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PURE SPINORS ON LIE GROUPS

by

A. Alekseev, H. Bursztyn & E. Meinrenken

Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday.

Abstract. — For any manifold M , the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a natural inner product given by the pairing of vectors and covectors. Differential forms on M may be viewed as spinors for the corresponding Clifford bundle, and in particular there is a notion of *pure spinor*. In this paper, we study pure spinors and Dirac structures in the case when $M = G$ is a Lie group with a bi-invariant pseudo-Riemannian metric, e.g. G semi-simple. The applications of our theory include the construction of distinguished volume forms on conjugacy classes in G , and a new approach to the theory of quasi-Hamiltonian G -spaces.

Résumé (Spineurs purs sur les groupes de Lie). — Pour toute variété lisse M , le fibré $\mathbb{T}M = TM \oplus T^*M$ est muni d'un produit scalaire naturel défini par la dualité entre vecteurs et co-vecteurs. Les formes différentielles sur M sont des spineurs pour le fibré de Clifford correspondant. On définit alors les *spineurs purs*. Dans cet article, nous étudions les spineurs purs et les structures de Dirac dans le cas où M est un groupe de Lie G muni d'une métrique pseudo-riemannienne bi-invariante, par exemple un groupe semi-simple. Comme applications de notre théorie, nous définissons une forme volume distinguée sur les classes de conjugaison de G , et nous proposons une nouvelle approche de la théorie des G -espaces quasi-hamiltoniens.

0. Introduction

For any manifold M , the direct sum $\mathbb{T}M = TM \oplus T^*M$ carries a non-degenerate symmetric bilinear form, extending the pairing between vectors and covectors. There is a natural Clifford action ρ of the sections $\Gamma(\mathbb{T}M)$ on the space $\Omega(M) = \Gamma(\wedge T^*M)$ of differential forms, where vector fields act by contraction and 1-forms by exterior multiplication. That is, $\wedge T^*M$ is viewed as a spinor module over the Clifford bundle $\text{Cl}(\mathbb{T}M)$. A form $\phi \in \Omega(M)$ is called a *pure spinor* if the solutions $w \in \Gamma(\mathbb{T}M)$ of

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$\varrho(w)\phi = 0$ span a Lagrangian subbundle $E \subset TM$. Given a closed 3-form $\eta \in \Omega^3(M)$, a pure spinor ϕ is called *integrable* (relative to η) [9, 28] if there exists a section $w \in \Gamma(TM)$ with

$$(d + \eta)\phi = \varrho(w)\phi.$$

In this case, there is a generalized foliation of M with tangent distribution the projection of E to TM . The subbundle E defines a *Dirac structure* [20, 50] on M , and the triple (M, E, η) is called a *Dirac manifold*.

The present paper is devoted to the study of Dirac structures and pure spinors on Lie groups G . We assume that the Lie algebra $\mathfrak{g}$ carries a non-degenerate invariant symmetric bilinear form B , and take $\eta \in \Omega^3(G)$ as the corresponding Cartan 3-form. Let $\bar{\mathfrak{g}}$ denote the Lie algebra $\mathfrak{g}$ with the opposite bilinear form $-B$. We will describe a trivialization

$$TG \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}}),$$

under which any Lagrangian Lie subalgebra $\mathfrak{s} \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$ defines a Dirac structure on G . There is also a similar identification of spinor bundles

$$\mathcal{R}: G \times \mathrm{Cl}(\mathfrak{g}) \xrightarrow{\cong} \wedge T^*G,$$

taking the standard Clifford action of $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ on $\mathrm{Cl}(\mathfrak{g})$, where the first summand acts by left (Clifford) multiplication and the second summand by right multiplication, to the Clifford action ϱ . This isomorphism takes the *Clifford differential* d_{Cl} on $\mathrm{Cl}(\mathfrak{g})$, given as Clifford commutator by a cubic element [4, 38], to the differential $d + \eta$ on $\Omega(G)$. As a result, pure spinors $x \in \mathrm{Cl}(\mathfrak{g})$ for the Clifford action of $\mathrm{Cl}(\mathfrak{g} \oplus \bar{\mathfrak{g}})$ on $\mathrm{Cl}(\mathfrak{g})$ define pure spinors $\phi = \mathcal{R}(x) \in \Omega(G)$, and the integrability condition for ϕ is equivalent to a similar condition for x . The simplest example $x = 1$ defines the *Cartan-Dirac structure* E_G [14, 50], introduced by Alekseev, Ševera and Strobl in the 1990's. In this case, the resulting foliation of G is just the foliation by conjugacy classes. We will study this Dirac structure in detail, and examine in particular its behavior under group multiplication and under the exponential map. When G is a complex semi-simple Lie group, it carries another interesting Dirac structure, which we call the *Gauss-Dirac structure*. The corresponding foliation of G has a dense open leaf which is the ‘big cell’ from the Gauss decomposition of G .

The main application of our study of pure spinors is to the theory of q-Hamiltonian actions [2, 3]. The original definition of a q-Hamiltonian G -space in [3] involves a G -manifold M together with an invariant 2-form ω and a G -equivariant map $\Phi: M \rightarrow G$ satisfying appropriate axioms. As observed in [14, 15], this definition is equivalent to saying that the ‘ G -valued moment map’ Φ is a suitable morphism of Dirac manifolds (in analogy with classical moment maps, which are morphisms $M \rightarrow \mathfrak{g}^*$ of Poisson manifolds). In this paper, we will carry this observation further, and develop all the basic results of q-Hamiltonian geometry from this perspective. A conceptual advantage of this alternate viewpoint is that, while the arguments in [3] required G to be compact, the Dirac geometry approach needs no such assumption, and in fact works in the complex (holomorphic) category as well. This is relevant for applications:

For instance, the symplectic form on a representation variety $\text{Hom}(\pi_1(\Sigma), G)/G$ (for Σ a closed surface) can be obtained by q-Hamiltonian reduction, and there are many interesting examples for noncompact G . (For instance, the case $G = \text{PSL}(2, \mathbb{R})$ gives the symplectic form on Teichmüller space.) Complex q-Hamiltonian spaces appear e.g. in the work of Boalch [13] and Van den Bergh [11].

The organization of the paper is as follows. Sections 1 and 2 contain a review of Dirac geometry, first on vector spaces and then on manifolds. The main new results in these sections concern the geometry of Lagrangian splittings $\text{TM} = E \oplus F$ of the bundle TM . If $\phi, \psi \in \Omega(M)$ are pure spinors defining E, F , then, as shown in [17, 19], the top degree part of $\phi^\top \wedge \psi$ (where $\top$ denotes the standard anti-involution of the exterior algebra) is nonvanishing, and hence defines a volume form μ on M . Furthermore, there is a bivector field $\pi \in \mathfrak{X}^2(M)$ naturally associated with the splitting, which satisfies

$$\phi^\top \wedge \psi = e^{-\iota(\pi)} \mu.$$

We will discuss the properties of μ and π in detail, including their behavior under Dirac morphisms.

In Section 3 we specialize to the case $M = G$, where G carries a bi-invariant pseudo-Riemannian metric, and our main results concern the isomorphism $\text{TG} \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$ and its properties. Under this identification, the Cartan-Dirac structure $E_G \subset \text{TG}$ corresponds to the diagonal $\mathfrak{g}_\Delta \subset \mathfrak{g} \oplus \bar{\mathfrak{g}}$, and hence it has a natural Lagrangian complement $F_G \subset \text{TG}$ defined by the anti-diagonal. We will show that the exponential map gives rise to a Dirac morphism $(\mathfrak{g}, E_{\mathfrak{g}}, 0) \rightarrow (G, E_G, \eta)$ (where $E_{\mathfrak{g}}$ is the graph of the linear Poisson structure on $\mathfrak{g} \cong \mathfrak{g}^*$), but this morphism does not relate the obvious complements $F_{\mathfrak{g}} = T\mathfrak{g}$ and F_G . The discrepancy is given by a ‘twist’, which is a solution of the *classical dynamical Yang-Baxter equation*. For G complex semi-simple, we will construct another Lagrangian complement of E_G , denoted by $\widehat{F}_G$, which (unlike F_G) is itself a Dirac structure. The bivector field corresponding to the splitting $E_G \oplus \widehat{F}_G$ is then a Poisson structure on G , which appeared earlier in the work of Semenov-Tian-Shansky [49].

In Section 4, we construct an isomorphism $\wedge T^*G \cong G \times \text{Cl}(\mathfrak{g})$ of spinor modules, valid under a mild topological assumption on G (which is automatic if G is simply connected). This allows us to represent the Lagrangian subbundles E_G , F_G and $\widehat{F}_G$ by explicit pure spinors ϕ_G , ψ_G , and $\widehat{\psi}_G$, and to derive the differential equations controlling their integrability. We show in particular that the Cartan-Dirac spinor satisfies

$$(d + \eta)\phi_G = 0.$$

Section 5 investigates the foundational properties of q-Hamiltonian G -spaces from the Dirac geometry perspective. Our results on the Cartan-Dirac structure give a direct construction of the fusion product of q-Hamiltonian spaces. On the other hand, we use the bilinear pairing of spinors to show that, for a q-Hamiltonian space (M, ω, Φ) , the top degree part of $e^\omega \Phi^* \psi_G \in \Omega(M)$ defines a volume form μ_M . This volume form was discussed in [8] when G is compact, but the discussion here applies

equally well to non-compact or complex Lie groups. Since conjugacy classes in G are examples of q -Hamiltonian G -spaces, we conclude that for any simply connected Lie group G with bi-invariant pseudo-Riemannian metric (e.g. G semi-simple), *any conjugacy class in G carries a distinguished invariant volume form*. If G is complex semi-simple, one obtains the same volume form μ_M if one replaces ψ_G with the Gauss-Dirac spinor $\widehat{\psi}_G$. However, the form $e^\omega \Phi^* \widehat{\psi}_G$ satisfies a nicer differential equation, which allows us to compute the volume of M , and more generally the measure $\Phi_* |\mu_M|$, by Berline-Vergne localization [12]. We also explain in this Section how to view the more general q -Hamiltonian q -Poisson spaces [2] in our framework.

Lastly, in Section 6, we revisit the theory of K^* -valued moment maps in the sense of Lu [42] and its connections with P -valued moment maps [3, Sec. 10] from the Dirac geometric standpoint.

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Notation. — Our conventions for Lie group actions are as follows: Let G be a Lie group (not necessarily connected), and $\mathfrak{g}$ its Lie algebra. A G -action on a manifold M is a group homomorphism $\mathcal{A}: G \rightarrow \text{Diff}(M)$ for which the action map $G \times M \rightarrow M$, $(g, m) \mapsto \mathcal{A}(g)(m)$ is smooth. Similarly, a $\mathfrak{g}$ -action on M is a Lie algebra homomorphism $\mathcal{A}: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ for which the map $\mathfrak{g} \times M \rightarrow TM$, $(\xi, m) \mapsto \mathcal{A}(\xi)_m$ is smooth. Given a G -action $\mathcal{A}$, one obtains a $\mathfrak{g}$ -action by the formula $\mathcal{A}(\xi)(f) = \frac{\partial}{\partial t} \Big|_{t=0} \mathcal{A}(\exp(-t\xi))^* f$, for $f \in C^\infty(M)$ (here vector fields are viewed as derivations of the algebra of smooth functions).

1. Linear Dirac geometry

The theory of Dirac manifolds was initiated by Courant and Weinstein in [20, 21]. We briefly review this theory, developing and expanding the approach via pure spinors advocated by Gualtieri [28] (see also Hitchin [32] and Alekseev-Xu [9]). All vector spaces in this section are over the ground field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We begin with some background material on Clifford algebras and spinors (see e.g. [19] or [47].)

1.1. Clifford algebras. — Suppose V is a vector space with a non-degenerate symmetric bilinear form B . We will sometimes refer to such a bilinear form B as an *inner product* on V . The *Clifford algebra* over V is the associative unital algebra

generated by the elements of V , with relations

$$vv' + v'v = B(v, v')1.$$

It carries a compatible $\mathbb{Z}_2$ -grading and $\mathbb{Z}$ -filtration, such that the generators $v \in V$ are odd and have filtration degree 1. We will denote by $x \mapsto x^\top$ the canonical anti-automorphism of exterior and Clifford algebras, equal to the identity on V . For any $x \in \text{Cl}(V)$, we denote by $l^{\text{Cl}}(x), r^{\text{Cl}}(x)$ the operators of graded left and right multiplication on $\text{Cl}(V)$:

$$l^{\text{Cl}}(x)x' = xx', \quad r^{\text{Cl}}(x)x' = (-1)^{|x||x'|}x'x.$$

Thus $l^{\text{Cl}}(x) - r^{\text{Cl}}(x)$ is the operator of graded commutator $[x, \cdot]_{\text{Cl}}$.

The *quantization map* $q: \wedge V \rightarrow \text{Cl}(V)$ is the isomorphism of vector spaces defined by $q(v_1 \wedge \cdots \wedge v_r) = v_1 \cdots v_r$ for pairwise orthogonal elements $v_i \in V$. Let

$$\text{str}: \text{Cl}(V) \rightarrow \det(V) := \wedge^{\text{top}}(V)$$

be the *super-trace*, given by q^{-1} , followed by taking the top degree part. It has the property $\text{str}([x, x']_{\text{Cl}}) = 0$.

A *Clifford module* is a vector space S together with an algebra homomorphism $\varrho: \text{Cl}(V) \rightarrow \text{End}(S)$. If S is a Clifford module, one has a dual Clifford module given by the dual space S^* with Clifford action $\varrho^*(x) = \varrho(x^\top)^*$.

Recall that $\text{Pin}(V)$ is the subgroup of $\text{Cl}(V)^\times$ generated by all $v \in V$ whose square in the Clifford algebra is $vv = \pm 1$. It is a double cover of the orthogonal group $\text{O}(V)$, where $g \in \text{Pin}(V)$ takes $v \in V$ to $(-1)^{|g|}gv g^{-1}$, using Clifford multiplication. The *norm homomorphism* for the Pin group is the group homomorphism

$$(1) \quad \text{N}: \text{Pin}(V) \rightarrow \{-1, +1\}, \quad \text{N}(g) = g^\top g = \pm 1.$$

Let $\{\cdot, \cdot\}$ be the graded Poisson bracket on $\wedge V$, given on generators by $\{v_1, v_2\} = B(v_1, v_2)$. Then $\wedge^2 V$ is a Lie algebra under the Poisson bracket, isomorphic to $\mathfrak{o}(V)$ in such a way that $\varepsilon \in \wedge^2 V$ corresponds to the linear map $v \mapsto \{\varepsilon, v\}$. The Lie algebra $\mathfrak{pin}(V) \cong \mathfrak{o}(V)$ is realized as the Lie subalgebra $q(\wedge^2(V)) \subset \text{Cl}(V)$.

A subspace $E \subset V$ is called *isotropic* if $E \subset E^\perp$ and *Lagrangian* if $E = E^\perp$. The set of Lagrangian subspaces is non-empty if and only if the bilinear form is *split*. If $\mathbb{K} = \mathbb{C}$, this just means that $\dim V$ is even, while for $\mathbb{K} = \mathbb{R}$ this requires that the bilinear form has signature (n, n) . From now on, we will reserve the letter W for a vector space with split bilinear form $\langle \cdot, \cdot \rangle$. We denote by $\text{Lag}(W)$ the Grassmann manifold of Lagrangian subspaces of W . It carries a transitive action of the orthogonal group $\text{O}(W)$.

Remark 1.1. — Suppose $\mathbb{K} = \mathbb{R}$, and identify $W \cong \mathbb{R}^{2n}$ with the standard bilinear form of signature (n, n) . The group $\text{O}(W) \cong \text{O}(n, n)$ has maximal compact subgroup $\text{O}(n) \times \text{O}(n)$. Already the subgroup $\text{O}(n) \times \{1\}$ acts transitively on $\text{Lag}(W)$, and in fact the action is free. It follows that $\text{Lag}(W)$ is diffeomorphic to $\text{O}(n)$. Further details may be found in [46].

1.2. Pure spinors. — An irreducible module S over the Clifford algebra $\text{Cl}(W)$ is called a *spinor module*. Any $E \in \text{Lag}(W)$ defines a spinor module $S = \text{Cl}(W)/\text{Cl}(W)E$. The choice of a Lagrangian complement F to E identifies $S = \wedge E^*$, where the generators in $E \subset W$ act by contraction and the generators in $F \subset W$ act by exterior multiplication. (Here F is identified with E^* , using the pairing defined by $\langle \cdot, \cdot \rangle$.) The dual spinor module is $S^* = \wedge E$, with generators in E acting by exterior multiplication and those in F by contraction.

For any non-zero element $\phi \in S$ of a spinor module, its null space

$$N_\phi = \{w \in W \mid \varrho(w)\phi = 0\}$$

is easily seen to be isotropic. The element $\phi \in S$ is a *pure spinor* [17] provided N_ϕ is Lagrangian. One can show that any Lagrangian subspace $E \in \text{Lag}(W)$ arises in this way: in fact, $S^E = \{\phi \in S \mid \varrho(E)\phi = 0\}$ is a one-dimensional subspace, with non-zero elements given by the pure spinors defining E . Any spinor module S admits a $\mathbb{Z}_2$ -grading (unique up to parity inversion) compatible with the Clifford action. Pure spinors always have a definite parity, either even or odd.

Example 1.2. — Let V be a vector space with inner product B . We denote by $\bar{V}$ the same vector space with the opposite bilinear form $-B$. Then $W = V \oplus \bar{V}$ is a vector space with split bilinear form. The space $S = \text{Cl}(V)$ is a spinor module over $\text{Cl}(W) = \text{Cl}(V) \otimes \text{Cl}(\bar{V})$, with Clifford action given on generators by $\varrho(v \oplus v') = l^{\text{Cl}}(v) - r^{\text{Cl}}(v')$. The element $1 \in \text{Cl}(V)$ is a pure spinor, with corresponding Lagrangian subspace the diagonal $V_\Delta \subset V \oplus \bar{V}$.

1.3. The bilinear pairing of spinors. — For any two spinor modules S_1, S_2 over $\text{Cl}(W)$, the space $\text{Hom}_{\text{Cl}(W)}(S_1, S_2)$ of intertwining operators is one-dimensional. Given a spinor module S , let

$$K_S = \text{Hom}_{\text{Cl}(W)}(S^*, S)$$

be the *canonical line*. There is a bilinear pairing [17]

$$S \otimes S \rightarrow K_S, \quad \phi \otimes \psi \mapsto (\phi, \psi)_S,$$

defined by the isomorphism $S \otimes S \cong S \otimes S^* \otimes \text{Hom}_{\text{Cl}(W)}(S^*, S)$ followed by the duality pairing $S \otimes S^* \rightarrow \mathbb{K}$. The pairing satisfies

$$(2) \quad (\varrho(x^\top)\phi, \psi)_S = (\phi, \varrho(x)\psi)_S, \quad x \in \text{Cl}(W),$$

and is characterized by this property up to a scalar. (2) implies the following invariance property under the action of the group $\text{Pin}(V)$, involving the norm homomorphism (1),

$$(g\phi, g\psi)_S = N(g)(\phi, \psi)_S, \quad g \in \text{Pin}(V).$$

Theorem 1.3 (E. Cartan [17]). — Let S be a spinor module over $\text{Cl}(W)$, and let $\phi, \psi \in S$ be pure spinors. Then the corresponding Lagrangian subspaces N_ϕ, N_ψ are transverse if and only if $(\phi, \psi)_S \neq 0$.

A simple proof of this result is given in Chevalley's book [19, III.2.4], see also [47, Section 3.5].

Example 1.4. — Suppose V is a space with inner product B , and take $S = \text{Cl}(V)$ as a spinor module over $\text{Cl}(V \oplus \bar{V})$ (cf. Example 1.2). Then $K_S = \det(V)$, with bilinear pairing on spinors given as

$$(3) \quad (x, x')_{\text{Cl}(V)} = \text{str}(x^\top x') \in \det(V).$$

Using the isomorphism $q : \wedge(V) \rightarrow \text{Cl}(V)$ to identify $S \cong \wedge(V)$, the bilinear pairing becomes

$$(4) \quad (y, y')_{\wedge(V)} = (y^\top \wedge y')^{[\text{top}]} \in \det(V).$$

1.4. Contravariant spinors. — For any vector space V , the direct sum $\mathbb{V} := V \oplus V^*$ carries a split bilinear form given by the pairing between V and V^* :

$$(5) \quad \langle w_1, w_2 \rangle = \langle \alpha_1, v_2 \rangle + \langle \alpha_2, v_1 \rangle, \quad w_i = v_i \oplus \alpha_i \in \mathbb{V}.$$

Every vector space W with split bilinear form is of this form, by choosing a pair of transverse Lagrangian subspaces V, V' , and using the bilinear form to identify $V' = V^*$. Then $S = \wedge V^*$, with Clifford action given on generators $w = v \oplus \alpha \in \mathbb{V}$ by

$$\varrho(w) = \epsilon(\alpha) + \iota(v)$$

(where $\epsilon(\alpha) = \alpha \wedge \cdot$), is a natural choice of spinor module for $\text{Cl}(\mathbb{V})$. The restriction of ϱ to $\wedge V^* \subset \text{Cl}(\mathbb{V})$ is given by exterior multiplication, while the restriction to $\wedge V \subset \text{Cl}(\mathbb{V})$ is given by contraction ⁽¹⁾. The line $K_S = \text{Hom}_{\text{Cl}(\mathbb{V})}(S^*, S)$ is canonically isomorphic to $\det(V^*) = \wedge^{\text{top}} V^*$, and the bilinear pairing on spinors is simply

$$(\phi, \psi)_{\wedge(V^*)} = (\phi^\top \wedge \psi)^{[\text{top}]} \in \det(V^*),$$

similar to Example 1.4. Theorem 1.3 shows that if ϕ, ψ are pure spinors for transverse Lagrangian subspaces, the pairing $(\phi, \psi)_{\wedge(V^*)}$ defines a *volume form* on V .

Remarks 1.5. — We mention the following two facts for later reference.

a. We have the identity

$$(-1)^{|\phi|} (-(\varrho(w)\phi)^\top \wedge \psi + \phi^\top \wedge (\varrho(w)\psi)) = \iota(v)(\phi^\top \wedge \psi), \quad w = v \oplus \alpha \in \mathbb{V},$$

which refines property (2) of the bilinear pairing.

b. One can also consider the *covariant spinor module* $\wedge(V)$, obtained by reversing the roles of V and V^* . Suppose $\mu \in \det(V)$ is non-zero, and let $\star : \wedge(V^*) \rightarrow \wedge(V)$ be the corresponding star operator, defined by $\star\phi = \iota(\phi)\mu$. Let μ^* be the dual generator defined by $\star((\mu^*)^\top) = 1$. Then $\star$ is an isomorphism of $\text{Cl}(\mathbb{V})$ -modules. Furthermore, using μ, μ^* to trivialize $\det(V), \det(V^*)$, the isomorphism intertwines the bilinear pairings:

$$(\phi, \psi)_{\wedge(V^*)} = (\star\phi, \star\psi)_{\wedge(V)}, \quad \phi, \psi \in \wedge(V^*).$$

⁽¹⁾ We are using the convention that $\iota : \wedge(V) \rightarrow \text{End}(\wedge V^*)$ is the extension of the map $v \mapsto \iota(v)$ as an algebra homomorphism. Note that some authors use the extension as an algebra anti-homomorphism.

Any 2-form $\omega \in \wedge^2 V^*$ defines a pure spinor $\phi = e^{-\omega}$, with N_ϕ the graph of ω :

$$\text{Gr}_\omega = \{v \oplus \alpha \mid v \in V, \alpha = \iota(v)\omega\}.$$

Note that, in accordance with Theorem 1.3, $\text{Gr}_\omega \cap V = \{0\}$ if and only if ω is non-degenerate, if and only if $(e^\omega)^{[\text{top}]}$ is non-zero. The most general pure spinor $\phi \in \wedge V^*$ can be written in the form

$$(6) \quad \phi = e^{-\omega_Q} \wedge \theta,$$

where $\omega_Q \in \wedge^2 Q^*$ is a 2-form on a subspace $Q \subset V$ and $\theta \in \det(\text{Ann}(Q)) \setminus \{0\}$ is a volume form on V/Q . To write (6), we have chosen an extension of ω_Q to a 2-form on V . (Clearly, ϕ does not depend on this choice.) The corresponding Lagrangian subspace is

$$N_\phi = \{v \oplus \alpha \mid v \in Q, \alpha|_Q = \iota(v)\omega_Q\}.$$

The triple (Q, ω_Q, θ) is uniquely determined by ϕ , see e.g. [19, III.1.9]. A simple consequence is that any pure spinor has definite parity, that is, ϕ is either even or odd depending on the parity of $\dim(V/Q)$. For any $E \in \text{Lag}(V)$ we define subspaces $\ker(E) \subset \text{ran}(E) \subset V$ by

$$\ker(E) = E \cap V, \quad \text{ran}(E) = \text{pr}_V(E),$$

where $\text{pr}_V: \mathbb{V} \rightarrow V$ is the projection along V^* . For any pure spinor ϕ , written in the form (6), we have $\text{ran}(E_\phi) = Q$ and $\ker(E_\phi) = \ker(\omega_Q)$. In particular, $\phi^{[\text{top}]}$ is non-zero if and only if $\ker(E_\phi) = 0$. Similarly, $\text{ran}(E_\phi) = V$ if and only if $\phi^{[0]}$ is non-zero, if and only if $\phi = e^{-\omega}$ for a global 2-form ω .

1.5. Action of the orthogonal group. — Recall the identification $\wedge^2(W) \cong \mathfrak{o}(W)$ (see Section 1.1). For any Lagrangian subspace $E \subset W$, the space $\wedge^2(E)$ is embedded as an Abelian subalgebra of $\mathfrak{o}(W)$. The inclusion map exponentiates to an injective group homomorphism,

$$(7) \quad \wedge^2(E) \rightarrow \text{O}(W), \quad \varepsilon \mapsto A^\varepsilon, \quad A^\varepsilon(v \oplus \alpha) = v \oplus (\alpha - \iota(v)\varepsilon),$$

with image the orthogonal transformations fixing E pointwise. The subgroup $\wedge^2(E)$ acts freely and transitively on the subset of $\text{Lag}(W)$ of Lagrangian subspaces transverse to E , which therefore becomes an affine space. Observe that A^ε has a distinguished lift $\tilde{A}^\varepsilon = \exp(\varepsilon) \in \text{Pin}(W)$ (exponential in the subalgebra $\wedge(E) \subset \text{Cl}(W)$).

For any spinor module S over $\text{Cl}(W)$, the induced representation of the group $\text{Pin}(W) \subset \text{Cl}(W)^\times$ preserves the set of pure spinors, and the map $\phi \mapsto N_\phi$ is equivariant. That is, if $\tilde{A} \in \text{Pin}(W)$ lifts $A \in \text{O}(W)$, then

$$N_{\ell(\tilde{A})\phi} = A(N_\phi).$$

Consider again the case $W = \mathbb{V}$. Then 2-forms $\omega \in \wedge^2 V^*$ and bivectors $\pi \in \wedge^2(V)$ define orthogonal transformations

$$A^{-\omega}(v \oplus \alpha) = v \oplus (\alpha + \iota_v \omega), \quad A^{-\pi}(v \oplus \alpha) = (v + \iota_\alpha \pi) \oplus \alpha.$$

Their lifts act in the spin representation as follows:

$$(8) \quad \varrho(\tilde{A}^{-\omega})\phi = e^{-\omega} \phi, \quad \varrho(\tilde{A}^{-\pi})\phi = e^{-\iota(\pi)}\phi.$$

1.6. Morphisms. — It is easy to see that the group of orthogonal transformations of $\mathbb{V}$ preserving the ‘polarization’

$$(9) \quad 0 \longrightarrow V^* \longrightarrow \mathbb{V} \longrightarrow V \longrightarrow 0$$

(i.e., taking the subspace V^* to itself) is the semi-direct product $\wedge^2 V^* \rtimes \mathrm{GL}(V) \subset \mathrm{O}(\mathbb{V})$, where $\omega \in \wedge^2 V^*$ acts as $A^{-\omega}$ and $\mathrm{GL}(V)$ acts in the natural way on V and by the conjugate transpose on V^* .

More generally, for vector spaces V and V' , we define the set of *morphisms from $\mathbb{V}$ to $\mathbb{V}'$* [33] to be

$$\mathrm{Hom}(V, V') \times \wedge^2 V^*,$$

with the following composition law:

$$(10) \quad (\Phi_1, \omega_1) \circ (\Phi_2, \omega_2) = (\Phi_1 \circ \Phi_2, \omega_2 + \Phi_2^* \omega_1).$$

Given $w = v \oplus \alpha \in \mathbb{V}$ and $w' = v' \oplus \alpha' \in \mathbb{V}'$, we write

$$w \sim_{(\Phi, \omega)} w' \Leftrightarrow v' = \Phi(v), \quad \Phi^* \alpha' = \alpha + \iota_v \omega.$$

In particular, taking $V' = V$ and $\Phi = \mathrm{id}$ we have $w \sim_{(\mathrm{id}, \omega)} w'$ if and only if $w' = A^{-\omega}(w)$. The *graph of a morphism* (Φ, ω) is the subspace

$$(11) \quad \Gamma_{(\Phi, \omega)} = \{(w', w) \in \mathbb{V}' \times \mathbb{V} \mid w \sim_{(\Phi, \omega)} w'\}.$$

We have $\Gamma_{(\Phi_1, \omega_1) \circ (\Phi_2, \omega_2)} = \Gamma_{(\Phi_1, \omega_1)} \circ \Gamma_{(\Phi_2, \omega_2)}$ under composition of relations. The morphisms (Φ, ω) are ‘isometric’, in the sense that

$$(12) \quad w_1 \sim_{(\Phi, \omega)} w'_1, \quad w_2 \sim_{(\Phi, \omega)} w'_2 \Rightarrow \langle w_1, w_2 \rangle = \langle w'_1, w'_2 \rangle.$$

Equivalently, $\Gamma_{(\Phi, \omega)}$ is Lagrangian in $\mathbb{V}' \oplus \overline{\mathbb{V}}$. We write

$$\begin{aligned} \ker(\Phi, \omega) &= \{w \in \mathbb{V} \mid w \sim_{(\Phi, \omega)} 0\}, \\ \mathrm{ran}(\Phi, \omega) &= \{w' \in \mathbb{V}' \mid \exists w \in \mathbb{V}: w \sim_{(\Phi, \omega)} w'\}. \end{aligned}$$

Thus $\ker(\Phi, \omega) = \{(v, -\iota_v \omega) \mid v \in \ker(\Phi)\}$ while $\mathrm{ran}(\Phi, \omega) = \mathrm{ran}(\Phi) \oplus (V')^*$.

Definition 1.6. — Let $(\Phi, \omega): \mathbb{V} \rightarrow \mathbb{V}'$ be a morphism, and $E \in \mathrm{Lag}(\mathbb{V})$. We define the *forward image* $E' \in \mathrm{Lag}(\mathbb{V}')$ to be the Lagrangian subspace

$$E' := \Gamma_{(\Phi, \omega)} \circ E = \{w' \in \mathbb{V}' \mid \exists w \in E: w \sim_{(\Phi, \omega)} w'\}.$$

Similarly, for $F' \in \mathrm{Lag}(\mathbb{V}')$ the *backward image* is defined as the Lagrangian subspace

$$F := F' \circ \Gamma_{(\Phi, \omega)} = \{w \in \mathbb{V} \mid \exists w' \in F': w \sim_{(\Phi, \omega)} w'\}.$$

The proof that forward and backward images of Lagrangian subspaces are Lagrangian is parallel to the similar statement in the symplectic category of Guillemin-Sternberg [30] (see also Weinstein [54]). It is simple to check that the composition $E' = \Gamma_{(\Phi, \omega)} \circ E$ is transverse if and only if $\ker(\Phi, \omega) \cap E = \{0\}$. Similarly, the composition $F = F' \circ \Gamma_{(\Phi, \omega)}$ is transverse if and only if $\text{ran}(\Phi, \omega) + F' = \mathbb{V}'$ (equivalently, if and only if $\text{ran}(\Phi) + \text{ran}(F') = V'$).

Remark 1.7. — As in the symplectic category [30, 54], one could consider morphisms given by arbitrary *Lagrangian relations*, i.e. Lagrangian subspaces $\Gamma \subset \mathbb{V}' \oplus \overline{\mathbb{V}}$ (see e.g. [16]). The graphs (11) of morphisms (Φ, ω) are exactly those Lagrangian relations preserving the ‘polarization’ (9), in the sense that $\Gamma \circ V^* = (V')^*$ (where the composition is transverse), see [33].

The (Φ, ω) -relation may also be interpreted in terms of the spinor representations of $\text{Cl}(\mathbb{V})$ and $\text{Cl}(\mathbb{V}')$:

Lemma 1.8. — Suppose $(\Phi, \omega): \mathbb{V} \rightarrow \mathbb{V}'$ is a morphism, and $w \in \mathbb{V}$, $w' \in \mathbb{V}'$. Then

$$(13) \quad w \sim_{(\Phi, \omega)} w' \Leftrightarrow \varrho(w)(e^\omega \Phi^* \psi') = e^\omega \Phi^*(\varrho(w')\psi'), \quad \psi' \in \wedge(V')^*.$$

Proof. — This follows from $(\epsilon(\alpha) + \iota_v)(e^\omega \Phi^* \psi') = e^\omega (\epsilon(\alpha + \iota_v \omega) + \iota_v) \Phi^* \psi'$, for $v \oplus \alpha \in \mathbb{V}$. $\square$

Lemma 1.9. — Suppose $(\Phi, \omega): \mathbb{V} \rightarrow \mathbb{V}'$ is a morphism, and ψ' is a pure spinor defining a Lagrangian subspace F' . Then $\psi = e^\omega \Phi^* \psi'$ is non-zero if and only if the composition $F = F' \circ \Gamma_{(\Phi, \omega)}$ is transverse, and in that case it is a pure spinor defining F .

