Astérisque

YVES ANDRÉ
LUCIA DI VIZIO

q-difference equations and p-adic local monodromy

Astérisque, tome 296 (2004), p. 55-111
<http://www.numdam.org/item?id=AST_2004__296__55_0>

Astérisque
296, 2004, p. 55–111

q-DIFFERENCE EQUATIONS AND p-ADIC LOCAL MONODROMY

by

Yves André & Lucia Di Vizio

Abstract. — We present a p-adic theory of q-difference equations over arbitrarily thin annuli of outer radius 1. After a detailed study of rank one equations, we consider higher rank equations and prove a local monodromy theorem (a q-analog of Crew's quasi-unipotence conjecture). This allows us to define, in this context, a canonical functor of "confluence" from q-difference equations to differential equations, which turns out to be an equivalence of categories (in the presence of Frobenius structures).

***Résumé* (Équations aux q-différences et monodromie p-adique).** — Nous présentons une théorie p-adique des équations aux q-différences sur des couronnes arbitrairement minces de rayon extérieur 1. Après une étude détaillée des équations de rang 1, nous nous penchons sur le cas de rang supérieur et nous démontrons un théorème de monodromie locale (un q-analogue de la conjecture de quasi-unipotence de Crew). Cela nous permet de définir, dans ce contexte, un foncteur canonique de « confluence » des équations aux q-différences vers les équations différentielles, qui s'avère être une équivalence de catégories (en présence de structures de Frobenius).

Introduction

In the context of p-adic differential equations, the expression "local theory" occurs in two different senses. In the naive sense, it refers to the study of the behaviour of solutions in a small punctured disk around a given singularity. This theory has been reasonably well-understood for a long time[(1)].

On the other hand, according to some insights of Dwork and Grothendieck, the geometrically relevant p-adic differential equations are those which admit analytic solutions in all non-singular open unit disks, and which extend a little inside the singular disks. They should be understood as objects (cohomological coefficients) belonging to geometry in characteristic p. It is then consistent with this viewpoint

2000 ***Mathematics Subject Classification.*** — Primary 39A13; Secondary 33D05, 12H50.
Key words and phrases. — p-adic monodromy, q-difference equations, p-adic representation.

[(1)] Although by no means completely understood, *cf.* for instance the problems raised by Ramero's theory [**Ra98**] in its differential variant.

to call "local theory" the study of the behaviour of solutions in arbitrarily thin annuli with outer radius 1 contained in singular open unit disks[(2)].

In this sense, the local theory of p-adic differential equations has been developped first by Robba (in rank one), then by Christol and Mebkhout (in arbitrary rank), and has recently reached full maturity with the proof of the so-called local monodromy theorem (Crew's quasi-unipotence conjecture) which provides a bridge toward the theory of p-adic Galois representations.

The objective of this paper is to set up a local theory of p-adic q-difference equations, parallel to the differential theory, and to put a link forward between the two theories.

* * *

In the history of the theory of p-adic differential equations, going from the rank 1 case to arbitrary rank has been a difficult step. This is due in part to the fact that the study of rank 1 p-adic differential equations indulges fairly down-to-earth methods (*cf.* for instance [**R85**], [**CC96**]). In the first part of the paper we develop an analogous theory for p-adic q-difference equations of rank 1. The techniques employed are inspired by the differential case and, due to their explicit and direct nature, bring to the fore the relationship with differential equations. In fact, we construct a *canonical deformation functor* from the category of p-adic differential equations of rank 1 to the category of p-adic q-difference equations, which we describe explicitly.

The first part is organized as follows. In § 1 we recall some basic facts of p-adic q-difference algebra proved in [**DV03**]. In § 2 we prove some properties of the q-exponential function which play a significant role in the sequel. Sections § 3 and § 4 contain a q-analog of Dwork-Robba's criterion of solvability and its application to q-difference equations of rank 1 with meromorphic coefficient. The results in § 4 are used in the next section to show that one can actually reduce the study of rank-one q-difference equations analytic over an arbitrary thin annulus of outer radius 1, to the study of rank 1 q-difference equations with polynomial coefficient. This reduction is crucial for the characterization of q-difference equations with Frobenius structure (*cf.* § 6). We finish the first part by proving that for a q-difference equation having a Frobenius structure is equivalent to being a "deformation" of a differential equation with strong Frobenius structure (*cf.* § 7). From there, we obtain the p-adic monodromy theorem in the rank 1 case and the deformation functor (*cf.* § 8).

There are two appendices, the first one being devoted to the Frobenius structure of the q-exponential series. In the second one, we give a q-analog of Dwork's approach to the p-adic gamma function via the Frobenius structure of so-called exponential modules.

* * *

(2) See the previous paper [**A**] for more detail and perspective, and for the apparatus of analogies which motivates the present paper.

In the second part, we consider q-difference modules M of arbitrary rank over the "Robba ring" $\mathcal{R}$ of analytic functions on an arbitrarily thin annulus of outer radius 1. We prove the local monodromy theorem for those q-difference modules which admit a Frobenius structure: there exists a finite étale extension $\mathcal{R}'/\mathcal{R}$ coming from characteristic p, such that $M \otimes_{\mathcal{R}} \mathcal{R}'[\log x]$ becomes a trivial q-difference module (*cf.* § 14.2, § 14.3 for various equivalent precise statements). We follow K. Kedlaya's approach to the p-adic local monodromy theorem in the differential case, proving along the way a q-analog of Tsuzuki's theorem on unit-root objects.

This second part is organized as follows. We first discuss finite étale extensions $\mathcal{R}'/\mathcal{R}$ coming from characteristic p, and how the q-difference operator d_q extends to $\mathcal{R}'$ (the lack of an explicit expression for this extended operator leads to many technical difficulties in the sequel). We then introduce and investigate two notions of Frobenius structures for q-difference modules: the strong Frobenius structure (analogous to its differential counterpart), and the confluent weak Frobenius structure (which yields a sequence of q^{p^n}-difference modules converging to a differential module with Frobenius structure).

In § 13, we analyse q-difference modules over $\mathcal{R}$ with overconvergent (strong) Frobenius structure of slope 0. As in Tsuzuki's theorem, they arise from finite p-adic representations of the inertia group of a local field of characteristic p.

We then prove three versions of the theorem of quasi-unipotence for q-difference modules over $\mathcal{R}$ which admit a strong Frobenius structure. We also show that such q-difference modules have a confluent weak Frobenius structure.

This gives rise to a canonical functor of "confluence" between such q-difference modules (M, Σ_q), and differential modules over $\mathcal{R}$ which admit a strong Frobenius structure, which has a canonical quasi-inverse (15.1, 15.2). More precisely, for any such (M, Σ_q), there is a canonical sequence of $q^{p^{is}}$-difference structures on the $\mathcal{R}$-module M (for fixed s and with $i \to \infty$, so that $q^{p^{is}} \to 1$), related to each other by Frobenius, and which converges to a differential structure on M.

PART I
RANK 1

1. Generalities on p-adic q-difference equations of rank 1

1.1. The q-difference algebra of analytic functions over an annulus

Let K be a field of characteristic zero, complete with respect to a non archimedean absolute value $|\ |$, with residue field k of characteristic $p > 0$. We denote by $\mathcal{O}_K$ its ring of integers and we assume that the absolute value is normalized by $|p| = p^{-1}$.

For any interval $I \subset \mathbb{R}_{\geqslant 0}$ we consider the K-algebra $\mathcal{A}_K(I)$ of analytic functions with coefficients in K on the annulus $\mathcal{C}_K(I) = \{x \in K : |x| \in I\}$:

$$\mathcal{A}_K(I) = \left\{ \textstyle\sum_{n\in\mathbb{Z}} a_n x^n : a_n \in K; \lim_{n\to\pm\infty} |a_n|\rho^n = 0 \ \forall\, \rho \in I \right\}.$$

We denote by $\mathcal{M}_K(I)$ its field of fractions (the field of meromorphic functions on $\mathcal{C}_K(I)$), and by $\mathcal{B}_K(I)$ the subring of bounded elements of $\mathcal{A}_K(I)$. The theory of Newton polygons shows that every invertible analytic function on $\mathcal{C}(I)$ is bounded, so that $\mathcal{A}_K(I)^* = \mathcal{B}_K(I)^*$. We will omit the subscript K when there is no ambiguity.

We fix once and for all an element $q \in K$, such that $|1-q| < 1$ and that q is not a root of unity. The algebra $\mathcal{A}(I)$ has a natural structure of a q-difference algebra. This means that the homeomorphism

$$\begin{aligned} \mathcal{C}(I) &\longrightarrow \mathcal{C}(I) \\ x &\longmapsto qx \end{aligned}$$

induces a K-algebra isomorphism

$$\begin{aligned} \sigma_q : \mathcal{A}(I) &\longrightarrow \mathcal{A}(I) \\ f(x) &\longmapsto f(qx) \end{aligned}$$

Similarly for $\mathcal{M}(I)$ and $\mathcal{B}(I)$.

