^{-1}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes A_V\Big), \tag{2.34}$$

the *cone operator*. Then the final result reads as follows.

Theorem 2.5. — 1. *We have*

$$\Psi^{-1}D^{\Lambda}_{U_\infty}\Psi = \gamma\Big(\frac{\partial}{\partial t} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \otimes A(t)\Big), \tag{2.35}$$

and

$$\Psi^{-1}\tau_{U_\infty}\Psi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.36}$$

$$\Psi^{-1}D^{\Lambda}_{N_t}\tau_{N_t}\Psi = \begin{pmatrix} A_0(t) & 0 \\ 0 & A_0(t) \end{pmatrix}, \tag{2.37}$$

such that

$$\Psi^{-1}D^{\mathrm{sign}}_{U_\infty}\Psi = \frac{\partial}{\partial t} + A(t). \tag{2.38}$$

2.

$$A_H(0)A_V + A_V A_H(0) =: A_{HV} \tag{2.39}$$

is a first order vertical operator.

3. *If A_V is invertible then for t sufficiently small we have the estimate*

$$A(t)^2 \geq Ct^{-2}A_V^2 \tag{2.40}$$

with a positive constant C.

Proof. — 1. The transformation formulas are again verified by straightforward computations.

2. To prove (2.39) we use Theorem 1.2 which, after the appropriate transformations, shows that we can modify $A_H(t)$ and $t^{-1}A_V$ by adding a bounded endomorphism multiplied by t to each term, such that their anticommutator becomes first order vertical. This, however, is an algebraic condition so that, after multiplication with t, all operator coefficients in the resulting polynomial have to be first order vertical, in particular the leading one which is A_{HV}.

3. The estimate (2.40) is an easy consequence of (2.39). □

3. Spectral decomposition of the cone coefficient

We want to deal with the existence of self-adjoint extensions of the cone operator, $D^{\Lambda}_{\mathrm{cone}}(b)$, defined in (2.34). According to [**12**, Thm. 3.1], this operator is *essentially self-adjoint* in $L^2(\mathbb{R}_+, \mathbb{C}^2 \otimes H^0)$ with domain $C^1_c((0,\infty), \mathbb{C}^2 \otimes H^1)$ if and only if

$$|A_V(b)| \geq \frac{1}{2}, \tag{3.1}$$

where $b \in B$. If the condition (3.1) is violated, then the self-adjoint extensions of $D^{\Lambda}_{\mathrm{cone}}$ are classified by the Lagrangian subspaces of

$$V := \sum_{|\lambda|<\frac{1}{2}} \ker(A(b)-\lambda) \oplus \sum_{|\lambda|<\frac{1}{2}} \ker(A(b)+\lambda) \tag{3.2}$$

with respect to the standard symplectic form

$$\omega_b\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = x_1y_2 - x_2y_1.$$

It is therefore necessary to determine the small eigenvalues of $A_V(b)$; in fact, we will describe the full spectral resolution in Theorem 3.1 below.

For its proof we recall some well known material from Hodge theory. In what follows, we fix $b \in B$ and write $Y := Y_b$, with metric $g := g^{TY} = g^{T_V N}$, the closed submanifold of N which is the fiber over b under the fibration $\pi : N \to B$; we will also suppress the index "Y" if no confusion is to be expected. Thus we consider the Hodge Laplacian

$$\Delta := \Delta_Y = d_Y(d_Y)^\dagger + (d_Y)^\dagger d_Y =: dd^\dagger + d^\dagger d,$$

which defines the harmonic forms,

$$\mathscr{H}^j := \mathscr{H}^j(Y) = \ker \Delta^j \subset \lambda^j(Y) =: \lambda^j,$$

and the Hodge decomposition

$$\lambda^j := \mathscr{H}^j \oplus \lambda^j_{\mathrm{cl}} \oplus \lambda^j_{\mathrm{ccl}}, \tag{3.3}$$

$$\Delta^j_{\mathrm{cl/ccl}} := \Delta|\lambda^j_{\mathrm{cl/ccl}}. \tag{3.4}$$

Here the subscripts "cl" and "ccl" refer to closed and coclosed forms, respectively; the eigenspaces of $\Delta^j_{\mathrm{cl/ccl}}$ with eigenvalue $\kappa > 0$ will be denoted by $E^j_{\mathrm{cl/ccl}}(\kappa)$.

We also recall the following definitions and relations, where $* := *_Y$ denotes the Hodge star operator on Y and $v(2)$ the remainder of v mod 2:

$$\varepsilon_V|\lambda^j =: \varepsilon|\lambda^j = (-1)^j, \tag{3.5}$$

$$\alpha_i|\lambda^j := (-1)^{[(j+i)/2]}, \tag{3.6}$$

$$\alpha_0\alpha_1 = \varepsilon, \tag{3.7}$$

$$d^\dagger = (-1)^{v+1} * d * \varepsilon^v, \tag{3.8}$$

$$\tau_V|Y =: \tau = \sqrt{-1}^{[(v+1)/2]} * (-1)^{[v/2]}\alpha_{v(2)}, \tag{3.9}$$

$$d\tau = (-1)^{v+1}\tau d^\dagger. \tag{3.10}$$

Then we have

$$A_V(b) =: A_V = -\varepsilon_H^{v+1}\tau_H \otimes \big(d\tau + (-1)^{v+1}\tau d\big). \tag{3.11}$$

Next we introduce some spaces which are invariant under A_V (here and below, $j \in \mathbb{N} \cap [1, (v+1)/2)$ if not stated otherwise):

$$\tilde{\lambda}^j_h := \mathcal{H}^j \oplus \mathcal{H}^{v-j}, \tag{3.12}$$

$$\tilde{\lambda}^j_{\mathrm{cl}} := \lambda^j_{\mathrm{cl}} \oplus \lambda^{v+1-j}_{\mathrm{cl}}, \tag{3.13}$$

$$\tilde{\lambda}^j_{\mathrm{ccl}} := \lambda^{j-1}_{\mathrm{ccl}} \oplus \lambda^{v-1-j}_{\mathrm{ccl}}, \tag{3.14}$$

$$F^j_h := \mathcal{H}^j \oplus \mathcal{H}^{v-j}, \tag{3.15}$$

$$F^j_{\mathrm{cl}}(\kappa) := E^j_{\mathrm{cl}}(\kappa) \oplus E^{v+1-j}_{\mathrm{cl}}(\kappa), \tag{3.16}$$

$$F^j_{\mathrm{ccl}}(\kappa) := E^{j-1}_{\mathrm{ccl}}(\kappa) \oplus E^{v-1-j}_{\mathrm{ccl}}(\kappa). \tag{3.17}$$

It is then convenient to put

$$A^j_{V,h} := A_V | \tilde{\lambda}^j_h, \tag{3.18}$$

$$A^j_{V,\mathrm{cl}} - \frac{1}{2} := (A_V - \frac{1}{2}) | \tilde{\lambda}^j_{\mathrm{cl}}$$

$$= \begin{pmatrix} j - \frac{v+1}{2} & -\varepsilon_H^{v+1} \tau_H \otimes d\tau \\ -\varepsilon_H^{v+1} \tau_H \otimes d\tau & -(j - \frac{v+1}{2}) \end{pmatrix}, \tag{3.19}$$

$$A^j_{V,\mathrm{ccl}} + \frac{1}{2} := (A_V + \frac{1}{2}) | \tilde{\lambda}^j_{\mathrm{ccl}}$$

$$= \begin{pmatrix} j - \frac{v+1}{2} & (-1)^v \varepsilon_H^{v+1} \tau_H \otimes \tau d \\ (-1)^v \varepsilon_H^{v+1} \tau_H \otimes \tau d & -(j - \frac{v+1}{2}) \end{pmatrix}. \tag{3.20}$$

Then the spectral resolution of A_V can be expressed as follows.

Theorem 3.1. — *1. $A^j_{V,h}$ has the eigenspaces $\mathcal{H}^j$ and $\mathcal{H}^{v-j}$ with eigenvalues $\pm(j - \frac{v}{2})$.*

2. For $\kappa \in \operatorname{spec} \Delta^j_{\mathrm{cl}} \setminus \{0\}$, $A^j_{V,\mathrm{cl}} - \frac{1}{2}$ has two eigenspaces in $F^j_{\mathrm{cl}}(\kappa)$, with eigenvalues

$$\mu^j_{\mathrm{cl},\pm}(\kappa) := \pm\sqrt{\kappa + (j - \frac{v+1}{2})^2},$$

and multiplicities $m^j_{\mathrm{cl},\pm}(\kappa)$.

3. For $\kappa \in \operatorname{spec} \Delta^j_{\mathrm{ccl}} \setminus \{0\}$, $A^j_{V,\mathrm{ccl}} + \frac{1}{2}$ has two eigenspaces in $F^j_{\mathrm{ccl}}(\kappa)$, with eigenvalues

$$\mu^j_{\mathrm{ccl},\pm}(\kappa) := \pm\sqrt{\kappa + (j - \frac{v+1}{2})^2},$$

and multiplicities $m^j_{\mathrm{ccl},\pm}(\kappa)$.

4. If v is odd, then, for $\kappa > 0$, there are two more eigenspaces of $A^{(v+1)/2}_{V,\mathrm{cl}} \oplus A^{(v-1)/2)}_{V,\mathrm{ccl}}$ in $E^{(v+1)/2}_{\mathrm{cl}}(\kappa) \oplus E^{(v-1)/2}_{\mathrm{ccl}}(\kappa)$ with eigenvalues $\pm\sqrt{\kappa}$.

5. For $\kappa > 0$, the four eigenvalues of A_V in $F^j_{\mathrm{cl}}(\kappa) \oplus F^j_{\mathrm{ccl}}(\kappa)$ have the common multiplicity $2 \dim E^j_{\mathrm{cl}}(\kappa)$.

Proof. — The first statement is obvious from Poincaré duality.

