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ON THE JACQUET-LANGLANDS CORRESPONDENCE IN THE COHOMOLOGY OF THE LUBIN-TATE DEFORMATION TOWER

by

Matthias Strauch

Abstract. — Let F be a local non-archimedean field, and let $\mathbb{X}$ be a one-dimensional formal $\mathfrak{o}_F$-module over $\overline{\mathbb{F}}_p$ of height n. The formal deformation schemes of $\mathbb{X}$ with Drinfeld level structures give rise to a projective system of rigid-analytic spaces $(M_K)_K$, where K runs through the compact-open subgroups of $G = GL_n(F)$. On the inductive limit H_c^* of the spaces $H_c^*(M_K \otimes \overline{F}^\wedge, \mathbb{Q}_\ell)$ $(\ell \neq p)$ there is a smooth action of $G \times B^\times$, B being a central division algebra over F with invariant $1/n$. For a supercuspidal representation π of G it follows from the work of Boyer resp. Harris-Taylor that in the Grothendieck group of admissible representations of $B^\times$ one has $\mathrm{Hom}_G(H_c^*, \pi) = n \cdot (-1)^{n-1} \mathcal{JL}(\pi)$, $\mathcal{JL}$ denoting the Jacquet-Langlands correspondence. In this paper we propose an approach that is based on a conjectural Lefschetz trace formula for rigid-analytic spaces, and we calculate the contribution coming from the fixed points.

Résumé **(Sur la correspondance de Jacquet-Langlands dans la cohomologie de la tour de deformations de Lubin-Tate)**

Soient F un corps local non-archimédien et $\mathbb{X}$ un $\mathfrak{o}_F$-module formel de hauteur n sur $\overline{\mathbb{F}}_p$. Les schémas de déformations de $\mathbb{X}$ munies de structures de niveau de Drinfeld fournissent un système projectif d'espaces analytiques rigides $(M_K)_K$, où K parcourt l'ensemble des sous-groupes compacts ouverts de $G = GL_n(F)$. La limite inductive H_c^* des espaces $H_c^*(M_K \otimes \overline{F}^\wedge, \mathbb{Q}_\ell)$ $(\ell \neq p)$ constitue une représentation virtuelle lisse du groupe $G \times B^\times$, B étant une algèbre à division sur F d'invariant $1/n$. Si π est une représentation supercuspidale de G, les travaux de Boyer et Harris-Taylor impliquent que dans le groupe de Grothendieck des représentations admissibles de $B^\times$ on a la relation $\mathrm{Hom}_G(H_c^*, \pi) = n \cdot (-1)^{n-1} \mathcal{JL}(\pi)$, $\mathcal{JL}$ désignant la correspondance de Jacquet-Langlands. Dans cet article nous proposons une approche de ce resultat fondé sur une formule des traces à la Lefschetz conjecturale, et nous calculons la contribution venant des points fixes.

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1. Introduction

Let F be a non-archimedean local field with ring of integers $\mathfrak{o} = \mathfrak{o}_F$. Let $\mathbb{X}$ be a formal $\mathfrak{o}$-module of F-height n over the algebraic closure of the residue field of $\mathfrak{o}$. The functor of deformations of $\mathbb{X}$ is representable by an algebra of formal power series in $n-1$ variables over $\widehat{\mathfrak{o}^{nr}}$. Associated to this algebra there is a rigid-analytic space: the open polydisc of dimension $n-1$. Introducing Drinfeld level structures gives rise to a tower of étale coverings of this space with pro-Galois group $GL_n(\mathfrak{o})$. Moreover, the automorphism group $\mathrm{Aut}_{\mathfrak{o}}(\mathbb{X})$ of $\mathbb{X}$ acts on the deformation space and its coverings, and this action commutes with the action of $GL_n(\mathfrak{o})$. It is convenient to work not only with deformations in the strict sense, *i.e.* ones equipped with an isomorphism of the special fibre to $\mathbb{X}$, but with deformations coming along with a quasi-isogeny of the special fibre to $\mathbb{X}$. In this way one obtains an infinite disjoint union of such towers (indexed by the height of the quasi-isogeny), all being non-canonically isomorphic, and on this tower there is then an action of $GL_n(F) \times B^\times$, where $B = \mathrm{End}_{\mathfrak{o}}(\mathbb{X}) \otimes_{\mathfrak{o}} F$ is a central division algebra over F of dimension n^2. The inductive limit H_c^i of the ℓ-adic étale cohomology groups with compact support of these spaces (after base change to an algebraic closure of F) furnish smooth/continuous representations of $GL_n(F) \times B^\times \times W_F$, where W_F is the Weil group of F, and $\ell \neq p$.

Carayol's conjecture predicts that for a supercuspidal representation π of $GL_n(F)$ the following relation holds true:

$$\mathrm{Hom}_{GL_n(F)}(H_c^{n-1}, \pi) = \mathcal{JL}(\pi) \otimes \sigma(\pi),$$

where $\mathcal{JL}(\pi)$ is the representation of $B^\times$ that is associated to π by the Jacquet-Langlands correspondence, and $\sigma(\pi)$ is, up to twist and dualization, the representation of W_F that is associated to π by the local Langlands correspondence for GL_n. *Cf.* [**Ca1**], sec. 3.3, for a more precise statement also covering the case of non-cuspidal discrete series representations.

In the equal characteristic case, this conjecture has been proven by P. Boyer [**Bo**]. In the mixed characteristic case it may be regarded as being true by the work of M. Harris and R. Taylor [**HT**]. Although they do not state it this way, it seems likely that Carayol's conjecture follows without difficulty from what has been proven in their book, *cf.* [**Ca2**]. Both proofs (equal and mixed characteristic case) use global methods.

In this paper we investigate the alternating sum

$$\mathrm{Hom}_{GL_n(F)}(H_c^*, \pi) := \sum_i (-1)^i \,\mathrm{Hom}_{GL_n(F)}(H_c^i, \pi)$$

as a virtual representation of $B^\times$ by a purely local method. We do not obtain any information about the Weil group representation, except its dimension. Moreover, we pay only attention to the part of the correspondence that concerns the supercuspidal representations. Our approach is based on a *conjectural* Lefschetz trace formula for

rigid analytic spaces, and has its origin in Faltings' paper [**Fa**]. Faltings investigated there the corresponding situation of Drinfeld's symmetric spaces and their coverings. In both cases the problem arises that the spaces under consideration are not proper. Hence we cannot expect to express the alternating sum of traces on the cohomology groups as a sum of fixed point mutiplicities. Indeed, simple calculations show that in general there will be an extra term coming from the "boundary". (In the case considered by Faltings however, the "boundary term" turns out to be zero; this is definitively not true in our case, and this is why the situation considered here seems to be more difficult.) In the case $n = 2$ one can use a trace formula for one-dimensional rigid curves proven by R. Huber, *cf.* [**Hu3**]. Huber's trace formula is applied to certain compactifications (in the category of adic spaces) of quasi-compact subspaces, and Huber's trace formula gives an expression of the trace in terms of usual fixed point multiplicities and a contribution from the finitely many compactifying points. While trying to extend Huber's formula to the higher-dimensional case, the author found out that there is another kind of canonical "quasi-compactification" in the category of adic spaces, namely the projective limit of all admissible blow-ups of the corresponding formal schemes representing the deformation functors. The advantage of this compactification is that one has an immediate modular interpretation of the boundary: the boundary has a natural stratification and the geometry and combinatorial structure of the strata can be related to parabolic subgroups of $GL_n(\mathfrak{o}/(\varpi^m))$. Unfortunately, this seems still to be not sufficient to prove that the boundary term (= actual trace minus the number of fixed points) is a sum of parabolically induced virtual characters. We are finally led to consider certain "tubular neighborhoods" of the strata in the boundary. These spaces are insofar interesting as they can be considered as examples of truly non-archimedean spaces over higher-dimensional local fields. But work in this direction has not yet been finished, and hence is not included in this paper. Here we therefore assume that the trace on the alternating sum of the cohomology groups has an appropriate shape, *cf.* sec. 3.5. This conjecture seems to be geometrically justifiable (*cf.* sec. 3.10), and it turns out that the correspondence between the representations of $GL_n(F)$ and $B^\times$ is then given by the number of fixed points (at least as long we consider supercuspidal representations of $GL_n(F)$).