1.2. The q-derivation. — To the operator σ_q one associates a "twisted derivation" d_q defined by

$$d_q(f)(x) = \frac{f(qx) - f(x)}{(q-1)x},$$

which satisfies the twisted Leibniz Formula:

$$d_q(fg)(x) = f(qx)d_q(g)(x) + d_q(f)(x)g(x). \tag{1}$$

For any pair of integers $n \geqslant i \geqslant 1$ and any $f, g \in \mathcal{M}(I)$ the q-derivation d_q verifies:

$$d_q x^n = [n]_q x^{n-1}, \text{ where } [n]_q = 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q-1}; \tag{2}$$

$$\frac{d_q^n}{[n]_q^!} x^i = \binom{n}{i}_q x^{n-i}, \text{ where } [0]_q^! = 1,\ [n]_q^! = [n]_q[n-1]_q^! \text{ and } \binom{n}{i}_q = \frac{[n]_q^!}{[i]_q^![n-i]_q^!}; \tag{3}$$

$$d_q^n(fg)(x) = \sum_{j=0}^{n} \binom{n}{j}_q d_q^{n-j}(f)(q^j x) d_q^j(g)(x). \tag{4}$$

1.3. q-difference equations. — Let us now consider a *q-difference equation of rank* 1 *with coefficients in* $\mathcal{M}(I)$:

$$y(qx) = a(x)y(x), \quad a(x) \in \mathcal{M}(I). \tag{5}$$

We shall often write (5) in the form

$$d_q y(x) = g(x)y(x), \text{ with } g(x) = \frac{a(x)-1}{(q-1)x}. \tag{6}$$

For any $u \in \mathcal{M}(I)^*$, $z(x) = u(x)^{-1}y(x)$ is a solution of the q-difference equation

$$z(qx) = \left[u(qx)^{-1}a(x)u(x)\right] z(x) \tag{7}$$

or equivalently of

$$d_q z(x) = \left[\frac{u(x)}{u(qx)}g(x) + \frac{d_q u(x)}{u(qx)}\right] z(x). \tag{8}$$

We shall say that equations (5) and (7) (or (6) and (8)) are $\mathcal{M}(I)$-equivalent[(3)].

From (6), one derives the following sequence of equations

$$d_q^n(y)(x) = g_n(x)y(x),$$

with $g_1(x) = g(x)$, $g_{n+1}(x) = g_n(qx)g_1(x) + d_q g_n(x)$. It is convenient to set $g_0(x) = 1$.

If $g(x)$ is analytic at 0, then $\sum_{n\geqslant 0} \frac{g_n(0)}{[n]_q^!}x^n$ is a formal solution of $y(qx) = a(x)y(x)$.

1.4. Generic points. — In order to apply the technique of generic points, we shall have to use an auxiliary extension of normed fields Ω/K, with the following properties (for the construction of such a field, see for instance [**Ro00**, § 3, 2]):

1) the field Ω is complete and algebraically closed;
2) the set of values of Ω is $\mathbb{R}_{\geqslant 0}$;
3) the residue field of Ω is a transcendental extension of the residue field of K.

For any $\rho \in \mathbb{R}_{\geqslant 0}$ the field Ω contains an element t_ρ, called a *generic point (at distance ρ from 0)*, such that t_ρ is transcendent over K and $|t_\rho| = \rho$, so that the norm induced over $K(t_\rho) \subset \Omega$ is defined by

$$\left|\frac{\sum a_i t_\rho^i}{\sum b_j t_\rho^j}\right| = \frac{\sup_i |a_i|\rho^i}{\sup_j |b_j|\rho^j}.$$

Definition 1.1. — For any $\rho \in I$, we call the number

$$R_\rho(\sigma_q - a(x)) = \inf\left(\rho, \liminf_{n\to\infty}\left|\frac{g_n(t_\rho)}{[n]_q^!}\right|^{-1/n}\right)$$

the generic radius of convergence of $y(qx) = a(x)y(x)$ at t_ρ. We will write simply R_ρ when no confusion is possible.

(3) We shall also use a similar terminology for other rings of functions.

1.5. Properties of the generic radius of convergence. — The following proposition summarizes some facts about the generic radius of convergence, which are proven in [**DV03**].

Proposition 1.2

1) **(Twisted Taylor expansion)** *Let $d_q y(x) = g(x)y(x)$ be a q-difference equation with coefficient $g(x) \in \mathcal{M}(I)$ and let $\xi \in \mathcal{C}(I)$. Suppose that $g(x)$ does not have any pole in $q^{\mathbb{N}}\xi$. Then $d_q y(x) = g(x)y(x)$ has an analytic solution in a neighborhood of ξ if and only if*

$$R := \liminf_{n\to\infty} \left|\frac{g_n(\xi)}{[n]_q^!}\right|^{-1/n} > |(q-1)\xi|.$$

In that case, the unique analytic solution $y(x)$ of $y(qx) = a(x)y(x)$ in the open disk $D(\xi, R^-)$ verifying $y(\xi) = 1$ coincides with the sum of the series

$$\sum_{n\geqslant 0} \frac{g_n(\xi)}{[n]_q^!}(x,\xi)_{n,q}, \quad \text{where } (x,\xi)_{n,q} = (x-\xi)(x-q\xi)\cdots(x-q^{n-1}\xi).$$

2) *Let $b(x) = u(qx)^{-1}a(x)u(x)$, with $u(x) \in \mathcal{M}(I)^*$. Then $R_\rho(\sigma - a(x)) = R_\rho(\sigma - b(x))$ for any $\rho \in I$, i.e., the generic radius of convergence is invariant under $\mathcal{M}(I)$-equivalence.*

3) **(q-analog of the Dwork-Robba effective bound)** *If $R_\rho > |q-1|\rho$, then*

$$\left|\frac{g_n(t_\rho)}{[n]_q^!}\right| \leqslant \frac{1}{R_\rho^n}, \text{ for any } n \geqslant 1.$$

4) **(Transfer to an ordinary disk)** *Let $g(x)$ be analytic over $D(\xi, \rho^-)$, with $\xi \in K$ and $|\xi| \leqslant \rho$, and let $R_\rho > |q-1|\rho$. Then $d_q y(x) = g(x)y(x)$ has an analytic solution in the disk $D(\xi, R_\rho^-)$. Moreover, the equation $d_q y(x) = g(x)y(x)$ has an analytic solution in the disk $D(\xi, \rho^-)$ if and only if $R_\rho = \rho$.*

5) **(Transfer to a regular singular disk)** *Let $a(x) \in \mathcal{A}(]0,1[)$ and $u(x) \in K[\![x]\!]$ be a formal power series with coefficients in K such that $u(qx)^{-1}a(x)u(x) \in K$. If $R_\rho = \rho$, the series $u(x)$ converges for $|x| < \rho$.*

Corollary 1.3. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation with $a(x) \in \mathcal{A}([0,1[)$ (resp. $\mathcal{A}([0,1[) \cap \mathcal{M}([0,1]))$. Then $y(qx) = a(x)y(x)$ has a solution $y(x)$ analytic and bounded in $\mathcal{C}([0,1[)$ if and only if $\lim_{\rho\to 1} R_\rho = 1$ (resp. $R_1 = 1$).*

Proof. — Let $a(x) \in \mathcal{A}([0,1[)$. It follows from the assertions in 4) of the previous proposition that the existence of a solution $y(x)$ analytic and bounded over $\mathcal{C}([0,1[)$ implies $R_\rho = \rho$ for any $\rho \in]0,1[$. Hence we conclude that $\lim_{\rho\to 1} R_\rho = 1$.

On the other hand, suppose $\lim_{\rho\to 1} R_\rho = 1$. Again, by the assertions of 4), that the formal solution

$$y(x) = \sum_{n\geqslant 0} \frac{g_n(0)}{[n]_q^!}x^n$$

of $y(qx) = a(x)y(x)$ converges in $D(0, R_\rho^-)$ for any $\rho \in]0,1[$. This proves that $y(x) \in \mathcal{A}([0,1[)$. Finally statement 3. implies that

$$\left|\frac{g_n(0)}{[n]_q^!}\right| \leqslant \left|\frac{g_n(t_\rho)}{[n]_q^!}\right| \leqslant \frac{1}{R_\rho^n}. \tag{9}$$

By letting ρ tend to 1, one proves that $y(x)$ is bounded.

If $a(x) \in \mathcal{A}([0,1[) \cap \mathcal{M}([0,1])$, the generic radius of convergence R_1 is defined. Assertion 4) of (1.2) states that $y(qx) = a(x)y(x)$ has a solution $y(x) \in \mathcal{A}([0,1[)$ if and only if $R_1 = 1$. Moreover the existence of the analytic solution $y(x) \in \mathcal{A}([0,1[)$ implies that $R_\rho = \rho$ for any $\rho \in]0,1[$, therefore the inequality (9) allows to conclude that $y(x)$ is bounded. □

It is customary in the theory of q-difference equations to assume that the coefficient a is invertible[(4)]. We shall follow this tradition, and consider mostly q-difference equations $y(qx) = a(x)y(x)$ with $a(x) \in \mathcal{A}(I)^* = \mathcal{B}(I)^*$. Written in the form $d_q(y) = gy$, this implies that $g \in \mathcal{B}(I)$. Actually most of the time, we shall not only have $g \in \mathcal{B}(I)$ as in the differential case (the logarithmic differential of any element of $\mathcal{A}(I)$ belongs to $\mathcal{B}(I)$), but also $|(q-1)g|_{\mathcal{B}(I)} < 1$, $|a|_{\mathcal{B}(I)} = 1$.