We compute next, using (3.18)

$$(A^j_{\mathrm{cl}} - \frac{1}{2})^2 = \begin{pmatrix} \Delta^j_{\mathrm{cl}} + (j - \frac{v+1}{2})^2 & 0 \\ 0 & \Delta^{v+1-j}_{\mathrm{cl}} + (j - \frac{v+1}{2})^2 \end{pmatrix}, \tag{3.21}$$

$$(A^j_{\mathrm{ccl}} + \frac{1}{2})^2 = \begin{pmatrix} \Delta^{j-1}_{\mathrm{ccl}} + (j - \frac{v+1}{2})^2 & 0 \\ 0 & \Delta^{v-1-(j-1)}_{\mathrm{cl}} + (j - \frac{v+1}{2})^2 \end{pmatrix}. \tag{3.22}$$

It follows that $F^j_{\mathrm{cl}}(\kappa) \oplus F^j_{\mathrm{ccl}}(\kappa)$ is invariant under A_V, and that A_V has the indicated eigenvalues on $F^j_{\mathrm{cl}}(\kappa) \oplus F^j_{\mathrm{ccl}}(\kappa)$. Moreover, we have unitary equivalences

$$\Delta^j_{\mathrm{cl}} \simeq \Delta^{v+1-j}_{\mathrm{cl}} \simeq \Delta^{j-1}_{\mathrm{ccl}} \simeq \Delta^{v-1-j}_{\mathrm{ccl}},$$

induced by the mappings $d\tau, \tau$, and τd, respectively. If we employ the bijective maps

$$\begin{pmatrix} 0 & -d\tau / -\tau d \\ d\tau/\tau d & 0 \end{pmatrix} : \tilde{\lambda}^j_{\mathrm{cl/ccl}} \mapsto \tilde{\lambda}^j_{\mathrm{cl/ccl}}, \tag{3.23}$$

$$\begin{pmatrix} 0 & (-1)^v \tau \\ \tau & 0 \end{pmatrix} : \tilde{\lambda}^j_{\mathrm{cl}} \mapsto \tilde{\lambda}^j_{\mathrm{ccl}}, \tag{3.24}$$

we see that the respective restrictions of A_V are unitarily equivalent under these maps up to the factor -1, which easily implies that the four eigenvalues on $F^j_{\mathrm{cl}}(\kappa) \oplus F^j_{\mathrm{ccl}}(\kappa)$ have the same multiplicities, and this must be $2 \dim E^j_{\mathrm{cl}}(\kappa)$, as asserted. This proves the assertions 2), 3), and 4), while 5) follows immediately from (3.18). □

4. A self-adjoint extension

With $D := D^{\Lambda}_M$ we associate the operators $D_{\min}$, i. e. the closure in $\lambda_{(2)}(M)$ of $D|\lambda_c(M)$, and $D_{\max} := D^*_{\min}$. In this section, we construct a suitable self-adjoint extension of the operator $D_{\min}$. For this, we introduce an operator family $G(\mu, D)$ for sufficiently large real μ, with $\operatorname{im} G(\mu, D)$ contained in the maximal domain of D, and

$$(D - i\mu) G(\mu, D)\tau = \tau, \ \tau \in L^2(M, \mathscr{E}). \tag{4.1}$$

Moreover, all the operators $G(\mu, D)$ map into a common domain on which D is symmetric. Hence this domain defines a self-adjoint extension of D, with resolvent $G(\mu, D)$. By a certain abuse of notation, we will denote this extension also by D.

We can naturally extend the conic fibers at hand to the infinite cones $C_{(0,\infty)}Y_b$, so we may and will assume that we are dealing with a fibration of infinite cones over B. The results can then be applied to U_ε by a standard cut-off procedure.

We obtain $G(\mu, D)$ as a pseudo-differential operator on B with operator valued symbol. For given $b_0 \in B$, choose $W_{b_0} := B_\delta(b_0)$, a ball on which the Hilbert bundle $\mathscr{E}$ is trivial. We identify forms $\tau \in C_c(W_{b_0}, \mathscr{E}|W_{b_0})$ with their representation in

$C_c(B_\delta^{\mathbb{R}^h}(0), \mathscr{E}_{b_0})$, and define a local parametrix $G_1(\mu, D, b_0)$ in the form

$$G_1(\mu, D, b_0)\tau(b) := \int_{\mathbb{R}^h} \exp(i\langle b, \beta\rangle) G(\mu, b, \beta)\hat{\tau}(\beta) d\!\!\bar{}\beta. \tag{4.2}$$

Here (b, β) are coordinates for $T^* W_{b_0}$, and $d\!\!\bar{}\beta := (2\pi)^{-h} d\beta$. These local parametrices are patched together in the usual way to make a global parametrix, $G_1(\mu, D)$, such that $(D - i\mu)G_1(\mu, D) - I$ decays in norm like $|\mu|^{-1}$, so that $G_1(\mu, D)$ serves as the leading term in a Neumann series for the resolvent $G(\mu, D)$.

We recall from Section 1 the decomposition $D = D_H + D_V$ and construct our operator $G(\mu, b, \beta)$ with the property that $\operatorname{im} G(\mu, b, \beta) \subset \mathscr{D}_{V,\max}(b)$, the domain of $D_{V,\max}(b)$, and

$$\big(D_{V,\max}(b) + i\,\mathrm{cl}(\beta^\sharp) - i\mu\big) G(\mu, b, \beta)\tau = \tau, \ \tau \in \Lambda T_b^* B \otimes \lambda_{(2)}(Y_b). \tag{4.3}$$

Just as above, $G(\mu, b, \beta)$ will define a self-adjoint extension of $D_V(b)$, with domain $\mathscr{D}_V(b)$. Note that, in view of Lemma 1.1, part 2, we have for $\sigma(b) \in \mathscr{D}_V(b)$

$$\begin{aligned} (4.4)\quad &\|\big((D_V(b) + i\,\mathrm{cl}(\beta^\sharp)) - i\mu\big)\sigma(b)\|^2_{\mathscr{D}_V(b)} \\ &\qquad = \|D_V(b)\sigma(b)\|^2_{\mathscr{D}_V(b)} + (|\mu|^2 + |\beta|_b^2)\|\sigma(b)\|^2_{\mathscr{D}_V(b)}, \end{aligned}$$

where $|\beta|_b^2 := g^{T^*B}(b)(\beta, \beta)$.

With (2.34) we now write $D_V(b)$ in the form

$$\begin{aligned} (4.5)\quad D_V(b) &:= \gamma\left(\frac{d}{dt} + t^{-1}\tilde{A}(b)\right) \\ &:= \varepsilon_H \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\frac{d}{dt} + \frac{1}{t}\begin{pmatrix} D_{Y_b}\alpha_{Y_b} + \nu & 0 \\ 0 & -(D_{Y_b}\alpha_{Y_b} + \nu) \end{pmatrix}\right). \end{aligned}$$

The trivialization of $\mathscr{E}$ identifies the fibers $\mathscr{E}_b, b \in W_{b_0}$, with

$$L^2\big((0, \infty), \Lambda T_{b_0}^* B \otimes \mathbb{C}^2 \otimes \lambda_{(2)}(Y_{b_0})\big) =: L^2((0, \infty), H).$$

We will need the following description of the singularities of elements in the maximal domain of $\mathscr{D}_{V,\max}(b)$ (see [**12**, Lem.3.2]).

Lemma 4.1. — 1. *Any σ in $\mathscr{D}_{V,\max}(b)$ has a representation of the form*

$$\sigma(t) = \sum_{\lambda \in \operatorname{spec} A, |\lambda| < 1/2} t^{-\lambda} C_\lambda(\sigma) + O_\sigma(t^{1/2}|\log t|), \ t \to 0, \tag{4.6}$$

with certain linear forms C_λ.

2. *Each closed extension of $D_{V,\max}(b)$ is determined by linear relations between the coefficients C_λ for $|\lambda| < \frac{1}{2}$.*

3. *$\sigma \in \mathscr{D}_{V,\min}$ if and only if*

$$\|\sigma(t)\|_H = O_\sigma(t^{1/2}|\log t|), \ t \to 0. \tag{4.7}$$

Now, to construct $G(\mu, b, \beta)$, we split the spectrum of the operator $\tilde{A}(b)$ from (4.5), and treat separately the high and low eigenvalues. Arguing as in [**12**, Lemma 1.1] and making U_{b_0} smaller if necessary, we may then assume that, for some $\Lambda \geq 1$ with the property that $\Lambda \notin \operatorname{spec} \tilde{A}(b)$ for all $b \in W_{b_0}$, the spectral projection

$$Q_{>} := Q_{|\lambda| \geq \Lambda}(\tilde{A}(b)) \tag{4.8}$$

does not depend on $b \in W_{b_0}$ (here and below we denote, for any Borel subset $I \subset \mathbb{R}$, the corresponding spectral projection of a self-adjoint operator, A, by $Q_I(A)$).

In constructing $G(\mu, b, \beta)$, consider first the high eigenvalues of $\tilde{A}(b)$. We reduce $D_{V,\min}(b)$ by the spectral projection $Q_{>}$, which is independent of $b \in W_{b_0}$, and denote the resulting objects by a subscript $'' >''$. Since $|\tilde{A}(b)_{>}| \geq 1$, Lemma 4.1 shows that

$$D_V(b)_{>} := D_{V,\min}(b)_{>}$$

is essentially self-adjoint on compactly supported sections. Moreover, from [**12**, Lem.3.1], by a proof as in Lemma 2.2 there, for $\sigma \in \mathcal{D}_V(b)_{>}$ we have

$$(1/t)\tilde{A}(b)_{>}\sigma(t) \in L^2\big((0,\infty), H\big) \text{ hence also } \sigma' \in L^2\big((0,\infty), H\big).$$

It follows from this and (4.4) that

$$G(\mu, b, \beta)_{>} := \big(D_V(b)_{>} + i\,\mathrm{cl}(\beta^\sharp) - i\mu\big)^{-1}$$

satisfies the estimates

$$\|\frac{\partial^j}{\partial \mu^j}\frac{\partial^{|\kappa|}}{\partial b^\kappa}\frac{\partial^{|\lambda|}}{\partial \beta^\lambda} G(\mu, b, \beta)_{>}\|_{\mathcal{L}(\mathcal{E})} \leq C_{l,\kappa,\lambda}|\mu|^{-1-j}, \tag{4.9}$$

while from Lemma 4.1 we see that

$$G(\mu, b, \beta)_{>}\sigma(t) = O(t^{1/2}|\log t|), t \to 0. \tag{4.10}$$

As usual, the low eigenvalue case needs more care. We note first that the reduction with $Q_{<} := I - Q_{>}$ leads to the matrix equation

$$D_{V<}(b) := \gamma(\frac{d}{dt} + t^{-1}\tilde{A}(b)_{<}) \tag{4.11}$$

in $L^2((0,\infty), H_{<})$, $H_{<} = Q_{<}(H)$. In view of Lemma 4.1 this operator is not essentially self-adjoint with domain $C_c^1((0,\infty), H_{<})$ if there are "small" eigenvalues with modulus less than $1/2$. Hence we will construct an operator function satisfying the conditions

$$\big(D_{V,<}(b) + i\,\mathrm{cl}(\beta^\sharp) - i\mu\big)G(\mu, b, \beta)_{<} = I; \tag{4.12}$$

$$D_{V,<}(b)_{\max} \text{ is symmetric on } \operatorname{im} G(\mu, b, \beta)_{<}; \tag{4.13}$$

$$\|\frac{\partial^j}{\partial \mu^j}\frac{\partial^{|\kappa|}}{\partial b^\kappa}\frac{\partial^{|\lambda|}}{\partial \beta^\lambda} G(\mu, b, \beta)_{<}\|_{\mathcal{L}(\mathcal{E})} \leq C_{l,\kappa,\lambda}|\mu|^{-1-j}. \tag{4.14}$$

From (4.12) and (4.13), $D_{V<}(b)$, on $\operatorname{im} G(\mu, b, \beta)_{<}$, is self-adjoint.

The estimates (4.14), together with the Calderón- Vaillancourt Theorem (cf.[**14**]), will provide the necessary norm estimates on our pseudo-differential operator.

