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THE q-ANALOGUE OF THE WILD FUNDAMENTAL GROUP (II)

by

Jean-Pierre Ramis & Jacques Sauloy

Abstract. — In a previous paper, we defined q-analogues of alien derivations and stated their basic properties. In this paper, we prove the density theorem and the freeness theorem announced there.

***Résumé* (Le q-analogue du groupe fondamental sauvage (II)).** — Dans un article précédent nous avons défini les q-analogues des dérivations étrangères et leurs propriétés de base. Dans cet article nous démontrons le théorème de densité et d'indépendance que nous y avions annoncé.

1. Introduction

1.1. The problem. — In this paper we shall continue the study of the local meromorphic classification of q-difference modules. In [**10**] we gave such a classification in Birkhoff style, using normal forms and index theorems; this classification is complete in the "integral slope case". (One could extend it to the general case using some results of [**3**].)

In [**6**] we introduced a new approach of the classification, using a "fundamental group" and its finite dimensional representations, in the style of the Riemann-Hilbert correspondence for linear differential equations. At some abstract level, such a classification is well known: the fundamental group is the tannakian Galois group of the tannakian category of local meromorphic q-modules. But we wanted more information: our essential aim was to get a *smaller* fundamental group which is Zariski dense in the tannakian Galois group and to describe it *explicitly*, in the spirit of what was done by the first author for the differential case [**5**].

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In [**6**] we built a family of elements of the Lie algebra of the tannakian group, the q-alien derivations, we achieved our program for the one-level case and we announced the main results in the general case. The aim of the present paper is to give some proofs omitted in [**6**] for the multi-level case. We will finally give a more precise algebraic formulation of our results in [**7**], which will end the series.

1.2. Contents of the paper. — General notations and conventions are explained in the next paragraph 1.3. In section 2, we recall basic properties of the category $\mathcal{E}_1^{(0)}$ of linear analytic q-difference equations with integral slopes, and the structure and action of its Galois group $G_1^{(0)}$. In section 3, we recall the unipotent structure of the Stokes subgroup $\mathfrak{St}$ of $G_1^{(0)}$, and the construction (taken from [**6**]) of elements of the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$, the so-called *q-alien derivations.* Our "q-analogue of the wild fundamental group" is the Lie subalgebra they generate. We then prove in 3.2 and 3.3 our main results: density and a freeness property of the q-alien derivations. In section 4, we summarize what remains to be solved, and will be the contents of [**7**].

The paper is written so as to be read widely independently of [**6**] - granted the reader is willing to take on faith some key points. The principle of the proofs is almost purely tannakian, but we have stated explicitly the underlying methods and prerequisites. Moreover, they are described in a concrete, computational form (with a systematic use of matrices). Since neither q-difference equations, nor even tannakian methods are so popular, this may help the reader to get familiarized with either domain. Note that, since we heavily rely on transcendental tools, the methods here are, to a large extent, independent of those of M. van der Put and his coauthors.

1.3. General notations. — The notations are the same as in [**6**]. Here are the most useful ones.

We let $q \in \mathbf{C}$ be a complex number with modulus $|q| > 1$. We write σ_q the q-dilatation operator, so that, for any map f on an adequate domain in $\mathbf{C}$, one has: $\sigma_q f(z) = f(qz)$. Thus, σ_q defines a ring automorphism in each of the following rings: $\mathbf{C}\{z\}$ (convergent power series), $\mathbf{C}[[z]]$ (formal power series), $\mathcal{O}(\mathbf{C}^*)$ (holomorphic functions over $\mathbf{C}^*$), $\mathcal{O}(\mathbf{C}^*, 0)$ (germs at 0 of elements of $\mathcal{O}(\mathbf{C}^*)$). Likewise, σ_q defines a field automorphism in each of their fields of fractions: $\mathbf{C}(\{z\})$ (convergent Laurent series), $\mathbf{C}((z))$ (formal Laurent series over), $\mathcal{M}(\mathbf{C}^*)$ (meromorphic functions over $\mathbf{C}^*$), $\mathcal{M}(\mathbf{C}^*, 0)$ (germs at 0 of elements of $\mathcal{M}(\mathbf{C}^*)$)

The σ_q-invariants elements of $\mathcal{M}(\mathbf{C}^*)$ can be considered as meromorphic functions on the quotient Riemann surface $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. Through the mapping $x \mapsto z = e^{2i\pi x}$, the latter is identified to the complex torus $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$, where $q = e^{2i\pi\tau}$. Accordingly, we shall identify the fields $\mathcal{M}(\mathbf{C}^*)^{\sigma_q}$ and $\mathcal{M}(\mathbf{E}_q)$. We shall write $a \mapsto \overline{a}$ the canonical projection map $\pi : \mathbf{C}^* \to \mathbf{E}_q$ and $[c; q] = \pi^{-1}(\overline{c}) = cq^{\mathbf{Z}}$ (a discrete logarithmic q-spiral).

Last, we shall have use for the function $\theta \in \mathcal{O}(\mathbf{C}^*)$, a Jacobi Theta function such that $\sigma_q\theta = z\theta$ and θ has simple zeroes along $[-1; q]$. One then puts $\theta_c(z) = \theta(z/c)$, so that $\theta_c \in \mathcal{O}(\mathbf{C}^*)$ satisfies $\sigma_q\theta_c = (z/c)\theta_c$ and θ_c has simple zeroes along $[-c; q]$.

2. Linear analytic q-difference equations

A linear analytic (resp. formal) q-difference equation (implicitly: at $0 \in \mathbf{C}$) is an equation:

$$\sigma_q X = AX, \tag{1}$$

where $A \in GL_n(\mathbf{C}(\{z\}))$ (resp. $A \in GL_n(\mathbf{C}((z)))$). There is an intrinsic description as a "q-difference module M_A", which runs as follows. We consider the operator Φ_A on $\mathbf{C}(\{z\})^n$ which maps a column vector X to $A^{-1}\sigma_q X$. This can be abstracted as a finite dimensional $\mathbf{C}(\{z\})$-vector space V endowed with a so-called "σ_q-linear automorphism" Φ. A q-difference module is such a pair $M = (V, \Phi)$. Here, we have $M_A = (\mathbf{C}(\{z\})^n, \Phi_A)$.

We shall here stick to the matrix model and, for all practical purposes, the reader may identify the equation (1), the matrix A and the q-difference module M_A with each other. For instance, we call *solution of A, or of (1), or of M_A* in some extension K of $\mathbf{C}(\{z\})$, on which σ_q operates, a column vector $X \in K^n$ such that $\sigma_q X = AX$. The *underlying space* of $A \in GL_n(\mathbf{C}(\{z\}))$ is $\mathbf{C}(\{z\})^n$.

2.1. Description of the tannakian structure. — We now proceed to describe the *tannakian category of analytic q-difference equations* $\mathcal{E}^{(0)}$. There is a similar description for the corresponding formal category $\hat{\mathcal{E}}^{(0)}$. The objects of $\mathcal{E}^{(0)}$ are linear analytic q-difference equations (1). A morphism from $A \in GL_n(\mathbf{C}(\{z\}))$ to $B \in GL_p(\mathbf{C}(\{z\}))$ is a matrix $F \in M_{p,n}(\mathbf{C}(\{z\}))$ such that:

$$(\sigma_q F)A = BF. \tag{2}$$

This just means that F sends any solution X of A to a solution FX of B. One can check that $\mathcal{E}^{(0)}$ is an abelian category. Indeed, it is the category of finite length left modules over the euclidean non-commutative ring $\mathcal{D}_{q,K}$ of q-difference operators over $K = \mathbf{C}(\{z\})$.

The abelian category $\mathcal{E}^{(0)}$ is endowed with a tensor structure. The tensor product $A_1 \otimes A_2$ of two objects (resp. the tensor product $F_1 \otimes F_2$ of two morphisms) is just the Kronecker product of the matrices; of course, we must define a consistent way of identifying $\mathbf{C}^n \otimes \mathbf{C}^p$ with $\mathbf{C}^{np}$, or $\mathbf{C}(\{z\})^n \otimes \mathbf{C}(\{z\})^p$ with $\mathbf{C}(\{z\})^{np}$ (see, for instance [**11**]).

The unit object $\underline{1}$ (which is neutral for the tensor product) is the matrix $(1) \in GL_1(\mathbf{C}(\{z\})) = \mathbf{C}(\{z\})^*$, with underlying space $\mathbf{C}(\{z\})$. The object $\underline{1}$ of course corresponds to the "trivial" equation [(1)] $\sigma x = x$. One easily checks that the space $\mathrm{Hom}(\underline{1}, A)$ of morphisms from $\underline{1}$ to A is exactly the space of solutions of A in $\mathbf{C}(\{z\})$, or, equivalently, the space of fixed points of Φ_A in $\mathbf{C}(\{z\})^n$. We shall write $\Gamma(A)$ or $\Gamma(M_A)$ that space, as it is similar to a space of global sections (and, indeed, can be realised as such, see [**14**]).

[(1)] In differential Galois theory, the matrix A of a system is in $M_n(\mathbf{C}(\{z\}))$ (rather than in GL_n), the trivial equation is $x' = 0$, etc. The theory of q-difference equations rather has a multiplicative character

The characterization (2) of morphisms can itself be seen as a q-difference equation $\sigma_q F = BFA^{-1}$. This means that there is an "internal Hom" object, which can be described in the following way. Consider the linear operator $F \mapsto BFA^{-1}$ on the vector space $M_{p,n}(\mathbf{C}(\{z\}))$. Through identification of $M_{p,n}(\mathbf{C}(\{z\}))$ with $\mathbf{C}(\{z\})^{np}$, this operator is described by a matrix in $GL_{np}(\mathbf{C}(\{z\}))$, which yields the desired equation. We shall write $\underline{\mathrm{Hom}}(A, B)$ the corresponding object. Thus, one gets:

$$\Gamma(\underline{\mathrm{Hom}}(A, B)) \simeq \mathrm{Hom}(\underline{1}, \underline{\mathrm{Hom}}(A, B)) \simeq \mathrm{Hom}(A, B). \tag{3}$$

Actually, this is a special case of the following canonical isomorphism::

$$\mathrm{Hom}(A, \underline{\mathrm{Hom}}(B, C)) \simeq \mathrm{Hom}(A \otimes B, C). \tag{4}$$

The reader will check that the object $\underline{\mathrm{Hom}}(A, \underline{1})$ has the following description. The underlying space is $M_{1,n}(\mathbf{C}(\{z\}))$, which we identify with $\mathbf{C}(\{z\})^n$. The corresponding matrix for the linear operator $F \mapsto FA^{-1}$ is the *contragredient matrix* $A^\vee = {}^tA^{-1}$. We call the object $A^\vee$ the *dual* of the object A. From this, we get yet another construction of the internal Hom:

$$\underline{\mathrm{Hom}}(A, B) \simeq A^\vee \otimes B. \tag{5}$$

We summarize these properties by saying that $\mathcal{E}^{(0)}$ is a tannakian category. This is halfway to showing that it is (isomorphic to) the category of representations of a proalgebraic group, our hoped for Galois group. To get further, one needs a *fiber functor on* $\mathcal{E}^{(0)}$. This was defined and, to some extent, studied in full generality in [**13**], [**12**] and [**6**]. However, for our strongest results, we need to restrict to the case of *integral slopes.*

2.2. Equations with integral slopes. — In [**13**], one defined the Newton polygon of a q-difference module (analytic or formal). This consists in slopes [(2)] $\mu_1 < \cdots < \mu_k$ (rational numbers) together with ranks (or multipicities) $r_1, \ldots, r_k$ (positive integers). We shall say that a module is *pure isoclinic* if it has only one slope and that it is *pure* [(3)] if it is a direct sum of pure isoclinic modules. We call *fuchsian* a pure isoclinic module with slope 0. The Galois theory of fuchsian modules was studied in [**11**]. Pure modules are irregular objects without *wild monodromy*, as follows from [**10**], [**12**] and [**6**].