Proof. — Suppose $w \in F$, i.e. $w \sim_{(\Phi, \omega)} w'$ with $w' \in F' = N_{\psi'}$. Then $w \in N_\psi$ by Equation (13). Thus $F \subset N_\psi$. For $\psi \neq 0$, this is an equality since F is Lagrangian. $\square$

Example 1.10. — Suppose $E, F \subset \mathbb{V}$ are Lagrangian, with defining pure spinors ϕ, ψ . Let $E^\top$ be the image of E under the map $v \oplus \alpha \mapsto v \oplus (-\alpha)$. Then $\phi^\top$ is a pure spinor defining $E^\top$. Consider the diagonal inclusion $\text{diag}: V \rightarrow V \times V$, so that $\text{diag}^*(\phi^\top \otimes \psi) = \phi^\top \wedge \psi$ is just the wedge product. The wedge product is non-zero if and only if the composition $E^\top \wedge F := (E^\top \times F) \circ \Gamma_{\text{diag}}$ is transverse. This is the case, for instance, if E and F are transverse (since the top degree part of $\phi^\top \wedge \psi$ is non-zero in this case). Explicitly,

$$E^\top \wedge F = \{v \oplus \alpha \mid \exists v \oplus \alpha_1 \in E, v \oplus \alpha_2 \in F: \alpha = \alpha_2 - \alpha_1\}.$$

Note that $\text{ran}(E^\top \wedge F) = \text{ran}(E) \cap \text{ran}(F)$, with 2-form the difference of the restrictions of the 2-forms on $\text{ran}(E)$ and $\text{ran}(F)$. Note also that $(A^{-\omega}(E))^\top \wedge (A^{-\omega}(F)) = E^\top \wedge F$ for all $\omega \in \wedge^2 V^*$.

This “wedge product” operation of Lagrangian subspaces was noticed independently by Gualtieri, see [29].

1.7. Dirac spaces. — A *Dirac space* is a pair (V, E) , where V is a vector space and $E \subset \mathbb{V}$ is a Lagrangian subspace. As remarked in Section 1.4, E determines a subspace $Q = \text{ran}(E) = \text{pr}_V(E) \subset V$ together with a 2-form $\omega_Q \in \wedge^2 Q^*$,

$$(14) \quad \omega_Q(v, v') = \langle \alpha, v' \rangle = -\langle \alpha', v \rangle$$

for arbitrary lifts $v \oplus \alpha, v' \oplus \alpha' \in E$ of $v, v' \in Q$. The kernel of ω_Q is the subspace $\ker(E) = E \cap V$. Conversely, any subspace Q equipped with a 2-form ω_Q determines a Lagrangian subspace $E = \{v \oplus \alpha \in \mathbb{V} \mid v \in Q, \alpha|_Q = \omega_Q(v, \cdot)\}$. The gauge transformation $A^{-\omega}$ by a 2-form $\omega \in \wedge^2 V^*$ preserves Q , while ω_Q changes by the pull-back of ω to Q .

Definition 1.11. — Let $(\mathbb{V}, E)$ and $(\mathbb{V}', E')$ be Dirac spaces. A *Dirac morphism* $(\Phi, \omega): (V, E) \rightarrow (V', E')$ is a morphism (Φ, ω) with $E' = \Gamma_{(\Phi, \omega)} \circ E$. It is called a *strong Dirac morphism*⁽²⁾ if this composition is transverse, i.e.,

$$\ker(\Phi, \omega) \cap E = \{0\}.$$

Clearly, the composition of strong Dirac morphisms is again a strong Dirac morphism. Note that the definition of a Dirac morphism $(\Phi, \omega): (V, E) \rightarrow (V', E')$ amounts to the existence of a linear map $\hat{\mathbf{a}}: E' \rightarrow E$, assigning to each $w' \in E'$ an element of E to which it is (Φ, ω) -related:

$$(15) \quad \hat{\mathbf{a}}(w') \sim_{(\Phi, \omega)} w' \quad \forall w' \in E'.$$

The map $\hat{\mathbf{a}}$ is completely determined by its V -component

$$\mathbf{a} = \text{pr}_V \circ \hat{\mathbf{a}}: E' \rightarrow V,$$

since $\hat{\mathbf{a}}(v' \oplus \alpha') = v \oplus (\Phi^* \alpha' + \iota_v \omega)$ where $v = \mathbf{a}(v' \oplus \alpha')$. Hence (Φ, ω) is a Dirac morphism if and only if there exists a map $\mathbf{a}: E' \rightarrow V$, such that the corresponding map $\hat{\mathbf{a}}$ takes values in E .

Lemma 1.12. — For a strong Dirac morphism $(\Phi, \omega): (V, E) \rightarrow (V', E')$, the map $\hat{\mathbf{a}}$ satisfying (15) is unique. Its range is given by

$$(16) \quad \text{ran}(\hat{\mathbf{a}}) = E \cap \ker(\Phi, \omega)^\perp.$$

Proof. — The map $\hat{\mathbf{a}}$ associated to a Dirac morphism is unique up to addition of elements in $E \cap \ker(\Phi, \omega)$. Hence, it is unique precisely if the Dirac morphism is strong. Its range consists of all $w \in E$ which are (Φ, ω) -related to some element of $w' \in E'$. By (12), the subspace $\{w \in \mathbb{V} \mid \exists w' \in E': w \sim_{(\Phi, \omega)} w'\}$ is orthogonal to $\ker(\Phi, \omega)$. Hence, by a dimension count it coincides with $\ker(\Phi, \omega)^\perp$. On the other hand, if $w \in E$ lies in this subspace, it is automatic that $w' \in E'$ since $E' = \Gamma_{(\Phi, \omega)} \circ E$. $\square$

⁽²⁾ In the particular case when $\omega = 0$, Dirac morphisms are also called *forward Dirac maps* [15, 16], and strong Dirac morphisms are called *Dirac realizations* [14].

Example 1.13. — Let $E \subset \mathbb{V}$ be a Lagrangian subspace, and let ω_Q be the corresponding 2-form on $Q = \text{ran}(E)$. Let $\iota_Q: Q \rightarrow V$ be the inclusion. Then $(\iota_Q, \omega_Q): (Q, Q) \rightarrow (V, E)$ is a strong Dirac morphism. Equivalently $(\iota_Q, 0): (Q, \text{Gr}_{\omega_Q}) \rightarrow (V, E)$ is a strong Dirac morphism. Here $\mathfrak{a}(v \oplus \alpha) = \iota_Q(v)$.

Example 1.14. — Suppose $\pi \in \wedge^2 V$ and $\pi' \in \wedge^2 V'$. Then $(\Phi, 0): (V, \text{Gr}_{\pi}) \rightarrow (V', \text{Gr}_{\pi'})$ is a Dirac morphism if and only if $\Phi(\pi) = \pi'$. It is automatically strong (since $\ker(\text{Gr}_{\pi}) = 0$), with $\mathfrak{a}(v' \oplus \alpha') = \pi^\sharp(\Phi^* \alpha')$.

Proposition 1.15. — Suppose $(\Phi, \omega): (V, E) \rightarrow (V', E')$ is a Dirac morphism, and that F' is a Lagrangian subspace transverse to E' . Let ϕ be a pure spinor defining E , and ψ' a pure spinor defining F' . Then $\psi := e^\omega \Phi^* \psi'$ is non-zero, and is a pure spinor defining the backward image $F = F' \circ \Gamma_{(\Phi, \omega)}$. Moreover, the following are equivalent:

- (Φ, ω) is a strong Dirac morphism,
- the backward image F is transverse to E ,
- The pairing $(\phi, \psi)_{\wedge(V^*)} \in \det(V^*)$ is non-zero, that is, it is a volume form on V .

Proof. — By (6), we may write $\psi' = e^{-\omega_{Q'}} \theta'$, where $\omega_{Q'}$ is a 2-form on $Q' = \text{ran}(F')$, and $\theta' \in \wedge^{\text{top}}(V'/\text{ran}(F'))^*$. Identifying $(V'/\text{ran}(F'))^*$ with the annihilator of $\text{ran}(F')$, this gives

$$\begin{aligned} \psi \neq 0 &\Leftrightarrow \Phi^* \theta' \neq 0 \\ &\Leftrightarrow \ker(\Phi^*) \cap \text{ann}(\text{ran}(F')) = 0 \\ &\Leftrightarrow \{w' \in F' \mid 0 \sim_{(\Phi, \omega)} w'\} = \{0\}. \end{aligned}$$

(Indeed, $0 \sim_{(\Phi, \omega)} w'$ if and only if $w' = 0 \oplus \alpha'$ with $\Phi^* \alpha' = \{0\}$. Moreover $w' \in F' = (F')^\perp$ if and only if $\alpha' \in \text{ann}(\text{ran}(F'))$.) But the condition $0 \sim_{(\Phi, \omega)} w'$ implies that $w' \in E'$. Since $E' \cap F' = 0$ it follows that $\{w' \in F' \mid 0 \sim_{(\Phi, \omega)} w'\} = \{0\}$, hence $\psi \neq 0$. Lemma 1.9 shows that it is a pure spinor defining the backward image F .

(a) $\Leftrightarrow$ (b). By definition, $E \cap F$ consists of all $w \in E$ such that $w \sim_{(\Phi, \omega)} w'$ for some $w' \in F'$. Since $E' = \Gamma_{(\Phi, \omega)} \circ E$, this element w' also lies in E' , and hence $w' = 0$. Thus,

$$E \cap F = E \cap \ker(\Phi, \omega),$$

which is zero precisely if the Dirac morphism (Φ, ω) is strong. (b) $\Leftrightarrow$ (c) is immediate from Theorem 1.3. $\square$

1.8. Lagrangian splittings. — Suppose W is a vector space with split bilinear form. By a *Lagrangian splitting* of W we mean a direct sum decomposition $W = E \oplus F$ into transverse Lagrangian subspaces.

Lemma 1.16. — Let W be a vector space with split bilinear form $\langle \cdot, \cdot \rangle$. There is a 1-1 correspondence between projection operators $\mathfrak{p} \in \text{End}(W)$ with the property $\mathfrak{p} + \mathfrak{p}^t = 1$, and Lagrangian splittings $W = E \oplus F$. (Here $\mathfrak{p}^t$ is the transpose with respect to the inner product on W .)

Proof. — A Lagrangian splitting of W into transverse Lagrangian subspaces is equivalent to a projection operator whose kernel and range are isotropic. For any projection operator $\mathfrak{p} = \mathfrak{p}^2$, the range $\text{ran}(\mathfrak{p})$ is isotropic if and only if $\mathfrak{p}^t \mathfrak{p} = 0$, while $\ker(\mathfrak{p}) = \text{ran}(1 - \mathfrak{p})$ is isotropic if and only if $(1 - \mathfrak{p})^t(1 - \mathfrak{p}) = 0$. If both the kernel and the range of $\mathfrak{p}$ are isotropic, then

$$1 - (\mathfrak{p} + \mathfrak{p}^t) = (1 - \mathfrak{p})^t(1 - \mathfrak{p}) - \mathfrak{p}^t \mathfrak{p} = 0.$$

Conversely, if $\mathfrak{p}$ is a projection operator with $\mathfrak{p} + \mathfrak{p}^t = 1$, then $\mathfrak{p}^t \mathfrak{p} = (1 - \mathfrak{p})\mathfrak{p} = 0$, and similarly $(1 - \mathfrak{p})^t(1 - \mathfrak{p}) = 0$. $\square$

Again, we specialize to the case $W = \mathbb{V}$. Suppose $\mathbb{V} = E \oplus F$ is a Lagrangian splitting, with associated projection operator $\mathfrak{p}$. The property $\mathfrak{p} + \mathfrak{p}^t = 1$ implies that there is a bivector $\pi \in \wedge^2 V$ defined by

$$(17) \quad \pi^\sharp(\alpha) = -\text{pr}_V(\mathfrak{p}(\alpha)), \quad \alpha \in V^*,$$

that is, $\pi(\alpha, \beta) = -\langle \mathfrak{p}(\alpha), \beta \rangle = \langle \alpha, \mathfrak{p}(\beta) \rangle$, $\alpha, \beta \in V^*$. If $\{e_i\}$ is a basis of E and $\{f^i\}$ is the dual basis of F , then

$$(18) \quad \pi = \frac{1}{2} \text{pr}_V(e_i) \wedge \text{pr}_V(f^i).$$

The graph of the bivector π was encountered in Example 1.10 above:

Proposition 1.17. — *The graph of the bivector π is given by*

$$(19) \quad \text{Gr}_\pi = E^\top \wedge F.$$

In particular, $\text{ran}(\pi^\sharp) = \text{ran}(E) \cap \text{ran}(F)$, and the symplectic 2-form on $\text{ran}(\pi^\sharp)$ is the difference of the restrictions of the 2-forms on $\text{ran}(E), \text{ran}(F)$. If ϕ, ψ are pure spinors defining E, F , then

$$\phi^\top \wedge \psi = e^{-\iota(\pi)}(\phi^\top \wedge \psi)^{[\text{top}]}$$

Proof. — Since both sides of (19) are Lagrangian subspaces, it suffices to prove the inclusion $\supset$. Let $v \oplus \alpha \in E^\top \wedge F$. Hence, there exist α_1, α_2 with $\alpha = \alpha_2 - \alpha_1$ and $v \oplus \alpha_1 \in E$, $v \oplus \alpha_2 \in F$. Thus $v \oplus \alpha_1 = -\mathfrak{p}(\alpha)$, which implies that $\pi^\sharp(\alpha) = -\text{pr}_V \mathfrak{p}(\alpha) = v$. The description of $\text{ran} \pi^\sharp = \text{ran}(\text{Gr}_\pi)$ is immediate from (19), see the discussion in Example 1.10. The formula for $\phi^\top \wedge \psi$ follows since both sides are pure spinors defining the Lagrangian subspace Gr_π , with the same top degree part. $\square$

Proposition 1.18. — *Suppose $\mathbb{V} = E \oplus F$ is a Lagrangian splitting, defining a bivector π . If $\varepsilon \in \wedge^2 E$, so that $F_\varepsilon = A^{-\varepsilon} F$ is a new Lagrangian complement to E , the bivector π_ε for the splitting $E \oplus F_\varepsilon$ is given by*

$$\pi_\varepsilon = \pi + \text{pr}_V(\varepsilon),$$

where $\text{pr}_V: \wedge E \rightarrow \wedge V$ is the algebra homomorphism extending the projection to V .

Proof. — Let ϕ, ψ be pure spinors defining E, F . Then F_ε is defined by the pure spinor $\psi_\varepsilon = \varrho(e^{-\varepsilon})\psi$. Using Remark 1.5(a), we obtain

$$\phi^\top \wedge \psi_\varepsilon = \phi^\top \wedge \varrho(e^{-\varepsilon})\psi = e^{-\iota(\mathrm{pr}_V(\varepsilon))}\phi^\top \wedge \psi.$$

The claim now follows from (1.17). $\square$

Proposition 1.19. — *Let $(\Phi, \omega): (V, E) \rightarrow (V', E')$ be a strong Dirac morphism. Suppose $F' \in \mathrm{Lag}(\mathbb{V}')$ is transverse to E' , and F is its backward image under (Φ, ω) . Then the bivectors for the Lagrangian splittings $\mathbb{V} = E \oplus F$ and $\mathbb{V}' = E' \oplus F'$ are Φ -related:*

$$\Phi(\pi) = \pi'.$$

Proof. — To prove $\Phi(\pi) = \pi'$, we have to show that $(\Phi, 0): (V, \mathrm{Gr}_\pi) \rightarrow (V', \mathrm{Gr}_{\pi'})$ is a Dirac morphism:

$$\Gamma_{(\Phi, 0)} \circ (E^\top \wedge F) = (E')^\top \wedge F'.$$

Since both sides are Lagrangian, it suffices to prove the inclusion $\supset$. If $v' \oplus \alpha' \in (E')^\top \wedge F'$, then $\alpha' = \alpha'_2 - \alpha'_1$, where $v' \oplus \alpha'_1 \in E'$ and $v' \oplus \alpha'_2 \in F'$. Since (Φ, ω) is a strong Dirac morphism for E, E' , there is a unique element $v \oplus \alpha_1 \in E$ such that $v' = \Phi(v)$, $\Phi^*(\alpha'_1) = \alpha_1 + \iota_v \omega$. Let $\alpha_2 = \Phi^*(\alpha'_2) - \iota_v \omega$. Then $v \oplus \alpha_2 \in F$ since $v \oplus \alpha_2 \sim_{(\Phi, \omega)} v' \oplus \alpha_2$. Hence $v \oplus \Phi^*(\alpha') = v \oplus (\alpha_2 - \alpha_1) \in E^\top \wedge F$, proving that $v' \oplus \alpha' \in \Gamma_{(\Phi, 0)} \circ (E^\top \wedge F)$. $\square$

We next explain how a splitting $\mathbb{V}' = E' \oplus F'$ may be ‘pulled back’ under a linear map $\Phi: V \rightarrow V'$, given a bivector $\pi \in \wedge^2 V$ and a linear map $\mathfrak{a}: E' \rightarrow V$ satisfying suitable compatibility relations.

Theorem 1.20. — *Suppose that $\Phi: V \rightarrow V'$ is a linear map and $\omega \in \wedge^2 V^*$ a 2-form. Given a Lagrangian splitting $\mathbb{V}' = E' \oplus F'$, with associated projection $\mathfrak{p}' \in \mathrm{End}(\mathbb{V}')$, there is a 1-1 correspondence between*

- (i) *Lagrangian subspaces $E \subset \mathbb{V}$ such that $(\Phi, \omega): (V, E) \rightarrow (V', E')$ is a strong Dirac morphism, and*
- (ii) *Bivectors $\pi \in \wedge^2 V$ together with linear maps $\mathfrak{a}: E' \rightarrow V$, satisfying $\Phi \circ \mathfrak{a} = \mathrm{pr}_{V'}|_{E'}$ and*

$$(20) \quad \pi^\sharp \circ \Phi^* = -\mathfrak{a} \circ \mathfrak{p}'|_{(V')^*}.$$

Under this correspondence, π is the bivector defined by the splitting $\mathbb{V} = E \oplus F$, where F is the backward image of F' , and $\mathfrak{a}$ is the linear map defined by the strong Dirac morphism (Φ, ω) (see (15)).

Proof. — “(i) $\Rightarrow$ (ii)”. By Proposition 1.15, we know that the backward image F of F' is transverse to E . Let $\mathfrak{p}$ and $\mathfrak{p}'$ be the projections defined by the Lagrangian splittings $\mathbb{V} = E \oplus F$ and $\mathbb{V}' = E' \oplus F'$, and π, π' the corresponding bivectors. As in (15), the strong Dirac morphism (Φ, ω) defines a linear map $\hat{\mathfrak{a}}: E' \rightarrow E$, taking

$w' \in E'$ to the unique element $w \in E$ such that $w \sim_{(\Phi, \omega)} w'$. We claim that for all $w \in \mathbb{V}$, $w' \in V'$,

$$(21) \quad w \sim_{(\Phi, \omega)} w' \Rightarrow \mathfrak{p}(w) = \widehat{\mathfrak{a}}(\mathfrak{p}'(w')).$$

Indeed, let $w_1 = \mathfrak{p}(w) \in E$, so that $w_2 = w - w_1 \in F$. There is a (unique) element $w'_2 \in F'$ with $w_2 \sim_{(\Phi, \omega)} w'_2$, so let $w'_1 = w' - w'_2$. Since $w_2 \sim_{(\Phi, \omega)} w'_2$, it follows that $w_1 \sim_{(\Phi, \omega)} w'_1$. Hence $w'_1 \in E'$ by definition of E' . It follows that $\mathfrak{p}(w) = w_1 = \widehat{\mathfrak{a}}(w'_1) = \widehat{\mathfrak{a}}(\mathfrak{p}'(w'))$, as claimed. In particular, since $\Phi^* \alpha' \sim_{(\Phi, \omega)} \alpha'$ for $\alpha' \in V'$, (21) implies that

$$\pi^\#(\Phi^* \alpha') = -\text{pr}_V(\mathfrak{p}(\Phi^* \alpha')) = -\text{pr}_V(\widehat{\mathfrak{a}}(\mathfrak{p}'(\alpha'))) = -\mathfrak{a}(\mathfrak{p}'(\alpha')), \quad \alpha' \in (V')^*$$

where $\mathfrak{a} = \text{pr}_V \circ \widehat{\mathfrak{a}}$.

“(i) $\Leftarrow$ (ii)”. Our aim is to construct the projection $\mathfrak{p}$ with kernel $F := F' \circ \Gamma_{(\Phi, \omega)}$ and range E . We define $\mathfrak{p}$ by the following equations, for $v, v_1, v_2 \in V$ and $\alpha, \alpha_1, \alpha_2 \in V^*$:

$$\begin{aligned} \langle \mathfrak{p}(v_1), v_2 \rangle &= \langle \mathfrak{p}'(\Phi(v_1)), \Phi(v_2) \rangle, \\ \langle \mathfrak{p}(\alpha_1), \alpha_2 \rangle &= -\pi(\alpha_1, \alpha_2), \\ \langle \mathfrak{p}(v), \alpha \rangle &= \langle \mathfrak{a}^* \alpha, \Phi(v) \rangle + \pi(\iota_v \omega, \alpha), \\ \langle \mathfrak{p}(\alpha), v \rangle &= \langle \alpha, v \rangle - \langle \mathfrak{a}^* \alpha, \Phi(v) \rangle - \pi(\iota_v \omega, \alpha), \end{aligned}$$

where $\mathfrak{a}^*: V^* \rightarrow (E')^* = F'$ is the dual map to $\mathfrak{a}$. The linear map $\mathfrak{p}$ defined in this way has the property $\mathfrak{p} + \mathfrak{p}^t = 1$. We claim that this linear map satisfies (21), where $\widehat{\mathfrak{a}}: E' \rightarrow \mathbb{V}$ is defined as follows,

$$\widehat{\mathfrak{a}}(w') = \mathfrak{a}(w') \oplus (\Phi^* \text{pr}_{(V')^*}(w') - \iota_{\mathfrak{a}(w')} \omega).$$

For $w = v \oplus \iota_v \omega$, $w' = \Phi(v) \oplus 0$, (21) is easily checked using the definition of $\mathfrak{p}$. Hence it suffices to consider the case $w = \Phi^* \alpha'$, $w' = \alpha'$ with $\alpha' \in (V')^*$. For all $v \in V$, using the definition of $\mathfrak{p}$ and $\Phi \circ \mathfrak{a} = \text{pr}_{V'}|_{E'}$, i.e., $\mathfrak{a}^* \circ \Phi^* = (\mathfrak{p}')^t|_{(V')^*}$, we have:

$$\begin{aligned} \langle \mathfrak{p}(\Phi^* \alpha'), v \rangle &= \langle \alpha', \Phi(v) \rangle - \langle (\mathfrak{p}')^t \alpha', \Phi(v) \rangle - \pi(\iota_v \omega, \Phi^* \alpha') \\ &= \langle \mathfrak{p}' \alpha', \Phi(v) \rangle + \pi(\Phi^* \alpha', \iota_v \omega) \\ \langle \widehat{\mathfrak{a}}(\mathfrak{p}'(\alpha')), v \rangle &= \langle \Phi^* \text{pr}_{(V')^*} \mathfrak{p}'(\alpha'), v \rangle - \omega(\mathfrak{a}(\mathfrak{p}'(\alpha')), v) \\ &= \langle \mathfrak{p}' \alpha', \Phi(v) \rangle + \omega(\pi^\#(\Phi^* \alpha'), v) \end{aligned}$$

which shows $\langle \mathfrak{p}(\Phi^* \alpha'), v \rangle = \langle \widehat{\mathfrak{a}}(\mathfrak{p}'(\alpha')), v \rangle$. Similarly, for $\beta \in V^*$ we have, by (20),

$$\langle \mathfrak{p}(\Phi^* \alpha'), \beta \rangle = -\langle \pi^\#(\Phi^* \alpha'), \beta \rangle = \langle \widehat{\mathfrak{a}}(\mathfrak{p}'(\alpha')), \beta \rangle.$$

This proves (21). Equation (21) applies in particular to all elements $w \in F$, since these are by definition (Φ, ω) -related to elements $w' \in F'$. We hence see that $\mathfrak{p}(w) = 0$ for all $w \in F$. This proves that $F \subset \ker(\mathfrak{p})$. Taking orthogonals, $\text{ran}(\mathfrak{p}^t) \subset F$. In particular, the range of $\mathfrak{p}^t$ is isotropic, i.e. $\mathfrak{p}\mathfrak{p}^t = 0$, and hence $\mathfrak{p} - \mathfrak{p}^2 = \mathfrak{p}(1 - \mathfrak{p}) = \mathfrak{p}\mathfrak{p}^t = 0$. Thus $\mathfrak{p}$ is a projection. As before, we see that $\ker \mathfrak{p} = \text{ran}(1 - \mathfrak{p})$ is isotropic as well, hence $F = \ker(\mathfrak{p})$ since F is maximal isotropic. It remains to show that the Lagrangian subspace $E := \text{ran}(\mathfrak{p})$ satisfies $\Gamma_{(\Phi, \omega)} \circ E \subset E'$. Suppose $w \sim_{(\Phi, \omega)} w'$ for some $w \in E$.

By (21), we also have $w = p(w) \sim_{(\Phi, \omega)} p'(w')$. Thus $0 \sim_{(\Phi, \omega)} (w' - p'(w')) = (p')^t(w')$. Observe that $\text{ran}(\Phi) \supset \Phi(\mathfrak{a}(E')) = \text{ran}(E')$. Hence $\ker(\Phi^*) \subset \text{ann}(\text{ran}(E'))$. Since $E' \cap F' = 0$, it follows that

$$(22) \quad \ker(\Phi^*) \cap \text{ann}(\text{ran}(F')) = 0.$$

Using Equation (22), the relation $0 \sim_{(\Phi, \omega)} (p')^t(w') \in F'$ implies that $(p')^t(w') = 0$, i.e. $w' \in E'$. $\square$

The proof shows that $p|_V = \widehat{\mathfrak{a}} \circ p' \circ \Phi$, whereas $h := p|_{V^*} : V^* \rightarrow E$ is given by

$$(23) \quad h(\alpha) = (-\pi^\sharp(\alpha)) \oplus (\alpha - \Phi^* \text{pr}_{(V')^*} \mathfrak{a}^*(\alpha) - \iota(\pi^\sharp(\alpha))\omega).$$

It follows that $E = \text{ran}(\widehat{\mathfrak{a}}) + \text{ran}(h)$. Projecting to V , it follows in particular that

$$(24) \quad \text{ran}(E) = \text{ran}(\mathfrak{a}) + \text{ran}(\pi^\sharp).$$

2. Pure spinors on manifolds

A pure spinor on a manifold is simply a differential form whose restriction to any point is a pure spinor on the tangent space. The following discussion is carried out in the category of real manifolds and C^∞ vector bundles, but works equally well for complex manifolds with holomorphic vector bundles.

2.1. Dirac structures. — For any manifold M , we denote by $\mathbb{T}M = TM \oplus T^*M$ the direct sum of the tangent and cotangent bundles, with fiberwise inner product $\langle \cdot, \cdot \rangle$. The fiberwise Clifford action defines a bundle map

$$(25) \quad \varrho : \text{Cl}(\mathbb{T}M) \rightarrow \text{End}(\wedge T^*M).$$

The same symbol will denote the action of sections of $\text{Cl}(\mathbb{T}M)$ on sections of $\wedge T^*M$, i.e. differential forms. The bilinear pairing will be denoted by

$$(26) \quad (\cdot, \cdot)_{\wedge T^*M} : \wedge T^*M \otimes \wedge T^*M \rightarrow \det(T^*M),$$

and the same notation will be used for sections. Thus $(\phi, \phi')_{\wedge T^*M} = (\phi^\top \wedge \phi')^{\text{[top]}}$ for differential forms $\phi, \phi' \in \Gamma(\wedge T^*M) = \Omega(M)$. An *almost Dirac structure* on M is a smooth Lagrangian subbundle $E \subset \mathbb{T}M$. The pair (M, E) is called an *almost Dirac manifold*. A *pure spinor defining* E is a nonvanishing differential form $\phi \in \Omega(M)$ such that $\phi|_m$ is a pure spinor defining E_m , for all m . Equivalently, ϕ is a nonvanishing section of the line bundle $(\wedge T^*M)^E$. Thus E is globally represented by a pure spinor if and only if the line bundle $(\wedge T^*M)^E$ is orientable. (Otherwise, one may still use pure spinors to describe E locally.)

Let $\eta \in \Omega^3(M)$ be a closed 3-form. A direct computation shows that the spinor representation defines a bilinear bracket $[\![\cdot, \cdot]\!]_\eta : \Gamma(\mathbb{T}M) \times \Gamma(\mathbb{T}M) \rightarrow \Gamma(\mathbb{T}M)$ by the condition:

$$(27) \quad \varrho([\![x_1, x_2]\!]_\eta)\psi = [[d + \eta, \varrho(x_1)], \varrho(x_2)]\psi, \quad \psi \in \Omega(M), \quad x_i \in \Gamma(\mathbb{T}M),$$

where the brackets on the right-hand side are graded commutators of operators on $\Omega(M)$. The bracket $[\![\cdot, \cdot]\!]_\eta$ is the η -twisted Courant bracket [35, 50].⁽³⁾ (For more on the definition of $[\![\cdot, \cdot]\!]_\eta$ as a ‘derived bracket’, see e.g. [9, 36, 48].) The operator on $\Omega(M)$ defined by

$$[\varrho(x_1), [\varrho(x_2), [\varrho(x_3), d + \eta]]]$$

is multiplication by a function

$$(28) \quad \Upsilon(x_1, x_2, x_3) = -\langle [\![x_3, x_2]\!]_\eta, x_1 \rangle \in C^\infty(M).$$

Given an almost Dirac structure $E \subset \mathbb{T}M$, let Υ^E denote the restriction of the trilinear form $(x_1, x_2, x_3) \mapsto \Upsilon(x_1, x_2, x_3)$ to the sections of E . In contrast to Υ , the trilinear form Υ^E is *tensorial* and *skew-symmetric*. The resulting element

$$\Upsilon^E \in \Gamma(\wedge^3 E^*)$$

is called the η -twisted Courant tensor of E .

Definition 2.1. — A *Dirac structure* on a manifold M is an almost Dirac structure E together with a closed 3-form η such that its η -twisted Courant tensor vanishes: $\Upsilon^E = 0$. The triple (M, E, η) is called a *Dirac manifold*.

For E an almost Dirac structure one can always choose a complementary almost Dirac structure F such that $E \oplus F = \mathbb{T}M$. (This is parallel to a well-known fact from symplectic geometry [51, Proposition 8.2], with a similar proof.) As a vector bundle, $F \cong E^*$ with pairing induced by the inner product on $\mathbb{T}M$. We have:

Proposition 2.2. — Let E be an almost Dirac structure on M , and F be a complementary almost Dirac structure. Suppose E is represented by a pure spinor $\phi \in \Omega(M)$. Then there is a unique section $\sigma^E \in \Gamma(E^*)$ (depending on ϕ) such that

$$(d + \eta)\phi = \varrho(-\Upsilon^E + \sigma^E)\phi.$$

Here we view Υ^E and σ^E as sections of $\wedge F \subset \text{Cl}(\mathbb{T}M)$.

Proof. — Choose a Lagrangian subbundle F complementary to E . Since

$$\Gamma(\wedge F) \rightarrow \Omega(M), \quad x \mapsto \varrho(x)\phi$$

is an isomorphism, there is a unique odd element $x \in \Gamma(\wedge F) \subset \Gamma(\mathbb{T}M)$ such that $(d + \eta)\phi = \varrho(x)\phi$. To see that x has filtration degree 3, let x_1, x_2, x_3 be three sections of E . Since $\varrho(x_i)\phi = 0$, it follows that

$$\begin{aligned} \varrho([x_1, [x_2, [x_3, x]]])\phi &= [[[\varrho(x_1), [\varrho(x_2), [\varrho(x_3), \varrho(x)]]]]\phi = \varrho(x_1 x_2 x_3) \varrho(x)\phi \\ &= \varrho(x_1 x_2 x_3) (d + \eta)\phi = [[[\varrho(x_1), [\varrho(x_2), [\varrho(x_3), d + \eta]]]]\phi = \Upsilon^E(x_1, x_2, x_3)\phi, \end{aligned}$$

⁽³⁾ This definition agrees with the non-skew symmetric version of the Courant bracket [40, 50], called the *Dorfman bracket* in [28]; the η -term in the bracket, however, differs from the one in [50] by a sign.

proving that the Clifford commutator $[x_1, [x_2, [x_3, x]]] = \iota(x_1)\iota(x_2)\iota(x_3)x$ (contraction of $x \in \Gamma(\wedge(E^*))$ with sections of E) is a scalar. This implies that x has filtration degree 3, and that the degree 3 part of x is $-\Upsilon^E$. $\square$

We hence see that an almost Dirac structure $E \subset TM$ is integrable if and only if

$$(d + \eta)\phi \in \varrho(TM)\phi,$$

for any pure spinor $\phi \in \Omega(M)$ (locally) representing E . The characterization of the integrability condition $\Upsilon^E = 0$ in terms of pure spinors was observed by Gualtieri [28], see also [9].