In order to carry out this construction, we now consider the following model case to which we will reduce our situation. We are given a finite dimensional complex

Hilbert space $(H, \langle,\rangle)$ and a Hermitian operator $A \in \mathscr{L}(H)$. Moreover, there are two self-adjoint involutions α_1, α_2 with the following properties:

$$\alpha_1\alpha_2 + \alpha_2\alpha_1 = 0, \tag{4.15}$$

$$\alpha_1 A - A\alpha_1 = 0, \tag{4.16}$$

$$\alpha_2 A + A\alpha_2 = 0. \tag{4.17}$$

We want to solve the equation

$$L(A)\sigma(t) := (\frac{d}{dt} + t^{-1}A + \mu\alpha_2)\sigma(t) = \tau(t),\ t > 0, \tag{4.18}$$

in $L^2(\mathbb{R}_+, H)$, for $\mu \in \mathbb{R}^*$. We transform H by introducing the subspaces $H^\pm := \frac{1}{2}(I \pm \alpha_1)(H)$ and the isomorphism $\mathbb{C}^2 \otimes H^+ \to H$ which is induced by

$$H^+ \oplus H^+ \ni (x_+, x_-) \mapsto x_+ + \alpha_2 x_- \in H.$$

Then our equation takes the form, with $A^+ := A|H^+$,

$$\left(\frac{d}{dt} + t^{-1}\begin{pmatrix} A^+ & 0 \\ 0 & -A^+ \end{pmatrix} + \mu\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\begin{pmatrix} \sigma_+ \\ \sigma_- \end{pmatrix}(t) = \begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(t). \tag{4.19}$$

If we multiply the operator occuring in (4.19) with its formal adjoint from the left, then we obtain the Bessel type operator

$$-\frac{d^2}{dt^2} + t^{-2}\begin{pmatrix} A^2 + A & 0 \\ 0 & A^2 - A \end{pmatrix} + \mu^2 I, \tag{4.20}$$

where we have now replaced A^+ by A to ease the notation, which should not cause confusion. Now we introduce the modified matrix Bessel functions in H^+ as solutions of the homogeneous equation associated with (4.20), following [**12**, Sec. 2]. Thus, if N is hermitian in $\mathscr{L}(H^+)$ with eigenvalues ν_j then we define the modified matrix Bessel function with respect to an orthonormal eigenbasis of N by

$$I_N(t)_{ij} := \delta_{ij} I_{\nu_j}(t),$$

and require that for any unitary operator U in H^+ we have

$$U^{-1} I_N(t) U =: I_{U^{-1}NU}(t),\ t > 0.$$

Likewise, we introduce

$$\frac{2}{\pi}\sin(\pi N) K_N(t) := I_{-N}(t) - I_N(t).$$

We can then prove the following result.

Theorem 4.2. — *For $\mu > 0$, the equation (4.18) admits the solution*

$$G(\mu, A)\begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(t) =$$
$$= \int_0^t \mu(ts)^{1/2}\begin{pmatrix} K_{A+1/2}(\mu t)I_{A-1/2}(\mu s) & K_{A+1/2}(\mu t)I_{A+1/2}(\mu s) \\ K_{A-1/2}(\mu t)I_{A-1/2}(\mu s) & K_{A-1/2}(\mu t)I_{A+1/2}(\mu s) \end{pmatrix}\begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(s)ds$$
$$- \int_t^\infty \mu(ts)^{1/2}\begin{pmatrix} I_{A+1/2}(\mu t)K_{A-1/2}(\mu s) & -I_{A+1/2}(\mu t)K_{A+1/2}(\mu s) \\ -I_{A-1/2}(\mu t)K_{A-1/2}(\mu s) & I_{A-1/2}(\mu t)K_{A+1/2}(\mu s) \end{pmatrix}\begin{pmatrix} \tau_+ \\ \tau_- \end{pmatrix}(s)ds$$
$$=: G_0(\mu, A)\tau(t) + G_\infty(\mu, A)\tau(t).$$

The operators $G_{0/\infty}(\mu, A)$ are bounded in $L^2(\mathbb{R}_+, H)$ and smooth functions of the variables $\mu \in [1, \infty)$ and $A \in \mathscr{L}_s(H)$, the space of Hermitian matrices on H, such that for $p, q \in \mathbb{Z}_+$

$$\text{(4.21)} \qquad ||D_A^p(\frac{\partial}{\partial \mu})^q G_{0/\infty}(\mu, A)||_{L^2(\mathbb{R}_+, H)} \leq C_{p,q,A}\ \mu^{-1}.$$

Moreover, for $\sigma \in \operatorname{im} G(\mu, A)$ and t sufficiently small we have the estimates

$$\text{(4.22)} \qquad ||\sigma_+(t)||_H \leq C_\varepsilon t^{1/2-\varepsilon}||\tau||_{L^2(\mathbb{R}_+, H)} \text{ for every } \varepsilon > 0,$$

$$\text{(4.23)} \qquad ||\sigma_-(t)||_H \leq C t^{-1/2+\delta}||\tau||_{L^2(\mathbb{R}_+, H)} \text{ for some } \delta > 0.$$

If $|A| \geq \frac{1}{2}$, then we have the better estimate

$$\text{(4.24)} \qquad ||\sigma(t)||_H \leq C t^{1/2}||\tau||_{L^2(\mathbb{R}_+, H)}.$$

Proof. — We begin with verifying that $G(\mu, A)\tau(t)$ is indeed a solution of (4.19). The well known conic scaling

$$\sigma(t) =: t^{1/2}\rho(\mu t)$$

transforms the homogeneous equation associated with (4.19) to

$$\text{(4.25)} \qquad \left(\frac{d}{dt} + t^{-1}\begin{pmatrix} A + 1/2 & 0 \\ 0 & -A + 1/2 \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)\begin{pmatrix} \rho_+ \\ \rho_- \end{pmatrix}(t) = 0.$$

The Bessel recursion relations (cf. [**12**, (2.5a,b)]),

$$I'_N(t) \pm t^{-1}NI_N(t) = I_{N\mp 1}(t),$$
$$K'_N(t) \pm t^{-1}NK_N(t) = -K_{N\mp 1}(t),$$

show at once that two solutions are given by

$$\rho_{c_+}(t) = \begin{pmatrix} I_{A+1/2}(t)c_+ \\ -I_{A-1/2}(t)c_+ \end{pmatrix}, \ \rho_{c_-}(t) = \begin{pmatrix} K_{A+1/2}(t)c_- \\ K_{A-1/2}(t)c_- \end{pmatrix}, \ c_\pm \in H^+.$$

It remains to note that (cf. [**24**, p. 68])

$$I_N K_{N+1}(t) + I_{N+1}K_N(t) = t^{-1},$$

from which we deduce that

$$\begin{pmatrix} I_{A+1/2}(t) & K_{A+1/2}(t) \\ -I_{A-1/2}(t) & K_{A-1/2}(t) \end{pmatrix} \begin{pmatrix} K_{A-1/2}(t) & -K_{A+1/2}(t) \\ I_{A-1/2}(t) & I_{A+1/2}(t) \end{pmatrix} = t^{-1} I_H.$$

Thus, $G(\mu, A)\tau$ is indeed a solution of (4.18).

To deduce the estimate (4.21), we perform some reductions of the operator $L(A)$. First, we select a number $\Lambda \leq \frac{1}{2}$ such that $|A| \leq \Lambda$, and we choose a number $\Lambda_1 \in [-1/2, 0], \Lambda_1 \notin \operatorname{spec} A$. Then we split, with obvious notation,

$$A = A_{>\Lambda_1} \oplus A_{<\Lambda_1}.$$

This splits $L(A) = L(A_{>\Lambda_1}) \oplus L(A_{<\Lambda_1})$, and conjugating with α_2 in the second summand allows us to assume that

$$A > -\frac{1}{2} \tag{4.26}$$

in what follows. By the same token, we can select numbers $\Lambda_j, j = 1, \ldots, N$, such that

$$\Lambda_j \notin \operatorname{spec} A, \ \Lambda_N > \Lambda; \tag{4.27}$$

$$\Lambda_j < \Lambda_{j+1} < \Lambda_j + 1. \tag{4.28}$$

Splitting $L(A)$ accordingly as a direct sum, we may further assume that for some $\Lambda^* \in [-\frac{1}{2}, \Lambda)$ we have

$$\Lambda^* < A < \Lambda^* + 1. \tag{4.29}$$

Under the assumption (4.29) we will next prove the estimates (4.21) using [**12**, Lemma 2.3], which is perfectly adapted to the situation at hand, at least for the operator $G_\infty(\mu, A)$. However, it is easily seen that $G_0(\mu, A)$ is essentially the adjoint operator to $G_\infty(\mu, A)$, up to permutations and sign changes of the matrix elements. Since we will base our estimate on estimates of the matrix elements, it is hence enough to deal with $G_\infty(\mu, A)$. These estimates for the modified matrix Bessel functions and their derivatives have been derived in [**12**, Lemmas 2.1, 2.2] and are combined in the statement that follows. We recall from loc. cit. that l denotes a positive function, defined for positive real numbers, which equals $-\log t$ for $t \leq 1/2$ and 1 for $t \geq 1$.

Lemma 4.3. — *The modified matrix Bessel functions $I_N(t), K_N(t)$ are smooth in $\mathscr{L}_s(H) \times (0, \infty)$, and if $N \in \mathscr{L}_s(H)$ satisfies the inequality*

$$-\infty < a \leq N \leq b < \infty,$$

then the estimates

$$||D_N^p (\frac{\partial}{\partial t})^q I_N(t)|| \leq C_{a,b,p,q} t^{a-q} (1+t)^{q-a-1/2} e^t \, l(t)^p, \tag{4.30}$$

$$||D_N^p (\frac{\partial}{\partial t})^q K_N(t)|| \leq C_{a,b,p,q} t^{-b-q} (1+t)^{b+q-1/2} e^{-t} \, l(t)^p, \tag{4.31}$$

hold for $p, q \in \mathbb{Z}_+$ and $t > 0$.

Now we use Lemma 4.3 in [**12**, Lemma 2.3] to derive the norm estimate (4.21) for $G_\infty(\mu, A)$ where, by the above reduction, we may assume that

$$-\frac{1}{2} < a \leq A \leq b < a+1. \tag{4.32}$$

The desired estimate follows from the following block matrix estimate for the kernel:

$$\begin{aligned}(4.33)\quad ||D_A^p(\frac{\partial}{\partial\mu})^q\big(\mu(ts)^{1/2}I_{A\pm1/2}(\mu t)K_{A\pm1/2}(\mu s)\big)||_{\mathscr{L}(H)} \\ \leq C_{pq}(\mu t)^a(\mu s)^{-b}(1+\mu t)^{-a}(1+\mu s)^b e^{\mu(t-s)}.\end{aligned}$$

As mentioned above, the same estimate gives the result for $G_0(\mu, A)$.

For the statement on the domain, we use again the estimates (4.30), (4.31), this time with $p = q = 0$. Moreover, since the operators $A \pm 1/2$ can be simultaneously diagonalized, we may assume that $A = \nu I_{H^+}$ where $\nu > -1/2$. We write for $\sigma \in \operatorname{im} G(\mu, A)$

$$\begin{aligned}\sigma(t) = G(\mu, A)\tau(t) &= G_0(\mu, A)\tau(t) + G_\infty(\mu, A)\tau(t)\\ &=: \sigma_0(t) + \sigma_\infty(t).\end{aligned}$$

Then we observe that for $\operatorname{supp}\tau \subset (1,\infty)$ Lemma 4.3 implies immediately that, with $\underline{\nu} := \inf\operatorname{spec} A > -1/2$,

$$\begin{aligned}||\sigma_+(t)||_H &= O(t^{\underline{\nu}+1}) = O(t^{1/2}),\ t \to 0,\\ ||\sigma_-(t)||_H &= O(t^{\underline{\nu}}),\ t \to 0,\end{aligned}$$

such that we may assume that $\operatorname{supp}\tau \subset (0,1]$. Next we have the estimate

$$\begin{aligned}||\sigma_0(t)||_H &\leq C_\sigma \int_0^t (s/t)^{\underline{\nu}}||\tau(s)||_H ds\\ &\leq C_\sigma(1+2\underline{\nu})^{-1/2}t^{1/2}||\tau||_{L^2(\mathbb{R}_+,H)},\end{aligned}$$

which proves (4.22) and (4.23) for σ_0.

