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QUESTIONS ABOUT SLOPES OF MODULAR FORMS

by

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Abstract. — We formulate a conjecture which predicts, in many cases, the precise p-adic valuations of the eigenvalues of the Hecke operator T_p acting on spaces of classical modular forms. The conjecture has very concrete consequences in the classical theory, but can also be thought of as saying that there is a lot of unexplained symmetry in many of the Coleman-Mazur eigencurves.

***Résumé* (Questions sur les pentes des formes modulaires).** — Nous formulons une conjecture prédisant, dans de nombreux cas, les valuations p-adiques exactes des valeurs propres de l'opérateur de Hecke T_p agissant sur les espaces de formes modulaires classiques. Cette conjecture a des conséquences très concrètes sur la théorie classique, mais elle suggère aussi de nombreuses symétries inexpliquées concernant les courbes de Coleman-Mazur.

Introduction

Let $N \geqslant 1$ be a fixed integer, and let p denote a fixed prime not dividing N. If $k \in \mathbf{Z}$ then there is a complex vector space $S_k(\Gamma_0(Np))$ of cusp forms of weight k and level Np. This space is finite-dimensional over the complex numbers and comes equipped with an action of the Hecke operator U_p, an endomorphism whose eigenvalues are non-zero complex numbers. The characteristic polynomial of U_p has integer coefficients, which implies that the eigenvalues are algebraic integers. Hence we can consider the eigenvalues as lying in $\mathbf{C}$ or in $\overline{\mathbf{Q}}_l$ for any prime l.

The U_p-eigenvalues fall naturally into two classes, p-old ones and p-new ones. The p-old eigenvalues are the roots of $X^2 - a_p X + p^{k-1}$, where a_p runs through the eigenvalues of T_p acting on $S_k(\Gamma_0(N))$. A deep theorem of Deligne says that the p-old eigenvalues all have complex absolute value $p^{(k-1)/2}$. The p-new eigenvalues are what is left, and it is well-known that these eigenvalues are square roots of p^{k-2}. Hence the complex valuations of these U_p-eigenvalues are known in every case. Moreover, from these definitions it is clear that if $l \neq p$ is a prime then the U_p-eigenvalues are all l-adic units.

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From this point of view, the question that remains about valuations of eigenvalues is:

Question. — *What can one say about the p-adic valuations of the eigenvalues of U_p?*

The term "slopes" is used nowadays to refer to these valuations. A study of the simplest special case, namely $N = 1$ and $p = 2$, shows that the answer is nowhere near as simple as the other cases. The forms which are 2-new at weight k will all have slope $\frac{k-2}{2}$ and this leaves us with the oldforms, whose slopes we can easily compute from the theory of the Newton Polygon, if we know the 2-adic valuations of the eigenvalues of T_2 acting on cusp forms of level 1. The smallest k for which non-zero level 1 cusp forms exist is $k = 12$; the space $S_{12}(\mathrm{SL}_2(\mathbf{Z}))$ is one-dimensional, and T_2 acts as multiplication by -24. Hence the 2-old eigenvalues of U_2 at weight 12 and level 2 are the two roots of $X^2 + 24X + 2^{11}$, and these two roots have 2-adic valuations equal to 3 and 8. Note that $3 \neq 8$, and so the story is already necessarily different to the complex and l-adic cases. We include a short table of valuations and slopes for small weights.

Weight	2-adic valuations of T_2-eigenvalues at level 1	Slopes of U_2 at level 2
12	3	3,8
14		6,6
16	3	3,7,12
18	4	4,8,13
20	3	3,9,9,16
22	5	5,10,10,16
24	3,7	3,7,11,16,20
26	4	4,12,12,12,21

From this table, one wonders whether there is any structure at all in the slopes. However, the purpose of this paper is to suggest that in fact there is a very precise structure here. In fact, in this paper we explain a completely elementary conjectural combinatorial recipe, recursive in the weight k, for generating the above table line by line. In fact, for a large class of pairs (N, p) (including $(1, p)$ for all primes $p < 100$ apart from 59 and 79) we give a conjectural recipe for the valuations of the T_p-eigenvalues at level N, and hence the slopes of U_p at level Np. We strongly believe that there should be a recipe for generating the slopes of U_p at level Np for any N and p, given as an input the slopes for level N and weights at most $p+2$. However we have not yet managed to formulate such a recipe at the present time. In this paper, we offer a recipe only in the case where p is $\Gamma_0(N)$-regular, a term that we shall define later.

Before we explain our conjectural recipe, we shall explain what is known about the slopes of U_p, and what has been conjectured before. The first observation, hinted at by the apparent randomness in the table above, is that to find structure in the slopes of U_p one should, contrary to the complex and l-adic cases, not consider the slopes at one fixed weight, but let the weight vary. There are well-known concrete examples of this phenomenon. For example, a theorem of Hida says that for fixed level, the number of U_p-eigenvalues with slope zero is bounded, and indeed for $k \geqslant 2$ this number depends only on k modulo $p-1$ (resp. modulo 2) for p odd (resp. $p=2$). As an example of this, we note that there are no slope zero forms in the table above, and we deduce from Hida's theorem that in fact for $N=1$ and $p=2$ there will never be any slope zero forms, however high the weight gets.

