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DON ZAGIER

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RAMANUJAN'S MOCK THETA FUNCTIONS AND THEIR APPLICATIONS
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by **Don ZAGIER**

INTRODUCTION

One of the most romantic stories in the history of mathematics is that of the friendship between Hardy and Ramanujan. It began and ended with two famous letters. The first, sent by Ramanujan to Hardy in 1913, presents its author as a penniless clerk in a Madras shipping office who has made some discoveries that "are termed by the local mathematicians as 'startling'." Hardy spent the night with Littlewood convincing himself that the letter was the work of a genius and not of a fraud and promptly invited Ramanujan to come to England for what was to become one of the most famous mathematical collaborations in history. The other letter was sent in 1920, also by Ramanujan to Hardy, just three months before his death at the age of 32 in India, to which he had returned after five years in England. Here he recovers briefly from his illness and depression to tell Hardy excitedly about a new class of functions that he has discovered and that he calls "mock theta functions."

This letter has become celebrated, not only because of the tragic circumstances surrounding it, but also because it was mathematically so mysterious and intriguing. Ramanujan gives no definition of mock theta functions but only a list of 17 examples and a qualitative description of the key property that he had noticed: that these functions have asymptotic expansions at every rational point of the same type as those of theta functions (Ramanujan used the word "theta functions" where we would say "modular forms" today, so that "mock theta functions" meant something like "fake modular forms"), but that there is no single theta function whose asymptotic expansion agrees at all rational points with that of the mock theta function. Obviously, this is a basic property, but far from a complete definition.

In the years since 1920, many papers have been written, including many by famous mathematicians like Watson, Selberg and Andrews, studying the 17 specific examples Ramanujan had given, proving the identities that he had stated, and finding further identities of the same type. But no natural definition was known that

described what these functions are intrinsically and hence could give a natural explanation of the identities between them and a method to construct further examples at will. The breakthrough came in 2002 with the thesis of a Dutch doctoral student, Sander Zwegers, who finally found the missing intrinsic characterization of mock theta functions. In fact, he did this in three different ways! Specifically, he observed that various known identities from the literature could be interpreted as saying that each of Ramanujan's examples belongs to at least one (and presumably to all, although probably not all 51 verifications have been carried out explicitly) of three infinite families of functions:

(A) "Lerch sums"

(B) "Quotients of indefinite binary theta series by unary theta series"

(C) "Fourier coefficients of meromorphic Jacobi forms"

(We will define and discuss these families in more detail below.) For each of these classes he was able to prove a specific type of near-modular behavior which therefore held in particular for Ramanujan's examples. What's more, this near-modularity property turned out to be the same for each of the three classes, so that the original problem was not only triply solved, but in a way that made it quite convincing that the essential property of these functions really had been correctly identified.

In this talk we will describe Ramanujan's letter and the 17 original examples, describe each of the classes (A) – (C) and the nature of their modularity, formulate a general definition of mock modular forms, and describe further examples. In the final section, we will also discuss some of the beautiful recent work of Kathrin Bringmann and Ken Ono, based on these ideas, that has led to the solution of several well-known open problems in combinatorics and the theory of q-series.

Before beginning the main story, there are two points that I would like to emphasize. The first is that one of the reasons for the great usefulness (or "unreasonable effectiveness," to coin a phrase) of classical modular forms in number theory is that each modular form has calculable invariants—its weight, level, and a (known) finite number of its first Fourier coefficients—that suffice to characterize it uniquely. This means that to prove any conjectured identity between modular forms, like the famous formulas

$$q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=1}^{\infty}\left(\frac{12}{n}\right)q^{n^2/24}, \qquad \left(\sum_{n\in\mathbb{Z}}(-q)^{n^2}\right)^4+\left(\sum_{n\in\mathbb{Z}+\frac{1}{2}}q^{n^2}\right)^4 = \left(\sum_{n\in\mathbb{Z}}q^{n^2}\right)^4$$

of Euler and Jacobi, respectively, it suffices to calculate the invariants on both sides and check that they are the same; one does not need to know any further properties of the functions involved or even where they come from. Precisely the same principle will apply also to the larger class of mock modular forms motivated by and containing

Ramanujan's examples, as soon as we know their modular transformation properties, so that here, too, identities which previously required lengthy computations and great ingenuity for their proofs can now be established by an essentially automatic procedure.

The second point is that all 17 of Ramanujan's mock theta functions were given in the form of q-hypergeometric series. (We recall that a q-hypergeometric series is a sum of the form $\sum_{n=0}^{\infty} A_n(q)$ where each $A_n(q) \in \mathbb{Q}(q)$ and $A_{n+1}(q)/A_n(q) = R(q, q^n)$ for all $n \geq 1$ for some fixed rational function $R(q, r) \in \mathbb{Q}(q, r)$.) Some modular forms are q-hypergeometric series, classical examples being the theta series and Eisenstein series

$$\frac{1}{2} + \sum_{n=1}^{\infty} q^{n^2} \qquad \text{and} \qquad \frac{1}{24} + \sum_{n=1}^{\infty} q^n \frac{1+q^{2n}}{(1-q^{2n})^2},$$

respectively, but this is very rare and there is no known criterion for deciding whether a given q-hypergeometric series is modular or not. (There are fascinating conjectures due to Werner Nahm relating this question to deep questions of conformal field theory and algebraic K-theory [**18, 25**].) Ramanujan loved and was a supreme connoisseur of q-hypergeometric series, and his examples all quite naturally belonged to this category, but it is a complete red herring from the point of view of understanding the intrinsic modular transformation properties that make these functions special. It is perhaps precisely for this reason that it took so long for these transformation properties to be found, just as the theory of ordinary modular forms would have developed much more slowly if for some reason one had focused only on the rare q-hypergeometric examples.

1. RAMANUJAN'S LETTER

Ramanujan divided his seventeen examples into four of order 3, ten of order 5, and three of order 7, though he gave no indication what these "orders" were. (We'll see later that they are related to the levels of the corresponding mock modular forms.) We will discuss most of these functions here to illustrate various points involved.

The mock theta functions of order 3 were denoted f, ϕ, ψ and χ. We give only the first three (changing q to $-q$ in ϕ and ψ in order to simplify the relations):

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2 \cdots (1+q^n)^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{(-q)^{n^2}}{(1+q^2)(1+q^4)\cdots(1+q^{2n})},$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{(-q)^{n^2}}{(1+q)(1+q^3)\cdots(1+q^{2n-1})} .$$

Ramanujan gives two relations among these functions (as well as a further relation involving f and χ), all proved later by Watson:

$$2\phi(q) - f(q) = f(q) + 4\psi(q) = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{(1+q)(1+q^2)(1+q^3)\cdots},$$

where the expression on the right-hand side is, up to a factor $q^{-1/24}$, a modular form of weight $\frac{1}{2}$. Already in this first example we see three points:

- there are linear relations among the mock theta functions (here, $\phi = f + 2\psi$);
- the space they span contains a subspace of ordinary modular forms;
- one must multiply by suitable powers of q to get the correct modular behavior.

Ramanujan also describes the asymptotics of $f(q)$ as q tends to any root of unity, a typical result being

$$e^{\pi t/24} f\bigl(-e^{-\pi t}\bigr) = -\frac{1}{\sqrt{t}} e^{\pi/24t} + 4 + o(1) \qquad \text{as } t \to 0 .$$

Notice that, as Ramanujan asserts, this is similar to the type of expansion which we would have if $q^{-1/24} f(q)$ were a true modular form of weight $\frac{1}{2}$, except that then the subleading terms would have a form like $t^{-1/2} \sum_{n\geq 0} a_n e^{-\pi n/24t}$ rather than $4 + o(1)$.