For $\sigma_\infty(t)$, we have again to distinguish the $\pm$-components. Arguing as before, we arrive at the estimates

$$\begin{aligned}||\sigma_{\infty,+}(t)||_{H^+} &\leq C_\sigma t^{\underline{\nu}+1}\int_t^1 s^{-\underline{\nu}}||\tau(s)||_H ds\\ &\leq C_\sigma t^{1/2},\\ ||\sigma_{\infty,-}(t)||_{H^+} &\leq C_\sigma t^{\underline{\nu}}\int_t^1 s^{-\underline{\nu}}||\tau(s)||_H ds\\ &\leq C_\sigma t^{\underline{\nu}}.\end{aligned}$$

The proof is complete. ∎

Now we apply Theorem 4.2 to construct the desired operator symbol, $G(\mu,b,\beta)_<$. Recall that we want

$$G(\mu,b,\beta)_< = \big(\gamma(\frac{d}{dt}+t^{-1}\tilde{A}(b)_<)+i\,\mathrm{cl}(\beta^\sharp)-i\mu\big)^{-1}, \tag{4.34}$$

$$= \big(D_V(b)_< + i\,\mathrm{cl}(\beta^\sharp)-i\mu)\big)^{-1}, \tag{4.35}$$

for a suitable self-adjoint extension, $D_V(b)_<$, of the conic operator. We will define this extension by solving the matrix equation on the right hand side of (4.34) using Theorem 4.2 appropriately. Let us recall from (4.5) that we now deal with the following data:

$$H := \Lambda T^*_{b_0}B\otimes\mathbb{C}^2\otimes Q_<(\lambda_{(2)}(Y_{b_0})), \tag{4.36}$$

$$\gamma = \varepsilon_H\otimes\begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}, \tag{4.37}$$

$$\tilde{A}(b)_< = \begin{pmatrix} A(b)_< & 0\\ 0 & -A(b)_<\end{pmatrix}, \tag{4.38}$$

$$A(b)_< = Q_<(D_{Y_b}\alpha_{Y_b}+\nu), \tag{4.39}$$

where $Q_< = I - Q_>$ and $Q_>$ is given by (4.8). Now we put

$$\tilde{\gamma} := i\gamma,\ \zeta = \zeta(\mu,\beta) := \mu\tilde{\gamma}-\tilde{\gamma}\,\mathrm{cl}(\beta^\sharp), \tag{4.40}$$

and noting that for $\beta\in T^*B$, $\tilde{\gamma}$ and $\mathrm{cl}(\beta^\sharp)$ anticommute while $\mathrm{cl}(\beta^\sharp)$ commutes with α_1 and A, one easily computes that

$$\zeta^\dagger = \zeta, \tag{4.41}$$

$$\zeta^2 = (\mu^2+|\beta|_b^2)I =: \tilde{\mu}(b,\beta)^2 I, \tag{4.42}$$

$$\zeta\tilde{A}(b)_< + \tilde{A}(b)_<\zeta = 0 \tag{4.43}$$

This allows us to introduce two anticommuting self-adjoint involutions, α_1,α_2, by

$$\alpha_1 := I\otimes\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \tag{4.44}$$

$$\tilde{\mu}\alpha_2 := \zeta. \tag{4.45}$$

Then we can state

Lemma 4.4. — *With this notation we have in $L^2(\mathbb{R}_+,H)$*

$$D_V(b)_< + i\,\mathrm{cl}(\beta^\sharp)-i\mu = \gamma\big(\frac{d}{dt}+t^{-1}\tilde{A}(b)_< + \tilde{\mu}\alpha_2\big), \tag{4.46}$$

and the following relations hold:

$$\alpha_1\alpha_2+\alpha_2\alpha_1 = 0, \tag{4.47}$$

$$\alpha_1\tilde{A}(b)_< - \tilde{A}(b)_<\alpha_1 = 0, \tag{4.48}$$

$$\alpha_2\tilde{A}(b)_< + \tilde{A}(b)_<\alpha_2 = 0. \tag{4.49}$$

Thus we are in the position to prove Theorem 0.1.

Proof of Theorem 0.1. — 1. We construct an operator $\hat{D}$ by the method of Theorem 4.2. The proof of Theorem 4.2 has to be modified somewhat since we have to verify the conditions (4.12) through (4.14) for the operator symbol

$$G(\mu,\beta,b)_< := \big(\gamma(\frac{d}{dt}+t^{-1}\tilde{A}(b)_< + \tilde{\mu}\alpha_2)\big)^{-1},$$

where now $\tilde{\mu}$ and α_2 depend on μ,β, and b. First we use [**12**, Lemma 1.1] to the effect that the spectral projections $Q_{(\Lambda_j,\Lambda_{j+1})}(\tilde{A}(b)_<)$ are locally independent of b. Observing next that $\tilde{\mu}$ as well as its b- derivatives are homogeneous in (μ,β) of degree one and using Lemma 4.3, we reduce the estimates (4.14) to (4.33) where μ is replaced by $\tilde{\mu}$.

(4.12) holds by construction, while for (4.13) we use the boundary conditions (4.22), (4.23) to calculate with $\sigma_1,\sigma_2 \in \operatorname{im} G(\mu,\beta,b)_<$

$$(4.50)\quad (D_{V,\max}(b)_<\sigma_1,\sigma_2) - (\sigma_1, D_{V,\max}(b)_<\sigma_2) = \lim_{t\to 0}\big(\langle\sigma_1^-,\sigma_2^+\rangle(t) - \langle\sigma_1^+,\sigma_2^-\rangle(t)\big) = 0.$$

That the operator $\hat{D}$ anticommutes with τ_M is obviously built into our construction. Finally, the discreteness is equivalent to the compactness of $G(\mu,D)$ which follows in turn from the compactness of the parametrix $G_1(\mu,D)$, by the form of the Neumann series. Now we choose $\psi \in C_c(M)$ with $\psi = 1$ on M_ε. Then $\psi G_1(\mu,D)$ is compact by interior regularity, while the estimate

$$||(1-\psi)G_1(\mu,D)|| \le C\varepsilon^{2\delta}$$

follows from (4.22) and (4.23) for the low eigenvalues; since the estimate (4.24) also holds for the large eigenvalues, by (4.10), $G_1(\mu,D)$ is a limit of compact operators and hence compact.

Finally, since $\tilde{D}$ is a symmetric extension of $\hat{D}$ the two operators coincide.

2. If $|A_V| \ge \frac{1}{2}$, then elements in the domain of D_M^Λ satisfy the estimate (4.24). Now the assertion follows as in [**12**, Lemma 5.1].

3. The assertion holds if $|A_V| \ge \frac{1}{2}$ since then $D_{\min}$ is essentially self-adjoint, by part 2, and the case of uniqueness holds by [**11**, Lemma 3.3].

In the general case, we construct a smooth family of metrics, $g(\alpha)^{TM}$, such that

$$(4.51)\qquad g(\alpha)^{TM} := \begin{cases} dt^2 \oplus g^{T_HN} \oplus \alpha^2t^2g^{T_VN} & \text{on } U_{\varepsilon_0/2},\\ g^{TM} & \text{on } M_{\varepsilon_0}.\end{cases}$$

We denote by $D^\Lambda(\alpha) = D(\alpha)$ the corresponding self-adjoint operator defined by the maximal de Rham complex and choose $\alpha_0 > 0$ such that $D(\alpha_0)$ is essentially self-adjoint. Since all metrics $g(\alpha)^{TM}$ are mutually quasi-isometric, the case of uniqueness holds for all of them since it is a quasi-isometry invariant.

4. We use the notation of part 3 and note that $D^{\mathrm{sign}}(\alpha)$ is well defined for all α. To prove the asserted equality we show first that

$$\operatorname{ind} D^{\mathrm{sign}}(\alpha) = \operatorname{ind} D(\alpha)^+. \tag{4.52}$$

Since $\operatorname{ind} D^{\mathrm{sign}}(\alpha)$ is constant in $[\alpha_0, 1]$, this identity will follow from [**22**, Thm.IV,5.17] if we prove an estimate of the form

$$\hat{\delta}\big(D^{\mathrm{sign}}(\alpha_1), D^{\mathrm{sign}}(\alpha_2)\big) \leq C_{\alpha_0}|\alpha_1 - \alpha_2|, \ \alpha_1, \alpha_2 \in [\alpha_0, 1], \tag{4.53}$$

where $\hat{\delta}$ denotes the gap function defined in [**22**, p.197]. One checks that for $\mu \geq 1$

$$\hat{\delta}\big(D^{\mathrm{sign}}(\alpha_1), D^{\mathrm{sign}}(\alpha_2)\big) \leq ||G\big(\mu, D(\alpha_1)\big) - G\big(\mu, D(\alpha_2)\big)||_{\lambda_{(2)}(M)},$$

such that (4.53) will follow if we show e. g. that the function

$$[\alpha_0, 1] \ni \alpha \to G(\mu, D(\alpha)) \in \mathcal{L}\big(\lambda_{(2)}(M)\big)$$

is continuously differentiable. We fix a large $\mu > 1$ and write with our parametrix $G_1(\alpha) := G_1(\mu, D)(\alpha)$

$$(D(\alpha) - i\mu)G_1(\alpha) =: I - R(\alpha),$$

where

$$||R(\alpha)|| \leq C < 1, \ \alpha \in [\alpha_0, 1].$$

Hence it is enough to prove the differentiability of $G_1(\alpha)$ and $R(\alpha)$. This is clear for the interior part, by interior regularity. For the boundary part involving high eigenvalues this is also clear from the Calderón-Vaillancourt Theorem since the image of $G_1(\alpha)_>$ does not depend on α. For the low eigenvalue part, however, we have to go back to the proof of Theorem 4.2.

Since $\tilde{A}(b, \alpha)_<$ depends smoothly on α and $G(\alpha)_<$ depends smoothly on $\tilde{A}(b, \alpha)_<$, we have to insure that the spectral splittings (Λ_j) can be made locally independent of α. This can be done for the spectral projections in many ways but using the spectral analysis of Sec. 3 we can take into account the special role of the eigenvalues $\pm 1/2$, as needed in the next step. The Hodge decomposition on Y_b can also be made locally independent of b and α, by conjugating the equation with a transformation function (cf. [**22**, II,§4.2]). Then the operator function splits into the harmonic, the closed, and the coclosed parts which have uniform spectral gaps around 0, 1/2, and -1/2, respectively, independent of the parameter values. Conjugating appropriately as before, we may reduce to the case $\tilde{A}(b, \alpha)_< > -1/2$ locally in b and α; since the corresponding solution operator is smooth in α, this completes the proof (4.52). □

Next we want to show that $D^{\mathrm{sign}}(\alpha)$ extends $\tilde{D}^+(\alpha)$ which will give the assertion in view of (4.52).