2. An example: the q-exponential function

2.1. The q-exponential $e_q(x)$. — The power series

$$e_q(x) = \sum_{n \geqslant 0} \frac{x^n}{[n]_q^!}$$

is a q-deformation of the exponential series and satisfies the q-difference equation

$$d_q e_q = e_q,$$

that is to say

$$e_q(qx) = (1 + (q-1)x)e_q(x).$$

Proposition 2.1. — *The series $e_q(x)$ has radius of convergence $\prod_{i \geqslant 0} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}}$.*

Proof. — Every positive integer n can be uniquely written in the form $n = p^s m + k$, where $m, s, k \in \mathbb{Z}$, m and p are coprime and $0 \leqslant k \leqslant p-1$. If $k > 0$ then $|[n]_q| = \left|q^{p^s m}[k]_q + [p^s m]_q\right| = 1$, in fact $|[k]_q| = 1$ and $|[p^s m]_q| = |[m]_{q^{p^s}}[p^s]_q| < 1$. Therefore

$$\left|[n]_q^!\right| = \prod_{i=1}^{[n/p]} |[p]_q[i]_{q^p}| = \prod_{i \geqslant 0} \left|[p]_{q^{p^i}}^{\left[n/p^{i+1}\right]}\right|,$$

(4) This convention is also in use in the higher rank case, where $a(x)$ is a matrix; it allows to define the dual system.

where the product on the right is actually finite. It follows that

$$\limsup_{n\to\infty} \left|[n]_q^!\right|^{1/n} = \prod_{i\geqslant 0} \left|[p]_{q^{p^i}}^{1/p^{i+1}}\right|. \qquad \square$$

Proposition 2.2. — *If*

$$\text{(10)} \qquad \operatorname{dist}(aq^{\mathbb{Z}_p}, bq^{\mathbb{Z}_p}) = \min_{\alpha\in\mathbb{Z}_p} |a - q^\alpha b| < \textit{radius of convergence of } e_q(x),$$

the analytic function $e_q(ax)/e_q(bx)$, with $a, b \in K$, $a \neq 0 \neq b$, is overconvergent, i.e., it has a radius of convergence > 1.

Remark 2.3. — Notice that $\exp(ax)/\exp(bx)$, with $a, b \in K$, is overconvergent if and only if $|a - b| < |\pi|$, hence the inequality above is actually a q-deformation of the analogous condition in the differential framework.

Proof. — The series $e_q(ax)/e_q(bx)$ is solution of the q-difference equation

$$y(qx) = \frac{1 + (q-1)ax}{1 + (q-1)bx}\, y(x).$$

Notice that $\min_{\alpha\in\mathbb{Z}_p} |q^\alpha a - b|$ is realized for $\alpha \in \mathbb{Z}$, $\alpha \geqslant 0$, hence, by multiplying $e_q(ax)/e_q(bx)$ by $(1, (q-1)ax)_{\alpha,q}$, we can assume that $\min_{\alpha\in\mathbb{Z}_p} |a - q^\alpha b| = |a - b|$. Observing that for any integer $n \geqslant 1$

$$d_q(1, (q-1)bx)_{n,q} = (q^n - 1)b(1, (q-1)bqx)_{n-1,q},$$

one verifies by induction that the series $e_q(ax)/e_q(bx)$ is solution of

$$d_q^n y(x) = \frac{(a,b)_{n,q}}{(1, (q-1)bx)_{n,q}} y(x).$$

This proves that

$$\frac{e_q(ax)}{e_q(bx)} = \sum_{n\geqslant 0} \frac{(a,b)_{n,q}}{[n]_q^!} x^n.$$

Let $r(e_q(x))$ be the radius of convergence of $e_q(x)$. The condition $|a - b| < r(e_q(x))$ implies that

$$\limsup_{n\to\infty} \left|\frac{(a,b)_{n,q}}{[n]_q^!}\right|^{1/n} = \frac{1}{r(e_q(x))} \limsup_{n\to\infty} \left|\prod_{i=0}^{n-1} (a - bq^i)\right|^{1/n} < 1. \qquad \square$$

2.2. The analytic function $\log e_q(x)$. — If $|e_q(x) - 1| < 1$, it makes sense to consider the analytic function $L_q(x) = \log e_q(x)$. From the q-difference equation satisfied by e_q, one derives immediately the equation:

$$L_q(qx) = L_q(x) + \log(1 + (q-1)x),$$

which can be rewritten in the form

$$d_q L_q(x) = \sum_{n\geqslant 0} (-1)^n (q-1)^n \frac{x^n}{n+1}.$$

We find that the analytic function $L_q(x)$ has the (well-known[5]) expansion (*cf.* [**HL46**, § 2], [**Q**]):

$$L_q(x) = \sum_{n \geqslant 1} \frac{(-1)^{n-1}(q-1)^{n-1}}{[n]_q n}\, x^n.$$

Proposition 2.4

1) *The series $L_q(x)$ converges for $|x| < |q-1|^{-1}$.*
2) *If $|x| < \dfrac{|p|^{1/(p-1)}}{|q-1|} \sup\left(|p|^{1/(p-1)}, |q-1|\right)$, then $|L_q(x)| < |x|$.*

Proof

1) It is enough to notice that

$$\liminf_{n\to\infty} \left|\frac{(q-1)^{n-1}}{[n]_q n}\right|^{-1/n} = \frac{1}{|q-1|} \liminf_{n\to\infty} \left|n^2 \log q\right|^{1/n} = \frac{1}{|q-1|}.$$

2) Since $L_q(x) = x\left(1 + \sum_{n\geqslant 1} \dfrac{(-1)^n (q-1)^n}{[n+1]_q (n+1)}\, x^n\right)$, we have to prove that $|x| < \frac{|p|^{1/(p-1)}}{|q-1|} \sup\left(|p|^{1/(p-1)}, |q-1|\right)$ implies

$$\sup_{n\geqslant 1} \left|\frac{(q-1)^n}{[n+1]_q (n+1)} x^n\right| < 1.$$

This follows from the inequalities $|p|^{1/(p-1)} \leqslant |n+1|^{1/n}$ and $\sup\left(|p|^{1/(p-1)}, |q-1|\right) \leqslant |[n+1]_q|^{1/n}$. □

In the sequel of this section, we assume that K *contains the p-th roots of unity.* It then also contains $p-1$ distinct non zero roots of the equation $\pi^p = -p\pi$. One picks one of them and denotes it by π (Dwork's constant). Notice that $|\pi| = |p|^{1/(p-1)}$.

Corollary 2.5. — *If $|q-1| < |\pi|$, the series $e_q(\pi x)/\exp(\pi x)$ is overconvergent.*

Proof. — Let us consider the series

$$L_q(\pi x) = \pi x + \sum_{n\geqslant 2} (-1)^{n-1} \frac{(1-q)^{n-1}}{[n]_q n} \pi^n x^n.$$

Notice that the assumption $|q-1| < |\pi|$ implies $|[n]_q| = |n|$. For $n = 2, \ldots, p-1$ and $|x| < |\pi(q-1)|^{-1/2}$ the following inequality holds:

$$\left|\frac{(1-q)^{n-1}}{[n]_q n} \pi^n x^n\right| < |\pi|.$$

On the other hand, for $n \geqslant p$ and $|x| < |q-1|^{-1}|\pi|^{1/2} \leqslant |q-1|^{-1}|\pi|^{-(n-3)/n}$ we have

$$\left|\frac{(1-q)^{n-1}}{[n]_q n} \pi^n x^n\right| = \left|\frac{(q-1)^n \pi^n}{n^2} x^n\right| < |\pi|.$$

[5] the complex analog is essentially the quantum dilogarithm

We conclude that there exists $\varepsilon > 0$ such that for $|x| < 1+\varepsilon$ we have $|L_q(\pi x)-\pi x| < |\pi|$. Hence the series $e_q(\pi x)/\exp(\pi x) = \exp\left(L_q(\pi x) - \pi x\right)$ is analytic for $|x| < 1 + \varepsilon$. □

2.3. Overconvergent solutions of ordinary q-difference equation at 0

In this subsection we are going to use the analytic function L_q to construct an overconvergent solution of a linear q-difference equation of rank 1, under a suitable hypothesis.

Lemma 2.6. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation such that $a(x)$ is an analytic function at 0, with $a(0) = 1$. Then write $a(x)$ as an infinite product $\prod_{i\geqslant 1}\left(1+\mu_i x^i\right)$. If there exists $\varepsilon > 0$ such that*

$$\sup_{i\geqslant 1} \frac{|\mu_i|}{|q^i-1|}(1+\varepsilon)^i < |\pi|$$

then the infinite product

$$\prod_{i\geqslant 1} e_{q^i}\left(\frac{\mu_i}{q^i-1}x^i\right)$$

converges to an overconvergent solution of $y(qx) = a(x)y(x)$.