The tannakian subcategory of $\mathcal{E}^{(0)}$ made up of pure modules is called $\mathcal{E}_p^{(0)}$. Modules with integral slopes also form tannakian subcategories, which we write $\mathcal{E}_1^{(0)}$ and $\mathcal{E}_{p,1}^{(0)}$. *From now on, we restrict to the case of integral slopes.* Our category of interest is therefore $\mathcal{E}_1^{(0)}$ and we shall now start its description.

(2) Note that here, as in [**6**], we have changed the definitions of slopes. Those used here are the opposites of those used in [**13**], [**8**]and [**12**].

(3) Here again, starting with [**6**], we changed our terminology: we now call pure isoclinic (resp. pure) what was previously called pure (resp. tamely irregular). The latter are called *split modules* in [**3**].

Any equation in $\mathcal{E}_1^{(0)}$ can be written in the following *standard form*:

$$A = \begin{pmatrix} z^{\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k}A_k \end{pmatrix}, \tag{6}$$

where the slopes $\mu_1 < \cdots < \mu_k$ are integers, $r_i \in \mathbf{N}^*$, $A_i \in GL_{r_i}(\mathbf{C})$ $(i = 1, \dots, k)$ (those μ_i and r_i make up the Newton polygon of A) and:

$$\forall i, j \text{ s.t. } 1 \leq i < j \leq k\ ,\ U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\})).$$

We actually can, and will, require the blocks $U_{i,j}$ to have all their coefficients in $\mathbf{C}[z, z^{-1}]$. Then any morphism $F : A \to B$ between two matrices in standard form is easily seen to be meromorphic at 0 (by definition) and holomorphic all over $\mathbf{C}^*$; this is because the equation $\sigma_q F = BFA^{-1}$ allows one to propagate the regularity near 0 to increasing neighborhoods.

We moreover say that A is in *polynomial* standard form if each block $U_{i,j}$ with $1 \leq i < j \leq k$ has coefficients in $\sum_{\mu_i \leq d < \mu_j} \mathbf{C}z^d$. It was proved in [**10**] that any object in $\mathcal{E}_1^{(0)}$ is analytically equivalent to one written in polynomial standard form (in essence, this is due to Birkhoff and Guenther). Last, we say that A is in *normalized* standard form is if all the eigenvalues of all the blocks A_i are in the fundamental annulus $\{z \in \mathbf{C}^* \mid 1 \leq |z| < |q|\}$. Any standard form can be normalized through shearing transformations. Note that polynomial standard form is stable under tensor product, while normalized standard form is not.

The standard form (6) above expresses the existence of a *filtration by the slopes* ([**13**]). The functoriality of the filtration moreover entails that a morphism $F : A \to B$ is also upper triangular (by blocks) in the following sense: if the slopes of $B \in GL_p(\mathbf{C}(\{z\}))$ are $\nu_1 < \cdots < \nu_l$, with ranks $s_1 < \cdots < s_l$, then the morphism $F \in M_{p,n}(\mathbf{C}(\{z\}))$ from A to B has only non null blocks $F_{i,j} \in M_{s_j,r_i}(\mathbf{C}(\{z\}))$, $1 \leq i \leq k$, $1 \leq j \leq l$ for $\nu_j \leq \mu_i$.

To the matrix A and module $M = M_A$ is associated the graded module $\mathrm{gr}M = M_0 = M_{A_0}$ with block diagonal matrix:

$$A_0 = \begin{pmatrix} z^{\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k}A_k \end{pmatrix}, \tag{7}$$

The graded module M_0 is the direct sum $P_1 \oplus \cdots \oplus P_k$, where each module P_i is pure of rank r_i and slope μ_i and corresponds to the matrix $z^{\mu_i}A_i$. The functor $M \rightsquigarrow \mathrm{gr}M$

also acts on morphisms. To F, it associates F_0 which has the same diagonal blocks as F, that is, $(F_0)_{i,j} = F_{i,j}$ if $\mu_i = \nu_j$. But all the $(F_0)_{i,j}$ such that $\mu_i \neq \nu_j$ are null.

By formalisation, *i.e.* base change $\mathbf{C}(\{z\}) \to \mathbf{C}((z))$, the slope filtration splits and the functor gr becomes isomorphic to the identity functor. In matrix terms, this translates as follows. There is a unique isomorphism $F : A_0 \to A$ with formal components $F_{i,j} \in M_{r_i,r_j}(\mathbf{C}((z)))$ (for $1 \leq i, j \leq k$) and the following shape:

$$F = \begin{pmatrix} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & I_{r_k} \end{pmatrix}. \tag{8}$$

To express that a matrix has such a shape and coefficients in some domain K, we shall write $F \in \mathfrak{G}_{A_0}(K)$. Thus, $\mathfrak{G}_{A_0}$ is a unipotent algebraic subgroup of the linear group and it can be realised above any field K: in the above case, one has $F \in \mathfrak{G}_{A_0}(\mathbf{C}((z)))$. For further use, we also give a notation for the corresponding Lie algebra $\mathfrak{g}_{A_0}$. An element $f \in \mathfrak{g}_{A_0}(K)$ has the shape:

$$\begin{pmatrix} 0_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & f_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0_{r_k} \end{pmatrix}, \tag{9}$$

where 0_r is the null $r \times r$ matrix and where each $f_{i,j} \in M_{r_i,r_j}(K)$.

We shall denote $\hat{F}_A$ the unique F mentioned above. Its blocks can be characterised as the unique formal solutions to the following recursive equations:

$$\forall 1 \leq i < j \leq k \ , \ \sigma_q F_{i,j} z^{\mu_j} A_j - z^{\mu_i} A_i F_{i,j} = \sum_{i<l<j} U_{i,l} F_{l,j} + U_{i,j}. \tag{10}$$

There are usually no analytic solutions (that is, with coefficients in $\mathbf{C}(\{z\})$) for equations (10). (The existence of analytic solutions is equivalent to M_A being pure.) There are, however, meromorphic solutions, to be considered as *resummations* of the formal solution $\hat{F}_A$ (section 3.1).

The graded counterpart F_0 of $F = \hat{F}_A$ satisfies simpler equations. From the above description, we know that $(F_0)_{i,j} = 0$ for any i, j such that $\mu_i \neq \mu_j$, that is, if $i \neq j$; if $i = j$:

$$\sigma_q (F_0)_{i,i} z^{\mu_i} A_i = z^{\mu_i} A_i (F_0)_{i,i}.$$

This implies that $\sigma_q (F_0)_{i,i} A_i = A_i (F_0)_{i,i}$, and it then follows from [**11**] that the coefficients of F_0 are Laurent polynomials (elements of $\mathbf{C}[z, z^{-1}]$); if moreover A is in normalized standard form, then these coefficients are in $\mathbf{C}$.

2.3. Description of the fiber functor. — In Tannaka theory, the Galois group is defined as the group of tensor automorphisms of a fiber functor. We now describe a fiber functor on $\mathcal{E}_1^{(0)}$. There is actually a whole family of these, indexed by $\mathbf{C}^*$, and one can therefore define a Galois groupoïd ([**6**]). Here, we shall first choose an arbitrary basepoint $a \in \mathbf{C}^*$. As a consequence, some constructions of 3.1 will be valid for most equations, but not all. This means that, to study a particular equation, one has to choose a basepoint compatible with it, which will be seen to be a generically true condition.

The fiber functor $\hat{\omega}_a^{(0)}$ goes from $\mathcal{E}_1^{(0)}$ to the category of finite dimensional $\mathbf{C}$-vector spaces. On the side of objects, to each matrix $A \in GL_n(\mathbf{C}(\{z\}))$ and module M_A, it associates the space $\hat{\omega}_a^{(0)}(A) = \mathbf{C}^n$. On the side of morphisms, to $F : A \to B \in GL_p(\mathbf{C}(\{z\}))$, it associates $F_0(a) : \mathbf{C}^n \to \mathbf{C}^p$. (The dimensions are right and it follows from the last remark in 2.2 that $F_0(a)$ is well defined).

Apart from functoriality, the properties of $\hat{\omega}_a^{(0)}$ which make it a fiber functor are the following: it is exact, faithful and $\otimes$-compatible. The latter means that, for any A, B, the natural map $t_{A,B} : \hat{\omega}_a^{(0)}(A) \otimes \hat{\omega}_a^{(0)}(B) \to \hat{\omega}_a^{(0)}(A \otimes B)$ is an isomorphism.