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2. Deformation spaces and their cohomology groups

2.1. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$. Fix a generator ϖ of the maximal ideal of $\mathfrak{o}$, and put $\mathbb{F}_q = \mathfrak{o}/(\varpi)$, q being the cardinality of the residue class field. Moreover, we denote by $\mathbb{F}$ the residue field of the maximal unramified extension $\mathfrak{o}^{nr}$ of $\mathfrak{o}$, and we let $v : F^\times \to \mathbb{Z}$ the valuation determined by $v(\varpi) = 1$.

Let $\mathbb{X}$ be a one-dimensional formal group over $\mathbb{F}$ that is equipped with an action of $\mathfrak{o}$, *i.e.* we assume given a homomorphism $\mathfrak{o} \to \mathrm{End}_{\mathbb{F}}(\mathbb{X})$ such that the action of $\mathfrak{o}$ on the tangent space is given by the reduction map $\mathfrak{o} \to \mathbb{F}_q \subset \mathbb{F}$. Such an object is called a *formal $\mathfrak{o}$-module* over $\mathbb{F}$. Moreover, we assume that $\mathbb{X}$ is of F-height n, meaning that the kernel of multiplication by ϖ is a finite group scheme of rank q^n over $\mathbb{F}$.

It is known that for each $n \in \mathbb{Z}_{>0}$ there exists a formal $\mathfrak{o}$-module of F-height n over $\mathbb{F}$, and that it is unique up to isomorphism [**Dr**], Prop. 1.6, 1.7.

Let $\mathcal{C}$ be the category of complete local noetherian $\widehat{\mathfrak{o}^{nr}}$-algebras with residue field $\mathbb{F}$. A *deformation* of $\mathbb{X}$ over an object R of $\mathcal{C}$ is a pair (X, ι), consisting of a formal $\mathfrak{o}$-module X over R which is equipped with an isomorphism $\iota : \mathbb{X} \to X_{\mathbb{F}}$ of formal $\mathfrak{o}$-modules over $\mathbb{F}$, where $X_{\mathbb{F}}$ denotes the reduction of X modulo the maximal ideal $\mathfrak{m}_R$ of R. Sometimes we will omit ι from the notation.

Following Drinfeld [**Dr**], sec. 4B, we define a *structure of level m* on a deformation X over R ($m \geqslant 0$) as an $\mathfrak{o}$-module homomorphism

$$\phi : (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n \longrightarrow \mathfrak{m}_R,$$

such that $[\varpi]_X(T)$ is divisible by

$$\prod_{a \in (\varpi^{-1}\mathfrak{o}/\mathfrak{o})^n} (T - \phi(a)).$$

Here, $\mathfrak{m}_R$ is given the structure of an $\mathfrak{o}$-module via X, and $[\varpi]_X(T)$ is the power series that gives multiplication by ϖ on X (after having fixed a coordinate T).

For each $m \geqslant 1$ let $K_m = 1 + \varpi^m M_n(\mathfrak{o})$ be the m'th principal congruence subgroup inside $K_0 = GL_n(\mathfrak{o})$. Define the following set-valued functor $\mathcal{M}^{(0)}_{K_m}$ on the category $\mathcal{C}$. For an object R of $\mathcal{C}$ put

$$\mathcal{M}^{(0)}_{K_m}(R) = \{(X, \iota, \phi) \mid (X, \iota) \text{ is a def. over } R,\ \phi \text{ is a level-}m\text{-structure on } X\}/\simeq,$$

where $(X, \iota, \phi) \simeq (X', \iota', \phi')$ iff there is an isomorphism $(X, \iota) \to (X', \iota')$ of formal $\mathfrak{o}$-modules over R, which is compatible with the level structures.

2.2. *Theorem* (Drinfeld, [Dr] Prop. 4.3). — *The functor $\mathcal{M}_{K_m}^{(0)}$ is representable by a regular local ring, which is a finite flat algebra over $\widehat{\mathfrak{o}^{nr}}[[u_1,\dots,u_{n-1}]]$ which itself represents $\mathcal{M}_{K_0}^{(0)}$.*

The fact that $\widehat{\mathfrak{o}^{nr}}[[u_1,\dots,u_{n-1}]]$ represents $\mathcal{M}_{K_0}^{(0)}$ is due to Lubin and Tate (for $F=\mathbb{Q}_p$) [**LT**]. For this reason $\mathcal{M}_{K_0}^{(0)}$, the deformation space without level structures, is sometines called the *Lubin-Tate moduli space*, *cf.* [**HG**], [**Ch**].

2.3. Let X be a formal $\mathfrak{o}$-module over $R\in\mathcal{C}$ such that $X_{\mathbb{F}}$ has F-height n, in which case we say that the formal $\mathfrak{o}$-module X has height n. As pointed out above, $X_{\mathbb{F}}$ is then isomorphic to $\mathbb{X}$. Denote $\mathrm{End}_{\mathfrak{o}}(\mathbb{X})$ by $\mathfrak{o}_B$; this $\mathfrak{o}$-algebra is the maximal compact subring of $B:=\mathfrak{o}_B\otimes_{\mathfrak{o}}F$, which is a central division algebra over F with invariant $1/n$.

Any non-zero element of $\mathrm{Hom}_{\mathfrak{o}}(\mathbb{X},X_{\mathbb{F}})\otimes_{\mathfrak{o}}F$ is called an *$\mathfrak{o}$-quasi-isogeny* from $\mathbb{X}$ to $X_{\mathbb{F}}$. For such an element ι we define its *F-height* by

$$F\text{-height}(\iota)=F\text{-height}(\varpi^r\iota)-nr,$$

where we choose some $r\in\mathbb{Z}$ such that $\varpi^r\iota$ lies in $\mathrm{Hom}_{\mathfrak{o}}(\mathbb{X},X_{\mathbb{F}})$, and for an element ι' of this latter set, its F-height is h if $\ker(\iota')$ is a group scheme of rank q^h over $\mathbb{F}$.

Define for $h\in\mathbb{Z}$ a set-valued functor $\mathcal{M}_{K_m}^{(h)}$ on $\mathcal{C}$ as follows: for $R\in\mathcal{C}$ the set $\mathcal{M}_{K_0}^{(h)}(R)$ consists of equivalence classes of triples (X,ι,ϕ), where X is a formal $\mathfrak{o}$-module of height n over R, ι is an $\mathfrak{o}$-quasi-isogeny from $\mathbb{X}$ to $X_{\mathbb{F}}$ of F-height h, and ϕ is a level-m-structure on X. Now put

$$\mathcal{M}_{K_m}=\coprod_{h\in\mathbb{Z}}\mathcal{M}_{K_m}^{(h)}.$$

By the uniqueness of $\mathbb{X}$ (up to isomorphism), we have $\mathcal{M}_{K_m}^{(h)}\simeq\mathcal{M}_{K_m}^{(0)}$, but there is no distinguished isomorphism.