Proof. — Since

$$\frac{|\mu_i|}{|q^i-1|}(1+\varepsilon)^i < |\pi| \frac{\sup\left(|\pi|, \left|q^i-1\right|\right)}{|q^i-1|} \quad \text{for any } i \geqslant 1,$$

we have

$$\sup_{|x|<1+\varepsilon}\left|L_{q^i}\left(\frac{\mu_i x^i}{q^i-1}\right)\right| < \sup_{|x|<1+\varepsilon}\left|\frac{\mu_i x^i}{q^i-1}\right| < |\pi|.$$

It follows that

$$z(x) = \sum_{i\geqslant 1} L_{q^i}\left(\frac{\mu_i x^i}{q^i-1}\right)$$

is an analytic function for $|x| < 1+\varepsilon$ and that $\sup_{|x|<1+\varepsilon}|z(x)| < |\pi|$. We conclude that

$$\exp z(x) = \prod_{i\geqslant 1} e_{q^i}\left(\frac{\mu_i x^i}{q^i-1}\right)$$

is an overconvergent solution of $y(qx) = a(x)y(x)$. □

Proposition 2.7. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation such that $a(x) = \prod_{i\geqslant 1}\left(1+\mu_i x^i\right)$ is an overconvergent analytic function. Then there exists a positive integer M and a positive real number ε such that $y(qx) = a(x)y(x)$ is $\mathcal{M}([0,1+\varepsilon[)$-equivalent to*

$$y(qx) = \prod_{i=1}^{M}\left(1+\mu_i x^i\right) y(x).$$

Proof. — Since $a(x)$ is overconvergent there exists a positive integer M and a real number $\varepsilon > 0$ such that $|\mu_i| > (1+\varepsilon)^i$ for any $i > M$. Let $0 < \varepsilon' < \varepsilon$. For any $|x| \leqslant 1+\varepsilon' < 1+\varepsilon$ we have

$$\lim_{i\to\infty} \frac{(1+\varepsilon)^i |x^i|}{|(q^i-1)|} = 0.$$

Let us fix an integer $M > 0$ such that

$$\sup_{i>M} \sup_{|x|<1+\varepsilon'} \frac{(1+\varepsilon)^i |x^i|}{|(q^i-1)|} < |\pi|$$

and set

$$\widetilde{a}(x) = \prod_{i>M} \left(1 + \frac{\mu_i}{x^i}\right).$$

It follows from the previous lemma that there exists $u(x) \in \mathcal{A}([0, 1+\varepsilon'[)$ such that $u(qx) = \widetilde{a}(x)u(x)$, *i.e.*, such that $\dfrac{u(qx)}{u(x)} a(x) = \prod_{i=1}^{M} \left(1 + \mu_i x^i\right)$. □

3. Solvability (at the boundary)

Recall that the Robba ring $\mathcal{R} = \mathcal{R}_x = \mathcal{R}_{K,x}$ is the ring

$$\mathcal{R} = \cup_{\varepsilon>0} \mathcal{A}(]1-\varepsilon, 1[)$$

of analytic functions on some thin annulus with the unit circle as outer boundary.

The subring

$$\mathcal{E}^\dagger = \mathcal{E}^\dagger_x = \mathcal{E}^\dagger_{K,x} = \cup_{\varepsilon>0} \mathcal{B}(]1-\varepsilon, 1[)$$

of bounded functions is endowed with the sup-norm $|\sum a_n x^n|_{\mathcal{E}^\dagger} = \sup |a_n|$ (caution: this is not a Banach ring). If the valuation of K is discrete, this is a Henselian field, with residue field $k((x))$.

We introduce the subrings

$$\mathcal{B} = \mathcal{B}_x = \mathcal{A}([0,1[) \cap \mathcal{E}^\dagger, \quad \mathcal{H}^\dagger = \mathcal{H}^\dagger_x = \cup_{\varepsilon>0} \mathcal{A}(]1-\varepsilon, \infty])$$

(on which the restriction of $|\ |_{\mathcal{E}^\dagger}$ is the sup-norm, according to the principle of the maximum).

In this section, we begin the study of q-difference equations $y(qx) = a(x)y(x)$ with $a(x) \in \mathcal{R}^* = (\mathcal{E}^\dagger)^*$.

By iteration of the operator d_q, we deduce from the q-difference equation $y(qx) = a(x)y(x)$ a sequence of equations

$$d_q^n y(x) = g_n(x) y(x),$$

with $g_n(x) \in \mathcal{E}^\dagger$ and $g_0(x) = 1$. Since $a(x) \in \mathcal{B}(]1-\varepsilon, 1[)$ for some $\varepsilon > 0$, it makes sense to consider $R_\rho(\sigma_q - a(x))$ for $\rho \in]1-\varepsilon, 1[$.

Definition 3.1. — The equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, is said to be *solvable*[(6)] (at the outer boundary) if

$$\lim_{\rho \to 1} R_\rho(\sigma_q - a(x)) = 1.$$

Remark 3.2. — It follows immediately from (1.2) that solvability is invariant under $\mathcal{E}^\dagger$-equivalence.

One can define the notion of solvability without using the generic radius of convergence:

Lemma 3.3. — $\lim_{\rho \to 1} R_\rho = \inf\left(1, \liminf_{n \to \infty} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}^{-1/n}\right)$.

Proof[(7)]. — Let us set

$$R_{\mathcal{E}^\dagger} = \inf\left(1, \liminf_{n \to \infty} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}^{-1/n}\right).$$

Notice that for any $\rho \in]1-\varepsilon, 1[$, we have

$$R_\rho = \liminf_{n \to \infty} \left(\sup_{0 \leqslant s \leqslant n} \left|\frac{g_n(t_\rho)}{[n]_q^!}\right|\right)^{-1/n} \text{ and } R_{\mathcal{E}^\dagger} = \liminf_{n \to \infty} \left(\sup_{0 \leqslant s \leqslant n} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger}\right)^{-1/n}.$$

Moreover (*cf.* [**CD94**, 2.1]) $h_\rho(n) = \sup_{0 \leqslant s \leqslant n} \left|g_n(t_\rho)/[n]_q^!\right|$ is a continuous function of ρ and

$$\lim_{\rho \to 1} h_\rho(n) = \sup_{0 \leqslant s \leqslant n} \left|\frac{g_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger},$$

hence it is enough to prove the uniform convergence of the sequence $h_\rho(n)^{1/n}$.

The proof of [**DV02**, 4.2.7] actually shows that for any positive integers $N \geqslant n > s \geqslant 0$ such that $N = [N/n]\, n + s$, the $h_\rho(n)$'s verify the inequality

$$h_\rho(N)^{1/N} \leqslant h_\rho(n)^{(1/n)+(1/N)} \left|\frac{[N]_q!}{([n]_q!)^{[N/n]}\, [s]_q!}\right|^{-1/N}.$$

Hence, letting $N \to \infty$, we obtain

$$\frac{1}{R_\rho} \leqslant h_\rho(n)^{1/n} \left(\left|[n]_q^!\right|^{1/n} \lim_{N \to \infty} \left|[N]_q^!\right|^{-1/N}\right).$$

(6) This terminology is very unsatisfactory (solvable in what?) but has been of constant use in the theory of p-adic differential equations since Robba's studies. For want of a better word, we shall adopt it here.

(7) The continuity of the function R_ρ is proved in [**CD94**]. The proof that follows uses an argument of uniform convergence and it is a q-analog of an unpublished proof by F. Baldassarri and L. Di Vizio.

It follows from (1.2) that

$$\frac{1}{R_\rho}\left(\left|[n]_q^!\right|^{-1/n}\lim_{N\to\infty}\left|[N]_q^!\right|^{1/N}\right)\leqslant h_\rho(n)^{1/n}\leqslant\frac{1}{R_\rho},$$

which finishes the proof. □

4. A characterization of solvability

The following characterization of solvability is a q-analog of a result by Dwork and Robba [**DR77**, 5.4] and will be used to prove that solvable q-difference equations of rank 1 are $\mathcal{E}^\dagger$-equivalent to solvable q-difference equations with coefficients in $K\,[1/x]$.

Proposition 4.1. *Let Ω/K be the extension introduced in subsection 1.4. The following assertions are equivalent:*

1) *The q-difference equation $y(qx)=a(x)y(x)$, $a(x)\in\mathcal{E}^\dagger$, is solvable.*
2) *There exists a sequence $R_n(x)\in\mathcal{E}_\Omega^\dagger$, such that*

$$\lim_{n\to\infty}\left|\frac{R_n(qx)}{R_n(x)}-a(x)\right|_{\mathcal{E}_\Omega^\dagger}=0.$$

3) *There exists a sequence $R_n(x)\in\mathcal{E}_\Omega^\dagger$ such that*

$$\lim_{n\to\infty}\left|\frac{d_q(R_n)(x)}{R_n(x)}-g(x)\right|_{\mathcal{E}_\Omega^\dagger}=0,\quad\text{where } g(x)=\frac{a(x)-1}{(q-1)x}.$$

Proof. The equivalence between *2.* and *3.* is straightforward.

Let us prove that *3.* implies *1.* We set $g_1(x)=g(x)$ and $g_{N+1}(x)=d_qg_N(x)+g_1(x)g_N(qx)$ and we fix $\varepsilon\in\,]0,1[$ and $n>0$. We claim that if the inequality

$$\left|\frac{d_q(R_n)(x)}{R_n(x)}-g(x)\right|_{\mathcal{E}_\Omega^\dagger}\leqslant\varepsilon$$

is satisfied then we have

$$\left|\frac{d_q^N(R_n)(x)}{R_n(x)}-g_N(x)\right|_{\mathcal{E}_\Omega^\dagger}\leqslant\varepsilon\text{ for any }N\geqslant1.$$

We prove our claim by induction. In fact, it follows from proposition 1.2 that

$$\left|\frac{d_q^N(R_n)(x)}{R_n(x)}\right|_{\mathcal{E}_\Omega^\dagger}\leqslant\left|[N]_q^!\right|$$

and hence that $|g_N(x)|_{\mathcal{E}^\dagger}\leqslant1$. Therefore, recursively, one obtains:

$$\begin{aligned}
&\left|d_q^{N+1}(R_n)(x)-g_{N+1}(x)R_n(x)\right|_{\mathcal{E}_\Omega^\dagger}\\
&\quad=\left|d_q\left(d_q^N(R_n)(x)-g_N(x)R_n(x)\right)+g_N(qx)\left(d_q(R_n)(x)-g_1(x)R_n(x)\right)\right|_{\mathcal{E}_\Omega^\dagger}\\
&\quad\leqslant\sup\left(\left|d_q^N(R_n)(x)-g_N(x)R_n(x)\right|_{\mathcal{E}_\Omega^\dagger},\left|g_N(qx)\left(d_q(R_n)(x)-g_1(x)R_n(x)\right)\right|_{\mathcal{E}_\Omega^\dagger}\right)\\
&\quad\leqslant\varepsilon\left|R_n(x)\right|_{\mathcal{E}_\Omega^\dagger}.
\end{aligned}$$

Let us fix $N > 0$ and let $\varepsilon < |[N]_q^!|$. Then there exists $n \gg 0$, depending on N, such that

$$\left|\frac{g_N(x)}{[N]_q^!}\right|_{\mathcal{E}_\Omega^\dagger} \leqslant \sup\left(\left|\frac{d_q^N(R_n)(x)}{[N]_q^! R_n(x)} - \frac{g_N(x)}{[N]_q^!}\right|_{\mathcal{E}_\Omega^\dagger}, \left|\frac{d_q^N R_n(x)}{[N]_q^! R_n(x)}\right|_{\mathcal{E}_\Omega^\dagger}\right) \leqslant \sup\left(\frac{\varepsilon}{[N]_q^!}, 1\right) \leqslant 1,$$

which implies that $y(qx) = a(x)y(x)$ is solvable.