We now define the Galois group of $\mathcal{E}_1^{(0)}$ (at base point a) as $G_1^{(0)} = Aut^{\otimes}(\hat{\omega}_a^{(0)})$. It would be more rigorous to write explicitly the index a indicating the basepoint, but this would make the notation heavier without true necessity. An element of the group $Aut^{\otimes}(\hat{\omega}_a^{(0)})$ is, by definition, a natural transformation $g : A \rightsquigarrow g(A) \in GL\left(\hat{\omega}_a^{(0)}(A)\right) = GL_n(\mathbf{C})$, subject to the following conditions:

1. Functoriality: for any morphism $F : A \to B$, one has $g(B) \circ F_0(a) = F_0(a) \circ g(A)$. Thus, the following diagram is commutative:

$$\begin{array}{ccc} \hat{\omega}_a^{(0)}(A) & \xrightarrow{F_0(a)} & \hat{\omega}_a^{(0)}(B) \\ {\scriptstyle g(A)}\downarrow & & \downarrow{\scriptstyle g(B)} \\ \hat{\omega}_a^{(0)}(A) & \xrightarrow{F_0(a)} & \hat{\omega}_a^{(0)}(B) \end{array}$$

2. Tensor compatibility: for any objects A, B, up to the natural identifications, one has an equality $g(A \otimes B) = g(A) \otimes g(B)$. Thus, the following diagram is commutative:

$$\begin{array}{ccc} \hat{\omega}_a^{(0)}(A) \otimes \hat{\omega}_a^{(0)}(B) & \xrightarrow{t_{A,B}} & \hat{\omega}_a^{(0)}(A \otimes B) \\ {\scriptstyle g(A)\otimes g(B)}\downarrow & & \downarrow{\scriptstyle g(A\otimes B)} \\ \hat{\omega}_a^{(0)}(A) \otimes \hat{\omega}_a^{(0)}(B) & \xrightarrow{t_{A,B}} & \hat{\omega}_a^{(0)}(A \otimes B) \end{array}$$

In [**11**] was completely described the Galois group $G_f^{(0)}$ of the subcategory $\mathcal{E}_f^{(0)}$ of $\mathcal{E}^{(0)}$ made up of fuchsian equations. From [**13**], one could (trivially) deduce the Galois group $G_{p,1}^{(0)}$ of the category $\mathcal{E}_{p,1}^{(0)}$ of pure objects with integral slopes. Here, we will describe the Galois group $G^{(0)}$ of $\mathcal{E}_1^{(0)}$. The extension of these results to the case of

non integral slopes should not involve new ideas on the analytic side, but will have to take in account the results of van der Put and Reversat in [**3**].

2.4. Galois group and Galois action

Theorem 2.1. — *The structure of the Galois group $G_1^{(0)}$ is as follows:*

$$\begin{aligned}
G_1^{(0)} &= \mathfrak{Gt} \rtimes G_{p_1}^{(0)} \quad \textit{(total Galois group with integral slopes)},\\
G_{p,1}^{(0)} &= T_1^{(0)} \times G_f^{(0)} \quad \textit{(pure Galois group with integral slopes)},\\
T_1^{(0)} &= \mathbf{C}^* \quad \textit{(theta torus with integral slopes)},\\
G_f^{(0)} &= G_{f,s}^{(0)} \times G_{f,u}^{(0)} \quad \textit{(fuchsian Galois group)},\\
G_{f,u}^{(0)} &= \mathbf{C} \quad \textit{(unipotent component of the fuchsian Galois group)},\\
G_{f,s}^{(0)} &= \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \quad \textit{(semisimple component of the fuchsian Galois group)}.
\end{aligned}$$

The structure and action of the prounipotent Stokes group $\mathfrak{Gt}$ are the subject matter of [**6**] and of section 3 of the present paper. We shall presently explain the structure and action of the pure group $G_{p,1}^{(0)}$. This means that we should associate to any object A a representation of $G_{p,1}^{(0)}$ in the space $\hat{\omega}_a^{(0)}(A)$; thus, for any $g \in G_{p,1}^{(0)}$ and any matrix $A \in GL_n(\mathbf{C}(\{z\}))$, we should realize $g(A) \in GL_n(\mathbf{C})$.

We start from the standard form (6). For each of the block matrices A_i, we write:

$$A_i = A_{i,s} A_{i,u}$$

its multiplicative Dunford decomposition: $A_{i,s}$ is semisimple, $A_{i,u}$ is unipotent and they commute.

1. Let $g = \gamma \in G_{f,s}^{(0)} = \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$. The latter is here identified with the group of morphisms from the abstract group $\mathbf{C}^*$ to itself that send q to 1. We let γ act on each $A_{i,s}$ through its eigenvalues: if $A_{i,s} = P\mathrm{diag}(c_1, \dots, c_r)P^{-1}$, then $\gamma(A_{i,s}) = P\mathrm{diag}(\gamma(c_1), \dots, \gamma(c_r))P^{-1}$ (it does not depend on the choice of a particular diagonalisation). Then:

$$g(A) = \begin{pmatrix} \gamma(A_{1,s}) & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & \gamma(A_{k,s}) \end{pmatrix},$$

2. Let $g = \lambda \in G^{(0)}_{f,u} = \mathbf{C}$. Since the $A_{i,u}$ are unipotent matrices, the $A^{\lambda}_{i,u}$ are well defined and we put:

$$g(A) = \begin{pmatrix} A^{\lambda}_{1,u} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A^{\lambda}_{k,u} \end{pmatrix},$$

3. Let $g = t \in T^{(0)}_1 = \mathbf{C}^*$. This *theta torus* is the analogue here of the *exponential torus* of the classical differential Galois theory. Then:

$$g(A) = \begin{pmatrix} t^{\mu_1} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & t^{\mu_k} I_{r_k} \end{pmatrix},$$

Note that all these depend on A_0 only. This is because the category $\mathcal{E}^{(0)}_{p,1}$ of pure modules with integral slopes is equivalent to the category of representations of $G^{(0)}_{p,1}$, so that giving a representation of the latter group is the same as giving an object in the former category. We leave as an exercise for the reader the reconstruction of A_0 from the representation described above. For further use, we shall now prove two lemmas about the action of $G^{(0)}_{p,1}$ on $\hat{\omega}^{(0)}_a(A)$. These lemmas actually express the "duality" of $G^{(0)}_{p,1}$ and $\mathcal{E}^{(0)}_{p,1}$.

Lemma 2.2. — *Let A be in normalized standard form (6). Let $X \in \hat{\omega}^{(0)}_a(A)$ be* covariant *under the action of $G^{(0)}_{p,1}$, that is, for all $g \in G^{(0)}_{p,1}$, the vectors X and $g(A)X$ are colinear. Then there exists $i \in \{1, \dots, k\}$ and $\alpha \in Sp(A_i)$ such that: $A_0 X = \alpha z^{\mu_i} X$.*

Proof. — First note that the block decomposition of A_0 (or, equivalently, the action of the theta torus) entails a splitting of vector spaces:

$$\hat{\omega}^{(0)}_a(A) = \mathbf{C}^n = \mathbf{C}^{r_1} \oplus \dots \oplus \mathbf{C}^{r_k},$$

each A_i acting upon the corresponding $\mathbf{C}^{r_i}$. We can accordingly write $X = (X_1, \dots, X_k)$ (in row form, instead of column form, for economy of space). Covariance under the action of $T^{(0)}_1$ say that $(t^{\mu_1} X_1, \dots, t^{\mu_k} X_k)$ and $(X_1, \dots, X_k)$ are colinear for all $t \in \mathbf{C}^*$, which implies that at most one component X_i is non trivial. Then, covariance under the action of $G^{(0)}_{f,u}$ says that X_i is fixed by $A_{i,u}$ (since the latter is unipotent). Last, covariance under $G^{(0)}_{f,s}$ implies that X_i is an eigenvector of $A_{i,s}$. Indeed, this comes from the fact that, if $\alpha \neq \alpha'$ are eigenvalues of A_i, then, by the normalization condition, $\alpha q^{\mathbf{Z}} \cap \alpha' q^{\mathbf{Z}} = \varnothing$; it is then easy to see that there exists

$\gamma \in \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$ such that $\gamma(\alpha) \neq \gamma(\alpha')$, so that X_i cannot have nontrivial components in both eigenspaces of A_i. The conclusion follows. □

Lemma 2.3. — *Let A be in normalized standard form (6). Let $X \in \hat{\omega}_a^{(0)}(A)$ be* invariant *under the action of $G_{p,1}^{(0)}$, that is, for all $g \in G_{p,1}^{(0)}$, the vectors X and $g(A)X$ are equal. Then $A_0 X = X$.*

Proof. — The proof is similar, with two adaptations. First, equality of $(t^{\mu_1}X_1, \ldots, t^{\mu_k}X_k)$ and $(X_1, \ldots, X_k)$ entails that at most one component X_i is non trivial and the corresponding slope is $\mu_i = 0$; second, invariance under $G_{f,s}^{(0)}$ implies that at most one component of X_i (in the eigenspace decomposition) is non trivial, that the corresponding $\alpha \in \mathrm{Sp}(A_i)$ is in the kernel of all elements of $\mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$, so it is in $q^{\mathbf{Z}}$, so equal to 1 by the normalisation condition. □

Again because of the duality of $G_{p,1}^{(0)}$ and $\mathcal{E}_{p,1}^{(0)}$, the conclusions of these two lemmas have useful interpretations. The conclusion of lemma 2.2 says that the column matrix $X \in M_{n,1}(\mathbf{C})$ is a morphism from the rank one object $(\alpha z^{\mu_i}) \in GL_1(\mathbf{C}(\{z\}))$ into A_0. The conclusion of lemma 2.3 says that the column matrix $X \in M_{n,1}(\mathbf{C})$ is a morphism from the unit object $\underline{1} = (1) \in GL_1(\mathbf{C}(\{z\}))$ into A_0, *i.e.* a section $X \in \Gamma(A_0)$.

3. The wild fundamental group

3.1. The action of the Stokes group. — An element $s \in \mathfrak{St}$ is characterized by the following properties:

1. To each A in standard form (6), it associates a matrix $s(A) \in \mathfrak{G}_{A_0}(\mathbf{C})$; recall that $\mathfrak{G}_{A_0}$ was described as the algebraic group of matrices of shape as in equation (8).
2. If $A = A_0$, that is, if A is pure, then $s(A) = I_n$.
3. Functoriality and tensor compatibility are defined as in section 2.3.

Since $\mathfrak{St}$ is a prounipotent proalgebraic group, it is convenient to study it through its Lie algebra $\mathfrak{st}$. (The underlying formalism is expounded in the appendix of [**2**].) An element $D \in \mathfrak{st}$ is also a natural transformation of $\hat{\omega}_a^{(0)}$. It associates to each object A an endomorphism $D(A) \in \mathcal{L}(\hat{\omega}_a^{(0)}(A)) = M_n(\mathbf{C})$, subject to the following conditions:

1. For each A in standard form (6), the matrix $D(A) \in M_n(\mathbf{C})$ is in $\mathfrak{g}_{A_0}(\mathbf{C})$; recall that $\mathfrak{g}_{A_0}$ was described as the Lie algebra of matrices of shape as in equation (9).
2. If $A = A_0$, that is, if A is pure, then $D(A) = 0_n$.
3. Functoriality is defined as in section 2.3.
4. Tensor compatibility is that of "Lie-like elements" (as in [**15**], §6): for any $A \in GL_n(\mathbf{C}(\{z\}))$ and $B \in GL_p(\mathbf{C}(\{z\}))$, one should have, up to natural identifications: $D(A \otimes B) = D(A) \otimes I_p + I_n \otimes D(B)$. Thus, D behaves like a derivation.