2.4. There is an action of $B^\times$ from the right on the functors $\mathcal{M}_{K_m}$ given by

$$[X,\iota,\phi].b=[X,\iota\circ b,\phi],$$

where we denote by $[X,\iota,\phi]$ the equivalence class of (X,ι,ϕ), and where $b\in B^\times$. If $[X,\iota,\phi]$ belongs to $\mathcal{M}_{K_m}^{(h)}(R)$, then $[X,\iota,\phi].b$ is an element of $\mathcal{M}_{K_m}^{(h+v(N(b)))}(R)$, where $N:B\to F$ denotes the reduced norm.

Next we will describe the "action" of the group $G=GL_n(F)$ on the tower $(\mathcal{M}_{K_m})_m$. Let $g\in G$ and suppose first that $g^{-1}\in M_n(\mathfrak{o})$. For integers $m\geqslant m'\geqslant 0$ such that

$$g\mathfrak{o}^n\subset\varpi^{-(m-m')}\mathfrak{o}^n$$

(this inclusion is meant to be inside F^n) we will define a natural transformation

$$g:\mathcal{M}_{K_m}\longrightarrow\mathcal{M}_{K'_m}.$$

Let $[X, \iota, \phi] \in \mathcal{M}_{K_m}(R)$, $R \in \mathcal{C}$. The following construction gives an element $[X', \iota', \phi']$ of $\mathcal{M}_{K_m}(R)$ that is the image under the corresponging map

$$g_R : \mathcal{M}_{K_m}(R) \longrightarrow \mathcal{M}_{K'_m}(R)$$

on R-valued points and it will be denoted by $[X, \iota, \phi] \cdot g$.

The conditions imposed on g show that $g\mathfrak{o}^n$ contains $\mathfrak{o}^n$ and that $g\mathfrak{o}^n/\mathfrak{o}^n$ can naturally be regarded as a subgroup of $\varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n$, so we may define a formal $\mathfrak{o}$-module X' over R by taking the quotient of X by the finite subgroup $\phi(g\mathfrak{o}^n/\mathfrak{o}^n)$ (*cf.* [**Dr**], Prop. 4.4):

$$X' = X/\phi(g\mathfrak{o}^n/\mathfrak{o}^n).$$

Moreover, left multiplication with g induces an injective homomorphism

$$\varpi^{-m'}\mathfrak{o}^n/\mathfrak{o}^n \xrightarrow{g} \varpi^{-m}\mathfrak{o}^n/g\mathfrak{o}^n = (\varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n)/(g\mathfrak{o}^n/\mathfrak{o}^n)$$

and the composition with the morphism induced by ϕ,

$$(\varpi^{-m}\mathfrak{o}^n/\mathfrak{o}^n)/(g\mathfrak{o}^n/\mathfrak{o}^n) \longrightarrow X/\phi(g\mathfrak{o}^n/\mathfrak{o}^n) = X',$$

gives by [**Dr**], Prop. 4.4, a level-m'-structure

$$\phi' : \varpi^{-m'}\mathfrak{o}^n/\mathfrak{o}^n \longrightarrow X'[\varpi^{m'}].$$

Finally define ι' to be the composition of ι with the projection

$$X_{\mathbb{F}} \longrightarrow (X')_{\mathbb{F}}.$$

One checks easily that this construction is independent of the representative (X, ι, ϕ) and gives indeed a morphism of functors. If $[X, \iota, \phi]$ lies in $\mathcal{M}^{(h)}_{K_m}(R)$ then $[X, \iota, \phi] \cdot g$ is an element of $\mathcal{M}^{(h-v(\det g))}_{K_m}(R)$.

For an arbitrary element $g \in G$, choose $r \in \mathbb{Z}$ such that $(\varpi^{-r}g)^{-1} \in M_n(\mathfrak{o})$. Then, for $m \geqslant m' \geqslant 0$ with

$$\varpi^{-r}g\mathfrak{o}^n \subset \varpi^{-(m-m')}\mathfrak{o}^n$$

and $[X, \iota, \phi] \in \mathcal{M}_{K_m}(R)$, define $[X', \iota', \phi'] = [X, \iota, \phi].(\varpi^{-r}g)$ as above and put

$$[X, \iota, \phi] \cdot g = [X', \iota' \circ \varpi^{-r}, \phi'].$$

This construction gives natural transformation

$$g : \mathcal{M}_{K_m} \longrightarrow \mathcal{M}_{K_{m'}},$$

which depends neither on ϖ nor on the integer r (among all r's with $\mathfrak{o}^n \subset \varpi^{-r}g\mathfrak{o}^n \subset \varpi^{-(m-m')}\mathfrak{o}^n$). In particular, one gets for each m an action of $GL_n(\mathfrak{o})$ on $\mathcal{M}_{K_m}$ which commutes with the action of $B^\times$.

2.5. The next step is to introduce the analytic spaces whose ℓ-adic étale cohomology groups we are going to study in this paper.

There are different possible methods how to construct such spaces, namely as rigid-analytic spaces, as non-archimedean analytic spaces as defined and studied by V.G. Berkovich [**Be1**], or finally as adic spaces in the sense of R. Huber [**Hu1**]. For each of these kinds of spaces there has been defined an étale cohomology theory ([**dJ-vdP**], [**Be2**], [**Hu2**]) and there are comparison theorems assuring that the resulting cohomology groups for the spaces considered by us are the same ([**Hu2**], sec. 8.3). For the purpose of this paper it is not important with which construction we actually work. The reader is invited to use the theory he feels most comfortable with. We will give brief references where the actual constructions have been carried out.

It follows from Theorem 2.2 that each of the functors $\mathcal{M}_{K_m}^{(h)}$ is representable by a regular local $\widehat{\mathfrak{o}^{nr}}$-algebra of Krull dimension n, $R_m^{(h)}$ say, which are for varying h (but fixed m) non-canonically isomorphic. We give $R_m^{(h)}$ the topology defined by the maximal ideal, and denote also by $\mathcal{M}_{K_m}^{(h)}$ the formal spectrum $\mathrm{Spf}(R_m^{(h)})$, and by $\mathcal{M}_{K_m}$ the disjoint union over all $h \in \mathbb{Z}$.

A construction due to P. Berthelot, generalizing Raynaud's construction for ϖ-adic formal schemes, associates a rigid-analytic space to $\mathcal{M}_{K_m}$ ([**Ber**], ch. 0, or [**RZ**], sec. 5.1). In the context of non-archimedean analytic spaces, Berkovich has given a construction of such spaces associated to formal schemes of this type [**Be3**]. Finally, R. Huber defines in [**Hu1**], sec. 4, an adic space $t(\mathcal{M}_{K_m}^{(h)})$ associated to $\mathcal{M}_{K_m}^{(h)}$ (and to $\mathcal{M}_{K_m}$). The set of points of the underlying topological space consists of all (equivalence classes of) continuous valuations $|\cdot|_v$ on $R_m^{(h)}$ such that $|f|_v \leqslant 1$ for all $f \in R_m^{(h)}$. The set of valuations $|\cdot|_v$ with $|\varpi|_v = 0$ is a closed subset which we denote by $V(\varpi)$. The open complement inherits the structure of an adic space and we put

$$M_{K_m}^{(h)} = t(\mathcal{M}_{K_m}^{(h)}) - V(\varpi), \text{ and } M_{K_m} = \coprod_{h\in\mathbb{Z}} M_{K_m}^{(h)}.$$