We choose $\sigma = \sigma^+ \in \operatorname{dom} \tilde{D}^+$ and may assume that $\operatorname{supp} \sigma \subset U_{\varepsilon_0}$. We decompose σ into its harmonic, closed, and coclosed part which all satisfy the estimate (4.22). By part 3 of Lemma 4.1 we see that all components of σ are in the minimal domain of the corresponding conic operator. Moreover, by the spectral decomposition described in part 3 of this proof all cone coefficients will not have $-\frac{1}{2}$ in their spectrum such that we can apply Lemma 5.12 in Section 5; we find that

$$\sigma' \in L^2\big((0,\varepsilon_0),H^0\big),\ t^{-1}\sigma \in L^2\big((0,\varepsilon_0),H^1\big).$$

The pseudodifferential construction of the parametrix shows next that

$$d_H\sigma, d_H^\dagger\sigma \in \lambda_{(2)}(M),$$

and Lemma 2.2 finally shows that

$$d_M\sigma \in \lambda_{(2)}(M)$$

and completes the proof. □

5. The index calculation

In this section, we want to compute the index of the signature operator, as constructed in Theorem 0.1. As noted there, the index is stable under scaling of the fiber metric; this rules out, according to Theorem 3.1, that small eigenvalues occur on the closed and coclosed subspaces, while we need an extra condition on the space $\mathcal{H}^{v/2}(Y)$ known as the *Witt condition*:

$$\mathcal{H}^{v/2}(Y) = 0. \tag{5.1}$$

Thus we may and will assume in what follows that

$$|A_V| \geq \frac{1}{2}, \tag{5.2}$$

which ensures, by Theorem 0.1 again, that we do not have to impose boundary conditions near the singularity. However, the crucial vanishing results we need will require in addition that

$$-\frac{1}{2} \notin \operatorname{spec} A_{V,\mathrm{cl}} \cup \operatorname{spec} A_{V,\mathrm{ccl}}. \tag{5.3}$$

In view of Theorem 3.1, this can also be achieved by scaling $g^{T_V N}$; thus we will assume in what follows (5.1) and

$$\operatorname{spec} |A_{V,\mathrm{cl}}| \cup \operatorname{spec} |A_{V,\mathrm{ccl}}| \subset [1/2 + C, \infty), \tag{5.4}$$

for some positive constant C.

We will reduce the index calculation to a problem of APS-Type, by splitting the operator as a sum at ∂U_ε, for a sufficiently small $\varepsilon \in (0,\varepsilon_0)$, using [**3**, Thm. H]. At ∂M_ε, we will introduce the boundary condition

$$Q_{\geq 0}(A(\varepsilon))\sigma(\varepsilon) = 0, \tag{5.5}$$

where A is the operator family from (2.32) and $Q_{>0}$ denotes the spectral projection onto the positive eigenspaces. At ∂U_ε, we impose the complementary boundary condition (cf. [**3**, Thm. 4.17]),

$$Q_{<0}(A(\varepsilon))\sigma(\varepsilon) = 0; \tag{5.6}$$

note that these boundary conditions are invariant under τ_M. These boundary conditions generate the operators $D^{\Lambda}_{U_\varepsilon,Q_{<0}(A(\varepsilon))}$ and $D^{\Lambda}_{M_\varepsilon,Q_{\geq 0}(A(\varepsilon))}$ by imposing the boundary conditions on the maximal domain of $D^{\mathrm{sign}}_{U_\varepsilon}$ and $D^{\mathrm{sign}}_{M_\varepsilon}$, respectively (note that no boundary condition is necessary at 0 in view of (5.2)). The boundary conditions are such that the following holds.

Theorem 5.1. — *$D^{\Lambda}_{U_\varepsilon,Q_{<0}(A(\varepsilon))}$ and $D^{\Lambda}_{M_\varepsilon,Q_{\geq 0}(A(\varepsilon))}$ are Fredholm operators, and we have the index identity*

$$\operatorname{ind} D^{\mathrm{sign}}_M = \operatorname{ind} D^{\Lambda,+}_{U_\varepsilon,Q_{<0}(A(\varepsilon))} + \operatorname{ind} D^{\Lambda,+}_{M_\varepsilon,Q_{\geq 0}(A(\varepsilon))}. \tag{5.7}$$

Proof. — The proof of (5.7) follows immediately from [**3**, Thm.4.17] (cf. Remark 5.17) with the following data for $0 < u < \varepsilon < \varepsilon_0/2$:

$$D_1^+ := \gamma(\frac{\partial}{\partial u} + A(\varepsilon + u)), \tag{5.8}$$

$$D_2^+ := -\gamma(\frac{\partial}{\partial u} - A(\varepsilon - u)), \tag{5.9}$$

$$B_1 := Q_{<0}(A(\varepsilon))(\operatorname{dom} |A(\varepsilon)|^{1/2}), \tag{5.10}$$

$$B_2 := Q_{\geq 0}(A(\varepsilon))\big(\operatorname{dom} |A(\varepsilon)|^{1/2}\big). \tag{5.11}$$

□

We show next that the index contribution from U_ε vanishes.

Theorem 5.2. — *Assume that* (5.3) *holds. Then for $\varepsilon \in (0, \varepsilon_0]$ and sufficiently small we have*

$$\operatorname{ind} D^{\Lambda,+}_{U_\varepsilon,Q_{<0}(A(\varepsilon))} = 0. \tag{5.12}$$

This theorem will be proved in Subsection 5.2.

Thus it remains to compute the index of an APS-type problem on the smooth compact manifold with boundary, M_ε. However, to apply [**1**, Thm. 3.10] we need to modify the metric on U_{ε_0}, making it cylindrical near $t = \varepsilon$. To this end we choose a smooth positive function ψ on $(0, \infty)$ such that $\psi(t) = t$ if $t \in (0, 1] \cup [4, \infty)$ and $\psi(t) = 1$ if $t \in [2, 3]$. Then we put for $\varepsilon < \varepsilon_0/4$

$$g_\varepsilon^{TU_{\varepsilon_0}} := dt^2 \oplus g^{T_H N} \oplus \varepsilon^2 \psi(t/\varepsilon)^2 g^{T_V N}, \tag{5.13}$$

$$g_\varepsilon^{TM} | M_{\varepsilon_0} := g^{TM} | M_{\varepsilon_0}, \tag{5.14}$$

$$g_\varepsilon^{TM} | U_{\varepsilon_0} := g_\varepsilon^{TU_{\varepsilon_0}}. \tag{5.15}$$

Moreover, we are not yet dealing with the correct boundary condition in order to apply the APS-Theorem. In fact, we have from (2.32)

$$A(t) = A_H(t) + t^{-1}(A_{0,V} + \nu) \tag{5.16}$$

$$= A_0(t) + t^{-1}\nu, \tag{5.17}$$

and it follows from (2.37) and Theorem 2.5 that $A_0(t) \simeq D^{\Lambda}_{N_t} \tau_{N_t}$ is the tangential operator corresponding to $D^{\mathrm{sign}}_{U_{\varepsilon_0}}$, acting in H^0 with domain H^1. The correct boundary condition can be achieved by applying the Agranovich-Dynin Theorem respectively its equivariant version, as stated e. g. in [**3**, Thm. 4.14]. Noting that $A_0(\varepsilon) = D_{N_\varepsilon}\tau_{N_\varepsilon}$ has even dimensional kernel, we obtain

Theorem 5.3. — *The pair of subspaces* $\big(Q_{<0}(A(\varepsilon))(H^0), Q_{\geq 0}(A_0(\varepsilon))(H^0)\big)$ *is a Fredholm pair in* H^0. *If we denote its Kato index by* $i(\varepsilon)$ *then*

$$\begin{aligned}\operatorname{ind} D^{\mathrm{sign}}_{M_\varepsilon, Q_{<0}(A(\varepsilon))(H^0)} &= \operatorname{ind} D^{\mathrm{sign}}_{M_\varepsilon, Q_{<0}(A_0(\varepsilon))(H^0)} + i(\varepsilon)\\ &=: \operatorname{ind} D^{\mathrm{sign}}_{(M_\varepsilon, g_\varepsilon^{TM}), Q_{<0}(A_0(\varepsilon))(H^0)} + \big(\tau(\varepsilon) + \frac{1}{2}\dim\ker A_0(\varepsilon)\big).\end{aligned}$$

This result will be proved in Subsection 5.1.

To obtain an explicit index formula, we need to identify the integer $\tau(\varepsilon)$. To do so, we use the generalized Thom space associated with the fibration (0.3), as introduced by Cheeger and Dai in [**17**] which we denote by T_π. Then we show using [**17**, Thm.1.1] (note our choice of orientation)

Theorem 5.4. — *For* ε *sufficiently small, we have*

$$\tau(\varepsilon) = \mathrm{sign}_{(2)} T_\pi =: \tau,$$

where τ *denotes the invariant introduced in* [**18**, Thm.0.3].

This theorem will be proved in Subsection 5.3.

Now we obtain our final local index formula by combining Theorem 5.3 and Theorem 5.4 with the APS-Theorem [**1**, Thm. 3.10] and the result of Dai [**18**, Thm. 0.3] which evaluates the adiabatic limit of the eta-invariant for the signature operator, to get

$$\mathrm{sign}_{(2)} M = \lim_{\varepsilon\to 0}\int_{M_\varepsilon} L(TM, g^{TM}) - \int_B L(TB, g^{TB}) \wedge \tilde{\eta} - \frac{1}{2}\eta(A_{0,\mathscr{H}}(0)), \tag{5.18}$$

where the operator $A_{0,\mathscr{H}}$, the Dirac operator on ΛT^*M twisted by the harmonic forms on the fibers, is defined in (5.53).

Remark 5.5. — 1. Using arguments as in [**7**, Sec.VI], it follows that the transgression term of the L-class from g_ε^{TM} to g^{TM} goes to zero with ε.

2. It is desirable to give a direct proof of the equality $\tau(\varepsilon) = \tau$, without using [**17**, Thm.1.1].

5.1. Perturbations of regular projections.

— We use the terminology introduced in [**3**, Sec. 2.1]. Thus we consider a self-adjoint operator A with domain H_A in the (complex) Hilbert space H which we assume to be discrete i. e. to have a compact resolvent. For a Borel subset $J \subset \mathbb{R}$ we denote by $Q_J := Q_J(A)$ the associated spectral projection, and we write $Q_{>0} := Q_{(0,\infty)}$ etc.

With A we associate its Sobolev chain $(H^s := H^s(A))_{s\in\mathbb{R}}$ restricting attention to $\{|s| \leq 1\}$. Thus for $s \in [0,1], H^s$ is the closure of H_A under the norm

$$||x||_s^2 := ||(I + A^2)^{s/2} x||_H^2 \tag{5.19}$$

and H^{-s} is its strong dual space under the norm (5.19).

An operator $S \in \mathcal{L}(H)$ will be called 1/2-smooth if it restricts to $H^{1/2}$, with restriction $\hat{S}$, and extends to $H^{-1/2}$, with extension $\tilde{S}$. S will be called (1/2-)smoothing if $\operatorname{im} \tilde{S} \subset H^{1/2}$. With these preparations we can define *regular* and *elliptic* projections for A which are introduced to characterize elliptic boundary conditions for the evolution operator associated with A (cf. [**3**, Secns. 1.4, 2.3]). If A comes with an anticommuting skew-adjoint unitary operator $\gamma \in \mathcal{L}(H)$,

$$\gamma A + A\gamma = 0, \tag{5.20}$$

then we can also define the Dirac operator associated with A, cf. [**3**, Sec.2.1]. The following formulation derives from [**3**, Prop.1.99].

Definition 5.6. — A 1/2-smooth orthogonal projection P in H is called *regular (with respect to A)* if and only if

$$\begin{aligned} &x \in H^{-1/2},\ \tilde{P}x = 0,\ Q_{\leq 0}x \in H^{1/2} \\ &\Rightarrow x \in H^{1/2}. \end{aligned}$$

A regular projection P is called *elliptic (with respect to A)* if (5.20) holds and

$$P_\gamma := \gamma^*(I - P)\gamma \tag{5.21}$$

is also regular.