In [**6**], we have produced many elements of $\mathfrak{St}$ and of $\mathfrak{st}$. However, for a given basepoint $a \in \mathbf{C}^*$, these do not operate on the whole of $\mathcal{E}_1^{(0)}$ but on a tannakian subcategory of it. Therefore, the way of using them is the following: given an equation A of interest, proposition 4.2 of *loc. cit.* yields an explicit criterion to select adequate basepoints (these are generically adequate). Then all the constructions that follow make sense in the tannakian subcategory of $\mathcal{E}_1^{(0)}$ generated by A. This means that each time we shall evaluate a meromorphic function at a, this will be possible. Henceforth, we shall not anymore discuss this matter. *We assume that the basepoint has been chosen so that all the objects we deal with are compatible with it.*

In [**12**] and [**6**], we defined an explicit finite subset Σ_{A_0} of $\mathbf{E}_q$ and proved:

Theorem 3.1. — *Let $\overline{c} \in \mathbf{E}_q \setminus \Sigma_{A_0}$. Then, there is a unique $F : A_0 \to A$ such that $F \in \mathfrak{G}_{A_0}(\mathcal{M}(\mathbf{C}^*))$, with poles only on $[-c; q] = -cq^{\mathbf{Z}}$ and such that, for $1 \leq i < j \leq k$, the poles of $F_{i,j}$ have multiplicity $\leq \mu_j - \mu_i$.*

We write this meromorphic isomorphism $S_{\overline{c}}\hat{F}_A$ and see it as some kind of *summation of $\hat{F}_A$ in the direction $\overline{c} \in \mathbf{E}_q$*. Therefore, changing direction of summation, we may define, for every $\overline{c}, \overline{d} \in \mathbf{E}_q \setminus \Sigma_{A_0}$:

$$S_{\overline{c},\overline{d}}\hat{F}_A = (S_{\overline{c}}\hat{F}_A)^{-1} S_{\overline{d}}\hat{F}_A,$$

some kind of "ambiguity of summation", that is, a Stokes operator. It is plainly a meromorphic automorphism of A_0. We also proved in *loc. cit.*:

Proposition 3.2. — *If moreover $\overline{a} \neq \overline{c}, \overline{d}$, then $A \rightsquigarrow S_{\overline{c},\overline{d}}\hat{F}_A(a)$ is an element of $\mathfrak{St}$. In particular, $S_{\overline{c},\overline{d}}\hat{F}_A(a) \in \mathfrak{St}(A)$. (Recall that we implicitly restrict ourselves to a subcategory of $\mathcal{E}_1^{(0)}$ where everything is defined.)*

For the following corollary, we fix an arbitrary direction of summation $\overline{c_0} \in \mathbf{E}_q$, again to be considered as a choice of basepoint (and inessential).

Corollary 3.3. — *Putting $LS_{\overline{c},a}(A) = \log(S_{\overline{c_0},\overline{c}}\hat{F}_A(a)) \in \mathfrak{st}(A)$ yields a family of elements of elements of $\mathfrak{st}(A)$. Moreover, $A \rightsquigarrow LS_{\overline{c},a}(A)$ is an element of $\mathfrak{st}$. (We omit $\overline{c_0}$ in the notation.)*

The above family is a meromorphic map from $\mathbf{E}_q$ to a vector space, hence one can take residues. Define the q-alien derivations by the formula:

$$\dot{\Delta}_{\overline{c}}(A) = Res_{\overline{d}=\overline{c}} LS_{\overline{d},a}(A).$$

(We do not mention the arbitrary basepoints $\overline{c_0}, a$ in the notation.) Of course, for $\overline{c} \notin \Sigma_{A_0}$, we have $\dot{\Delta}_{\overline{c}}(A) = 0$. Another result we need from [**6**] is:

Theorem 3.4. — *One has $\dot{\Delta}_{\overline{c}}(A) \in \mathfrak{st}(A)$. More precisely, $A \rightsquigarrow \dot{\Delta}_{\overline{c}}(A)$ is an element of $\mathfrak{st}$.*

Since $\mathfrak{St}$ is a normal subgroup of $G_1^{(0)}$, it admits a conjugation action by $G_{p,1}^{(0)}$. This can be transferred to the Lie algebra $\mathfrak{st}$. Because of the action by the theta torus $T_1^{(0)} = \mathbf{C}^*$, we thus have a spectral decomposition:

$$\mathfrak{st} = \bigoplus_{\delta \geq 1} \mathfrak{st}^\delta,$$

and each alien derivation admits a canonical decomposition:

$$\dot{\Delta}_{\overline{c}} = \bigoplus_{\delta \geq 1} \dot{\Delta}_{\overline{c}}^{(\delta)},$$

where $\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in \mathfrak{st}^\delta(A)$ has only non null blocks for $\mu_j - \mu_i = \delta$. Each $t \in T_1^{(0)}$ acts on $\mathfrak{st}^\delta$ by multiplication by t^δ, and carries $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ to $t^\delta \dot{\Delta}_{\overline{c}}^{(\delta)}(A)$.

Remarks

1. The theta torus actually operates on each $\hat{\omega}_a^{(0)}(A) = \hat{\omega}_a^{(0)}(A_0)$ and, being semi-simple, splits it into the direct sum of its eigenspaces: one for each slope μ, with rank $r(\mu)$. The corresponding increasing filtration comes from the filtration by the slopes:

$$\hat{\omega}_a^{(0)}(A)_{\geq \mu} = \hat{\omega}_a^{(0)}(A_{\geq \mu}).$$

   The elements of the group $\mathfrak{G}_{A_0}(\mathbf{C})$ are the automorphisms of $\hat{\omega}_a^{(0)}(A)$ which respect that filtration and are trivial (*i.e.* the identity) on the associated graded space. The elements of the algebra $\mathfrak{G}_{A_0}(\mathbf{C})$ are the endomorphisms of $\hat{\omega}_a^{(0)}(A)$ which respect that filtration and are trivial (*i.e.* null) on the associated graded space.
2. From this, we deduce a spectral decomposition:

$$\mathfrak{g}_{A_0} = \bigoplus_{\delta \geq 1} {\mathfrak{g}_{A_0}}^\delta,$$

   from which the decomposition of $\mathfrak{st}$ follows.
3. Putting ${\mathfrak{g}_{A_0}}^{\geq \delta} = \sum_{\delta' \geq \delta} {\mathfrak{g}_{A_0}}^{\delta'}$ defines a filtration of the Lie algebra $\mathfrak{g}_{A_0}$ by ideals. Putting ${\mathfrak{G}_{A_0}}^{\geq \delta} = I_n + {\mathfrak{g}_{A_0}}^{\geq \delta} = \exp {\mathfrak{g}_{A_0}}^{\geq \delta}$ then defines a filtration of $\mathfrak{G}_{A_0}$ by normal subgroups.
4. Similarly, we can decompose each (i,j) block of $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ (where $\mu_j - \mu_i = \delta$) into subblocks indexed by pairs $(\alpha, \beta) \in \mathrm{Sp}(A_i) \times \mathrm{Sp}(A_j)$. Each $\gamma \in G_{f,s}^{(0)}$ then multiplies the corresponding subblock of $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ by $\dfrac{\gamma(\alpha)}{\gamma(\beta)}$. This may be deduced as above from the action of $G_{f,s}^{(0)}$ on $\hat{\omega}_a^{(0)}(A_0)$ and a corresponding splitting of each of the ${\mathfrak{g}_{A_0}}^\delta$.

3.2. The density theorem

3.2.1. *Plain density theorem.* — The wild monodromy group actually is the Lie subalgebra of $\mathfrak{st}$ generated by the q-alien derivations $\dot{\Delta}_{\overline{c}}$. The justification of the name is that its definition has a transcendental character and the following result.

***Theorem 3.5* (Density theorem).** — *(i) The subgroup of $\mathfrak{St}$ associated with the wild monodromy group (as defined above), together with the pure group $G^{(0)}_{p,1}$, generate a Zariski-dense subgroup of the whole Galois group $G^{(0)}_1$. (ii) The $\dot{\Delta}_{\overline{c}}$ together with all their conjugates under the action of $G^{(0)}_{p,1}$ generate a Zariski-dense Lie subalgebra of $\mathfrak{st}$.*

Proof. — Actually, (i) is but a rephrasing of (ii) and we shall prove the latter. We shall use Chevalley's criterion in the following form:
For a subset $H \subset G^{(0)}_1$ to generate a Zariski-dense subgroup of $G^{(0)}_1$, it is sufficient that, for each object A and each line $D \subset \hat{\omega}^{(0)}_a(A)$ which is invariant under the action of all elements of H, then D is actually invariant under the action of $G^{(0)}_1$. Our way of using it is similar to that in [**11**] (2.2.3.3 and 3.1.2.3).

We take $H = G^{(0)}_{p,1} \cup \exp\left(\{\dot{\Delta}_{\overline{c}} \mid c \in \mathbf{C}^*\}\right)$. If we choose a generator X of the line D, the assumption is that X is covariant under $G^{(0)}_{p,1}(A) = G^{(0)}_{p,1}(A_0)$ on the one hand, under all the $\dot{\Delta}_{\overline{c}}(A)$ on the other hand. Since the latter are nilpotent, this means that all $\dot{\Delta}_{\overline{c}}(A)X = 0$. Then, must prove that for all $D \in \mathfrak{st}$, one has $D(A)X = 0$.

Using lemma 2.2, along with its proof and notations, we may write (in row form) $X = (0, \dots, X_i, \dots, 0)$, where the components have sizes $r_1, \dots, r_k$ and where $A_iX_i = cX_i$ for some $c \in \mathrm{Sp}(A_i)$, so that $A_0X = cz^{\mu_i}X$.

Now, we note that components of slopes $> \mu_i$ are neither involved in the assumptions nor in the conclusion, so that one may as well assume from start that $i = k$. Indeed, write $n' = r_1 + \cdots + r_i$ the size of the components corresponding to slopes $\leq \mu_i$ (equivalently, the rank of the submodule $M'_{\leq \mu_i} \subset M = M_A$ of slopes $\leq \mu_i$ in the slope filtration), A' the corresponding submatrix of A (so that $M' = M_{A'}$) and $X' = (0, \dots, X_i)$ the corresponding subvector of X. The matrix $\Phi = \begin{pmatrix} I_{n'} \\ 0 \end{pmatrix} \in M_{n,n'}(\mathbf{C})$ is a morphism $\Phi : A' \to A$ (it is the inclusion $M' \subset M$) and $\Phi X' = X$. For all $g \in G^{(0)}_1$, one has (by functoriality) $g(A)\Phi = \Phi g(A')$ (here, one has $\Phi(a) = \Phi$). The reader will check that A' and X' satisfy the same assumption as A and X, and that it is enough to prove the conclusions for them.