There are obvious canonical maps given by restricting the level structure

$$M_{K_m} \longrightarrow M_{K_{m'}},$$

for $m \geqslant m'$, which are étale and galois with Galois group $K_{m'}/K_m$. In particular the Galois group of M_{K_m} over M_{K_0} is $GL_n(\mathfrak{o}/(\varpi^m))$. By 2.2 each space $\mathcal{M}_{K_0}^{(h)}$ is isomorphic to an open polydiscs of dimension $n-1$; in particular:

$$M_{K_0}^{(h)}(\overline{F}^{\wedge}) \simeq \{(z_1, \ldots, z_{n-1}) \in (\overline{F}^{\wedge})^{n-1} \mid \text{for all } i : |z_i| < 1\}.$$

For an open subgroup $K \subset K_0$ we choose a positive integer m such that K_m is normal in K, and we define

$$M_K = M_{K_m}/(K/K_m).$$

Note that the action of K/K_m on M_{K_m} respects the components $M_{K_m}^{(h)}$, hence we let $M_K^{(h)}$ be the quotient of $M_{K_m}^{(h)}$ by K/K_m. Let $K \subset K_0$ be as above, and let $g \in G$ such that $g^{-1}Kg \subset K_0$. Choose $m \geqslant m' \geqslant 0$ such there is a morphism

$$g : \mathcal{M}_{K_m} \longrightarrow \mathcal{M}_{K_{m'}},$$

as defined in the preceding section. Assume K_m to be normal in K, $K_{m'}$ to be normal in $g^{-1}Kg$, and $g^{-1}Kg \subset K_{m'}$. Then we have an induced morphism

$$g : M_K = M_{K_{m'}}/(K/K_m) \longrightarrow M_{K_{m'}}/(g^{-1}Kg/K_{m'}) = M_{g^{-1}Kg},$$

which is in fact an isomorphism and does not depend on the specific choices. This allows us to define M_K for arbitrary compact-open subgroups K of G: choose a normal subgroup K' of K which lies in K_0, then for each $g \in K$ we have just defined an isomorphism $g : M_{K'} \to M_{K'}$, hence we put

$$M_K = M_{K'}/(K/K').$$

Again we can define $M_K^{(h)}$ as the quotient of $M_{K'}^{(h)}$ by K/K'. Consequently, for any compact-open subgroup K of G and any $g \in G$ there is an isomorphism

$$g : M_K \longrightarrow M_{g^{-1}Kg}.$$

Via this construction, the tower $(M_K \mid K \subset G$ compact-open$)$ is equipped with a natural action of $G \times B^\times$ from the right.

2.6. Finally, we introduce the cohomology groups. We use the étale cohomology theory as developed by Huber [**Hu2**], respectively Berkovich [**Be2**]. Because of the comparison theorems in [**Hu2**], sec. 8.3, we can and will use results of V.G. Berkovich for the étale cohomology of non-archimedean analytic spaces. So far, the cohomology theories and the results concern mostly the cohomology of torsion sheaves, and a general theory of ℓ-adic cohomology has not been developed yet. Nevertheless, for the spaces considered by us, it is not difficult to show the finiteness of

$$H_c^i(M_K^{(h)} \otimes_{\widehat{F^{nr}}} \overline{F}^\wedge, \mathbb{Q}_\ell) := \Big(\varprojlim_r H_c^i(M_K^{(h)} \otimes_{\widehat{F^{nr}}} \overline{F}^\wedge, \mathbb{Z}/l^r\mathbb{Z})\Big) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

as a $\mathbb{Q}_\ell$-vector space (*cf.* [**Be4**]). The essential ingredient in showing this is the fact that $\mathcal{M}_{K_m}^{(h)}$ is a formal scheme which is the completion of a scheme of finite type over $\widehat{\mathfrak{o}^{nr}}$ at a closed point of the special fibre. This in turn follows from the very proof of the representability result 2.2, *cf.* [**Dr**], Prop. 4.3. Moreover, these cohomology groups are non-zero only in degree i for $n-1 \leqslant i \leqslant 2(n-1)$ by [**Be3**], Th. 6.1, Cor. 6.2.

Next we put

$$H_c^i(M_K) = H_c^i(M_K \otimes_{\widehat{F^{nr}}} \overline{F}^\wedge, \mathbb{Q}_\ell) = \bigoplus_{h \in \mathbb{Z}} H_c^i(M_K^{(h)} \otimes_{\widehat{F^{nr}}} \overline{F}^\wedge, \mathbb{Q}_\ell).$$

On each $\mathbb{Q}_\ell$-vector space $H^i_c(M_K)$ there is an induced action of $K_0 \times B^\times$ and for each $g \in G$ there is an isomorphism

$$H^i_c(M_{g^{-1}Kg}) \longrightarrow H^i_c(M_K).$$

These give rise to a representation of $G \times B^\times$ on

$$H^i_c := \varinjlim_K H^i_c(M_K),$$

where the limit is taken over all compact-open subgroups K of G.

***2.7. Theorem* (Berkovich).** — *The action of $G \times B^\times$ on H^i_c is smooth.*

Proof. — Any element of H^i_c lies in a cohomology group $H^i_c(M_K)$ on which the action of $K \subset G$ is trivial. Therefore G acts smoothly. It is a non-trivial result due to Berkovich that the action of $B^\times$ on $H^i_c(M_K)$ is smooth, *cf.* [**Be3**], introduction. □

Remark. — The inertia group $\mathrm{Gal}(F^{\mathrm{sep}}/F^{nr})$ acts also on $H^i_c(M_K)$, and this action can be extended to an action of the Weil group W_F, *cf.* [**Bo**], Prop. 2.3.2, [**RZ**], sec. 3.48. Then one gets a smooth/continuous action of $G \times B^\times \times W_F$ on H^i_c. In this paper however we pay only attention to the representations of G and $B^\times$.

Let π be a supercuspidal representation of G, and let $\mathcal{JL}(\pi)$ be the representation of $B^\times$ associated to π by the Jacquet-Langlands correspondence. The following theorem is implied by Boyer's Theorem, [**Bo**], Th. 3.2.4, in the equal characteristic case, and it follows from the work of Harris and Taylor [**HT**] in the mixed characteristic case.

2.8. Theorem. — *For each i the representation $\mathrm{Hom}_G(H^i, \pi)$ of $B^\times$ is finite-dimensional and smooth, and in the Grothendieck group of admissible representations of $B^\times$ the following equality holds:*

$$\sum_i (-1)^i \,\mathrm{Hom}_G(H^i_c, \pi) = n \cdot (-1)^{n-1} \cdot \mathcal{JL}(\pi).$$

As a definition, we put

$$\mathrm{Hom}_G(H^*_c, \pi) := \sum_i (-1)^i \,\mathrm{Hom}_G(H^i_c, \pi),$$

where we consider the right hand side as an element of the Grothendieck group of admissible representations of $B^\times$.

2.9. In [**HT**] this result comes only as a by-product of a detailed study of the cohomology groups of certain Shimura varieties attached to unitary groups coming from division algebras, and the precise investigation of the reduction of these varieties at bad primes. Similarly, Boyer's proof ([**Bo**]) in the equal characteristic case is based on the study of the bad reduction of Drinfeld modular varieties.

In the next paragraph, we discuss a purely local way towards this theorem, that is based on a Lefschetz type trace formula.