The ten mock theta functions of order 5 have similar features, but are considerably more complicated. We discuss this case in more detail since it is quite typical. The functions come in five groups of two each, denoted f_j, ϕ_j, ψ_j, χ_j and F_j with $j \in \{1, 2\}$. (These are Ramanujan's notations, except that he omits the indices.) The five functions with index $j = 1$ are given by

$$\begin{aligned}
f_1(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)\cdots(1+q^n)} , \\
\phi_1(q) &= \sum_{n=0}^{\infty} q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1}) , \\
\psi_1(q) &= \sum_{n=1}^{\infty} q^{n(n+1)/2}(1+q)(1+q^2)\cdots(1+q^{n-1}) , \\
\chi_1(q) &= \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})\cdots(1-q^{2n})} , \\
F_1(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(1-q)(1-q^3)\cdots(1-q^{2n-1})} ,
\end{aligned}$$

and the five with index $j = 2$ are very similar, e.g.,

$$f_2(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)\cdots(1+q^n)},$$

$$\chi_2(q) = \sum_{n=0}^{\infty} \frac{q^n}{(1-q^{n+1})\cdots(1-q^{2n+1})}.$$

Again Ramanujan gives a number of linear relations among these functions or between them and classical modular forms (multiplied by suitable powers of q). These relations, later proved by Watson, can be summarized in the form

$$\begin{pmatrix} f_1(\sqrt{q}) & f_1(-\sqrt{q}) & \chi_1(q)-2 & \phi_1(-q) & \psi_1(\sqrt{q}) & \psi_1(-\sqrt{q}) & F_1(q)-1 \\ f_2(\sqrt{q}) & -f_2(-\sqrt{q}) & \chi_2(q)\sqrt{q} & -\phi_2(-q)/\sqrt{q} & \psi_2(\sqrt{q}) & -\psi_2(-\sqrt{q}) & F_2(q)\sqrt{q} \end{pmatrix}$$
$$= \begin{pmatrix} U_1(q) & V_1(q) & W_1(q) \\ U_2(q) & V_2(q) & W_2(q) \end{pmatrix} \begin{pmatrix} -1 & 1 & 2 & 0 & 1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & -2 & -3 & 1 & -1 & 1 & -1 \end{pmatrix},$$

where U_j and V_j, multiplied by $q^{-1/120}$ for $j = 1$ and by $q^{11/120}$ for $j = 2$, are quotients of classical theta series and only W_1 and W_2 are functions of the new "mock" type. We thus see the same points as above, but in a more complicated setting: we have seven vectors, each consisting of two q-hypergeometric series, which span a space of dimension only 3 rather than 7, and this 3-dimensional space contains a 2-dimensional subspace of (weakly holomorphic) classical modular forms after multiplication by suitable rational powers of q. Again there are also asymptotic formulas as q tends to any root of unity.

Finally, the three mock theta functions of order 7 are much simpler, since they form in a natural way a single 3-vector, with no linear relations. The three functions are

$$\mathcal{F}_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})},$$

$$\mathcal{F}_2(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q^n)(1-q^{n+1})\cdots(1-q^{2n-1})},$$

$$\mathcal{F}_3(q) = \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(1-q^n)(1-q^{n+1})\cdots(1-q^{2n-1})}.$$

Since there are no relations, either among these functions or between them and classical modular forms, it is less apparent here than in the other cases what is special about these particular q-hypergeometric series. One answer (which was Ramanujan's) is that they again satisfy asymptotic formulas at roots of unity of the same

type as for order 3 and 5. Another will appear later when we state identities relating the functions $\mathcal{F}_j(q)$ to indefinite theta series and to mock Eisenstein series. But a third answer, which we can already state here, is simply at the level of the q-expansions themselves. If we calculate to high order, we find that the coefficients of these expansions grow very rapidly, the coefficient of q^{5000} in $\mathcal{F}_1(q)$, for instance, being 19452249375718841362777729 66. But if we multiply any of the series $\mathcal{F}_j(q)$ by the infinite product $\prod_{n=1}^{\infty}(1-q^{7n})$, which up to a rational power of q is a modular form, then in each case the first 5000 coefficients are all at most 10 in absolute value, suggesting that the functions $\mathcal{F}_j(q)$ are indeed related in some non-trivial way with modular forms.

2. LERCH–APPELL SUMS AND MORDELL INTEGRALS

In his famous lecture "The Final Problem" given on the occasion of his retirement as president of the London Mathematical Society in 1935, Watson [**23**] considered the mock theta functions from Ramanujan's last letter and in particular proved all of the identities and asymptotic expansions which Ramanujan had given for the functions of order 3. To do this, he first established a number of new identities—not actually all that new, as it transpired when Ramanujan's "lost notebooks" were discovered later—relating the mock theta functions to q-hypergeometric series of a much simpler form, a typical example being the identity

$$\prod_{n=1}^{\infty}(1-q^n)\cdot f(q) \;=\; 2\sum_{n=-\infty}^{\infty}(-1)^n\,\frac{q^{n(3n+1)/2}}{1+q^n}$$

for the first mock theta function of order 3.

In the first chapter of his thesis [**28**], Zwegers studies sums of the type appearing on the right-hand side of this formula, which he calls "Lerch sums" after M. Lerch, who studied functions of this kind in two papers [**15, 16**] (one in Czech and one in German) in 1892, though in fact they had been introduced some years earlier by Appell [**3**]. The transformation properties of these functions were studied by both Lerch and Appell and also by modern authors [**19, 21**], but Zwegers's analysis is very complete and we will follow his exposition here.

It turns out to be convenient to normalize the Lerch sums, which are objects of weight 1, by dividing them by theta series of weight $\frac{1}{2}$, since the mock theta functions will eventually be expressed as linear combinations of such quotients. For fixed $\tau \in \mathfrak{H}$ (= complex upper half plane) we define a function of two complex variables u, v by

$$\mu(u,v) \;=\; \mu(u,v;\tau) \;=\; \frac{a^{1/2}}{\theta(v)}\sum_{n\in\mathbb{Z}}\frac{(-b)^n\,q^{n(n+1)/2}}{1-aq^n},$$

where $q = e^{2\pi i\tau}$, $a = e^{2\pi iu}$, $b = e^{2\pi iv}$ (we will use these abbreviations throughout, and will omit the variable τ when it is not varying) and $\theta(v)$ is the Jacobi theta series

$$\theta(v) = \theta(v;\tau) = \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} (-1)^{\nu-1/2}\, b^\nu\, q^{\nu^2/2} = q^{1/8}\, b^{1/2} \prod_{n=1}^{\infty} \bigl(1-q^n\bigr)\bigl(1-bq^n\bigr)\bigl(1-b^{-1}q^{n-1}\bigr).$$

(The last equality is the famous triple product identity of Jacobi.) Zwegers shows that the function μ has the symmetry property

$$\mu(u,v) = \mu(v,u)\,,$$

the elliptic transformation properties

$$\begin{aligned} \mu(u+1,v) &= -\mu(u,v)\,, \\ a^{-1}bq^{-1/2}\mu(u+\tau,v) &= -\mu(u,v) + a^{-1/2}\, b^{1/2}\, q^{-1/8}\,, \end{aligned}$$

and the modular transformation properties

$$\begin{aligned} \mu(u,v;\tau+1) &= \zeta_8^{-1}\mu(u,v) \qquad \bigl(\zeta_N := e^{2\pi i/N}\bigr), \\ (\tau/i)^{-1/2}\, e^{\pi i(u-v)^2/\tau}\, \mu\bigl(\frac{u}{\tau}, \frac{v}{\tau}; \frac{-1}{\tau}\bigr) &= -\mu(u,v) + \frac{1}{2}\, h(u-v;\tau)\,, \end{aligned}$$

where $h(z;\tau) = \int_{-\infty}^{\infty} \frac{e^{\pi i x^2\tau - 2\pi xz}\, dx}{\cosh \pi x}$, an integral of a kind first introduced by Mordell [**17**]. These properties show that μ behaves nearly like a Jacobi form of weight $\frac{1}{2}$ in two variables (a Jacobi form being a function of a modular variable τ and one or more elliptic variables u, v, ... with appropriate transformation properties; the exact definition in the one-variable case will be recalled in §4), and that its failure to transform exactly like a Jacobi form depends only on the difference $u - v$. [(1)] Zwegers then constructs a second, but now non-holomorphic, function (of $u - v$ only) whose "non-Jacobiness" exactly matches that of μ, so that the difference of these two functions transforms in the correct way, though at the expense of no longer being holomorphic:

