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POLAR PENCIL OF CURVES AND FOLIATIONS

by

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Abstract. — The polar pencil $\Lambda_{\mathcal{F}}$ of a singular foliation $\mathcal{F}$ is the pencil of curves formed by the polar curves of $\mathcal{F}$. We study the relationship between the behaviour of $\Lambda_{\mathcal{F}}$ under blowing-up and the invariants associated to $\mathcal{F}$. The main result here describes a resolution of singularities of $\Lambda_{\mathcal{F}}$ in terms of the equireduction invariants of $\mathcal{F}$, for a Zariski-general foliation $\mathcal{F}$.

***Résumé* (Pinceau polaire de courbes et feuilletages).** — Le pinceau polaire $\Lambda_{\mathcal{F}}$ d'un feuilletage singulier $\mathcal{F}$ est le pinceau de courbes composé par les courbes polaires de $\mathcal{F}$. Nous allons étudier la relation entre le comportement de $\Lambda_{\mathcal{F}}$ par éclatement et les invariants associés à $\mathcal{F}$. Le résultat principal ici donne une description d'une résolution de singularités de $\Lambda_{\mathcal{F}}$ en termes des invariants d'équiréduction de $\mathcal{F}$ lorsque $\mathcal{F}$ est un feuilletage général de Zariski.

1. Introduction

Let A, B be two germs of holomorphic functions at $(\mathbb{C}^2, 0)$ with no common component and consider the pencil of curves $\Lambda = \{aA + bB = 0 \ : \ a, b \in \mathbb{C}\}$. Classically, these pencils of curves have been studied in relation to the reduction of singularities of $A = 0$ and $B = 0$ (see for instance [**14, 4, 8**]). Here we propose a different approach: we consider Λ as the *polar pencil* $\Lambda_{\mathcal{F}}$ associated to a singular foliation $\mathcal{F}$ defined by the 1-form $\omega = A(x,y)dx + B(x,y)dy$. Our objective is to describe properties of $\Lambda_{\mathcal{F}}$ in terms of the invariants associated to $\mathcal{F}$.

Let $\mathcal{G}_\omega$ be the *Gauss map* associated to $\mathcal{F}$ which is given by

$$\begin{array}{rccl} \mathcal{G}_\omega : & (\mathbb{C}^2, 0) \setminus \{0\} & \longrightarrow & \mathbb{P}^1_{\mathbb{C}} \\ & (x,y) & \longmapsto & [-B(x,y) : A(x,y)]. \end{array}$$

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A curve $\Gamma_{[a:b]}$ of $\Lambda_{\mathcal{F}}$ is the closure in $(\mathbb{C}^2,0)$ of the fiber $\mathcal{G}_\omega^{-1}([a:b])$ for $[a:b]\in\mathbb{P}^1_{\mathbb{C}}$. There is a maximal non-empty Zariski open set of $\Omega\subset\mathbb{P}^1_{\mathbb{C}}$ such that all the curves $\Gamma_{[a:b]}$ with $[a:b]\in\Omega$ are equisingular: they are the *generic curves* of $\Lambda_{\mathcal{F}}$.

Let $\sigma : X\to(\mathbb{C}^2,0)$ be a finite sequence of punctual blow-ups. We say that σ is an *elimination of indeterminations* of $\mathcal{G}_\omega$ (or *a resolution of singularities* of $\Lambda_{\mathcal{F}}$) iff the map $\tilde{\mathcal{G}}_\omega=\mathcal{G}_\omega\circ\sigma : X\to\mathbb{P}^1_{\mathbb{C}}$ is well-defined. Such σ gives an embedded reduction of singularities of the union $\Gamma\cup\Gamma'$ of two different generic fibers, then σ is a resolution of singularities of $\Lambda_{\mathcal{F}}$ (see [**14**]).

An irreducible component D of $\sigma^{-1}(0)$ is called *dicritical* if the restriction $\tilde{\mathcal{G}}_\omega|_D : D\to\mathbb{P}^1_{\mathbb{C}}$ is not constant. The *degree* of a dicritical component D is the degree of the map $\tilde{\mathcal{G}}_\omega|_D : D\to\mathbb{P}^1_{\mathbb{C}}$; this number coincides with the number of intersection points between D and the strict transform $\sigma^*\Gamma$ of Γ by σ, for any generic fiber Γ.

The curves of the polar pencil $\Lambda_{\mathcal{F}}$ can also be seen as the separatrices of a singular foliation: the *polar foliation* $\mathcal{P}_{\mathcal{F}}$ defined by $d(A/B)=0$. The minimal resolution $\sigma_\Lambda : X\to(\mathbb{C}^2,0)$ of $\Lambda_{\mathcal{F}}$ gives a *partial reduction* [**12**] of $\mathcal{P}_{\mathcal{F}}$ in the sense that the minimal reduction of singularities $\pi_{\mathcal{P}} : \mathfrak{X}\to(\mathbb{C}^2,0)$ of $\mathcal{P}_{\mathcal{F}}$ factorizes as $\pi_{\mathcal{P}}=\sigma_\Lambda\circ\tau$, where $\tau : \mathfrak{X}\to X$ is a finite sequence of punctual blow-ups which are non-dicritical for $\mathcal{P}_{\mathcal{F}}$.

Let $C\subset(\mathbb{C}^2,0)$ be a plane curve. We shall work in the space of foliations $\mathbb{G}_C$ of non-dicritical generalized curves over C (see [**2**]). It is known that the minimal reduction of singularities $\pi_C : M_C\to(\mathbb{C}^2,0)$ of C gives a reduction of singularities of any $\mathcal{F}\in\mathbb{G}_C$. But in general π_C does not give a desingularization of a generic fiber Γ of $\Lambda_{\mathcal{F}}$. This occurs essentially in the case that C has a *kind equisingularity type* and $\mathcal{F}$ is Zariski-general (in the sense of the exponents of the logarithmic model) as we have shown in [**6, 7**].

Take $\mathcal{F}\in\mathbb{G}_C$ and let $\sigma_{\Lambda,C} : M_{\Lambda,C}\to(\mathbb{C}^2,0)$ be the minimal reduction of singularities of $\Lambda=\Lambda_{\mathcal{F}}$ that factorizes through π_C. The main result of this paper provides a precise description of $\sigma_{\Lambda,C}$ for kind singularities and Zariski-general foliations. Let us state it.

Let $G(C)$ be the dual graph of C oriented by the first divisor E_1. For each divisor E, let $m(E)$ be the multiplicity of any E-"curvette" and $v(E)$ be the coincidence of two E-curvettes. Denote by b_E the number of edges and arrows which leave from E. Thus E is a *bifurcation divisor* if $b_E\geq 2$ and a *terminal divisor* if $b_E=0$. A *dead arc* joins a bifurcation divisor with a terminal divisor, with no other bifurcations. We say that the equisingularity type $\epsilon(C)$ of C is *kind* if $m(E_b)=2m(E_t)$, for each dead arc of $G(C)$ starting at E_b and ending at E_t.

The main result here can be stated as

Theorem 1. — *Let $C\subset(\mathbb{C}^2,0)$ be a plane curve with kind equisingularity type. Consider a Zariski-general foliation $\mathcal{F}\in\mathbb{G}^*_C$ and take any generic curve Γ of $\Lambda_{\mathcal{F}}$. Then $\sigma_{\Lambda,C}$ is obtained from π_C by blowing-up α_E times in a free way at each point $\pi^*_C\Gamma\cap E$*

with

$$(1) \qquad \alpha_E = \begin{cases} m(E)(v(E)-1), & \textit{if } E \textit{ is a bifurcation divisor;} \\ m(E)(v(E)-1)-1, & \textit{if } E \textit{ is the terminal divisor of a dead arc,} \end{cases}$$

for each irreducible component E of $\pi_C^{-1}(0)$. Moreover, the first divisor E_1 is dicritical for $\Lambda_{\mathcal{F}}$ if and only if $b_{E_1} > 1$, and the degree of E_1 as a dicritical component of $\Lambda_{\mathcal{F}}$ is equal to $b_{E_1} - 1$. The degree of the other dicritical components of $\Lambda_{\mathcal{F}}$ is equal to one.

Observe that, under the hypothesis of theorem above, the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$ belong either to a bifurcation divisor or to the terminal divisor of a dead arc ([**6**]). Moreover, the points of $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$ are non-singular points of $\pi_C^*\mathcal{F}$ and $\pi_C^*\Gamma$ cuts transversally $\pi_C^{-1}(0)$. Consequently $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$ where σ_1 is obtained by blowing-up free infinitely near points of $\pi_C^*\Gamma$, i.e., the centers of the blow-ups to obtain σ_1 are not corners of the corresponding exceptional divisor. Hence $\sigma_{\Lambda,C}$ is obtained from π_C by "blowing-up in a free way" as it is stated in the theorem above.