These theorems about U_p-eigenvalues of slope 0 were generalised by Gouvêa and Mazur to an explicit conjecture in [**11**] about the number of eigenvalues of arbitrary slope as the weight varies. The Gouvêa-Mazur conjecture says that if $M \geqslant 0$ is any integer, then for k and k' sufficiently large (which nowadays means at least $M+2$) and congruent modulo $(p-1)p^M$, the number of U_p-eigenvalues of slope α at weight k and weight k' should be the same, for any $\alpha \leqslant M$. Experimental evidence for this conjecture was supplied by Mestre in the case where $N=1$ and p is small. A few years after this conjecture was made, ground-breaking work of Coleman in [**6**] showed that cuspidal eigenforms naturally lay in p-adic analytic families, and an analysis by Wan [**17**] of Coleman's methods showed that one could deduce a weaker version of the Gouvêa-Mazur conjectures, namely that for k and k' sufficiently large, and congruent modulo $(p-1)p^M$, the number of eigenvalues with slope α at these two weights were equal, if $\alpha \leqslant O(\sqrt{M})$. The constants here are all explicit.

Note added in proof: For a few years the gap between the conjecture and the theoretical results was a mystery, but in some sense the mystery was resolved when a counterexample to the Gouvêa-Mazur conjecture was found by the author and F. Calegari in the case $N=1$ and $p=59$. This paper was written before the counterexample was found and in fact it was the results in this paper which led the author and Calegari to a study of the particular case $p=59$, which is the smallest prime for which (at level 1) the results of this paper do not apply. Note that for $N=1$, although the Gouvêa-Mazur conjecture is false for $p=59$, it may well still be true for $N=1$ and $p<59$, and indeed perhaps the results of this paper are an indication that it is true if p is $\Gamma_0(N)$-regular (see later for the definition). This paper is not about the counterexample at $p=59$ but about the extra structure discovered for $p<59$. The counterexample at $p=59$ is explained in [**4**].

The families in Coleman's work were beautifully interpolated into a mysterious geometric object, constructed by Coleman and Mazur, called an "eigencurve", whose very existence implies deep results about modular forms. One can compute what are essentially local equations for small pieces of these eigencurves for explicit p and N,

and computations of this nature have been undertaken by Emerton in [**9**] and Coleman, Stevens and Teitelbaum in [**7**], where for $N = 1$ and $p = 2, 3$ respectively the authors manage to compute the majority of the part of the eigencurve with smallest slope. Computations like this have concrete consequences in the theory—for example, Emerton deduced from his computations that when $N = 1$, the smallest slope of U_2 was periodic as the weight increased, repeating the pattern $3, 6, 3, 4, 3, 5, 3, 4$ indefinitely (one can see the first instance of this pattern in the table above, which already indicates that the table is much too small to be able to indicate what is going on).

The computations of Mestre concerning the Gouvêa-Mazur conjecture were done about ten years ago, and because computers are currently increasing vastly in speed, it was clear that one could go much further nowadays. The author's motivations for actually going further were several—firstly, Wan's results, and unpublished analogous theorems of the author for automorphic forms on definite quaternion algebras, both gave a version of the Gouvêa-Mazur conjecture with $\alpha \leqslant O(\sqrt{M})$ rather than $\alpha \leqslant M$, and this led us to believe that perhaps the Gouvêa-Mazur conjectures were too optimistic. Hence we thought we would make a concerted effort to search for a counterexample (Note added in proof: see [**4**] for the counterexample that we ultimately found). Secondly, several years ago we had come up with an (again unpublished) fast algorithm for computing a matrix representing T_2 on $S_k(\mathrm{SL}_2(\mathbf{Z}))$ and we felt that this would help us with the project. Thirdly, it seemed that a serious computation would be a way to get a "feeling" for the Coleman-Mazur eigencurves. Finally, William Stein has recently written a package that computes spaces of modular forms, and a serious computation seemed like a good way of testing his programs. We should remark that Gouvêa also did many computations since [**11**] was written, and the reader that wants to see the current status of things is strongly recommended to refer to [**10**] or to [**16**].

Our extensive numerical calculations did not (initially) reveal any counterexamples to the Gouvêa-Mazur conjecture (Note added in proof: however they did lead us to the observation that $p = 59$ was somehow different to other primes $p < 59$ and this is what ultimately led to the counterexample). On the contrary, to our surprise, they revealed what in many cases seemed to be far more structure. The Gouvêa-Mazur conjectures predict local constancy of slopes, in some sense, whereas, with the help of the numerical data, we were able to formulate in many cases a new conjecture, which predicted all slopes precisely.

Our investigations of the phenomenon of patterns in slopes were inspired by the aforementioned computations of Emerton, and also by results in Lawren Smithline's 1999 UC Berkeley thesis. We are grateful to both Smithline and Emerton for several helpful remarks. Smithline proves in his thesis that there is some structure to the set of slopes of weight zero 3-adic overconvergent modular forms of level 1, and this structure was one of the reasons why we were inspired to do these computations. We

are also grateful to William Stein and Tamzin Cuming for providing many spare CPU cycles, and to the referee for several helpful comments.

Although the conjectural formula that is the heart of this paper is of a purely elementary nature, it seems very complicated to explain. The structure of this paper is as follows. In the first section we explain what we mean by the notion of $\Gamma_0(N)$-regularity above. In the second, we formulate the conjecture. The third section is an attempt to explain heuristically our motivation behind the precise details of the conjecture. Finally, the fourth section raises further related questions. In particular, the finiteness questions 4.4 and 4.5 do not apparently appear to have been raised before.