THEOREM 2.1. — *For $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$ define $R(z;\tau)$ by the convergent series*

$$R(z;\tau) = \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} (-1)^{\nu-1/2} \bigl[\operatorname{sgn}(\nu) - E\bigl((\nu + \Im(z)/y)\sqrt{2y}\bigr)\bigr]\, e^{-2\pi i\nu z}\, q^{-\nu^2/2}\,,$$

where $y = \Im(\tau)$ and $E(z)$ is the odd entire function $2\int_0^z e^{-\pi u^2}\, du$. Then the function

$$\widehat{\mu}(u,v;\tau) = \mu(u,v;\tau) - \frac{1}{2}\, R(u-v;\tau)$$

(1) The fact that the "non-Jacobiness" of $\mu(u,v)$ depends only on $u - v$ can be explained by the fact that $\mu(u,v)$ has a decomposition of the form $\mu(u,v) = \frac{\zeta(u)-\zeta(v)-\zeta(u-v)}{\theta(u-v)} + \lambda(u-v)$, where $\zeta(z) = (2\pi i)^{-1}\theta'(z)/\theta(z)$ is the Weierstrass ζ-function, in which the first term is a true (meromorphic) Jacobi form and the second a function of $u - v$ alone. The details will be given in a later publication.

is symmetric in u and v and satisfies the elliptic transformation properties

$$\widehat{\mu}(u+1,v) \;=\; a^{-1}\,b\,q^{-1/2}\,\widehat{\mu}(u+\tau,v) \;=\; -\widehat{\mu}(u,v)$$

and the modular transformation properties

$$\zeta_8\,\widehat{\mu}(u,v;\tau+1) \;=\; -(\tau/i)^{-1/2}\,e^{\pi i(u-v)^2/\tau}\,\widehat{\mu}\Big(\frac{u}{\tau},\frac{v}{\tau};\frac{-1}{\tau}\Big) \;=\; \widehat{\mu}(u,v)\,.$$

Now it is well-known that specializing the "elliptic" variables of a Jacobi form to torsion points (= points of $\mathbb{Q}\tau+\mathbb{Q}$) gives functions of τ which are modular forms times rational powers of q. If we combine this with the two other facts that some or all mock theta functions can be written as linear combinations of the function $\mu(u,v;\tau)$ with u and v in $\mathbb{Q}\tau+\mathbb{Q}$, and that the function μ can be modified in a simple way to give a function $\widehat{\mu}$ which transforms like a Jacobi form, we deduce that a mock theta function, multiplied by a suitable rational power of q and corrected by the addition of a simple and explicit non-holomorphic function of τ, becomes a modular form. For instance, using the identity of Watson for the mock theta function $f(q)$ of order 3 given at the beginning of this section and the transformation properties of the Lerch sums, Zwegers [**27**] shows that the function $h_3(\tau) = q^{-1/24}f(q)$ $(\tau\in\mathfrak{H},\ q=e^{2\pi i\tau})$ can be "corrected" by adding to it the non-holomorphic unary theta series

$$R_3(\tau) \;=\; \sum_{n\equiv 1 \pmod 6} \operatorname{sgn}(n)\,\beta(n^2y/6)\,q^{-n^2/24} \qquad (y=\Im(\tau))\,,$$

where $\beta(x)$ is the complementary error function (or incomplete gamma function)

$$\beta(x) \;=\; \int_x^\infty u^{-1/2}\,e^{-\pi u}\,du \;=\; 2\int_{\sqrt{x}}^\infty e^{-\pi t^2}\,dt \;=\; 1 \;-\; E\big(\sqrt{x}\big) \qquad (x\ge 0),$$

and that the corrected function $\widehat{h}_3(\tau) = h_3(\tau)+R_3(\tau)$ transforms like a modular form of weight $\frac{1}{2}$ with respect to the congruence group $\Gamma(2)$.

Finally, we mention that the correction terms for the mock theta functions can be written in a different form. For instance, we can write the definition of $R_3(\tau)$ as

$$R_3(\tau) \;=\; \frac{i}{\sqrt{3}}\int_{-\overline{\tau}}^\infty \frac{g_3(z)}{\sqrt{(z+\tau)/i}}\,dz\,,$$

where $g_3(z)$, a holomorphic modular form of weight $\frac{3}{2}$, is the unary theta series

$$g_3(z) \;=\; \sum_{n\equiv 1 \pmod 6} n\,q^{n^2/24} \;=\; \sum_{n=1}^\infty \Big(\frac{-12}{n}\Big)\,n\,q^{n^2/24}\,.$$

This type of formula will play a role in §5 when we give the general definitions of mock theta functions and mock modular forms.

3. INDEFINITE THETA SERIES

Let $\langle\ ,\ \rangle$ be a $\mathbb{Z}$-valued bilinear form on $\mathbb{Z}^r$ and $Q(x) = \frac{1}{2}\langle x, x\rangle$ the associated quadratic form. If Q is positive definite, then it is a classical fact that the theta series $\Theta(\tau) = \sum_{\nu\in\mathbb{Z}^r} q^{Q(\nu)}$, or more generally $\Theta_{a,b}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)}$ for any a, $b \in \mathbb{Q}^r$, is a modular form of weight $r/2$ (and of known level and character). For indefinite theta series there is a well-known theory of non-holomorphic theta series due to Siegel, but no standard way to obtain holomorphic functions with arithmetic Fourier coefficients having any kind of modular transformation behavior. In the second chapter of his thesis, Zwegers shows how to do this when the quadratic form Q has signature $(r-1, 1)$. Since many (presumably, all) of the mock theta functions have representations as the quotient of a theta series associated to a quadratic form of signature $(1, 1)$ by a theta series associated to a positive definite quadratic form of rank 1, this has an immediate application to the transformation behavior of mock theta functions. We will describe the general result first and then give some of the applications to mock theta functions at the end of the section.

For Q indefinite, the theta series $\Theta_{a,b}$ as defined above is divergent, since its terms are unbounded (because there are vectors $\nu \in \mathbb{Z}^r$ with $Q(\nu) < 0$) and all occur with infinite multiplicity (because there is an infinite group of units permuting the terms). However, we can make it convergent by restricting the summation to the set of lattice points lying between two appropriately chosen hyperplanes in $\mathbb{R}^r$. More precisely, let C be one of the two components of the double cone $\{x \in \mathbb{R}^r \mid Q(x) < 0\}$, and for a, $b \in \mathbb{Q}^r$ and c, $c' \in C$ define

$$\Theta_{a,b}^{c,c'}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} \big(\operatorname{sgn}\big(\langle c,\nu\rangle\big) - \operatorname{sgn}\big(\langle c',\nu\rangle\big)\big)\, e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)}.$$

This series now contains only positive powers of q and is absolutely convergent (although this isn't obvious), so it defines a holomorphic function of τ, but of course it is not in general modular. To remedy this, Zwegers introduces the modified function

$$\widehat{\Theta}_{a,b}^{c,c'}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} \left(E\Big(\frac{\langle c,\nu\rangle\sqrt{y}}{\sqrt{-Q(c)}}\Big) - E\Big(\frac{\langle c',\nu\rangle\sqrt{y}}{\sqrt{-Q(c')}}\Big)\right) e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)} \qquad (y = \Im(\tau))$$

with $E(z)$ as in Theorem 2.1. Then from the relation $E(x) = \operatorname{sgn}(x)\,(1 - \beta(x^2))$ we get $\widehat{\Theta}_{a,b}^{c,c'}(\tau) = \Theta_{a,b}^{c,c'}(\tau) - \Phi_{a,b}^{c}(\tau) + \Phi_{a,b}^{c'}(\tau)$ with

$$\Phi_{a,b}^{c}(\tau) = \sum_{\nu\in\mathbb{Z}^r+a} \operatorname{sgn}\big(\langle c,\nu\rangle\big)\, \beta\Big(\frac{\langle c,\nu\rangle^2 y}{-Q(c)}\Big)\, e^{2\pi i\langle b,\nu\rangle} q^{Q(\nu)}$$