Examples of Dirac structures (for a given η) include graphs of 2-forms $\omega \in \Omega(M)$ with $d\omega = \eta$, as well as graphs of bivector fields $\pi \in \mathfrak{X}^2(M)$ defining η -twisted Poisson structures [35, 50] in the sense that $\frac{1}{2}[\pi, \pi]_{\text{Sch}} + \pi^\sharp(\eta) = 0$. One may also consider complex Dirac structures on M , given by complex Lagrangian subbundles $E \subset TM^{\mathbb{C}}$ satisfying $\Upsilon^E = 0$. The defining pure spinors are complex-valued differential forms ϕ on M , given as nonvanishing sections of $(\wedge T^*M^{\mathbb{C}})^E$. If E is a Dirac structure, then its image E^c under the complex conjugation mapping is a Dirac structure defined by the complex conjugate spinor ϕ^c . E is called a *generalized complex structure* [28, 32] if $E \cap E^c = 0$.

Suppose $E \subset TM$ is a Dirac structure. The vanishing of the Courant tensor implies that E is a *Lie algebroid*, with anchor given by the natural projection on TM , and Lie bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$ given by the restriction of the Courant bracket $[\![\cdot, \cdot]\!]_\eta$. From the theory of Lie algebroids, it follows that the generalized distribution $\text{ran}(E)$ is integrable (in the sense of Sussmann) [24]. The generalized foliation having $\text{ran}(E)$ as its tangent distribution is called the *Dirac foliation*. For any leaf $Q \subset M$ of the Dirac foliation, the collection of 2-forms on $T_m Q$ (defined as in (14)) defines a smooth 2-form $\omega_Q \in \Omega^2(Q)$ with

$$d\omega_Q = i_Q^* \eta,$$

where $i_Q: Q \rightarrow M$ is the inclusion (for a proof, see e.g. [47, Proposition 6.10]). If E is the graph of a Poisson bivector π (with $\eta = 0$), this is the usual symplectic foliation.

2.2. Dirac morphisms. — Suppose $\Phi: M \rightarrow M'$ is a smooth map, and $\omega \in \Omega^2(M)$ is a 2-form. As in the linear case, we view the pair (Φ, ω) as a ‘morphism’, with composition rule (10). Given sections $x \in \Gamma(TM)$ and $x' \in \Gamma(TM')$, we will write

$$x \sim_{(\Phi, \omega)} x' \Leftrightarrow \forall m \in M: x_m \sim_{((d\Phi)_m, \omega_m)} x'_{\Phi(m)}.$$

In terms of the spinor representation, this is equivalent to the condition

$$e^\omega \Phi^*(\varrho(x')\psi') = \varrho(x)(e^\omega \Phi^*(\psi')), \quad \psi' \in \Omega(M').$$

Using the definition (27) of the Courant bracket as a derived bracket, one obtains:

Lemma 2.3 (Stienon-Xu). — [53, Lemma 2.2] *Let M, M' be manifolds with closed 3-forms η, η' , $\Phi: M \rightarrow M'$ a smooth map, and $\omega \in \Omega^2(M)$ a 2-form such that $\Phi^*\eta' =$*

$\eta + d\omega$. Then

$$x_i \sim_{(\Phi, \omega)} x'_i, \quad i = 1, 2 \Rightarrow \llbracket x_1, x_2 \rrbracket_\eta \sim_{(\Phi, \omega)} \llbracket x'_1, x'_2 \rrbracket_{\eta'}.$$

That is, the morphism $(\Phi, \omega): M \rightarrow M'$ intertwines both the inner product and the (η - resp. η' -twisted) Courant brackets on $\mathbb{T}M$ and $\mathbb{T}M'$.

Definition 2.4. — a. Suppose (M, E) and (M', E') are almost Dirac manifolds. A morphism $(\Phi, \omega): M \rightarrow M'$ is called a *(strong) almost Dirac morphism* $(\Phi, \omega): (M, E) \rightarrow (M', E')$ if $((d\Phi)_m, \omega_m): (T_m M, E_m) \rightarrow (T_{\Phi(m)} M', E'_{\Phi(m)})$ is a linear (strong) Dirac morphism for all $m \in M$.
b. Suppose (M, E, η) and (M', E', η') are Dirac manifolds. A (strong) almost Dirac morphism $(\Phi, \omega): M \rightarrow M'$ is called a *(strong) Dirac morphism* $(\Phi, \omega): (M, E, \eta) \rightarrow (M', E', \eta')$ if $\eta + d\omega = \Phi^* \eta'$.

For $\omega = 0$, strong Dirac morphisms coincide with the *Dirac realizations* of [14].

Example 2.5. — If (M, E, η) is a Dirac manifold, then so is $(M, A^{-\omega}(E), \eta + d\omega)$, for any 2-form ω , and (id_M, ω) is a Dirac morphism between the two. The Dirac structures E and $A^{-\omega}(E)$ are isomorphic as Lie algebroids; in particular, they define the same Dirac foliation. However, the 2-forms on the leaves of this foliation change by the pull-back of ω .

Example 2.6. — Any manifold M can be trivially viewed as a Dirac manifold $M = (M, \mathbb{T}M, 0)$. A strong Dirac morphism from M to pt is then the same thing as a symplectic 2-form on M . More generally, strong Dirac morphisms $M \rightarrow N$ are (special types of) symplectic fibrations.

Example 2.7. — If (M, E, η) is a Dirac manifold, and $Q \subset M$ is a leaf of the associated foliation of M , then the inclusion map defines a strong Dirac morphism $(\iota_Q, \omega_Q): (Q, \mathbb{T}Q, 0) \rightarrow (M, E, \eta)$.

From the linear case, it follows that a strong almost Dirac morphism gives rise to a bundle map

$$\hat{\mathbf{a}}: \Phi^* E' \rightarrow E.$$

This is indeed a smooth bundle map: the projection $\mathbb{T}M \oplus \Phi^* \mathbb{T}M' \rightarrow \Phi^* \mathbb{T}M'$ restricts to a bundle isomorphism $\Gamma_{\Phi} \cap (E \oplus \Phi^* \mathbb{T}M') \rightarrow \Phi^* E'$, and $\hat{\mathbf{a}}$ is the inverse of this bundle isomorphism followed by the projection to $\mathbb{T}M$. We let

$$(29) \quad \mathbf{a} = \text{pr}_{\mathbb{T}M} \circ \hat{\mathbf{a}}: \Phi^* E' \rightarrow \text{ran}(E) \subset \mathbb{T}M$$

Proposition 2.8. — Suppose $(\Phi, \omega): (M, E, \eta) \rightarrow (M', E', \eta')$ is a strong Dirac morphism. Then the induced bundle map $\hat{\mathbf{a}}: \Phi^* E' \rightarrow E$ is a comorphism of Lie algebroids [43]. That is, it is compatible with the anchor maps in the sense that

$$d\Phi \circ \mathbf{a} = \text{pr}_{\Phi^* \mathbb{T}M'}|_{\Phi^* E'},$$

and the induced map on sections

$$\widehat{\mathfrak{a}}: \Gamma(E') \rightarrow \Gamma(E), \quad (\widehat{\mathfrak{a}}(x'))_m = \widehat{\mathfrak{a}}(x'_{\Phi(m)})$$

preserves brackets.

Proof. — Compatibility with the anchor is obvious. If x'_1, x'_2 are sections of E' , then (using Lemma 2.3) both $\widehat{\mathfrak{a}}(\Phi^*[x'_1, x'_2]_{E'})$ and $[\widehat{\mathfrak{a}}(\Phi^*x'_1), \widehat{\mathfrak{a}}(\Phi^*x'_2)]_E$ are sections of E which are (Φ, ω) -related to $[x'_1, x'_2]_{E'}$. Hence their difference is (Φ, ω) -related to 0. Since (Φ, ω) is a strong Dirac morphism, it follows that the difference is in fact 0. $\square$

The second part of Proposition 2.8 shows that (29) defines a Lie algebra homomorphism $\mathfrak{a}: \Gamma(E') \rightarrow \mathfrak{X}(M)$. That is, the strong Dirac morphism defines an ‘action’ of the Lie algebroid E' on the manifold M .

2.3. Bivector fields. — From the linear theory, we see that any Lagrangian splitting $TM = E \oplus F$ defines a bivector field π on M . Furthermore,

$$e^{-\iota(\pi)}(\phi^\top \wedge \psi)^{[\text{top}]} = \phi^\top \wedge \psi$$

for any pure spinors ϕ, ψ defining E, F . Recall that $(\phi^\top \wedge \psi)^{[\text{top}]}$ is a volume form on M .

For an arbitrary volume form μ on M , and any bivector field $\pi \in \mathfrak{X}^2(M)$, one has the formula [26]

$$(30) \quad d(e^{-\iota(\pi)}\mu) = \iota\left(-\frac{1}{2}[\pi, \pi]_{\text{Sch}} + X_\pi\right)(e^{-\iota(\pi)}\mu).$$

Here $[\cdot, \cdot]_{\text{Sch}}$ is the Schouten bracket on multivector fields, and X_π is the vector field on M defined by $d\iota(\pi)\mu = -\iota(X_\pi)\mu$. If π is a Poisson bivector field, then $X_\pi \in \mathfrak{X}(M)$ is called the *modular vector field* of π with respect to the volume form μ [56]. (See [37] for modular vector fields for *twisted* Poisson structures.)

Theorem 2.9. — *Let π be the bivector field defined by the Lagrangian splitting $TM = E \oplus F$. Let $\Upsilon^E \in \Gamma(\wedge^3 F)$ and $\Upsilon^F \in \Gamma(\wedge^3 E)$ be the Courant tensor fields of E, F .*

a) *The Schouten bracket of π with itself is given by the formula*

$$\frac{1}{2}[\pi, \pi]_{\text{Sch}} = \text{pr}_{TM}(\Upsilon^E) + \text{pr}_{TM}(\Upsilon^F),$$

where $\text{pr}_{TM}: \wedge E \rightarrow \wedge TM$ is the algebra homomorphism extending the projection $E \rightarrow TM$, and similarly for $\text{pr}_{TM}: \wedge F \rightarrow \wedge TM$.

b) *Given pure spinors $\phi, \psi \in \Omega(M)$ defining E, F , let $\sigma^E \in \Gamma(F)$ and $\sigma^F \in \Gamma(E)$ be the unique sections such that*

$$(d + \eta)\phi = \varrho(-\Upsilon^E + \sigma^E)\phi, \quad (d + \eta)\psi = \varrho(-\Upsilon^F + \sigma^F)\psi.$$

Then the vector field X_π defined using the volume form $\mu = (\phi^\top \wedge \psi)^{[\text{top}]}$ is given by

$$X_\pi = \text{pr}_{TM}(\sigma^F) - \text{pr}_{TM}(\sigma^E).$$

Proof. — We may assume that E, F are globally defined by pure spinors ϕ, ψ . Using Remark 1.5(a), we have

$$\begin{aligned} d(\phi^\top \wedge \psi) &= (-1)^{|\phi|} (\phi^\top \wedge d\psi + (d\phi)^\top \wedge \psi) \\ &= (-1)^{|\phi|} (\phi^\top \wedge (d + \eta)\psi + ((d + \eta)\phi)^\top \wedge \psi) \\ &= (-1)^{|\phi|} (\phi^\top \wedge (\varrho(-\Upsilon^F + \sigma^F)\psi) + (\varrho(-\Upsilon^E + \sigma^E)\phi)^\top \wedge \psi) \\ &= \iota(\text{pr}_{TM}(-\Upsilon^F + \sigma^F) + \text{pr}_{TM}(-\Upsilon^E - \sigma^E))(\phi^\top \wedge \psi). \end{aligned}$$

On the other hand, $\phi^\top \wedge \psi = e^{-\iota(\pi)}\mu$ gives, by (30),

$$d(\phi^\top \wedge \psi) = \iota(-\tfrac{1}{2}[\pi, \pi]_{\text{Sch}} + X_\pi)(\phi^\top \wedge \psi).$$

Applying the star operator $\star$ for μ , and using that $\star(\phi^\top \wedge \psi)$ is invertible, it follows that

$$\text{pr}_{TM}(-\Upsilon^F + \sigma^F) + \text{pr}_{TM}(-\Upsilon^E - \sigma^E) = -\tfrac{1}{2}[\pi, \pi]_{\text{Sch}} + X_\pi. \quad \square$$

As a special case, if both E, F are Dirac structures (i.e. integrable), then the corresponding bivector field π satisfies $[\pi, \pi]_{\text{Sch}} = 0$, i.e., it is a Poisson structure. The symplectic leaves of π are the intersections of the leaves of the Dirac structures E with those of F . The fact that transverse Dirac structures (or equivalently *Lie bialgebroids*) define Poisson structures goes back to Mackenzie-Xu [44].

Proposition 2.10. — *Suppose $(\Phi, \omega): (M, E) \rightarrow (M', E')$ is an almost Dirac morphism, and let $F' \subset \mathbb{T}M'$ be a Lagrangian subbundle complementary to E' . Then there is a smooth Lagrangian subbundle $F \subset \mathbb{T}M$ complementary to E , with the property that for all $m \in M$, F_m is the backward image of $F'_{\Phi(m)}$ under $(d_m\Phi, \omega_m)$. Furthermore:*

- a. *The bivector fields π, π' defined by the splittings $\mathbb{T}M = E \oplus F$ and $\mathbb{T}M' = E' \oplus F'$ satisfy*

$$\pi \sim_\Phi \pi',$$

i.e. $(d\Phi)_m \pi_m = \pi'_{\Phi(m)}$ for all $m \in M$.

- b. *The Courant tensors $\Upsilon^F \in \Gamma(\wedge^3 E)$ and $\Upsilon^{F'} \in \Gamma(\wedge^3 E')$ are related by*

$$\Upsilon^F = \widehat{\mathbf{a}}(\Phi^* \Upsilon^{F'}),$$

using the extension of $\widehat{\mathbf{a}}: \Gamma(\Phi^ E') \rightarrow \Gamma(E)$ to the exterior algebras.*

- c. *The bivector field π satisfies*

$$\tfrac{1}{2}[\pi, \pi]_{\text{Sch}} = \mathbf{a}(\Phi^* \Upsilon^{F'}) + \text{pr}_{TM}(\Upsilon^E),$$

using the extension of $\mathbf{a}: \Gamma(\Phi^ E') \rightarrow \Gamma(TM)$ to the exterior algebras.*

- d.

$$\pi^\sharp \circ \Phi^* = -\mathbf{a} \circ \mathbf{p}': T^*M' \rightarrow TM,$$

where $\mathbf{p}': \mathbb{T}M' \rightarrow E'$ is the projection along F' .

- e. If ψ' is a pure spinor defining F' , and $\psi = e^\omega \Phi^* \psi'$ the corresponding pure spinor defining F , the sections $\sigma^F, \sigma^{F'}$ are related by $\sigma_F = \widehat{\alpha}(\Phi^* \sigma_{F'})$, that is,

$$\sigma^F \sim_{(\Phi, \omega)} \sigma^{F'}.$$

Proof. — Let $\psi' \in \Omega(M')$ be a pure spinor (locally) representing F' . From the linear case (Proposition 1.15), it follows that $\psi = e^\omega \Phi^* \psi'$ is non-zero everywhere, and is a pure spinor representing a Lagrangian subbundle $F \subset \mathbb{T}M$ transverse to E . Now (a) follows from the linear case, see Proposition 1.19. We next verify (b), at any given point $m \in M$. Let $m' = \Phi(m)$. Given $(x_i)_m \in F_m$ for $i = 1, 2, 3$, let $(x'_i)_{m'} \in F'_{m'}$ with

$$(x_i)_m \sim_{((d\Phi)_m, \omega_m)} (x'_i)_{m'}.$$

Choose sections $x_i \in \Gamma(F)$, $x'_i \in \Gamma(F')$ extending the given values at m, m' . We have to show $\Upsilon^F(x_1, x_2, x_3)|_m = \Upsilon^{F'}(x'_1, x'_2, x'_3)|_{m'}$. We calculate:

$$\Upsilon^F(x_1, x_2, x_3) \psi = \varrho(x_1 x_2 x_3) (d + \eta)(e^\omega \Phi^* \psi') = \varrho(x_1 x_2 x_3) e^\omega \Phi^* (d + \eta') \psi'$$

On the other hand,

$$(\Phi^* \Upsilon^{F'}(x'_1, x'_2, x'_3)) \psi = e^\omega \Phi^* \Upsilon^{F'}(x'_1, x'_2, x'_3) \psi' = e^\omega \Phi^* \varrho(x'_1 x'_2 x'_3) (d + \eta') \psi'$$

These two expressions coincide at m , proving (b). Theorem 2.9 together with (b) implies the statement (c). Part (d) follows from Proposition 1.20. Part (e) follows from (b) together with the definition of $\sigma^F, \sigma^{F'}$. $\square$

Part (b) shows in particular that if F' is a Dirac structure, transverse to E' , then its backward image is again a Dirac structure.

2.4. Dirac cohomology. — In this Section, we will discuss certain cohomology groups associated with any pair of transverse Dirac structures $E, F \subset \mathbb{T}M$ and a given volume form μ on M . We assume that E, F are given by pure spinors ϕ, ψ , normalized by the condition $(\phi, \psi)_{\wedge T^*M} = \mu$. Let $\sigma^E \in \Gamma(F), \sigma^F \in \Gamma(E)$ be sections defined as in Theorem 2.9, and denote

$$\sigma = \sigma^F - \sigma^E \in \Gamma(\mathbb{T}M).$$

Replacing ϕ, ψ with $\tilde{\phi} = f\phi, \tilde{\psi} = f^{-1}\psi$, for f a nonvanishing function on M , this section changes by a closed 1-form:

$$(31) \quad \tilde{\sigma} = \sigma - f^{-1}df.$$

Indeed, letting let $\mathbf{p}$ be the projection from $\mathbb{T}M$ to E along F we have $\tilde{\sigma}^F = \sigma^F - \mathbf{p}(f^{-1}df)$, $\tilde{\sigma}^E = \sigma^E + (I - \mathbf{p})(f^{-1}df)$.

We define the *Dirac cohomology groups* associated to a triple (E, F, μ) as the cohomology of the operators

$$\not\partial_+ = d + \eta + \varrho(\sigma), \quad \not\partial_- = d + \eta - \varrho(\sigma)$$

on $\Omega(M)$, restricted to the subspace on which they square to zero:

$$(32) \quad H_{\pm}(E, F, \mu) := \ker(\not\partial_{\pm}) / \ker(\not\partial_{\pm}) \cap \text{im}(\not\partial_{\pm}) \equiv H(\ker \not\partial_{\pm}^2, \not\partial_{\pm}).$$

The pure spinors ϕ, ψ define classes in $H_+(E, F, \mu)$ and $H_-(E, F, \mu)$, respectively, since $\tilde{\phi}_+ \phi = 0$ and $\tilde{\phi}_- \psi = 0$. The Dirac cohomology groups are independent of the choice of defining spinors ϕ, ψ : Changing the pure spinors by a function f as above, (31) shows that the operators $\tilde{\phi}_\pm$ change by conjugation, $\tilde{\phi}_+ = f \tilde{\phi}_+ f^{-1}$ and $\tilde{\phi}_- = f^{-1} \tilde{\phi}_- f$.

Example 2.11. — Let M be a manifold with volume form μ . Consider transverse Dirac structures $E = \text{Gr}_\omega$ for some closed 2-form ω , and $F = T^*M$. In this case, one can choose $\phi = e^{-\omega}, \psi = \mu$. We obtain $\eta = 0, \sigma = 0, \tilde{\phi}_\pm = d$, and the Dirac cohomology groups $H_\pm(TM, T^*M, \mu)$ coincide with the de Rham cohomology of M .

Example 2.12. — Let M be a manifold with volume form μ and with a Poisson bivector π . Let $E = TM, F = \text{Gr}_\pi$. The choice $\phi = 1, \psi = e^{-\iota(\pi)}\mu$ gives $\tilde{\phi}_- = d - \iota(X_\pi)$, where X_π is the modular vector field. The operator $\tilde{\phi}_-^2 = -\mathcal{L}(X_\pi)$ vanishes on differential forms invariant under the flow generated by X_π . The Dirac cohomology $H_-(TM, \text{Gr}_\pi, \mu) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi))$ resembles the Cartan model of equivariant cohomology for circle actions.

Let π be the Poisson structure defined by the splitting $\mathbb{T}M = E \oplus F$, and $X_\pi = \text{pr}_{TM} \sigma$ the modular vector field. Let

$$(33) \quad H_\pi(M) = H(\Omega(M)^{X_\pi}, d - \iota(X_\pi)).$$

By Remark 1.5(a) there is a pairing

$$H_+(E, F, \mu) \otimes H_-(E, F, \mu) \rightarrow H_\pi(M)$$

given on representatives by the formula $u \otimes v \mapsto u^\top \wedge v$. The pure spinors ϕ, ψ define cohomology classes $[\phi] \in H_+(E, F, \mu), [\psi] \in H_-(E, F, \mu)$, and $[\phi^\top \wedge \psi] \in H_\pi(M)$. If M is compact, the integration map $\int_M: \Omega(M)^{X_\pi} \rightarrow \mathbb{R}$ descends to $H_\pi(M)$. Hence

$$\int_M \phi^\top \wedge \psi = \int_M \mu > 0$$

shows that the cohomology classes $[\phi] \in H_+(E, F, \mu), [\psi] \in H_-(E, F, \mu)$ are both nonzero.

There is the following version of functoriality with respect to strong Dirac morphisms for Dirac cohomology.

Proposition 2.13. — Let $(\Phi, \omega): (M, E, \eta) \rightarrow (M', E', \eta')$ be a strong Dirac morphism, and let $F' \subset \mathbb{T}M'$ be a Dirac structure transverse to E' , with backward image F . Assume that E, E' are defined by pure spinors ϕ, ϕ' such that the corresponding sections σ^E and $\sigma^{E'}$ vanish. Let ψ' and $\psi = e^\omega \Phi^* \psi'$ be pure spinors defining F' and F , and let μ' and μ be the resulting volume forms. Then $e^\omega \circ \Phi^*$ intertwines $\tilde{\phi}_-$ and $\tilde{\phi}'_-$, and hence induces a map in Dirac cohomology $e^\omega \Phi^*: H_-(E', F', \mu') \rightarrow H_-(E, F, \mu)$ taking $[\psi']$ to $[\psi]$.

Proof. — Since $\sigma^E, \sigma^{E'}$ vanish we have $\sigma = \sigma^F$ and $\sigma' = \sigma^{F'}$. By Proposition 2.10 (e), the map $e^\omega \Phi^*$ intertwines the Clifford actions of σ^F and $\sigma^{F'}$, while on the other hand this map also intertwines $d + \eta$ with $d + \eta'$. Hence it intertwines $\not\partial_-$ with $\not\partial'_-$. $\square$

2.5. Classical dynamical Yang-Baxter equation. — The following result describes the Courant tensor of Lagrangian subbundles defined by elements in $\Gamma(\wedge^2 E)$.

Proposition 2.14 (Liu-Weinstein-Xu [40]). — *Let $\mathbb{T}M = E \oplus F$ be a splitting into Lagrangian subbundles, where both E, F are integrable relative to the closed 3-form η , and let us identify $F^* = E$. Given a section $\varepsilon \in \Gamma(\wedge^2 E)$, defining a section $A^{-\varepsilon} \in \Gamma(\mathcal{O}(\mathbb{T}M))$, let $F_\varepsilon = A^{-\varepsilon}(F)$ be the Lagrangian subbundle spanned by the sections $x + \iota_x \varepsilon$ for $x \in \Gamma(F) = \Gamma(E^*)$. Then the Courant tensor $\Upsilon_\varepsilon \in \Gamma(\wedge^3 E)$ of F_ε is given by the formula:*

$$\Upsilon_\varepsilon = d_F \varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_E.$$

Here $[\cdot, \cdot]_E$ is the Lie algebroid bracket of E , and $d_F: \Gamma(\wedge^\bullet F^*) \rightarrow \Gamma(\wedge^{\bullet+1} F^*)$ is the Lie algebroid differential of F .

Remark 2.15. — The result in [40] is stated only for $\eta = 0$. However, since the statement is local, one may use a gauge transformation by a local primitive of η to reduce to this case.

We are interested in the following special case: Let $M = \mathfrak{g}^*$, with its standard linear Poisson structure $\pi_{\mathfrak{g}^*} \in \Gamma(\wedge^2 T\mathfrak{g}^*) = C^\infty(\mathfrak{g}^*) \otimes \wedge^2 \mathfrak{g}^*$, and put $F = T\mathfrak{g}^*$ and $E = \text{Gr}_{\pi_{\mathfrak{g}^*}}$. The bundle E is spanned by sections $\mathcal{U}_0(\xi) \oplus \langle \theta_0, \xi \rangle$ for $\xi \in \mathfrak{g}$, where $\mathcal{U}_0(\xi)$ is the generating vector fields for the co-adjoint action, and $\langle \theta_0, \xi \rangle \in \Omega^1(\mathfrak{g}^*)$ is the ‘constant’ 1-form defined by ξ . The trivialization $E = \mathfrak{g}^* \times \mathfrak{g}$ defined by these sections identifies E with the action algebroid for the co-adjoint action: The bracket on $\Gamma(E) = C^\infty(\mathfrak{g}^*, \mathfrak{g})$ is defined by the Lie bracket on $\mathfrak{g}$ via the Leibniz rule, and the anchor map is given by the action map $\mathcal{U}_0: \mathfrak{g} \rightarrow T\mathfrak{g}^*$. For $\varepsilon \in \Gamma(\wedge^2 E)$, the bracket $[\varepsilon, \varepsilon]_E$ is given by the Schouten bracket on $\wedge \mathfrak{g}$. On the other hand we may view $\varepsilon \in C^\infty(\mathfrak{g}^*, \wedge^2 \mathfrak{g})$ as a 2-form on $\mathfrak{g}^*$, and then $d\varepsilon = d_F \varepsilon$ is just its exterior differential. The resulting equation reads

$$d\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_{\text{Sch}} = \Upsilon_\varepsilon.$$

If Υ_ε is a multiple of the structure constants tensor, this is a special case of the *classical dynamical Yang-Baxter equation* (CDYBE) [5, 25]. We will see below how a solution arises from the Cartan-Dirac structure on G .

For more information on the relation between Dirac structures and the CDYBE, see the work of Liu-Xu [41] and Bangoura-Kosmann-Schwarzbach [10].

3. Dirac structures on Lie groups

In this Section, we will study Dirac structures over Lie groups G with bi-invariant pseudo-Riemannian metrics. This will be based on the existence of a canonical isomorphism

$$\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$$

preserving scalar products and Courant brackets. In the subsequent section, we will describe a corresponding isomorphism of spinor modules.

3.1. The isomorphism $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$. — Let G be a Lie group (not necessarily connected), and let $\mathfrak{g}$ be its Lie algebra. We denote by $\xi^L, \xi^R \in \mathfrak{X}(G)$ the left-, right-invariant vector fields on G which are equal to $\xi \in \mathfrak{g} = T_e G$ at the group unit. Let $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ be the left-, right-Maurer-Cartan forms, i.e. $\iota(\xi^L)\theta^L = \iota(\xi^R)\theta^R = \xi$. They are related by $\theta_g^R = \text{Ad}_g(\theta_g^L)$, for all $g \in G$. The adjoint action of G on itself will be denoted $\mathcal{A}_{\text{ad}}$ (or simply $\mathcal{A}$, if there is no risk of confusion). The corresponding infinitesimal action is given by the vector fields

$$\mathcal{A}_{\text{ad}}(\xi) = \xi^L - \xi^R.$$

Suppose that the Lie algebra $\mathfrak{g}$ of G carries an *invariant inner product*. By this we mean an Ad -invariant, non-degenerate symmetric bilinear form B , not necessarily positive definite. Equivalently, B defines a bi-invariant pseudo-Riemannian metric on G . Given B , we can define the bi-invariant 3-form $\eta \in \Omega^3(G)$,

$$\eta := \frac{1}{12} B(\theta^L, [\theta^L, \theta^L])$$

Since η is bi-invariant, it is closed, and so it defines an η -twisted Courant bracket $[\cdot, \cdot]_\eta$ on G . The conjugation action $\mathcal{A}_{\text{ad}}$ extends to an action of $D = G \times G$ on G , by

$$(34) \quad \mathcal{A}: D \rightarrow \text{Diff}(G), \quad \mathcal{A}(a, a') = l_{a'} \circ r_{a^{-1}},$$

where $l_a(g) = ag$ and $r_a(g) = ga$. The corresponding infinitesimal action

$$\mathcal{A}: \mathfrak{d} \rightarrow \mathfrak{X}(G), \quad \mathcal{A}(\xi, \xi') = \xi^L - (\xi')^R$$

lifts to a map

$$(35) \quad \mathfrak{s}: \mathfrak{d} \rightarrow \Gamma(\mathbb{T}G), \quad \mathfrak{s}(\xi, \xi') = \mathfrak{s}^L(\xi) + \mathfrak{s}^R(\xi'),$$

where

$$\mathfrak{s}^L(\xi) = \xi^L \oplus \frac{1}{2} B(\theta^L, \xi), \quad \mathfrak{s}^R(\xi') = -(\xi')^R \oplus \frac{1}{2} B(\theta^R, \xi').$$

Let us equip $\mathfrak{d}$ with the bilinear form $B_{\mathfrak{d}}$ given by $+B$ on the first $\mathfrak{g}$ -summand and $-B$ on the second $\mathfrak{g}$ -summand. Thus $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is an example of a Lie algebra with invariant split bilinear form.

Proposition 3.1. — *The map $\mathfrak{s}: \mathfrak{d} \rightarrow \Gamma(\mathbb{T}G)$ is D -equivariant, and satisfies*

$$(36) \quad \langle \mathfrak{s}(\zeta_1), \mathfrak{s}(\zeta_2) \rangle = B_{\mathfrak{d}}(\zeta_1, \zeta_2), \quad [\mathfrak{s}(\zeta_1), \mathfrak{s}(\zeta_2)]_\eta = \mathfrak{s}([\zeta_1, \zeta_2])$$

for all $\zeta_1, \zeta_2 \in \mathfrak{d}$. Furthermore,

$$(37) \quad \Upsilon(\mathfrak{s}(\zeta_1), \mathfrak{s}(\zeta_2), \mathfrak{s}(\zeta_3)) = B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3])$$

for all $\zeta_i \in \mathfrak{d}$, where $\Upsilon: \Gamma(\mathbb{T}G)^{\otimes 3} \rightarrow C^\infty(G)$ was defined in (28).

Proof. — The D -equivariance of the map $\mathfrak{s}$ is clear. Let ϱ be the Clifford action of $\mathbb{T}G$ on $\wedge T^*G$. We have $[\varrho(\mathfrak{s}^L(\xi)), d + \eta] = \mathcal{L}(\xi^L)$ and $[\varrho(\mathfrak{s}^R(\xi)), d + \eta] = -\mathcal{L}(\xi^R)$, thus

$$[d + \eta, \varrho(\mathfrak{s}(\zeta))] = \mathcal{L}(\mathcal{U}(\zeta))$$

for all $\zeta \in \mathfrak{d}$. This proves the second Equation in (36), while the first Equation is obvious. Finally, (37) follows from (36) and the definition of Υ . Hence,

$$\varrho([\mathfrak{s}(\zeta_1), \mathfrak{s}(\zeta_2)]_\eta) = [[d + \eta, \varrho(\mathfrak{s}(\zeta_1))], \varrho(\mathfrak{s}(\zeta_2))] = \varrho(\mathfrak{s}([\zeta_1, \zeta_2])).$$

□

Put differently, the map $\mathfrak{s}$ defines a D -equivariant isometric isomorphism

$$(38) \quad \mathbb{T}G \cong G \times \mathfrak{d},$$

identifying the η -twisted Courant bracket on $\mathbb{T}G$ with the unique Courant bracket on $G \times \mathfrak{d}$ which agrees with the Lie bracket on $\mathfrak{d}$ on constant sections.

3.2. η -twisted Dirac structures on G . — Using (38), we see that any Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$ defines a Lagrangian subbundle

$$E^{\mathfrak{s}} \cong G \times \mathfrak{s},$$

spanned by the sections $\mathfrak{s}(\zeta)$ with $\zeta \in \mathfrak{s}$. The Lagrangian subbundle $E^{\mathfrak{s}}$ is invariant under the action of the subgroup of D preserving $\mathfrak{s}$. Let $\Upsilon^{\mathfrak{s}} \in \wedge^3 \mathfrak{s}^*$ be defined as

$$(39) \quad \Upsilon^{\mathfrak{s}}(\zeta_1, \zeta_2, \zeta_3) = B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3]), \quad \zeta_i \in \mathfrak{s}.$$

By (37), the Courant tensor $\Upsilon^{E^{\mathfrak{s}}}$ is just $\Upsilon^{\mathfrak{s}}$, using the sections $\mathfrak{s}$ to identify $(E^{\mathfrak{s}})^* \cong G \times \mathfrak{s}^*$. In particular, we see that $\mathfrak{s}$ defines a Dirac structure if and only if $\mathfrak{s}$ is a Lie subalgebra. To summarize:

Any Lagrangian subalgebra $\mathfrak{s} \subset \mathfrak{d}$ defines an η -twisted Dirac structure $E^{\mathfrak{s}}$.