Although we shall not mention overconvergent forms in the main body of this paper, we should perhaps mention that the original reason we were motivated to do these computations was to try and understand the geometry of the Coleman-Mazur eigencurves in some specific cases. Closely related to conjectures about the values of slopes of classical modular forms are analogous conjectures about the values of slopes of overconvergent forms, as one can see by a simple continuity argument and the theorem of Coleman that overconvergent forms of small slope are classical. In fact these conjectures below about slopes of classical forms could be entirely rephrased in terms of overconvergent forms. On the other hand, this rephrasing seemed equally complicated, if not more, and so we have not mentioned it below. However, in the specific case of $p = 2$ and $N = 1$, the author and F. Calegari have managed to come up with a conjecture for both classical and overconvergent forms that is much simpler to state, and have furthermore have succeeded in proving it for overconvergent forms of weight 0—we can prove that the valuation of the nth slope of U_2 is $1 + 2v_2\left(\frac{(3n)!}{n!}\right)$. In particular, all slopes are positive odd integers, which could perhaps be regarded as some very weak evidence towards Question 4.2 below. See the forthcoming [**3**] for more details.

Note added in proof: the forthcoming Northwestern thesis [**12**] of Graham Herrick attempts to explain the main conjecture of this paper in a much more conceptual manner.

1. $\Gamma_0(N)$-regularity

Let N be a fixed positive integer and let p be a fixed prime not dividing N. For $k \geqslant 2$ an even integer, define v_k to be the sequence of p-adic valuations of the eigenvalues of T_p acting on $S_k(\Gamma_0(N))$, with multiplicities, arranged in increasing order. For example, if $N = 1$ and $p = 2$ then we see from the table above that $v_{24} = [3, 7]$, where we use square brackets to denote a sequence.

We firstly remark that there is probably no elementary combinatorial formula for predicting v_k in general. For example, when $k = 12$ and $N = 1$ one finds that T_p acts

as a p-adic unit for most primes, but occasionally (for example for $p = 2, 3, 5, 7$ and 2411) the eigenvalue of T_p is divisible by p. Our goals are thus slightly more modest. Define $k_p = \frac{p+3}{2}$ if $p > 2$, and define $k_2 = 4$.

Question 1.1. — *Is there an elementary combinatorial recipe which, given v_k for all $k \leqslant k_p$, predicts v_k for all $k \geqslant 2$?*

We have substantial numerical evidence suggesting that the answer to the question above is "yes", although we have not really made the question precise. Indeed, we shall not make this question precise in general, but only in the case where the prime p is $\Gamma_0(N)$-regular. We now give a definition of $\Gamma_0(N)$-regularity.

Definition 1.2 **($\Gamma_0(N)$-regularity: p odd)**. — If $p > 2$ then we say that p is $\Gamma_0(N)$-*regular* if the eigenvalues of T_p acting on $S_k(\Gamma_0(N))$ are all p-adic units, for all even integers $2 \leqslant k \leqslant k_p$.

If $p = 2$ then this definition is not a good idea in general, because by Hida theory we see that the number of unit eigenvalues of T_p at weight 4 is bounded above by $\dim(S_2(\Gamma_0(2N))) - \dim(S_2(\Gamma_0(N)))$, which is almost always less than $\dim(S_4(\Gamma_0(N)))$.

Definition 1.3 **($\Gamma_0(N)$-regularity: $p = 2$)**. — We say that the prime $p = 2$ is $\Gamma_0(N)$-*regular* if

(1) The eigenvalues of T_2 on $S_2(\Gamma_0(N))$ are 2-adic units.

(2) There are exactly $\dim(S_2(\Gamma_0(2N))) - \dim(S_2(\Gamma_0(N)))$ eigenvalues of T_2 on $S_4(\Gamma_0(N))$ which are 2-adic units, and all the others (if any) have 2-adic valuation equal to 1.

The reader who would like a uniform definition should perhaps think of the definition as saying that p is $\Gamma_0(N)$-regular if the valuations of the eigenvalues of T_p for $k \leqslant k_p$ are as small as possible. This definition for $p = 2$ is a little ad-hoc, and is based on the fact that a computation in the case of $p = 2$ and $N = 5$ showed that we did not want 2 to be $\Gamma_0(5)$-regular. The modification is motivated by the following consequence of (one form of) the Gouvêa-Mazur conjecture: if $p = 2$ and all eigenvalues of T_2 on $S_2(\Gamma_0(N))$ are units, then there should be no eigenvalues of T_2 on $S_4(\Gamma_0(N))$ with valuation strictly between 0 and 1. This justifies the phrase "as small as possible" above.

Assume for the rest of this section that $p > 3$. Then any continuous odd irreducible Galois representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ which has determinant equal to an integer power of the cyclotomic character, and which is modular, has a twist coming from a characteristic zero form of weight at most $p + 1$, level equal to the conductor of ρ, and trivial character. Furthermore, one can read off whether the eigenvalue of T_p on such a form is a p-adic unit by the local behaviour of ρ at p. Details of these results can be found for example in [**15**] and [**8**]. Finally, by the theory of theta cycles, if

there is a mod p eigenform of level N and weight k with $\frac{p+3}{2} < k \leqslant p+1$ which is in the kernel of T_p, then there is another such form of weight $p+3-k \leqslant \frac{p+3}{2}$. From these facts, one can easily deduce

Lemma 1.4. — *$p > 3$ is $\Gamma_0(N)$-regular if and only if any irreducible modular Galois representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ with conductor dividing N and determinant a power of the mod p cyclotomic character is necessarily reducible when restricted to a decomposition group at p.*

The restriction to $p > 3$ is because of technical problems lifting mod p forms with trivial character to characteristic zero forms with trivial character, and could perhaps be avoided if we worked with $\Gamma_1(N)$, or with mod p modular forms.