(which is rapidly convergent, with summands bounded by $e^{-A\|\nu\|^2}$ for some $A > 0$). If c belongs to $C \cap \mathbb{Q}^r$, then $\Phi_{a,b}^{c}(\tau)$ is a finite linear combination $\sum_j R_j(\tau)\theta_j(\tau)$ where each $R_j(\tau)$ is a sum of the same sort as occurred in §2 as the correction needed to make

mock theta functions modular (i.e., $R_j(\tau) = \sum_{n\in\mathbb{Z}+\alpha_j} \operatorname{sgn}(n)\,\beta(4\kappa_j n^2 y)\,q^{-\kappa_j n^2}$ for some $\alpha_j \in \mathbb{Q}$ and $\kappa_j \in \mathbb{Q}_{>0}$) and each $\theta_j(\tau)$ is an ordinary theta series associated to the quadratic form $Q|\langle c\rangle^\perp$ and hence is a holomorphic modular form of weight $(r-1)/2$. (In the case of mock theta functions, one can choose $\theta_j(\tau) = \theta(\tau)$ independent of j, so that $\Phi^c_{a,b}(\tau)$ factors as $\theta(\tau)R^c(\tau)$, and moreover the theta function $\theta(\tau)$, here of weight $\frac{1}{2}$ because $r = 2$, is the same for c and c'. The mock theta function is then, up to a power of q, the quotient $h(\tau) = \Theta^{c,c'}_{a,b}(\tau)/\theta(\tau)$ and its completed version is $\widehat{h}(\tau) = h(\tau) - R^c(\tau) + R^{c'}(\tau)$.) Zwegers now shows ([**28**], Cor. 2.9):

THEOREM 3.1. — *The non-holomorphic function* $\widehat{\Theta}_{a,b} = \widehat{\Theta}^{c,c'}_{a,b}$ $(a,\, b \in \mathbb{Q}^r,\ c,\, c' \in C)$ *satisfies the same transformation equations (expressing* $\widehat{\Theta}_{a,b}(\tau+1)$ *and* $\widehat{\Theta}_{a,b}(-1/\tau)$ *as finite linear combinations of functions* $\widehat{\Theta}_{a',b'}(\tau)$*) as in the positive definite case, and in particular is a non-holomorphic modular form of weight* $r/2$.

Note that this theorem can also be used to get theta series associated to Q which are *holomorphic* modular forms. For instance, let $\mathrm{O}(Q)^+$ be the component of the orthogonal group of Q mapping C to itself and Γ^+ congruence subgroup of all $\gamma \in \mathrm{O}(Q)^+$ preserving $\mathbb{Z}^r + a$ and the function $\nu \mapsto e^{2\pi i\langle b,\nu\rangle}$ on $\mathbb{Z}^r + a$. Then $\Phi^c_{a,b} = \Phi^{\gamma c}_{a,b}$ for any $c \in C$ and $\gamma \in \Gamma^+$, so the function $\Theta^{c,\gamma c}_{a,b} = \widehat{\Theta}^{c,\gamma c}_{a,b}$ (which is independent of c, as one can easily check) is both holomorphic and modular.

We now give examples of the applications of these results to mock theta functions. In [**1**], Andrews found representations for all of Ramanujan's fifth order mock theta functions except $\chi_1(\tau)$ and $\chi_2(\tau)$ as quotients $\Theta(\tau)/\theta(\tau)$ with $\theta(\tau)$ modular of weight $\frac{1}{2}$ and $\Theta(\tau)$ a theta series associated to a binary quadratic form of signature (1,1), a typical formula being

$$f_1(q) \;=\; \frac{1}{\prod_{n=1}^\infty (1-q^n)} \left(\sum_{n\ge 0} \sum_{|j|\le n} - \sum_{n<0} \sum_{|j|<|n|} \right) (-1)^j\, q^{n(5n+1)/2 - j^2}\,.$$

Similar formulas for the seventh order functions were proved by Hickerson [**12**], e.g.,

$$\mathcal{F}_1(q) \;=\; \frac{1}{\prod_{n=1}^\infty (1-q^n)} \left(\sum_{r,\,s\ge 0} - \sum_{r,\,s<0} \right) (-1)^{r+s}\, q^{(3r^2+8rs+3s^2+r+s)/2}\,.$$

Using these formulas and Theorem 3.1, we can find the modular properties of all of these mock theta functions. For example, from the first identity just given and its companion for f_2 we find that the sum of the holomorphic vector-valued function $\binom{q^{-1/60} f_1(q)}{q^{11/60} f_2(q)}$ and the non-holomorphic correction term $\binom{R_{5,1}(\tau)}{R_{5,2}(\tau)}$, where

$$R_{p,j}(\tau) \;=\; \sum_{n\,\equiv\, j \pmod p} \left(\frac{12}{n}\right) \operatorname{sgn}(n)\,\beta\Big(\frac{n^2 y}{6p}\Big)\, q^{-n^2/24p} \qquad (6 \nmid p,\ j \in \mathbb{Z}/p\mathbb{Z}),$$

transforms like a modular form of weight $\frac{1}{2}$ with respect to the congruence subgroup $\Gamma(5)$ of $\mathrm{SL}(2,\mathbb{Z})$, while from the second one we find the even nicer result for the seventh order functions that, if we define

$$M_7(\tau) = \begin{pmatrix} q^{-1/168}\,\mathcal{F}_1(q) \\ -q^{-25/168}\,\mathcal{F}_2(q) \\ q^{47/168}\,\mathcal{F}_3(q) \end{pmatrix}, \qquad \widehat{M}_7(\tau) = M_7(\tau) + \begin{pmatrix} R_{7,1}(\tau) \\ R_{7,2}(\tau) \\ R_{7,3}(\tau) \end{pmatrix},$$

then $\widehat{M}_7$ transforms in a modular way with respect to the full modular group $\mathrm{SL}(2,\mathbb{Z})$:

$$\begin{aligned} \widehat{M}_7(\tau+1) &= \mathrm{diag}\big(\zeta_{168}^{-1},\, \zeta_{168}^{-25},\, \zeta_{168}^{-121}\big)\, \widehat{M}_7(\tau)\,, \\ \widehat{M}_7(-1/\tau) &= \sqrt{\tau/7i}\ \big(2\sin 6\pi jk/7\big)_{1\le j,\,k\le 3}\, \widehat{M}_7(\tau)\,. \end{aligned}$$

4. FOURIER COEFFICIENTS OF MEROMORPHIC JACOBI FORMS

We recall that a Jacobi form is a holomorphic function $\varphi(\tau,z)$ of two variables $\tau\in\mathfrak{H}$ and $z\in\mathbb{C}$ which transforms like a modular form with respect to the first and like an elliptic function with respect to the second. More precisely, a Jacobi form of weight k and index $N/2$ on the full modular group is a holomorphic function $\varphi:\mathfrak{H}\times\mathbb{C}\to\mathbb{C}$ satisfying $\varphi\big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\big)=(c\tau+d)^k\,e^{\pi iNcz^2/(c\tau+d)}\,\varphi(\tau,z)$ for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ and $\varphi(\tau,z+r\tau+s)=e^{-\pi iN(r^2\tau+2rz)}\,\varphi(\tau,z)$ for all $\binom{r}{s}\in\mathbb{Z}^2$, as well as certain growth conditions which we omit. (For these and further details and explanations, see [**9**].) Examples are given by classical theta functions and by the Fourier coefficients of Siegel modular forms of degree 2, and an important property, already mentioned in §2 in connection with the mock Jacobi forms $\mu(u,v;\tau)$, is that for any $a,\,b\in\mathbb{Q}$ the specialization $\varphi(\tau,a\tau+b)$, multiplied by a suitable rational power of q, is a modular form with respect to τ.