The paper is organized as follows. Section 2 is devoted to introduce notations relative to the dual graph and the equisingularity data of a plane curve. In section 3 we remind some results concerning the generic fiber of the polar pencil and we also prove some technical lemmas. Section 4 deals with the base points of the pencil $\Lambda_{\mathcal{F}}$. In section 5 we state some results describing the dicritical components of a resolution of $\Lambda_{\mathcal{F}}$. The proof of the main result is given in section 6. We finish the paper with a list of examples showing different behaviours in the non Zariski-general cases.

2. Notations

In this section we introduce some notations concerning the dual graph and the equisingularity data of a plane curve $C = \cup_{i=1}^r C_i \subset (\mathbb{C}^2, 0)$ that will be used from now on. For each irreducible component C_i of C, denote by $n^i = m_0(C_i)$ the multiplicity of C_i at the origin. Let $y^i(x) = \sum_{j \geq n^i} a_j^i x^{j/n^i}$ be a Puiseux series of C_i, for $i = 1, \ldots, r$. The *characteristic exponents* $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ of C_i are given by

$$\beta_0^i = m_0(C_i) = n^i$$
$$\beta_q^i = \min\{j \;:\; a_j^i \neq 0 \text{ and } j \not\equiv 0 \mod (\gcd(\beta_0^i, \ldots, \beta_{q-1}^i))\}$$

for $q = 1, \ldots, g_i$, where g_i is the first integer such that $\gcd(\beta_0^i, \ldots, \beta_{g_i}^i) = 1$. An equivalent data to the characteristic exponents of C_i are the *Puiseux pairs* $\{(m_k^i, n_k^i)\}_{k=1}^{g_i}$ of C_i defined by

$$\gcd(m_k^i, n_k^i) = 1 \quad \text{and} \quad \frac{\beta_k^i}{n^i} = \frac{m_k^i}{n_1^i \cdots n_k^i} \quad \text{for} \quad k = 1, \ldots, g_i.$$

In particular, we have that $n^i = n_1^i \cdots n_{g_i}^i$ and $\beta_k^i = m_k^i n_{k+1}^i \cdots n_{g_i}^i$ for $k = 1, \ldots, g_i$.

Let us denote by $\pi_C : M_C \to (\mathbb{C}^2, 0)$ the minimal reduction of singularities of C. We recall briefly the construction of the *dual graph* $G(C) = G(\pi_C)$ of C. Each irreducible component E of $\pi_C^{-1}(0)$ is represented by a vertex in $G(C)$. Two vertices are joined by an edge if their associated divisors intersect. An irreducible component of C is represented by an arrow attached to the only divisor that it meets. The dual graph weighted with the self-intersection of each divisor $E \subset M_C$ determines the equisingularity type $\epsilon(C)$ of the curve C.

It is also possible to construct in a similar way the dual graph of a resolution of singularities of a pencil or a dicritical foliation by marking the dicritical components. If σ is any finite sequence of blow-ups, we denote by $G(\sigma, \Lambda)$ the graph constructed from the transform of a pencil Λ by σ.

Denote by E_1 the irreducible component of $\pi_C^{-1}(0)$ obtained after blowing-up the origin. The dual graph $G(C)$ is oriented by the first divisor E_1. The *geodesic* of a divisor E is the path which joins E_1 with E and the geodesic of a curve C_i is the geodesic of the divisor that meets the strict transform $\pi_C^* C_i$ of C_i. Thus, there is a partial order in the set of vertices of $G(C)$ given by $E < E'$ if, and only if, the geodesic of E' goes through E. Given a divisor E of $G(C)$, we denote by I_E the set of indices $i \in \{1, \dots, r\}$ such that E belongs to the geodesic of C_i.

A *curvette* $\tilde{\gamma}$ of a divisor E is a non-singular curve transversal to E at a non-singular point of $\pi_C^{-1}(0)$. The projection $\gamma = \pi_C(\tilde{\gamma})$ is a germ of plane curve in $(\mathbb{C}^2, 0)$ and γ is called an E-curvette. We denote by $m(E)$ the multiplicity at the origin of any E-curvette and by $v(E)$ the coincidence $\mathcal{C}(\gamma_E, \gamma'_E)$ of two E-curvettes γ_E, γ'_E which cut E in different points; observe that $v(E) < v(E')$ if $E < E'$. Recall that the *coincidence* $\mathcal{C}(\gamma, \delta)$ between two irreducible curves γ and δ is defined as

$$\mathcal{C}(\gamma, \delta) = \sup_{\substack{1 \le i \le m_0(\gamma) \\ 1 \le j \le m_0(\delta)}} \{ ord_x(y_i^\gamma(x) - y_j^\delta(x)) \}$$

where $\{y_i^\gamma(x)\}_{i=1}^{m_0(\gamma)}$, $\{y_j^\delta(x)\}_{j=1}^{m_0(\delta)}$ are the Puiseux series of γ and δ respectively.

Denote by b_E the number of edges and arrows which leave from a divisor E in $G(C)$. We say that E is a *bifurcation divisor* if $b_E \ge 2$ and a *terminal divisor* if $b_E = 0$. A *dead arc* is a path which joins a bifurcation divisor with a terminal one, without passing through other bifurcation divisors. We denote by $B(C)$ the set of bifurcation vertices of $G(C)$.

Let E be an irreducible component of the exceptional divisor $\pi_C^{-1}(0)$. The *reduction* $\pi_E : M_E \to (\mathbb{C}^2, 0)$ *of* π_C *to* E is the morphism satisfying that

- there is a factorization $\pi_C = \pi'_E \circ \pi_E$ where π'_E and π_E are composition of punctual blow-ups;
- the divisor E is the strict transform by π'_E of an irreducible component E_{red} of $\pi_E^{-1}(0)$ and $E_{red} \subset M_E$ is the only component of $\pi_E^{-1}(0)$ with self-intersection equal to -1.

The morphism π_E is obtained from π_C by blowing-down successively the divisors different from E with self-intersection is equal to -1. Given any curvette $\tilde{\gamma}_E$ of E, the curve $\pi'_E(\tilde{\gamma}_E)$ is also a curvette of $E_{red} \subset M_E$. Let $\{\beta_0^E, \beta_1^E, \dots, \beta_{g(E)}^E\}$ be the characteristic exponents of $\gamma_E = \pi_C(\tilde{\gamma}_E)$. It is clear that $m(E) = m_0(\gamma_E) = \beta_0^E$. If E is a bifurcation divisor of $G(C)$, there are two possibilities for the value $v(E)$:

1. either π_E is the minimal reduction of singularities of γ_E and then $v(E) = \beta^E_{g(E)}/\beta^E_0$. We say that E is a *Puiseux divisor* for π_C.
2. or π_E is obtained by blowing-up $q \geq 1$ times after the minimal reduction of singularities of γ_E and in this situation $v(E) = (\beta^E_{g(E)} + q)/\beta^E_0$. We say that E is a *contact divisor* for π_C.

Observe that $m(E) = m(E_{red})$ and $v(E) = v(E_{red})$. Moreover, a bifurcation divisor E can belong to a dead arc only if it is a Puiseux divisor.

Consider a bifurcation divisor E of $G(C)$ and let $\{(m^E_1, n^E_1), (m^E_2, n^E_2), \ldots, (m^E_{g(E)}, n^E_{g(E)})\}$ be the Puiseux pairs of an E-curvette γ_E, we denote

$$n_E = \begin{cases} n_{g(E)}, & \text{if } E \text{ is a Puiseux divisor;} \\ 1, & \text{otherwise,} \end{cases}$$

and $\underline{n}_E = m(E)/n_E$. Observe that, if E is a bifurcation divisor which belongs to a dead arc with terminal divisor F, then $m(F) = \underline{n}_E$. We define k_E to be

$$k_E = \begin{cases} g(E) - 1, & \text{if } E \text{ is a Puiseux divisor;} \\ g(E), & \text{if } E \text{ is a contact divisor.} \end{cases}$$

Thus, we have that $\underline{n}_E = n^E_1 \cdots n^E_{k_E}$.

To finish this section, we recall a lemma which gives the relationship between the intersection multiplicity $(\gamma, \delta)_0$ and the coincidence $\mathcal{C}(\gamma, \delta)$ (see Zariski [**15**], prop. 6.1 or Merle [**11**], prop. 2.4):

***Lemma* 2.** — *Let γ and δ be two germs of irreducible plane curves of $(\mathbb{C}^2, 0)$. If $\{\beta_0, \beta_1, \ldots, \beta_g\}$ are the characteristic exponents of γ and α is a rational number such that $\beta_q \leq \alpha < \beta_{q+1}$ ($\beta_{g+1} = \infty$), then the following statements are equivalent:*

1. $\mathcal{C}(\gamma, \delta) = \dfrac{\alpha}{m_0(\gamma)}$,
2. $\dfrac{(\gamma, \delta)_0}{m_0(\delta)} = \dfrac{\bar{\beta}_q}{n_1 \cdots n_{q-1}} + \dfrac{\alpha - \beta_q}{n_1 \cdots n_q}$,

where $\{(m_i, n_i)\}_{i=1}^g$ are the Puiseux pairs of γ ($n_0 = 1$) and $\{\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_q\}$ is a minimal system of generators of the semigroup $S(\gamma)$ of γ.