If one now assumes Serre's conjecture on modularity of continuous irreducible odd mod p Galois representations, then one can deduce a purely representation-theoretic criteria for $\Gamma_0(N)$-irregularity, because the word "modular" in the lemma above can then be replaced by "continuous and odd". This formulation of $\Gamma_0(N)$-regularity can perhaps be thought of as an analogue to the representation-theoretic criteria for irregular (in the classical sense) primes—if a prime is irregular in the classical sense then there is an upper-triangular 2-dimensional mod p Galois representation which is non-split, unramified away from p and tamely ramified at p.

The first few $\mathrm{SL}_2(\mathbf{Z})$-irregular primes are $59, 79, 107, 131, 139, 151, 173, \ldots$.

2. The conjecture

Recall that we have fixed N and p, where now $p \geqslant 2$ is back to being an arbitrary prime not dividing N. Below, we shall give a recipe for constructing sequences s_2, s_4, $s_6, \ldots$. These sequences depend only on k, p and the dimension of various space of cusp forms of level N and Np. The main conjecture of this paper is

Conjecture 2.1. — *Assume that p is $\Gamma_0(N)$-regular. Then the sequences s_2, $s_4, \ldots$ of integers are precisely the sequences v_2, $v_4, \ldots$ of p-adic valuations of T_p acting on $S_k(\Gamma_0(N))$.*

The recipe defining the s_k is messy, and it seems to us that its ideal presentation is as a computer program. The recipe depends on the dimension of various spaces of cusp forms, sometimes with non-trivial character, and the only package of which we are currently aware that has these things built in is the MAGMA package [**1**]. We have implemented our conjecture in MAGMA, and the source is available at the author's web page [**2**]. We have also implemented our conjecture in pari-gp [**14**], but this was a little messier because we also had to implement some of the theory of Dirichlet characters, and also routines for computing dimensions of spaces of cusp forms with non-trivial character. Again, the source is available at [**2**]. The reader

may well find playing with these programs a lot more enjoyable than attempting to read the description of the conjecture below.

Firstly, some notation. A *sequence* denotes a finite sequence $[a_1, a_2, \ldots, a_n]$ of integers. The square brackets are merely notational. If $s = [a_1, a_2, \ldots, a_n]$ is a sequence, then we let $l(s) := n$ denote the length of s and we let $s[i] := a_i$ denote its ith term. We say that a sequence s is *increasing* if $s[i] \leqslant s[i+1]$ for all i with $1 \leqslant i < l(s)$. The *union* $a \cup b$ of two sequences is the sequence of length $l(a) + l(b)$ defined as the sequence a followed by the sequence b. Note that this is of course not commutative in general. If a and b are sequences of the same length l, then $\mathrm{Min}(a, b)$ denotes the sequence of length l whose ith term is $\mathrm{Min}(a[i], b[i])$.

For k an integer, write $d(k)$ for the dimension of $S_k(\Gamma_0(N))$, write $d_p(k)$ for the dimension of $S_k(\Gamma_0(Np))$, and for ε a Dirichlet character of level p, write $d_{p,\varepsilon}(k)$ for the dimension of $S_k(\Gamma_0(N) \cap \Gamma_1(p); \varepsilon)$. For $n, r \geqslant 0$ we define $\kappa(n, r)$ to be the constant sequence $[r, r, \ldots, r]$ of length n. If v is a sequence of length l and e is an integer, we define $v + e$ to be the sequence $[v[1] + e, v[2] + e, \ldots, v[l] + e]$ and $e - v$ to be the sequence $[e - v[l], e - v[l-1], \ldots, e - v[1]]$ (note the reversal of order). If v is a sequence and $0 \leqslant \delta \leqslant l(v)$, we define $\sigma(v, \delta)$ to be the truncation $[v[1], v[2], \ldots, v[\delta]]$ of v. More generally, if $1 \leqslant \delta_1, \delta_2 \leqslant l(v)$, we define $\sigma(v, \delta_1, \delta_2)$ to be $[v[\delta_1], v[\delta_1 + 1], \ldots, v[\delta_2]]$, where this is interpreted as the empty sequence if $\delta_2 < \delta_1$. For $\alpha \in \mathbf{Q}$, we write $\lfloor \alpha \rfloor$ for the largest integer which is at most α.

We begin by defining sequences $t_2, t_4, \ldots$ of integers; note that later on we will define s_k to be t_k for $k > 2$, and hence in particular for $k > 2$ we are conjecturing that t_k is going to be the sequence of slopes at weight k. We will define the first few t_k "by hand", and then proceed recursively to define t_k for all positive even integers k.