The Dirac structure $E^{\mathfrak{s}}$ is invariant under the action of any Lie subgroup normalizing $\mathfrak{s}$, and in particular under the action of the subgroup $S \subset D$ integrating $\mathfrak{s}$. As a Lie algebroid, $E^{\mathfrak{s}}$ is just the action algebroid for this S -action. In particular, its leaves are just the components of the S -orbits on G . The 2-form on the orbit $\mathcal{O} = \mathcal{U}(S)g$ of an element $g \in G$ is the S -invariant form $\omega_{\mathcal{O}}$ given as follows: for $\zeta_i = (\xi_i, \xi'_i) \in \mathfrak{s}$,

$$(40) \quad \begin{aligned} \omega_{\mathcal{O}}(\mathcal{U}(\zeta_1), \mathcal{U}(\zeta_2))|_g &= \frac{1}{2} \langle B(\theta^L, \xi_1) + B(\theta^R, \xi'_1), \xi_2^L - (\xi'_2)^R \rangle \\ &= \frac{1}{2} B(\xi_2 - \text{Ad}_{g^{-1}} \xi'_2, \xi_1 + \text{Ad}_{g^{-1}} \xi'_1) \\ &= \frac{1}{2} (B(\text{Ad}_g \xi_2, \xi'_1) - B(\xi'_2, \text{Ad}_g \xi_1)), \end{aligned}$$

using $B(\xi_1, \xi_2) = B(\xi'_1, \xi'_2)$ since $\mathfrak{s}$ is Lagrangian. By the general theory from Section 2.1, these 2-forms satisfy $d\omega_{\mathcal{O}} = \iota_{\mathcal{O}}^* \eta$, where $\iota_{\mathcal{O}}: \mathcal{O} \rightarrow G$ is the inclusion. The kernel

of ω_θ equals $\ker(E^s)$, i.e. it is spanned by all $\mathcal{U}(\zeta)$ such that the T^*G -component of $s(\zeta)$ is zero:

$$(41) \quad \ker(\omega_\theta|_g) = \{\mathcal{U}(\zeta)|_g \mid \zeta = (\xi, \xi') \in \mathfrak{s}, \text{ Ad}_g \xi + \xi' = 0\}.$$

Remark 3.2. — For $\mathfrak{g}$ a complex semi-simple Lie algebra, a complete classification of Lagrangian subalgebras of $\mathfrak{d}$ was obtained by Karolinsky [34]. The Poisson geometry of the variety of Lagrangian subalgebras of $\mathfrak{d}$ was studied in detail by Evens–Lu [27].

Remark 3.3. — If $\mathfrak{d} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$ is a splitting into two Lagrangian subalgebras (i.e., $(\mathfrak{d}, \mathfrak{s}_1, \mathfrak{s}_2)$ is a Manin triple), one obtains two transverse Dirac structures $E^{\mathfrak{s}_1}, E^{\mathfrak{s}_2}$. As discussed after Theorem 2.9, such a pair of transverse Dirac structures gives rise to a Poisson structure on G , with symplectic leaves the intersections of the orbits of S_1, S_2 . For $\mathfrak{g}$ a complex semi-simple Lie algebra, the Manin triples were classified by Delorme [22]. See Evens–Lu [27] for a wealth of information regarding Poisson structures obtained from Lagrangian subalgebras. An example will be worked out in Section 3.6 below.

Remark 3.4. — We may also use this construction to obtain generalized complex (and Kähler) structures [28] on even-dimensional real Lie groups K , with complexification $G = K^\mathbb{C}$. Indeed, let $\mathfrak{s} \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ be a Lagrangian subalgebra such that

$$(42) \quad \mathfrak{s} \cap \mathfrak{s}^c = \{0\},$$

where $\mathfrak{s}^c$ denotes the complex conjugate of $\mathfrak{s}$. Then the associated Dirac structure $E^s \subset TG$ satisfies $E^s \cap (E^s)^c = \{0\}$ along K . Hence it defines a generalized complex structure on K . For a concrete example, suppose K is compact, and let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$ be a triangular decomposition. (That is, $\mathfrak{t} = \mathfrak{t}_K^\mathbb{C}$ is the complexification of a maximal Abelian subalgebra, and $\mathfrak{n}_+, \mathfrak{n}_-$ are the sums of the positive, negative root spaces). Then

$$\mathfrak{s} = (\mathfrak{n}^+ \oplus 0) \oplus \mathfrak{l} \oplus (0 \oplus \mathfrak{n}^-) \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$$

has the desired property, for any Lagrangian subspace $\mathfrak{l} \subset \mathfrak{t} \oplus \bar{\mathfrak{t}}$ with $\mathfrak{l} \cap \mathfrak{l}^c = \{0\}$ (i.e., $\mathfrak{l}$ is a linear generalized complex structure on the vector space $\mathfrak{t}_K$). The generalized complex structures on Lie groups considered in Gualtieri [28, Example 6.39] are examples of this construction.

3.3. The Cartan-Dirac structure. — The simplest example of a Lagrangian subalgebra is the diagonal $\mathfrak{s} = \mathfrak{g}_\Delta \hookrightarrow \mathfrak{d}$, with corresponding S the diagonal subgroup $G_\Delta \subset D$. The associated Dirac structure E_G is spanned by the sections $e(\xi) := s(\xi, \xi)$:

$$(43) \quad E_G = \text{span}\{e(\xi) \mid \xi \in \mathfrak{g}\} \subset TG,$$

$$e(\xi) = (\xi^L - \xi^R, B(\frac{\theta^L + \theta^R}{2}, \xi)).$$

We call E_G the *Cartan-Dirac structure*, see [15, 39, 50]. This Dirac structure was introduced independently by Alekseev, Ševera, and Strobl in the mid-1990's. The $G_\Delta \cong G$ -action is just the action by conjugation on G , hence the Dirac foliation is

given by the conjugacy classes $\mathcal{C} \subset G$. The formula (40) specializes to the 2-form on conjugacy classes introduced in [31]:

$$\omega_{\mathcal{C}}(\mathcal{O}_{\text{ad}}(\xi_1), \mathcal{O}_{\text{ad}}(\xi_2)) = -\frac{1}{2}B((\text{Ad}_g - \text{Ad}_{g^{-1}})\xi_1, \xi_2),$$

The kernel at $g \in \mathcal{C}$ is the span of vector fields $\mathcal{O}_{\text{ad}}(\xi)|_g$ with $\text{Ad}_g \xi + \xi = 0$. The anti-diagonal in $\mathfrak{g} \oplus \bar{\mathfrak{g}}$ is a G -invariant Lagrangian complement to the diagonal, and hence defines a G -invariant Lagrangian subbundle F_G complementary to E_G , spanned by $\mathbf{f}(\xi) = \mathbf{s}(\xi/2, -\xi/2)$:

$$(44) \quad F_G = \text{span}\{\mathbf{f}(\xi) \mid \xi \in \mathfrak{g}\} \subset \mathbb{T}G,$$

$$\mathbf{f}(\xi) = \left(\frac{\xi^L + \xi^R}{2}, B\left(\frac{\theta^L - \theta^R}{4}, \xi\right) \right).$$

The $1/2$ factors in the definition of $\mathbf{f}(\xi)$ are introduced so that $\langle \mathbf{e}(\xi), \mathbf{f}(\xi') \rangle = B(\xi, \xi')$.

Let $\Xi \in \wedge^3(\mathfrak{g})$ be the structure constants tensor of $\mathfrak{g}$, normalized as follows:

$$(45) \quad \iota(\xi_3)\iota(\xi_2)\iota(\xi_1)\Xi = \frac{1}{4}B(\xi_1, [\xi_2, \xi_3]_{\mathfrak{g}}).$$

Let $\mathbf{e}: \wedge \mathfrak{g} \rightarrow \Gamma(\wedge E_G)$ be the extension of $\mathbf{e}: \mathfrak{g} \rightarrow \Gamma(E_G)$ as an algebra homomorphism. Thus $\mathbf{e}(\Xi)$ is a section of $\wedge^3(E_G)$.

Lemma 3.5. — *The Courant tensor of F_G is given by :*

$$\Upsilon^{F_G} = \mathbf{e}(\Xi).$$

Proof. — This follows from (37) since $B_{\mathfrak{d}}(\zeta_1, [\zeta_2, \zeta_3]_{\mathfrak{d}}) = \frac{1}{4}B(\xi_1, [\xi_2, \xi_3]_{\mathfrak{g}})$ for $\zeta_i = (\xi_i/2, -\xi_i/2)$. $\square$

The element Ξ also defines a trivector field, $\mathcal{O}_{\text{ad}}(\Xi) \in \mathfrak{X}^3(G)$. Theorem 2.9 implies that the bivector field $\pi_G \in \mathfrak{X}^2(G)$ defined by the Lagrangian splitting $\mathbb{T}G = E \oplus F$ satisfies

$$\frac{1}{2}[\pi_G, \pi_G]_{\text{Sch}} = \mathcal{O}_{\text{ad}}(\Xi).$$

To give an explicit formula for π_G , let v_i, v^i be B -dual bases of $\mathfrak{g}$, i.e. $B(v_i, v^j) = \delta_i^j$.

Proposition 3.6. — *The bivector field π_G is given by*

$$(46) \quad \pi_G = \frac{1}{2} \sum_i v^{i,L} \wedge v_i^R.$$

Proof. — By (18), we have

$$\pi_G = \frac{1}{2} \sum_i ((v_i)^L - (v_i)^R) \wedge \frac{(v^i)^L + (v^i)^R}{2}.$$

Since $\sum_i v^{i,L} \wedge v_i^L = \sum_i v^{i,R} \wedge v_i^R$, this simplifies to the expression in (46). $\square$

The bivector field π_G was first considered in [1, 2].

3.4. Group multiplication. — In this Section, we will examine the behavior of the Cartan-Dirac structure under group multiplication,

$$\text{Mult}: G \times G \rightarrow G, \quad (a, b) \mapsto ab.$$

For any differential form $\beta \in \Omega(G)$, we will denote by $\beta^i \in \Omega(G \times G)$ its pull-back to the i 'th factor, for $i = 1, 2$. We will use similar notation for vector fields on $G \times G$, and for sections of the bundle $\mathbb{T}(G \times G)$. Let $\varsigma \in \Omega^2(G \times G)$ denote the 2-form

$$(47) \quad \varsigma = -\frac{1}{2}B(\theta^{L,1}, \theta^{R,2}).$$

A direct computation shows that

$$(48) \quad \text{Mult}^* \eta = \eta^1 + \eta^2 + d\varsigma,$$

hence we have a multiplication morphism

$$(\text{Mult}, \varsigma): (G, \eta) \times (G, \eta) = (G \times G, \eta^1 + \eta^2) \rightarrow (G, \eta).$$

Remark 3.7. — This is expressed more conceptually in terms of the simplicial model $B_p G = G^p$ of the classifying space BG . Let $\partial_i: G^p \rightarrow G^{p-1}$, $0 \leq i \leq p$ be the ‘face maps’ given as $\partial_i(g_1, \dots, g_p) = (g_1, \dots, g_i g_{i+1}, \dots, g_p)$, while ∂_0 omits the first entry g_1 , and ∂_p omits the last entry g_p . Let $\delta = \sum_{i=0}^p \partial_i^*: \Omega^\bullet(G^{p-1}) \rightarrow \Omega^\bullet(G^p)$. Then δ commutes with the de-Rham differential, turning $\bigoplus_{p,q} \Omega^q(G^p)$ into a double complex. The total differential on $\Omega^q(G^p)$ is $d + (-1)^q \delta$. Then $\eta \in \Omega^3(G)$ and $\varsigma \in \Omega^2(G^2)$ define a cocycle of degree 4 (see [55]):

$$(49) \quad d\eta = 0, \quad \partial\eta = -d\varsigma, \quad \partial\varsigma = 0.$$

(If G is compact, simple, and simply connected, and B the basic inner product, this pair is the Bott-Shulman representative of the generator of $H^4(BG) \cong H^3(G)$.) The second condition is just the property (48) used above. Using the third property, one may verify that the multiplication morphism is associative, in the sense that

$$(\text{Mult}, \varsigma) \circ ((\text{Mult}, \varsigma) \times (\text{id}_G, 0)) = (\text{Mult}, \varsigma) \circ ((\text{id}_G, 0) \times (\text{Mult}, \varsigma)).$$

We will compare the morphism (Mult, ς) with the groupoid multiplication of $\mathfrak{d}$, viewed as the pair groupoid over $\mathfrak{g}$: writing $\zeta = (\xi, \xi')$, $\zeta_i = (\xi_i, \xi'_i)$, $i = 1, 2$, the groupoid multiplication is

$$\zeta = \zeta_2 \circ \zeta_1 \Leftrightarrow \xi = \xi_2, \quad \xi' = \xi'_1, \quad \xi'_2 = \xi_1.$$

Proposition 3.8. — *The isomorphism $G \times \mathfrak{d} \rightarrow \mathbb{T}G$ defined by s intertwines the groupoid multiplication of $\mathfrak{d}$ with the morphism (Mult, ς) , in the sense that*

$$(50) \quad \zeta_2 \circ \zeta_1 = \zeta \Leftrightarrow s^1(\zeta_1) + s^2(\zeta_2) \sim_{(\text{Mult}, \varsigma)} s(\zeta),$$

for $\zeta, \zeta_1, \zeta_2 \in \mathfrak{d}$.

Proof. — Spelling out the relations (50), we have to show that, for all $\xi \in \mathfrak{g}$,

$$(51) \quad \begin{aligned} \mathfrak{s}^{R,1}(\xi) &\sim_{(\text{Mult}, \varsigma)} \mathfrak{s}^R(\xi), & \mathfrak{s}^{L,2}(\xi) &\sim_{(\text{Mult}, \varsigma)} \mathfrak{s}^L(\xi), \\ \mathfrak{s}^{L,1}(\xi) + \mathfrak{s}^{R,2}(\xi) &\sim_{(\text{Mult}, \varsigma)} 0. \end{aligned}$$

The equivariance properties

$$\begin{aligned} \text{Mult}(ga, b) &= g \text{Mult}(a, b), & \text{Mult}(a, bg^{-1}) &= \text{Mult}(a, b)g^{-1}, \\ \text{Mult}(ag^{-1}, gb) &= \text{Mult}(a, b) \end{aligned}$$

of the multiplication map imply the following relations of generating vector fields:

$$-\xi^{R,1} \sim_{\text{Mult}} -\xi^R, \quad \xi^{L,2} \sim_{\text{Mult}} \xi^L, \quad \xi^{L,1} - \xi^{R,2} \sim_{\text{Mult}} 0.$$

This proves the ‘vector field part’ of the relations (51). The 1-form part is equivalent to the following three identities, which are verified by a direct computation:

$$\begin{aligned} \frac{1}{2}B(\theta^{R,1}, \xi) + \iota(-\xi^{R,1})\varsigma &= \frac{1}{2}\text{Mult}^* B(\theta^R, \xi), \\ \frac{1}{2}B(\theta^{L,2}, \xi) + \iota(\xi^{L,2})\varsigma &= \frac{1}{2}\text{Mult}^* B(\theta^L, \xi), \\ \frac{1}{2}B(\theta^{L,1} + \theta^{R,2}, \xi) + \iota(\xi^{L,1} - \xi^{R,2})\varsigma &= 0. \end{aligned} \quad \square$$

Theorem 3.9. — *The multiplication map $\text{Mult}: G \times G \rightarrow G$ extends to a strong Dirac morphism*

$$(\text{Mult}, \varsigma): (G, E_G, \eta) \times (G, E_G, \eta) \rightarrow (G, E_G, \eta),$$

with $\varsigma \in \Omega^2(G \times G)$ as defined above. In terms of the trivialization $E_G = G \times \mathfrak{g}$, the map $\hat{\mathbf{a}}: \text{Mult}^* E_G \rightarrow E_G \times E_G$ associated with the strong Dirac morphism is given by the diagonal embedding $\mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$. Similarly, the inversion map $\text{Inv}: G \rightarrow G$, $g \mapsto g^{-1}$ extends to a Dirac morphism

$$(\text{Inv}, 0): (G, E_G, \eta) \rightarrow (G, E_G^\top, -\eta).$$

Proof. — By Proposition 3.8, the sections $\mathbf{e}(\xi) = \mathfrak{s}(\xi, \xi)$ satisfy

$$\mathbf{e}^1(\xi) + \mathbf{e}^2(\xi) \sim_{(\text{Mult}, \varsigma)} \mathbf{e}(\xi).$$

This shows that (Mult, ς) is a Dirac morphism. For any given point $(a, b) \in G \times G$, no non-trivial linear combination of $\mathbf{e}^1(\xi)|_a$, $\mathbf{e}^2(\xi')|_b$ is (Mult, ς) -related to 0. Hence, the Dirac morphism (Mult, ς) is strong.

We have $\text{Inv}^* B(\theta^L + \theta^R, \xi) = -B(\theta^L + \theta^R, \xi)$ and $\xi^L - \xi^R \sim_{\text{Inv}} (\xi^L - \xi^R)$. Hence

$$\mathbf{e}(\xi) \sim_{(\text{Inv}, 0)} \mathbf{e}(\xi)^\top$$

where $\mathbf{e}(\xi)^\top$ is the image of $\mathbf{e}(\xi)$ under the map $(v, \alpha) \rightarrow (v, -\alpha)$. Since $\text{Inv}^* \eta = -\eta$, this shows that $(\text{Inv}, 0): (G, E_G, \eta) \rightarrow (G, E_G^\top, -\eta)$ is a Dirac morphism. $\square$

Remark 3.10. — More generally, suppose that $\mathfrak{s} \subset \mathfrak{d}$ is a Lagrangian subalgebra, defining a Dirac structure $E^{\mathfrak{s}}$. Since $\mathfrak{g}_\Delta \circ \mathfrak{s} = \mathfrak{s}$, the same argument as in the proof above shows that (Mult, ς) is a strong Dirac morphism from $(G, E_G, \eta) \times (G, E^{\mathfrak{s}}, \eta)$ to $(G, E^{\mathfrak{s}}, \eta)$.

Let $\widetilde{F}_{G \times G} \subset \mathbb{T}(G \times G)$ be the backward image of the Lagrangian subbundle F_G under (Mult, ς) . Since F_G is spanned by the sections $f(\xi) = \frac{1}{2}(s^L(\xi) - s^R(\xi))$, (51) shows that $\widetilde{F}_{G \times G}$ is spanned by the sections

$$(52) \quad \frac{1}{2}(s^{L,2}(\xi) - s^{R,1}(\xi)), \quad \frac{1}{2}(s^{L,1}(\xi) + s^{R,2}(\xi)).$$

Since F_G is a complement to E_G , its backward image $\widetilde{F}_{G \times G}$ is a complement to $E_G^1 \oplus E_G^2$ (see Proposition 1.15). Let us describe the element of $\Lambda^2(E_G^1 \oplus E_G^2)$ relating $\widetilde{F}_{G \times G}$ to the standard complement $F_G^1 \oplus F_G^2$. Let $v_i \in \mathfrak{g}$ and $v^i \in \mathfrak{g}$ be B -dual bases, and put

$$(53) \quad \gamma = \frac{1}{2}(v_i)^1 \wedge (v^i)^2 \in \Lambda^2(\mathfrak{g} \oplus \mathfrak{g}).$$

Let

$$(54) \quad e(\gamma) = \frac{1}{2} \sum_i e^1(v_i) \wedge e^2(v^i) \in \Gamma(\Lambda^2(E_G^1 \oplus E_G^2))$$

be the corresponding section.

Proposition 3.11. — *The Lagrangian complement $\widetilde{F}_{G \times G} = F_G \circ \Gamma_{(\text{Mult}, \varsigma)}$ is obtained from $F_G^1 \oplus F_G^2$ by the bivector $e(\gamma)$:*

$$\widetilde{F}_{G \times G} = A^{-e(\gamma)}(F_G^1 \oplus F_G^2).$$

Proof. — We compute $\iota(f^1(\xi))e(\gamma) = e(\iota^1(\xi)\gamma) = \frac{1}{2}e^2(\xi) = \frac{1}{2}(s^{L,2}(\xi) + s^{R,2}(\xi))$. Thus

$$f^1(\xi) + \iota(f^1(\xi))e(\gamma) = \frac{1}{2}(s^{L,1}(\xi) - s^{R,1}(\xi) + s^{L,2}(\xi) + s^{R,2}(\xi))$$

is the sum of the sections in (52). Similarly, we find that $f^2(\xi) + \iota(f^2(\xi))e(\gamma)$ is the difference of the sections in (52). $\square$

The bivector field on $G \times G$ corresponding to the splitting $(E_G^1 \times E_G^2) \oplus A^{-e(\gamma)}(F_G^1 \times F_G^2)$ of $\mathbb{T}(G \times G)$ is given by (see Proposition 1.18(i)),

$$(55) \quad \widetilde{\pi} = \pi_G^1 + \pi_G^2 + \mathcal{U}_{\text{ad}}^{12}(\gamma),$$

where π_G is the bivector field for the splitting $\mathbb{T}G = E_G \oplus F_G$, and $\mathcal{U}_{\text{ad}}^{12} = \mathcal{U}_{\text{ad}}^1 \oplus \mathcal{U}_{\text{ad}}^2: \mathfrak{g} \oplus \mathfrak{g} \rightarrow \mathfrak{X}(G \times G)$. By Proposition 2.10(c) we have $\widetilde{\pi} \sim_{\text{Mult}} \pi$. Furthermore, Proposition 2.10(c) and Lemma 3.5, imply that the Schouten bracket $\frac{1}{2}[\widetilde{\pi}, \widetilde{\pi}]_{\text{Sch}}$ equals the trivector field $\mathcal{U}_{\text{ad}}^{\text{diag}}(\Xi)$, where $\mathcal{U}_{\text{ad}}^{\text{diag}}$ is the diagonal action on $G \times G$.

3.5. Exponential map. — We will now discuss the behavior of the Cartan-Dirac structure under the exponential map,

$$\exp: \mathfrak{g} \rightarrow G.$$

Let $\mathfrak{g}_{\text{r}} \subset \mathfrak{g}$ denote the set of regular points of the exponential map, that is, all points where $d\exp$ is an isomorphism. We begin with some preliminaries concerning $\mathbb{T}\mathfrak{g}^*$, not using the inner product on $\mathfrak{g}$ for the time being. Let $\mathcal{U}_0$ be the action of $D_0 := \mathfrak{g}^* \rtimes G$ on $\mathfrak{g}^*$ by

$$\mathcal{U}_0(\beta, g)\nu = (\text{Ad}_{g^{-1}})^*\nu - \beta.$$

This action lifts to an action by automorphisms of $\mathbb{T}\mathfrak{g}^*$, preserving the inner product as well as the (untwisted) Courant bracket. Let $\mathfrak{d}_0 = \mathfrak{g}^* \rtimes \mathfrak{g}$ be the Lie algebra of D_0 , equipped with the canonical inner product defined by the pairing, and let $\mathcal{U}_0: \mathfrak{d}_0 \rightarrow \mathfrak{X}(\mathfrak{g}^*)$ be the infinitesimal action. To simplify notation, we denote the constant vector field defined by $\beta \in \mathfrak{g}^*$ by $\beta_0 = \mathcal{U}_0(\beta, 0)$, and write $\mathcal{U}_0(\xi) = \mathcal{U}_0(0, \xi)$. Let $\theta_0 \in \Omega^1(\mathfrak{g}^*) \otimes \mathfrak{g}^*$ be the tautological 1-form, defined by $\iota(\beta_0)\theta_0 = \beta$. Consider the D_0 -equivariant map

$$(56) \quad s_0: \mathfrak{d}_0 \rightarrow \Gamma(\mathbb{T}\mathfrak{g}^*), \quad s_0(\beta, \xi) = \mathcal{U}_0(\beta, \xi) \oplus \langle \theta_0, \xi \rangle.$$

Then $\langle s_0(\zeta), s_0(\zeta') \rangle = B_{\mathfrak{d}_0}(\zeta, \zeta')$, showing that s_0 defines a D_0 -equivariant isometric isomorphism

$$\mathbb{T}\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{d}_0.$$

A direct computation shows that this isomorphism is compatible with the Courant bracket $[\cdot, \cdot]_0$ on $\mathbb{T}\mathfrak{g}^*$ and the Lie bracket on $\mathfrak{d}_0$.

Since $\mathfrak{g} \subset \mathfrak{d}_0$ is a Lagrangian Lie subalgebra, the sections $e_0(\xi) := s_0(0, \xi)$ span a Dirac structure $E_{\mathfrak{g}^*} \subset \mathbb{T}\mathfrak{g}^*$. Since $E_{\mathfrak{g}^*} \cap T\mathfrak{g}^* = 0$, this Dirac structure is of the form $E_{\mathfrak{g}^*} = \text{Gr}_{\pi_{\mathfrak{g}^*}}$ for a Poisson bivector field $\pi_{\mathfrak{g}^*}$ satisfying

$$(57) \quad \iota(\langle \theta_0, \xi \rangle) \pi_{\mathfrak{g}^*} = \mathcal{U}_0(\xi), \quad \xi \in \mathfrak{g}.$$

The Poisson structure $\pi_{\mathfrak{g}^*}$ is just the standard linear Poisson structure on $\mathfrak{g}^*$. Similarly, the sections $f_0(\beta) := s_0(\beta, 0)$ span the Lagrangian subspace $F_{\mathfrak{g}^*} = T\mathfrak{g}^*$, which is complementary to $E_{\mathfrak{g}^*}$.

Let us now use the invariant inner product B on $\mathfrak{g}$ to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let

$$(58) \quad \varpi \in \Omega^2(\mathfrak{g}), \quad d\varpi = \exp^* \eta$$

be the primitive of $\exp^* \eta \in \Omega^3(\mathfrak{g})$ defined by the de Rham homotopy operator for the radial homotopy.

Proposition 3.12. — *The sections $e_0(\xi)$ and $e(\xi)$ are $(\exp, \varpi)$ -related:*

$$(59) \quad e_0(\xi) \sim_{(\exp, \varpi)} e(\xi).$$

Similarly, over the subset $\mathfrak{g}_{\mathfrak{h}} \subset \mathfrak{g}$, one has

$$(60) \quad f_0(\xi) + e_0(C\xi) \sim_{(\exp, \varpi)} f(\xi),$$

where $C: \mathfrak{g}_{\mathfrak{h}} \rightarrow \text{End}(\mathfrak{g})$ is given by the formula:

$$(61) \quad C|_{\nu} = (1/2 \coth(z/2) - 1/z)|_{z=\text{ad}_{\nu}}, \quad \nu \in \mathfrak{g}_{\mathfrak{h}}.$$

Proof. — Recall that β_0 denotes the ‘constant vector field’ $\mathcal{U}_0(\beta, 0)$. We extend the notation $(\cdot)_0$ to $\mathfrak{g}^* \cong \mathfrak{g}$ -valued functions on $\mathfrak{g}^* \cong \mathfrak{g}$: For instance, the vector field corresponding to the function $\nu \mapsto -\text{ad}_{\xi} \nu = \text{ad}_{\nu} \xi$ is $(\text{ad}_{\nu} \xi)_0 = \mathcal{U}_0(\xi)$.

The vector field part of the relation (59) says that $\mathcal{U}_0(\xi) \sim_{\exp} \xi^L - \xi^R = \mathcal{U}_{\text{ad}}(\xi)$, which follows by the G -equivariance of $\exp$. The 1-form part of (59) is equivalent to the following property [3] of ϖ :

$$\iota(\mathcal{U}_0(\xi))\varpi = \frac{1}{2} \exp^* B(\theta^L + \theta^R, \xi) - B(\theta_0, \xi).$$

Since $\exp$ is a local diffeomorphism over $\mathfrak{g}_{\mathfrak{h}}$, the section $f(\xi)$ of $\mathbb{T}G$ is $(\exp, \varpi)$ -related to a unique section $\tilde{f}(\xi)$ of $\mathbb{T}\mathfrak{g}|_{\mathfrak{g}_{\mathfrak{h}}}$. Since inner products are preserved under the $(\exp, \varpi)$ -relation (see (12)) we have

$$\langle e_0(\xi'), \tilde{f}_0(\xi) \rangle = \langle e(\xi'), f_0(\xi) \rangle = B(\xi', \xi) = \langle e_0(\xi'), f_0(\xi) \rangle$$

for all $\xi' \in \mathfrak{g}$, showing that the $F_{\mathfrak{g}}$ -component of $\tilde{f}_0(\xi)$ is equal to $f_0(\xi)$. It follows that $\tilde{f}_0(\xi) = f_0(\xi) + e_0(C(\xi))$, where C is defined by $B(\xi', C(\xi)) = \langle f_0(\xi'), \tilde{f}_0(\xi) \rangle$. To compute C , we re-write (60) in the equivalent form (using (59)):

$$f_0(\xi) \sim_{(\exp, \varpi)} f(\xi) - e(C(\xi))$$

Again, we write out the vector field and 1-form parts of this relation:

$$(62) \quad \begin{aligned} \xi_0 &= \frac{1}{2} \exp^*(\xi^L + \xi^R) - \mathcal{Q}_0(C(\xi)), \\ \iota(\xi_0)\varpi &= \frac{1}{4} \exp^* B(\theta^L - \theta^R, \xi) - \frac{1}{2} \exp^* B(\theta^L + \theta^R, C(\xi)). \end{aligned}$$

We now verify that C given by (61) satisfies these two equations. Let $T, U^L, U^R: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ be the functions defined by

$$\iota(\xi_0)\varpi = B(\theta_0, T\xi), \quad \exp^* \theta^L = U^L \theta_0, \quad \exp^* \theta^R = U^R \theta_0.$$

It is known that (for the first identity, see e.g. [45])

$$T|_{\nu} = \left(\frac{\sinh(z) - z}{z^2} \right) \Big|_{z=\text{ad}_{\nu}}, \quad U^L|_{\nu} = \left(\frac{1 - e^{-z}}{z} \right) \Big|_{z=\text{ad}_{\nu}}, \quad U^R|_{\nu} = \left(\frac{e^z - 1}{z} \right) \Big|_{z=\text{ad}_{\nu}}.$$

Note that U^L and U^R are transposes relative to the inner product on $\mathfrak{g}$, and that they are invertible over $\mathfrak{g}_{\mathfrak{h}}$. Their definitions imply that

$$\exp^* \xi^L = ((U^L)^{-1} \xi)_0, \quad \exp^* \xi^R = ((U^R)^{-1} \xi)_0.$$

The first equation in (62) becomes

$$\xi_0 = \left(\left(\frac{(U^L)^{-1} + (U^R)^{-1}}{2} - \text{ad}_{\nu} C \right) \xi \right)_0$$

which follows from the identity

$$1 = \frac{1}{2} \left(\frac{z}{1 - e^{-z}} + \frac{z}{e^z - 1} \right) - z \left(\frac{1}{2} \coth\left(\frac{z}{2}\right) - \frac{1}{z} \right).$$

In a similar fashion, the second equation in (62) follows from the identity

$$\frac{\sinh(z) - z}{z^2} = \frac{1}{4} \left(\frac{e^z - 1}{z} - \frac{1 - e^{-z}}{z} \right) - \frac{1}{2} \left(\frac{e^z - 1}{z} + \frac{1 - e^{-z}}{z} \right) \left(\frac{1}{2} \coth\left(\frac{z}{2}\right) - \frac{1}{z} \right). \quad \square$$

As an immediate consequence of (59), we obtain

Theorem 3.13. — *The exponential map and the 2-form ϖ define a Dirac morphism*

$$(\exp, \varpi): (\mathfrak{g}, E_{\mathfrak{g}}, 0) \rightarrow (G, E_G, \eta).$$

It is a strong Dirac morphism over the open subset $\mathfrak{g}_{\mathfrak{h}} \subset \mathfrak{g}$.

Let $\widetilde{F}_{\mathfrak{g}}$ be the backward image (defined over $\mathfrak{g}_{\mathfrak{h}}$) of F_G under $(\exp, \varpi)$, and let

$$\varepsilon \in C^\infty(\mathfrak{g}_{\mathfrak{h}}, \wedge^2 \mathfrak{g})$$

be the unique map such that the associated orthogonal transformation $A^{-\varepsilon_0(\varepsilon)} \in \Gamma(\mathrm{O}(\mathrm{T}\mathfrak{g}_{\mathfrak{h}}))$ takes $F_{\mathfrak{g}}$ to $\widetilde{F}_{\mathfrak{g}}$. By (60), this section is given by $\iota_{\xi}\varepsilon = C(\xi)$, with C given by (61).

Let $[\varepsilon, \varepsilon]_{\mathrm{Sch}} \in C^\infty(\mathfrak{g}_{\mathfrak{h}}, \wedge^3 \mathfrak{g})$ be defined using the Schouten bracket on $\wedge \mathfrak{g}$, and $d\varepsilon \in C^\infty(\mathfrak{g}_{\mathfrak{h}}, \wedge^3 \mathfrak{g})$ the exterior differential of ε , viewed as a 2-form on $\mathfrak{g}_{\mathfrak{h}}$.

Proposition 3.14. — *The map ε satisfies the classical dynamical Yang-Baxter equation:*

$$(63) \quad d\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_{\mathrm{Sch}} = \Xi.$$

Proof. — Proposition 2.14 and the discussion following it show that $d\varepsilon + \frac{1}{2}[\varepsilon, \varepsilon]_{\mathrm{Sch}}$ equals the Courant tensor of $\widetilde{F}_{\mathfrak{g}}$ (relative to the complementary subbundle $E_{\mathfrak{g}}$). By Lemma 3.5, together with Proposition 2.10, $\Upsilon^{\widetilde{F}_{\mathfrak{g}}} = \Xi$. $\square$

This solution of the classical dynamical Yang-Baxter equation was obtained in [5], using a different argument. As a special case of Proposition 1.18, the map ε relates the linear Poisson bivector $\pi_{\mathfrak{g}}$ on $\mathfrak{g} \cong \mathfrak{g}^*$ with the pull-back $\exp^* \pi_G \in \mathfrak{X}^2(\mathfrak{g}_{\mathfrak{h}})$ of the bivector field (46) on G :

$$\exp^* \pi_G = \pi_{\mathfrak{g}} + \mathcal{Q}_0(\varepsilon).$$

3.6. The Gauss-Dirac structure. — In this Section we assume that $G = K^{\mathbb{C}}$ is a complex Lie group, given as the complexification of a compact, connected Lie group K of rank l . Thus the Cartan-Dirac structure E_G will be regarded as a holomorphic Dirac structure on the complex Lie group G . We will show that G carries another interesting Dirac structure besides the Cartan-Dirac structure. An important feature of this Dirac structure is that the corresponding Dirac foliation has an open dense leaf.