So we assume from now on that $X = (0, \dots, X_k)$, that $A_kX_k = cX_k$ for some $c \in \mathrm{Sp}(A_k)$, so that $A_0X = cz^{\mu_k}X$. Then $X : (cz^{\mu_k}) \to A_0$ is an analytic morphism, and therefore $G = \hat{F}_AX : (cz^{\mu_k}) \to A$ is a formal morphism. We shall prove below (lemma 3.6) that it is actually an analytic morphism. Therefore, taking $D \in \mathfrak{st}$ and

using functoriality, we get the commutative diagram:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{G_0(a)} & \hat{\omega}_a^{(0)}(A) \\ {\scriptstyle D(cz^{\mu_k})}\downarrow & & \downarrow{\scriptstyle D(A)} \\ \mathbf{C} & \xrightarrow{G_0(a)} & \hat{\omega}_a^{(0)}(A) \end{array}$$

The matrix G_0 is the graded part of the column matrix $\hat{F}_A X$, that is, in row notation, X itself. Since $D \in \mathfrak{st}$ and since the source object (cz^{μ_k}) is pure, one has $D(cz^{\mu_k}) = 0$. Hence we get $D(A)X = 0$ as wanted. □

Lemma 3.6. — *The matrix $\hat{F}_A X$ is analytic and the summations $S_{\overline{c}}\hat{F}_A$ do not depend on the direction $\overline{c} \in \mathbf{E}_q$, and they are all equal to $\hat{F}_A X$ (that is, its "classical summation", as a* convergent *power series).*

Proof. — First fix a direction $\overline{c} \in \mathbf{E}_q$ and write $F = S_{\overline{c}}\hat{F}_A$. Likewise, write $G = \hat{F}_A$ for short. The components $F_{i,j}$ and $G_{i,j}$ satisfy equations (10). We are interested in the $F_{i,k}X_k$ and the $G_{i,k}X_k$ for $1 \leq i < k$ (for $i = k$, both equal X_k).

We shall put: $Y_i = F_{i,k}X_k$, $Z_i = G_{i,k}X_k$ and $V_i = U_{i,k}X_k$. Then, for $1 \leq i < k$, multiplying (10) by X_k on the right and taking in account the equalities $\sigma_q X_k = X_k$ and $A_k X_k = cX_k$, one gets:

$$cz^{\mu_k}(\sigma_q Y_i) - z^{\mu_i} A_i Y_i = \sum_{i<j<k} U_{i,j}Y_j + V_i \quad \text{and} \quad cz^{\mu_k}(\sigma_q Z_i) - z^{\mu_i} A_i Z_i = \sum_{i<j<k} U_{i,j}Z_j + V_i.$$

On the other hand, we shall have to use the assumptions: $\dot{\Delta}_{\overline{c}}(A)X = 0$. Since $X = (0, \ldots, X_k)$, this means that, for each $i < k$, one has $(\dot{\Delta}_{\overline{c}}(A))_{i,k}X_k = 0$. Writing for short $F_d = S_{\overline{d}}\hat{F}_A(a)$, $F_0 = S_{\overline{c_0}}\hat{F}_A(a)$ and $L = LS_{\overline{d},a}(A) = \log(S_{\overline{c_0},\overline{d}}\hat{F}_A(a))$ (see the notations of section 3.1), we shall see in lemma 3.7 below that, for $i < k$:

$$L_{i,k} = (F_d)_{i,k} - (F_0)_{i,k} + \sum_{i<j<k} M_{i,j,k}\big((F_d)_{j,k} - (F_0)_{j,k}\big),$$

where the $M_{i,j,k}$ are some arbitrary matrices (their values are inessential here).

We use a downward induction on i. For $i = k-1$:

$$cz^{\mu_k}(\sigma_q Y_{k-1}) - z^{\mu_{k-1}} A_{k-1} Y_{k-1} = V_{k-1} \quad \text{and} \quad cz^{\mu_k}(\sigma_q Z_{k-1}) - z^{\mu_{k-1}} A_{k-1} Z_{k-1} = V_{k-1}.$$

Thus, Z_{k-1} is the formal solution and Y_{k-1} the solution summed in direction $\overline{c}$ of the equation $cz^{\mu_k}(\sigma_q Y) - z^{\mu_{k-1}} A_{k-1} Y = V_{k-1}$. On the other hand, from the formula above, one has: $L_{k-1,k} = (F_d)_{k-1,k} - (F_0)_{k-1,k}$. Taking residue at $\overline{d} = \overline{c}$ and multiplying at right by the constant vector X_k, one gets:

$$(\dot{\Delta}_{\overline{c}}(A))_{k-1,k}X_k = Res_{\overline{d}=\overline{c}} LS_{k-1,k}X_k = Res_{\overline{d}=\overline{c}}(F_d)_{k-1,k}X_k = Res_{\overline{d}=\overline{c}} Y_{k-1}.$$

This means that the residues of resummed solutions of the equation just mentioned are all 0. According to the results of section 4.2 of [**6**], this implies that Y_{k-1} is analytic near 0 (it has no poles other than 0), that it does not depend on the direction of

summation $\overline{d}$, and that it is equal to Z_{k-1}. This completes the first step of the induction.

Now take $i < k-1$ and assume that the property has been proved for all $j > i$. Consider the equation of which Y_i is solution. Its right hand member $\sum_{i<j<k} U_{i,j}Y_j + V_i$ is analytic, by the induction hypothesis (analyticity part). The residue at $\overline{d} = \overline{c}$ of Y_i is equal to, $Res_{\overline{d}=\overline{c}}(F_d)_{i,k}X_k$, thus to:

$$Res_{\overline{d}=\overline{c}} LS_{i,k}X_k = (\dot{\Delta}_{\overline{c}}(A))_{i,k}X_k = 0.$$

This is because all other terms in the formula taken from lemma 3.7 have at right a factor $\big((F_d)_{j,k} - (F_0)_{j,k}\big)X_k = Y_j - Y_j$, since Y_j does not depend on the direction of summation. Thus we have again a solution Y_i with all residues null, so that it is analytic and independent of the direction of summation by *loc. cit.*. □

Lemma 3.7. — *With the notations of section 3.1, one has, for $i < k$:*

$$\begin{aligned}&\log(S_{\overline{c_0},\overline{d}}\hat{F}_A(a))_{i,k}\\ &\quad = (S_{\overline{d}}\hat{F}_A(a))_{i,k} - (S_{\overline{c_0}}\hat{F}_A(a))_{i,k} + \sum_{i<j<k} M_{i,j,k}\big((S_{\overline{d}}\hat{F}_A(a))_{j,k} - (S_{\overline{c_0}}\hat{F}_A(a))_{j,k}\big),\end{aligned}$$

where the $M_{i,j,k}$ are some arbitrary matrices.

Proof. — We write $A = S_{\overline{d}}\hat{F}_A(a)$, $B = S_{\overline{c_0}}\hat{F}_A(a)$ and $C = A - B$, which is strictly upper triangular by blocks. Then:

$$\log(B^{-1}A) = \log(I_n + B^{-1}C) = \sum_{p\geq 1} \frac{(-1)^{p-1}}{p}(B^{-1}C)^p,$$

from which the equality of blocks:

$$\big(\log(B^{-1}A)\big)_{i,k} = C_{i,k} + \sum_{i<j<k} M_{i,j,k}C_{j,k}$$

follows easily. □

3.2.2. *Functorial density theorem.* — In section 3.3, we shall describe how the Zariski generators $\dot{\Delta}_{\overline{c}}$ of $\mathfrak{st}$ (theorem 3.5) are related. For that, we shall first give a more functorial version of the density theorem.

Since $\mathcal{E}^{(0)}_{p,1}$ and $\mathcal{E}^{(0)}_1$ are respectively isomorphic to the category of (finite dimensional complex) representations of $G^{(0)}_{p,1}$ and $G^{(0)}_1 = \mathfrak{St} \rtimes G^{(0)}_{p,1}$, and since finite dimensional representations of the prounipotent proalgebraic group $\mathfrak{St}$ are equivalent to finite dimensional representations of the pronilpotent proalgebraic Lie algebra $\mathfrak{st}$, the tannakian category $\mathcal{E}^{(0)}_1$ admits an alternative "mixed" description, which runs as follows:

1. Objects are pairs $A = (A_0, (D(A))_{D\in\mathfrak{st}})$, where A_0 is some object of $\mathcal{E}^{(0)}_{p,1}$, *e.g.* a matrix in pure standard form (7), and where each $D(A) \in \mathfrak{g}_{A_0}(\mathbf{C})$.

2. Morphisms from $A = (A_0, (D(A))_{D\in\mathfrak{st}})$ to $B = (B_0, (D(B))_{D\in\mathfrak{st}})$ are morphisms $F_0 : A_0 \to B_0$ in $\mathcal{E}^{(0)}_{p,1}$ such that, for each $D \in \mathfrak{st}$, one has $D(B)F_0(a) = F_0(a)D(A)$. (Recall that an arbitrary basepoint $a \in \mathbf{C}^*$ has been chosen once for all.)
3. The tensor product of $A = (A_0, (D(A))_{D\in\mathfrak{st}})$ and $B = (B_0, (D(B))_{D\in\mathfrak{st}})$ is the object $C = (C_0, (D(C))_{D\in\mathfrak{st}})$, with the previous rule $C_0 = A_0 \otimes B_0$ from $\mathcal{E}^{(0)}_{p,1}$, and with the "Lie-like element" rule $D(A\otimes B) = D(A)\otimes I_p + I_n \otimes D(B)$. The unit is $\underline{1} = ((1), (0)_{D\in\mathfrak{st}})$. The dual of A is $A^\vee = (A_0^\vee, (-{}^tD(A))_{D\in\mathfrak{st}})$. The space of sections of A is $\Gamma(A) = \mathrm{Hom}(\underline{1}, A) = \{X_0 \in \Gamma(A_0) \mid \forall D \in \mathfrak{st}\ ,\ D(A)X_0 = 0\}$. (Recall that $\Gamma(A_0) = \{X_0 \in \mathbf{C}(\{z\})^n \mid \sigma_q X_0 = A_0 X_0\}$.)
4. There is a a fiber functor $\hat{\omega}^{(0)}_a(A) \underset{def}{=} \hat{\omega}^{(0)}_a(A_0)$.

To be complete, such a description should take in account the adjoint action of $G^{(0)}_{p,1}$ on $\mathfrak{st}$, which is, for all A, the restriction of the action of $G^{(0)}_{p,1}$ on $\mathfrak{g}_{A_0}(\mathbf{C})$. For instance, from the action of the theta torus, one draws the graduation $\mathfrak{st} = \bigoplus_{\delta\geq 1} \mathfrak{st}^\delta$, whence decompositions $D(A) = \sum_{\delta\geq 1} D^\delta(A)$, where each $D^\delta(A) \in \mathfrak{g}_{A_0}{}^\delta(\mathbf{C})$. We shall take in account the adjoint action of $G^{(0)}_f$ later.