Recall that the *semigroup* $S(\gamma)$ of γ is defined as

$$S(\gamma) = \{(\gamma, \delta)_0 : \gamma \text{ is not an irreducible component of } \delta\}.$$

There is a minimal system of generators $\{\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_g\}$ of $S(\gamma)$ whose elements are defined by

$$(2) \quad \bar{\beta}_0 = \beta_0 = m_0(\gamma), \quad \bar{\beta}_1 = \beta_1, \quad \bar{\beta}_l = n_{l-1}\bar{\beta}_{l-1} + \beta_l - \beta_{l-1}, \quad \text{for } l = 2, \ldots, g,$$

where $\{\beta_0, \beta_1, \ldots, \beta_g\}$ are the characteristic exponents of γ (see [**1**] or [**16**]). It is clear that $S(\gamma)$ is determined by the equisingularity type of γ and reciprocally.

3. Generic curves of the pencil

This section is devoted to describe some properties of a generic curve of the polar pencil $\Lambda_{\mathcal{F}}$ of a singular foliation $\mathcal{F}$. The reader may refer to [**5, 7**] for a more detailed description.

Consider a plane curve $C = \cup_{i=1}^{r} C_i \subset (\mathbb{C}^2, 0)$. Let $f = f_1 \cdots f_r$ be a reduced equation of C and $\pi_C : M_C \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of C. Denote by $\mathbb{G}_C$ the space of generalized curve foliations [**2**] having C as curve of separatrices. Let $\mathbb{G}_C^*$ be the sub-space of $\mathbb{G}_C$ defined as follows: a foliation $\mathcal{F}$ is in $\mathbb{G}_C^*$ iff the logarithmic model $\mathcal{L}_\lambda$ of $\mathcal{F}$ avoids a finite set of resonances $R_{\epsilon(C)} \subset (\mathbb{Z}_{\geq 0})^r$. More precisely, each foliation $\mathcal{F} \in \mathbb{G}_C$ has a unique logarithmic model $\mathcal{L}_\lambda$ given by $f_1 \cdots f_r \sum_{i=1}^{r} \lambda_i df_i/f_i = 0$ with $\lambda = \lambda(\mathcal{F}) = (\lambda_1, \ldots, \lambda_r) \in \mathbb{P}_{\mathbb{C}}^{r-1}$ (see [**5**]). The logarithmic foliation $\mathcal{L}_\lambda$ has the same reduction of singularities as $\mathcal{F}$ and the same Camacho-Sad indices [**3**] at the final points of the reduction. Thus, a foliation $\mathcal{F}$ belongs to $\mathbb{G}_C^*$ iff $\sum_{i=1}^{r} k_i \lambda_i \neq 0$ for each $k = (k_1, \ldots, k_r) \in R_{\epsilon(C)}$ where $R_{\epsilon(C)} \subset (\mathbb{Z}_{\geq 0})^r$ is a finite set which depends only on the equisingularity type $\epsilon(C)$ of C (see [**5, 7**] for a detailed construction of it).

Remark 3. — Note that a foliation $\mathcal{F}$ avoids the resonances of the set $R_{\epsilon(C)}$ if and only if there is no corner in the reduction of singularities of $\rho^*\mathcal{F}$ with Camacho-Sad equal to -1, where $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is any ramification transversal to C such that $\rho^{-1}C$ has only non-singular irreducible components (see [**5**]).

Consider a generic fiber Γ of the pencil $\Lambda_{\mathcal{F}}$. A first result describing some properties of the equisingularity type $\epsilon(\Gamma)$ of Γ in terms of the equisingularity type $\epsilon(C)$ of C is the following one:

***Theorem 4* (of decomposition [10, 11, 9, 5])**. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and a generic curve Γ of $\Lambda_{\mathcal{F}}$. There is a decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$ such that:*

(i) $m_0(\Gamma^E) = \begin{cases} \underline{n}_E n_E (b_E - 1), & \text{if } E \text{ does not belong to a dead arc;} \\ \underline{n}_E n_E (b_E - 1) - \underline{n}_E, & \text{otherwise.} \end{cases}$

(ii) *For each irreducible component γ of Γ^E we have that*
- $\mathcal{C}(C_i, \gamma) = v(E)$ *if E belongs to the geodesic of C_i;*
- $\mathcal{C}(C_j, \gamma) = \mathcal{C}(C_j, C_i)$ *if E belongs to the geodesic of C_i but not to the one of C_j.*

It is clear that the result above does not determine $\epsilon(\Gamma)$. However, there is a Zariski-open set $U_C \subset \mathbb{P}_{\mathbb{C}}^{r-1}$ such that $\epsilon(\Gamma)$ is completely determined by $\epsilon(C)$ if $\lambda(\mathcal{F}) \in U_C$. The set U_C depends on the analytic type of C and it is a non-empty set if, and only if, the curve C has a kind equisingularity type. We say that a curve C has *kind equisingularity type* if $m(E_b) = 2m(E_t)$ for each dead arc of $G(C)$ with bifurcation divisor E_b and terminal divisor E_t. Using the notations introduced in section 2, the curve C has a kind equisingularity type if and only if $n_{E_b} = 2$ for each bifurcation divisor E_b of $G(C)$ which belongs to a dead arc since $m(E_b) = n_{E_b} m(E_t)$.

In particular, this implies that each dead arc in $G(C)$ has only two vertices: the bifurcation divisor and the terminal divisor.

A foliation $\mathcal{F}$ is called *Zariski-general* when $\lambda(\mathcal{F}) \in U_C$ and in this case $\epsilon(\Gamma)$ is described as follows:

Theorem 5. — [**6, 7**] *Let C be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}_C^*$. If Γ is a generic curve of the pencil $\Lambda_{\mathcal{F}}$, then π_C gives a reduction of singularities of $\Gamma \cup C$. Moreover, the branches of Γ intersect an irreducible component E of the exceptional divisor $\pi_C^{-1}(0)$ as follows:*

- *If E is a bifurcation divisor of $G(C)$, the number of branches of Γ cutting E equals to $b_E - 2$ if E belongs to a dead arc and to $b_E - 1$ otherwise.*
- *If E is a terminal divisor of a dead arc of $G(C)$, there is exactly one branch of Γ through E.*
- *Otherwise, no branches of Γ intersect E.*

In particular, the characteristic exponents of the branches of Γ can be completely determined in terms of the equisingularity data of C. Denote by $\{\beta_0^i, \beta_1^i, \ldots, \beta_{g_i}^i\}$ the characteristic exponents of an irreducible component C_i of C. Given a bifurcation divisor E of $G(C)$, let I_E^* be the set of indices $i \in I_E$ such that $v(E) = \beta_{k_E+1}^i/\beta_0^i$; note that if $i \in I_E \setminus I_E^*$ then there exists $j \in I_E$ such that $v(E) = \mathcal{C}(C_i, C_j)$. Hence, if E is a contact divisor $I_E^* = \varnothing$. Moreover, if C has a kind equisingularity type and E is a bifurcation divisor belonging to a dear arc of $G(C)$, then the corresponding Puiseux pair $(m_{k_E+1}^i, n_{k_E+1}^i)$ satisfies $n_{k_E+1}^i = 2$ for each $i \in I_E = I_E^*$.

Lemma 6. — [**7**] *Consider a curve C with kind equisingularity type and a Zariski general foliation $\mathcal{F} \in \mathbb{G}_C^*$. Let Γ be a generic curve of $\Lambda_{\mathcal{F}}$ with decomposition $\Gamma = \cup_{E \in B(C)} \Gamma^E$. Then, for each $E \in B(C)$, we have that*

(i) *If E is a contact divisor, the curve Γ^E has $b_E - 1$ irreducible components. Each branch γ of Γ^E with characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E, \quad \nu_l^\gamma = \underline{n}_E \beta_l^i/\beta_0^i, \quad l = 1, 2, \ldots, k_E,$$

for any $i \in I_E$.

(ii) *If E is a Puiseux divisor which belongs to a dead arc, the curve Γ^E has one irreducible component γ_0 with characteristic exponents $\{\nu_0^{\gamma_0}, \nu_1^{\gamma_0}, \ldots, \nu_{k_E}^{\gamma_0}\}$ given by*

$$\nu_0^{\gamma_0} = m_0(\gamma_0) = \underline{n}_E, \quad \nu_l^{\gamma_0} = \underline{n}_E \beta_l^i/\beta_0^i, \quad l = 1, 2, \ldots, k_E,$$

and $b_E - 2$ irreducible components such that each branch $\zeta \subset \Gamma^E \setminus \gamma_0$ has characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma, \nu_{k_E+1}^\gamma\}$ given by

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E n_E, \quad \nu_l^\gamma = \underline{n}_E n_E \beta_l^i/\beta_0^i, \quad l = 1, 2, \ldots, k_E + 1,$$

for any $i \in I_E^*$.