3. The approach via a Lefschetz trace fromula

3.1. Let π be a supercuspidal representation of $G = GL_n(F)$. By the fundamental result of Bushnell-Kutzko [**BK**] and Corwin [**Co**], we know that π is induced from a (finite-dimensional) irreducible representation λ of some open subgroup $K_\pi \subset G$ that contains and is compact modulo the centre of G, *cf.* [**BK**], Th. 8.4.1, for a more precise statement. Hence we may write

$$\pi = \text{c-Ind}_{K_\pi}^G(\lambda) = \text{Ind}_{K_\pi}^G(\lambda),$$

where the second equality holds by [**Bu**], Th. 1. Moreover, the character of π is a locally constant function on the set of elliptic regular elements in G (*i.e.* those whose characteristic polynomial is separable and irreducible), and for such an element $g \in G$ we have

$$\chi_\pi(g) = \sum_{\substack{g' \in G/K_\pi \\ (g')^{-1}gg' \in K_\pi}} \chi_\lambda((g')^{-1}gg').$$

For regular elliptic g the number of elements $g' \in G/K_\pi$ such that $(g')^{-1}gg' \in K_\pi$ is finite. This formula is due to Harish-Chandra, proofs can be found in [**He**] and [**Sa**]. For the rest of this section we fix π, K_π, and λ with this property.

3.2. For π as above, the representation $\rho = \mathcal{JL}(\pi)$ is characterized by the following identity. Let $g \in G$ and $b \in B^\times$ be regular elliptic elements with the same characteristic polynomial. Then the following character relation holds

$$\chi_\rho(b) = (-1)^{n-1} \cdot \chi_\pi(g),$$

cf. [**DKV**], introduction, [**Ro**]. Th. 5.8., [**Ba**].

3.3. For π as in (3.1) we will analyze $\text{Hom}_G(H_c^*, \pi)$ as a representation of $B^\times$. Note first that by Frobenius reciprocity

$$\text{Hom}_G(H_c^*, \pi) = \text{Hom}_{K_\pi}(H_c^*, \lambda).$$

Choose $c \in \overline{\mathbb{Q}}_\ell$ such that $\lambda(\varpi) = c^n$, and define a character ζ of G by $\zeta(g) = c^{-v(\det(g))}$. Then:

$$\begin{aligned}\text{Hom}_{K_\pi}(H_c^*, \lambda) &= \text{Hom}_{K_\pi}(H_c^* \otimes \zeta, \lambda \otimes \zeta) \\ &= \text{Hom}_{K_\pi}((H_c^* \otimes \zeta)/\langle v - c^{-n} \cdot \varpi \cdot v \mid v \in H_c^* \rangle, \lambda \otimes \zeta),\end{aligned}$$

where $\varpi \cdot v$ denotes the action of ϖ, considered as an element of G, on v, considered as an element of H_c^*.

Next, $(H_c^* \otimes \zeta)/\langle v - c^{-n} \cdot \varpi \cdot v \mid v \in H_c^* \rangle$ is isomorphic, as a representation of $G \times B^\times$, to the natural representations of $G \times B^\times$ on

$$\Big(\varinjlim_{K'} H_c^*(M_{K'}/\varpi^{\mathbb{Z}})\Big) \otimes \xi,$$

where the limit is taken over all compact-open subgroups K' of G, and ξ is the character of $B^\times$ given by $\xi(b) = c^{-v(Nrd(b))}$. The map is defined as follows: an element $\alpha \in H_c^*(M_K^{(h)}, \overline{\mathbb{Q}}_\ell)$ is mapped to $c^h \varpi^{-k}\alpha \in H_c^*(M_K^{(h_0)}, \overline{\mathbb{Q}}_\ell)$, where $h = h_0 + nk$ with $0 \leqslant h_0 < n$. It is not difficult to check that this is a $G \times B^\times$-equivariant isomorphism. Hence we get the following identity of representation of $B^\times$:

$$\mathrm{Hom}_G(H_c^*, \pi) = \mathrm{Hom}_{K_\pi}\left(\varinjlim_{K'} H_c^*(M_{K'}/\varpi^{\mathbb{Z}}), \lambda \otimes \zeta\right) \otimes \xi^\vee.$$

Let $K \subset K_0$ be an open subgroup that is normal in K_π and such that $\lambda|_K$ is a multiple of the trivial representation of K. Then

$$\left(\varinjlim_{K'} H_c^*(M_{K'}/\varpi^{\mathbb{Z}})\right)^K = \varinjlim_{K'} H_c^*(M_{K'}/\varpi^{\mathbb{Z}})^K = H_c^*(M_K/\varpi^{\mathbb{Z}}),$$

and therefore

$$\mathrm{Hom}_{K_\pi}\left(\varinjlim_{K'} H_c^*(M_{K'}/\varpi^{\mathbb{Z}}), \lambda \otimes \zeta\right) \otimes \xi^\vee = \mathrm{Hom}_\Gamma\left(H_c^*(M_K/\varpi^{\mathbb{Z}}), \lambda \otimes \zeta\right) \otimes \xi^\vee,$$

where we have put

$$\Gamma = K_\pi/\varpi^{\mathbb{Z}} K;$$

this is finite group. Note that we finally arrived at an expression for $\mathrm{Hom}_G(H_c^*, \pi)$ that involves only finite-dimensional vector spaces.

3.4. The next step is to compute the trace of a regular elliptic element $b \in B^\times$ on this space. Using the result of the preceding section, the identity below is elementary:

$$\mathrm{tr}(b \mid \mathrm{Hom}_G(H_c^*, \pi)) = \frac{\xi(b)^{-1}}{\#\Gamma} \sum_{\gamma \in \Gamma} \mathrm{tr}((\gamma, b^{-1}) \mid H_c^*(M_K/\varpi^{\mathbb{Z}})) \cdot \chi_{\lambda \otimes \zeta}(\gamma^{-1}).$$

Of course, this is the point where the Lefschetz trace formula comes in, because we would like to replace the trace of $(\gamma, b^{-1}) \in \Gamma \times B^\times$ on the virtual representation $H_c^*(M_K/\varpi^{\mathbb{Z}})$ by an expression involving the number of fixed points. Our spaces, like $M_K/\varpi^{\mathbb{Z}}$, are not proper however, so we cannot expect to get an expression involving only the number of fixed points, and indeed, there is in general an additional term. In the case $n = 2$ one can use a trace formula which has been proved by R. Huber [**Hu3**] to get a manageable description of this "boundary term". For the general case we have the following

3.5. Conjecture. — In the setting and with the notations introduced above, there is a trace formula of the following form:

$$\mathrm{tr}((\gamma, b^{-1}) \mid H_c^*(M_K/\varpi^{\mathbb{Z}})) = \mathrm{Fix}_K(\gamma, b^{-1}) + \beta_K(\gamma, b^{-1}),$$

where $\mathrm{Fix}_K(\gamma, b^{-1})$ denotes the number (counted with multiplicity) of fixed points of (γ, b^{-1}) on $(M_K/\varpi^{\mathbb{Z}})(\overline{F}^\wedge)$ and $\beta_K(\gamma, b^{-1})$ has the property that for λ as above

$$\sum_{\gamma \in \Gamma} \beta_K(\gamma, b^{-1}) \cdot \chi_{\lambda \otimes \zeta}(\gamma^{-1}) = 0.$$

3.6. Remark. — At the end of this section will give some heuristic arguments justifying this conjecture. For the moment, let us mention that the last formula has the following representation theoretic meaning. Firstly, for fixed b the function $\gamma \mapsto \beta_K(\gamma, b^{-1})$ is a class function on the finite group Γ, and can hence be written as a sum

$$\beta_K(\cdot, b^{-1}) = \sum_\tau \alpha_\tau \chi_\tau,$$

where τ runs over the set of equivalence classes of irreducible representations of Γ. Those τ with non-zero α_τ may be called the *representations that occur in the boundary.* Therefore the preceding formula signifies that:
No representation that gives rise to a supercuspidal representation occurs in the boundary.