We would like to consider the $\dot{\Delta}^{(\delta)}_{\bar{c}}(A)$ as encoding a Lie algebra representation from the free Lie algebra L generated by the family of symbols $(\dot{\Delta}^{(\delta)}_{\bar{c}})_{\delta\geq 1, \bar{c}\in\mathbf{E}_q}$, and so describe $\mathcal{E}^{(0)}_1$ as the category of representations of $L \rtimes G^{(0)}_{p,1}$ in a way similar to that above. This would require some other tools (see the conclusion of the paper). As a substitute, we define a new tannakian category $\mathcal{E}'$ as follows:

1. Objects are pairs:
$$A = \left(A_0, (\dot{\Delta}^{(\delta)}_{\bar{c}}(A))_{\delta\geq 1, \bar{c}\in\mathbf{E}_q}\right),$$
where A_0 is in pure standard form (7), and where each $\dot{\Delta}^{(\delta)}_{\bar{c}}(A) \in \mathfrak{g}_{A_0}{}^\delta(\mathbf{C})$.
2. Morphisms from A to $B = \left(B_0, (\dot{\Delta}^{(\delta)}_{\bar{c}}(B))_{\delta\geq 1, \bar{c}\in\mathbf{E}_q}\right)$ are morphisms $F_0 : A_0 \to B_0$ in $\mathcal{E}^{(0)}_{p,1}$ such that, for each $\delta \geq 1, \bar{c} \in \mathbf{E}_q$, one has $\dot{\Delta}^{(\delta)}_{\bar{c}}(B)F_0(a) = F_0(a)\dot{\Delta}^{(\delta)}_{\bar{c}}(A)$.
3. The tensor product of A and B is the object $C = \left(C_0, (\dot{\Delta}^{(\delta)}_{\bar{c}}(B))_{\delta\geq 1, \bar{c}\in\mathbf{E}_q}\right)$, with $C_0 = A_0 \otimes B_0$ and $\dot{\Delta}^{(\delta)}_{\bar{c}}(A \otimes B) = \dot{\Delta}^{(\delta)}_{\bar{c}}(A) \otimes I_p + I_n \otimes \dot{\Delta}^{(\delta)}_{\bar{c}}(B)$. The unit and dual are described as before. The space of sections of A is $\Gamma(A) = \mathrm{Hom}(\underline{1}, A) = \{X_0 \in \Gamma(A_0) \mid \forall \delta \geq 1, \bar{c} \in \mathbf{E}_q\ ,\ \dot{\Delta}^{(\delta)}_{\bar{c}}(A)X_0 = 0\}$.
4. There is a a fiber functor $\hat{\omega}^{(0)}_a(A) \underset{def}{=} \hat{\omega}^{(0)}_a(A_0)$.

For the time being, we do not take in account the action of $G^{(0)}_f$.

We now consider the functor $A \rightsquigarrow \mathcal{F}(A) = \left(A_0, (\dot{\Delta}^{(\delta)}_{\bar{c}}(A))_{\delta\geq 1, \bar{c}\in\mathbf{E}_q}\right)$ from $\mathcal{E}^{(0)}_1$ to $\mathcal{E}'$. It is plainly an exact faithful $\otimes$-functor.

***Theorem 3.8* (Functorial density theorem).** — *The functor $\mathcal{F}$ is fully faithful.*

Proof. — To prove that $\mathrm{Hom}(A,B) \to \mathrm{Hom}(\mathcal{F}(A),\mathcal{F}(B))$ is onto, we draw on the identifications $\mathrm{Hom}(A,B) = \Gamma(A^\vee \otimes B)$ and $\mathrm{Hom}(\mathcal{F}(A),\mathcal{F}(B)) = \Gamma(\mathcal{F}(A)^\vee \otimes \mathcal{F}(B))$. Since $\mathcal{F}$ is a $\otimes$-functor, the latter is identified with $\Gamma(\mathcal{F}(A^\vee \otimes B))$, so that we are left check that, for any A, the map $\Gamma(A) \to \Gamma(\mathcal{F}(A))$ is onto.

That map sends a vector $X \in \mathbf{C}(\{z\})^n$ such that $\sigma_q X = AX$ to its graded part $X_0 \in \mathbf{C}(\{z\})^n$. The vector X_0 has the same null slope component as X and is zero elsewhere. It satisfies $\sigma_q X_0 = A_0 X_0$ and $\forall \delta \geq 1, \bar{c} \in \mathbf{E}_q$, $\dot{\Delta}_{\bar{c}}^{(\delta)}(A)X_0 = 0$. If we start from such a vector X_0, lemma 2.3 tells us that it comes indeed from some $X \in \Gamma(A)$. □

3.3. A freeness theorem. — We now shall describe the (essential) image of the functor $\mathcal{F}$, or, what amounts to the same, which families $(\dot{\Delta}_{\bar{c}}^{(\delta)}(A))_{\delta\geq 1,\bar{c}\in\mathbf{E}_q}$ can be realized for a given A_0 in $\mathcal{E}_{p,1}^{(0)}$. To understand what is going on, we start with the first level, which is easier.

3.3.1. *The first level.* — In theorem 3.1, we obtained the $F_{i,j}$ blocks of $S_{\bar{c}}\hat{F}_A$ as solutions of the following equations:

$$(\sigma_q F_{i,j}) z^{\mu_j} A_j - z^{\mu_i} A_i F_{i,j} = \sum_{i<l<j} U_{i,l} F_{l,j} + U_{i,j}.$$

We consider the first non trivial level in the computation of $S_{\bar{c}}\hat{F}_A$, that is: $\delta_0 = \min_{i<j}(\mu_j - \mu_i)$. For a block $F_{i,j}$ of level $\mu_j - \mu_i = \delta_0$, there is no $F_{l,j} \neq 0$ for $i < l < j$, so that the second hand member in the equation above is $U_{i,j}$, which is analytic near 0. In that case, there is a solution $F_{i,j}$ with poles on $[-c;q]$ and multiplicity $\leq \delta_0$ for any $\bar{c} \in \mathbf{E}_q$ which satisfies the non-resonancy condition:

$$\forall \alpha \in \mathrm{Sp}(A_i)\ ,\ \forall \beta \in \mathrm{Sp}(A_j)\ ,\ \alpha c^{\mu_i} \not\equiv \beta c^{\mu_j} \pmod{q^{\mathbf{Z}}}.$$

We recall briefly, from [**6**], how this was computed. One puts $F_{i,j} = \dfrac{G_{i,j}}{\theta_c^{\delta_0}}$, (the function θ_c has been defined in section 1.3). We thus look for $G_{i,j}$ holomorphic on $\mathbf{C}^*$ and satisfying:

$$c^{\delta_0}(\sigma_q G_{i,j}) A_j - A_i G_{i,j} = z^{-\mu_i} U_{i,j} \theta_c^{\delta_0} = \sum_{n\in\mathbf{Z}} v_n z^n.$$

Writing the Laurent series $G_{i,j} = \sum g_n z^n$, we are left to solve, for each $n \in \mathbf{Z}$:

$$c^{\delta_0} q^n g_n A_j - A_i g_n = v_n \in \mathrm{Mat}_{r_i,r_j}(\mathbf{C}).$$

If $\mathrm{Sp}(c^{\delta_0} q^n A_j) \cap \mathrm{Sp}(A_i) = \varnothing$, which is is just the non-resonancy condition above, then, for each n, this admits a unique solution.

Using the notations given at the end of 3.1, we see that, for any $\bar{d} \notin \Sigma_{A_0}$, one has $S_{\bar{d}}\hat{F}_A(a) \in \mathfrak{G}_{A_0}{}^{\geq\delta_0}(\mathbf{C})$. Provisionally call $f_{\bar{d}}^{\delta}$ its component at level δ. A small

computation shows that $LS_{\overline{d},a}(A)$ is in ${\mathfrak{g}_{A_0}}^{\geq\delta_0}(\mathbf{C})$ and that its component at level δ_0 is $f_{\overline{d}}^{\delta_0} - f_{\overline{c_0}}^{\delta}$. Thus, $\dot{\Delta}_{\overline{c}}^{(\delta_0)}(A) = Res_{\overline{d}=\overline{c}} f_{\overline{d}}^{\delta_0}$.

From the previous computation, we now conclude that, for $\mu_j - \mu_i = \delta_0$, the (i,j) block of $\dot{\Delta}_{\overline{c}}^{(\delta_0)}(A)$ is trivial for non-resonant directions, *i.e.* if $\mathrm{Sp}(c^{\delta_0} q^n A_j) \cap \mathrm{Sp}(A_i) = \varnothing$. This is the necessary conditions we were looking for. It is not hard to see (and it will come as a particular case of the following sections) that these are indeed the only conditions on the first level.

3.3.2. *Structure of the q-alien derivations at an arbitrary level.* — We are led to introduce some more notations. We first refine the spectral decomposition of $\hat{\omega}_a^{(0)}(A)$ under the action of the theta torus by taking in account the action of $G_{f,s}^{(0)}$, the semi-simple component of the fuchsian group. From the equivalence:

$$\alpha \equiv \beta \pmod{q^{\mathbf{Z}}} \Longleftrightarrow \forall \gamma \in G_{f,s}^{(0)}\ ,\ \gamma(\alpha) = \gamma(\beta),$$

we see that the action of $G_{f,s}^{(0)}$ splits each eigenspace under $\mathbf{C}^*$ corresponding to the slope μ_i into a sum indexed by the $\overline{\alpha} \in \overline{\mathrm{Sp}(A_i)}$. Precisely, if $V = \hat{\omega}_a^{(0)}(A)$, then one may write:

$$V = \bigoplus V^{(\mu)},$$

where μ runs through the set of slopes of A, and, for each μ:

$$V^{(\mu)} = \bigoplus V^{(\mu,\overline{\alpha})},$$

where α runs through $\mathrm{Sp}(A_i)$ if $\mu = \mu_i$ in our usual notations.