For example, the spectral projections $Q_{>\Lambda}(A)$ are regular with respect to A for any $\Lambda \in \mathbb{R}$ since A is discrete, and since $(Q_{>\Lambda}(A))_\gamma = Q_{\geq\Lambda}(A)$ they are also elliptic.

Now we want to study perturbations of A in the sense of Kato, i. e. operators of the form $\tilde{A} := A + B$ where B is a symmetric operator in H defined on H_A with estimate

$$||Bx||_H \leq a||x||_H + b||Ax||_H,\ x \in H_A \tag{5.22}$$

for some constants $a, b \in \mathbb{R}_+$ with $b < 1$. Then $\tilde{A}$ is self-adjoint and discrete in H with domain H_A, by the Kato-Rellich Theorem, and the projection $Q_{>0}(\tilde{A})$ is elliptic with respect to $\tilde{A}$. We want to know under what conditions it is also elliptic with respect to A. To answer this question we need two preliminary results; the first one parallels [**3**, Prop.1.93].

Lemma 5.7. — *Let P be a 1/2-smooth orthogonal projection in H such that*

$$P = Q_{>0}(A) + R_1 + R_2, \tag{5.23}$$

where R_1 is smoothing and

$$\max\{||\hat{R}_2||, ||\tilde{R}_2||\} < 1. \tag{5.24}$$

Then P is elliptic with respect to A, and $\big(\operatorname{im}(I - P), \operatorname{im} Q_{>0}(A)\big)$ is a Fredholm pair in H.

Proof. — Consider $x \in H^{-1/2}$ with $\tilde{P}x = 0$ and $Q_{\leq 0}(A)x \in H^{1/2}$. We write $x = Q_{>0}(A)x + Q_{\leq 0}(A)x =: x_> + x_\leq$ and obtain from (5.23)

$$(I + \tilde{R}_2)x_> =: y \in H^{1/2},$$

hence from (5.24)

$$x_> = (I + \hat{R}_2)^{-1}y \in H^{1/2},$$

such that P is regular. It follows from (5.20) that γ induces a unitary operator in each H^s; since $Q_0(A)$ is smoothing, we infer that the representation (5.22) also holds for P_γ such that P is elliptic.

We see next that

$$P : H_> = \operatorname{im} Q_{>0}(A) \to H, x_> \mapsto (I + R_2 + R_1)x_>,$$

is a left Fredholm operator, by [3, Lemma A.11] and the compactness of R_1, hence, from [3, Lemma A.12], $\big(\operatorname{im}(I - P), H_>\big)$ is a left Fredholm pair. By the same token, we see that $(\operatorname{im} P, H_\leq)$ is a left Fredholm pair, too, which completes the proof of the lemma. □

The second lemma addresses smoothing perturbations.

Lemma 5.8. — *Assume that A and $A + B$ are both invertible. If B is smoothing then so is*

$$R := Q_{>0}(A + B) - Q_{>0}(A).$$

Proof. — For any invertible and discrete self-adjoint operator $\tilde{A}$ in H we have from [**22**, p.359] the strongly convergent integral representation

$$\frac{1}{2}\big(I - 2\,Q_{>0}(\tilde{A})\big) = \frac{1}{2\pi i}\int_{\operatorname{Re} z=0} (\tilde{A} - z)^{-1}dz. \tag{5.25}$$

This implies that

$$R = \frac{1}{2\pi i}\int_{\operatorname{Re} z=0} (A + B - z)^{-1}B(A - z)^{-1}dz. \tag{5.26}$$

By construction, R is 1/2-smooth; to show that R is smoothing, we need to show the boundedness in H of the operator

$$\begin{aligned}
\tilde{R} &:= (I + A^2)^{1/4}R(I + A^2)^{1/4} \\
&= \big((I + A^2)^{1/4}|A + B|^{-1/2}\big)\big(|A + B|^{1/2}R|A|^{1/2}\big)\big(|A|^{-1/2}(I + A^2)^{1/4}\big) \\
&=: V\big(|A|^{1/2}|A + B|^{-1/2}\big)\big(|A + B|^{1/2}R|A|^{1/2}\big)V \\
&=: VW\big(|A + B|^{1/2}R|A|^{1/2}\big)V.
\end{aligned}$$

In view of (5.22), A and $A+B$ generate the same Hilbert spaces, with equivalent norms, in their respective Sobolev chains implying the boundedness in H of the operators V and W. From (5.26) we obtain for the remaining part the representation

$$
\begin{aligned}
(5.27)\qquad |A+B|^{1/2}R|A|^{1/2}
&= \frac{1}{2\pi i}\int_{\operatorname{Re} z=0} (A+B-z)^{-1}|A+B|^{1/2}B|A|^{1/2}(A-z)^{-1}dz\\
&=: \frac{1}{2\pi i}\int_{\operatorname{Re} z=0} (A+B-z)^{-1}\tilde{B}(A-z)^{-1}dz.
\end{aligned}
$$

Now if B is smoothing then $\tilde{B} = |A+B|^{1/2}B|A|^{1/2}$ is bounded in H. Thus we may apply [**2**, Lemma A.1] to complete the proof. □

Now we can deduce the desired perturbation result.

Theorem 5.9. — *Assume that $A+B$ is a Kato perturbation of A with*

$$(5.28)\qquad b < \frac{2}{3}.$$

Then $Q_{>0}(A+B)$ is elliptic with respect to A and the subspaces $Q_{\leq 0}(A)(H)$ and $Q_{>0}(A+B)(H)$ form a Fredholm pair.

If B is bounded and $|A| \geq \mu$ where

$$(5.29)\qquad \mu > \sqrt{2}||B||_H,$$

then

$$(5.30)\qquad \operatorname{ind}\big(Q_{\leq 0}(A)(H), Q_{>0}(A+B)(H)\big) = 0.$$

Proof. — We show first that we may assume that

$$(5.31)\qquad |A| \geq \Lambda$$

for any $\Lambda > 0$. Indeed, if we put

$$(5.32)\qquad f_\Lambda(t) := \begin{cases} t & \text{if } |t| \geq \Lambda,\\ \Lambda & \text{if } 0 < t < \Lambda,\\ -\Lambda & \text{if } -\Lambda < t \leq 0,\end{cases}$$

then for the operator

$$A_\Lambda := f_\Lambda(A)$$

the following properties are easily verified:

(5.33) A_Λ is discrete and commutes with A,

(5.34) $A - A_\Lambda$ is smoothing,

(5.35) $|A| \leq |A_\Lambda|$,

(5.36) $|A_\Lambda| \geq \Lambda \Rightarrow ||A_\Lambda^{-1}|| \leq \Lambda^{-1}$,

(5.37) $Q_{>0}(A_\Lambda) = Q_{>0}(A)$.

We have

$$A + B = A_\Lambda + (A - A_\Lambda + B) =: A_\lambda + B_\Lambda,$$

such that $A_\Lambda + B_\Lambda$ is a Kato perturbation of A_Λ with the same constant b in (5.21) as for A and B. Hence, by (5.36) it is enough to prove the theorem under the assumption (5.30).

Next we note that (5.21) implies that

$$b' := ||BA^{-1}|| \tag{5.38}$$

can be chosen arbitrarily close to b. Thus we may also assume that both A and $A+B$ are invertible.

Now we want to show that for Λ sufficiently large and b satisfying the condition (5.28), the 1/2-smooth operator

$$R_2 := Q_{>0}(A+B) - Q_{>0}(A)$$

satisfies the estimate (5.24). To do so, we proceed as in the proof of Lemma 5.8. We observe first that from the symmetry of R_2 in H and the obvious identity

$$\tilde{S} = (\widehat{S^*})',$$

where S' denotes the dual in $H^{-1/2}$ and S^* the adjoint operator in H, for any 1/2-smooth operator S, it is enough to estimate $||R_2||_{1/2}$ or equivalently, the norm in H of the operator

$$(I + A^2)^{1/4} R_2 (I + A^2)^{-1/4} = VW|A + B|^{1/2} R_2 |A|^{-1/2} V^{-1}, \tag{5.39}$$

where V and W are the operators introduced in the proof of Lemma 5.8.

From the Spectral Theorem we see that for any $\delta > 0$ we may choose Λ so large that

$$\sup\{||V||_H, ||V^{-1}||_H\} \leq 1 + \delta. \tag{5.40}$$

Next we estimate the H-norm of W by the maximum principle applied to the holomorphic function

$$z \mapsto e^{-z(1-z)} \langle |A|^z |A + B|^{-z} x, y \rangle_H \in \mathbb{C}, \ x, y \in H_A,$$

in the strip $\{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ which reduces us to an estimate for $\operatorname{Re} z = 1$. Clearly, for $b' < 1$ we have

$$||\,|A|\,|A + B|^{-1}||_H \leq (1 - b')^{-1},$$

hence also

$$||W||_H \leq (1 - b')^{-1}. \tag{5.41}$$

It remains to estimate the norm of $|A+B|^{1/2} R_2 |A|^{-1/2}$ for which we invoke again [**2**, Lemma A.1]. There we choose $A_1 := A + B, A_2 := A, \alpha_1 := \alpha_2 := 1/2$ and find with $B(z) = |A + B|^{1/2} B |A|^{-1/2}$

$$|||A + B|^{1/2} R_2 |A|^{-1/2}||_H \leq \frac{1}{2} ||B|A|^{-1}||_H \leq \frac{1}{2} b'. \tag{5.42}$$

Combining (5.40), (5.41), and (5.42) we arrive at

$$\|R_2\|_{H^{1/2}} \leq \frac{1}{2} b'(1-b')^{-1}(1+\delta). \tag{5.43}$$

This can be made smaller than 1 if $b < \frac{2}{3}$.

For the proof of (5.30) we observe that this will follow from the estimate

$$\|Q_{>0}(A) - Q_{>0}(A+B)\|_H < 1, \tag{5.44}$$

which again is an easy consequence of [**2**, Lemma A.1], this time applied with $\alpha_1 := \alpha_2 := \frac{1}{2}$. □

From the proof of the theorem we get

Corollary 5.10. — *Lemma 5.8 holds without the assumption that A and $A+B$ are invertible.*

Remark 5.11. — Theorem 5.9 is stronger than needed for our application but it is useful in other situations and does not seem to be known.

Proof of Theorem 5.3. — From (2.31) we have

$$A(\varepsilon) = A_0(\varepsilon) + \varepsilon^{-1}\nu,$$

and since ν is bounded it follows from Lemma 5.7 and Theorem 5.9 that $Q_{<0}(A(\varepsilon))$ is elliptic with respect to $A_0(\varepsilon)$ and that $\big(Q_{<0}(A(\varepsilon)(H^0), Q_{\geq 0}(A_0(\varepsilon))(H^0)\big)$ is a Fredholm pair in H.

The index formula follows from [**3**, Thm.4.14]. □

5.2. A vanishing theorem. — The purpose of this subsection is the proof of Theorem 5.2. We abbreviate

$$D^{\mathrm{sign}}_\varepsilon := D^{\mathrm{sign}}_{U_\varepsilon, Q_{<0}(A(\varepsilon))}$$

and note that, in view of (5.2) and Theorem 0.1, a core for $D^{\mathrm{sign}}_\varepsilon$ is given by

$$\mathscr{D}^{\mathrm{sign}}_\varepsilon := \{\sigma \in C^1_c((0,\varepsilon], H^1) : Q_{<0}(A(\varepsilon))\sigma(\varepsilon) = 0\}.$$

We prove the theorem first in a special case.