Let us now prove that *1.* implies *3.* Let us consider the sequence of elements of $\mathcal{E}^\dagger$:

$$\beta_0 = 1,\ \beta_1(x) = -g(x),\ \beta_{N+1}(qx) = d_q\beta_N(x) + \beta_1(x)\beta_N(x).$$

We choose a generic point $t_1 \in \Omega$ such that $|t_1| = 1$ and we set

$$R_n(x) = \sum_{N=0}^{n} \frac{\beta_N(x)}{[N]_q^!}(x, t_1)_{N,q} \in \mathcal{E}^\dagger[t_1] \subset \mathcal{E}_\Omega^\dagger, \quad \text{for any } n \geqslant 1.$$

Notice that $R_n(x)$ satisfies the inhomogeneous q-difference equation

$$\begin{aligned} d_q R_n(x) - g(x)R_n(x) &= \sum_{N=0}^{n} \frac{d_q\beta_N(x) + \beta_1(x)\beta_N(x)}{[N]_q^!}(x, t_1)_{N,q} - \sum_{N=1}^{n} \frac{\beta_N(qx)}{[N-1]_q^!}(x, t_1)_{N-1,q} \\ &= [n+1]_q \frac{\beta_n(x)}{[n]_q^!}(x, t_1)_{n-1,q}. \end{aligned}$$

The following lemma allows to conclude the proof by considering the subsequence $(R_{p^n-1}(x))_{n\in\mathbb{Z}_{>0}}$. □

Lemma 4.2. $\quad \left|\beta_n(x)/[n]_q^!\right|_{\mathcal{E}^\dagger} \leqslant |R_n(x)|_{\mathcal{E}_\Omega^\dagger}.$

Proof. — Consider the polynomial ring $\mathcal{E}^\dagger[z]$. We have an embedding of valued K-algebras $(\mathcal{E}^\dagger, |\ |_{\mathcal{E}^\dagger}) \subset (\mathcal{E}^\dagger[z], |\ |_{\mathcal{E}^\dagger,z})$, where $|\ |_{\mathcal{E}^\dagger,z}$ is defined by

$$\left|\textstyle\sum_i a_i(x)z^i\right|_{\mathcal{E}^\dagger,z} = \sup_i |a_i(x)|_{\mathcal{E}^\dagger}, \quad \text{for any } \textstyle\sum_i a_i(x)z^i \in \mathcal{E}^\dagger[z].$$

Let $d_{q,z}$ be a q-difference derivation acting on $\mathcal{E}^\dagger[z]$, in the following way

$$d_{q,z}\left(\textstyle\sum_{i\geqslant 0} a_i(x)z^i\right) = \sum_{i\geqslant 1} a_i(x)[i]_q z^{i-1}, \quad \text{for any } \textstyle\sum_{i\geqslant 0} a_i(x)z^i \in \mathcal{E}^\dagger[z].$$

Observe that

$$\left|\frac{d_{q,z}^n}{[n]_q^!}\left(\textstyle\sum_{i\geqslant 0} a_i(x)z^i\right)\right|_{\mathcal{E}^\dagger,z} \leqslant \left|\textstyle\sum_{i\geqslant 0} a_i(x)z^i\right|_{\mathcal{E}^\dagger,z}$$

and that

$$\left|\textstyle\sum_{i\geqslant 0} a_i(x)z^i\right|_{\mathcal{E}^\dagger,z} = \left|\textstyle\sum_{i\geqslant 0} a_i(x)t_1^i\right|_{\mathcal{E}_\Omega^\dagger}.$$

We set $R_n(x,z) = \sum_{N=0}^n \frac{\beta_N(x)}{[N]_q^!}(x,z)_{N,q} \in \mathcal{E}^\dagger[z]$. Since

$$\frac{d_{q,z}^n}{[n]_q^!}(x,z)_{N,q} = \begin{cases} 0 & \text{for any pair of integers } n > N > 0, \\ (-1)^n q^{n(n-1)/2} & \text{if } n = N, \end{cases}$$

we conclude that

$$\left|\frac{\beta_n(x)}{[n]_q^!}\right|_{\mathcal{E}^\dagger} = \left|\frac{d_{q,z}^n(R_n)}{[n]_q^!}(x,z)\right|_{\mathcal{E}^\dagger,z} \leqslant |R_n(x,z)|_{\mathcal{E}^\dagger,z} = |R_n(x,t_1)|_{\mathcal{E}_\Omega^\dagger}\,. \qquad \square$$

Remark 4.3. — Notice that if $g(x) \in K[x]$ then $R_n(x) \in \Omega[x]$.

4.1. Solvability of q-difference equations with constant coefficients

Corollary 4.4. — *The q-difference equation $y(qx) = ay(x)$, with constant coefficient $a \in K$, is solvable if and only if $a \in q^{\mathbb{Z}_p}$.*

Proof. — It follows from proposition 4.1 and remark 4.3 that $y(qx) = ay(x)$ is solvable if and only if a is the limit in $\mathcal{E}_\Omega^\dagger$ of a sequence $R_n(qx)R_n(x)^{-1}$, with $R_n(x) \in \Omega[x]$. Therefore if $y(qx) = ay(x)$ is solvable, the coefficient a is the limit of a sequence in $q^{\mathbb{Z}}$, *i.e.*, $a \in q^{\mathbb{Z}_p}$. Conversely, if $a = q^\alpha$ with $\alpha \in \mathbb{Z}_p$, then there exists a sequence of integers $\alpha_n \in \mathbb{Z}$ such that $\alpha_n \to \alpha$ and hence that $|\frac{(qx)^{\alpha_n}}{x^{\alpha_n}} - a|_{\mathcal{E}^\dagger} \to 0$. $\square$

Remark 4.5. — By induction on $n \geqslant 1$ (*cf.* [**DV03**, 1.2.4]), one can prove that the solutions of the equation $y(qx) = ay(x)$ are necessarily solutions of the sequence of equations:

$$\frac{d_q^n}{[n]_q^!}y(x) = \frac{(a-1)(a-q)\cdots(a-q^{n-1})}{(q-1)(q^2-1)\cdots(q^n-1)q^{n(n-1)/2}x^n}\,y(x).$$

Therefore the previous corollary implies that the series

$$\sum_{n\geqslant 0} \frac{(a-1)(a-q)\cdots(a-q^{n-1})}{(q-1)(q^2-1)\cdots(q^n-1)}\,x^n \tag{11}$$

has radius of convergence 1 if and only if $a \in q^{\mathbb{Z}_p}$, generalizing [**DV03**, 8.2], where the radius of convergence of (11) was calculated under the assumption $|1-q| < |p|^{1/(p-1)}$. This should allow to drop the assumption $|q-1| < |\pi|$ in [**DV03**, § 3].

4.2. Solvability of q-difference equations meromorphic at zero. —

The next corollary concerns rank 1 q-difference equations whose coefficient has at worst a pole at 0 and is analytic in $\mathcal{C}(]0,1[)$.