We set $t_2 = \kappa(d_p(2) - d(2), 0)$. If $p = 2$ then we set $t_4 = t_2 \cup \kappa(d(4) - l(t_2), 1)$ and define $k_{\min} = 6$. If $p > 2$ then for $4 \leqslant k \leqslant p + 1$ even we set $t_k = \kappa(d(k), 0)$ and set $k_{\min} = p + 3$. For $p > 2$ what we are doing here is assuming that all slopes are 0 for all weights $k \leqslant p + 3$, and in particular for all $k \leqslant k_p$, which is our $\Gamma_0(N)$-regularity condition.

Now let us assume that $k \geqslant k_{\min}$ is even and that we have defined t_l for all even $l < k$. We will now define t_k. The definition depends on three parameters a, b and c, defined thus. Let a be the unique element of $\mathbf{Z}_{\geqslant 1}$ such that $p^a < k - 1 \leqslant p^{a+1}$. Let b be the unique integer with $1 \leqslant b \leqslant p - 1$ such that $p^a b < k - 1 \leqslant p^a(b + 1)$. Set $c = 1 + \lfloor \frac{(k-2-p^a b)}{p^{a-1}} \rfloor$. Then $1 \leqslant c \leqslant p$. Also, let m be the number of cusps on $X_0(N)$.

We will firstly define a sequence V which will be the "first few slopes" of t_k. The algorithm used for the definition of V will depend on b and c. More precisely, the definition of V will depend on which of the following cases we are in: $b + c \leqslant p - 1$, $b < p - 1 < b + c$ or $b = p - 1$. Note that if $p = 2$ then the third case is the only one that can arise. We will attempt to give some explanation of what is happening

at this point in the algorithm, in particular the motivation behind the definitions of the k_i, in the next section.

Case 1: $b+c \leqslant p-1$. — We set $k_1 = k-b(p-1)p^{a-1}$ and $k_2 = k-(b-1)(p-1)p^{a-1}-2(b+c-1)p^{a-1}$. We set $v_1 = t_{k_1}$ and $v_2 = t_{k_2}$. Define $B = p^a b + p^{a-1}(c-1)+1$, set $e = k - B$ and let ε denote χ^{B-1}, where χ is any Dirichlet character of conductor p and order $p-1$ (note that $p > 2$). Finally set $s = 1 + d_{p,\varepsilon}(1+e)$.

If $l(v_1) \geqslant s-1$ then we set $V_1 = \sigma(v_1, s-1)$. Otherwise we set $V_1 = v_1 \cup (e - \sigma(v_2, s-1-l(v_1)))$. Finally, we set $V = V_1 \cup \kappa(m, e)$.

Case 2: $b < p-1 < b+c$. — We set $k_1 = k - ((b+1)p^{a-1}(p-1))$ and $k_2 = k - p^{a-1}(p-1)$. We set $v_1 = t_{k_1}$ and $v_2 = t_{k_2}$. We define $B = (b+1)p^{a-1}(p-1)+1$ and set $e = k-B$. Finally, set $s = 1+d_p(1+e)$, let $s_2 = \lfloor (s-1)/2 \rfloor$ and let $e_2 = \lfloor e/2 \rfloor$.

If $l(v_1) \geqslant s-1$ then we set $V_1 = \sigma(v_1, s-1)$. If $s-1 \leqslant 2l(v_1) < 2(s-1)$ then we set $V_1 = v_1 \cup (e - \sigma(v_1, s-1-l(v_1)))$. If however $2l(v_1) < s-1$ then define $w = \sigma(v_2, l(v_1)+1, s_2)$, and our definition of V_1 depends on the parity of s. If s is even then we set $V_1 = v_1 \cup w \cup [e_2] \cup (e-1-w) \cup (e-v_1)$ and if s is odd then we set $V_1 = v_1 \cup w \cup (e-1-w) \cup (e-v_1)$. Note here that $[e_2]$ denotes the sequence with one element, e_2.

Finally, if $e = 1$ then we define $V = V_1 \cup \kappa(m-1, 1)$ and otherwise we set $V = V_1 \cup \kappa(m, e)$.

Case 3: $b = p-1$. — This is the only case that occurs when $p = 2$. It is similar to case 2 but w is slightly modified. We set $k_1 = k - p^a(p-1)$ and $k_2 = k - p^{a-1}(p-1)$. We set $v_1 = t_{k_1}$ and $v_2 = t_{k_2}$. We set $B = p^a(p-1)+1$ and $e = k-B$. Next, set $s = 1 + d_p(1+e)$, set $s_2 = \lfloor (s-1)/2 \rfloor$ and set $e_2 = \lfloor e/2 \rfloor$.

Again, if $l(v_1) \geqslant s-1$ then we set $V_1 = \sigma(v_1, s-1)$, and if $s-1 \leqslant 2l(v_1) < 2(s-1)$ then we set $V_1 = v_1 \cup (e - \sigma(v_1, s-1-l(v_1)))$. If however $2l(v_1) < s-1$ then define $w_0 = \sigma(v_2, l(v_1)+1, s_2)$ and set $w = \mathrm{Min}(w_0 + 1, \kappa(l(w_0), e_2))$ (recall that this minimum is taken pointwise). Now we proceed as in case 2. If s is even then we set $V_1 = v_1 \cup w \cup [e_2] \cup (e-1-w) \cup (e-v_1)$, and if s is odd then we set $V_1 = v_1 \cup w \cup (e-1-w) \cup (e-v_1)$.