(iii) *If E is a Puiseux divisor which does not belong to a dead arc, then Γ^E has $b_E - 1$ irreducible components. Each irreducible component γ of Γ^E with characteristic exponents $\{\nu_0^\gamma, \nu_1^\gamma, \ldots, \nu_{k_E}^\gamma, \nu_{k_E+1}^\gamma\}$ given by*

$$\nu_0^\gamma = m_0(\gamma) = \underline{n}_E n_E, \quad \nu_l^\gamma = \underline{n}_E n_E \beta_l^i / \beta_0^i, \quad l = 1, 2, \ldots, k_E + 1,$$

for any $i \in I_E^*$.

The last part of the section is devoted to prove some technical lemmas which will be useful in the sequel. The first one is a general result concerning intersection multiplicities of polar curves:

Lemma 7. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C$ and let Γ, Γ' be any two generic curves of $\Lambda_{\mathcal{F}}$. For any irreducible component γ of Γ, we have that*

$$(\Gamma', \gamma)_0 + m_0(\gamma) = (C, \gamma)_0. \tag{3}$$

Proof. — Consider a 1-form $\omega = A(x,y)dx + B(x,y)dy$ which defines $\mathcal{F}$ and assume that $\Gamma = \Gamma_{[a:b]}$, $\Gamma' = \Gamma_{[a':b']}$. Take an irreducible component γ of $\Gamma_{[a:b]}$ and let $\phi_\gamma(t) = (x_\gamma(t), y_\gamma(t))$ be a parametrization of γ. Since $\mathcal{F}$ is a generalized curve foliation, then

$$(C, \gamma)_0 = \mathrm{ord}_t(\phi_\gamma^* \omega) + 1$$

(see [**13**], lemma 3.7). The intersection multiplicity $(\Gamma_{[a':b']}, \gamma)_0$ is given by

$$(\Gamma_{[a':b']}, \gamma)_0 = \mathrm{ord}_t\{a' A(\phi_\gamma(t)) + b' B(\phi_\gamma(t))\}.$$

Moreover, since γ is an irreducible component of $\Gamma_{[a:b]}$, then $aA(\phi_\gamma(t)) + bB(\phi_\gamma(t)) \equiv 0$. Assume that $a \neq 0$, a similar argument holds if $b \neq 0$. In this case, we have that either $\mathrm{ord}_t(A(\phi_\gamma(t))) = \mathrm{ord}_t(B(\phi_\gamma(t)))$ when $b \neq 0$ or $A(\phi_\gamma(t)) \equiv 0$ otherwise. In both situations, the following equalities to compute $\mathrm{ord}_t(\phi_\gamma^*\omega)$ hold:

$$\begin{aligned}
\mathrm{ord}_t(\phi_\gamma^*\omega) &= \mathrm{ord}_t\{A(\phi_\gamma(t))\,\dot{x}_\gamma(t) + B(\phi_\gamma(t))\,\dot{y}_\gamma(t)\} \\
&= \mathrm{ord}_t\left\{-\frac{b}{a}B(\phi_\gamma(t))\,\dot{x}_\gamma(t) + B(\phi_\gamma(t))\,\dot{y}_\gamma(t)\right\} \\
&= \mathrm{ord}_t(B(\phi_\gamma(t))) + \mathrm{ord}_t(-b\dot{x}_\gamma(t) + a\dot{y}_\gamma(t)) \\
&= \mathrm{ord}_t(a'A(\phi_\gamma(t)) + b'B(\phi_\gamma(t))) + (\gamma, -bx + ay = 0)_0 - 1 \\
&= (\Gamma_{[a':b']}, \gamma)_0 + (\gamma, \ell_{[a:b]})_0 - 1,
\end{aligned}$$

where $\ell_{[a:b]}$ is the line given by $-bx + ay = 0$. In particular, this implies that the formula (3) holds for all $[a:b]$ such that $\ell_{[a:b]}$ is not tangent to $\Gamma_{[a:b]}$ which is the case when $\Gamma_{[a:b]}$ is a generic curve of $\Lambda_{\mathcal{F}}$. □

Let us introduce some notations in order to simplify the proofs of the following lemmas. Given a bifurcation divisor E of $G(C)$, we denote

$$d_E^1 = \begin{cases} b_E & \text{if } E \text{ is a contact divisor;} \\ 1, & \text{if } E \text{ is a Puiseux divisor which does not belong to a dead arc;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$d_E^2 = \begin{cases} 0, & \text{if } E \text{ is a contact divisor;} \\ b_E - 1, & \text{otherwise.} \end{cases}$$

Hence, if Γ is a generic curve of $\Lambda_{\mathcal{F}}$ with decomposition $\Gamma = \cup_{E\in B(C)}\Gamma^E$, then $m_0(\Gamma^E) = \underline{n}_E(d_E^1 + d_E^2 n_E - 1)$.

Lemma 8. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and a generic curve $\Gamma = \cup_{E\in B(C)}\Gamma^E$ of $\Lambda_{\mathcal{F}}$. Then, for each bifurcation divisor E of $G(C)$, we have that*

$$m_0(\bigcup_{i\in I_E} C_i) - m_0(\bigcup_{E'>E} \Gamma^{E'}) = \underline{n}_E(d_E^1 + n_E d_E^2). \tag{4}$$

Proof. — Let ℓ_E be the size of the largest chain of divisors in $B(C)$ starting at E. We prove the lemma by induction on ℓ_E. If $\ell_E = 1$, then E is a maximal bifurcation divisor of $G(C)$. In this case, the equality (4) turns into

$$m_0(\cup_{i\in I_E} C_i) = \underline{n}_E(d_E^1 + n_E d_E^2)$$

and it can be directly deduced from the properties of $G(C)$. Assume now that $\ell_E > 1$ and let $E_1, \ldots, E_s$ be the bifurcation vertices of $G(C)$ which are consecutive to E, that is, $E < E_i$ without any other bifurcation divisor between E and E_i. Put $J_E = I_E \setminus \cup_{i=1}^s I_{E_i}$ and $t = \sharp J_E$. Note that $t + s = d_E^1 + d_E^2$. Then we have the following equalities

$$\begin{aligned} m_0(\bigcup_{i\in I_E} C_i) - m_0(\bigcup_{E'>E}\Gamma^{E'}) = \\ &= \sum_{j\in J_E} m_0(C_j) + \sum_{i=1}^{s} m_0(\bigcup_{j\in I_{E_i}} C_j) - \left[\sum_{i=1}^{s} m_0(\bigcup_{E'>E_i}\Gamma^{E'}) + \sum_{i=1}^{s} m_0(\Gamma^{E_i})\right] \\ &= \sum_{i\in J_E} m_0(C_i) + \sum_{i=1}^{s}\left[m_0(\bigcup_{j\in I_{E_i}} C_j) - m_0(\bigcup_{E'>E_i}\Gamma^{E'})\right] - \sum_{i=1}^{s} m_0(\Gamma^{E_i}). \end{aligned}$$

For each $i = 1, \ldots, s$, we have that $m_0(\cup_{j\in I_{E_i}} C_j) - m_0(\cup_{E'>E_i}\Gamma^{E'}) = \underline{n}_{E_i}(d_{E_i}^1 + d_{E_i}^2 n_{E_i})$ by the induction hypothesis and $m_0(\Gamma^{E_i}) = \underline{n}_{E_i}(d_{E_i}^1 + d_{E_i}^2 n_{E_i} - 1)$ by theorem 4. Hence, we deduce that

$$m_0(\bigcup_{i\in I_E} C_i) - m_0(\bigcup_{E'>E}\Gamma^{E'}) = \sum_{j\in J_E} m_0(C_j) + \sum_{i=1}^{s} \underline{n}_{E_i}.$$

Now three situations may happen:

- If E is a contact divisor, then $n_E = 1$, $\underline{n}_{E_i} = \underline{n}_E$ for $i = 1, \ldots, s$ and $m_0(C_j) = \underline{n}_E$ for $j \in J_E$. Moreover, $d_E^2 = 0$ and $t + s = d_E^1$.
- If E is a Puiseux divisor which belongs to a dead arc, then $\underline{n}_{E_i} = \underline{n}_E n_E$ with $n_E > 1$ for each $i = 1, \ldots, s$ and $m_0(C_j) = \underline{n}_E n_E$ for $j \in J_E$. In this case, $d_E^1 = 0$ and $t + s = d_E^2$.