To be able to carry this splitting to matrices, we fix an arbitrary linear order on $\mathbf{E}_q$ and assume the order on indices is compatible with that arbitrary order. The corresponding adjoint action of $G_{f,s}^{(0)}$ on $\mathfrak{g}_{A_0}(A)$ then allows one to define the eigenspaces: ${\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}(\mathbf{C}) = \{M \in \mathfrak{g}_{A_0}(\mathbf{C}) \mid M$ is trivial out of the $(\mu_i, \overline{\alpha}, \mu_j, \overline{\beta})$ components such that $\alpha c^{\mu_i} \equiv \beta c^{\mu_j} \pmod{q^{\mathbf{Z}}}\}$. This can be non-trivial only if $\overline{c} \in \Sigma_{A_0}^{\delta}$, where:

$$\Sigma_{A_0}^{\delta} = \{\overline{c} \in \mathbf{E}_q \mid \exists i < j \text{ such that } \mu_j - \mu_i = \delta \text{ and } \frac{\overline{\alpha}}{\overline{\beta}} = \overline{c^{\delta}}\}.$$

By definition, $\Sigma_{A_0} = \bigcup_{\delta \geq 1} \Sigma_{A_0}^{\delta}$. then:

$${\mathfrak{g}_{A_0}}^{\delta}(\mathbf{C}) = \bigoplus_{\overline{c} \in \Sigma_{A_0}^{\delta}} {\mathfrak{g}_{A_0}}^{(\delta,\overline{c})}(\mathbf{C}).$$

Example. From the previous paragraph, it follows that on the first non trivial level, $\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in {\mathfrak{g}_{A_0}}^{\delta,\overline{c}}(\mathbf{C})$. The difficulty is to properly generalize this fact to upper levels.

Remark. The equality $\overline{\alpha}\overline{c}^{\mu_i} = \overline{\beta}\overline{c}^{\mu_j}$ is equivalent to: $\forall \gamma \in G_{f,s}^{(0)}$, $\gamma(\alpha c^{\mu_i}) = \gamma(\beta c^{\mu_j})$. Thus, ${\mathfrak{g}_{A_0}}^{\delta,\overline{c}}(\mathbf{C})$ can be characterized as the common fixed space of all the $\big(\gamma(\overline{c})^{-1}, \gamma, 0\big) \in G_{p,1}^{(0)}$, where γ runs through $G_{f,s}^{(0)}$.

Now let A, A' be matrices in standard form with the same graded part A_0. From [**12**] and [**6**], we have the following generalisation of theorem 3.1: for each $\overline{c} \in \mathbf{E}_q \backslash \Sigma_{A_0}$,

there exists a unique meromorphic morphism $F : A \to A'$ in $\mathfrak{G}_{A_0}(\mathcal{M}(\mathbf{C}^*))$, with poles on $[-c; q]$ and with multiplicities prescribed as in the theorem. We write it $S_{\overline{d}}\hat{F}_{A,A'}$. One then has:

$$S_{\overline{d}}\hat{F}_{A,A'} = S_{\overline{d}}\hat{F}_{A'}\left(S_{\overline{d}}\hat{F}_A\right)^{-1}.$$

Assume now that $A \equiv A' \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C}(\{z\}))}$, that is, A and A' have the same over-diagonals at levels $< \delta$. The components $F_{i,j}$ of $S_{\overline{d}}\hat{F}_{A,A'}$ for $0 < \mu_j - \mu_i < \delta$ are solutions of the equations:

$$(\sigma_q F_{i,j})z^{\mu_j}A_j - z^{\mu_i}A_iF_{i,j} = 0.$$

Therefore, they are null (*cf. loc. cit.*). This implies:

$$S_{\overline{d}}\hat{F}_{A,A'} \in \mathfrak{g}_{A_0}{}^{\geq\delta}(\mathcal{M}(\mathbf{C}^*)).$$

From the equality: $S_{\overline{d}}\hat{F}_{A'} = S_{\overline{d}}\hat{F}_{A,A'}S_{\overline{d}}\hat{F}_A$, we deduce:

$$S_{\overline{d}}\hat{F}_A \equiv S_{\overline{d}}\hat{F}_{A'} \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathcal{M}(\mathbf{C}^*))}.$$

Proposition 3.9. — *Let $f_{A,A',\overline{d}}$ be the component at level δ of $S_{\overline{d}}\hat{F}_{A,A'}(a)$. Then:*

$$\dot{\Delta}_{\overline{c}}^{(\delta)}(A') = \dot{\Delta}_{\overline{c}}^{(\delta)}(A) + Res_{\overline{d}=\overline{c}}f_{A,A',\overline{d}}.$$

Proof. — To alleviate notations, we omit the evaluation at a and the direction $\overline{d}$ in the notations; to indicate summation along the arbitrary fixed direction $\overline{c}_0$, we just add the index 0. Thus, we respectively write:

$$\begin{array}{rcl} F_A \text{ for } S_{\overline{d}}\hat{F}_A(a) & \text{and} & F_{A,0} \text{ for } S_{\overline{c_0}}\hat{F}_A(a) \\ F_{A'} \text{ for } S_{\overline{d}}\hat{F}_{A'}(a) & \text{and} & F_{A',0} \text{ for } S_{\overline{c_0}}\hat{F}_{A'}(a) \\ F_{A,A'} \text{ for } S_{\overline{d}}\hat{F}_{A,A'}(a) & \text{and} & F_{A,A',0} \text{ for } S_{\overline{c_0}}\hat{F}_{A,A'}(a) \\ f_{A,A'} \text{ for } f_{A,A',\overline{d}} & \text{and} & f_{A,A',0} \text{ for } f_{A,A',\overline{c_0}}. \end{array}$$

From the previous remark:

$$F_{A'} = F_{A,A'}F_A \equiv (I_n + f_{A,A'})F_A \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})},$$

so that:

$$F_{A',0}^{-1}F_{A'} \equiv F_{A,0}^{-1}F_{A,0} + f_{A,A'} - f_{A,A',0} \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}.$$

The conclusion then comes by taking logarithms, applying the following lemma and then taking residues. □

Lemma 3.10. — *Let $M \in \mathfrak{G}_{A_0}(\mathbf{C})$ and $N \in \mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})$. Then:*

$$\log(M + N) \equiv (\log M) + N \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}.$$

Proof. — Write $M = I_n + M'$. Then:

$$\log(M+N) \equiv \sum_{m\geq 1} \frac{(-1)^{m-1}}{m}(M'+N)^m \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}$$
$$\equiv \sum_{m\geq 1} \frac{(-1)^{m-1}}{m} M'^m + N \pmod{\mathfrak{g}_{A_0}{}^{\geq\delta}(\mathbf{C})}. \qquad \square$$

Corollary 3.11. — *Under the assumptions of the proposition, we have:*

$$\dot{\Delta}_{\overline{c}}^{(\delta)}(A') - \dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in \mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C}).$$

We are going to prove that these are, in some sense, the only conditions on the q-alien derivations at a given level δ.

3.3.3. *Interpolating categories.* — There are two equivalent ways of defining $\mathcal{E}_{p,1}^{(0)}$ from $\mathcal{E}_1^{(0)}$: the first is by restriction to a subclass of objects, the pure ones; the second is by formalisation, *i.e.* extension of the base field $\mathbf{C}(\{z\}) \to \mathbf{C}((z))$. The former way amounts to shrinking the Galois group $G_1^{(0)}$ to its *quotient* $G_{p,1}^{(0)}$. The latter way amounts to extending the class of morphisms (indeed, there are no really new objects), and therefore to shrinking the Galois group $G_1^{(0)}$ to its *subgroup* $G_{p,1}^{(0)}$.

The existence of a natural filtration on the Stokes group $\mathfrak{St}$ suggests that it should be possible to interpolate between $\mathcal{E}_{p,1}^{(0)}$ and $\mathcal{E}_1^{(0)}$. We shall presently do so by extending the class of morphisms; the interpretation by restriction to subobjects is a bit more complicated.

We first define intermediate fields between $\mathbf{C}(\{z\})$ and $\mathbf{C}((z))$, for all levels $\delta \in \mathbf{N}$:

$$\mathbf{C}((z))^{(\delta)} = \left\{\sum f_n z^n \in \mathbf{C}((z)) \mid \exists R > 0 \ : \ f_n = \mathrm{O}(R^n q^{n^2/2\delta})\right\}.$$

Thus $\mathbf{C}((z))^{(+\infty)} \underset{def}{=} \mathbf{C}(\{z\}) \subset \mathbf{C}((z))^{(\delta)} \subset \mathbf{C}((z))^{(\delta-1)} \subset \mathbf{C}((z))^{(0)} \underset{def}{=} \mathbf{C}((z))$ [(4)].
The following is standard ([**4**], [**16**],[**12**]):

Lemma 3.12. — *If $\nu - \mu = \delta \geq 1$, then, the following equation:*

$$(\sigma_q F)(z^\nu B) - (z^\mu A)F = U, \quad A \in GL_r(\mathbf{C}) \ , \ B \in GL_s(\mathbf{C}) \ , \ U \in Mat_{r,s}(\mathbf{C}(\{z\}))$$

has a unique solution $F \in Mat_{r,s}(\mathbf{C}((z))^{(\delta)})$. If moreover $F \in Mat_{r,s}(\mathbf{C}((z))^{(\delta')})$ for some $\delta' > \delta$, then $F \in Mat_{r,s}(\mathbf{C}(\{z\}))$.

Write $\mathbf{C}((z))^{>\delta} = \bigcup_{\delta'>\delta} \mathbf{C}((z))^{(\delta')}$. Then we call $\mathcal{C}^\delta$ the category with the same objects as $\mathcal{E}_1^{(0)}$ (seen in matrix form) and with morphisms satisfying the same conditions, but with $F \in GL_n(\mathbf{C}((z))^{>\delta})$. (Actually, since we deal only with integral slopes, we could as well take $\mathbf{C}((z))^{(\delta+1)}$ instead of $\mathbf{C}((z))^{>\delta}$.)

(4) This $\mathbf{C}((z))^{(\delta)}$ is the field of fraction of the algebra of q-Gevrey series of level δ, which was introduced in [**1**] and denoted $\mathbf{C}[[z]]_{q,s}$ with $s = 1/\delta$ (the q-Gevrey order) in [**4**].

It is then clear that the $\mathcal{C}^\delta$ are tannakian categories, and that embeddings are natural exact faithful $\otimes$-functors $\mathcal{C}^\delta \to \mathcal{C}^{\delta-1}$. Moreover, $\mathcal{C}^0 = \mathcal{E}_{p,1}^{(0)}$, because equations with integral slopes can be solved in $\mathbf{C}((z))^{(1)}$ by the lemma, so that $\hat{F}_A$ is an isomorphism from A_0 to A in $\mathcal{C}^0$. In the opposite direction, we have $\mathcal{C}^\infty = \mathcal{E}_1^{(0)}$. Actually, if A has slopes $\mu_1 < \cdots < \mu_k$, then it is entirely determined by its image in $\mathcal{C}^\delta$ for any $\delta \geq \mu_k - \mu_1$.

From the composite functor $\mathcal{C}^\delta \to \mathcal{C}^0 = \mathcal{E}_{p,1}^{(0)}$, we draw that objects in $\mathcal{C}^\delta$ have a well defined Newton polygon, that there is on $\mathcal{C}^\delta$ a "graded module" functor, and that $\hat{\omega}_a^{(0)}$ defines a fiber functor on $\mathcal{C}^\delta$.