Finally, if $e = 1$ then we set $V = V_1 \cup \kappa(m-1, 1)$ and otherwise we set $V = V_1 \cup \kappa(m, e)$.

We are finally ready to define t_k. If $l(V) \geqslant d(k)$ then we simply let t_k be $\sigma(V, d(k))$. Otherwise, we set $k_3 = 2\boldsymbol{B} - k$, $v_3 = t_{k_3}$, and define $t_k = \sigma(V \cup (e+v_3), d(k))$.

This gives us an infinite sequence of sequences t_2, $t_4, \ldots$. The definition of s_k is now simple: $s_k = t_k$ if $k > 2$, and $s_2 = \kappa(d(2), 0)$. Having now defined s_k we remind the reader that the conjecture is that s_k should be the slopes of T_p on modular forms of level N.

Note that although the definition of s_k is messy, it is elementary to implement on a computer, and in particular it is much easier to compute s_k than to compute

actual slopes of modular forms. For example when $p = 2$ it takes under a second to compute $s_{1,000,000}$ (note that one does not need to compute s_k for all $k < 1,000,000$ to compute $s_{1,000,000}$; indeed one only needs to compute 49 other s_k) but computing the characteristic polynomial of a matrix acting on weight 1,000,000 modular forms would be beyond modern computers.

We remark finally that the fact that s_2 differs from t_2 indicates that perhaps one should work with slopes of U_p at level Np rather than T_p at level N.

3. Remarks on the conjecture

Although the conjecture made above has some interesting consequences (see the next section) and raises some related interesting questions, the author feels that the precise form of the conjecture itself is deeply unsatisfactory. The conjecture is basically saying that there is a very precise and unproven structure amongst slopes, but it seems to us that when this structure is discovered and proved, it will probably not prove the conjecture as it stands—it is much more likely to explain how the conjecture should have been formulated. F. Calegari and the author in fact have a much more readable form of the conjecture in the special case $p = 2$ and $N = 1$, and have proved several cases of it (see [**3**]).

There was a lot of motivation behind the recipe in the conjecture. The recipe was formulated by firstly assuming that the Gouvêa-Mazur conjectures were in fact much too weak, and seeing what kind of consequences this assumption had. We take the time here to explain a little about the motivation behind the details that we understand.

Let us consider, for example, the definition of V_1 in Case 1. What is going on is that V_1 should in fact be the vector of U_p-slopes on $S_{1+e}(\Gamma_0(N) \cap \Gamma_1(p); \varepsilon)$. The number e is chosen so that $e+1$ is congruent to k modulo p^{a-1}. The power of the Teichmüller character chosen is to ensure that the weight-characters $x \mapsto x^k$ and $x \mapsto x^{1+e}\varepsilon(x)$ are in the same component of weight space. Hence one should expect small slopes in $S_{1+e}(\Gamma_0(N) \cap \Gamma_1(p); \varepsilon)$ and $S_k(\Gamma_0(Np))$ to be close, and we are predicting that many of them coincide. The w_p operator will send a form of slope s in $S_{1+e}(\Gamma_0(N) \cap \Gamma_1(p); \varepsilon^{-1})$ to a form of slope $e - s$ in $S_{1+e}(\Gamma_0(N) \cap \Gamma_1(p); \varepsilon)$. So to explain the higher slopes in V_1 we should look at small slopes in $S_{1+e}(\Gamma_0(N) \cap \Gamma_1(p); \varepsilon^{-1})$. The weight-character corresponding to these forms is close to $x \mapsto x^r$ if r is an integer which is congruent to $k + 2 - 2B$ modulo $(p-1)$ and to $1 + e$ modulo a high power of p. The integer k_2 has this property, because B is congruent to $b + c$ modulo $p - 1$. This then completely explains the motivation behind the definition of V_1 in case 1.

As another example, we explain the motivation for the final $e + v_3$. Let us assume that we are in case 3. If f is an eigenform of weight k_3 and slope σ then the Hodge-Tate weights of the associated Galois representation are 0 and $k_3 - 1$. Tate twisting this representation e times gives a Galois representation with Hodge-Tate weights

$e = k - 1 - p^a(p-1)$ and $k_3 - 1 + e = p^a(p-1)$. Hence p-adically these Hodge-Tate weights are close to $k-1$ and 0 respectively, and the conjecture is predicting that there is a modular Galois representation which does have Hodge-Tate weights 0 and $k-1$ and which is highly congruent to this Tate twist. The associated modular form will have slope $e + \sigma$ and will presumably be highly congruent to the p-adic modular form $\theta^e f$ of weight $k_3 + 2e = k$.

It is a pleasant exercise, if one really wants to understand the nuts and bolts of the conjecture, to try and explain all the combinatorics involved in this way. However there is one step that the author cannot explain in this conceptual manner, and that is the construction of w in the middle of case 3. The fact that one sometimes has to add precisely 1 to an entry of w_0 seems to say geometrically that the eigencurve looks less "flat" near a p-newform, but is varying in a very precise way. What seems to be happening is that families of overconvergent eigenforms that do not contain any classical p-newforms seem to be a lot flatter in general than families containing newforms. Here we use the word flat in a non-technical sense, to mean that the slope tends to vary a lot less as one moves through the family.