3.7. Let us assume that such a trace formula exists. Then the condition on the boundary term β_K implies that

$$\operatorname{tr}(b \mid \operatorname{Hom}_G(H_c^*, \pi)) = \frac{\xi(b)^{-1}}{\#\Gamma} \sum_{\gamma \in \Gamma} \operatorname{Fix}_K(\gamma, b^{-1}) \cdot \chi_{\lambda \otimes \zeta}(\gamma^{-1}).$$

The final step to establish the identity of characters is the

***3.8. Theorem* (Fixed point theorem)**. — *Let g_b be in the conjugacy class corresponding to b. Then*

$$\operatorname{Fix}_K(\gamma, b^{-1}) = n \cdot \#\{g \in G/\varpi^{\mathbb{Z}} K \mid g^{-1} g_b g = \gamma^{-1}\}.$$

The identity $g^{-1} g_b g = \gamma^{-1}$ means that for some representative $\dot{g}$ of $g \in G/\varpi^{\mathbb{Z}} K$ we have $\dot{g}^{-1} g_b \dot{g} \in K_\pi$ and the class of $\dot{g}^{-1} g_b \dot{g}$ in $\Gamma = K_\pi / \varpi^{\mathbb{Z}} K$ is γ^{-1}. By the fact pointed out in 3.1, the number of such $g \in G/\varpi^{\mathbf{Z}} K$ is always finite.

This formula will be proven in the next section. Putting the expression for the number of fixed points in the identity derived in 3.7 we get

3.9. Theorem. — *Suppose the conjecture 3.5 on the Lefschetz trace formula is fulfilled. Let π be a supercuspidal representation of $G = GL_n(F)$. Then the following holds:*

(i) *for every i the representation of $B^\times$ on $\operatorname{Hom}_G(H_c^i, \pi)$ is admissible;*
(ii) *in the Grothendieck group of admissible representation of $B^\times$ we have:*

$$\operatorname{Hom}_G(H_c^*, \pi) = n \cdot (-1)^{n-1} \mathcal{JL}(\pi),$$

where $\operatorname{Hom}_G(H_c^*, \pi)$ is $\sum_i (-1)^i \operatorname{Hom}_G(H_c^i, \pi)$, as defined after theorem 2.8.

Proof. — The first statement follows from the identity

$$\operatorname{Hom}_G(H_c^i, \pi) = \operatorname{Hom}_\Gamma(H_c^i(M_K/\varpi^{\mathbb{Z}}), \lambda \otimes \zeta) \otimes \xi^\vee,$$

and the fact that $H_c^i(M_K/\varpi^{\mathbb{Z}})$ is a finite-dimensional smooth representation of $B^\times$, *cf.* paragraph 2.6 and [**Be3**], introduction. To prove the second statement, let $b \in B^\times$

be regular elliptic. Then the preceding discussion immediately gives

$$\begin{aligned}\operatorname{tr}(b \mid \operatorname{Hom}_G(H_c^*, \pi)) &= \frac{\xi(b)^{-1}}{\#\Gamma} \sum_{\substack{g \in G/\varpi^{\mathbb{Z}} K \\ g^{-1} g_b g \in \Gamma}} n \cdot \chi_{\lambda \otimes \zeta}(g^{-1} g_b g) \\ &= n \sum_{\substack{g \in G/K_\pi \\ g^{-1} g_b g \in K_\pi}} \chi_\lambda(g^{-1} g_b g) \\ &= n \cdot \chi_\pi(g_b) = n \cdot (-1)^{n-1} \chi_{\mathcal{JL}(\pi)}(b),\end{aligned}$$

and this proves the assertion. □

3.10. On the conjectural shape of the trace formula. — The natural approach to prove a trace formula for $M := M_{K_m}^{(0)}$, is to consider admissible blow-ups $\mathcal{M}' \to \mathcal{M}$ of the corresponding formal scheme $\mathcal{M} := \mathcal{M}_{K_m}^{(0)} = \operatorname{Spf}(R_m^{(0)})$, and to use the Grothendieck-Verdier Lefschetz formula on the special fibre of $\mathcal{M}'$ with the complex of ℓ-adic nearby-cycles as coefficients. To do this, we only consider blow-ups to which the group action extends. So we have to study the fixed point locus on the special fibre of such blown-up models. The idea is to investigate the group action on all blow-ups simultaneously to eventually derive assertions about the existence of models where the connected components of the fixed point locus lie in different strata (see below for the definition of this stratification). Thus we are led to consider the projective limit $\varprojlim \mathcal{M}'$ over all admissible blow-ups. This space can in fact be given the structure of an adic space; it is, in Huber's notation, $t(\mathcal{M}) - V(\mathfrak{m}_{R_m^{(0)}})$ (where $\mathfrak{m}_{R_m^{(0)}}$ denotes the maximal ideal of $R_m^{(0)}$) and it contains $M = t(\mathcal{M}) - V(\varpi)$ as an open subspace. Because it is a projective limit of formal schemes with proper special fibres, we consider it as a kind of compactification of M, and we put

$$\overline{M} := t(\mathcal{M}) - V(\mathfrak{m}_{R_m^{(0)}}).$$

The underlying set is the set of all (equivalence classes) of continuous valuations of $R_m^{(0)}$ which do not factor through the residue field of this local ring.

Now we define a stratification on this space. For $m = 0$ and $0 \leqslant i \leqslant n-1$ let $\partial_i \overline{M}$ be the subspace where the connected part of the universal formal $\mathfrak{o}$-module has height i. We have in particular: $\partial_0 \overline{M} = M$. For $m > 0$ and any proper $\mathfrak{o}$-submodule $A \subset (\varpi^{-m}\mathfrak{o}/\mathfrak{o})^n$ which is free over $\mathfrak{o}/\varpi^m$ and a direct summand, define $\partial_A \overline{M}$ to be the subspace where the the kernel of the universal level structure is equal to A. So we have in particular $\partial_0 \overline{M} = M$. The stabilizer in $GL_n(\mathfrak{o}/\varpi^m)$ of the strata are maximal proper parabolic subgroups, the corresponding unipotent radical acts trivially on the stratum. Moreover, for any model $\mathcal{M}'$ of M the images of the strata induce a stratification of the special fibre: consider first the image of the zero-dimensional stratum, then remove this subscheme from the image of the one-dimensional stratum etc.

The conjecture 3.5 can be proven, if there is a model of M such that

(i) the strata of the special fibre of the model are in bijection with the strata of $\overline{M}$;

(ii) the connected components of the fixed point locus on the special fibre of the model are contained in the relative interior of the strata;

(iii) the connected components of the fixed point locus which lie in the open stratum correspond (via the specialisation map) one-to-one to the fixed points in the interior, and the local term attached to such a component is one;

(iv) the local term attached to a connected component lying in a non-open stratum is invariant under the unipotent radical of the stabilizer of the corresponding stratum of $\overline{M}$.