Lemma 5.12. — *Assume that A satisfies the further condition*

$$-\frac{1}{2} \notin \operatorname{spec} A_V. \tag{5.45}$$

Then for sufficiently small ε and $\sigma \in \mathscr{D}^{\mathrm{sign}}_\varepsilon$ we have the a priori estimate

$$\|D^{\mathrm{sign}}_\varepsilon \sigma\|_{H^0} \geq (2\varepsilon)^{-1} \|(A_V + \frac{1}{2})\sigma\|_{H^0}. \tag{5.46}$$

Proof. — We write $H := H^0, A_{HV}(t) := A_H(t)A_V + A_VA_H(t)$, and compute for $\sigma \in \mathscr{D}_\varepsilon^{\mathrm{sign}}$

$$(5.47)\quad ||D_\varepsilon^{\mathrm{sign}}\sigma(t)||_H^2 = ||\sigma'(t)||_H^2 + ||A_H\sigma(t)||_H^2 + t^{-2}||A_V\sigma(t)||_H^2 \\ + t^{-1}\langle A_{HV}\sigma(t), \sigma(t)\rangle_H + 2\,\mathrm{Re}\langle\sigma'(t), A\sigma(t)\rangle.$$

Next we verify that

$$(5.48)\quad 2\,\mathrm{Re}\langle\sigma'(t), A\sigma(t)\rangle = \\ \frac{d}{dt}\langle\sigma(t), A\sigma(t)\rangle_H + t^{-2}\langle\sigma(t), A_V\sigma(t)\rangle_H - \langle\sigma(t), A_H'(0)\sigma(t)\rangle_H.$$

The assumption (5.45) implies that $A_V + \frac{1}{2}$ is invertible while (2.39) and (2.30) imply that $A_{VH}(t)$ is a first order vertical operator on $\lambda(N)$. Hence there is a constant $C_1 > 0$ such that

$$(5.49)\qquad ||A_{HV}(t)(A_V + \frac{1}{2})^{-1}||_H \leq C_1,\ t \in (0, \varepsilon].$$

Combining (5.47), (5.48), and (5.49) and abbreviating $B := A_{HV}(t)(A_V + \frac{1}{2})^{-1}$ we arrive at the inequality

$$(5.50)\quad ||D_\varepsilon^{\mathrm{sign}}\sigma(t)||_H^2 = \big(||\sigma'(t)||_H^2 - \frac{1}{4}t^{-2}||\sigma(t)||_H^2\big) + \frac{d}{dt}\langle\sigma(t), A\sigma(t)\rangle_H \\ + t^{-2}||(A_V + \frac{1}{2})\sigma(t)||_H^2 \\ + t^{-1}\langle(A_V + \frac{1}{2})\sigma(t), B^*\sigma(t)\rangle_H - \langle\sigma(t), A_H'(0)\sigma(t)\rangle_H.$$

Hardy's inequality and the boundary condition at ε imply that the first two terms are nonnegative after integration over $(0, \varepsilon]$. Thus for sufficiently small ε we obtain

$$||D_\varepsilon^{\mathrm{sign}}\sigma||_{L^2((0,\varepsilon],H)}^2 \geq (4\varepsilon)^{-2}||(A_V + \frac{1}{2})\sigma||_{L^2(0,\varepsilon],H)}^2. \qquad \square$$

Now we can give the

Proof of Theorem 5.2. — The condition of Lemma 5.12 is satisfied, in view if of assumption (5.4), either v is even or

$$\mathscr{H}^{(v-1)/2}(Y) = 0,$$

in which case we can actually assert that

$$\ker D_\varepsilon^\Lambda = 0.$$

In the general case, we have to work differently since this assertion will no longer be true. If (5.12) does not hold then v must be odd hence h must be even. In this case, A_V is invertible and we can deform the operator $D_\varepsilon^{\mathrm{sign}}$ to an operator with vanishing index as follows.

As in Section 1, we view the Hilbert space $H^0 = \lambda_{(2)}(N_1)$ as a Hilbert bundle $\mathcal{E} \to B$ where

$$\mathcal{E} = \Lambda T^*B \otimes \lambda_{(2)}(F),$$
$$\lambda_{(2)}(F)_b = \lambda_{(2)}(Y_b).$$

In $\lambda_{(2)}(Y_b)$, we define a smooth family of projections, $P_{\mathcal{H}}(b)$, by

$$P_{\mathcal{H}} := \frac{1}{2\pi i}\int_{|z|=\delta} (\Delta_b - z)^{-1} dz,\ b \in B, \tag{5.51}$$

which splits

$$\lambda_{(2)}(F) =: \mathcal{H}_F \oplus \mathcal{H}_F^{\perp}. \tag{5.52}$$

Here $\mathcal{H}_F$ is the finite dimensional vector bundle over B formed by the harmonic forms on the fibers. We note next that the projection $I \otimes P_{\mathcal{H}}$ commutes with A_V and with the principal symbol of $A_H(t)$. Hence the operator

$$A^\delta(t) := I \otimes P_{\mathcal{H}} A(t) I \otimes P_{\mathcal{H}} + I \otimes (I - P_{\mathcal{H}}) A(t) I \otimes (I - P_{\mathcal{H}}) \tag{5.53}$$
$$=: A_{\mathcal{H}}(t) + A_{\mathcal{H}^\perp}(t) \tag{5.54}$$

differs from $A(t)$ by an operator of uniformly bounded norm,

$$A^\delta(t) =: A(t) + C(t),\ ||C(t)||_H \le C, t \in (0, \varepsilon].$$

It follows that $A^\delta(t)$ satisfies the estimate (2.40), possibly with a different constant; in particular, $A^\delta(t)$ is invertible and $Q_{\le 0}(A^\delta(t)) = Q_{<0}(A^\delta(t))$. We now deform the operator $D_\varepsilon^{\mathrm{sign}}$ to the operator $D_{\varepsilon,\delta}^{\mathrm{sign}}$ which is given on the core

$$\mathcal{D}_{\varepsilon,\delta}^{\mathrm{sign}} := \{\sigma \in C_c^1((0,\varepsilon], H^1) : Q_{<0}(A^\delta(\varepsilon))\sigma(\varepsilon) = 0\} \tag{5.55}$$

by

$$D_{\varepsilon,\delta}^{\mathrm{sign}} \sigma(t) = (\frac{\partial}{\partial t} + A^\delta(t))\sigma(t). \tag{5.56}$$

Since $D_\varepsilon^{\mathrm{sign}}$ and $D_{\varepsilon,\delta}^{\mathrm{sign}}$ differ by a uniformly bounded operator we obtain from Theorem 5.9 and [**3**, Thm.4.14] the identity

$$\begin{aligned} \operatorname{ind} D_\varepsilon^{\mathrm{sign}} &= \operatorname{ind} D_{\varepsilon,\delta}^{\mathrm{sign}} + \operatorname{ind}\big(Q_{<0}(A(\varepsilon))(H^0), Q_{>0}(A^\delta(\varepsilon))(H^0)\big) \\ &= \operatorname{ind} D_{\varepsilon,\mathcal{H}}^{\mathrm{sign}} + \operatorname{ind} D_{\varepsilon,\mathcal{H}^\perp}^{\mathrm{sign}} + \operatorname{ind}\big(Q_{<0}(A(\varepsilon))(H^0), Q_{>0}(A^\delta(\varepsilon))(H^0)\big). \end{aligned} \tag{5.57}$$

Here the operators $D_{\varepsilon,\mathcal{H}}^{\mathrm{sign}}$ and $D_{\varepsilon,\mathcal{H}^\perp}^{\mathrm{sign}}$ are formed as $D_\varepsilon^{\mathrm{sign}}$ above, by replacing A^δ in (5.55) and (5.56) by $A_{\mathcal{H}}$ and $A_{\mathcal{H}^\perp}$, respectively.

Now $D_{\varepsilon,\mathcal{H}^\perp}^{\mathrm{sign}}$ satisfies the assumptions of Lemma 5.12 such that

$$\operatorname{ind} D_{\varepsilon,\mathcal{H}^\perp}^{\mathrm{sign}} = 0. \tag{5.58}$$

Next we observe that $A_{\mathscr{H}}(t)$ anticommutes with τ_B up to a uniformly bounded operator since it has the same principal symbol as the canonical Dirac operator on B with coefficients in $\mathscr{H}_F$, that is

$$\tau_B A_{\mathscr{H}}(t)\tau_B =: -A_{\mathscr{H}}(t) + \tilde{C}(t), \tag{5.59}$$

where $||\tilde{C}(t)||_H \leq \tilde{C},\ t \in (0, \varepsilon_0]$. Thus we find that $\tau_B D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}}\tau_B$ is given on the core

$$\mathscr{D}^{\mathrm{sign}}_{\varepsilon,\mathscr{H}^\perp} := \{\sigma \in C^1_c((0,\varepsilon], H^1) : Q_{<0}(\tau_B A_{\mathscr{H}}(\varepsilon)\tau_B)\sigma(\varepsilon) = 0\} \tag{5.60}$$

by the operator

$$(\frac{\partial}{\partial t} + \tau_B A_{\mathscr{H}}(t)\tau_B)\sigma(t). \tag{5.61}$$

We compare this with the adjoint operator $(D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^*$ which is given on its core

$$(\mathscr{D}^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^* = \{\sigma \in C^1_c((0,\varepsilon], H^1) : Q_{>0}(A_{\mathscr{H}}(\varepsilon))\sigma(\varepsilon) = 0\} \tag{5.62}$$

by

$$(D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^*\sigma(t) = (-\frac{\partial}{\partial t} + A_{\mathscr{H}}(t))\sigma(t). \tag{5.63}$$

Using (5.59) and the invertibility of $A(t)$, and applying Theorem 5.9 once more, we see that

$$\operatorname{ind} D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}} = \operatorname{ind} \tau_B D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}}\tau_B = \operatorname{ind}(D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}})^* = -\operatorname{ind} D^{\mathrm{sign}}_{\varepsilon,\mathscr{H}} = 0. \tag{5.64}$$

A final application of (2.40) in Theorem 5.9 shows that

$$\operatorname{ind}\big(Q_{<0}(A(\varepsilon))(H^0), Q_{>0}(A^\delta(\varepsilon))(H^0)\big) = 0$$

and completes the proof of Theorem 5.2. □

5.3. Generalized Thom spaces. — In this subsection we compute the L^2-signature of a generalized Thom space, as introduced in [**17**], and identify it as a normalized spectral flow associated with the family $A(t)$ introduced in (2.32). We describe the generalized Thom space associated with the fibration (0.3) as the cylinder $T := T_\pi := (0,2) \times N$ with its product orientation and equipped with a family of metrics depending on a parameter $\varepsilon \in (0, 1/2)$ as follows. We write the metric on T_π in the form

$$g^{TT_\pi}_\varepsilon = dt^2 \oplus g^{TN}_\varepsilon(t), \tag{5.65}$$

where $g^{TN}_\varepsilon(t)$ is a smooth family of Riemannian metrics on N with the property

$$g^{TN}_\varepsilon(t) := \begin{cases} g^{T_H N} \oplus t^2 g^{T_V N} & \text{if } 0 < t \leq 1/2, \\ (2-t)^2(\varepsilon^{-2} g^{T_H N} \oplus g^{T_V N}) & \text{if } 3/2 < t < 2. \end{cases} \tag{5.66}$$