The elliptic transformation property of φ implies that it has an expansion

$$\varphi(\tau,z)=\sum_{\ell\ (\mathrm{mod}\ N)} h_\ell(\tau)\,\theta_{N,\ell}(\tau,z)\,,$$

where $h_\ell(\tau)$ $(\ell\in\mathbb{Z})$ is the Fourier coefficient

$$h_\ell(\tau) = e^{-\pi i\ell^2\tau/N}\int_{z_0}^{z_0+1} e^{-2\pi i\ell z}\,\varphi(\tau,z)\,dz \qquad (\text{any } z_0\in\mathbb{C})$$

and $\theta_{N,\ell}(\tau,z)$ is the unary theta series

$$\theta_{N,\ell}(\tau,z) = \sum_{n\equiv\ell\ (\mathrm{mod}\ N)} e^{2\pi inz}\,q^{n^2/2N}$$

(which is itself a Jacobi form of weight $\frac{1}{2}$ and index $N/2$ on a subgroup of $\mathrm{SL}(2,\mathbb{Z})$). The modular transformation property of φ then implies that the vector-valued function $h = (h_1, \dots, h_N) : \mathfrak{H} \to \mathbb{C}^N$ transforms like a modular form of weight $k - \frac{1}{2}$ on $\mathrm{SL}(2,\mathbb{Z})$.

Zwegers observed that certain representations of the fifth order mock theta functions found by Andrews [**2**] can be reinterpreted as saying that these functions are the Fourier coefficients of a *meromorphic* Jacobi form (= quotient of two holomorphic Jacobi forms), and proceeds to find a general theory of the transformation behavior of the vector-valued function $h(\tau)$, defined by the same integral formula as above, when the Jacobi form φ is allowed to have poles. The complete result (Theorem 3.9 in [**28**]) is a little too long and technical to be included here. Briefly (in the simplest case when φ has only simple poles), it says that h can be completed to a non-holomorphic vector-valued modular form $\widehat{h}(\tau) = \big(\widehat{h}_\ell(\tau)\big)_{\ell \pmod N}$ of weight $k - \frac{1}{2}$ by the addition of a vector of functions which are linear combinations of functions $R_\nu(\tau)$ of the same type as we encountered in §2 and §3, the coefficients in their turn being modular forms of weight $k-1$ if the poles of φ are at torsion points $z = a\tau + b$, a, $b \in \mathbb{Q}$. In particular, if $k = 1$ and the poles of φ are at torsion points, then the $\widehat{h}_\ell$ are $\mathbb{C}$-linear combinations of the functions R_ν and the functions h_ℓ are mock theta functions of precisely the same kind as Ramanujan's. Moreover (now again for general k), the meromorphic Jacobi form has an expansion of the form $\varphi(\tau, z) = \sum_\ell h_\ell(\tau)\, \theta_{N,\ell}(\tau, z) + \mathrm{Res}$ with the same $\theta_{N,\ell}$ as above and with "Res" being given as an explicit finite sum over the residues of $\varphi(\tau, z)$ in the fundamental domain $z_0 + [0,1)\tau + [0,1)$ for the action of $\mathbb{Z}\tau + \mathbb{Z}$ on $\mathbb{C}$.

One peculiarity of the expansion just described is that the individual terms change as the base-point z_0 used to compute the Fourier coefficients $h_\ell(\tau)$ moves across certain lines in $\mathbb{C}$, namely, those where $\varphi(\tau, \cdot)$ has a singularity on the boundary of the parallelogram $z_0 + [0,1]\tau + [0,1] \subset \mathbb{C}$. This is related to the so-called "wall-crossing phenomenon" in the theory of Donaldson invariants (which were in turn related to the theory of indefinite theta functions in [**10**]) and also to the similar wall-crossing phenomenon which has appeared more recently in the theory of black holes [**22**].

5. MOCK THETA FUNCTIONS AND MOCK MODULAR FORMS

In §§2–4 we have seen that each of Ramanujan's mock theta functions $H(q) \in \mathbb{Z}[[q]]$ acquires modularity transformation properties after carrying out the following three steps: (i) multiply $H(q)$ by a suitable rational power q^λ of q, e.g., $q^{-1/24}$ for the mock theta function $f(q)$ of order 3 or $q^{47/168}$ for the mock theta function

$\mathcal{J}_3(q)$ of order 7; (ii) change the variable from $q = e^{2\pi i\tau}$ to τ with $\tau \in \mathfrak{H}$, setting $h(\tau) = e^{2\pi i\lambda\tau} H(e^{2\pi i\tau})$; (iii) add a simple (but non-holomorphic) correction term to $h(\tau)$ so that the corrected function $\widehat{h}(\tau)$ transforms like a modular form of weight $\frac{1}{2}$ for some congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. The correction has the form $g^*(\tau) = \sum_{n\in\mathbb{Z}+\alpha} \mathrm{sgn}(n)\,\beta(4\kappa n^2 y)\,q^{-\kappa n^2}$ for some $\alpha \in \mathbb{Q}$ and $\kappa \in \mathbb{Q}_{>0}$, with $\beta(t)$ as in Sections 2 and 3, and is in turn associated (in a way which we will make precise in a moment) to the theta series $g(\tau) = \sum_{n\in\mathbb{Z}+\alpha} n\,q^{\kappa n^2}$, which is a true modular form of weight $\frac{3}{2}$. Notice that steps (i) and (ii) would also be necessary in the case of q-series which are attached to true modular forms; it is only in the final step that the "mock" aspect comes into play.

One can therefore say that each mock theta function $H(q)$ has two secret invariants: a rational number λ such that $H(q)$ must be multiplied by q^λ in order to have any kind of modularity properties, and a "shadow" $g(\tau)$ which is a unary theta series of weight $\frac{3}{2}$ such that the holomorphic function $h(\tau) = q^\lambda H(q)$ becomes a non-holomorphic modular form of weight $\frac{1}{2}$ when we complete it by adding a correction term $g^*(\tau)$ associated to $g(\tau)$. This picture generalizes immediately to other weights and leads to the notion of a *mock modular form of weight* k, which we now describe. The space $\mathbb{M}_k$ of all such forms contains as a subspace the space M_k of classical modular forms of weight k (and arbitrary level and character), but since—as we already saw for Ramanujan's original mock theta functions—we will in general need to allow negative powers of q in the Fourier expansions at infinity or other cusps, we will define $\mathbb{M}_k$ in such a way that it contains the larger space $M_k^!$ of weakly holomorphic modular forms of weight k (= functions which transform like modular forms of weight k and are holomorphic in $\mathfrak{H}$, but may have singularities of type $q^{-\mathrm{O}(1)}$ at cusps). The space $\mathbb{M}_k$ is of course infinite dimensional, but becomes finite dimensional when one adds conditions specifying the level of the form (i.e., the discrete group $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ and the character or finite-dimensional representation of Γ describing the modularity properties of the completed form) and the order of poles which we allow at the cusps. Each mock modular form $h \in \mathbb{M}_k$ has a "shadow" $g = \xi[h]$ which is an ordinary modular form of weight $2-k$. This "shadow" depends $\mathbb{R}$-linearly on h and vanishes if and only if h is a modular form (which is then weakly holomorphic, since in $\mathbb{M}_k$ we impose exponential growth conditions at the cusps), so that we have an exact sequence over $\mathbb{R}$

$$0 \longrightarrow M_k^! \longrightarrow \mathbb{M}_k \xrightarrow{\xi} M_{2-k}\ .$$

In fact the last map is also surjective, so that we have a short exact sequence and $\mathbb{M}_k$ can be seen as the extension of one space of classical modular forms by another.