4. Consequences of the conjecture and related questions

In this last section we raise some consequences and probable consequences of the conjecture, and the thoughts behind it. We start by emphasizing that we strongly believe that $\Gamma_0(N)$-regularity is a red herring, and that there should be a recipe which gives either the valuations of the eigenvalues of T_p at level N, or the slopes of U_p at level Np, in all cases. This recipe should take as input the slopes at weight $k \leqslant k_p$, or perhaps the slopes at weight $k \leqslant p+1$. However, if p is not $\Gamma_0(N)$-regular the situation is more complicated. As an example of why it is more complicated we present the first consequence of our conjecture:

Consequence 4.1. — *If p is $\Gamma_0(N)$-regular, and Conjecture 2.1 is true, then for any $k \geqslant 2$ the eigenvalues of T_p on $S_k(\Gamma_0(N))$ are all non-zero, and the valuation of any such eigenvalue is an integer.*

Wan has asked whether in the general case the denominators of the valuations are bounded by a constant depending on N and p, but independent of k. One may ask a stronger question (recall that $k_2 = 4$ and $k_p = \frac{p+3}{2}$ for $p > 2$, and also that p is prime to N):

Question 4.2. — *Let M be the lowest common multiple of the denominators of the slopes of U_p on forms of level Np and weight k, with $2 \leqslant k \leqslant k_p$. Does the denominator of any slope at level Np at any weight divide M?*

Related to these questions are questions about fields of definitions of modular forms. Let f_k denote the characteristic polynomial of T_2 acting on the space $S_k(\mathrm{SL}_2(\mathbf{Z}))$.

Maeda has conjectured that this polynomial is always irreducible over $\mathbf{Q}$, and various authors have checked both this statement and the stronger statement that the Galois group of its splitting field is the full symmetric group. For example, the author checked this for all $k \leqslant 2048$. On the other hand, if one looks at the factorization of f_k over $\mathbf{Q}_2$ for small values of k, one cannot help but notice that the irreducible factors are always linear or quadratic. This is related to the fact that our conjectures are frequently forcing slopes to be spread out, making it more difficult for them to repeat. We remark that the corresponding extensions of $\mathbf{Q}_2$ are sometimes ramified, even though we are conjecturing that the valuations are always integral. This raises the specific question

Question 4.3. — *Let f be a normalised eigenform of level 1. Does the extension of $\mathbf{Q}_2$ generated by the coefficients of f always have degree at most 2 over $\mathbf{Q}_2$?*

More generally, we have

Question 4.4. — *Let N be a positive integer, and let p be a prime not dividing N. Is there a bound $B = B(N,p)$ such that if f is any normalised eigenform of level N, then the coefficients of f generate an extension of $\mathbf{Q}_p$ of degree at most B? Equivalently, is there a subfield $F \subseteq \overline{\mathbf{Q}}_p$, finite over $\mathbf{Q}_p$, and depending only on N and p, such that any normalised eigenform $f \in S_k(\Gamma_0(N); \overline{\mathbf{Q}}_p)$ has q-expansion in $F[[q]]$?*

One might even consider the case where p divides N but we have not done any computations at all in this case.

A remark related to these questions: it is a recent theorem of Kilford (see [**13**]) that if f is a normalised eigenform of level $\Gamma_1(4)$ and any odd weight, the coefficients of f necessarily lie in $\mathbf{Q}_2$. This result could have been noticed over 50 years ago, and the author finds it interesting that it was proved before it was conjectured. This might reflect on the current ease with which one can compute spaces of forms, thanks to Stein. Kilford's proof relies strongly on Coleman's theory of overconvergent modular forms, and explicit computations of matrices related to the U_2 operator. Note added in proof: these results have now been generalised to level $\Gamma_1(2^n)$ in [**5**].

Motivated by the Fontaine-Mazur conjecture, one can move completely away from the theory of modular forms. If $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is continuous, irreducible, unramified at a finite set of primes, and crystalline at p, then it has a conductor $N(\rho)$, which is a positive integer prime to p. As before, let N be any positive integer and let p be a prime not dividing N.

Question 4.5. — *Is there a subfield $F \subset \overline{\mathbf{Q}}_p$, finite over $\mathbf{Q}_p$ and depending only on N and p, such that if $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is irreducible, crystalline at p, and has conductor N, then the trace of $\rho(g)$ lies in F for all $g \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$?*

One could relax the crystalline condition to a potentially semi-stable one, and let p divide N, for an even stronger question—here one has to define the p-part of

the conductor of such a representation using Fontaine theory. One could even ask n-dimensional analogues of this question, but we shall leave this to the optimistic reader.

We now move onto a rather amusing consequence of Conjecture 2.1. The dependence of the recipe in the conjecture on N is only via the dimension of various spaces of cusp forms of level N and Np. There are cases where these dimensions happen to coincide for different N. For example, $\dim(S_k(\Gamma_0(6))) = \dim(S_k(\Gamma_0(8)))$ for all N, as can be seen from classical formulae for these dimensions. In these cases, the recipe might produce the same results for different N. For example, $p = 5$ is both $\Gamma_0(6)$-regular and $\Gamma_0(8)$-regular, and as a consequence one gets the following rather strange result:

Consequence 4.6. — *If Conjecture 2.1 is true, then the* 5*-adic valuations of the eigenvalues of T_5 on $S_k(\Gamma_0(6))$ coincide, with multiplicities, with the* 5*-adic valuations of T_5 on $S_k(\Gamma_0(8))$. Similarly the slopes of U_5 on $S_k(\Gamma_0(30))$ coincide with the slopes of U_5 on $S_k(\Gamma_0(40))$.*