Here $g^{TN} = g^{T_H N} \oplus g^{T_V N}$ denotes again the metric introduced in (0.6) where we assume that $g^{T_V N}$ is approriately scaled, as detailed below. Note that $g^{TN}_\varepsilon(\varepsilon) = g^{TN}_\varepsilon(2-\varepsilon) = g^{T_H N} \oplus \varepsilon^2 g^{T_V N}$; note also that we use the opposite orientation as in

[**17**]. Since any two metrics in the family $(g_\varepsilon^{TT_\pi})_{0<\varepsilon<1/2}$ are quasi-isometric, they all compute the same L^2-signature and we find

$$\operatorname{sign}_{(2)} T_\pi = \operatorname{ind} D^{\mathrm{sign}}_{T_\pi, g_\varepsilon^{TT_\pi}}. \tag{5.67}$$

The computation of $\operatorname{sign}_{(2)} T_\pi$ is now a special case of our general index computation with two singular strata of dimension h and 0, respectively. We split the computation at $t = \varepsilon$ and $t = 2 - \varepsilon$ and obtain three parts, the cone bundle U_ε over B, the metric cone $C_\varepsilon N := C_{(2,2-\varepsilon)}(N, \varepsilon^{-2} g_\varepsilon^{TN}(\varepsilon))$ over $(N, \varepsilon^{-2} g_\varepsilon^{TN}(\varepsilon))$, and the cylinder $Z_\varepsilon := (\varepsilon, 2-\varepsilon) \times N$ equipped with a nonsingular metric. We are ready for the

Proof of Theorem 5.4. — Arguing as before we see that on $U_{\varepsilon_0} \cup C_{(2,2-\varepsilon_0)}(N, \varepsilon^{-2} g_\varepsilon^{TN})$, D^{sign} is unitarily equivalent to $\frac{\partial}{\partial t} + A_\varepsilon(t)$ acting in $L^2\big((0,\varepsilon_0) \cup (2-\varepsilon_0, 2), \lambda_{(2)}(N_1)\big)$, where

$$A_\varepsilon(t) = \begin{cases} A(t), & t \in (0, \varepsilon_0), \\ (2-t)^{-1} \varepsilon\big(A_0(\varepsilon) + \varepsilon^{-1}(\mathrm{td} - \frac{n}{2})\big), & t \in (2-\varepsilon_0, 2). \end{cases} \tag{5.68}$$

To formulate our boundary conditions conveniently we introduce the spaces

$$H_{(0),I}(\varepsilon/2-\varepsilon) := Q_I\big(A_{(0)}(\varepsilon/2-\varepsilon)\big)(H^0). \tag{5.69}$$

Next we want to apply Theorem 5.2 to the operator $D^{\mathrm{sign}}_{l/r,\varepsilon}$ which is defined by $\frac{\partial}{\partial t} + A_\varepsilon(t)$ on its core

$$\mathscr{D}^{\mathrm{sign}}_{l,\varepsilon} := \{\sigma \in C_c^1((0,\varepsilon], H^1) : Q_{<0}(A(\varepsilon))\sigma(\varepsilon) = 0\},$$

and

$$\mathscr{D}^{\mathrm{sign}}_{r,\varepsilon} := \{\sigma \in C_c^1([2-\varepsilon, 2), H^1) : Q_{>0}(A(2-\varepsilon))\sigma(2-\varepsilon) = 0\},$$

respectively. Theorem 5.2 obviously applies to $D^{\mathrm{sign}}_{l,\varepsilon}$ if the condition (5.4) is satisfied. For $D^{\mathrm{sign}}_{r,\varepsilon}$, we note that the role of A_V is now taken by the operator $A_{r,V} := \varepsilon\big(A_0(\varepsilon) + \varepsilon^{-1}(\mathrm{td} - n/2)\big)$, such that the analogue of (5.4) can be verified by a straightforward estimate using (2.39), (2.19), (2.30), and (2.31), after the approriate scaling of $g^{T_V N}$.

Consequently, we obtain

$$\operatorname{sign}_{(2)} T_\pi = D^{\mathrm{sign}}_{\varepsilon, Z_\varepsilon}, \tag{5.70}$$

where $D^{\mathrm{sign}}_{\varepsilon, Z_\varepsilon}$ denotes the signature operator on the cylinder $(Z_\varepsilon, g_\varepsilon^{TT_\pi})$ with core

$$\{\sigma \in H^1(Z_\varepsilon, \Lambda T^* Z_\varepsilon) : \sigma|\partial U_\varepsilon \in H_{<0}(\varepsilon), \sigma|\partial C_\varepsilon N \in H_{>0}(2-\varepsilon)\}. \tag{5.71}$$

Clearly, if we replace the boundary conditions in (5.71) by $H_{0,<0}(\varepsilon)$ and $H_{0,\geq 0}(\varepsilon)$, respectively, then the resulting operator on Z_ε will have index 0. Thus we obtain from [**3**, Thm.4.14] again

$$\begin{aligned} \operatorname{sign}_{(2)} T_\pi = {} & \operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,>0}(\varepsilon)\big) \\ & - \operatorname{ind}\big(H_{0,<0}(\varepsilon), H_{0,>0}(\varepsilon)\big) + \operatorname{ind}(H_{\geq 0}(2-\varepsilon), H_{0,<0}(\varepsilon)). \end{aligned} \tag{5.72}$$

Now we recall that

$$\begin{aligned}(5.73)\quad & \operatorname{ind}(H_{>0}(2-\varepsilon), H_{0,<0}(\varepsilon)) \\ &= \operatorname{ind}\Big(Q_{>0}\big(A_0(\varepsilon)+\varepsilon^{-1}(\mathrm{td}+n/2)\big)(H^0), Q_{<0}(A_0(\varepsilon))(H^0)\Big).\end{aligned}$$

We recall also that $A_0(\varepsilon)$ is unitarily equivalent to $D^{\Lambda}_{N_\varepsilon}\tau_{N_\varepsilon}$ such that, by Theorem 3.1, the eigenspaces of $A_0(\varepsilon)$ coincide with those of $A_0(\varepsilon)+\varepsilon^{-1}(\mathrm{td}+n/2)$. Hence the explicit computations of loc.cit. give

$$\operatorname{ind}(H_{\geq 0}(2-\varepsilon), H_{0,<0}(\varepsilon)) = -\frac{1}{2}\dim\ker A_0(\varepsilon). \tag{5.74}$$

Finally, we use [**3**, Prop.A.13] to see that

$$\operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,>0}(\varepsilon)\big) - \operatorname{ind}\big(H_{0,<0}(\varepsilon), H_{0,>0}(\varepsilon)\big) = \operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,\geq 0}(\varepsilon)\big),$$

which gives finally

$$\operatorname{sign}_{(2)}T_\pi = \operatorname{ind}\big(H_{\leq 0}(\varepsilon), H_{0,\geq 0}(\varepsilon)\big) - \frac{1}{2}\dim\ker A_0(\varepsilon) \tag{5.75}$$

$$=: \tau(\varepsilon). \tag{5.76}$$

This completes the proof of Theorem 5.4. □

References

[1] M. F. Atiyah, V. K. Patodi & I. M. Singer – "Spectral asymmetry and Riemannian geometry. I", *Math. Proc. Cambridge Philos. Soc.* **77** (1975), p. 43–69.

[2] W. Ballmann & J. Brüning – "On the spectral theory of manifolds with cusps", *J. Math. Pures Appl.* **80** (2001), p. 593–625.

[3] W. Ballmann, J. Brüning & G. Carron – "Regularity and index theory for Dirac-Schrödinger systems with Lipschitz coefficients", *J. Math. Pures Appl.* **89** (2008), p. 429–476.

[4] N. Berline, E. Getzler & M. Vergne – *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer, 2004.

[5] J.-M. Bismut & J. Cheeger – "η-invariants and their adiabatic limits", *J. Amer. Math. Soc.* **2** (1989), p. 33–70.

[6] ______, "Families index for manifolds with boundary, superconnections, and cones. I. Families of manifolds with boundary and Dirac operators", *J. Funct. Anal.* **89** (1990), p. 313–363.

[7] ______, "Families index for manifolds with boundary, superconnections and cones. II. The Chern character", *J. Funct. Anal.* **90** (1990), p. 306–354.

[8] ______, "Remarks on the index theorem for families of Dirac operators on manifolds with boundary", in *Differential geometry*, Pitman Monogr. Surveys Pure Appl. Math., vol. 52, Longman Sci. Tech., 1991, p. 59–83.

[9] J.-M. Bismut & D. S. Freed – "The analysis of elliptic families. I. Metrics and connections on determinant bundles", *Comm. Math. Phys.* **106** (1986), p. 159–176.

[10] ______, "The analysis of elliptic families. II. Dirac operators, eta invariants, and the holonomy theorem", *Comm. Math. Phys.* **107** (1986), p. 103–163.

[11] J. Brüning & M. Lesch – "Hilbert complexes", *J. Funct. Anal.* **108** (1992), p. 88–132.

[12] J. Brüning & R. Seeley – "An index theorem for first order regular singular operators", *Amer. J. Math.* **110** (1988), p. 659–714.

[13] ———, "The expansion of the resolvent near a singular stratum of conical type", *J. Funct. Anal.* **95** (1991), p. 255–290.

[14] A.-P. Calderón & R. Vaillancourt – "On the boundedness of pseudo-differential operators", *J. Math. Soc. Japan* **23** (1971), p. 374–378.

[15] J. Cheeger – "Spectral geometry of singular Riemannian spaces", *J. Differential Geom.* **18** (1983), p. 575–657.

[16] ———, "η-invariants, the adiabatic approximation and conical singularities. I. The adiabatic approximation", *J. Differential Geom.* **26** (1987), p. 175–221.

[17] J. Cheeger & X. Dai – "L^2-cohomology of spaces with non-isolated conical singularities and non-multiplicativity of the signature", in *Riemannian Topology and Geometric Structures on Manifolds*, Progress in Mathematics, vol. 271, 2008, p. 1–24.

[18] X. Dai – "Adiabatic limits, nonmultiplicativity of signature, and Leray spectral sequence", *J. Amer. Math. Soc.* **4** (1991), p. 265–321.

[19] T. Hausel, E. Hunsicker & R. Mazzeo – "Hodge cohomology of gravitational instantons", *Duke Math. J.* **122** (2004), p. 485–548.

[20] E. Hunsicker – "Hodge and signature theorems for a family of manifolds with fibre bundle boundary", *Geom. Topol.* **11** (2007), p. 1581–1622.

[21] E. Hunsicker & R. Mazzeo – "Harmonic forms on manifolds with edges", *Int. Math. Res. Not.* **2005** (2005), p. 3229–3272.

[22] T. Kato – *Perturbation theory for linear operators*, Grund. math. Wiss., vol. 132, Springer, 1966.

[23] H. B. J. Lawson & M.-L. Michelsohn – *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, 1989.

[24] W. Magnus, F. Oberhettinger & R. P. Soni – *Formulas and theorems for the special functions of mathematical physics*, Grund. math. Wiss., vol. 52, Springer, 1966.

[25] E. Witten – "Global gravitational anomalies", *Comm. Math. Phys.* **100** (1985), p. 197–229.

J. Brüning, Institut für Mathematik der Humboldt–Universität zu Berlin, Rudower Chaussee 25, 12489 Berlin, Germany • *E-mail :* `bruening@mathematik.hu-berlin.de`