Take the bilinear form B on $\mathfrak{g}$ to be the complexification of a positive definite invariant inner product on $\mathfrak{k}$. Let T_K be a maximal torus in K , with complexification $T = T_K^{\mathbb{C}}$. Let

$$(64) \quad \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$$

be the triangular decomposition relative to some choice of positive Weyl chamber, where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the nilpotent subalgebra given as the sum of positive (resp. negative) root spaces. For every root α , let e_{α} be a corresponding root vector, with the normalization $B(\bar{e}_{\alpha}, e_{\alpha}) = 1$ and $e_{-\alpha} = \bar{e}_{\alpha}$. The unipotent subgroups corresponding to $\mathfrak{n}_{\pm}$ are denoted $N_{\pm}$. Recall that the multiplication map

$$(65) \quad j: N_- \times T \times N_+ \rightarrow G, \quad (g_-, g_0, g_+) \mapsto g_- g_0 g_+$$

is a diffeomorphism onto its image $\mathcal{O} \subset G$, called the *big Gauss cell*. The big Gauss cell is open and dense in G , and the inverse map $j^{-1}: \mathcal{O} \rightarrow N_- \times T \times N_+$ is known as the Gauss decomposition. Consider $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ as Section 3.1. Then

$$(66) \quad \mathfrak{s} = \{(\xi_+ + \xi_0) \oplus (\xi_- - \xi_0) \in \mathfrak{d} \mid \xi_- \in \mathfrak{n}_-, \xi_0 \in \mathfrak{t}, \xi_+ \in \mathfrak{n}_+\}$$

is a Lagrangian subalgebra of $\mathfrak{d}$, corresponding to the subgroup

$$S = \{(g_+ t, g_- t^{-1}) \in G \times G \mid g_- \in N_-, t \in T, g_+ \in N_+\}$$

of $D = G \times G$. Since $\mathfrak{s}$ is transverse to the diagonal $\mathfrak{g}_\Delta$, the corresponding Lagrangian subbundle $\widehat{F}_G := E^\mathfrak{s}$ is transverse to the Cartan-Dirac structure E_G :

$$\mathbb{T}G = E_G \oplus \widehat{F}_G.$$

We shall refer to it as to *Gauss-Cartan* splitting.

Unlike the complement F_G defined by the anti-diagonal, $\widehat{F}_G$ is integrable (since $\mathfrak{s}$ is a subalgebra), and it defines a Dirac manifold $(G, \widehat{F}_G, \eta)$. We refer to $\widehat{F}_G$ as the *Gauss-Dirac structure*. Its leaves are the orbits of S as a subgroup of D ,

$$(67) \quad \mathcal{O}(g_+ t, g_- t)(g) = g_- t^{-1} g t^{-1} g_+^{-1}.$$

The S -orbit of the group unit is exactly the big Gauss cell. Let $\omega_\mathcal{O}$ be the 2-form on $\mathcal{O}$, and $j^* \omega_\mathcal{O}$ its pull-back to $N_- \times T \times N_+$.

Proposition 3.15. — *The pull-back of the 2-form $\omega_\mathcal{O}$ on the big Gauss cell $N_- \times T \times N_+$ is given by:*

$$(68) \quad j^* \omega_\mathcal{O} = -\frac{1}{2} B(\theta_-^L, \text{Ad}_{g_0} \theta_+^R).$$

Here $\theta_\pm^L$, $\theta_\pm^R$ are the Maurer-Cartan-forms on $N^\pm$, and g_0 is the T -component (i.e. projection of $N_- \times T \times N_+$ to the middle factor).

Proof. — Let $\omega \in \Omega^2(N_- \times T \times N_+)$ denote the 2-form on the right hand side of (68). Since both ω and $\omega_\mathcal{O}$ are S -invariant, it suffices to check that $j^* \omega_\mathcal{O} = \omega$ at the group unit $g = e$. At the group unit, the formula (40) for $\omega_\mathcal{O}$ simplifies to

$$(69) \quad \omega_\mathcal{O}(\mathcal{U}(\zeta_1), \mathcal{U}(\zeta_2))|_e = \frac{1}{2} (B(\xi'_1, \xi_2) - B(\xi'_2, \xi_1)),$$

for $\zeta_1 = (\xi_1, \xi'_1)$, $\zeta_2 = (\xi_2, \xi'_2) \in \mathfrak{s} \subset \mathfrak{d}$. Its kernel is

$$\ker(\omega_\mathcal{O}|_e) = \{\mathcal{U}(\zeta)|_e \mid \zeta = (\xi_0, -\xi_0), \xi_0 \in \mathfrak{t}\} = T_e(T)$$

which coincides with the kernel of $-\frac{1}{2} B(\theta_-^L, \theta_+^R)|_e$. Moreover, it is clear that $T_e(N_+)$ and $T_e(N_-)$ are isotropic subspaces for both 2-forms. Hence it is enough to compare on tangent vectors $\mathcal{U}(\zeta_1)$, $\mathcal{U}(\zeta_2)$ for ζ_i of the form $\zeta_1 = (0, \xi_-)$ with $\xi_- \in \mathfrak{n}_-$, and $\zeta_2 = (\xi_+, 0)$ with $\xi_+ \in \mathfrak{n}_+$. (69) gives,

$$\omega_\mathcal{O}(\mathcal{U}(0, \xi_-), \mathcal{U}(\xi_+, 0))|_e = \frac{1}{2} B(\xi_+, \xi_-).$$

Since $j^* \mathcal{U}(\xi_+, 0)|_e = (0, 0, \xi_+) \in \mathfrak{n}_+ \subset \mathfrak{g} = T_e G$ and $j^* \mathcal{U}(0, \xi_-)|_e = (-\xi_-, 0, 0)$, the right hand side of (68) gives exactly the same answer. $\square$

Since F_G and $\widehat{F}_G$ are both complements to the Cartan-Dirac structure E_G , they are related by an element in $\Gamma(\wedge^2 E_G)$. To compute this element, let $\mathfrak{p}$ be the anti-diagonal in $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, and let $\mathfrak{g}_\Delta \cong \mathfrak{g}$ be the diagonal. Let

$$(70) \quad \mathfrak{r} = \sum e_{-\alpha} \wedge e_\alpha \in \wedge^2 \mathfrak{g}$$

be the classical r -matrix.

Lemma 3.16. — *The bivector taking $\mathfrak{p}$ to $\mathfrak{s}$ is the image $\mathfrak{r}_\Delta \in \wedge^2 \mathfrak{g}_\Delta$ of the classical $\mathfrak{r}$ -matrix under the diagonal embedding $\mathfrak{g} \rightarrow \mathfrak{g}_\Delta \subset \mathfrak{d}$.*

Proof. — Let $\mathfrak{g} \oplus \mathfrak{g}^*$ carry the bilinear form defined by the pairing, and consider the isometric isomorphism

$$\mathfrak{g} \oplus \mathfrak{g}^* \rightarrow \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}, \quad \xi \oplus \mu \mapsto (\xi + \frac{B^\sharp(\mu)}{2}) \oplus (\xi - \frac{B^\sharp(\mu)}{2}).$$

This isomorphism takes $\mathfrak{g} = \mathfrak{g} \oplus 0$ to the diagonal $\mathfrak{g}_\Delta$, and $\mathfrak{g}^*$ to the anti-diagonal, $\mathfrak{p}$. The graph $\text{Gr}_\mathfrak{r} \subset \mathfrak{g} \oplus \mathfrak{g}^*$ of the bivector $\mathfrak{r}$ is spanned by vectors of the form

$$0 \oplus B^\flat(\xi_0), \quad e_\alpha \oplus B^\flat(e_\alpha), \quad e_{-\alpha} \oplus (-B^\flat(e_{-\alpha})),$$

for $\xi_0 \in \mathfrak{t}$ and positive roots α . The isomorphism $\mathfrak{g} \oplus \mathfrak{g}^* \cong \mathfrak{d}$ takes these vectors to

$$\xi_0/2 \oplus (-\xi_0/2), \quad 0 \oplus e_{-\alpha}, \quad e_\alpha \oplus 0.$$

Hence, it defines an isomorphism $\text{Gr}_\mathfrak{r} \cong \mathfrak{s}$. □

Corollary 3.17. — *The orthogonal transformation $A^{-\mathfrak{e}(\mathfrak{r})} \in \Gamma(\text{O}(\text{T}G))$ takes F_G to $\widehat{F}_G$.*

Proof. — This follows from Lemma 3.16 and the isomorphism $\text{T}G \cong G \times \mathfrak{d}$. □

The Gauss-Cartan splitting $\text{T}G = E_G \oplus \widehat{F}_G$ also defines a bivector field $\widehat{\pi}_G$, and Proposition 1.18 implies that it is related to the bivector field π_G (46) by

$$\widehat{\pi}_G = \pi_G + \mathcal{A}_{\text{ad}}(\mathfrak{r}).$$

Since $\widehat{F}_G$ is integrable, this bivector field is in fact a Poisson structure on G – see the remarks before Proposition 2.10. (On the other hand, unlike π_G , the Poisson structure is not invariant under the full adjoint action, but is only T -invariant.)

Proposition 3.18. — *The Poisson structure $\widehat{\pi}_G$ associated with the Gauss-Cartan splitting $\text{T}G = E_G \oplus \widehat{F}_G$ is given by the formula:*

$$\widehat{\pi}_G = \frac{1}{2} \sum_i e_i^L \wedge (e^i)^R - \sum_{\alpha > 0} e_{-\alpha}^L \wedge e_\alpha^R + \frac{1}{2} \mathfrak{r}^L + \frac{1}{2} \mathfrak{r}^R.$$

Here e_i is a basis of $\mathfrak{t}$, with B -dual basis e^i , and $\mathfrak{r}^L$, $\mathfrak{r}^R$ are the left-, right-invariant bivector fields defined by $\mathfrak{r}$. The symplectic leaves of this Poisson structure are the connected components of the intersections of conjugacy classes in G with the orbits of the action (67).

This Poisson structure was first defined by Semenov-Tian-Shansky, see [49].

Proof. — The vectors

$$\frac{1}{2}(e_i \oplus (-e_i)), \quad 0 \oplus (-e_{-\alpha}), \quad e_\alpha \oplus 0$$

form basis of $\mathfrak{s}$ that is dual (relative to the bilinear form on $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$) to the basis

$$e^i \oplus e^i, \quad e_\alpha \oplus e_\alpha, \quad e_{-\alpha} \oplus e_{-\alpha}$$

of the diagonal. Using the formula (18) for the bivector field, we obtain

$$\begin{aligned} \hat{\pi}_G &= \frac{1}{2} \sum_i ((e^i)^L - (e^i)^R) \wedge \frac{e_i^L + e_i^R}{2} + \frac{1}{2} \sum_{\alpha > 0} (e_\alpha^L - e_\alpha^R) \wedge (-e_{-\alpha}^L) + \frac{1}{2} \sum_{\alpha > 0} (e_{-\alpha}^L - e_{-\alpha}^R) \wedge (-e_\alpha^R) \\ &= \frac{1}{2} \sum_i e_i^L \wedge (e^i)^R - \sum_{\alpha > 0} e_{-\alpha}^L \wedge e_\alpha^R + \frac{1}{2} \mathfrak{t}^L + \frac{1}{2} \mathfrak{t}^R. \end{aligned}$$

Here we have used that the left- and right-invariant bivector fields generated by

$$\sum_i e_i \wedge e^i = \sum_i e_i \wedge e^i + \sum_{\alpha > 0} e_{-\alpha} \wedge e_\alpha + \sum_{\alpha > 0} e_\alpha \wedge e_{-\alpha}$$

coincide. □

Remark 3.19. — The Lagrangian subalgebra $\mathfrak{s}$ defines a Manin triple $(\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}, \mathfrak{g}_\Delta, \mathfrak{s})$, which induces a Poisson-Lie group structure on the double $D = G \times G$. The Poisson structure $\hat{\pi}_G$ is the push-forward image of this Poisson-Lie structure under the natural projection $D \rightarrow D/G \cong G$, see e.g. [1, Sec. 3.6].

4. Pure spinors on Lie groups

In the previous section we identified $\mathbb{T}G \cong G \times \mathfrak{d}$ as Courant algebroids. In particular, we have an identification $\text{Cl}(\mathbb{T}G) \cong G \times \text{Cl}(\mathfrak{d})$ of Clifford algebra bundles. In this section, we will complement this isomorphism of Clifford bundles by an isomorphism of spinor modules,

$$\wedge T^*G \cong G \times \text{Cl}(\mathfrak{g}),$$

where $\text{Cl}(\mathfrak{g})$ is given the structure of a spinor module over $\text{Cl}(\mathfrak{d})$. The differential $d + \eta$ on $\Omega(G)$ intertwines with a certain differential d_{Cl} on $\text{Cl}(\mathfrak{g})$. Hence, given a pure spinor $x \in \text{Cl}(\mathfrak{g})$ defining a Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, one directly obtains a pure spinor $\phi^{\mathfrak{s}} \in \Omega(G)$ defining $E^{\mathfrak{s}}$. We will also obtain expressions for $(d + \eta)\phi^{\mathfrak{s}}$ from the properties of x .

4.1. $\text{Cl}(\mathfrak{g})$ as a spinor module over $\text{Cl}(\mathfrak{g} \oplus \bar{\mathfrak{g}})$. — Recall from Examples 1.2 and 1.4 that for any vector space V with inner product B , the Clifford algebra $\text{Cl}(V)$ may be viewed as a spinor module over $\text{Cl}(V \oplus \bar{V})$. In the special case that $V = \mathfrak{g}$ is a Lie algebra, with B an invariant inner product, there is more structure that we now discuss.

Let $\lambda: \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ be the map defined by the condition $-\iota(\xi_2)\lambda(\xi_1) = [\xi_1, \xi_2]_{\mathfrak{g}}$ (see Section 1.1), and let $\Xi \in \wedge^3 \mathfrak{g}$ be the structure constants tensor (45). Then

$$\{\lambda(\xi_1), \lambda(\xi_2)\} = \lambda([\xi_1, \xi_2]_{\mathfrak{g}}), \quad \{\lambda(\xi_1), \xi_2\} = [\xi_1, \xi_2]_{\mathfrak{g}}$$

$$\{\Xi, \xi\} = -\frac{1}{4}\lambda(\xi), \quad \{\Xi, \Xi\} = 0$$

for all $\xi_1, \xi_2, \xi \in \mathfrak{g}$. The quantizations of these elements have similar properties: Let

$$(71) \quad \tau: \mathfrak{g} \rightarrow \text{Cl}(\mathfrak{g}), \quad \tau(\xi) = q(\lambda(\xi)).$$

Then

$$[\tau(\xi_1), \tau(\xi_2)]_{\text{Cl}} = \tau([\xi_1, \xi_2]_{\mathfrak{g}}), \quad [\tau(\xi_1), \xi_2]_{\text{Cl}} = [\xi_1, \xi_2]_{\mathfrak{g}},$$

$$[q(\Xi), \xi]_{\text{Cl}} = -\frac{1}{4}\tau(\xi), \quad [q(\Xi), q(\Xi)]_{\text{Cl}} \in \mathbb{K}.$$

(One can show (cf. [4]) that the constant $[q(\Xi), q(\Xi)]_{\text{Cl}}$ is $\frac{1}{24}$ times the trace of the Casimir operator in the adjoint representation.) This last identity implies that the derivation

$$(72) \quad d^{\text{Cl}} = -4[q(\Xi), \cdot]_{\text{Cl}}: \text{Cl}(\mathfrak{g}) \rightarrow \text{Cl}(\mathfrak{g})$$

squares to 0. We call d^{Cl} the *Clifford differential* [4, 38].

For the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$, with bilinear form $B \oplus (-B)$, the corresponding elements $\Xi_{\mathfrak{d}}$ and $\lambda_{\mathfrak{d}}$ in $\wedge \mathfrak{d} = \wedge \mathfrak{g} \otimes \wedge \bar{\mathfrak{g}}$ are given by

$$\Xi_{\mathfrak{d}} = \Xi \otimes 1 + 1 \otimes \Xi, \quad \lambda_{\mathfrak{d}}(\xi, \xi') = \lambda(\xi) \otimes 1 - 1 \otimes \lambda(\xi'), \quad \text{for } (\xi, \xi') \in \mathfrak{d}.$$

Note also that $q(\Xi_{\mathfrak{d}})^2 = 0$. Consider the Clifford algebra $\text{Cl}(\mathfrak{g})$ as a spinor module over $\text{Cl}(\mathfrak{d})$, with Clifford action given on generators $\zeta = (\xi, \xi') \in \mathfrak{d}$ by

$$\varrho^{\text{Cl}}(\xi, \xi') = l^{\text{Cl}}(\xi) - r^{\text{Cl}}(\xi').$$

Then the Clifford differential d^{Cl} is implemented as a Clifford action:

$$d^{\text{Cl}} = -4\varrho^{\text{Cl}}(q(\Xi_{\mathfrak{d}})).$$

The elements $\tau_{\mathfrak{d}}(\zeta) = q(\lambda_{\mathfrak{d}}(\zeta))$ generate a $\mathfrak{d}$ -action on $\text{Cl}(\mathfrak{g})$, with generators

$$\mathcal{L}^{\text{Cl}}(\zeta) = l^{\text{Cl}}(\tau(\xi)) - r^{\text{Cl}}(\tau(\xi')) = \varrho^{\text{Cl}}(\tau(\zeta)).$$

Note that

$$(73) \quad \mathcal{L}^{\text{Cl}}(\zeta) = [\varrho^{\text{Cl}}(\zeta), d^{\text{Cl}}],$$

which implies that

$$[\varrho^{\text{Cl}}(\zeta_1), [\varrho^{\text{Cl}}(\zeta_2), d^{\text{Cl}}]] = \varrho^{\text{Cl}}([\zeta_1, \zeta_2]).$$

Let $\mathfrak{s} \subset \mathfrak{d}$ be a Lagrangian subspace, and recall the definition of $\Upsilon^{\mathfrak{s}}$ given in (39). Given a Lagrangian complement $\mathfrak{p}$ to $\mathfrak{s}$, let $\text{pr}_{\mathfrak{s}}: \mathfrak{d} \rightarrow \mathfrak{s}$ be the projection along $\mathfrak{p}$, and define a linear functional $\sigma^{\mathfrak{s}} \in \mathfrak{s}^*$ by

$$(74) \quad \langle \sigma^{\mathfrak{s}}, \xi \rangle = \frac{1}{2} \text{trace}(\text{pr}_{\mathfrak{s}} \circ \text{ad}_{\xi} \big|_{\mathfrak{s}}), \quad \xi \in \mathfrak{s}.$$

If $\mathfrak{s}$ is a Lagrangian subalgebra (i.e. $\Upsilon^{\mathfrak{s}} = 0$), we may omit $\text{pr}_{\mathfrak{s}}$ in this formula; in this case $\sigma^{\mathfrak{s}}$ equals $-\frac{1}{2}$ times the modular character of the Lie algebra $\mathfrak{s}$.

Proposition 4.1. — *Let $\mathfrak{s} \subset \mathfrak{d}$ be a Lagrangian subspace, with defining pure spinor $x \in \text{Cl}(\mathfrak{g})$. Choose a Lagrangian complement $\mathfrak{p} \cong \mathfrak{s}^*$ to $\mathfrak{s}$ to view $\Upsilon^{\mathfrak{s}}$ as an element of the Clifford algebra $\text{Cl}(\mathfrak{d})$. Then*

$$d^{\text{Cl}}x = \varrho^{\text{Cl}}(-\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}})x.$$

In particular, $\mathfrak{s}$ is a Lie subalgebra if and only if the defining pure spinor x is ‘integrable’, in the sense that

$$d^{\text{Cl}}x \in \varrho^{\text{Cl}}(\mathfrak{d})x.$$

Proof. — The choice of a Lagrangian complement identifies $\mathfrak{d} = \mathfrak{s} \oplus \mathfrak{s}^*$, with bilinear form given by the pairing. Using a basis e_i of $\mathfrak{s}$ and a dual basis f^i of $\mathfrak{s}^*$, we have

$$\begin{aligned} 4\Xi_{\mathfrak{d}} = & \frac{1}{6} \sum_{ijk} B_{\mathfrak{d}}([e_i, e_j], e_k) f^i \wedge f^j \wedge f^k + \frac{1}{2} \sum_{ijk} B_{\mathfrak{d}}([e_j, e_k], f^i) e_i \wedge f^j \wedge f^k \\ & + \frac{1}{2} \sum_{ijk} B_{\mathfrak{d}}([f^j, f^k], e_i) e_j \wedge e_k \wedge f^i + \frac{1}{6} \sum_{ijk} B_{\mathfrak{d}}([f^j, f^k], f^i) e_j \wedge e_k \wedge e_i. \end{aligned}$$

The quantization map takes the last two terms into the left ideal $\text{Cl}(\mathfrak{d})\mathfrak{s}$, and it takes the second term to

$$\frac{1}{2} \sum_{ik} B_{\mathfrak{d}}([e_i, e_k], f^j) f^k + \frac{1}{2} \sum_{ijk} B_{\mathfrak{d}}([e_j, e_k], f^i) f^j f^k e_i = -\sigma^{\mathfrak{s}} \pmod{\text{Cl}(\mathfrak{d})\mathfrak{s}}.$$

This gives

$$-4q(\Xi_{\mathfrak{d}}) = -\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}} \pmod{\text{Cl}(\mathfrak{d})\mathfrak{s}},$$

from which the result is immediate. $\square$

Let us now assume that the adjoint action $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ lifts to a group homomorphism

$$(75) \quad \tau: G \rightarrow \text{Pin}(\mathfrak{g}) \subset \text{Cl}(\mathfrak{g})$$

to the double cover $\text{Pin}(\mathfrak{g}) \rightarrow \text{O}(\mathfrak{g})$. If G is connected, this is automatic if $\pi_1(G)$ is torsion free. Note that (75) is consistent with our previous notation $\tau(\xi) = q(\lambda(\xi))$, since [4]

$$\tau(\xi) = \left. \frac{d}{dt} \right|_{t=0} \tau(\exp t\xi).$$

We will write $\text{N}(g) = \text{N}(\tau(g)) = \pm 1$ for the image under the norm homomorphism, and $|g| = |\tau(g)|$ for the parity of $\tau(g)$. Since $\tau(g)$ lifts Ad_g , one has $(-1)^{|g|} = \det(\text{Ad}_g)$. The definition of the Pin group implies that conjugation by $\tau(g)$ is the *twisted adjoint action*,

$$(76) \quad \tau(g)x\tau(g^{-1}) = \widetilde{\text{Ad}}_g(x) := (-1)^{|g||x|} \text{Ad}_g(x)$$

(using the extension of $\text{Ad}_g \in \text{O}(\mathfrak{g})$ to an automorphism of the Clifford algebra). This twisted adjoint action extends to an action of the group D on $\text{Cl}(\mathfrak{g})$,

$$(77) \quad \mathcal{A}^{\text{Cl}}(a, a')(x) = \tau(a)x\tau((a')^{-1}).$$

4.2. The isomorphism $\wedge T^*G \cong G \times \text{Cl}(\mathfrak{g})$. — Let us now fix a generator $\mu \in \det(\mathfrak{g})$, and consider the corresponding star operator $\star: \wedge \mathfrak{g}^* \rightarrow \wedge \mathfrak{g}$, see Remark 1.5(b). The star operator satisfies

$$(78) \quad \text{Ad}_g \circ \star = (-1)^{|g|} \star \circ \text{Ad}_{g^{-1}}^*.$$

We use the trivialization by left-invariant forms to identify $\wedge T^*G \cong G \times \wedge \mathfrak{g}^*$. Applying $\star$ pointwise, we obtain an isomorphism $q \circ \star: \wedge T_g^*G \xrightarrow{\sim} \text{Cl}(\mathfrak{g})$ for each $g \in G$. Let us define the linear map

$$(79) \quad \mathcal{R}: \text{Cl}(\mathfrak{g}) \rightarrow \Omega(G), \quad \mathcal{R}(x)|_g = (q \circ \star)^{-1}(x\tau(g)).$$

We denote by $\mu^* \in \det(\mathfrak{g}^*)$ the dual generator, defined by $\iota((\mu^*)^\top)\mu = 1$, and let μ_G be the left-invariant volume form on G defined by μ^* .

Proposition 4.2. — *The map (79) has the following properties:*

a. $\mathcal{R}$ intertwines the Clifford actions, in the sense that

$$\mathcal{R}(\varrho^{\text{Cl}}(\zeta)x) = \varrho(s(\zeta))\mathcal{R}(x), \quad \forall x \in \text{Cl}(\mathfrak{g}), \zeta \in \mathfrak{d}.$$

Up to a scalar function, $\mathcal{R}$ is uniquely characterized by this property.

b. $\mathcal{R}$ intertwines differentials:

$$\mathcal{R}(d^{\text{Cl}}(x)) = (d + \eta)\mathcal{R}(x), \quad \forall x \in \text{Cl}(\mathfrak{g}).$$

c. $\mathcal{R}$ satisfies has the following D -equivariance property: For any $h = (a, a') \in D$, and at any given point $g \in G$,

$$\mathcal{A}(h^{-1})^* \mathcal{R}(x) = (-1)^{|a|(|g|+|x|)} \mathcal{R}(\mathcal{A}^{\text{Cl}}(h)x).$$

d. $\mathcal{R}$ relates the bilinear pairings on the Clifford modules $\text{Cl}(\mathfrak{g})$ and $\Omega(G)$ as follows: At any given point $g \in G$, and for all $x, x' \in \text{Cl}(\mathfrak{g})$,

$$(80) \quad (\mathcal{R}(x), \mathcal{R}(x'))_{\wedge T^*G} = (-1)^{|g|(\dim G + 1)} \mathbf{N}(g) (x, x')_{\text{Cl}(\mathfrak{g})} \mu_G.$$

Here the pairing $(\cdot, \cdot)_{\text{Cl}(\mathfrak{g})}$ is viewed as scalar-valued, using the trivialization of $\det(\mathfrak{g})$ defined by μ . (Cf. Remark 1.5.)

Notice that the signs in part (c), (d) disappear if G is connected.

Proof. — (a) Given $\xi \in \mathfrak{g}$, let $\epsilon(\xi): \wedge \mathfrak{g} \rightarrow \wedge \mathfrak{g}$ be defined by $\epsilon(\xi)\xi' = \xi \wedge \xi'$. Then

$$l^{\text{Cl}}(\xi) \circ q = q \circ (\epsilon(\xi) + \tfrac{1}{2}\iota(B^b(\xi))), \quad r^{\text{Cl}}(\xi) \circ q = q \circ (\epsilon(\xi) - \tfrac{1}{2}\iota(B^b(\xi))).$$

Since the star operator exchanges exterior multiplication and contraction, we have

$$\star^{-1} \circ q^{-1} \circ \varrho^{\text{Cl}}(\xi, \xi') = \left(\iota(\xi - \xi') + \epsilon \left(B^b \left(\frac{\xi + \xi'}{2} \right) \right) \right) \circ \star^{-1} \circ q^{-1}.$$

On the other hand,

$$(\varrho^{\text{Cl}}(\xi, \xi')x)\tau(g) = (\xi x - (-1)^{|x|}x\xi')\tau(g) = \varrho^{\text{Cl}}(\xi, \text{Ad}_{g^{-1}}\xi')(x\tau(g)).$$

This implies that, at $g \in G$,

$$\mathcal{R}(\varrho^{\text{Cl}}(\xi, \xi')x) = \left(\iota(\xi - \text{Ad}_{g^{-1}}\xi') + \epsilon \left(B^b \left(\frac{\xi + \text{Ad}_{g^{-1}}\xi'}{2} \right) \right) \right) \mathcal{R}(x),$$

which is precisely the Clifford action of $\mathfrak{s}(\xi, \xi')$ since

$$\mathfrak{s}(\xi, \xi') = (\xi - \text{Ad}_{g^{-1}} \xi') \oplus B^b \left(\frac{\xi + \text{Ad}_{g^{-1}} \xi'}{2} \right)$$

under left-trivialization $\mathbb{T}G \cong G \times (\mathfrak{g} \oplus \mathfrak{g}^*)$. This shows that $\mathcal{R}$ intertwines the Clifford actions of $\text{Cl}(\mathfrak{d}) \cong \text{Cl}(\mathbb{T}_g G)$. By the uniqueness properties of spinor modules, $\mathcal{R}$ is uniquely characterized by this property up to a scalar.

(b) From the global equivariance property in (c), verified below, we obtain the infinitesimal equivariance: $\mathcal{L}(\mathcal{U}(\zeta))\mathcal{R}(x) = \mathcal{R}(\mathcal{L}^{\text{Cl}}(\zeta)x)$. Since $[\varrho(\mathfrak{s}(\zeta)), \mathfrak{d} + \eta] = \mathcal{L}(\mathcal{U}(\zeta))$ and $[\varrho^{\text{Cl}}(\zeta), \mathfrak{d}^{\text{Cl}}] = \mathcal{L}^{\text{Cl}}(\zeta)$, this gives

$$\begin{aligned} \varrho(\mathfrak{s}(\zeta))((\mathfrak{d} + \eta)\mathcal{R}(x) - \mathcal{R}(\mathfrak{d}^{\text{Cl}}x)) &= \mathcal{L}(\mathcal{U}(\zeta))\mathcal{R}(x) - \mathcal{R}(\varrho^{\text{Cl}}(\zeta)\mathfrak{d}^{\text{Cl}}x) \\ &= \mathcal{L}(\mathcal{U}(\zeta))\mathcal{R}(x) - \mathcal{R}(\mathcal{L}^{\text{Cl}}(\zeta)x). \end{aligned}$$

That is, the map $(\mathfrak{d} + \eta) \circ \mathcal{R} - \mathcal{R} \circ \mathfrak{d}^{\text{Cl}}: \text{Cl}(\mathfrak{g}) \rightarrow \Gamma(\mathbb{T}G)$ intertwines the Clifford actions, and hence agrees with $\mathcal{R}$ up to a scalar function. Since its parity is opposite to that of $\mathcal{R}$, that function is zero.

(c) We have to show that for all $a \in G$,

$$(81) \quad l_a^* \mathcal{R}(x) = \mathcal{R}(x\tau(a)), \quad r_a^* \mathcal{R}(x) = (-1)^{|a|(|g|+|x|)} \mathcal{R}(\tau(a)x).$$

In terms of the left-trivialization $\wedge T^*G = G \times \wedge \mathfrak{g}^*$,

$$(l_a^* \mathcal{R}(x))|_g = \mathcal{R}(x)|_{ag}, \quad (r_a^* \mathcal{R}(x))|_g = \text{Ad}_{a^{-1}}^*(\mathcal{R}(x))|_{ga}.$$

(Here $\text{Ad}_{a^{-1}}^*$ stands for the contragredient action on $\wedge \mathfrak{g}^*$, not for a pull-back on $\Omega(G)$.) We compute, using (76) and (78):

$$\begin{aligned} \text{Ad}_{a^{-1}}^*(\mathcal{R}(x))|_{ga} &= (-1)^{|a|} \star^{-1} q^{-1} \text{Ad}_a(x\tau(ga)) \\ &= (-1)^{|a|} (-1)^{|a|(|x|+|g|+|a|)} \star^{-1} q^{-1} (\tau(a)x\tau(g)) \\ &= (-1)^{|a|(|x|+|g|)} \mathcal{R}(\tau(a)x)|_g \end{aligned}$$

The equivariance property with respect to left translations is immediate from the definition.

(d) Use the generator $\mu \in \det(\mathfrak{g})$ and μ_G to trivialize both $\det(\mathfrak{g})$ and $\det(\wedge T^*G)$. By Remark 1.5(b) and Example 1.4 we have, at $g \in G$,

$$(\mathcal{R}(x), \mathcal{R}(x'))_{\wedge T^*G} = (x\tau(g), x'\tau(g))_{\text{Cl}(\mathfrak{g})}.$$

This is computed as follows:

$$\begin{aligned} \text{str}(\tau(g)^\top x^\top x' \tau(g)) &= (-1)^{|g|(|g|+|x|+|x'|)} \text{str}(\tau(g)\tau(g)^\top x^\top x') \\ &= \mathbf{N}(g) (-1)^{|g|(1+|x|+|x'|)} \text{str}(x^\top x') \end{aligned}$$

Finally, replace $|x| + |x'|$ with $\dim G$, using that $(x, x')_{\text{Cl}(\mathfrak{g})}$ vanishes unless $|x| + |x'| = \dim G \pmod 2$. $\square$

As an immediate consequence of Propositions 4.1 and 4.2, we have

Corollary 4.3. — *If $x \in \text{Cl}(\mathfrak{g})$ is a pure spinor defining a Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, then the differential form $\phi^{\mathfrak{s}} := \mathcal{R}(x) \in \Omega(G)$ is a pure spinor defining the Lagrangian subbundle $E^{\mathfrak{s}}$. It satisfies the differential equation*

$$(82) \quad (d + \eta)\phi^{\mathfrak{s}} = \varrho(\mathfrak{s}(-\Upsilon^{\mathfrak{s}} + \sigma^{\mathfrak{s}}))\phi^{\mathfrak{s}},$$

where $\sigma^{\mathfrak{s}} \in \mathfrak{s}^*$ is defined as in (74) (using a complementary Lagrangian subspace $\mathfrak{p} \cong \mathfrak{s}^* \subset \mathfrak{d}$). Let $H \subset D$ be a subgroup preserving $\mathfrak{s}$, and define the character $u^{\mathfrak{s}}: H \rightarrow \mathbb{K}^{\times}$ by $\mathcal{U}^{\text{Cl}}(h)x = u^{\mathfrak{s}}(h)x$. Then

$$(83) \quad \mathcal{U}(h^{-1})^*\phi^{\mathfrak{s}} = (-1)^{|a|(|g|+|x|)} u^{\mathfrak{s}}(h)\phi^{\mathfrak{s}}$$

for all $h = (a, a') \in H$, and at any given point $g \in G$.