Corollary 4.6. — *Consider a solvable q-difference equation $y(qx) = a(x)y(x)$, with $x^N a(x) \in \mathcal{B}$ for some positive integer N. Let $a_\infty(x) \in K(x)$ be a rational function such that all the finite zeros and poles of $a_\infty(x)$ are in $\mathcal{C}([0,1[)$, and that $a(x)/a_\infty(x)$ is an invertible analytic function in $\mathcal{B}$ having value 1 at 0. Then the q-difference equations $y(qx) = a_\infty(x)y(x)$ and $y(qx) = \frac{a(x)}{a_\infty(x)}y(x)$ are both solvable.*

Proof. — It follows from (4.1) and its proof that there exists a sequence $R_n(x) \in \mathcal{E}_\Omega^\dagger \cap \mathcal{A}_\Omega(]0,1[)$ such that

$$\lim_{n\to\infty}\left|\frac{R_n(qx)}{R_n(x)} - a(x)\right|_{\mathcal{E}_\Omega^\dagger} = 0.$$

Let $G_n(x) \in \Omega(x)$ be a rational function such that $R_n(x)/G_n(x)$ is an invertible analytic function over $\mathcal{C}([0,1[)$, having value 1 at 0, and that all the poles and zeros of $G_n(x)$ are in $\mathcal{C}([0,1[) \cup \{\infty\}$. Hence the Taylor expansion at ∞ of $G_n(qx)/G_n(x)$ defines an invertible element of $\mathcal{H}^\dagger_\Omega$ and $|G_n(qx)/G_n(x)|_{\mathcal{E}^\dagger_\Omega} = 1$. Then,

$$\begin{aligned}
&\left|\frac{R_n(qx)}{R_n(x)} - a(x)\right|_{\mathcal{E}^\dagger_\Omega} \\
&= \left|\left(\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right)\frac{G_n(qx)}{G_n(x)} + \frac{a(x)}{a_\infty(x)}\left(\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right)\right|_{\mathcal{E}^\dagger_\Omega} \\
&= \left|\frac{a(x)}{a_\infty(x)}\right|_{\mathcal{E}^\dagger_\Omega} \left|\left(\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right)\frac{a_\infty(x)}{a(x)} + \frac{G_n(x)}{G_n(qx)}\left(\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right)\right|_{\mathcal{E}^\dagger_\Omega}.
\end{aligned}$$

Since $\left(\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right)\frac{a_\infty(x)}{a(x)} \in x\mathcal{B}$ and $\frac{G_n(x)}{G_n(qx)}\left(\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right) \in \mathcal{H}^\dagger$ we obtain the equality

$$\left|\frac{R_n(qx)}{R_n(x)} - a(x)\right|_{\mathcal{E}^\dagger_\Omega} = \sup\left(\left|\frac{R_n(qx)G_n(x)}{R_n(x)G_n(qx)} - \frac{a(x)}{a_\infty(x)}\right|_{\mathcal{E}^\dagger_\Omega}, \left|\frac{G_n(qx)}{G_n(x)} - a_\infty(x)\right|_{\mathcal{E}^\dagger_\Omega}\right).$$

By proposition 4.1 we conclude that both $y(qx) = a_\infty(x)y(x)$ and $y(qx) = \frac{a(x)}{a_\infty(x)}y(x)$ are solvable q-difference equations. □

5. Reduction to the case of q-difference equations with polynomial coefficient

In [**R85**], Robba has shown that any rank one differential equation over $\mathcal{E}^\dagger$ is equivalent to a differential equation with coefficient in $K[1/x]$[8]. His method uses a kind of additive decomposition of the coefficient (using logarithmic derivatives) and cannot be translates into the q-difference context. In this section we prove a q-analog of Robba's result using a kind of multiplicative decomposition of the coefficient.

Proposition 5.1. — *Any solvable q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, is $\mathcal{E}^\dagger$-equivalent to a solvable q-difference equation of the form*

$$y(qx) = q^{l_0}\prod_{i=1}^{M}\left(1 + \frac{\mu_i}{x^i}\right)y(x), \tag{12}$$

where $l_0 \in \mathbb{Z}_p$, M is a positive integer, $\mu_i \in K$ and $|\mu_i| \leqslant |q-1|$ for $i = 1, \ldots, M$.

Proof. — The proof is divided into several steps.

[8] Actually he considered only differential equations with rational coefficients, but his argument extends in general.

Step 0. — There exists an unique multiplicative decomposition

$$a(x) = \frac{\lambda}{x^N}\, l(x)\, m(x)$$

of $a(x)$ in $\mathcal{E}^\dagger$ such that
- $\lambda \in K$, $\lambda \neq 0$;
- $N \in \mathbb{Z}$;
- $l(x)$ is an invertible analytic function in $1 + x\mathcal{B}$;
- $m(x)$ is invertible function in $1 + \frac{1}{x}\mathcal{H}^\dagger$.

Proof of Step 0. — [**CM02**, 6.5] and [**C81a**]. □

Step 1. — The q-difference equation $y(qx) = a(x)y(x)$ is $\mathcal{H}^\dagger$-equivalent to a q-difference equation of the form

$$y(qx) = \frac{\lambda}{x^N}\, l(x) \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x), \tag{13}$$

where $\lambda \in K$, $\lambda \neq 0$, $N, M \in \mathbb{Z}$, $M > 0$, $\mu_i \in K$ for any $i = 1, \ldots, M$ and $l(x)$ is an invertible analytic function in $1 + x\mathcal{B}$.

Proof of Step 1. — The analytic function $m(x) \in \mathcal{H}^\dagger$, considered in Step 0, can be uniquely written as a convergent infinite product

$$m(x) = \prod_{i=1}^{\infty} \left(1 + \frac{\mu_i}{x^i}\right). \tag{14}$$

It follows from proposition 2.7 that there exists $z(x) \in \mathcal{H}^\dagger$ such that $z(qx) = \widetilde{m}(x)z(x)$ and hence that $\frac{z(qx)}{z(x)}a(x)$ has the form (13). □

Step 2. — The equation (13) is $\mathcal{B}$-equivalent to the solvable q-difference equation

$$y(qx) = \frac{\lambda}{x^N} \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x), \tag{15}$$

with $\lambda \in K$, $\lambda \neq 0$, $N, M \in \mathbb{Z}$, $M > 0$ and $|\mu_i| < 1$ for any $i = 1, \ldots, M$.

Proof of Step 2. — Let us write $l(x)$ as an infinite convergent product

$$l(x) = \prod_{i=1}^{\infty} \left(1 + \lambda_i x^i\right).$$

Since $l(x)$ does not have any zero in $\mathcal{C}([0, 1[)$, we have $|\lambda_i| \leqslant 1$ for any $i > 0$. As far as the μ_i's are concerned, it is enough to recall that the analytic function $m(x)$ (*cf.* (14)) is invertible in $\mathcal{H}^\dagger$ to conclude that $|\mu_i| < 1$ for any $i = 1, \ldots, M$. Hence it follows from (4.6) that the q-difference equations

$$y(qx) = \prod_{i=1}^{\infty} \left(1 + \lambda_i x^i\right) y(x) \tag{16}$$

and

$$y(qx) = \frac{\lambda}{x^N} \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^i}\right) y(x)$$

are both solvable. Since $l(x) \in \mathcal{B}$, the equation (16) has a solution $u(x) \in \mathcal{B}$ (*cf.* 1.3), which establishes the $\mathcal{B}$-equivalence between (13) and (15). □

Step 3. The solvability of (15) implies that $\lambda \in q^{\mathbb{Z}_p}$, $N = 0$ and $|\mu_i| < |q-1|$ for any $i = 1, \dots, M$.

Proof of Step 3. Let $b(x) = (\lambda/x^N) \prod_{i=1}^{M} \left(1 + \mu_i/x^i\right)$. Since $|(b(x)-1)/(q-1)x|_{\mathcal{E}^\dagger} \leqslant 1$, necessarily $|b(x)|_{\mathcal{E}^\dagger} = 1$ and hence $|\lambda| = 1$.

Let us write inductively $d_q^n y(x) = g_n(x) y(x)$ for any $n \geqslant 1$. Writing $g_n(x)$ explicitely in terms of $b(x)$ (*cf.* [**DV03**, 1.2.4]) we obtain

$$\begin{aligned} \left|\frac{g_n}{[n]_q^!}\right|_{\mathcal{E}^\dagger} &= \left|\frac{(-1)^n}{[n]_q^!(q-1)^n x^n} \sum_{j=0}^{n} (-1)^j \binom{n}{j}_{q^{-1}} q^{-j(j-1)/2} b(x) b(qx) \cdots b(q^{j-1}x)\right|_{\mathcal{E}^\dagger} \\ &\geqslant \left|\frac{(-1)^n}{[n]_q^!(q-1)^n} \frac{\lambda^n}{x^{(N+1)n}}\right|_{\mathcal{E}^\dagger} \\ &> \frac{1}{|q-1|^n}. \end{aligned} \tag{17}$$

This shows that if $N \neq 0$, the equation $y(qx) = b(x)y(x)$ is not solvable, in contradiction to the hypothesis. Therefore it is enough to prove that $\lambda \in \mathbb{Z}_p$ assuming that $N = 0$.

Since $y(qx) = b(x)y(x)$ is solvable, there exists $R_n(x) \in \Omega\,[x, 1/x]$ (*cf.* Remark 4.3) such that

$$\left|\frac{R_n(qx)}{R_n(x)} - b(x)\right|_{\mathcal{E}^\dagger_\Omega} \longrightarrow 0,$$

hence λ is a limit of a sequence in $q^{\mathbb{Z}}$. This proves that $\lambda \in q^{\mathbb{Z}_p}$ and that the equation $y(qx) = b(x)\lambda^{-1}y(x)$ is also solvable.

So we are reduced to proving the statement for a solvable q-difference equation of the form

$$y(qx) = \prod_{i=1}^{M} \left(1 + \frac{\mu_i}{x^{n_i}}\right) y(x),$$

or equivalently of the form

$$d_q y(x) = \left(\sum_{i=1}^{M} \frac{\mu_i (q-1)^{-1}}{x^{n_i+1}} \prod_{j>i} \left(1 + \frac{\mu_j}{x^{n_j}}\right)\right) y(x).$$

Since $|\mu_i| < 1$, the solvability (*cf.* 1.2) implies that $|\mu_i| \leqslant |q-1|$ for any $i = 1, \dots, M$.