– If E is a Puiseux divisor without dead arc, then $d_E^1 = 1$ and $t + s - 1 = d_E^2$. Moreover $n_E > 1$ and there is:
  - either a divisor E_{i_0}, with $i_0 \in \{1, \dots, s\}$, such that $\underline{n}_{E_{i_0}} = \underline{n}_E$ and $\underline{n}_{E_i} = \underline{n}_E n_E$ for $i \neq i_0$; in this situation $m_0(C_j) = \underline{n}_E n_E$ for all $j \in J_E$.
  - or a curve C_{j_0} with $j_0 \in J_E$ such that $m_0(C_{j_0}) = \underline{n}_E$ and $m_0(C_j) = \underline{n}_E n_E$ if $j \neq j_0$; in this case $\underline{n}_{E_i} = \underline{n}_E n_E$ for all $i \in \{1, \dots, s\}$.

It follows that $\sum_{j \in J_E} m_0(C_j) + \sum_{i=1}^{s} \underline{n}_{E_i} = \underline{n}_E(d_E^1 + d_E^2 n_E)$ and the result is straightforward. □

Take a bifurcation divisor E of $G(C)$. Let $F_1 < F_2 < \cdots < F_m < F_{m+1} = E$ be the bifurcation vertices in the geodesic of E in $G(C)$ and denote $\mathcal{B}_i = \{E' \in B(C) : E' \geq F_i\}$. Then we have the following result

Lemma 9. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and let Γ, Υ be two generic curves of $\Lambda_{\mathcal{F}}$ with decompositions $\Gamma = \cup_{E \in B(C)} \Gamma^E$ and $\Upsilon = \cup_{E \in B(C)} \Upsilon^E$. Let γ be an irreducible component of $\Gamma^E \subset \Gamma$. Denote by $\{\nu_0^\gamma, \nu_1^\gamma, \dots, \nu_{g_\gamma}^\gamma\}$ the characteristic exponents of γ, by $\{(m_i^\gamma, n_i^\gamma)\}_{i=1}^{g_\gamma}$ the Puiseux pairs of γ and by $\{\bar{\nu}_0^\gamma, \bar{\nu}_1^\gamma, \dots, \bar{\nu}_{g_\gamma}^\gamma\}$ the minimal set of generators of the semigroup of γ. Then we have that*

$$\sum_{l=1}^{m} \left[\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 \right] = \nu_{k_E}^\gamma - n_{k_E}^\gamma \bar{\nu}_{k_E}^\gamma. \tag{5}$$

Proof. — By the properties of the decomposition of a generic curve of $\Lambda_{\mathcal{F}}$ given in theorem 4, we have that:

- $\mathcal{C}(C_i, \gamma) = v(F_l)$ if $i \in I_{F_l} \setminus I_{F_{l+1}}$;
- $\mathcal{C}(\zeta^{E'}, \gamma) = v(F_l)$ if $\zeta^{E'}$ is an irreducible component of $\Upsilon^{E'}$ with $E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}$.

For each $l \in \{1, \dots, m\}$, let $t(l)$ be the integer in $\{0, 1, \dots, g_\gamma\}$ such that

$$\nu_{t(l)}^\gamma \leq m_0(\gamma) v(F_l) < \nu_{t(l)+1}^\gamma$$

($\nu_{g_\gamma+1}^\gamma = +\infty$). Note that $t(l) \leq k_E \leq g_\gamma$ for $l = 1, \dots, m$ and $t(m) = k_E$. We use now the relationship between the coincidence and the intersection multiplicity given in lemma 2 to compute $(C_i, \gamma)_0$ and $(\zeta^{E'}, \gamma)_0$. We have that

$$\frac{(C_i, \gamma)_0}{m_0(C_i)} = \frac{\bar{\nu}_{t(l)}^\gamma \cdot n_{t(l)}^\gamma + m_0(\gamma) v(F_l) - \nu_{t(l)}^\gamma}{n_1^\gamma \cdots n_{t(l)}^\gamma}, \text{ for } i \in I_{F_l} \setminus I_{F_{l+1}},$$

and

$$\frac{(\zeta^{E'}, \gamma)_0}{m_0(\zeta^{E'})} = \frac{\bar{\nu}_{t(l)}^\gamma \cdot n_{t(l)}^\gamma + m_0(\gamma) v(F_l) - \nu_{t(l)}^\gamma}{n_1^\gamma \cdots n_{t(l)}^\gamma},$$

for each irreducible component $\zeta^{E'}$ of $\Upsilon^{E'}$ with $E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}$, Consequently, we obtain that

$$\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 =$$
$$\left(\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} m_0(C_i) - \sum_{E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}} m_0(\Upsilon^{E'}) \right) \frac{\bar{\nu}^{\gamma}_{t(l)} \cdot n^{\gamma}_{t(l)} + m_0(\gamma) v(F_l) - \nu^{\gamma}_{t(l)}}{n^{\gamma}_1 \cdots n^{\gamma}_{t(l)}}.$$

By lemma 8, we have that

$$\sum_{i \in I_{F_l}} m_0(C_i) - \sum_{E' \in \mathcal{B}_l} m_0(\Upsilon^{E'}) = \underline{n}_{F_l}(d^1_{F_l} + d^2_{F_l} n_{F_l}) - m_0(\Upsilon^{F_l}) = \underline{n}_{F_l},$$

and hence it follows that

$$\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} m_0(C_i) - \sum_{E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}} m_0(\Upsilon^{E'}) = \underline{n}_{F_l} - \underline{n}_{F_{l+1}} = \underline{n}_{F_l}(1 - n_{F_l}).$$

By definition n_{F_l} is given by

$$n_{F_l} = \begin{cases} 1, & \text{if } F_l \text{ is a contact divisor;} \\ n^{\gamma}_{t(l)}, & \text{if } F_l \text{ is a Puiseux divisor.} \end{cases}$$

Moreover, $m_0(\gamma) v(F_l) = \nu^{\gamma}_{t(l)}$ and $\underline{n}_{F_l} = n_1 \cdots n_{t(l)-1}$ if F_l is a Puiseux divisor. Therefore, we deduce that

$$\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 = \begin{cases} 0, & \text{if } F_l \text{ is a contact divisor;} \\ (1 - n^{\gamma}_{t(l)}) \bar{\nu}^{\gamma}_{t(l)}, & \text{if } F_l \text{ is a Puiseux divisor.} \end{cases}$$

To finish the proof we use the relationship between the characteristic exponents of γ and the minimal system of generators of the semigroup $S(\gamma)$ given in equation (2). The following computations complete the proof:

$$\sum_{i=1}^{m} \left[\sum_{i \in I_{F_l} \setminus I_{F_{l+1}}} (C_i, \gamma)_0 - \sum_{E' \in \mathcal{B}_l \setminus \mathcal{B}_{l+1}} (\Upsilon^{E'}, \gamma)_0 \right] = \sum_{j=1}^{k_E} (1 - n^{\gamma}_j) \bar{\nu}^{\gamma}_j$$
$$= \bar{\nu}^{\gamma}_1 - n^{\gamma}_{k_E} \bar{\nu}^{\gamma}_{k_E} + \sum_{j=1}^{k_E - 1} (\bar{\nu}^{\gamma}_{j+1} - n^{\gamma}_j \bar{\nu}^{\gamma}_j) = \bar{\nu}^{\gamma}_1 - n^{\gamma}_{k_E} \bar{\nu}^{\gamma}_{k_E} + \sum_{j=1}^{k_E - 1} (\nu^{\gamma}_{j+1} - \nu^{\gamma}_j)$$
$$= \bar{\nu}^{\gamma}_1 - n^{\gamma}_{k_E} \bar{\nu}^{\gamma}_{k_E} + \nu^{\gamma}_{k_E} - \nu^{\gamma}_1 = \nu^{\gamma}_{k_E} - n^{\gamma}_{k_E} \bar{\nu}^{\gamma}_{k_E}.$$ □

4. Base points of the polar pencil

Consider a morphism $\sigma : N \to (\mathbb{C}^2, 0)$ composition of a finite number of punctual blow-ups. A point $p \in \sigma^{-1}(0)$ is a *base point* of the pencil $\Lambda_{\mathcal{F}}$ if p is an infinitely near point of each generic curve of $\Lambda_{\mathcal{F}}$. More precisely, p is a base point of $\Lambda_{\mathcal{F}}$ if and only if, there is an irreducible component γ of Γ such that $\sigma^*\gamma \cap \sigma^{-1}(0) = \{p\}$, for each generic fiber Γ of $\Lambda_{\mathcal{F}}$.