The definition of the shadow map is as follows. For $g(\tau)$ a modular form of weight $2-k$ we define a new function $g^*(\tau)$ with $\partial g^*/\partial\bar\tau$ proportional to $y^{-k}\,\overline{g(\tau)}$ by

$$g^*(\tau) = (i/2)^{k-1}\int_{-\bar\tau}^{\infty}(z+\tau)^{-k}\,g^c(z)\,dz \;=\; \sum_{n>0} n^{k-1}\,\overline{b_n}\,\beta_k(4ny)\,q^{-n}\,,$$

if $g = \sum_{n>0} b_n q^n$ (sum over some arithmetic progression in $\mathbb{Q}$), where $g^c(\tau) = \overline{g(-\bar\tau)} = \sum \overline{b_n}\,q^n$ and $\beta_k(t)$ is the incomplete gamma function $\int_t^\infty u^{-k}\,e^{-\pi u}\,du$. (Here we are assuming for convenience that g vanishes at infinity. For mock theta functions, the unary theta series g is in fact a cusp form.) Then to say that g is the shadow $\mathcal{S}[h]$ of h means that the non-holomorphic function $\widehat{h}(\tau) = h + g^*$ transforms like a modular form of weight k, i.e., $\widehat{h}(\gamma\tau) = \rho(\gamma)(c\tau+d)^k\widehat{h}(\tau)$ for all $\gamma = \left(\begin{smallmatrix}\cdot&\cdot\\ c&d\end{smallmatrix}\right)$ in a suitable subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, where ρ is a character (or, for vector-valued h, representation) of Γ.

We can say this in another way. There is a canonical isomorphism between the space $\mathbb{M}_k$ of mock modular forms, which are holomorphic, but not quite modular, and a second space $\widehat{\mathbb{M}}_k$ of functions which are modular, but not quite holomorphic, the situation here being exactly analogous to the isomorphism (see [**13**] or §5 of [**26**]) between the space $\widetilde{M}_k$ of "quasimodular forms" (which are again holomorphic, but not quite modular) and the space $\widehat{M}_k$ of "almost holomorphic modular forms". The isomorphism $\mathbb{M}_k \cong \widehat{\mathbb{M}}_k$ sends the mock modular form h with shadow g to the completed function $\widehat{h} = h + g^*$, so to find the definition of the image space $\widehat{\mathbb{M}}_k$ we must see how to recover h from $\widehat{h}$. But this is easy: since h is holomorphic, we can just apply the Cauchy–Riemann operator $\partial/\partial\bar\tau$ to get $\partial\widehat{h}/\partial\bar\tau = \partial g^*/\partial\bar\tau \doteq y^{-k}\overline{g}$, and from $\widehat{h}$ and g we recover h as $\widehat{h} - g^*$. The direct definition of $\widehat{\mathbb{M}}_k$ is therefore as follows. Let $\mathfrak{M}_k$ be the space of real-analytic functions $F(\tau)$ in $\mathfrak{H}$ which transform like modular forms of weight k (i.e., $F(\gamma\tau) = \rho(\gamma)(c\tau+d)^kF(\tau)$ for all γ in some modular group $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$ and some character or representation ρ of Γ) and have at most exponential growth at the cusps. More generally, let $\mathfrak{M}_{k,\ell}$ be the space of functions which transform under some modular group Γ by $F(\gamma\tau) = \rho(\gamma)(c\tau+d)^k(c\bar\tau+d)^\ell F(\tau)$. Since the derivative of a modular form of weight 0 is a modular form of weight 2, the map $\partial/\partial\bar\tau$ sends $\mathfrak{M}_k = \mathfrak{M}_{k,0}$ to $\mathfrak{M}_{k,2}$ and the map $\partial/\partial\tau$ sends $\mathfrak{M}_{0,\ell}$ to $\mathfrak{M}_{2,\ell}$. Also, the function $y := \Im(\tau)$ belongs to $\mathfrak{M}_{-1,-1}$, so we have isomorphisms $\cdot y^r : \mathfrak{M}_{k,\ell} \to \mathfrak{M}_{k-r,\ell-r}$ for any $r \in \mathbb{Z}$. This gives a commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{M}_k & = & \mathfrak{M}_{k,0} & \xrightarrow{\partial/\partial\bar\tau} & \mathfrak{M}_{k,2} & \xrightarrow[\cong]{y^k} & \mathfrak{M}_{0,2-k} \\
 & & & & & & \cup\uparrow \\
 & & & & & & \overline{M_{2-k}}
\end{array}
\qquad
\begin{array}{ccc}
 & \xrightarrow{\partial/\partial\tau} & \mathfrak{M}_{2,2-k}\,, \\
 & \nearrow_{0} & \\
 & &
\end{array}$$

and $\widehat{\mathbb{M}}_k$ is the space of functions $F \in \mathfrak{M}_k$ for which $y^k F_{\bar\tau} = y^k\, \partial F/\partial\bar\tau$ belongs to $\overline{M_{2-k}}$, or equivalently (since it already transforms like a modular form of weight $(0, 2-k)$ by the above diagram), for which $y^k F_{\bar\tau}$ is anti-holomorphic:

$$\widehat{\mathbb{M}}_k = \left\{ F \in \mathfrak{M}_k \;\middle|\; \frac{\partial}{\partial\tau}\Big(y^k \frac{\partial F}{\partial\bar\tau}\Big) = 0 \right\}.$$

Note that the composite map $\mathfrak{M}_k \overset{y^k\,\partial/\partial\bar\tau}{\longrightarrow} \mathfrak{M}_{0,2-k} \overset{y^{2-k}\partial/\partial\tau}{\longrightarrow} \mathfrak{M}_k$ is, up to a factor of 4 and an additive constant $k(2-k)$, the Laplace (or Casimir) operator in weight k, so that the elements of $\widehat{\mathbb{M}}_k$ are in particular *weak Maass forms* (= non-holomorphic modular forms of at most exponential growth at the cusps which are eigenfunctions of the Laplace operator), but they are very special weak Maass forms since the eigenvalue under the Laplacian is only allowed to have the particular value $\frac{k}{2}\big(1-\frac{k}{2}\big)$. Following a suggestion of Bruinier and Funke, these functions are called *harmonic weak Maass forms.*

The whole discussion can be summarized by the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_k^! & \xrightarrow{\subset} & \widehat{\mathbb{M}}_k & \xrightarrow{y^k\,\partial/\partial\bar\tau} & \overline{M_{2-k}} & \longrightarrow & 0\,, \\
 & & \Big\| & & \Big\downarrow\cong & & \Big\downarrow\cong & & \\
 & & & & & & \text{complex conjugation} & & \\
0 & \longrightarrow & M_k^! & \longrightarrow & \mathbb{M}_k & \xrightarrow{\not\partial} & M_{2-k} & \longrightarrow & 0
\end{array}$$

We observe that the map $\not\partial$ is linear over $\mathbb{R}$ but antilinear over $\mathbb{C}$, so that the sequence $0 \to M_k^! \to \mathbb{M}_k \to M_{2-k} \to 0$ is exact only over $\mathbb{R}$. We could make it exact over $\mathbb{C}$ by replacing the last term by $\overline{M_{2-k}}$; this would be more natural, but less aesthetic since holomorphic modular forms are more familiar than antiholomorphic ones.

General Principle. A mock theta function is by definition a q-series $H(q) = \sum_{n\geq 0} a_n\, q^n$ such that $q^\lambda H(q)$ for some $\lambda \in \mathbb{Q}$ is a mock modular form of weight $\frac{1}{2}$ whose shadow is a unary theta series of weight $\frac{3}{2}$, i.e., a function of the form $\sum_{n\in\mathbb{Z}} \varepsilon(n)\, n\, q^{\kappa n^2}$ with $\kappa \in \mathbb{Q}_{>0}$ and ε an odd periodic function. It follows that if $\mathcal{A} \subset \mathbb{Q}$ is any arithmetic progression containing no number of the form $-\kappa n^2 - \lambda$ with $n \in \mathbb{Z}$ and $\varepsilon(n) \neq 0$, then $\sum_{n\in\mathcal{A}} a_n\, q^{n+\lambda}$ is a true (though in general only weakly holomorphic) modular form of weight $\frac{1}{2}$. This principle has many applications, one of which will be described in §7.

6. NEW IDENTITIES AND NEW EXAMPLES

At the end of the introduction we mentioned that one application of Zwegers's theory is that it now becomes as easy to prove identities among mock theta functions (or more generally, among mock modular forms) as it previously was for modular forms.