3.3.3.1. *An alternative description of $\mathcal{C}^\delta$.*— We also see that, if two objects A and B in $\mathcal{E}_1^{(0)}$ have isomorphic images in $\mathcal{C}^\delta$, then they have isomorphic images in $\mathcal{C}^0 = \mathcal{E}_{p,1}^{(0)}$ and they can be written in standard form (6) with the same block diagonal A_0. Of course, we may moreover assume A_0 to be in *normalized* standard form and A, B to be in *polynomial* standard form (section 2.2).

Proposition 3.13. — *Let A and B in $\mathcal{E}_1^{(0)}$ be in normalized polynomial standard form with the same block diagonal A_0. Then they have isomorphic images in $\mathcal{C}^\delta$ if, and only if, there exists $F_0 \in GL_n(\mathbf{C})$ such that $B \equiv F_0 A F_0^{-1}$ (mod $\mathfrak{g}_{A_0}{}^{>\delta}(\mathbf{C}(\{z\})))$.*

Proof. — Here, of course, we have put $\mathfrak{g}_{A_0}{}^{>\delta} = \sum_{\delta'>\delta} \mathfrak{g}_{A_0}{}^{\delta'}$ (which is the same as $\mathfrak{g}_{A_0}{}^{\geq\delta+1}$ since we deal with integral slopes) and the condition just means that B and $F_0 A F_0^{-1}$ have the same over-diagonals up to level δ.

The diagonal part F_0 of any formal morphism F from A to B is an automorphism of A_0, thus constant (because of normalisation). Up to composing F with F_0^{-1} and replacing A by $F_0 A F_0^{-1}$, we may assume that $F_0 = I_n$, so that $F = \hat{F}_{A,B}$. The condition then means that $\hat{F}_{A,B}$ has its coefficients in $\mathbf{C}((z))^{(\delta)}$. From the lemma, we draw, by induction on the level, that all over-diagonals up to level δ are analytic, therefore 0 because of results in [**10**]. □

Corollary 3.14. — *One can define $\mathcal{C}^\delta$ in the following alternative way:*

1. *Objects of $\mathcal{C}^\delta$ are matrices in $\mathcal{E}_1^{(0)}$ modulo the equivalence relation $A \equiv B$ (mod $\mathfrak{g}_{A_0}{}^{>\delta}(\mathbf{C}(\{z\})))$.*
2. *Morphisms from (the class of) A to (the class of) B are matrices $F \in Mat_{p,n}(\mathbf{C}(\{z\}))$ such that $(\sigma_q F)A$ and BF differ only in levels $> \delta$.*

Corollary 3.15. — *The Galois group of $\mathcal{C}^\delta$ is $\mathfrak{St}(\delta) \rtimes G_{p,1}^{(0)}$ for some unipotent subgroup $\mathfrak{St}(\delta)$ of $\mathfrak{St}$. For $i \leq \delta$, the $\dot{\Delta}_{\bar{c}}^{(i)}$ are well defined on $\mathcal{C}^\delta$ and belong to the Lie algebra $\mathfrak{st}(\delta)$ of $\mathfrak{St}(\delta)$.*

3.3.4. *A freeness theorem.* — We now describe precisely the essential image of the functor $\mathcal{F}$, that is, given A_0 in $\mathcal{E}_{p,1}^{(0)}$, the exact conditions on $(\dot{\Delta}_{\bar{c}}^{(\delta)}(A))_{\delta\geq 1, \bar{c}\in\mathbf{E}_q}$ that

allow the reconstruction of A. The reconstruction will be done inductively, using q-alien derivations of levels up to δ to reconstruct the over-diagonals of A up to level δ, that is (after the previous paragraph) an object in $\mathcal{C}^\delta$.

In the same spirit as the definition of isoformal analytic classes in [**10**], we consider classes of objects A in $\mathcal{C}^\delta$ above an object B of $\mathcal{C}^{\delta-1}$ under the equivalence induced by gauge transform $F \equiv I_n \pmod{\mathfrak{g}_{A_0}{}^{\leq\delta}}$. Using polynomial standard normal form and the results of *loc. cit.*, we see that these classes make up a vector space of dimension:

$$\mathrm{irr}^\delta(A_0) = \sum_{\mu_j-\mu_i=\delta} r_i r_j(\mu_j-\mu_i) = \delta \sum_{\mu_j-\mu_i=\delta} r_i r_j.$$

Moreover, to see if two objects $A, A'$are in the same class, one computes $\hat{F}_{A,A'} \in \mathfrak{G}_{A_0}{}^\delta(\mathbf{C}((z)))$; if its over-diagonal at level δ has null q-Borel invariants, then we have the same class.

***Theorem 3.16* (Freeness theorem).** — *Let B be an object of $\mathcal{C}^{\delta-1}$. Then, there is an affine space $V_{\overline{c}}(B)$ of direction $\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C})$ such that:*
(i) The $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ for A an object of $\mathcal{C}^\delta$ above B belong to $V_{\overline{c}}(B)$.
(ii) The mapping which sends an object A of $\mathcal{C}^\delta$ above B to the family of all $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ induces a one-to-one correspondance between classes of such objects (as defined above) and $\prod V_{\overline{c}}(B)$.

Proof. — (i) It follows from paragraph 3.3.2 that all $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ (where A is fixed and $\overline{c}$ varies) belong to a unique affine space of direction $\mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C})$. Call it $V_{\overline{c}}(B)$. It is easily seen that the product space $\prod V_{\overline{c}}(B)$ has dimension $\mathrm{irr}^\delta(A_0)$.
(ii) The map from the set of polynomial representatives of a class, as described above, onto the above affine space, is affine. Up to the choice of an arbitrary basepoint, it is equivalent, after the results of [**7**] (section 3.2), to the parametrisation of the isoformal class by q-Borel transform, which is one-to-one after [**10**]. □

Corollary 3.17. — *The following algorithm allows one to reconstruct A in $\mathcal{E}_1^{(0)}$ from A_0 and the $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$:*

1. *Reconstruct the first over-diagonal using the q-derivations of lowest level (this is the linear situation and it rests on* [**6**].
2. *Having reconstructed the over-diagonals up to level $\delta-1$ (using q-alien derivations up to level $\delta-1$), call A' the matrix with these over-diagonals and 0 above; then compute the $\dot{\Delta}_{\overline{c}}^{(\delta)}(A')$.*
3. *Use the relation $\dot{\Delta}_{\overline{c}}^{(\delta)}(A')-\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in \mathfrak{g}_{A_0}{}^{(\delta,\overline{c})}(\mathbf{C})$ to find the level δ over-diagonal of $\hat{F}_{A,A'}$, then the level δ over-diagonal of A.*

In [**7**], we shall give a representation-theoretic formulation of the theorem, and a description of the nonlinear part of $\dot{\Delta}_{\overline{c}}^{(\delta)}(A)$ (that part which depends on the lower level q-derivations) in terms of convolution.

4. Conclusion

Write $H = \mathbf{C}^* \times \mathrm{Hom}_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*)$ and $\nu = (t, \gamma) \in H$. We saw how the group H acts upon the diagonal of A_0: for $1 \leq i \leq k$ and $\alpha \in \mathrm{Sp}(A_i$, positions corresponding to slope μ_i and the eigenspace of $A_{i,s}$ for α are multiplied by $t^{\mu_i}\gamma(\alpha)$. Now let $i < j$ be indices of slopes $\mu_i < \mu_j$ and $\alpha \in \mathrm{Sp}(A_i$, $\beta \in \mathrm{Sp}(A_j)$ be corresponding exponents. The adjoint action of ν on the $(\mu_i, \overline{\alpha}, \mu_j, \overline{\beta})$ block is the multiplication by:

$$t^{\mu_i - \mu_j} \frac{\gamma(\alpha)}{\gamma(\beta)} = \left(t^{-1}\gamma(\overline{c})\right)^{\delta}$$

for each "resonant" $\overline{c}$, *i.e.* $\overline{c} \in \mathbf{E}_q$ such that $c^{\delta} \equiv \dfrac{\alpha}{\beta} \pmod{q^{\mathbf{Z}}}$. For any Galois derivation $D \in \mathfrak{st}$, we now put:

$$\Phi_{\nu}^{(\delta,\overline{c})}(D) = \nu(D) - \left(t^{-1}\gamma(\overline{c})\right)^{\delta} D \in \mathfrak{st},$$

where $\nu(D)$ comes from the adjoint action of H on $\mathfrak{st}$. From the remark on page 318 and from paragraph 3.3.2, one draws that, for two objects A, A' of $\mathcal{C}^{\delta}$ above the same object B of $\mathcal{C}^{\delta-1}$, $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A)\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A') = 0$. In other words, $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A)$ depends only on the lower levels $\delta' < \delta$ of A. Moreover it is trivial on the first level. Actually, with methods similar to those used here, one can prove that $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})(A)$ is in the Lie algebra generated by the q-alien derivations at lower levels. So it is natural to conjecture that $\Phi_{\nu}^{(\delta,\overline{c})}(\dot{\Delta}_{\overline{c}}^{(\delta)})$ belong to the free Lie algebra generated by the $\dot{\Delta}_{\overline{d}}^{(\delta')}$ ($\delta' < \delta$, $\overline{d} \in \mathbf{E}_q$), and even that there is a universal explicit formula. This would allow us to define a semi-direct product by a free Lie algebra, and to definitely "free" the q-alien derivations.

All the problems comes from the fact that points come from two distinct origins: elements of the dual of H on the one hand, packs of points of $\mathbf{E}_q$ on the other hand, and from the interplay of the corresponding games of localisation. Comparing with the differential case, where one localizes geometrically on the circle of directions S^1, *then* one takes a Log, here, we take a Log, *then* we localise on Eq; whence an embroilment with plenty of Campbell-Haussdorff formulas between the two approaches [(5)].

We shall also give in [**7**] various applications, to the abelianisation of the tannakian π_1 and to the inverse problem for the local Galois group. For the latter problem, we shall state a list of necessary conditions; we don't know for the time being if they are sufficient.

Last, we built our alien q-derivations by tannakian methods. One can ask what happens for *solutions*. There, one meets the usual difficulty about constants, since one wishes operators defined over $\mathbf{C}$ and acting upon solutions (while constants are here elliptic functions). That problem maybe has no solution; however, one could perhaps, in analogy with the differential case, "unpoint" the q-alien derivations and

(5) Actually, we think that, in the end, we'll have a simpler description with a denumerable family of q-alien derivations, freed by the mere action of the theta torus.

build operators acting upon adequate spaces of formal power series. This seems related to a "q-convolution" mechanism presently studied by Changgui Zhang.

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