A first property concerning the resolution of singularities of the polar foliation, and hence of the polar pencil, is the property of "separation of the separatrices" (see [**12**]). Let Π be a morphism which is a partial reduction of $\mathcal{P}_{\mathcal{F}}$ and also a reduction of singularities of $\mathcal{F}$. We say that $\mathcal{F}$ satisfies the *property of separation of the separatrices* if the geodesic in $G(\Pi)$ of any separatrix of $\mathcal{F}$ does not go through a dicritical component of $\mathcal{P}_{\mathcal{F}}$, except maybe E_1. We proved [**5**] that the foliations in $\mathbb{G}_C^*$ satisfy the property of separation of the separatrices. From this property we can deduce the following result:

Lemma 10. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and take any generic curve Γ of $\Lambda_{\mathcal{F}}$. If E is a bifurcation divisor of $G(C)$, $E \neq E_1$, then the points $\pi_C^*\Gamma \cap E$ are base points of the polar pencil $\Lambda_{\mathcal{F}}$.*

Proof. — The result is a direct consequence of the property of separation of the separatrices since E cannot be a dicritical component and hence the points of the set $\pi_C^*\Gamma \cap E$ are base points of $\Lambda_{\mathcal{F}}$. □

Remark 11. — Note that, if E_1 is a bifurcation divisor, the points $\pi_C^*\Gamma \cap E_1$ are not base points of the polar pencil. In fact, if $\Gamma = \Gamma_{[a:b]}$, then the set $\pi_C^*\Gamma_{[a:b]} \cap E_1$ has exactly $b_{E_1} - 1$ points which depend on $[a : b]$ (see [**7**]).

Let $\sigma_{\Lambda,C} : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$ be the minimal reduction of singularities of $\Lambda_{\mathcal{F}}$ that factorizes by π_C. The next result describes how to construct $\sigma_{\Lambda,C}$ from π_C.

Proposition 12. — *Assume that C is a curve with kind equisingularity type and let $\mathcal{F} \in \mathbb{G}_C^*$ be a Zariski-general foliation. There is a morphism $\sigma_1 : M_{\Lambda,C} \to M_C$ composition of a finite number of punctual blow-ups such that $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$. Moreover, the centers of the blow-ups to obtain σ_1 are not singular points of $\pi_C^*\mathcal{F}$.*

Proof. — Let Γ, Γ' be two generic curves of $\Lambda_{\mathcal{F}}$. If the morphism π_C is also a reduction of singularities of $\Gamma \cup \Gamma'$, we take $\sigma_1 : M_C \to M_C$ to be the identity map id_{M_C} on M_C and hence $\sigma_{\Lambda,C} = \pi_C$. Otherwise, let $\{R_1, \ldots, R_s\}$ be the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$; observe that these points are not singular points of $\pi_C^*\mathcal{F}$ since π_C is a reduction of singularities of $C \cup \Gamma$. By theorem 5, there is a unique irreducible component γ_i of Γ such that $\pi_C^*\gamma_i$ cuts transversally $\pi_C^{-1}(0)$ at R_i for $i = 1, \ldots, s$. Moreover, a point R_i belongs either to a bifurcation divisor of $G(C)$ or to the terminal divisor of a dead arc in $G(C)$. There are three possible situations:

- If R_i belongs to E_1, then R_i is not a base point of $\Lambda_{\mathcal{F}}$ by remark 11.
- If R_i belongs to a bifurcation divisor E, $E \neq E_1$, then R_i is a base point of $\Lambda_{\mathcal{F}}$ by lemma 10. Hence, there is a unique irreducible component γ_i' of Γ' such that $\pi_C^*\gamma_i' \cap E = \{R_i\}$ by theorem 5.
- If R_i belongs to the terminal divisor E of a dead arc, then there is a unique irreducible component γ_i' of Γ' such that $\pi_C^*\gamma_i' \cap E \neq \varnothing$. In this case, the point

R_i can be either a base point or not. If it is a base point, then $\pi_C^*\gamma_i' \cap E = \pi_C^*\gamma_i \cap E = \{R_i\}$. Otherwise, $\pi_C^*\gamma_i' \cap E \neq \{R_i\}$ and E is a dicritical component for $\Lambda_{\mathcal{F}}$.

Put $X_1 = M_C$ and consider the morphism $\tau_i : (X_{i+1}, R_{i+1}) \to (X_i, R_i)$, for $i = 1, \ldots, s$, defined by

- $\tau_i = id_{X_i}$ if R_i is not a base point of $\Lambda_{\mathcal{F}}$;
- τ_i is the minimal reduction of singularities of the strict transform of $\pi_C^*\gamma_i \cup \pi_C^*\gamma_i'$ by $\tau_1 \circ \tau_2 \circ \cdots \circ \tau_{i-1}$ when R_i is a base point of $\Lambda_{\mathcal{F}}$.

The morphism $\sigma_1 : X_{s+1} \to M_C$ with $\sigma_1 = \tau_1 \circ \cdots \circ \tau_s$ fulfils the requirements of the statement because $\pi_C \circ \sigma_1$ is a reduction of singularities of $\Gamma \cup \Gamma'$. Moreover, it is clear by construction that $\pi_C \circ \sigma_1$ is the minimal resolution of $\Lambda_{\mathcal{F}}$ which factorizes by π_C; hence $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1 : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$ with $M_{\Lambda,C} = X_{s+1}$. □

5. Dicritical components

In this section we give some characteristics of the dicritical components which appear in a resolution of singularities of $\Lambda_{\mathcal{F}}$. Note that the degree and the valence $v(D)$ of a dicritical component D do not depend on the choice of the resolution. Hence to determine these values it is enough to consider the morphism $\sigma_{\Lambda,C} : M_{\Lambda,C} \to (\mathbb{C}^2, 0)$. Next lemma gives the degree of the dicritical components

Lemma 13. — *Consider a foliation $\mathcal{F} \in \mathbb{G}_C^*$ and let $\sigma : X \to (\mathbb{C}^2, 0)$ be any resolution of singularities of $\Lambda_{\mathcal{F}}$. Then*

1. *The divisor E_1 of $G(C)$ is dicritical for $\Lambda_{\mathcal{F}}$ if and only if $b_{E_1} \geq 2$. Moreover, in that case, the degree of E_1 as a dicritical component of $\Lambda_{\mathcal{F}}$ is equal to $b_{E_1} - 1$.*
2. *If $\mathcal{F}$ is a Zariski-general foliation, each dicritical component D of $\sigma^{-1}(0)$, $D \neq E_1$, has degree equal to 1.*

Proof. — The first assertion is a direct consequence of remark 11. The second one follows straightforward from the construction of the morphism $\sigma_{\Lambda,C}$ given in proposition 12. □

Next result determines the valence $v(D)$ of a dicritical component D of $\Lambda_{\mathcal{F}}$ in terms of the data in $G(C)$. It is a key result in the proof of theorem 1.

Theorem 14. — *Let $\mathcal{F} \in \mathbb{G}_C^*$ be a Zariski-general foliation and let $\sigma : X \to (\mathbb{C}^2, 0)$ be any resolution of singularities of the polar pencil $\Lambda_{\mathcal{F}}$. Given any dicritical component D of $\sigma^{-1}(0)$ and any D-curvette γ, we have that*

$$v(D) = 2 \sup_{1 \leq i \leq r} \{\mathcal{C}(C_i, \gamma)\} - 1. \tag{6}$$

If Γ, Υ are two generic curves of $\Lambda_{\mathcal{F}}$, then $v(D)$ is equal to $\mathcal{C}(\gamma_D, \zeta_D)$ where γ_D, ζ_D are irreducible components of Γ and Υ respectively such that $\sigma^*\gamma_D \cap D \neq \varnothing$ and $\sigma^*\zeta_D \cap D \neq \varnothing$. Moreover, if we denote by E_D the bifurcation divisor of $G(C)$ such that γ_D is a branch of the curve Γ^{E_D} of the decomposition of Γ (and also $\zeta_D \subset \Upsilon^{E_D}$), then $\sup_{1\leq i\leq r}\{\mathcal{C}(C_i,\gamma_D)\} = v(E_D)$. Consequently, equation (6) can be written as follows

$$v(D) = 2v(E_D) - 1. \tag{7}$$

Proof of theorem 14. — Consider two generic curves Γ, Υ of $\Lambda_{\mathcal{F}}$ with decompositions given by $\Gamma = \cup_{E\in B(C)}\Gamma^E$ and $\Upsilon = \cup_{E\in B(C)}\Upsilon^E$. Let D be a dicritical component of $\sigma^{-1}(0)$. If D is equal to the first divisor E_1 of $G(C)$, then $E_D = E_1$ and equation (6) is held. Assume now that $D \neq E_1$. Let γ, ζ be irreducible components of Γ and Υ respectively, with $\sigma^*\gamma \cap D \neq \varnothing$ and $\sigma^*\zeta \cap D \neq \varnothing$; note that they are unique by lemma 13 and $m_0(\gamma) = m_0(\zeta)$. Let us compute $(\gamma,\zeta)_0$. By lemma 7, we have that

$$(\Upsilon^{E_D},\gamma)_0 + \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 + m_0(\gamma) = \sum_{i=1}^{r}(C_i,\gamma)_0. \tag{8}$$

The intersection multiplicity $(\Upsilon^{E_D},\gamma)_0$ can be computed using the decomposition of Υ^{E_D} into irreducible components:

$$(\Upsilon^{E_D},\gamma)_0 = (\gamma,\zeta)_0 + \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0. \tag{9}$$