We are mainly interested in pure spinors defining the Cartan-Dirac structure E_G and its Lagrangian complement F_G . These are obtained by taking $x = 1$ and $x = q(\mu)$ in the above:

Proposition 4.4. — *Let $\phi_G, \psi_G \in \Omega(G)$ be the differential forms*

$$(84) \quad \phi_G = \mathcal{R}(1), \quad \psi_G = \mathcal{R}(q(\mu)).$$

Then ϕ_G, ψ_G are pure spinors defining the Lagrangian subbundles E_G, F_G . They satisfy the differential equations,

$$(85) \quad (d + \eta)\phi_G = 0, \quad (d + \eta)\psi_G = -\varrho(\mathfrak{e}(\Xi))\psi_G.$$

The equivariance properties under the adjoint action of G read

$$\mathcal{U}_{\text{ad}}(a^{-1})^*\phi_G = (-1)^{|a||g|}\phi_G, \quad \mathcal{U}_{\text{ad}}(a^{-1})^*\psi_G = (-1)^{|a|(|g|+1)}\psi_G.$$

We will refer to ϕ_G as the *Cartan-Dirac spinor*.

Proof. — It is clear that the diagonal $\mathfrak{g}_{\Delta} \subset \mathfrak{d}$ is defined by the pure spinor $x = 1$. Similarly, the anti-diagonal $\mathfrak{p} \subset \mathfrak{d} = \mathfrak{g} \oplus \bar{\mathfrak{g}}$ is defined by the pure spinor $q(\mu) \in \text{Cl}(\mathfrak{g})$:

$$\varrho^{\text{Cl}}(\xi, -\xi)q(\mu) = \xi q(\mu) + (-1)^{\dim G} q(\mu)\xi = 0.$$

Hence ϕ_G, ψ_G are pure spinors defining E_G, F_G . The equivariance properties are special cases of (83), since both $\mathfrak{g}_{\Delta}$ and $\mathfrak{p}$ are preserved under G_{Δ} . Here we are using $|1| = 0$, $|q(\mu)| = \dim G \bmod 2$, while $u^{\mathfrak{p}}(a) = (-1)^{|a|(1+\dim G)}$ by the calculation:

$$\tau(a)q(\mu)\tau(a^{-1}) = (-1)^{|a|\dim G} q(\text{Ad}_a(\mu)) = (-1)^{|a|(1+\dim G)} q(\mu).$$

The differential equation for ϕ_G follows since $d^{\text{Cl}}(1) = 0$. It remains to check the differential equation for ψ_G . Since the anti-diagonal satisfies $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{d}} \subset \mathfrak{g}_{\Delta}$, the element $\sigma^{\mathfrak{p}} \in \mathfrak{p}^*$ is just zero. On the other hand, the element $\Upsilon^{\mathfrak{p}}$ is given by Ξ_{Δ} , the image of Ξ under the map $\wedge \mathfrak{g} \xrightarrow{\sim} \wedge \mathfrak{g}_{\Delta}$. Hence $\mathfrak{s}(\Xi_{\Delta}) = \mathfrak{e}(\Xi)$, confirming that ψ_G satisfies (85). $\square$

Remarks 4.5. — a. The map $\mathcal{R}$ depends on the choice of generator $\mu \in \det(\mathfrak{g})$, via the star operator: Replacing μ with $t\mu$ changes $\mathcal{R}$ to $t^{-1}\mathcal{R}$. Hence, the definition of $\psi_G = \mathcal{R}(q(\mu))$ is independent of the choice of μ .

- b. Since $(1, q(\mu))_{\text{Cl}(\mathfrak{g})} = \mu$, the bilinear pairing between ϕ_G, ψ_G equals the volume form, up to a sign:

$$(\phi_G, \psi_G)_{\wedge T^*G} = N(g)(-1)^{|g|(\dim G + 1)} \mu_G.$$

Proposition 4.6. — *Over the open subset $\mathcal{U}$ of G where $1 + \text{Ad}_g$ is invertible, the pure spinor ψ_G is given by the formula:*

$$\psi_G = \det^{1/2}\left(\frac{1 + \text{Ad}_g}{2}\right) \exp\left(\frac{1}{4} B\left(\frac{1 - \text{Ad}_g}{1 + \text{Ad}_g} \theta^L, \theta^L\right)\right),$$

at any given point $g \in \mathcal{U}$. (The square root depends on the choice of lift $\tau: G \rightarrow \text{Pin}(\mathfrak{g})$.)

Note that the exponent in this formula becomes singular where $1 + \text{Ad}_g$ fails to be invertible, but these singularities are compensated by the zeroes of the factor $\det^{1/2}\left(\frac{1 + \text{Ad}_g}{2}\right)$. One proof of this formula is given in [47]; here is an outline of an alternative approach.

Sketch of proof. — One easily checks that over $\mathcal{U}$, F_G coincides with the graph of the 2-form $\omega_F := -\frac{1}{4} B\left(\frac{1 - \text{Ad}_g}{1 + \text{Ad}_g} \theta^L, \theta^L\right)$. Hence $\psi_G|_{\mathcal{U}} = f \exp(-\omega_F)$ for some nonvanishing function $f \in C^\infty(\mathcal{U})$, with $f(e) = 1$. Equation (85) reads, after dividing by $f \exp(-\omega_F)$,

$$d \log(f) + \eta + \exp(\omega_F) \varrho(e(\Xi)) (\exp(-\omega_F)) = 0.$$

Taking the form degree 1 parts of both sides of this equation, one obtains the following condition on f :

$$d \log(f) + \left(\exp(\omega_F) \varrho(e(\Xi)) (\exp(-\omega_F)) \right)_{[1]} = 0.$$

f is uniquely determined by this Equation with the initial condition $f(e) = 1$. It is straightforward (though slightly cumbersome) to verify that $f(g) = \det^{1/2}\left(\frac{1 + \text{Ad}_g}{2}\right)$ solves this equation. $\square$

If G is connected, one has $\det(1 + \text{Ad}_g) \neq 0$ on a dense open subset of G . However, for a disconnected group G it vanishes on the components with $\det(\text{Ad}_g) = -1$.

Example 4.7. — Let $G = \text{O}(2)$. Here $\text{O}(\mathfrak{g}) = \mathbb{Z}_2$ and $\text{Pin}(\mathfrak{g}) = \mathbb{Z}_4$. There are two possible lifts $\text{O}(\mathfrak{g}) \rightarrow \text{Pin}(\mathfrak{g})$. Let $\theta \in \Omega^1(G)$ be the left-invariant Maurer-Cartan form (using the isomorphism $\mathfrak{g} = \mathbb{R}$ defined by a generator $\mu \in \det(\mathfrak{g}) = \mathfrak{g}$). One finds that on $\text{SO}(2) \subset \text{O}(2)$, $\phi_G = \theta$, while $\psi_G = 1$. On the non-identity component $\text{O}(2) \setminus \text{SO}(2)$, the roles are reversed: $\psi_G = \pm \theta$ and $\phi_G = \pm 1$. (The signs depend on the choice of lift.) Observe that ϕ_G, ψ_G given by these formulas have the correct equivariance properties.

4.3. Group multiplication. — In this section, we will examine the composition of the map $\mathcal{R}: \mathrm{Cl}(\mathfrak{g}) \rightarrow \Omega(G)$ with the pull-back under group multiplication. It will be convenient to work with the element $\Lambda \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G)$, defined by the property

$$\mathcal{R}(x) = \mathrm{str}(x\Lambda)$$

where we have extended $\mathrm{str}: \mathrm{Cl}(\mathfrak{g}) \rightarrow \wedge^{[\mathrm{top}]}(\mathfrak{g}) = \mathbb{K}$ to the tensor product with $\Omega(G)$. The properties of $\mathcal{R}$ under the Clifford action translate into

$$(l^{\mathrm{Cl}}(\xi) + \varrho(s^R(\xi)))\Lambda = 0, \quad (-r^{\mathrm{Cl}}(\xi) + \varrho(s^L(\xi)))\Lambda = 0.$$

Thus Λ is itself a pure spinor for the action of $\mathrm{Cl}(\mathfrak{d}) \times \mathrm{Cl}(\mathbb{T}G)$, defining a Lagrangian subbundle of $\mathfrak{d} \times \mathbb{T}G$. The equivariance properties (81) of $\mathcal{R}$ translate into

$$l_a^* \Lambda = \tau(a^{-1})\Lambda, \quad r_{a^{-1}}^* \Lambda = \Lambda\tau(a)$$

The first identity is immediate, while for the second identity is obtained by the calculation:

$$\begin{aligned} \mathrm{str}(x r_{a^{-1}}^* \Lambda) &= r_{a^{-1}}^* \mathcal{R}(x) = (-1)^{|a|(|x|+|g|)} \mathcal{R}(\tau(a)x) \\ &= (-1)^{|a|(|x|+|g|)} \mathrm{str}(\tau(a)x\Lambda) \\ &= \mathrm{str}(x\Lambda\tau(a)). \end{aligned}$$

(Note that $|a| = |g|$ at $g \in G$.) We finally observe that the pull-back of Λ to the group unit is simply

$$(86) \quad i_e^* \Lambda = 1 \in \mathrm{Cl}(\mathfrak{g}).$$

Let $\Lambda^1, \Lambda^2 \in \mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G \times G)$ be the pull-back to the first, second G -factor, and recall the 2-form $\varsigma \in \Omega^2(G \times G)$ from (47).

Proposition 4.8. — *The pull-back of Λ under group multiplication satisfies*

$$(87) \quad e^\varsigma \mathrm{Mult}^* \Lambda = \Lambda^1 \Lambda^2,$$

using the product in the algebra $\mathrm{Cl}(\mathfrak{g}) \otimes \Omega(G \times G)$.

Proof. — Using (51), we find that both sides of (87) are annihilated by the following operators:

$$l^{\mathrm{Cl}}(\xi) + \varrho(s^{R,1}(\xi)), \quad -r^{\mathrm{Cl}}(\xi) + \varrho(s^{L,2}(\xi)), \quad \varrho(s^{L,1}(\xi) + s^{R,2}(\xi)).$$

Hence the two sides of (87) are pure spinors, defining the same Lagrangian subbundle of $\mathfrak{d} \times \mathbb{T}(G \times G)$. So the two sides agree up to a scalar function.

The 2-form ς is invariant under $l_{a,1}$ (left multiplication by a on the first factor) and $r_{a^{-1},2}$ (right multiplication by a^{-1} on the second factor). From the equivariance of Λ , and since $\mathrm{Mult} \circ l_{a,1} = l_a \circ \mathrm{Mult}$ and $\mathrm{Mult} \circ r_{a^{-1},2} = r_{a^{-1}} \circ \mathrm{Mult}$, we obtain the following equivariance property of $e^\varsigma \mathrm{Mult}^* \Lambda$:

$$\begin{aligned} (l_{a,1})^*(e^\varsigma \mathrm{Mult}^* \Lambda) &= \tau(a^{-1}) (e^\varsigma \mathrm{Mult}^* \Lambda), \\ (r_{a^{-1},2})^*(e^\varsigma \mathrm{Mult}^* \Lambda) &= (e^\varsigma \mathrm{Mult}^* \Lambda) \tau(a). \end{aligned}$$

The product $\Lambda^1 \Lambda^2$ has a similar equivariance property. Hence, to verify (87) it suffices to compare the two sides at $(e, e) \in G \times G$. But by (86), both sides of (87) pull back to 1 at (e, e) . $\square$

We will use Proposition 4.8 to obtain a formula for the pull-back of $\psi_G = \mathcal{R}(q(\mu))$, the pure spinor defining the Lagrangian subbundle $F_G \subset \mathbb{T}G$. Recall the element $\gamma \in \wedge^2(\mathfrak{g} \oplus \mathfrak{g})$ from (53).

Theorem 4.9. — *The pull-back of ψ_G under group multiplication is given by the formula*

$$e^\varsigma \text{Mult}^* \psi_G = \varrho(\exp(-\mathbf{e}(\gamma))) (\psi_G^1 \otimes \psi_G^2)$$

Note that up to a scalar function, this identity follows from Proposition 3.11.

Proof. — The element γ enters the following formula (cf. [4, Lemma 3.1]), relating the product Mult^{Cl} in $\text{Cl}(\mathfrak{g})$ with the wedge product $\text{Mult}^\wedge$ in $\wedge(\mathfrak{g})$:

$$q^{-1} \circ \text{Mult}^{\text{Cl}} = \text{Mult}^\wedge \circ \exp(-\iota^\wedge(\gamma)) \circ q^{-1}: \text{Cl}(\mathfrak{g} \oplus \mathfrak{g}) \rightarrow \wedge(\mathfrak{g}).$$

Since $\text{str} \circ l^{\text{Cl}}(q(\mu)) \circ q: \wedge \mathfrak{g} \rightarrow \mathbb{K}$ is simply the augmentation map, we have

$$\psi_G = \mathcal{R}(q(\mu)) = \text{str}(q(\mu)\Lambda) = q^{-1}(\Lambda)_{[0]},$$

where the subscript indicates the degree 0 part with respect to $\wedge \mathfrak{g}$. Using (87), we calculate:

$$\begin{aligned} e^\varsigma \text{Mult}^* \psi_G &= q^{-1}(\Lambda^1 \Lambda^2)_{[0]} \\ &= q^{-1} \circ (\text{Mult}^{\text{Cl}}(\Lambda^1 \otimes \Lambda^2))_{[0]} \\ &= (\text{Mult}^\wedge \circ \exp(-\iota^\wedge(\gamma)) \circ q^{-1}(\Lambda^1 \otimes \Lambda^2))_{[0]} \\ &= \exp(-\mathbf{e}(\gamma)) \circ (\text{Mult}^\wedge \circ q^{-1}(\Lambda^1 \otimes \Lambda^2))_{[0]} \\ &= \exp(-\mathbf{e}(\gamma)) \circ (\psi_G^1 \otimes \psi_G^2). \end{aligned}$$

Here we used that $(\iota^{\text{Cl}}(\xi) + \varrho(\mathbf{e}(\xi)))\Lambda = 0$, hence $(\iota^\wedge(\gamma) - \varrho(\mathbf{e}(\gamma)))q^{-1}(\Lambda^1 \otimes \Lambda^2) = 0$. $\square$

4.4. Exponential map. — Let us return to our description (Section 3.5) $\mathbb{T}\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{d}_0$ of the Courant algebroid over $\mathfrak{g}^*$, where $\mathfrak{d}_0 = \mathfrak{g}^* \rtimes \mathfrak{g}$.

Let $\wedge \mathfrak{g}^*$ be the contravariant spinor module over $\text{Cl}(\mathfrak{d}_0)$ (cf. Section 1.4), with Clifford action denoted $\varrho^\wedge$. Let $d^\wedge$ be the exterior algebra differential. For all $w = (\beta, \xi) \in \mathfrak{d}_0$ one has

$$L^\wedge(w) := [d^\wedge, \varrho^\wedge(w)] = d^\wedge \beta - (\text{ad}_\xi)^*.$$

One easily checks that $L^\wedge(w)$ defines an action of the Lie algebra $\mathfrak{d}_0$. This action exponentiates to an action of the group D_0 , given as

$$\mathcal{U}^\wedge(\beta, g)y = \exp(d^\wedge \beta) \wedge (\text{Ad}_{g^{-1}})^* y,$$

The function

$$\tau_0: \mathfrak{g}^* \rightarrow \wedge \mathfrak{g}^*, \quad \tau_0(\beta) = \exp(d^\wedge \beta) \in \wedge \mathfrak{g}^*$$

is the counterpart to the function $\tau: G \rightarrow \text{Cl}(\mathfrak{g})$. The D_0 -action commutes with the differential, and it is straightforward to check that the Clifford action is D_0 -equivariant:

$$\mathcal{U}^\wedge(\beta, g)(\varrho^\wedge(w)y) = \varrho^\wedge(\text{Ad}_{(\beta, g)} w)(\mathcal{U}^\wedge(\beta, g)y),$$

for $w \in \mathfrak{d}_0$, $(\beta, g) \in D_0$, $y \in \wedge \mathfrak{g}^*$.

Choose a generator $\mu \in \det(\mathfrak{g}^*)$, and let $\star: \wedge \mathfrak{g} \rightarrow \wedge \mathfrak{g}^*$ be the associated star operator⁽⁴⁾. Let X_π denote the modular vector field of the Kirillov-Poisson structure $\pi_{\mathfrak{g}^*}$, relative to the translation-invariant volume form $\mu_{\mathfrak{g}^*} \in \Gamma(\det(T^*\mathfrak{g}^*))$ defined by the dual generator $\mu^* \in \det(\mathfrak{g})$. (Recall that $X_\pi = 0$ if $\mathfrak{g}$ is unimodular.) Define a linear map

$$\mathcal{R}_0: \wedge \mathfrak{g}^* \rightarrow \Omega(\mathfrak{g}^*),$$

given at any point $\nu \in \mathfrak{g}^*$ by

$$\mathcal{R}_0(y) = \star^{-1}(y \wedge \tau_0(\nu)) \in \wedge \mathfrak{g} = \wedge T_\nu^* \mathfrak{g}^*.$$

Parallel to Proposition 4.2, we have,

Proposition 4.10. — a. *The map $\mathcal{R}_0$ intertwines the Clifford actions of $\mathfrak{d}_0$:*

$$\mathcal{R}_0 \circ \varrho^\wedge(w) = \varrho(\mathfrak{s}_0(w)) \circ \mathcal{R}_0, \quad w \in \mathfrak{d}_0.$$

It is uniquely determined by this property, up to a scalar function.

b. *The map $\mathcal{R}_0$ intertwines the differentials, up to contraction by the modular vector field:*

$$\mathcal{R}_0 \circ d^\wedge = (d - \iota(X_\pi)) \circ \mathcal{R}_0.$$

c. *$\mathcal{R}_0$ has the equivariance property, for all $h = (\beta, a) \in D_0 = \mathfrak{g}^* \rtimes G$,*

$$(\mathcal{U}_0(h^{-1}))^* \mathcal{R}_0(y) = \det(\text{Ad}_a) \mathcal{R}_0(\mathcal{U}^\wedge(h)y).$$

d. *$\mathcal{R}_0$ preserves the bilinear pairings on the spinor modules $\wedge \mathfrak{g}^*$, $\Omega(\mathfrak{g}^*)$, in the sense that*

$$(\mathcal{R}_0(y), \mathcal{R}_0(y'))_{\wedge T^* \mathfrak{g}^*} = (y, y')_{\wedge \mathfrak{g}^*} \mu_{\mathfrak{g}^*}$$

for all $y, y' \in \wedge \mathfrak{g}^$.*

Proof. — Each of the statements (a),(c),(d) is proved by a direct computation, parallel to those in Proposition 4.2. To prove (b), we first note that (c) implies the infinitesimal equivariance, for $(\beta, \xi) \in \mathfrak{d}_0$,

$$(88) \quad (\mathcal{L}(\mathcal{U}_0(\beta, \xi)) - \text{tr}(\text{ad}_\xi)) \mathcal{R}_0(y) = \mathcal{R}_0(\mathcal{L}_0^\wedge(\beta, \xi)y).$$

Since $\iota(X_\pi)\langle\theta_0, \xi\rangle = \text{tr}(\text{ad}_\xi)$, we have

$$\mathcal{L}(\mathcal{U}_0(\beta, \xi)) - \text{tr}(\text{ad}_\xi) = [(d - \iota(X_\pi)), \varrho(\mathfrak{s}_0(\beta, \xi))].$$

⁽⁴⁾ Note that in the previous Section, μ denoted a generator of $\det(\mathfrak{g})$, and hence the star operator went from $\wedge \mathfrak{g}^* \rightarrow \wedge \mathfrak{g}$. This change in notation is intended, since our aim is to compare the Poisson manifold $\mathfrak{g}^*$ with the Dirac manifold G .

Hence we can re-write (88) as

$$[(d - \iota(X_\pi)), \varrho(s_0(\beta, \xi))] \mathcal{R}_0(y) = \mathcal{R}_0([d^\wedge, \varrho^\wedge(\beta, \xi)]).$$

Together with (a), this implies that the linear map

$$(89) \quad (d - \iota(X_\pi)) \circ \mathcal{R}_0 - \mathcal{R}_0 \circ d^\wedge: \wedge \mathfrak{g}^* \rightarrow \Omega(\mathfrak{g}^*)$$

intertwines the Clifford actions of $\mathfrak{d}_0$. Since the parity of this map is opposite to that of $\mathcal{R}_0$, the uniqueness assertion in (a) implies that (89) is zero. $\square$

As before, we may use this map to construct pure spinors $\mathcal{R}_0(y) \in \Omega(\mathfrak{g}^*)$ from pure spinors $y \in \wedge \mathfrak{g}^*$.

The element $y = 1$ is the pure spinor defining the Lagrangian subspace $\mathfrak{g} \subset \mathfrak{d}_0$, and its image $\phi_{\mathfrak{g}^*} = \mathcal{R}_0(1)$ defines the Lagrangian subbundle $E_{\mathfrak{g}^*}$ (spanned by the sections $e_0(\xi)$). The pure spinor $y = \mu \in \wedge \mathfrak{g}^*$ defines a Lagrangian complement $\mathfrak{g}^* \subset \mathfrak{d}_0$, and its image $\psi_{\mathfrak{g}^*} = \mathcal{R}_0(\mu) = 1$ defines the Lagrangian subbundle $F_{\mathfrak{g}^*} = T\mathfrak{g}^*$ (spanned by the sections $f_0(\beta)$). For the bilinear pairing between these pure spinors, we obtain

$$(\phi_{\mathfrak{g}^*}, \psi_{\mathfrak{g}^*})_{\wedge T^*\mathfrak{g}^*} = \mu_{\mathfrak{g}^*}.$$

since $(1, \mu)_{\wedge \mathfrak{g}^*} = \mu$.

Lemma 4.11. — *The pure spinor $\phi_{\mathfrak{g}^*}$ is given by the formula*

$$\phi_{\mathfrak{g}^*} = (-1)^{n(n-1)/2} e^{-\iota(\pi_{\mathfrak{g}^*})} \mu_{\mathfrak{g}^*}$$

where $n = \dim G$.

Proof. — The Kirillov-Poisson bivector on $\mathfrak{g}^*$ is given by $\pi_{\mathfrak{g}^*}|_\nu = -d^\wedge \nu \in \wedge^2 \mathfrak{g}^* = \wedge^2 T_\nu \mathfrak{g}^*$. That is, $\tau_0 = \exp(-\pi_{\mathfrak{g}^*})$. The Lemma follows since $\star$ intertwines exterior product with contractions, and since $\star^{-1}(1) = (\mu^*)^\top = (-1)^{n(n-1)/2} \mu^*$. $\square$

Let us now return to our original setting where $\mathfrak{g}$ carries an invariant inner product B , used to identify $\mathfrak{g} \cong \mathfrak{g}^*$. We take the generators $\mu \in \det(\mathfrak{g})$ (from the last section) and $\mu \in \det(\mathfrak{g}^*)$ (from the present section) to be equal under this identification.

Let $\mu_{\mathfrak{g}}$ be the translation invariant volume form on $\mathfrak{g} \cong \mathfrak{g}^*$, and μ_G the corresponding left-invariant volume form on G . Let $J \in C^\infty(\mathfrak{g})$ be the Jacobian of the exponential map, defined by $\exp^* \mu_G = J \mu_{\mathfrak{g}}$. Recall that $\mathfrak{g}_{\mathfrak{h}} \subset \mathfrak{g}$ is the dense open subset where $\exp$ is a local diffeomorphism, i.e where $J \neq 0$. With $\varpi \in \Omega^2(\mathfrak{g})$ as in Section 3.5, we have:

Proposition 4.12. — *Over the subset $\mathfrak{g}_{\mathfrak{h}}$, the maps $\mathcal{R}_0: \wedge \mathfrak{g} \rightarrow \Omega(\mathfrak{g})$ and $\mathcal{R}: \text{Cl}(\mathfrak{g}) \rightarrow \Omega(G)$ are related as follows:*

$$(90) \quad \exp^*(\mathcal{R}(x)) = J^{1/2} e^{-\varpi} \varrho(\tilde{A}^{-e_0(\varepsilon)})(\mathcal{R}_0(y)),$$

for $x = q(y)$. Here $\varepsilon \in C^\infty(\mathfrak{g}_{\mathfrak{h}}, \wedge^2 \mathfrak{g})$ is the solution of the classical dynamical Yang-Baxter equation, cf. Proposition 3.14, and $J^{1/2} \in C^\infty(\mathfrak{g})$ is a smooth square root of J , equal to 1 at the origin.

Proof. — The map $\widetilde{\mathcal{R}}_0: \wedge \mathfrak{g} \rightarrow \Omega(\mathfrak{g}_{\mathfrak{h}})$ given as

$$\widetilde{\mathcal{R}}_0(y) = e^{-\varpi} \varrho(\widetilde{A}^{-e_0(\varepsilon)}) \exp^* \mathcal{R}(q(y))$$

intertwines the $\mathrm{Cl}(\mathfrak{d}_0)$ -actions, hence it coincides with $\widetilde{\mathcal{R}}_0 = f \mathcal{R}_0$ for a scalar function. To find f , we consider bilinear pairings. Note that

$$\begin{aligned} (\widetilde{\mathcal{R}}_0(y), \widetilde{\mathcal{R}}_0(y'))_{\wedge T^* \mathfrak{g}} &= (\exp^* \mathcal{R}(q(y)), \exp^* \mathcal{R}(q(y')))_{\wedge T^* \mathfrak{g}} \\ &= \exp^* (\mathcal{R}(q(y)), \mathcal{R}(q(y')))_{\wedge T^* G}. \end{aligned}$$

Taking $y' = 1$, $y = \mu$ we obtain

$$f^2 \mu_{\mathfrak{g}} = f^2 (\mathcal{R}_0(\mu), \mathcal{R}_0(1))_{\wedge T^* \mathfrak{g}} = (\widetilde{\mathcal{R}}_0(\mu), \widetilde{\mathcal{R}}_0(1))_{\wedge T^* \mathfrak{g}} = \exp^* \mu_G = J \mu_{\mathfrak{g}}.$$

This shows that $f^2 = J$. $\square$

Remark 4.13. — Of course, $\exp^*(R(x))$ is defined globally on all of $\mathfrak{g}$, not only on $\mathfrak{g}_{\mathfrak{h}}$. It follows from the Proposition that $J^{1/2} \exp(e_0(\varepsilon))$ extends smoothly to all of $\mathfrak{g}$. Hence, the expression

$$J^{1/2} \exp(\varepsilon)$$

extends smoothly to a global function $\mathfrak{g} \rightarrow \wedge \mathfrak{g}$. For a direct proof, see [5].

Applying the proposition to $y = 1$ and $y = \mu$, we find in particular that

$$\begin{aligned} (91) \quad \exp^* \phi_G &= J^{1/2} e^{-\varpi} \phi_{\mathfrak{g}}, \\ \exp^* \psi_G &= J^{1/2} e^{-\varpi} \varrho(\widetilde{A}^{-e_0(\varepsilon)})(1). \end{aligned}$$

4.5. The Gauss-Dirac spinor. — We return to the set-up of Section 3.6, with $G = K^{\mathbb{C}}$ denoting the complexification of a compact Lie group, with Cartan subgroup $T = T_K^{\mathbb{C}}$. Recall that the Gauss-Dirac structure $\widehat{F}_G$ is defined by the Lagrangian subspace $\mathfrak{s} \subset \mathfrak{d}$, with basis the collection of all $e_{\alpha} \oplus 0$, $0 \oplus e_{-\alpha}$, $e_i \oplus (-e_i)$ where $\alpha \succ 0$ are positive roots and $i = 1, \dots, l = \mathrm{rank}(G)$. The element

$$(92) \quad x = \prod_{\alpha \succ 0} e_{\alpha} e_{-\alpha} \prod_i e_i \in \mathrm{Cl}(\mathfrak{g})$$

is non-zero and is annihilated by the Clifford action of $\mathfrak{s}$; hence it is a pure spinor defining $\mathfrak{s}$. Note that x satisfies

$$\tau(h_+)x = x, \quad x\tau(h_-^{-1}) = x, \quad \tau(h_0)x\tau(h_0) = h_0^{2\rho}x$$

for all $h_+ \in N_+$, $h_- \in N_-$, $h_0 \in T$. Here $\rho = \frac{1}{2} \sum_{\alpha \succ 0} \alpha$, and $t \mapsto t^{2\rho} \in \mathbb{C}^{\times}$ is the character of T defined by the weight 2ρ . Hence,

$$\widehat{\psi}_G = \mathcal{R}(x) \in \Omega(G)$$

is a pure spinor defining $\widehat{F}_G$. We refer to $\widehat{\psi}_G$ as the *Gauss-Dirac spinor*. Its equivariance properties are:

$$l_{h_+}^* \widehat{\psi}_G = \widehat{\psi}_G, \quad r_{h_-^{-1}}^* \widehat{\psi}_G = \widehat{\psi}_G, \quad l_{h_0}^* r_{h_0}^* \widehat{\psi}_G = h_0^{2\rho} \widehat{\psi}_G.$$

That is, $\widehat{\psi}_G$ is invariant up to the character, given by the group homomorphism $S \rightarrow T$ followed by the 2ρ -character.

Since the big Gauss cell $\mathcal{O} = N_-TN_+ \subset G$ is dense in G , the equivariance property, together with the fact that the pull-back of ψ_G to the group unit is equal to $\text{str}(x) = 1$, completely characterizes the pure spinor ψ_G , and allows us to give an explicit formula. Recall the 2-form $\omega_{\mathcal{O}}$ on the big Gauss cell, given by (68):

Proposition 4.14. — *The restriction of the pure spinor $\widehat{\psi}_G$ to the big Gauss cell $\mathcal{O} = j(N_- \times T \times N_+)$ is given by the formula,*

$$\widehat{\psi}_G|_{\mathcal{O}} = g_0^{\rho} \exp(-\omega_{\mathcal{O}}).$$

Here $g_0: \mathcal{O} \rightarrow T$ is the composition of the Gauss decomposition $j^{-1}: \mathcal{O} \rightarrow N_- \times T \times N_+$ with projection to the middle factor.

Proof. — Both sides are pure spinors defining the Gauss-Dirac structure over $\mathcal{O}$, with the same equivariance property under S , and both sides pull back to 1 at the group unit e . $\square$

We now compare the Gauss-Dirac spinor $\widehat{\psi}_G$ with the pure spinor ψ_G from Proposition 4.4.

Proposition 4.15. — *The pure spinors $\psi_G, \widehat{\psi}_G$ are related by a twist by the r -matrix τ :*

$$\widehat{\psi}_G = \varrho(\exp(-\mathbf{e}(\tau)))\psi_G.$$

Proof. — Let $\tau_{\Delta} \in \wedge^2 \mathfrak{d}$ be the image of τ under the diagonal inclusion $\mathfrak{g} \hookrightarrow \mathfrak{d}$. We will show that

$$(93) \quad x = \varrho^{\text{Cl}}(\exp(-\tau_{\Delta}))q(\mu).$$

The proposition follows from this identity by applying the map $\mathcal{R}$. Up to a scalar, (93) holds since both sides are pure spinors defining the same Lagrangian subspace. To determine the scalar, we apply the super-trace to both sides. Recall that the spinor action of elements $\xi_{\Delta} \in \mathfrak{g}_{\Delta} \subset \mathfrak{g}$ is given by Clifford commutator with the corresponding element $\xi \in \mathfrak{g}$. Since the super-trace vanishes on Clifford commutators, it follows that

$$\text{str}(\varrho^{\text{Cl}}(\exp(-\tau))q(\mu)) = \text{str}(q(\mu)) = 1 = \text{str}(x). \quad \square$$

Let us next compute the Clifford differential $d^{\text{Cl}} = -4[q(\Xi), \cdot]$ of the element (92). Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \in \mathfrak{t}^*$ be the half-sum of positive (real) roots.

Lemma 4.16. — *The quantization of the structure constant tensor satisfies,*

$$-4q(\Xi) = 2\pi\sqrt{-1}\rho \mod \mathfrak{n}_- \text{Cl}(\mathfrak{g})\mathfrak{n}_+.$$

Here B is used to identify $\mathfrak{g}^* \cong \mathfrak{g}$.