The author has checked the above consequence in MAGMA for $k \leqslant 60$. The reader who knows about Coleman's results and the overconvergent theory will realise that as another consequence of the conjecture, the overconvergent (finite) slopes of U_5 at tame levels 6 and 8 must coincide for any weight-character in the closure of $\mathbf{Z}$. This result is surely not explained by a morphism between the two eigencurves, and the author has no idea of a more conceptual explanation of this phenomenon. Perhaps it is just a numerical coincidence. Even more unnerving is that it is very easy to find many more examples where coincidences at small weight imply equalities at all weights. The author does not know, unfortunately, of an example where the set of slopes coming from two levels are the same at all small weights but where p is not regular (regularity in the sense of this paper).

We next raise some combinatorial problems, which can presumably be attacked using only elementary techniques, and are almost certainly accessible.

Question 4.7. — *Is the conjecture well-defined, in the sense that every time a sequence is implicitly assumed to have at least a given length, it does have this length?*

Question 4.8. — *The sequences v_k are by definition increasing. Are the sequences s_k produced by the conjecture always increasing?*

One baulks at the combinatorics behind these questions, although they are surely both accessible. We believe that the answers are affirmative, in both cases, but have only checked the details in the case $p = 2$ and $N = 1$.

Question 4.9. — *Does Conjecture 2.1 imply that if p is $\Gamma_0(N)$-regular then the valuation of any eigenvalue of T_p on $S_k(\Gamma_0(N))$ is at most $\frac{k-1}{p+1}$?*

This phenomenon, that slopes tend to be very small, was explicitly noted by Gouvêa. The author again convinced himself that the conjecture did indeed imply that all slopes were at most $\frac{k-1}{3}$ in the case $p = 2$ and $N = 1$.

Gouvêa also considered the following: if one divides the sequence of slopes of U_p at weight k by a factor of $k-1$, one gets a sequence of rationals in the closed interval $[0, 1]$, and this sequence can be thought of as giving a (finite) probability measure on this closed interval.

Question 4.10. — *Does Conjecture 2.1 imply that, as k increases, the measures tend to a limit, and if so then what is this limit?*

Numerical experiments with $p = 2$ and $N = 1$ suggest to the author that in this case measures were tending to a limit, which gave the point $\frac{1}{2}$ a mass of $\frac{1}{3}$, and which distributed the remaining mass of $\frac{2}{3}$ uniformly on $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. This points to a natural conjecture in the general case.

There remains the very natural

Question 4.11. — *Does Conjecture 2.1 imply the Gouvea-Mazur conjectures in the $\Gamma_0(N)$-regular case?*

Again, the author convinced himself that this was the case when $p = 2$ and $N = 1$. We will not trouble the reader with the excruciating details.

References

[1] W. BOSMA, J. CANNON & C. PLAYOUST – The Magma algebra system I: The user language, *J. Symb. Comput.* **24** (1997), no. 3-4, p. 235–265, `http://www.maths.usyd.edu.au:8000/u/magma/`.

[2] K. BUZZARD – Web page, `http://www.ma.ic.ac.uk/~buzzard`.

[3] K. BUZZARD & F. CALEGARI – Slopes of overconvergent 2-adic modular forms, *Compositio Math.*, to appear.

[4] ———, A counterexample to the Gouvea-Mazur conjecture, *Comptes-rendus Mathematiques* **338** (2004), no. 10, p. 751–753.

[5] K. BUZZARD & L.J.P. KILFORD – The 2-adic eigencurve at the boundary of weight space, *Compositio Math.*, to appear.

[6] R. COLEMAN – p-adic Banach Spaces and families of modular forms, *Invent. Math.* **127** (1997), no. 3, p. 417–479.

[7] R. COLEMAN, G. STEVENS & J. TEITELBAUM – Numerical experiments on families of p-adic modular forms, in *Computational Perspectives on Number Theory, Proceedings of a conference in Honor of A.O.L. Atkin*, AMS/IP studies in advanced mathematics, vol. 7, American Mathematical Society, 1998, p. 143–158.

[8] S. EDIXHOVEN – The weight in Serre's conjectures on modular forms, *Invent. Math.* **109** (1992), p. 563–594.

[9] M. EMERTON – Ph.D. Thesis, Harvard University, 1998.

[10] F. GOUVÊA – Web page, `http://www.colby.edu/personal/f/fqgouvea/`.

[11] F. GOUVÊA & B. MAZUR – Families of modular eigenforms, *Math. Comp.* **58** (1992), no. 198, p. 793–805.

[12] G. HERRIUCK – Ph.D. Thesis, Northwestern University, forthcoming.

[13] L. KILFORD – Ph.D. Thesis, University of London, 2002.

[14] The Pari computer algebra system – currently available at `http://pari.math.u-bordeaux.fr/`.

[15] K. RIBET – Report on mod l representations of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, in *Motives (Seattle, WA, 1991), vol. II*, Proc. Sympos. Pure Math., vol. 55, American Mathematical Society, Providence, RI, 1994, p. 639–676.

[16] W. STEIN – Web page, currently `http://modular.fas.harvard.edu/`.

[17] D. WAN – Dimension variation of Classical and p-adic Modular Forms, *Invent. Math.* **133** (1998), no. 2, p. 449–463.

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