From equalities (8) and (9) we deduce that $(\gamma,\zeta)_0$ is given by

$$\begin{aligned}(\gamma,\zeta)_0 &= \sum_{i=1}^{r}(C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 - m_0(\gamma)\\ &= \sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 + \sum_{i\notin I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{E\in B(C)\\ E\neq E_D}} (\Upsilon^E,\gamma)_0 - m_0(\gamma).\end{aligned}$$

Denote by $F_1 < F_2 < \cdots < F_m < F_{m+1} = E_D$ the bifurcation vertices in the geodesic of E_D in $G(C)$ and put $\mathcal{B}_i = \{E' \in B(C) \ : \ E' \geq F_i\}$ for $i = 1,\ldots,m$. Thus we have that

$$\begin{aligned}(\gamma,\zeta)_0 = &\sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E> E_D}} (\Upsilon^E,\gamma)_0\\ &+\sum_{i=1}^{m}\left[\sum_{i\in I_{F_l}\setminus I_{F_{l+1}}} (C_i,\gamma)_0 - \sum_{E\in\mathcal{B}_l\setminus\mathcal{B}_{l+1}} (\Upsilon^E,\gamma)_0\right] - m_0(\gamma).\end{aligned}$$

We shall use lemmas 8 and 9 to compute the right side of the equality above. Note that

- $\mathcal{C}(C_i,\gamma) = v(E_D)$ for each $i \in I_{E_D}$, by theorem 4.
- $\mathcal{C}(\zeta',\gamma) = v(E_D)$ for each branch ζ' of Υ^E, with $E > E_D$.
- $\mathcal{C}(\zeta',\gamma) = v(E_D)$ for each branch ζ' of Υ^{E_D}, with $\zeta' \neq \zeta$, by theorem 5, since $\mathcal{F}$ is a Zariski-general foliation.

Let $\{\nu_0^\gamma, \nu_1^\gamma, \dots, \nu_{g_\gamma}^\gamma\}$ be the characteristic exponents of γ, $\{(m_i^\gamma, n_i^\gamma)\}_{i=1}^{g_\gamma}$ the Puiseux pairs of γ and $\{\bar\nu_0^\gamma, \bar\nu_1^\gamma, \dots, \bar\nu_{g_\gamma}^\gamma\}$ the minimal set of generators of the semigroup $S(\gamma)$ of γ. From lemma 6, we deduce that $\nu_{g_\gamma}^\gamma \leq m_0(\gamma)v(E_D)$. Consequently, applying lemmas 2 and 8, we get that

$$\sum_{i\in I_{E_D}} (C_i,\gamma)_0 - \sum_{\substack{\zeta'\subset\Upsilon^{E_D}\\ \zeta'\neq\zeta}} (\zeta',\gamma)_0 - \sum_{\substack{E\in B(C)\\ E>E_D}} (\Upsilon^E,\gamma)_0 =$$

$$= \left(\sum_{i\in I_{E_D}} m_0(C_i) - \sum_{E>E_D} m_0(\Upsilon^E) - m_0(\Upsilon^{E_D}\setminus\zeta)\right)\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)}$$

$$= (\underline{n}_{E_D} + m_0(\zeta))\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)}.$$

We use now the equality above and the result given in lemma 9 to compute $(\gamma,\zeta)_0$. We obtain that

$$(\gamma,\zeta)_0 = ((\underline{n}_{E_D} + m_0(\zeta))\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma}{m_0(\gamma)} + \nu_{k_{E_D}}^\gamma - n_{k_{E_D}}^\gamma \bar\nu_{k_{E_D}}^\gamma - m_0(\gamma).$$

To finish the computation of $(\gamma,\zeta)_0$ we consider the different possibilities for the bifurcation divisor E_D and we use the expression of the characteristic exponents of the irreducible components of the generic curves of $\Lambda_{\mathcal{F}}$ given in lemma 6.

- If E is a contact divisor, then $m_0(\gamma) = m_0(\zeta) = \underline{n}_{E_D} = n_1^\gamma \cdots n_{g_\gamma}^\gamma$ with $g_\gamma = k_{E_D}$. Then

$$\begin{aligned}(\gamma,\zeta)_0 &= 2[\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)v(E_D) - \nu_{g_\gamma}^\gamma] + \nu_{g_\gamma}^\gamma - \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma - m_0(\gamma)\\ &= 2m_0(\gamma)v(E_D) + \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma - \nu_{g_\gamma}^\gamma - m_0(\gamma).\end{aligned}$$

  Moreover, by lemma 2, the relationship between $(\gamma,\zeta)_0$ and $\mathcal{C}(\gamma,\zeta)_0$ is given by $(\gamma,\zeta)_0 = \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)\mathcal{C}(\gamma,\zeta) - \nu_{g_\gamma}^\gamma$. Taking into account that $\mathcal{C}(\gamma,\zeta) = v(D)$, we conclude that

$$v(D) = 2v(E_D) - 1.$$

- Assume now that E_D is a Puiseux divisor which belongs to a dead arc. By lemma 6, the multiplicity $m_0(\gamma)$ can be either $\underline{n}_{E_D}$ or $\underline{n}_{E_D}n_{E_D}$ with $n_{E_D} > 1$. If $m_0(\gamma) = \underline{n}_{E_D}$, the same computations as in the previous case give the result. Consider now the case $m_0(\gamma) = \underline{n}_{E_D}n_{E_D}$. Thus we have that $m_0(\gamma)v(E_D) = \nu_{g_\gamma}^\gamma$, $g_\gamma = k_{E_D} + 1$ and $n_{E_D} = n_{g_\gamma}^\gamma$. Hence we get that

$$\begin{aligned}(\gamma,\zeta)_0 &= [\underline{n}_{E_D} + \underline{n}_{E_D}n_{E_D}]\frac{\bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma}{\underline{n}_{E_D}n_{E_D}} + \nu_{g_\gamma-1}^\gamma - \bar\nu_{g_\gamma-1}^\gamma n_{g_\gamma-1}^\gamma - m_0(\gamma)\\ &= (1+n_{g_\gamma})\bar\nu_{g_\gamma}^\gamma + \nu_{g_\gamma}^\gamma - \bar\nu_{g_\gamma}^\gamma - m_0(\gamma) = n_{g_\gamma}^\gamma\bar\nu_{g_\gamma}^\gamma + \nu_{g_\gamma}^\gamma - m_0(\gamma).\end{aligned}$$

  By lemma 2, we have that $(\gamma,\zeta)_0 = \bar\nu_{g_\gamma}^\gamma n_{g_\gamma}^\gamma + m_0(\gamma)\mathcal{C}(\gamma,\zeta) - \nu_{g_\gamma}^\gamma$. We obtain that

$$\mathcal{C}(\gamma,\zeta) = 2\frac{\nu_{g_\gamma}^\gamma}{m_0(\gamma)} - 1 = 2v(E_D) - 1.$$

– If E_D is a Puiseux divisor which does not belong to a dead arc, then $m_0(\gamma)v(E_D) = \nu^{\gamma}_{g_\gamma}$, $g_\gamma = k_{E_D} + 1$ and $n_{E_D} = n^{\gamma}_{g_\gamma}$. Hence the computations in the previous case give the result. □

6. Resolution of singularities

In this section we give the proof of the main result of the paper and some consequences than can be deduced from it.

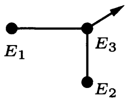
Proof of theorem 1. — In proposition 12 we have shown that $\sigma_{\Lambda,C}$ is obtained from π_C by a finite number of punctual blow-ups with centers at non-singular points of $\pi_C^*\mathcal{F}$. Recall that $\sigma_{\Lambda,C} = \pi_C \circ \sigma_1$, where σ_1 is obtained by blowing-up following the infinitely near points of the irreducible components of a generic curve Γ of $\Lambda_{\mathcal{F}}$. Moreover, since $\pi_C^*\Gamma$ is non-singular, then the centers of the blow-ups to get σ_1 are free infinitely near points of Γ.

Let $\{R_1, \ldots, R_s\}$ be the points of the set $\pi_C^*\Gamma \cap \pi_C^{-1}(0)$. By theorem 5, there is a unique irreducible component γ_i of Γ such that $\pi_C^*\gamma_i$ cuts transversally $\pi_C^{-1}(0)$ at R_i for $i = 1, \ldots, s$. Let D_i be the dicritical component of $\sigma_{\Lambda,C}^{-1}(0)$ such that $\sigma_{\Lambda,C}^*\gamma_i \cap D_i \neq \emptyset$ and denote by E_{R_i} the irreducible component of $\pi_C^{-1}(0)$ such that $\pi_C^*\gamma_i \cap E_{R_i} = \{R_i\}$. Note that it is possible that $E_{R_i} = E_{R_j}$ for $i \neq j$. Moreover, E_{R_i} is either a bifurcation divisor of $G(C)$ or the terminal divisor of a dead arc in $G(C)$.