For example, the so-called "Mock theta conjectures" for the mock theta functions of order 5, which were stated by Ramanujan in his "Lost Notebook", were proved only in 1988 by D. Hickerson [**11**] after heroic efforts, but now with the knowledge of the transformation properties of the mock theta functions the proof becomes automatic: one only has to verify that the left- and right-hand sides of the identities become modular after the addition of the same non-holomorphic correction term and that the first few coefficients of the q-expansions agree. Moreover, knowing the transformation behavior also allows one to find new identities in a systematic way. For instance, we mentioned in §3 that representations as quotients of a binary by a unary theta series were found in [**1**] for only four of the five vector-valued mock theta functions of order 5, but using the mock modular transformation properties one easily gets such a representation also in the missing case, and it actually turns out to be the best one, since it is the only one of the five functions whose completed version transforms under the full modular group $\mathrm{SL}(2,\mathbb{Z})$: if we set $M_5(\tau) = \binom{M_{5,1}(\tau)}{M_{5,2}\tau)} = -\frac{2}{3}\binom{q^{-1/120}\,(2-\chi_1(q))}{q^{71/120}\,\chi_2(q)}$, then we have

$$-\frac{3}{2}\,M_{5,j}(\tau) \;=\; \frac{1}{\eta(\tau)} \sum_{\substack{|a|>5|b| \\ a+b\equiv 2 \pmod 4 \\ a\equiv 2j \pmod 5}} (-1)^a \left(\frac{-3}{a^2-b^2}\right) \operatorname{sgn}(a)\, q^{(a^2-5b^2)/120}$$

and the completed function $\widehat{M}_5 = M_5 + \binom{R_{5,1}}{R_{5,2}}$, where $R_{p,j}$ is defined as in §3, satisfies

$$\widehat{M}_5(\tau+1) = \begin{pmatrix} \zeta_{120}^{-1} & 0 \\ 0 & \zeta_{120}^{-49} \end{pmatrix} \widehat{M}_5(\tau)\,, \qquad \widehat{M}_5(-1/\tau) = \sqrt{\tau/5i}\, \begin{pmatrix} -2\sin\frac{\pi}{5} & 2\sin\frac{2\pi}{5} \\ 2\sin\frac{2\pi}{5} & 2\sin\frac{\pi}{5} \end{pmatrix} \widehat{M}_5(\tau)\,.$$

Similarly, for the mock theta functions of order 7, as well as Hickerson's identity for $\eta(\tau)M_{7,j}(\tau)$ as an indefinite binary theta series, we find the representation

$$\eta(7\tau)\,M_{7,j}(\tau) \;=\; \sum_{\substack{|r|>|s|,\ rs>0 \\ 2r\equiv -2s\equiv j \pmod 7}} \operatorname{sgn}(r)\,\big(2\varepsilon_6(s) - \varepsilon_2(r)\varepsilon_3(s) - \varepsilon_3(r)\varepsilon_2(s)\big)\, q^{rs/42}$$

(where $\varepsilon_N(s) = 1$ if $s \equiv 0 \pmod N$ and 0 otherwise) of the product of $M_{7,j}(\tau)$ with $\eta(7\tau)$ as a "mock Eisenstein series" of weight 1 (explaining the smallness of the Fourier coefficients of this product that was mentioned in §1), and also the representation

$$\eta(\tau)^3\,M_{7,j}(\tau) \;=\; \sum_{\substack{m>2|n|/9 \\ n\equiv j \pmod 7}} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\Big(m\operatorname{sgn}(n) - \frac{3n}{14}\Big)\, q^{m^2/8 - n^2/168}$$

of the product of $M_{7,j}(\tau)$ with $\eta(\tau)^3$ as an indefinite theta series of weight 2. What's more, by methods obtained in a reasonably straightforward way by generalizing methods from standard modular form theory (holomorphic projection, Rankin–Cohen

brackets, etc.), one can produce infinitely many new examples of mock theta functions or of more general types of mock modular forms. In particular, we can construct vector-valued mock modular forms $M_p(\tau) = \big(M_{p,j}(\tau) = -M_{p,-j}(\tau)\big)_{j \pmod p}$ of length $(p-1)/2$ of order $p > 3$ for any prime p by a formula like the one just given for M_7, e.g.,

$$M_{11,j}(\tau) = \frac{1}{\eta(\tau)^3} \sum_{\substack{m > 2|n|/11 \\ n \equiv j \pmod{11}}} \left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)\left(m\,\mathrm{sgn}(n) - \frac{n}{6}\right) q^{m^2/8 - n^2/264}$$

for $p = 11$, in such a way that the completed function $\widehat{M}_p(\tau) = \big(M_{p,j}(\tau)\big)_{j \pmod p}$ with $\widehat{M}_{p,j}(\tau) = M_{p,j}(\tau) + R_{p,j}(\tau)$ transforms like a vector-valued modular form of weight $\frac{1}{2}$ on $\mathrm{SL}(2,\mathbb{Z})$, thus directly generalizing the previous two cases $p = 5$ and $p = 7$.

There are also many examples of other types. For instance, there is a family of scalar-valued functions having completions that transform like modular forms of every even integral weight k on the full modular group. The kth function $F_k = F_k(\tau)$ is defined as

$$F_k = \sum_{n \neq 0} (-1)^n \left(\frac{-3}{n-1}\right) n^{k-1} \frac{q^{n(n+1)/6}}{1-q^n} = -\sum_{r>s>0} \left(\frac{12}{r^2-s^2}\right) s^{k-1} q^{rs/6}$$

(i.e., as a Lerch-like sum or as a mock Eisenstein series), the first two values being

$$\begin{aligned} F_2 &= q + 2q^2 + q^3 + 2q^4 - q^5 + 3q^6 - \cdots, \\ F_4 &= 7q + 26q^2 + 7q^3 + 26q^4 - 91q^5 + \cdots. \end{aligned}$$

Then the function

$$f(\tau) = \frac{E_2(\tau) - 12F_2(q)}{\eta(\tau)} = q^{-1/24}\left(1 - 35q - 130q^2 - 273q^3 - 595q^4 - \cdots\right),$$

where $E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n$ is the usual quasimodular Eisenstein series of weight 2, is a mock modular form of weight $\frac{3}{2}$ on the full modular group with shadow $\eta(\tau)$, and for each integer $n > 0$ the sum of $12F_{2n+2}(\tau)$ and $24^n \binom{2n}{n}^{-1} [f,\eta]_n$ (where $[f,g]_n$ denotes the n-th Rankin–Cohen bracket, here in weight $(\frac{3}{2}, \frac{1}{2})$), is a modular form of weight $2n+2$ on $\mathrm{SL}(2,\mathbb{Z})$. In a different direction, the Eichler integral $\widetilde{f} = \sum_{n=1}^{\infty} n^{-k+1} a(n)\, q^n$ of a classical cusp form $f = \sum a(n)\, q^n$ of weight k is a mock modular form of weight $2-k$, but of a somewhat generalized kind in which the "shadow" is allowed to be a weakly holomorphic modular form. (This latter fact was observed independently by K.-H. Fricke in Bonn.) Yet another example—actually the oldest—is the generating function of class numbers of imaginary quadratic fields (more precisely, of Hurwitz–Kronecker class numbers), which was shown in [**24**] to be a mock modular form of weight $\frac{3}{2}$ and level 4 with shadow $\sum q^{n^2}$, although the notion had not yet been formulated at that time.

7. APPLICATIONS

Since the appearance of Zwegers's thesis, Kathrin Bringmann and Ken Ono and their collaborators have developed the theory further and given a number of beautiful applications, a sampling of which we describe in this final section.