Proof. — By definition,

$$-4q(\Xi) = \frac{1}{6} \sum B([e^a, e^b], e^c) e_a e_b e_c,$$

using a basis e_a of $\mathfrak{g}$, with B -dual basis e^a . Take this basis to be the Cartan-Weil basis, and use the Clifford relations to write factors $e_{-\alpha}$ to the left and factors e_{α} to the right. Then

$$-4q(\Xi) \in \mathrm{Cl}(\mathfrak{g})^T \subset \mathfrak{n}_- \mathrm{Cl}(\mathfrak{g}) \mathfrak{n}_+ \oplus \mathrm{Cl}(\mathfrak{t}).$$

(For a T -equivariant element in $\mathrm{Cl}(\mathfrak{g})$, the T -weight of the $\mathfrak{n}_-$ -factors must be compensated by the T weights of the $\mathfrak{n}_+$ -factors.) Since $-4q(\Xi)$ is an odd element of filtration degree 3, and since Ξ has no component in $\wedge^3 \mathfrak{t}$, it follows that

$$-4q(\Xi) \in \mathfrak{t} \oplus \mathfrak{n}_- \mathrm{Cl}(\mathfrak{g}) \mathfrak{n}_+.$$

To compute the $\mathfrak{t}$ -component, we calculate the constant component of

$$[\xi, -4q(\Xi)]_{\mathrm{Cl}} = d^{\mathrm{Cl}} \xi = q(\lambda(\xi))$$

for any $\xi \in \mathfrak{t}$. We have

$$\lambda(\xi) = - \sum_{\alpha \succ 0} [\xi, e_{-\alpha}] \wedge e_{\alpha} = 2\pi\sqrt{-1} \sum_{\alpha \succ 0} \langle \alpha, \xi \rangle e_{-\alpha} \wedge e_{\alpha},$$

hence (see Sternberg [52, Equation (9.25)])

$$q(\lambda(\xi)) = 2\pi\sqrt{-1} \sum_{\alpha \succ 0} \langle \alpha, \xi \rangle e_{-\alpha} e_{\alpha} + 2\pi\sqrt{-1} \langle \rho, \xi \rangle. \quad \square$$

As a consequence, we obtain,

Proposition 4.17. — *The element $x = \prod_{\alpha \succ 0} e_{\alpha} e_{-\alpha} \prod_i e_i$ satisfies,*

$$(d^{\mathrm{Cl}} - 2\pi\sqrt{-1} \iota^{\mathrm{Cl}}(\rho))x = 0.$$

Proof. — d^{Cl} is given as the Clifford commutator with $-4q(\Xi)$. Since x is annihilated under both left and right multiplication by elements of $\mathfrak{n}_- \mathrm{Cl}(\mathfrak{g}) \mathfrak{n}_+$, it follows that

$$d^{\mathrm{Cl}}(x) = 2\pi\sqrt{-1} [\rho, x]_{\mathrm{Cl}}. \quad \square$$

As a consequence, the Gauss-Dirac spinor satisfies the differential equation:

$$(94) \quad (d + \eta - 2\pi\sqrt{-1} \varrho(e(\rho))) \hat{\psi}_G = 0.$$

In fact, there is a more general version of this Equation, stated in the following Proposition. For any (real) dominant weight λ of G (not to be confused with the map λ above), let $\Delta_{\lambda} \in C^{\infty}(G)$ be the function

$$\Delta_{\lambda}(g) = \frac{\langle v_{\lambda}, g \cdot v_{\lambda} \rangle}{\langle v_{\lambda}, v_{\lambda} \rangle},$$

where v_λ is a highest weight vector in the irreducible unitary representation $(V_\lambda, \langle \cdot, \cdot \rangle)$ of highest weight λ . The function Δ_λ is invariant under the left-action of N_- , under the right-action of N_+ , and under the T -action it satisfies

$$(95) \quad \Delta_\lambda(tg) = \Delta_\lambda(gt) = t^\lambda \Delta_\lambda(g).$$

Since $\Delta_\lambda(e) = 1$, it follows that $\Delta_\lambda \neq 0$ on the big Gauss cell. We are interested in the product $\Delta_\lambda \hat{\psi}_G$. Away from the zeroes of Δ_λ , this is a pure spinor defining the Gauss-Dirac structure. Similar to $\hat{\psi}_G$, it is invariant under the left-action of N_- and the right-action of N_+ , and satisfies

$$(96) \quad l_t^*(\Delta_\lambda \hat{\psi}_G) = r_t^*(\Delta_\lambda \hat{\psi}_G) = t^{\lambda+\rho}(\Delta_\lambda \hat{\psi}_G)$$

for all $t \in T$.

Proposition 4.18. — *For any dominant weight λ , the product $\Delta_\lambda \hat{\psi}_G$ satisfies the differential equation:*

$$(97) \quad (d + \eta - 2\pi\sqrt{-1}\varrho(e(\lambda + \rho)))\Delta_\lambda \hat{\psi}_G = 0,$$

where B is used to identify $\mathfrak{g}^* \cong \mathfrak{g}$.

Proof. — Let $\mathfrak{s} \subset \mathfrak{d}$ be the Lagrangian subalgebra (66) defining the Gauss-Dirac structure. We have, for all $\zeta = (\xi, \xi') \in \mathfrak{s}$,

$$\begin{aligned} & \varrho(s(\zeta))(d + \eta - 2\pi\sqrt{-1}\varrho(e(\lambda + \rho)))\Delta_\lambda \hat{\psi}_G \\ &= \left[\varrho(s(\zeta)), d + \eta - 2\pi\sqrt{-1}\varrho(e(\lambda + \rho)) \right] \Delta_\lambda \hat{\psi}_G \\ &= (\mathcal{L}(\xi^L - (\xi')^R) - 2\pi\sqrt{-1}B(\xi - \xi', \lambda + \rho))\Delta_\lambda \hat{\psi}_G = 0, \end{aligned}$$

where the last equality follows from the equivariance properties (96) of $\Delta_\lambda \hat{\psi}_G$. (Note that for the elements of the form $\zeta = (\xi, 0)$ with $\xi \in \mathfrak{n}_+$ or $\zeta = (0, \xi)$ with $\xi \in \mathfrak{n}_-$, the inner product with $\lambda + \rho \in \mathfrak{t}$ vanishes.) Hence, the left hand side of (97) is annihilated by all $s(\zeta)$, for $\zeta \in \mathfrak{s}$. Hence it is a function times $\hat{\psi}_G$, and thus vanishes since it has parity opposite to that of $\hat{\psi}_G$. $\square$

Remark 4.19. — The holomorphic Dirac structure $\hat{F}_G$ on $G = K^\mathbb{C}$ restricts to a complex Dirac structure $\hat{F}_G|_K = \hat{F}_K$ on the real Lie group K , with defining pure spinor the pull-back (restriction) $\hat{\psi}_K$ of $\hat{\psi}_G$. On the other hand, $E_G|_K = (E_K)^\mathbb{C}$. In the notation of Section 2.4, applied to the Gauss-Cartan-splitting $(\mathbb{T}K)^\mathbb{C} = E_K^\mathbb{C} \oplus \hat{F}_K$, we have $\sigma = 2\pi\sqrt{-1}e(\rho) \in \Gamma((\mathbb{T}K)^\mathbb{C})$, thus $\hat{\phi}_\pm = d + \eta \pm 2\pi\sqrt{-1}\varrho(e(\rho))$. As usual, $\hat{\phi}_+ \phi_K = 0$, $\hat{\phi}_- \hat{\psi}_K = 0$ (the second equation is the pull-back of (94) to K). Let μ be the bi-invariant (real) volume form on K defined by $\phi_K, \hat{\psi}_K$. Since $\hat{\phi}_\pm^2 = \pm 2\pi\sqrt{-1}\mathcal{L}(\mathcal{U}_{\text{ad}}(\rho))$, the Dirac cohomology groups $H_\pm(E_K^\mathbb{C}, \hat{F}_K, \mu)$ are the cohomology groups of $\hat{\phi}_\pm$ on the space of $\mathcal{U}_{\text{ad}}(\rho)$ -invariant complex-valued differential forms on K . These may be computed by the standard localization argument ([12], see also [33]): The set of zeroes of the vector field $\mathcal{U}_{\text{ad}}(\rho)$ on K is just the maximal torus T_K , and the pull-back to T_K intertwines $\hat{\phi}_\pm$ with $d \pm 2\pi\sqrt{-1}B(\theta_T, \rho)$, with θ_T

the Maurer-Cartan form on T_K . Hence, by localization the pull-back to T_K induces an isomorphism,

$$H_{\pm}(E_K^{\mathbb{C}}, \widehat{F}_K, \mu) \cong H(\Omega(T_K)^{\mathbb{C}}, d \pm 2\pi\sqrt{-1}B(\theta_T, \rho))$$

Since ρ is a weight, it defines a T_K -character t^{ρ} , and the operators $d \pm 2\pi\sqrt{-1}B(\theta_T, \rho)$ are obtained from d by conjugation by $t^{\pm\rho}$. Hence $H_{\pm}(E_K^{\mathbb{C}}, \widehat{F}_K, \mu) \cong H(T_K)^{\mathbb{C}}$.

5. q-Hamiltonian G -manifolds

In this section, we use the techniques developed in this paper to extend the theory of group-valued moment maps, as developed in [3, 8] for the case of compact Lie groups, to more general settings.

5.1. Dirac morphisms and group-valued moment maps. — We briefly recall the definitions.

Definition 5.1. — A *quasi-Hamiltonian $\mathfrak{g}$ -manifold* (or simply *q-Hamiltonian $\mathfrak{g}$ -manifold*) is a manifold M with a Lie algebra action $\mathcal{U}_M: \mathfrak{g} \rightarrow \mathfrak{X}(M)$, a 2-form ω , and a $\mathfrak{g}$ -equivariant *moment map* $\Phi: M \rightarrow G$ such that

$$\begin{aligned} d\omega &= \Phi^*\eta \\ (98) \quad \iota(\mathcal{U}_M(\xi))\omega &= \Phi^*B(\xi, \frac{\theta^L + \theta^R}{2}) \quad (\text{moment map condition}) \\ \ker(\omega_m) &= \{\mathcal{U}_M(\xi)_m \mid \text{Ad}_{\Phi(m)}\xi = -\xi\} \quad (\text{minimal degeneracy condition}). \end{aligned}$$

If the action of $\mathfrak{g}$ extends to an action of the Lie group G , and if ω and Φ are equivariant for the action of G , we speak of a *q-Hamiltonian G -manifold*.

The first two conditions in (98) imply that ω is $\mathfrak{g}$ -invariant (see [3]). As shown by Bursztyn-Crainic [14], the definition of a q-Hamiltonian space may be restated in Dirac geometric terms (see also Xu [57] for another interpretation).

Theorem 5.2. — *There is a 1-1 correspondence between q-Hamiltonian $\mathfrak{g}$ -manifolds, and manifolds M together with a strong Dirac morphism*

$$(99) \quad (\Phi, \omega): (M, TM, 0) \rightarrow (G, E_G, \eta).$$

More precisely, $(M, \mathcal{U}_M, \omega, \Phi)$ satisfies the first two conditions if and only if (Φ, ω) is a Dirac morphism, and in this case the third condition is equivalent to this Dirac morphism being strong.

Proof. — Let $(M, \mathcal{U}_M, \omega, \Phi)$ be a q-Hamiltonian $\mathfrak{g}$ -space. Given $m \in M$, let $E'_{\Phi(m)}$ be the forward image of $T_m M$ under $((d\Phi)_m, \omega_m)$:

$$E'_{\Phi(m)} = \{(d\Phi(v), \alpha) \mid v \in T_m M, (d\Phi)_m^* \alpha = \iota(v)\omega_m\}.$$

Taking v of the form $\mathcal{Q}_M(\xi)_m$ for $\xi \in \mathfrak{g}$, and using the moment map condition, we see $E'_{\Phi(m)} \supset (E_G)_{\Phi(m)}$. In fact, one has equality since both are Lagrangian subspaces. This shows that (Φ, ω) is a Dirac morphism. In particular,

$$(d\Phi)_m(\ker(\omega_m)) = \ker((E_G)_{\Phi(m)}) = \{\mathcal{Q}_{\text{ad}}(\xi)_{\Phi(m)} \mid \text{Ad}_{\Phi(m)} \xi = -\xi\}.$$

Hence, the minimal degeneracy condition holds if and only if $(d\Phi)_m$ restricts to an isomorphism on $\ker(\omega_m)$, i.e. if and only if (Φ, ω) is a strong Dirac morphism. Conversely, given a strong Dirac morphism (99), the associated map $\mathfrak{a}$ defines a $\mathfrak{g}$ -action $\mathcal{Q}_M(\xi) = \mathfrak{a}(\Phi^*e(\xi))$ on M , for which the map Φ is $\mathfrak{g}$ -equivariant. The above argument then shows that $(M, \mathcal{Q}_M, \omega, \Phi)$ is a q -Hamiltonian $\mathfrak{g}$ -space. $\square$

Remark 5.3. — As a consequence of this result (or rather its proof), we see that if $(M, \mathcal{Q}_M, \omega, \Phi)$ satisfies the first two conditions in (98), then the third condition (minimal degeneracy) is equivalent to the transversality property [15, 57]

$$\ker(\omega) \cap \ker(d\Phi) = \{0\}.$$

Remark 5.4. — There is a similar result for q -Hamiltonian G -manifolds. Here, it is necessary to assume the existence of a G -action on M for which the Dirac morphism (Φ, ω) is equivariant, and such that the infinitesimal action coincides with that defined by $\mathfrak{a}$.

Example 5.5. — By Example 2.7, the inclusion of the conjugacy classes $\mathcal{C}$ in G , with 2-forms defined by the Cartan-Dirac structure, defines a strong Dirac morphism $(\iota_{\mathcal{C}}, \omega_{\mathcal{C}})$. Thus, conjugacy classes are q -Hamiltonian G -manifolds.

Using our results on the Cartan-Dirac structure, it is now straightforward to deduce the basic properties of q -Hamiltonian spaces $(M, \mathcal{Q}_M, \omega, \Phi)$. In contrast with the original treatment in [3], the discussion works equally well for non-compact Lie groups, and also in the holomorphic category.

Theorem 5.6 (Fusion). — *Let $(M, \mathcal{Q}_M, \Phi, \omega)$ be a q -Hamiltonian $G \times G$ -manifold. Let $\mathcal{Q}_{\text{fus}}$ be the diagonal G -action, $\Phi_{\text{fus}} = \text{Mult} \circ \Phi$, and $\omega_{\text{fus}} = \omega + \Phi^*\varsigma$, with $\varsigma \in \Omega^2(G^2)$ the 2-form defined in (47). Then $(M, \mathcal{Q}_{\text{fus}}, \Phi_{\text{fus}}, \omega_{\text{fus}})$ is a q -Hamiltonian G -manifold. (An analogous statement holds for q -Hamiltonian $\mathfrak{g} \times \mathfrak{g}$ -manifolds.)*

Proof. — Since

$$(\Phi_{\text{fus}}, \omega_{\text{fus}}) = (\text{Mult}, \varsigma) \circ (\Phi, \omega)$$

is a composition of two strong Dirac morphism, it is itself a strong Dirac morphism from $(M, TM, 0)$ to (G, E_G, η) . The induced map $M \times \mathfrak{g} = \Phi_{\text{fus}}^* E_G \rightarrow TM$ is a composition of the map $\text{Mult}^* E_G \rightarrow E_{G \times G}$ defined by the strong Dirac morphism (Mult, ς) , with the map $\Phi^* E_G \times E_G \rightarrow TM$ given by the strong Dirac morphism (Φ, ω) . If we use the sections $e(\xi)$ to identify $E_G \cong G \times \mathfrak{g}$, the latter map is the $\mathfrak{g} \times \mathfrak{g}$ -action on M , while the former is the diagonal inclusion $\mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g}$. This confirms that the resulting action is just the diagonal action. $\square$

If $M = M_1 \times M_2$ is a direct product of two q -Hamiltonian manifolds, the quadruple $(M, \mathcal{A}_{\text{fus}}, \Phi_{\text{fus}}, \omega_{\text{fus}})$ is called the *fusion product* of M_1, M_2 . In particular we obtain products of conjugacy classes as new examples of q -Hamiltonian G -spaces.

Suppose $(M, \mathcal{A}_M, \omega_0, \Phi_0)$ is a Hamiltonian $\mathfrak{g}$ -manifold: That is, ω_0 is symplectic, and $\Phi_0: M \rightarrow \mathfrak{g}^*$ is the moment map for a Hamiltonian $\mathfrak{g}$ -action on M . As is well-known, this is equivalent to Φ_0 being a Poisson map from the symplectic manifold (M, ω_0) to the Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}^*})$. But this is also equivalent to

$$(\Phi_0, \omega_0): (M, TM, 0) \rightarrow (\mathfrak{g}^*, \text{Gr}_{\pi}, 0)$$

being a strong Dirac morphism. A *Hamiltonian G -manifold* comes with a G -action on M integrating the $\mathfrak{g}$ -action, and such that the Dirac morphism (Φ_0, ω_0) is equivariant. Given an invariant inner product B on $\mathfrak{g}$, used to identify $\mathfrak{g}^* \cong \mathfrak{g}$, we may compose the Dirac morphism (Φ_0, ω_0) with the Dirac morphism $(\exp, \varpi)$ from Theorem 3.13, and obtain:

Theorem 5.7 (Exponentials). — *Suppose $(M, \mathcal{A}_M, \omega_0, \Phi_0)$ is a Hamiltonian G -manifold, and let $\omega = \omega_0 + \Phi_0^* \varpi$, $\Phi = \exp \circ \Phi_0$. Then $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98). On $M_{\mathfrak{h}} = \Phi_0^{-1}(\mathfrak{g}_{\mathfrak{h}})$, the third condition (minimal degeneracy) holds as well, thus $(M_{\mathfrak{h}}, \mathcal{A}_M, \omega, \Phi)$ is a q -Hamiltonian G -manifold. (Similar statements hold for q -Hamiltonian $\mathfrak{g}$ -manifolds.)*

5.2. Volume forms. — Any symplectic manifold (M, ω) carries a distinguished volume form, given as the top degree component $\exp(\omega)^{[\dim M]} = \frac{1}{n!} \omega^n$. For a q -Hamiltonian G -manifold $(M, \mathcal{A}_M, \omega, \Phi)$, the 2-form ω is usually degenerate, hence $\exp(\omega)^{[\text{top}]}$ will have zeroes. Nevertheless, any q -Hamiltonian G -manifold carries a distinguished volume form, provided the adjoint action $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ lifts to $\text{Pin}(\mathfrak{g})$:

Theorem 5.8 (Volume forms). — *Suppose the adjoint action $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ lifts to $\text{Pin}(\mathfrak{g})$, and let $\psi_G \in \Omega(G)$ be the pure spinor defined by this lift. For any q -Hamiltonian G -manifold $(M, \mathcal{A}_M, \omega, \Phi)$, the differential form*

$$(100) \quad \mu_M = (\exp(\omega) \wedge \Phi^* \psi_G)^{[\dim M]}$$

is a volume form. It has the equivariance property $\mathcal{A}_M(g)^ \mu_M = \det(\text{Ad}_g) \mu_M$. More generally, if $(M, \mathcal{A}_M, \omega, \Phi)$ satisfies the first two conditions in (98), the form μ_M is non-zero exactly at those points where ω satisfies the minimal degeneracy condition.*

Of course, the factor $\det(\text{Ad}_g) = \pm 1$ is trivial if G is connected.

Proof. — Since ψ_G is a pure spinor defining the complementary Lagrangian subbundle F_G , and since (Φ, ω) is a strong Dirac morphism, the pull-back $\Phi^* \psi_G$ is non-zero everywhere. Furthermore, $\exp(\omega) \Phi^* \psi_G$ is a pure spinor defining the backward image F of F_G under the Dirac morphism (Φ, ω) . Since F is transverse to TM (see Proposition 1.15), the top degree part of $\exp(\omega) \Phi^* \psi_G$ is nonvanishing. More generally, if $(M, \mathcal{A}_M, \omega, \Phi)$ only satisfies the first two conditions in (98), then the above argument

applies at all points of M where (Φ, ω) is a strong Dirac morphism. But these are exactly the points where $\Phi^*\psi_G$ is non-zero.

The equivariance property of μ_M is a direct consequence of the equivariance properties of ϕ_G and ψ_G described in Proposition 4.4. $\square$

The volume form μ_M is called the *Liouville volume form* of the q -Hamiltonian G -manifold $(M, \mathcal{A}_M, \omega, \Phi)$. Let $|\mu_M|$ be the associated measure. If the moment Φ is proper, the push-forward $\Phi_*|\mu_M|$ is a well-defined measure on G , called the *Duistermaat-Heckman measure*.

Remark 5.9. — For the case of compact Lie groups, the q -Hamiltonian Liouville forms and Duistermaat-Heckman measures were introduced in [8]. The fact that μ_M is a volume form was verified by ‘direct computation’. However, the argument in [8] does not extend to non-compact Lie groups.

Remark 5.10. — The expression $\exp(\omega)\Phi^*\psi_G$ entering the definition of the volume form μ_M satisfies the differential equation

$$(101) \quad (d + \iota(\mathcal{A}_M(\Xi)))\left(\exp(\omega)\Phi^*\psi_G\right) = 0.$$

This follows from the differential equation (85) for ψ_G together with Remark 1.5(a).

Proposition 5.11. — Suppose $(M, \mathcal{A}_M, \omega, \Phi)$ is a q -Hamiltonian G -manifold, and that Ad lifts to the Pin group. Then M is even-dimensional if $\det(\text{Ad}_\Phi) = +1$, and odd-dimensional if $\det(\text{Ad}_\Phi) = -1$. In particular, it is even-dimensional when G is connected, and in this case M carries a canonical orientation.

Proof. — The construction of ψ_G in terms of the map $\mathcal{R}$ (see Proposition 4.4) shows that the form ψ_G has even degree at points $g \in G$ with $\det(\text{Ad}_g) = 1$, and odd degree at points with $\det(\text{Ad}_g) = -1$. Hence, the parity of the volume form μ_M is determined by the parity of $\det(\text{Ad}_\Phi)$. If G is connected, the lift of Ad (which exists by assumption) is unique, and $\det(\text{Ad}_g) \equiv 1$. $\square$

Without the existence of a lift to $\text{Pin}(\mathfrak{g})$, the form ψ_G is only defined locally, up to sign. That is, we still obtain a G -invariant *measure* on M , given locally as $(e^\omega \Phi^* \psi_G)^{[\text{top}]}$. It is interesting to specialize these results to conjugacy classes:

Theorem 5.12. — Suppose G is a connected Lie group, whose Lie algebra carries an invariant inner product B . Then:

- Every conjugacy class $\mathcal{C} \subset G$ carries a distinguished invariant measure (depending only on B).
- The conjugacy class $\mathcal{C}$ of $g \in G$ is even-dimensional if and only if $\det(\text{Ad}_g) = +1$.
- If the adjoint action $G \rightarrow \text{O}(\mathfrak{g})$ lifts to $\text{Pin}(\mathfrak{g})$, then every conjugacy class carries a distinguished orientation.

Example 5.13. — Consider the conjugacy classes of $G = \mathrm{O}(2)$: If $g \in \mathrm{SO}(2)$, the conjugacy class of g is zero-dimensional, consisting of either one or two points. On the other hand, the circle $\mathrm{O}(2) \setminus \mathrm{SO}(2) \cong S^1$ forms a single conjugacy class. Similarly, for $G = \mathrm{O}(3)$, the elements $g \in G$ with $\det(g) = -1$ have $\det(\mathrm{Ad}_g) = 1$. Each of these form a single 2-dimensional conjugacy class. The group $\mathrm{SO}(3)$ is the simplest example where the adjoint action $G \rightarrow \mathrm{SO}(\mathfrak{g})$ (which in this case is just the identity map) does not lift to the spin group. Indeed the conjugacy class of rotations by 180° is isomorphic to $\mathbb{R}P(2)$, hence non-orientable.

Example 5.14. — Suppose G carries an involution σ , such that the corresponding involution of $\mathfrak{g}$ preserves B . Form the semi-direct product $G \rtimes \mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is generated by the involution σ . The $G \rtimes \mathbb{Z}_2$ -conjugacy class of the element (e, σ) is isomorphic to the homogeneous space $M = G/G^\sigma$, which therefore is an example of a q -Hamiltonian $G \rtimes \mathbb{Z}_2$ -space. The 2-form on M is just zero. Let us compute the Liouville measure on M , for the case that the restriction of B to $\mathfrak{g}^\sigma = \ker(\sigma - 1)$ is still non-degenerate. Let $e_1, \dots, e_n$ be a basis of $\mathfrak{g}$, with $B(e_i, e_j) = \pm \delta_{ij}$, such that $e_1 \dots, e_k$ are a basis of $\mathfrak{g}^\sigma$. Then

$$\tilde{\sigma} = 2^{(n-k)/2} e_{k+1} \cdots e_n \in \mathrm{Pin}(\mathfrak{g})$$

is a lift of σ . Note that $\tilde{\sigma}^2 = \pm 1$, with sign depending on $n-k$. Taking $\mu = e_1 \wedge \cdots \wedge e_n$ as the Riemannian volume form on $\mathfrak{g}$, we have

$$\tilde{\sigma}q(\mu) = \pm 2^{(n-k)/2} e_1 \cdots e_k$$

so $\star q^{-1}(\tilde{\sigma}q(\mu)) = \pm 2^{(n-k)/2} e_{k+1} \wedge \cdots \wedge e_n$. We conclude that the Liouville measure on $M = G/G^\sigma$ coincides with the G -invariant measure defined by the metric on $(\mathfrak{g}^\sigma)^\perp \subset \mathfrak{g}$.

Proposition 5.15 (Volume form for ‘fusions’). — *The volume form of a q -Hamiltonian $G \times G$ -manifold $(M, \mathcal{A}_M, \omega, \Phi)$ (as in Theorem 5.6) coincides with the volume form of its fusion $(M, \mathcal{A}_{\mathrm{fus}}, \omega_{\mathrm{fus}}, \Phi_{\mathrm{fus}})$:*

$$(\exp(\omega) \Phi^* \psi_{G \times G})^{[\dim M]} = (\exp(\omega_{\mathrm{fus}}) \Phi_{\mathrm{fus}}^* \psi_G)^{[\dim M]}.$$

Proof. — Using Theorem 4.9, we have

$$\begin{aligned} \exp(\omega_{\mathrm{fus}}) \Phi_{\mathrm{fus}}^* \psi_G &= \exp(\omega + \Phi^* \varsigma) \Phi^* \mathrm{Mult}^* \psi_G \\ &= \exp(\omega) \Phi^* (\varrho(\exp(-e(\gamma))) \psi_G^1 \otimes \psi_G^2) \\ &= \exp(-\iota(\mathcal{A}_M(\gamma))) (\exp(\omega) \Phi^* \psi_{G \times G}), \end{aligned}$$

where we used Remark 1.5(a) for the last equality. Since the operator $\exp(-\iota(\mathcal{A}_M(\gamma)))$ does not affect the top degree part, the proof is complete. $\square$

Example 5.16. — An important example of a q -Hamiltonian G -space is the double $D(G) = G \times G$, with moment map the commutator $\Phi(a, b) = aba^{-1}b^{-1}$. As explained in [8] the double is obtained by fusion, as follows: Start by viewing the Lie group G as a homogeneous space $G = G \times G/G_\Delta$, where G_Δ is the diagonal subgroup. Since G_Δ is the fixed point set for the involution σ of $G \times G$ switching the two factors, we

see as in Example 5.14 that G is a q -Hamiltonian $(G \times G) \rtimes \mathbb{Z}_2$ -space, with moment map $a \mapsto (a, a^{-1}, \sigma)$. The Liouville measure is simply the Haar measure on G . Fusing two copies, the direct product $G \times G$ becomes a q -Hamiltonian $G \times G$ -space. Finally, passing to the diagonal action one arrives at the double $D(G)$. By Proposition 5.15, the resulting Liouville measure on $D(G)$ is just the Haar measure.

Proposition 5.17 (Volume form for ‘exponentials’). — *Let $(M, \mathcal{U}_M, \Phi_0, \omega_0)$ be a Hamiltonian G -space, and $(M, \mathcal{U}_M, \Phi, \omega)$ its ‘exponential’, as in Theorem 5.7. Then*

$$(\exp(\omega)\Phi^*\psi_G)^{[\dim M]} = \Phi_0^* J^{1/2} \exp(\omega_0)^{[\dim M]}.$$

Proof. — Using the relation (91) between $\exp^*\psi_G$ and $\psi_g = 1$, we find

$$\begin{aligned} \exp(\omega)\Phi^*\psi_G &= \exp(\omega_0 + \Phi_0^*\varpi)\Phi_0^*\exp^*\psi_G \\ &= \exp(\omega_0)\Phi_0^* J^{1/2} \varrho(\tilde{A}^{-e_0(\varepsilon)})(1) \\ &= \Phi_0^* J^{1/2} \exp(-\iota(\mathcal{U}_M(\varepsilon))) \exp(\omega_0). \end{aligned}$$

Since $\exp(-\iota(\mathcal{U}_M(\varepsilon)))$ does not affect the top degree part, the proof is complete. $\square$

5.3. The volume form in terms of the Gauss-Dirac spinor. — Suppose now that K is a compact Lie group, with complexification $G = K^{\mathbb{C}}$, and let $B: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ be the complexification of a positive definite inner product on $\mathfrak{k}$. In this case, as discussed in Section 3.6, E_G has a second Lagrangian complement $\widehat{F}_G$, defined by the Gauss-Dirac spinor $\widehat{\psi}_G$. Its pull-back to $K \subset G$, denoted by $\widehat{\psi}_K$, is thus a complex-valued pure spinor defining a (complex) Lagrangian complement $\widehat{F}_K \subset (\mathbb{T}K)^{\mathbb{C}}$.

Given a q -Hamiltonian K -space $(M, \mathcal{U}_M, \Phi, \omega)$, the complex differential form $\exp(\omega)\Phi^*\widehat{\psi}_K$ is related to $\exp(\omega)\Phi^*\psi_K$ by the r -matrix,

$$\exp(\omega)\Phi^*\widehat{\psi}_K = \exp(-\iota(\mathcal{U}_M(\tau))) \left(\exp(\omega)\Phi^*\psi_K \right).$$

Since $\exp(-\iota(\mathcal{U}_M(\tau)))$ does not affect the top degree part, it follows that we can write our volume form also in terms of $\widehat{\psi}_K$:

$$\mu_M = \left(\exp(\omega)\Phi^*\widehat{\psi}_K \right)^{[\dim M]}.$$

Remark 5.18. — Let $\tilde{F}_M$ be the backward image of $\widehat{F}_K$ under the strong Dirac morphism $(\Phi, \omega): (M, TM, 0) \rightarrow (K, E_K, \eta)$. Since $\tilde{F}_M$ is transverse to $TM^{\mathbb{C}}$, it is given by a graph of a (complex-valued) bivector π , and $H_-(TM^{\mathbb{C}}, \tilde{F}_M, \mu_M) \cong H_{\pi}(M) = H(\Omega(M)^{X_{\pi}}, d - \iota(X_{\pi}))$. A simple calculation shows that $X_{\pi} = 2\pi\sqrt{-1}\mathcal{U}_M(\rho)$ (where B is used to identify $\mathfrak{k}^* \cong \mathfrak{k}$).

The pure spinors $\phi_M = 1$ and ϕ_K satisfy $d\phi_M = 0$ and $(d + \eta)\phi_K = 0$. Hence, by Proposition 2.13 the map $e^{\omega}\Phi^*$ descends to Dirac cohomology, $H_-(E_K^{\mathbb{C}}, \widehat{F}_K, \mu_K) \rightarrow H_{\pi}(M)$. In particular, $\phi_- \widehat{\psi}_K = 0$ implies that $\exp(\omega)\Phi^*\widehat{\psi}_K$ is closed under $d - 2\pi\sqrt{-1}\iota(\mathcal{U}_M(\rho))$. For M is compact, the class $[e^{\omega}\Phi^*\widehat{\psi}_K]$ in $H_{\pi}(M)$ is nonvanishing because its integral is $\int_M \mu_M > 0$.

Let $\Delta_\lambda: G \rightarrow \mathbb{C}$ be the holomorphic functions introduced in Section 4.5.

Proposition 5.19. — *For any dominant weight λ , the complex differential form $\exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K)$ satisfies the differential equation*

$$(102) \quad (d - 2\pi\sqrt{-1}\iota(\mathcal{U}_M(\lambda + \rho)))\left(\exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K)\right) = 0.$$

Her B_K is used to identify $\mathfrak{k}^* \cong \mathfrak{k}$.

Proof. — This follows from the differential equation for the Gauss-Dirac spinor, Proposition 4.18, together with Remark 1.5(a). $\square$

As remarked in [6], the orthogonal projection of $\dim V_\lambda \Delta_\lambda|_K$ to the K -invariant functions on K coincides with the irreducible character χ_λ of highest weight λ . Thus,

$$\begin{aligned} \int_M \exp(\omega)\Phi^*(\Delta_\lambda\widehat{\psi}_K) &= \int_M |\mu_M| \Phi^* \Delta_\lambda \\ &= \int_K \Phi_* |\mu_M| \Delta_\lambda = (\dim V_\lambda)^{-1} \int_K \chi_\lambda \Phi_* |\mu_M|. \end{aligned}$$

On the other hand, by (102) the integral may be computed by localization [12] to the zeroes of the vector field $\mathcal{U}_M(\lambda + \rho)$. As shown in [6], the 2-form ω pulls back to symplectic forms $\omega_Z = \iota_Z^* \omega$ on the components Z of the zero set, and the restriction $\Phi_Z = \iota_Z^* \Phi$ takes values in T . Since $\iota_T^*(\Delta_\lambda\widehat{\psi}_K)(t) = t^{\lambda+\rho}$ for $t \in T$, one obtains the following formula for the Fourier coefficients of the q-Hamiltonian Duistermaat-Heckman measure:

$$\int_K \chi_\lambda \Phi_* |\mu_M| = \dim V_\lambda \sum_{Z \subset \mathcal{U}_M(\lambda+\rho)^{-1}(0)} \int_Z \frac{\exp(\omega_Z)(\Phi_Z)^{\lambda+\rho}}{\text{Eul}(\nu_Z, 2\pi\sqrt{-1}(\lambda + \rho))}.$$

Here $\text{Eul}(\nu_Z, \cdot)$ is the T -equivariant Euler form of the normal bundle. This formula was proved in [6], using a more elaborate argument. Taking $\lambda = 0$, one obtains a formula for the volume $\int_M |\mu_M|$ of M .