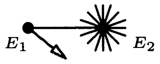
Let $\alpha_i = \alpha_{E_{R_i}}$ be the number of blow-ups needed to obtain D_i from E_{R_i}. Let us show that the value of α_i is given by equation (1). We consider separately the different possibilities for E_{R_i}:

– E_{R_i} is the first divisor E_1 of $\pi_C^{-1}(0)$, then it is a dicritical component for $\Lambda_{\mathcal{F}}$. Hence, $\alpha_i = 0$ and the equality $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1)$ holds since $v(E_1) = 1$.
– E_{R_i} is a bifurcation divisor different from E_1, then R_i is a base point of $\Lambda_{\mathcal{F}}$. The valuation $v(D_i)$ is equal to
$$v(D_i) = \frac{m(E_{R_i})v(E_{R_i}) + \alpha_i}{m(E_{R_i})}.$$
By theorem 14, we have that $v(D_i) = 2v(E_{R_i}) - 1$. Hence, we deduce that $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1)$.
– E_{R_i} is the terminal divisor of a dead arc with bifurcation divisor E. Using the fact that C has a kind equisingularity type, we get that
$$m(E_{R_i}) = m(E)/2; \quad v(E_{R_i}) = (m(E)v(E) + 1)/m(E). \tag{10}$$
By theorem 14, we have that $v(D_i) = 2v(E) - 1$. Thus we obtain the following equality
$$\frac{m(E_{R_i})v(E_{R_i}) + \alpha_i}{m(E_{R_i})} = \frac{2m(E_{R_i})v(E_{R_i}) - 1}{m(E_{R_i})} - 1,$$
and we conclude that $\alpha_i = m(E_{R_i})(v(E_{R_i}) - 1) - 1$. □

Note that, in general, the minimal resolution of singularities σ_Λ of $\Lambda_\mathcal{F}$ is not a reduction of singularities of the foliation $\mathcal{F}$. Consider, for instance, the foliation $\mathcal{F}$ given by $d(y^2 - x^3) = 0$. The generic curves of $\Lambda_\mathcal{F}$ are the parabolas $\{2by - 3ax^2 = 0\}$; the minimal resolution of singularities σ_Λ of $\Lambda_\mathcal{F}$ is a composition of two blow-ups whereas the separatrix of $\mathcal{F}$ is a $(3,2)$-cusp. The dual graphs $G(C)$ and $G(\sigma_\Lambda, \Lambda_\mathcal{F})$ are given by



$G(C)$



$G(\sigma_\Lambda, \Lambda_\mathcal{F})$

Next result characterizes the curves C such that $\sigma_{\Lambda,C}$ coincides with the minimal reduction of singularities of $\Lambda_\mathcal{F}$.

Corollary 15. — *Let C be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}_C^*$. The following statements are equivalent:*

1. *The morphism $\sigma_{\Lambda,C}$ is the minimal resolution of singularities of $\Lambda_\mathcal{F}$.*
2. *There is no maximal bifurcation divisor of $G(C)$ which belongs to the geodesic of only one irreducible component of C.*

Proof. — Let $\Gamma = \cup_{E \in B(C)} \Gamma^E$ be a generic curve of $\Lambda_\mathcal{F}$. Assume that $\sigma_{\Lambda,C}$ is the minimal resolution of singularities of $\Lambda_\mathcal{F}$. If there is a maximal bifurcation vertex E of $G(C)$ which belongs to a dead arc and with $b_E = 2$, then Γ^E is irreducible and Γ^E cuts the terminal divisor F of the dead arc starting at E (by theorem 5). Hence, π_C is not the minimal reduction of singularities of Γ and consequently $\sigma_{\Lambda,C}$ cannot be the minimal resolution of $\Lambda_\mathcal{F}$.

Assume now that $G(C)$ satisfies the conditions in the second statement. This implies that, for each maximal bifurcation divisor E of $G(C)$, there is an irreducible component γ of Γ with $\pi_C^*\gamma \cap E \neq \varnothing$. If $E \neq E_1$, then $\pi_C^*\gamma \cap E$ is a base point of $\Lambda_\mathcal{F}$ and hence the minimal resolution of singularities of $\Lambda_\mathcal{F}$ factorizes by π_C. If $E = E_1$, then π_C is a resolution of $\Lambda_\mathcal{F}$. We conclude that $\sigma_{\Lambda,C}$ is the minimal resolution of $\Lambda_\mathcal{F}$. □

Finally we characterize when a terminal divisor of a dead arc is a dicritical component for the pencil $\Lambda_\mathcal{F}$.

Corollary 16. — *Let C be a curve with kind equisingularity type and consider a Zariski-general foliation $\mathcal{F} \in \mathbb{G}_C^*$. Let F be terminal divisor of a dead arc in $G(C)$ starting at the bifurcation divisor E. The divisor F is dicritical for $\Lambda_\mathcal{F}$ if and only if $v(E) = 3/2$.*

Proof. — If $v(E) = 3/2$, then $v(F) = 2$ and $m(F) = 1$ because C has kind equisingularity type. Thus, by theorem 1, $\alpha(F) = 0$ and hence F is a dicritical component for $\Lambda_\mathcal{F}$.

Conversely, assume that F is a dicritical divisor for $\Lambda_{\mathcal{F}}$ and then $v(F) = 1+1/m(F)$ by theorem 1. Since C has a kind equisingularity type, the relationship between $v(F)$ and $v(E)$ is given by equation (10), thus $v(E) = 1 + 1/m(E)$.

Let $\{(m_l^i, n_l^i)\}_{l=1}^{g_i}$ be the Puiseux pairs of an irreducible component C_i of C. We have that $m(E) = n_1^i \cdots n_{k_E}^i n_{k_E+1}^i$ and $v(E) = m_{k_E+1}^i / m(E)$ for $i \in I_E$ because E is a Puiseux divisor. Consequently, the dicriticalness of F implies that $m_{k_E+1}^i = 1 + n_1^i \cdots n_{k_E}^i n_{k_E+1}^i$. But

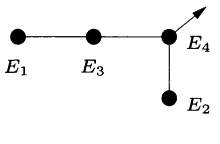
$$1 < \frac{m_{k_E}^i}{n_1^i \cdots n_{k_E}^i} < \frac{m_{k_E+1}^i}{n_1^i \cdots n_{k_E}^i n_{k_E+1}^i}$$

by the properties of the Puiseux pairs. This implies that $n_1^i \cdots n_{k_E}^i n_{k_E+1}^i < m_{k_E}^i n_{k_E+1}^i < m_{k_E+1}^i = 1 + n_1^i \cdots n_{k_E}^i n_{k_E+1}^i$. The previous inequalities hold only if $k_E = 0$, i.e., $m_{k_E}^i = 0$. Consequently $v(E) = (1 + n_1^i)/n_1^i$ and the result follows since $n_E = n_1^i = 2$. □

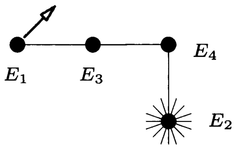
7. Examples

We illustrate here some different behaviours of a polar pencil $\Lambda_{\mathcal{F}}$ when $\mathcal{F}$ is not a Zariski-general foliation.

Example 1. — There can be dicritical components of $\Lambda_{\mathcal{F}}$ with degree ≥ 2, which are different from E_1. Consider the foliation $\mathcal{F}$ given by $d(y^3 - x^5) = 0$; note that C has not a kind equisingularity type. The pencil $\Lambda_{\mathcal{F}}$ has a dicritical component of degree 2 which corresponds to the terminal divisor E_2 of the unique dead arc in $G(C)$. In this case, π_C gives a resolution of singularities of $\Lambda_{\mathcal{F}}$ but it is not the minimal resolution of $\Lambda_{\mathcal{F}}$.

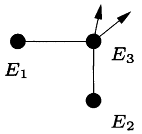


$G(C)$

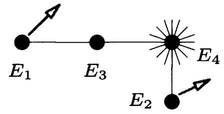


$G(\pi_C, \Lambda_{\mathcal{F}})$

Example 2. — Consider the foliation $\mathcal{F}$ given by $\omega = x^5 dx - y^3 dy = 0$. The minimal reduction of singularities π_C of $\mathcal{F}$ is not a reduction of singularities of a generic fiber $\Gamma_{[a:b]} = \{ax^5 - by^3 = 0\}$. It is necessary to blow-up the corner $E_3 \cap E_2$ of $\pi_C^{-1}(0)$ to obtain an elimination of indeterminations σ_Λ of $\Lambda_{\mathcal{F}}$; hence we need to blow-up a singular point of $\pi_C^* \mathcal{F}$.



$G(\mathcal{F})$



$G(\sigma_\Lambda, \Lambda_{\mathcal{F}})$

Notice that $v(E_4) = 5/3$ and $v(E_3) = 3/2$, thus equation (7) is not true for this foliation. In this example, the curve of separatrices C has a kind equisingularity type but the foliation $\mathcal{F}$ is not Zariski-general.

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