Step 3 finishes the proof. □

6. Frobenius structure in rank 1: existence criterion

From now on, we assume that the *residue field k of K is perfect.*

In [**CC96**, § 2], it is proven that a differential equation of the form

$$\frac{y'(x)}{y(x)} = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_m}{x^m} \in \frac{1}{x} K\left[\frac{1}{x}\right]$$

has a strong Frobenius structure if and only if it is solvable and there exists a positive integer s such that $(p^s - 1)a_1 \in \mathbb{Z}$. In this section we prove an analogous result for q-difference equations. Some steps of our proof use methods that can be adapted to the differential case, simplifying some technical details in [**CC96**].

Let us consider a Frobenius automorphism τ of K, *i.e.*, a continuous automorphism of the field K lifting the Frobenius automorphism of the residue field k. Let s be a positive integer and let us assume that q is τ^s-invariant. Usually one considers the semilinear endomorphism $\phi = \phi_s$ of $\mathcal{E}^\dagger$ defined by

$$\phi_s\left(\sum_{n\in\mathbb{Z}} a_n x^n\right) = \sum_{n\in\mathbb{Z}} \tau^s(a_n) x^{p^s n}.$$

An analogous endomorphism ϕ can be defined over the q-difference algebra of analytic functions over a disk or an annulus, centered at 0 or at ∞.

Definition 6.1. — We say that a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, has a *(strong) Frobenius structure* if there exists $u(x) \in \left(\mathcal{E}^\dagger\right)^*$ such that

$$\frac{u(qx)}{u(x)} a(x) = a(x)^\phi a(qx)^\phi \cdots a(q^{p^s-1}x)^\phi \tag{18}$$

for a suitable choice of s.

Remark 6.2. — Notice that $v(x)^{\phi\sigma_q^{p^s}} = v(x)^{\sigma_q\phi}$ for any $v(x) \in \mathcal{E}^\dagger$ (here $v(x)^{\sigma_q\phi}$ means $\phi\circ\sigma_q(v(x))$). Let $y(x)$ be a solution of the equation $y(qx) = a(x)y(x)$ in an extension of $\mathcal{E}^\dagger$. If $y(qx) = a(x)y(x)$ has a Frobenius structure then $y(x)^\phi = u(x)y(x)$ with $u(x) \in \left(\mathcal{E}^\dagger\right)^*$ and (18) can be written in the following way:

$$y(x)^{\phi\sigma_q^{p^s}} = y(x)^{\phi\sigma_{q^{p^s}}} = y(x)^{\sigma_q\phi}.$$

Lemma 6.3. — *If a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, has a Frobenius structure, then it is solvable.*

Proof. — Let $\varepsilon > 0$ be such that $a(x) \in \mathcal{A}(]1-\varepsilon, 1[)$ and let $z(x) = \sum_{n\geqslant 0} \alpha_n (x, t_\rho)_{n,q}$, with $\alpha_n \in K$, be a solution of $y(qx) = a(x)y(x)$ at the generic point t_ρ, for $\rho \in]1-\varepsilon, 1[$.

It follows from the assumption that there exists $u(x) \in \left(\mathcal{E}^\dagger\right)^*$ such that $z(x)^\phi = u(x)z(x)$ is a solution of $y(qx) = a_{p^s}(x)^\phi y(x)$ at $t_\rho^{\tau^s}$. Hence we obtain

$$R_{\rho^{p^s}}(\sigma_q - a_{p^s}(x)^\phi) \leqslant R_\rho(\sigma_q - a(x))^{p^s} \leqslant \rho^{p^s} \leqslant 1. \tag{19}$$

Since $y(qx) = a(x)y(x)$ and $y(qx) = a_{p^s}(x)^\phi y(x)$ are $\mathcal{E}^\dagger$-equivalent, we have

$$\lim_{\rho\to 1} R_\rho(\sigma_q - a(x)) = \lim_{\rho\to 1} R_\rho(\sigma_q - a_{p^s}(x)^\phi).$$

This forces $y(qx) = a(x)y(x)$ to be solvable. □

We recall (*cf.* proof of (5.1), Step 0) that any invertible $a(x) \in \mathcal{E}^\dagger$ can be uniquely written as a product $a(x) = (\lambda/x^N)\, l(x)\, m(x)$, with $\lambda \in K$, $N \in \mathbb{Z}$, $l(x) \in (1 + x\mathcal{B})^*$ and $m(x) \in \left(1 + \frac{1}{x}\mathcal{H}^\dagger\right)^*$. The main result of this section is:

Theorem 6.4. — *A q-difference equation of rank* 1 *with coefficient in* $\mathcal{E}^\dagger$, *i.e.*,

$$y(qx) = \frac{\lambda}{x^N}\, l(x)\, m(x)\, y(x),$$

has a Frobenius structure if and only if it is solvable and there exists a positive integer s *such that* $\lambda^{p^s-1} \in q^{\mathbb{Z}}$.

6.1. Idea of the proof of theorem 6.4. — It follows from (6.3) and (5.1) that it is enough to prove the statement:

Proposition 6.5. — *A q-difference equation*

$$y(qx) = q^{l_0} \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x), \tag{20}$$

with $l_0 \in \mathbb{Z}_p$ *and* $\mu_1, \ldots, \mu_M \in K$, *has a Frobenius structure if and only if it is solvable and there exists an integer* $s \geqslant 0$ *such that* $l_0 \in \mathbb{Z}/(p^s - 1)$.

First of all let us remark that:

Lemma 6.6. — *A q-difference equation* $y(qx) = q^{l_0}y(x)$, *with* $l_0 \in \mathbb{Z}_p$ *(cf. (4.4)), has a Frobenius structure if and only if* $l_0 \in \mathbb{Z}/(p^s - 1)$.

Proof. — The equation $y(qx) = q^{l_0}y(x)$ is $\mathcal{E}^\dagger$-equivalent to $y(qx) = q^{l_0 p^s}y(x)$ if and only if $(p^s - 1)l_0 \in \mathbb{Z}$. □

Eventually, the proposition 6.5 is a consequence of the following proposition:

Proposition 6.7. — *A q-difference equation of the form*

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x), \tag{21}$$

with $\mu_1, \ldots, \mu_M \in K$, *has a Frobenius structure if and only if it is solvable.*

In fact:

Proof of proposition 6.5 (assuming proposition 6.7). Suppose that (20) has a Frobenius structure, which implies that it is solvable. Since $l_0 \in \mathbb{Z}_p$, the equation $y(qx) = q^{l_0} y(x)$ is solvable. This implies that the equation

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x)$$

is also solvable, and hence that it has a Frobenius structure. Moreover it proves that also $y(qx) = q^{l_0} y(x)$ has a Frobenius structure, *i.e.*, that $l_0 \in \mathbb{Z}/(p^s - 1)$ (*cf.* (6.6)).

On the other hand, if $l_0 \in \mathbb{Z}/(p^s - 1)$ and (20) is solvable, the equation $y(qx) = q^{l_0} y(x)$ has a Frobenius structure and $y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\frac{\mu_i}{x^i}\right) y(x)$ is solvable. Then it follows from (6.7) that (20) has a Frobenius structure. □

6.2. Proof of proposition 6.7. — First we prove a lemma, which is a fundamental step in the proof of (6.7). It is a q-analog of a particular case of [**Mo77**, Prop. 1]:

Lemma 6.8. — *Let $u(qx) = v(x)u(x)$ be a q-difference equation such that $v(x)$ is an analytic element on $\mathcal{C}([0, 1])$, without zeros and poles in $\mathcal{C}([0, 1])$ and $u(x)$ is a non zero analytic element on $\mathcal{C}([0, 1[)$. Then $u(x)$ is an analytic element on $\mathcal{C}([0, 1])$.*

Proof. One has to show (*cf.* [**Mo77**, Th. 4]) that for any $a \in K$, $|a| = 1$, there exists an analytic element on $D(0, 1^-) \cup D(a, 1^-)$ whose restriction to $D(0, 1^-)$ coincides with $u(x)$. Let us fix $a \in K$, $|a| = 1$. It follows from [**Mo77**, Th. 1] that there exists $g_a(x)$, called *the singular factor* of $u(x)$ with respect to $D(a, 1^-)$, such that

1) $g_a(x)$ is an analytic element on $\mathbb{P}^1 \smallsetminus D(a, 1^-)$, without zeros in $\mathbb{P}^1 \smallsetminus D(a, 1^-)$;

2) there exists an integer m such that $\lim_{x \to \infty} (x - a)^m g_a(x) = 1$;

3) $h_a(x) = u(x)/g_a(x)$ is an analytic element on $D(0, 1^-) \cup D(a, 1^-)$, with no zeros in $D(a, 1^-)$.

It follows from our hypothesis on

$$v(x) = \frac{u(qx)}{u(x)} = \frac{g_a(qx)}{g_a(x)} \frac{h_a(qx)}{h_a(x)}$$

and the properties of $g_a(x)$ and $h_a(x)$ that:

1) $g_a(qx)/g_a(x)$ is an analytic element on $\mathbb{P}^1 \smallsetminus D(a, 1^-)$, without zeros in $\mathbb{P}^1 \smallsetminus D(a, 1^-)$;

2) $\lim_{x \to \infty} g_a(qx)/g_a(x) = q^m$;

3) $h_a(qx)/h_a(x)$ is an analytic element without zeros on $D(a, 1^-)$. Since both $u(qx)/u(x)$ and $g_a(qx)/g_a(x)$ are analytic elements on $D(0, 1^-)$, without zeros in $D(0, 1^-)$, the same is true for $h_a(qx)/h_a(x)$. This proves that $h_a(qx)/h_a(x)$ is an analytic element on $D(0, 1^-) \cup D(a, 1^-)$, with no zeros in $D(a, 1^-)$.