5.4. q-Hamiltonian q-Poisson g-manifolds. — Just as any symplectic 2-form determines a Poisson bivector π , any q-Hamiltonian G -manifold carries a distinguished bivector field π . However, since ω is not non-degenerate π is not simply obtained as an inverse, and also π is not generally a Poisson structure.

Suppose $(M, \mathcal{U}_M, \omega, \Phi)$ is a q-Hamiltonian g-manifold, or equivalently that (Φ, ω) is a strong Dirac morphism $(M, TM, 0) \rightarrow (G, E_G, \eta)$. Let $\widetilde{F} \subset \mathbb{T}M$ be the backward image of F_G under this Dirac morphism. It is a complement to TM , hence it is of the form $\widetilde{F} = \text{Gr}_\pi$ for some g-invariant bivector field $\pi \in \mathfrak{X}^2(M)$. By Proposition 2.10(c), the Schouten bracket of this bivector field with itself satisfies

$$(103) \quad \frac{1}{2}[\pi, \pi]_{\text{Sch}} = \mathcal{U}_M(\Xi).$$

Let $\mathfrak{p}': \mathbb{T}G \rightarrow E_G$ be the projection along F_G . Let $\{v_a\}$ and $\{v^a\}$ be bases of $\mathfrak{g}$ with $B(v_a, v^b) = \delta_a^b$. Then $\mathfrak{p}'(x') = \sum_a \langle x, \mathfrak{f}(v_a) \rangle \mathfrak{e}(v^a)$ for all $x' \in \Gamma(\mathbb{T}G)$. For

$\alpha' \in \Omega^1(G) \subset \Gamma(\mathbb{T}G)$, we have $\langle \alpha', f(v^a) \rangle = \frac{1}{2} \langle \alpha', v_a^L + v_a^R \rangle e(v^a)$. Hence, (20) shows that

$$(104) \quad \pi^\# \Phi^* \alpha' = - \sum_a \Phi^* \langle \alpha', \frac{v_a^L + v_a^R}{2} \rangle \mathcal{U}_M(v^a), \quad \alpha' \in \Omega^1(G),$$

and, by (24), we have:

$$(105) \quad \text{ran}(\mathcal{U}_M) + \text{ran}(\pi^\#) = TM.$$

This last condition can be viewed as a counterpart to the invertibility of a Poisson bivector defined by a symplectic form. Dropping this condition, one arrives at the following definition:

Definition 5.20. — [1, 2] A *q-Hamiltonian q-Poisson $\mathfrak{g}$ -manifold* is a manifold M , together with a Lie algebra action $\mathcal{U}_M: \mathfrak{g} \rightarrow \mathfrak{X}(M)$, a $\mathfrak{g}$ -invariant bivector field π , and a $\mathfrak{g}$ -equivariant *moment map* $\Phi: M \rightarrow G$, such that conditions (103) and (104) are satisfied. If the $\mathfrak{g}$ -action on M integrates to a G -action, such that π, Φ are G -equivariant, we speak of a *q-Hamiltonian q-Poisson G -manifold*.

Example 5.21. — The basic example of a Hamiltonian Poisson G -manifold is provided by the coadjoint action on $M = \mathfrak{g}^*$, with $\pi = \pi_{\mathfrak{g}^*}$ the Kirillov bivector and moment map the identity map. Similarly, the quadruple $(G, \mathcal{U}_{\text{ad}}, \pi_G, \text{id})$, with π_G the bivector field (46), is a q-Hamiltonian q-Poisson G -manifold.

The techniques in this paper allow us to give a much simpler proof to the following theorem from [14]:

Theorem 5.22. — *There is a 1-1 correspondence between q-Hamiltonian q-Poisson $\mathfrak{g}$ -manifolds $(M, \mathcal{U}_M, \pi, \Phi)$, and Dirac manifolds (M, E_M, η_M) equipped with a strong Dirac morphism*

$$(106) \quad (\Phi, 0): (M, E_M, \eta_M) \rightarrow (G, E_G, \eta).$$

Under this correspondence, $\text{ran}(E_M) = \text{ran}(\mathcal{U}_M) + \text{ran}(\pi^\#)$.

Proof. — Suppose $(\Phi, 0): (M, E_M, \eta_M) \rightarrow (G, E_G, \eta)$ is a strong Dirac morphism. Consider the bundle map $\mathfrak{a}: \Phi^* E_G \rightarrow TM$ defined by Φ (see Section 2.2). By Proposition 2.10(c), the vector fields $\mathcal{U}_M(\xi) = \mathfrak{a}(e(\xi)) \in \mathfrak{X}(M)$ define a Lie algebra action of $\mathfrak{g}$ on M for which Φ is equivariant. Note also that since $\text{ran}(\mathfrak{a}) \subset \text{ran}(E_M)$, this action preserves the leaves $Q \subset M$ of E_M . In fact, the bundle E_M is $\mathfrak{g}$ -invariant: If $E_M = \text{Gr}_\omega$ this follows from the $\mathfrak{g}$ -invariance of ω (see comment after Def. 5.1), and in the general case it follows since $E_M|_Q$ is invariant, for any leaf Q . Let F_M be the backward image of F_G under $(\Phi, 0)$, and $\pi \in \mathfrak{X}^2(M)$ be the bivector field defined by the splitting $TM = E_M \oplus F_M$. Then π is $\mathfrak{g}$ -invariant (since E_M, F_M are). Equation (103) follows from Proposition 2.10(d), while Equation (104) is a consequence of Theorem 1.20, Equation (20).

Conversely, given a quasi-Poisson $\mathfrak{g}$ -manifold $(M, \mathcal{U}_M, \pi, \Phi)$, let $\mathfrak{a}: \Phi^* E_G \rightarrow TM$ be the bundle map given on sections by $\Phi^* e(\xi) \mapsto \mathcal{U}_M(\xi)$. The $\mathfrak{g}$ -equivariance of Φ

implies that $\Phi \circ \alpha = \text{pr}_{\Phi^*TG} |_{\Phi^*E_G}$. Theorem 1.20 provides a Lagrangian splitting $TM = E_M \oplus F_M$ such that F_M is the backward image of F_G and E_G is the forward image of E_M . It remains to check the integrability condition of E_M relative to the 3-form $\eta_M = \Phi^*\eta$. Let $\Upsilon^E \in \Gamma(\wedge^3 F_M)$ be the Courant tensor of E_M . We have to show that $\Upsilon^E = 0$, or equivalently that $\Gamma(E_M)$ is closed under the η_M -twisted Courant bracket. Recall that E_M is spanned by the sections of two types:

$$\widehat{\mathcal{U}}_M(\xi) := \widehat{\alpha}(\Phi^*e(\xi)) = \mathcal{U}_M(\xi) \oplus \Phi^*B(\frac{\theta^L + \theta^R}{2}, \xi)$$

for $\xi \in \mathfrak{g}$, and sections $h(\alpha)$, for $\alpha \in \Omega^1(M)$, where the map h is defined as in (23), with $\mathbb{V}$ replaced with TM , and with $\omega = 0$. Since $\widehat{\alpha}$ is a comorphism of Lie algebroids (cf. Proposition 2.8), we have

$$(107) \quad \llbracket \widehat{\mathcal{U}}_M(\xi_1), \widehat{\mathcal{U}}_M(\xi_2) \rrbracket_{\eta_M} = \widehat{\mathcal{U}}_M([\xi_1, \xi_2]).$$

Furthermore, since π is $\mathfrak{g}$ -invariant, it follows from (23) that the map h is $\mathfrak{g}$ -equivariant, and therefore

$$[\varrho(h(\alpha)), [\varrho(\widehat{\mathcal{U}}_M(\xi)), d + \Phi^*\eta]] = [\varrho(h(\alpha)), \mathcal{L}(\mathcal{U}_M(\xi))] = -\varrho(h(\mathcal{L}(\mathcal{U}_M(\xi))\alpha)).$$

Thus

$$(108) \quad \llbracket \widehat{\mathcal{U}}_M(\xi), h(\alpha) \rrbracket_{\eta_M} = h(\mathcal{L}(\mathcal{U}_M(\xi))\alpha)$$

by definition of the Courant bracket. Equations (107) and (108) show that $\llbracket \widehat{\mathcal{U}}_M(\xi), \cdot \rrbracket_{\eta_M}$ preserves $\Gamma(E_M)$. Thus $\Upsilon^E(x_1, x_2, x_3)$ vanishes if one of the three sections $x_i \in \Gamma(E_M)$ lies in the range of $\widehat{\mathcal{U}}_M$. It remains to show that $\Upsilon^E(h(\alpha_1), h(\alpha_2), h(\alpha_3)) = 0$ for all 1-forms α_i , or equivalently that $h^*\Upsilon^E = 0$, where $h^*: F_M \rightarrow TM$ is the dual map to $h: T^*M \rightarrow E_M = F_M^*$. Since $h = \text{p}|_{T^*M}$, where $\text{p}: TM \rightarrow E_M$ is the projection along F_M (see (23)), we have $h^* = \text{pr}_{TM}|_{F_M}$. Thus, we must show that $\text{pr}_{TM}\Upsilon^E = 0$. By Proposition 2.10(b), and the defining property of q -Hamiltonian q -Poisson spaces, we have

$$\text{pr}_{TM}(\Upsilon^F) = \alpha(\Phi^*\Upsilon^{F_G}) = \mathcal{U}_M(\Xi) = \frac{1}{2}[\pi, \pi]_{\text{Sch}}.$$

On the other hand, Theorem 2.9(a) gives $\text{pr}_{TM}(\Upsilon^E) + \text{pr}_{TM}(\Upsilon^F) - \frac{1}{2}[\pi, \pi]_{\text{Sch}} = 0$. Taking the two results together, we obtain $\text{pr}_{TM}(\Upsilon^E) = 0$ as desired. $\square$

As an immediate consequence, the data $(M, \mathcal{U}_M, \pi, \Phi)$ defining a q -Hamiltonian q -Poisson G -manifold are equivalent to the data of a G -equivariant Dirac manifold (M, E_M, η_M) , equipped with a G -equivariant Dirac morphism $(\Phi, 0)$, for which the G -action on M integrates the $\mathfrak{g}$ -action defined by the Dirac morphism.

Proposition 5.23 (Fusion). — *Suppose $(M, \mathcal{U}_M, \pi, \Phi)$ is a q -Hamiltonian q -Poisson $\mathfrak{g} \times \mathfrak{g}$ -manifold. Let $\mathcal{U}_{\text{fus}}$ be the diagonal $\mathfrak{g}$ -action, $\Phi_{\text{fus}} = \text{Mult} \circ \Phi$, and $\pi_{\text{fus}} = \pi + \mathcal{U}_M(\gamma)$. Then $(M, \mathcal{U}_{\text{fus}}, \pi_{\text{fus}}, \Phi_{\text{fus}})$ is a q -Hamiltonian q -Poisson $\mathfrak{g}$ -manifold.*

Proof. — By Theorem 5.22, the given q -Poisson $\mathfrak{g} \times \mathfrak{g}$ -manifold corresponds to a Dirac manifold (M, E_M, η_M) such that $(\Phi, 0)$ is a Dirac morphism into $(G, E_G, \eta) \times (G, E_G, \eta)$. Thus, $\eta_M = \Phi^*(\eta_G^1 + \eta_G^2)$. The bivector field π is defined by the Lagrangian

splitting $TM = E_M \oplus F_M$, where F_M is the backward image of $F_G^1 \oplus F_G^2$ under $(\Phi, 0)$. Composing with (Mult, ς) (cf. Thm. 3.9), we obtain a strong Dirac morphism,

$$(\Phi_{\text{fus}}, \Phi^* \varsigma): (M, E_M, \eta_M) \rightarrow (G, E_G, \eta),$$

which in turn defines a q-Hamiltonian q-Poisson $\mathfrak{g}$ -manifold. Let $\widetilde{F}_M$ be the backward image of F_G under this Dirac morphism. By Proposition 3.11, $\widetilde{F}$ is related to F by the section $\widehat{\mathcal{U}}_M(\gamma) \in \Gamma(\wedge^2 E_M)$, where $\widehat{\mathcal{U}}_M: \mathfrak{g} \times \mathfrak{g} \rightarrow E_M$ is the map defined by the Dirac morphism $(\Phi, 0)$. Hence, by Proposition 1.18, the bivector for the new splitting $TM = E_M \oplus \widetilde{F}_M$ is $\pi_{\text{fus}} = \pi + \mathcal{U}_M(\gamma)$. $\square$

Proposition 5.24 (Exponentials). — *Suppose $(M, \mathcal{U}_M, \pi_0, \Phi_0)$ is a Hamiltonian Poisson $\mathfrak{g}$ -manifold. That is, $\mathcal{U}_M$ is a $\mathfrak{g}$ -action on M , π_0 is a $\mathfrak{g}$ -invariant Poisson structure, and $\Phi_0: M \rightarrow \mathfrak{g}$ is a $\mathfrak{g}$ -equivariant moment map generating the given action on M . Assume that $\Phi_0(M) \subset \mathfrak{g}_{\mathfrak{h}}$, and let*

$$\Phi = \exp \circ \Phi_0, \quad \pi = \pi_0 + \mathcal{U}_M(\Phi_0^* \varepsilon)$$

where $\varepsilon \in C^\infty(\mathfrak{g}_{\mathfrak{h}}, \wedge^2 \mathfrak{g})$ is the solution of the CDYBE defined in Section 3.5. Then $(M, \mathcal{U}_M, \pi, \Phi)$ is a q-Hamiltonian q-Poisson $\mathfrak{g}$ -manifold.

Proof. — It is well-known that $(M, \mathcal{U}_M, \pi_0, \Phi_0)$ is a Hamiltonian $\mathfrak{g}$ -manifold if and only if $\Phi_0: M \rightarrow \mathfrak{g}^*$ is a Poisson map, i.e., if and only if

$$(\Phi_0, 0): (M, E_M, 0) \rightarrow (\mathfrak{g}^*, E_{\pi_{\mathfrak{g}^*}}, 0)$$

is a strong Dirac morphism, with $E_M = \text{Gr}_{\pi_0}$ and $E_{\mathfrak{g}^*} = \text{Gr}_{\pi_{\mathfrak{g}^*}}$. Using B to identify $\mathfrak{g}^* \cong \mathfrak{g}$, and composing with the strong Dirac morphism $(\exp, \varpi)$, one obtains the strong Dirac morphism

$$(\Phi, \Phi_0^* \varpi): (M, E_M, 0) \rightarrow (G, E_G, \eta),$$

which in turn gives rise to a q-Hamiltonian q-Poisson $\mathfrak{g}$ -manifold $(M, \mathcal{U}_M, \pi, \Phi)$. The backward image $\widetilde{F}_M \subset TM$ of F_G under the Dirac morphism $(\Phi, \Phi_0^* \varpi)$ is a Lagrangian complement to $E_M = \text{Gr}_{\pi}$. Let $\widehat{\alpha}: \Phi_0^* E_{\mathfrak{g}} \rightarrow E_M$ be defined by the Dirac morphism $(\Phi_0, 0)$, and put $\widehat{\mathcal{U}}_M(\xi) = \widehat{\alpha} \circ \Phi_0^* e_0(\xi)$. As explained in Section 3.5, $\widetilde{F}_M$ is related to the Lagrangian complement $F_M = TM$ by the section $\widehat{\mathcal{U}}_M(\Phi_0^* \varepsilon)$. Hence, $\pi = \pi_0 + \mathcal{U}_M(\Phi_0^* \varepsilon)$. $\square$

5.5. $\mathfrak{k}^*$ -valued moment maps. — Let K be any Lie group. An ordinary Hamiltonian Poisson K -manifold is a triple (M, π, Φ) where M is a K -manifold, $\pi \in \mathfrak{X}^2(M)$ is an invariant Poisson structure, and $\Phi: M \rightarrow \mathfrak{k}^*$ is a K -equivariant map satisfying the moment map condition,

$$\pi^\sharp(d\langle \Phi, \xi \rangle) = \mathcal{U}_M(\xi).$$

The moment map condition is equivalent to Φ being a Poisson map. The following result implies that $\mathfrak{k}^*$ -valued moment maps can be viewed as special cases of $G = \mathfrak{k}^* \rtimes K$ -valued moment maps. Let $\mathfrak{g} = \mathfrak{k}^* \rtimes \mathfrak{k}$ carry the invariant inner product given by the pairing.

Proposition 5.25. — *The inclusion map $j: \mathfrak{k}^* \hookrightarrow \mathfrak{k}^* \rtimes K = G$ is a strong Dirac morphism $(j, 0)$, as well as a backward Dirac morphism, relative to the Kirillov-Poisson structure on $\mathfrak{k}^*$ and the Cartan-Dirac structure on G . The backward image of F_G under this Dirac morphism is $F_{\mathfrak{k}^*} = T\mathfrak{k}^*$. The pure spinor ψ_G on $G = \mathfrak{k}^* \rtimes K$ satisfies*

$$j^*\psi_G = 1.$$

Proof. — The Cartan-Dirac structure E_G is spanned by the sections $e(w)$ for $w = (\beta, \xi) \in \mathfrak{g}$, while $E_{\mathfrak{k}^*}$ is spanned by the sections $e_0(\xi)$ for $\xi \in \mathfrak{k}$. The first part of the Proposition will follow once we show that $s_0(\beta, \xi) \sim_{(j,0)} s(\beta, \xi)$, i.e.

$$(109) \quad e_0(\xi) \sim_{(j,0)} e(\beta, \xi), \quad f_0(\beta) \sim_{(j,0)} f(\beta, \xi).$$

The vector field part of the first relation follows since the inclusion $j: \mathfrak{k}^* \hookrightarrow \mathfrak{k}^* \rtimes K$ is equivariant for the conjugation action of $G = \mathfrak{k}^* \rtimes K$. (Here, the $\mathfrak{k}^*$ -component of G acts trivially on $\mathfrak{k}^*$, while the K -component acts by the co-adjoint action.) For the 1-form part, we note that the pull-back of the Maurer-Cartan forms $\theta^L, \theta^R \in \Omega^1(G) \otimes \mathfrak{g}$ to the subgroup $\mathfrak{k}^* \subset G$ is the Maurer-Cartan form for additive group $\mathfrak{k}^*$, i.e.

$$j^*\theta^L = j^*\theta^R = \theta_0$$

where the ‘tautological 1-form’ $\theta_0 \in \Omega^1(\mathfrak{k}^*) \otimes \mathfrak{k}^*$ is defined as in Section 3.5. Thus

$$j^*B\left(\frac{\theta_G^L + \theta_G^R}{2}, (\beta, \xi)\right) = B(\theta_0, (\beta, \xi)) = \langle \theta_0, \xi \rangle.$$

This verifies the first relation in (109); the second one is checked similarly.

Since the adjoint action $\text{Ad}: G \rightarrow \text{O}(\mathfrak{g})$ is trivial over $\mathfrak{k}^*$, the lift $\tau: G \rightarrow \text{Pin}(\mathfrak{g}) \subset \text{Cl}(\mathfrak{g})$ satisfies $\tau|_{\mathfrak{k}^*} = 1$. It follows that the pure spinor $\psi_G = \mathcal{R}(q(\mu))$ satisfies $j^*\psi_G = 1$. $\square$

Corollary 5.26. — *Let (M, π) be a Poisson manifold. Then $\Phi: M \rightarrow \mathfrak{k}^*$ is a Poisson map if and only if the composition $j \circ \Phi: M \rightarrow G$ is a strong Dirac morphism*

$$(j \circ \Phi, 0): (M, \text{Gr}_\pi, 0) \rightarrow (G, E_G, \eta).$$

Put differently, Hamiltonian Poisson K -manifolds are q-Hamiltonian q-Poisson $\mathfrak{k}^* \rtimes K$ -manifolds for which the moment map happens to take values in $\mathfrak{k}^*$.

As a special case, a Hamiltonian K -manifold (M, ω, Φ) (with ω a symplectic 2-form, and Φ satisfying the moment map condition $\iota(\mathcal{Q}_M(\xi))\omega = d\langle \Phi, \xi \rangle$) is equivalent to a q-Hamiltonian $G = \mathfrak{k}^* \rtimes K$ -space for which the moment map takes values in $\mathfrak{k}^*$. Since $j^*\psi_G = 1$, its q-Hamiltonian volume form coincides with the usual Liouville form $(\exp \omega)^{[\text{top}]}$.

6. K^* -valued moment maps

For a Poisson Lie group K , J.-H. Lu [42] introduced another type of group-valued moment map, taking values in the dual Poisson Lie group K^* . For a compact Lie group K , with its standard Poisson structure, this moment map theory turns out

to be equivalent to the usual $\mathfrak{k}^*$ -valued one. In this Section, we will re-examine this equivalence using the techniques developed in this paper.

6.1. Review of K^* -valued moment maps. — The theory of Poisson-Lie groups were introduced by Drinfeld in [23], see e.g. [18] for an overview and bibliography. Suppose K is a connected Poisson Lie group, with Poisson structure defined by a Manin triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{k}')$. (That is, $\mathfrak{g}$ is a Lie algebra with an invariant split inner product, and $\mathfrak{k}, \mathfrak{k}'$ are complementary Lagrangian subalgebras.) Use the pairing to identify $\mathfrak{k}' = \mathfrak{k}^*$, and let K^* be the associated dual Poisson Lie group. We assume that $\mathfrak{g}$ integrates to a Lie group G (the double) such that K, K^* are subgroups and the product map $K \times K^* \rightarrow G$ is a diffeomorphism. The left action of K on G descends to a *dressing action* $\mathcal{A}_{K^*}$ on K^* (viewed as a homogeneous space G/K). The Poisson structure on K^* , or equivalently its graph $E_{K^*} = \text{Gr}_{\pi_{K^*}} \subset \mathbb{T}K^*$, may be expressed in terms of the infinitesimal dressing action, as the span of sections

$$e_{K^*}(\xi) = \mathcal{A}_{K^*}(\xi) \oplus \langle \theta_{K^*}^R, \xi \rangle$$

for $\xi \in \mathfrak{k}$. Here $\theta_{K^*}^R \in \Omega^1(K^*) \otimes \mathfrak{k}^*$ is the right-invariant Maurer-Cartan form for K^* . Note that as a Lie algebroid, E_{K^*} is just the action algebroid.

For the remainder of this Section 6, we will assume that K is a compact real Lie group. The *standard Poisson structure* on K is described as follows. Let $G = K^{\mathbb{C}}$ be the complexification, with Lie algebra $\mathfrak{g}$, and let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad G = KAN$$

be the Iwasawa decompositions. Here $\mathfrak{a} = \sqrt{-1}\mathfrak{k}_K$, $A = \exp \mathfrak{a}$ and $N = N_+$ (using the notation from Section 3.6). We denote by B_K an invariant inner product on $\mathfrak{k}$, and let $\langle \cdot, \cdot \rangle$ be the imaginary part of $2B_K^{\mathbb{C}}$. Then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{a} \oplus \mathfrak{n})$ (where $\mathfrak{g}$ is viewed as a real Lie algebra) is a Manin triple. Thus K becomes a Poisson Lie group, with dual Poisson Lie group $K^* = AN$.

A K^* -valued Hamiltonian $\mathfrak{k}$ -manifold, as defined by Lu [42], is a symplectic manifold (M, ω) together with a Poisson map $\Phi: M \rightarrow K^*$. Equivalently, $(\Phi, \omega): (M, TM, 0) \rightarrow (K^*, E_{K^*}, 0)$ is a strong Dirac morphism. The Poisson map Φ induces a $\mathfrak{k}$ -action on M , and if this action integrates to an action of K we speak of a K^* -valued Hamiltonian K -manifold. An interesting feature is that ω is not K -invariant, in general: Instead, the action map $K \times M \rightarrow M$ is a Poisson map. Accordingly, the volume form $(\exp \omega)^{[\text{top}]}$ is not K -invariant. However, let $\Phi^A: M \rightarrow A$ be the composition of Φ with projection $K^* = AN \rightarrow A$, and $(\Phi^A)^{2\rho}: M \rightarrow \mathbb{R}_{>0}$ its image under the homomorphism $T \rightarrow \mathbb{C}^\times$, $t \mapsto t^{2\rho}$ defined by the sum of positive roots. By [7, Theorem 5.1], the product

$$(110) \quad (\Phi^A)^{2\rho} (\exp \omega)^{[\text{top}]}$$

is a K -invariant volume form. The proof in [7] uses a tricky argument; one of the goals of this Section is to give a more conceptual explanation.

6.2. P -valued moment maps.— To explain the origin of the volume form (110), we will use the notion of a P -valued moment map introduced in [3, Section 10]. Let $g \mapsto g^c$ denote the complex conjugation map on G , and let

$$I(g) \equiv g^\dagger = (g^{-1})^c.$$

On the Lie algebra level, let $\xi \mapsto \xi^c$ denote conjugation, and $\xi^\dagger = -\xi^c$. We have $K = \{g \in G \mid g^\dagger = g^{-1}\}$. Let

$$P = \{g^\dagger g \mid g \in G\}$$

denote the subset of ‘positive definite’ elements in G . Then P is a submanifold fixed under I , and the product map defines the Cartan decomposition $G = KP$. Let E_G be the (holomorphic) Dirac structure on G defined by the inner product

$$B := \frac{1}{\sqrt{-1}} B_K^c.$$

Since $(\theta^L)^\dagger = I^* \theta^R$, $(\theta^R)^\dagger = I^* \theta^L$, the Cartan 3-form on G satisfies $\eta^c = I^* \eta$, thus $\eta_P := \iota_P^* \eta$ is real-valued. Similarly, the pull-backs of the 1-forms $B(\frac{\theta^L + \theta^R}{2}, \xi)$ for $\xi \in \mathfrak{k}$ are real-valued. It follows that the sections

$$e_P(\xi) := e(\xi)|_P$$

are real-valued. Letting $E_P \subset TP$ be the subbundle spanned by these sections, it follows that (P, E_P, η_P) is a real Dirac manifold, with $(E_P)^c = E_G|_P$. As a Lie algebroid, E_P is just the action algebroid for the K -action on P . Similarly, the sections $f_P(\xi) := f(\xi)|_P$ are real-valued, defining a complement F_P to E_P . The bundle F_P is defined by the (real-valued) pure spinor, $\psi_P := \iota_P^* \psi_G \in \Omega(P)$.

Remark 6.1. — Since $\det(\text{Ad}_g + 1) > 0$ for $g \in P$ (all eigenvalues of Ad_g are strictly positive), one finds that $\ker(E_P) = \{0\}$. Hence E_P is the graph of a bivector π_P with $\frac{1}{2}[\pi_P, \pi_P] = \pi_P^\sharp(\eta_P)$.

A P -valued Hamiltonian $\mathfrak{k}$ -manifold [3, Section 10] is a manifold M together with a strong Dirac morphism $(\Phi_1, \omega_1): (M, TM, 0) \rightarrow (P, E_P, \eta_P)$. For any such space we obtain, as for the q-Hamiltonian setting, an invariant volume form

$$(111) \quad (\exp(\omega_1) \wedge \Phi_1^* \psi_P)^{[\text{top}]}$$

Here ψ_P may be replaced by $\widehat{\psi}_P$, the pull-back of the Gauss-Dirac spinor.⁽⁵⁾ By Proposition 5.19, the expression $\exp \omega_1 \wedge \Phi_1^*(\Delta_\lambda \widehat{\psi}_P)$ is closed under the differential $d - 2\pi\iota(\mathcal{U}_M(\lambda + \rho))$, for any dominant weight λ .

⁽⁵⁾ In Section 5.3, B was taken as the complexification of B_K , while here we have an extra factor $\sqrt{-1}$. This amounts to a simple rescaling of the bilinear form B_K^c , not affecting any of the results.

6.3. Equivalence between K^* -valued and P -valued moment maps. — To relate the K^* -valued theory with the P -valued theory, we use the K -equivariant diffeomorphism

$$\kappa: K^* \rightarrow P, g \mapsto g^\dagger g.$$

Note that this map takes values in the big Gauss cell, $\theta = N_-KN \subset G$. Let ϖ_θ denote the (complex) 2-form on the big Gauss cell, and $\varpi_{K^*} = \kappa^*\varpi_\theta$. It is easy to check that ϖ_{K^*} is real-valued. One can check that

$$e_{K^*}(\xi) \sim_{(\kappa, \varpi_{K^*})} e_P(\xi)$$

for all $\xi \in \mathfrak{k}$: The vector field part of this relation is equivalent to the $\mathfrak{k}$ -equivariance, while the 1-form part is verified in [3, Section 10]. It follows that (κ, ϖ_{K^*}) is a Dirac isomorphism from $(K^*, E_{K^*}, 0)$ onto (P, E_P, η_P) .

Thus, if (M, ω, Φ) is a K^* -valued Hamiltonian $\mathfrak{k}$ -manifold, then (M, ω_1, Φ_1) with $\omega_1 = \omega + \Phi^*\varpi_{K^*}$ and $\Phi_1 = \kappa \circ \Phi$ is a P -valued Hamiltonian $\mathfrak{k}$ -manifold. In particular, we obtain an invariant volume form on M ,

$$(\exp(\omega + \Phi^*\varpi_{K^*}) \wedge \Phi^*\kappa^*\widehat{\psi}_P)^{[\text{top}]}$$

Using the explicit formula (Proposition 4.14) for the Gauss-Dirac spinor, we obtain

$$\kappa^*\widehat{\psi}_P = a^{2\rho} \exp(-\varpi_{K^*}),$$

where $a: K^* \rightarrow A$ is projection to the A -factor. Hence,

$$\exp(\omega + \Phi^*\varpi_{K^*}) \wedge \Phi^*\kappa^*\psi_P = (\Phi^A)^{2\rho} \exp(\omega),$$

identifying the volume form for the associated P -valued space with the volume form (110).

Proposition 6.2. — *For any K^* -valued Hamiltonian $\mathfrak{k}$ -space (M, ω, Φ) , the volume form $(\Phi^A)^{2\rho}(\exp \omega)^{[\text{top}]}$ is $\mathfrak{k}$ -invariant. Moreover, for all dominant weights λ the differential form*

$$(\Phi^A)^{2(\lambda+\rho)} \exp(\omega)$$

is closed under the differential $d - 2\pi \mathcal{Q}_M(B_K^\sharp(\lambda + \rho))$.

Proof. — Invariance follows from the identification with the volume form for the associated P -valued space. The second claim follows from Proposition 5.19, since the function Δ_λ from Section 4.5 satisfies $\kappa^*\Delta_\lambda = a^{2\lambda}$. $\square$

The differential equation permits a computation of the integrals $\int_M (\Phi^A)^{2(\lambda+\rho)} (\exp(\omega))^{[\text{top}]}$ by localization [12] to the zeroes of the vector field $\mathcal{Q}_M(B_K^\sharp(\lambda + \rho))$, similar to the formula in 5.3.

6.4. Equivalence between P -valued and $\mathfrak{k}^*$ -valued moment maps. — Finally, let us express the correspondence [3, Section 10] between P -valued moment maps and $\mathfrak{k}^*$ -valued moment maps in terms of Dirac morphisms. The exponential map for $G = K^{\mathbb{C}}$ restricts to a diffeomorphism

$$\exp_{\mathfrak{p}} : \mathfrak{p} := \sqrt{-1}\mathfrak{k} \rightarrow P := \exp(\sqrt{-1}\mathfrak{k}).$$

Let $\varpi \in \Omega^2(\mathfrak{g})$ be the primitive of $\exp^* \eta$ defined in (58), and $\varpi_{\mathfrak{p}}$ its pull-back to $\mathfrak{p}$. Since η_P is real-valued, so is $\varpi_{\mathfrak{p}}$, and $d\varpi_{\mathfrak{p}} = (\exp|_{\mathfrak{p}})^* \eta_P$. Similarly, $J_{\mathfrak{p}} := J|_{\mathfrak{p}} > 0$. The formulas for $\varpi_{\mathfrak{p}}$ and $J_{\mathfrak{p}}$ are similar to those for the Lie algebra $\mathfrak{k}$, but with $\sinh$ functions replaced by $\sin$ functions. Use $B^{\sharp} = \sqrt{-1}B_K^{\sharp}$ to identify $\mathfrak{k}^* \cong \mathfrak{p}$. By Proposition 3.12,

$$e_0(\xi) \sim_{(\exp_{\mathfrak{p}}, \varpi_{\mathfrak{p}})} e_P(\xi), \quad \xi \in \mathfrak{k}.$$

Hence $(\exp_{\mathfrak{p}}, \varpi_{\mathfrak{p}})$ is a Dirac (iso)morphism from $(\mathfrak{k}^*, E_{\mathfrak{k}^*}, 0)$ to (P, E_P, η_P) . This sets up a 1-1 correspondence between P -valued and $\mathfrak{k}^*$ -valued Hamiltonian $\mathfrak{k}$ -spaces. Thinking of the latter as given by strong Dirac morphisms (Φ_0, ω_0) to $(\mathfrak{k}^*, E_{\mathfrak{k}^*}, 0)$, the correspondence reads

$$(\Phi_1, \omega_1) = (\exp_{\mathfrak{p}}, \varpi_{\mathfrak{p}}) \circ (\Phi_0, \omega_0).$$

The volume forms are related by $(\exp(\omega_1) \wedge \Phi_1^* \psi_P)^{[\text{top}]} = J_{\mathfrak{p}}^{1/2} \exp(\omega_0)^{[\text{top}]}$.

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