In proving the first condition one uses the fact that the images $\phi(e_1), \ldots, \phi(e_n)$ of the universal Drinfeld base ϕ generate the maximal ideal of $R_m^{(0)}$. Assuming the second condition, the fourth condition should be provable, using a result of Berkovich, *cf.* [**Be2**], Th. 4.1. The fourth condition implies that the boundary term is a sum of characters of parabolically induced representations, and these characters are orthogonal to characters which give rise to supercuspidal representations (*cf.* [**Bu**]). Using Faltings methods ([**Fa**]), it seems possible to prove the existence of a model satisfying the third condition. (But the fixed point locus on the special fibre is not the one in the naive sense; one has to compute it using a model of $M \times M$ which is not the product of a model of M with itself). This means that we can separate the interior fixed points from fixed points at the boundary. To prove the second condition means to separate the fixed point locus on the boundary according to the strata. And to do this, one has to understand the group action on the boundary. In the interior, *i.e.* on M, this is done via the period map of Gross and Hopkins. This period map has analogues, not on the boundary itself, but on a "tubular neighborhood" of the boundary. We don't want to go into the definition of these spaces here, but it is hoped that the study of these maps on the tubular neighborhood of the boundary provides enough information about the group action on the boundary to finally prove the second condition.

4. Fixed points and the period morphism

4.1. To count fixed points we will use the period map from the moduli spaces M_K to a projective space of dimension $n-1$. This map was first studied by M. Hopkins and B. Gross [**HG**], and some results in connection with this map have been obtained before by G. Laffaille [**La**]. Later on, M. Rapoport and Th. Zink introduced these morphisms for moduli spaces for p-divisible groups [**RZ**], thereby giving a unified account of p-adic period maps that have been studied before (here one should mention Dworks period map from the deformation space of an ordinary elliptic curve to the affine line, *cf.* [**Ka**]). The set-up of Gross and Hopkins is insofar closer to our situation as they

work with formal $\mathfrak{o}$-modules (hence treat the mixed and equal characteristic case simultaneously), herein following Drinfeld. On the other hand, Gross and Hopkins only work with one component of the moduli space $\mathcal{M}_{K_0}$, namely the component $\mathcal{M}_{K_0}^{(0)}$ where the quasi-isogeny on the special fibre has height zero. After recalling the main results of [**HG**] in the next section, we will explain how to define the period map on the whole space M_{K_0}.

4.2. Let $\mathcal{X}$ be the universal formal $\mathfrak{o}$-module over the formal scheme $\mathcal{M}_{K_0}^{(0)}$, and denote by $\mathcal{E}$ the universal extension of $\mathcal{X}$ with additive kernel. This is a formal $\mathfrak{o}$-module of dimension n which sits in an exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{E} \longrightarrow \mathcal{X} \longrightarrow 0,$$

where $\mathcal{V} = \mathbb{G}_a \otimes \mathrm{Hom}_R(\mathrm{Ext}(\mathcal{X}, \mathbb{G}_a), R)$ and $\mathcal{M}_{K_0}^{(0)} = \mathrm{Spf}(R)$, so $R = R_0^{(0)}$ with the notation of section 2.5. This exact sequence furnishes an exact sequence

$$0 \longrightarrow \mathcal{L}ie(\mathcal{V}) \longrightarrow \mathcal{L}ie(\mathcal{E}) \longrightarrow \mathcal{L}ie(\mathcal{X}) \longrightarrow 0,$$

of vector bundles on the formal scheme $\mathcal{M}_{K_0}^{(0)}$, and an analogous sequence

$$0 \longrightarrow \mathcal{L}ie(\mathcal{V})^{\mathrm{rig}} \longrightarrow \mathcal{L}ie(\mathcal{E})^{\mathrm{rig}} \longrightarrow \mathcal{L}ie(\mathcal{X})^{\mathrm{rig}} \longrightarrow 0,$$

on the generic fibre of this formal scheme, *i.e.* on the space $M_{K_0}^{(0)}$.

4.3. Proposition **([HG], Prop. 22.4, 23.2, 23.4)**

(i) *There is a basis $c_0, \ldots, c_{n-1}$ of $\mathcal{L}ie(\mathcal{E})^{\mathrm{rig}}$ such that the $\widehat{F^{nr}}$-subspace generated by these global sections is stable by the action of $\mathfrak{o}_B^\times$. More precisely, the canonical map of vector bundles on $M_{K_0}^{(0)}$*

$$\langle c_0, \ldots, c_{n-1} \rangle_{\widehat{F^{nr}}} \otimes \mathcal{O}_{M_{K_0}^{(0)}} \longrightarrow \mathcal{L}ie(\mathcal{E})^{\mathrm{rig}}$$

is an $\mathfrak{o}_B^\times$-equivariant isomorphism, where $\mathfrak{o}_B^\times$ acts diagonally on the left hand side. The representation of $\mathfrak{o}_B^\times$ on $\langle c_0, \ldots, c_{n-1} \rangle_{\widehat{F^{nr}}}$ is equivalent to the representation of $\mathfrak{o}_B^\times$ on $B \otimes_{F_n} \widehat{F^{nr}}$ given by left multiplication (where F_n is the unramified extension of degree n in $\widehat{F^{nr}}$).

(ii) *Let w_i be the image of c_i in $\mathcal{L}ie(\mathcal{X})^{\mathrm{rig}}$, $i = 0, \ldots, n-1$, and denote by W the space generated by these global sections over $\widehat{F^{nr}}$. Then, the sections w_i have no common zeroes, and they are linearly independent over $\widehat{F^{nr}}$.*

(iii) *Denote by $\mathbb{P}(W)$ the projective space of hyperplanes in W, and by $\mathbb{P}(W)^{\mathrm{rig}}$ the associated analytic space. Define*

$$\pi_{K_0}^{(0)} : M_{K_0}^{(0)} \longrightarrow \mathbb{P}(W)^{\mathrm{rig}}$$

by sending $x \in M_{K_0}^{(0)}$ to the hyperplane

$$\{w = \alpha_0 w_0 + \cdots + \alpha_{n-1} w_{n-1} \in W \otimes \widehat{F^{nr}}(x) \mid \alpha_0 w_0(x) + \cdots + \alpha_{n-1} w_{n-1}(x) = 0\}.$$

This map is a rigid-analytic étale morphism. It is $\mathfrak{o}_B^\times$-equivariant and surjective on $\overline{F}^\wedge$.

4.4. Choose an element $\varpi_B \in \mathfrak{o}_B$ whose reduced norm is a uniformizer of F. The action of $B^\times$ on $\mathcal{M}_{K_0}$ furnishes for each $h \in \mathbb{Z}$ an isomorphism

$$\varpi_B^h : \mathcal{M}_{K_0}^{(0)} \longrightarrow \mathcal{M}_{K_0}^{(h)}.$$

Define $\pi_{K_0}^{(h)} : M_{K_0}^{(h)} \to \mathbb{P}(W)$ by $\pi_{K_0}^{(h)} = \varpi_B^h \circ \pi_{K_0}^{(0)} \circ \varpi_B^{-h}$. Because of the $\mathfrak{o}_B^\times$-equivariance of $\pi_{K_0}^{(0)}$, this map is does not depend on the choice of ϖ_B. Finally we get the *period map* on the whole space M_{K_0} by putting

$$\pi_{K_0} = \coprod_{h\in\mathbb{Z}} \pi_{K_0}^{(h)} : M_{K_0} = \coprod_{h\in\mathbb{Z}} M_{K_0}^{(h)} \longrightarrow \mathbb{P}(W)^{\mathrm{rig}}.$$

More generally, for any open subgroup $K \subset K_0$ we let π_K be the composition of the projection $M_K \to M_{K_0}$ with π_{K_0}, and refer to π_K as a period morphism. The proposition above gives immediately the following assertion about the morphisms π_K.