Define the *rank* of a partition to be its largest part minus the number of its parts, and for n, $t \in \mathbb{N}$ and $r \in \mathbb{Z}/t\mathbb{Z}$ let $N(r,t;n)$ denote the number of partitions of n with rank congruent to r modulo t. The rank was introduced by Dyson [**8**] to explain in a natural way the first two of Ramanujan's famous congruences

$$p(5\ell+4) \equiv 0 \pmod 5, \quad p(7\ell+5) \equiv 0 \pmod 7, \quad p(11\ell+6) \equiv 0 \pmod{11}$$

for the partition function $p(n)$: he conjectured (and Atkin and Swinnerton-Dyer [**4**] later proved) that the ranks of the partitions of an integer congruent to 4 (mod 5) or to 5 (mod 7) are equidistributed modulo 5 or 7, respectively, so that $N(r,5;5\ell+4) = \frac{1}{5}p(5\ell+4)$, $N(r,7;7\ell+5) = \frac{1}{7}p(7\ell+5)$. (He also conjectured the existence of a further invariant, which he dubbed the "crank," which would explain Ramanujan's third congruence in the same way; this invariant was constructed later by Garvan and Andrews.) The generating function that counts the number of partitions of given size and rank is given by

$$\mathcal{R}(w;q) := \sum_{\lambda} w^{\mathrm{rank}(\lambda)}\, q^{\|\lambda\|} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-wq^m)(1-w^{-1}q^m)},$$

where the first sum is over all partitions and $\|\lambda\| = n$ means that λ is a partition of n. Clearly knowing the functions $n \mapsto N(r,t;n)$ for all $r \pmod t$ is equivalent to knowing the specializations of $\mathcal{R}(w;q)$ to all t-th roots of unity $w = e^{2\pi i a/t}$. For $w=-1$, the function $\mathcal{R}(w;q)$ specializes to $f(q)$, the first of Ramanujan's mock theta functions, which is $q^{1/24}$ times a mock modular form of weight $\frac{1}{2}$. Bringmann and Ono [**7**] generalize this to other roots of unity:

THEOREM 7.1. — *If $\xi \neq 1$ is a root of unity, then $q^{-1/24}\mathcal{R}(\xi;q)$ is a mock modular form of weight $\frac{1}{2}$ with shadow proportional to $\left(\xi^{1/2}-\xi^{-1/2}\right)\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right) n\,\xi^{n/2}\,q^{n^2/24}$.*

Remarks. 1. Note that the choice of square root of ξ in the formula for the shadow does not matter, since n in the non-vanishing terms of the sum is odd.

2. In fact Bringmann and Ono prove the theorem only if the order of ξ is odd. (If it is even, they prove a weaker result showing the modularity only for a group of in general infinite index in $\mathrm{SL}(2,\mathbb{Z})$.) Also, both the formulation and the proof of the theorem in [**7**] are considerably more complicated than the ones given here.

Proof. An elementary identity stated in [8] and quoted in [4] and [11] says, after a slight rewriting, that:

$$\mathcal{R}(w;q) \;=\; \frac{1-w}{\prod_{n\geq 1}(1-q^n)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n\, q^{(3n^2+n)/2}}{1-q^n w}\,.$$

Using the identity $\frac{1}{1-x} = \frac{1+x+x^2}{1-x^3}$ we deduce from this that

$$\frac{q^{-1/24}\,\mathcal{R}(e^{2\pi i\alpha};q)}{e^{-\pi i\alpha}-e^{\pi i\alpha}} \;=\; \frac{\eta(3\tau)^3/\eta(\tau)}{\theta(3\alpha;3\tau)} \;+\; q^{1/6} \sum_{\varepsilon\,\in\,\{\pm 1\}} \varepsilon\, e^{-2\pi\varepsilon\alpha}\mu(3\alpha,\varepsilon\tau;3\tau)$$

with $\theta(v;\tau)$ and $\mu(u,v;\tau)$ as in §2. The first term on the right is a weakly holomorphic modular form of weight $\frac{1}{2}$ and the other two terms are mock modular forms of weight $\frac{1}{2}$, with shadow proportional to $\sum_{n=1}^{\infty}\left(\frac{12}{n}\right) n\, q^{n^2/24}\,\sin(\pi n\alpha)$, by Theorem 2.1. □

As a corollary of Theorem 7.1 we see that for all $t>0$ and all $r\in\mathbb{Z}/t\mathbb{Z}$ the function

$$\sum_{n\geq 0}\Big(N(r,t;n) \;-\; \frac{1}{t}\,p(n)\Big)\, q^{n-1/24}$$

is a mock modular form of weight $\frac{1}{2}$, with shadow proportional to

$$\Bigg(\sum_{n\equiv 2r+1 \pmod{2t}} \;-\; \sum_{n\equiv 2r-1 \pmod{2t}}\Bigg)\left(\frac{12}{n}\right) n\, q^{n^2/24}\,.$$

Applying the general principle formulated at the end of §5, one deduces that the sum

$$\sum_{n\in\mathcal{A},\; n\geq 0}\Big(N(r,t;n) \;-\; \frac{1}{t}\,p(n)\Big)\, q^{n-1/24}$$

is a (weakly holomorphic) modular form for any arithmetic progression $\mathcal{A}\subset\mathbb{Z}$ not containing any number of the form $(1-h^2)/24$ with $h\equiv 2r\pm 1 \pmod{2t}$. In particular, this holds if $\mathcal{A}$ is the set of n with $\left(\frac{1-24n}{p}\right) = -1$ for some prime $p>3$, and using this and methods from classical modular form theory the authors deduce the following nice result (stated there only for t odd and Q prime to t) about divisibility of the Dyson counting function $N(r,t;n)$:

THEOREM 7.2. — *Let $t>0$ and Q a prime power prime to 6. Then there exist $A>0$ and $B\in\mathbb{Z}/A\mathbb{Z}$ such that $N(r,t;n)\equiv 0 \pmod{Q}$ for all $n\equiv B \pmod{A}$ and $r\in\mathbb{Z}/t\mathbb{Z}$.*

In a different direction, knowing the modularity properties of mock theta functions permits one to obtain asymptotic results, as well as congruences, for their coefficients. We give two examples. In §2 we described the weak Maass form $\widehat{h}_3(\tau)$ associated to Ramanujan's order 3 mock theta function $f(q)$. In [6], Bringmann and Ono construct a weak Maass–Poincaré series that they can identify (essentially by comparing the

modular transformation properties and the asymptotics at cusps) with $\widehat{h}_3(\tau)$, and from this they deduce a Rademacher-type closed formula for the coefficient $\alpha(n)$ of q^n in $f(q)$ of the form

$$\alpha(n) = \frac{1}{\sqrt{n-1/24}} \sum_{k=1}^{\infty} c_k(n) \sinh\Big(\frac{\pi}{12k}\sqrt{24n-1}\Big),$$

where $c_k(n)$ is an explicit finite exponential sum depending only on n modulo $2k$, e.g., $c_1(n) = (-1)^{n-1}$. This formula had been conjectured by Andrews and Dragonette in 1966 (after Ramanujan had stated, and Dragonette and Andrews had proven, weaker asymptotic statements corresponding to keeping only the first term of this series), but had resisted previous attempts at proof because the circle method, which is the natural tool to use, requires having a very precise description of the behavior of $f(q)$ as q approaches roots of unity, and this in turn requires knowing the modular transformation properties of $h_3(\tau) = q^{-1/24}f(q)$. As a second example, Bringmann [**5**] was able to use this type of explicit formulas for the coefficients of mock theta functions, combined with Theorem 7.1, to prove an inequality that had been conjectured earlier by Andrews and Lewis, saying that $N(0,3;n)$ is larger than $N(1,3;n)$ for all $n \equiv 1 \pmod 3$ and smaller for all other values of n (except $n = 3$, 9 or 21, where they are equal).

We close by mentioning that mock theta functions (both in the guises of Appell–Lerch sums and of indefinite theta series) also arise in connection with characters of infinite-dimensional Lie superalgebras and conformal field theory [**21**], and that they also occur in connection with certain quantum invariants of special 3-dimensional manifolds [**14**]. This suggests that mock modular forms may have interesting applications even outside the domain of pure combinatorics and number theory.

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Don ZAGIER

Collège de France
3 rue d'Ulm
F–75005 Paris
and
Max-Planck-Institut für Mathematik
Vivatsgasse 7
D-53111 Bonn
E-mail : `don.zagier@college-de-france.fr`