4.5. Proposition. — *For any open subgroup $K \subset K_0$*

$$\pi_K : M_K \longrightarrow \mathbb{P}(W)^{\mathrm{rig}}$$

is an étale morphism of analytic spaces over $\widehat{F^{nr}}$. Moreover, π_K is equivariant with respect to the action of $N_G(K) \times B^\times$, where the normalizer $N_G(K)$ of K in G acts trivially on $\mathbb{P}(W)^{\mathrm{rig}}$ and the action of $B^\times$ on $\mathbb{P}(W)^{\mathrm{rig}}$ is the one that is induced by the action of $B^\times$ on W.

4.6. Now we are in a position to count fixed points. Let $b \in B^\times$ be an element which is regular elliptic. Hence b has n distinct simple fixed points on $\mathbb{P}(W)^{\mathrm{rig}} \otimes \overline{F}^\wedge$. Let K be a compact-open subgroup of G that is contained in K_0, and let γ be an element of the normalizer of K in G. By Proposition 4.5, the action of the pair (γ, b^{-1}) on $M_K \otimes \overline{F}^\wedge$ stabilizes the fibre of π_K over a fixed point of b on $\mathbb{P}(W)^{\mathrm{rig}} \otimes \overline{F}^\wedge$. Hence we need a description of the fibres of π_K together with the action of (γ, b^{-1}). The next proposition gives such a description.

4.7. Proposition

(i) *Let $x \in M_{K_0}(\overline{F}^\wedge)$, and let $[X, \iota]$ be the deformation of $\mathbb{X}$ corresponding to x. Then, the fibre of π_{K_0} through x consists of all deformations which are quasi-isogenous to X. More precisely, it consists of those pairs $[X', \iota']$ such that there exists a quasi-isogeny $f : X' \to X$ with the property that $f_{\mathbb{F}} \circ \iota' = \iota$, where $f_{\mathbb{F}}$ is the the reduction of f.*

(ii) *The fibre of π_{K_0} through x can be identified with the set of lattices in the rational Tate module $V(X) = T(X) \otimes_{\mathfrak{o}} F$, where*

$$T(X) = \varprojlim X[\varpi^m](\overline{F}^\wedge).$$

By fixing an isomorphism $\phi : F^n \to V(X)$, this set gets identified with G/K_0. More generally, let $K \subset K_0$ be an open subgroup, and let $[X, \iota, \phi]$ be a point of $M_K(\overline{F}^\wedge)$. Then, the fibre of π_K through this point can be identified with the coset G/K.

(iii) *Consider an $\overline{F}^\wedge$-valued fixed point of b on $\mathbb{P}(W)^{\mathrm{rig}}$, and choose a base point of the set of $\overline{F}^\wedge$-valued points of the fibre of π_K over this point. Using this fixed point, identify this set with G/K, as in part ii). Then there exists $g_b \in G$ with the same characteristic polynomial as b such that the action of (γ, b^{-1}) on the (set of $\overline{F}^\wedge$-valued points of the) fibre is given, in terms of this identification, by*

$$gK \longmapsto g_b g \gamma K.$$

Proof. — The first assertion follows from Prop. 23.28 of [**HG**]. The relationship between lattices in the rational Tate module and quasi-isogenies in the mixed characteristic case can be found in Lubin's paper [**Lu**], Theorem 2.2. The same holds true also in the equal characteristic case, *cf.* [**Yu**], sec. 3. The second assertion of ii) follows immediately.

Now we are going to prove part iii). Fix an $\overline{F}^\wedge$-valued point of M_K, given by a triple $[X, \iota, \phi]$. We can consider ϕ as an isomorphism $\mathfrak{o}^n \to T(X)$ which is determined up to multiplication (from the right) by elements from K. Suppose this point is mapped by π_K onto a fixed point of b. Then it follows from [**HG**], Prop. 23.28, that b lifts to an endomorphism $\widetilde{b} : X \to X$ of the formal $\mathfrak{o}$-module X such that $\widetilde{b}_{\mathbb{F}} \circ \iota = \iota \circ b$, where $\widetilde{b}_{\mathbb{F}}$ is the quasi-isogeny induced on the special fibre. $\widetilde{b}$ is mapped to b under the canonical map $\mathrm{End}_{\mathfrak{o}}(X) \otimes F \hookrightarrow \mathrm{End}_{\mathfrak{o}}(\mathbb{X}) \otimes F$. Therefore the caracteristic polynomial of $\widetilde{b}$ is the same as that of b. Let $g_b \in G$ be such that the following diagram is commutative:

$$\begin{tikzcd} F^n \arrow[r, "\phi"] \arrow[d, "g_b"'] & V(X) \arrow[d, "V(\widetilde{b})"] \\ F^n \arrow[r, "\phi"] & V(X) \end{tikzcd}$$

Let $[X', \iota', \phi']$ be an element in the fibre of π_K. Hence there is a quasi-isogeny $f : X' \to X$ and an element $g \in G$ such that the following diagram commutes:

$$\begin{tikzcd} F^n \arrow[r, "\phi'"] \arrow[d, "g"'] & V(X') \arrow[d, "V(f)"] \\ F^n \arrow[r, "\phi"] & V(X) \end{tikzcd}$$

The class $gK \in G/K$ corresponds to the point $[X', \iota', \phi']$. This point is mapped by b^{-1} to $[X', \iota' \circ b^{-1}, \phi']$. The map $\widetilde{b} \circ f : X' \to X$ is then a quasi-isogeny, if we equip X' with the map $\iota' \circ b^{-1} : \mathbb{X} \to (X')_{\mathbb{F}}$. Moreover, it is easily checked that the

following diagram commutes:

$$\begin{array}{ccc} F^n & \xrightarrow{\phi'} & V(X') \\ {\scriptstyle g_b g}\downarrow & & \downarrow{\scriptstyle V(\widetilde{b})V(f)} \\ F^n & \xrightarrow{\phi} & V(X) \end{array}$$

The action of b^{-1} on the fibre of π_K is thus given by sending gK to $g_b gK$. It is straightforward to check that the action of some $\gamma \in N_G(K)$ on this fibre is given by sending gK to $g\gamma K$. This proves the third assertion. □

4.8. Proof of the Fixed point theorem 3.8. — Let $b \in B^\times$ be regular elliptic and consider the fibre of the induced map

$$(M_K/\varpi^{\mathbb{Z}})(\overline{F}^\wedge) \longrightarrow \mathbb{P}(W)(\overline{F}^\wedge)$$

over a fixed point of b. By the preceding proposition, we may identify this set with $G/\varpi^{\mathbb{Z}}K$ and the action of (γ, b^{-1}), γ in the normalizer of K in G, is given by

$$g\varpi^{\mathbb{Z}}K \longmapsto gb_g\gamma\varpi^{\mathbb{Z}}K,$$

where $g_b \in G$ has the same characteristic polynomial as b. Hence the number of fixed points on such a fibre is

$$\#\{g \in G/\varpi^{\mathbb{Z}}K \mid g^{-1}g_b g = \gamma^{-1}\}.$$

Because there are n simple fixed points of b on $\mathbb{P}(W)$ and the morphism π_K is étale, all fixed points are simple and the total number of fixed points is

$$n \cdot \#\{g \in G/\varpi^{\mathbb{Z}}K \mid g^{-1}g_b g = \gamma^{-1}\}.$$ □

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