We conclude that $q^{-m} g_a(qx)/g_a(x)$ is the singular factor of $v(x)$ with respect to $D(a, 1^-)$. This implies that $g_a(qx)/g_a(x) = q^m$ and hence $g_a(x) = \lambda x^m$, for some $\lambda \in K$ and $m \in \mathbb{Z}$. By the definition of singular factor, $g_a(x)$ does not have any

zero in $\mathbb{P}^1 \smallsetminus D(a, 1^-)$, which implies that $m = 0$. Eventually, $u(x) = \lambda h_a(x)$, which means that $u(x)$ is the restriction of an analytic element on $D(0, 1^-) \cup D(a, 1^-)$, as claimed. □

Now we are ready to prove (6.7). Remark that one implication is a particular case of (6.3). So let us suppose that (21) is solvable and prove that it has a Frobenius structure. The proof is divided into steps:

Step 0. — It is enough to prove that there exists a solution $y(x) \in \mathcal{B}$ of the q-difference equation

$$y(qx) = \prod_{i=1}^{M} \left(1 + (q-1)\mu_i x^i\right) y(x), \tag{22}$$

with $|\mu_i| \leqslant 1$ for any $i = 1, \ldots, M$ and $|q-1| < |\pi|$, and a Frobenius endomorphism ϕ such that $y(x)/y(x)^\phi$ is overconvergent.

Proof of Step 0. — Notice that the fact that (21) is solvable implies that $|\mu_i| \leqslant 1$ for any $i = 1, \ldots, M$ (*cf.* (5.1)). Moreover (21) has a solution $y(x)$ analytic and bounded over the disk $\mathcal{C}(]1, \infty])$ (*cf.* (1.3)). We have to prove that there exists a positive integer s such that $y(x)/y(x)^\phi \in \mathcal{E}^\dagger$. By iteration we may replace q by an integral power of q and hence suppose that $|q-1| < |\pi|$: this doesn't change the solution $y(x)$, which is still solution of the iterated equation, and the inequality $|\mu_i| \leqslant 1$ is still verified after a reduction of the type (5.1, Step1), since the iterated equation is necessarily solvable. A change of variable of the form $x \mapsto 1/x$ allows to conclude. □

Step 1. — There exists $h > 0$ such that $y(x)^{p^h}$ is overconvergent, hence algebraic over the field E of analytic elements with coefficients in K, (*i.e.* the completion of the field of rational functions $K(x)$ with respect to the norm induced by $\mathcal{E}^\dagger$).

Proof of Step 1. — The solution $y(x)$ can be written as a product of q-exponentials:

$$y(x) = \prod_{i=1}^{M} e_{q^i}\left(\frac{\mu_i}{[i]_q} x^i\right).$$

The analytic function

$$\log y(x) = \sum_{i=1}^{M} L_{q^i}\left(\frac{\mu_i}{[i]_q} x^i\right)$$

converges for

$$|x| < \inf_{i=1,\ldots,M} \left|\frac{[i]_q}{(q^i-1)\mu_i}\right|^{1/i} = \inf_{i=1,\ldots,M} \left|\frac{1}{(q-1)\mu_i}\right|^{1/i}.$$

Therefore there exists $\varepsilon > 0$ such that $\log y(x)$ is analytic and bounded over the disk $|x| < 1+\varepsilon$. We deduce that there exists and integer $h > 0$ such that $|p^h \log y(x)| < |\pi|$ for any $|x| < 1 + \varepsilon$, and hence, taking the exponential of $\log y(x)^{p^h}$, that $y(x)^{p^h}$ converges for $|x| < 1 + \varepsilon$. □

Step 2. — There exists $s > 0$ such that $y(x)/y(x)^{\phi} \in E$.

Proof of Step 2. — It follows from Step 1. that the q-difference algebra $E[y(x)]$ is a finite extension of E. Since $y(qx)^i = a(x)^i y(x)^i$ for any $i \in \mathbb{N}$, the q-difference module $E[y(x)] = \oplus_{i=0}^{d} E\, y(x)^i$, with $d = \deg E[y(x)]/E$, is semisimple. Moreover the Frobenius ϕ_1 stabilizes $E[y(x)]$ as a subalgebra of $\mathcal{B}$ (*cf.* [**C86**, Th. 5.2]). It follows that $E\,\phi_s(y(x))$, $s \in \mathbb{N}$, is a finite family of sub-q-difference modules of rank 1 of $E[y(x)]$, and hence that there exists $s \in \mathbb{N}$ such that $E\,\phi_s(y(x)) = E\,y(x)$ (*cf.* [**C81b**. 10.1] and [**CC96**, proof of th. 2.3.1]). □

Step 3. — $y(x)/y(x)^{\phi}$ is an analytic element on $\mathcal{C}([0,1])$.

Proof of Step 3. — The analytic element $y(x)/y(x)^{\phi}$ is solution of the q-difference equation:

$$\frac{u(qx)}{u(x)} = v(x), \text{ with } v(x) = \prod_{i=1}^{M} \frac{1 + (q-1)\mu_i x^i}{\prod_{j=0}^{p^s-1}(1 + (q-1)\mu_i^{\tau^s} q^{ij} x^{p^s i})} \in K(x).$$

Since $|\mu_i| \leqslant 1$, $v(x)$ is an analytic element on $\mathcal{C}([0,1])$, without zeros and poles in $\mathcal{C}([0,1])$. We deduce that $u(x)$ is an analytic element over $\mathcal{C}([0,1])$ from lemma 6.8. □

Step 4. — $y(x)/y(x)^{\phi}$ is overconvergent.

Proof of Step 4. — This statement is proved in the second part for q-difference systems of any rank (*cf.* 13.3). We will give here a simplified proof under the assumption $|1-q| < |\pi|^{p^s} M$. This implies that there exists $\eta > |\pi|^{-1}$ such that

$$v(x) = \prod_{i=1}^{M} \frac{1 + (q-1)\mu_i x^i}{\prod_{j=0}^{p^s-1}(1 + (q-1)\mu_i^{\tau^s} q^{ij} x^{p^s i})}$$

is analytic for $|x| \leqslant \eta$. Let $d_q^n u(x) = g_n(x)u(x)$ for any $n \geqslant 1$, with $g_1(x) = v(x) - 1/(q-1)x$. We have

$$\sup_{|x| \leqslant \eta} |g_1(x)| \leqslant 1.$$

Therefore the recursive relation $g_{n+1}(x) = g_n(qx)g_1(x) + d_q g_n(x)$ implies that

$$\sup_{|x| \leqslant \eta} |g_{n+1}(x)| \leqslant \frac{1}{\eta^n}.$$

In particular $|g_{n+1}(0)| \leqslant \eta^{-n}$. Since $u(x) = 1 + \sum_{n \geqslant 1} \frac{g_n(0)}{[n]_q^!} x^n$, we conclude that $u(x)$ converges for $|x| < \eta|\pi|$, with $\eta|\pi| > 1$. The same is true for $y(x)/y(x)^{\phi}$ since $u(x)$ and $y(x)/y(x)^{\phi}$ coincide up to a non zero constant factor. □

6.3. Remark. — Notice that the Step 1 above combined with (5.1) proves following statement:

Corollary 6.9. — *Let $y(qx) = a(x)y(x)$ be a q-difference equation with Frobenius structure. Then, there exists a non negative integer h and a solution $y(x)$ of $y(qx) = a(x)y(x)$ in a finite extension of $\mathcal{E}^\dagger$ such that $y(x)^{p^h} \in \mathcal{E}^\dagger$.*

In the next sections we will show that the solution $y(x)$ is actually in an unramified extension of $\mathcal{E}^\dagger$, which is a much stronger statement.

7. q-deformation of differential equations with strong Frobenius structure

In the previous section we have given a *naïve* definition of the (strong) Frobenius structure for q-difference equations of rank 1. In the higher rank case we are going to consider another Frobenius structure that we call confluent weak Frobenius structure. Proposition 7.3 below establishes the equivalence between the two definitions for q-difference equations of rank 1.

Definition 7.1. — We say that $y(qx) = a(x)y(x)$ has *confluent weak Frobenius structure* if there exists a sequence of $q^{p^{sn}}$-difference equations $y(q^{p^{sn}}x) = a_n(x)y(x)$, with $q^{p^s} = q$ and $a_0(x) = a(x)$, such that

1) for any $n \geqslant 1$ the equations

$$y(q^{p^{s(n-1)}}x) = a_{n-1}(x)y(x) \text{ and } y(q^{p^{s(n-1)}}x) = a_n(x)^{\phi}y(x)$$

are $\mathcal{E}^\dagger$-equivalent via $u_n(x) \in \left(\mathcal{E}^\dagger\right)^*$:

2) the sequences $(a_n(x) - 1)/\left(q^{p^{sn}} - 1\right)x$ and $u_n(x)$ converge in $\mathcal{E}^\dagger$.

Remark 7.2. — In the notation of the previous definition, let $(a_n(x) - 1)/\left(q^{p^{sn}} - 1\right)x \to g(x)$ and $u_n(x) \to u(x)$. Then the differential equation $\dfrac{dy}{dx}(x) = g(x)y(x)$ has a strong Frobenius structure:

$$g(x) + \frac{u'(x)}{u(x)} = p^s x^{p^s - 1} g(x)^{\phi}.$$

In other words, there exists a discrete family of difference equations

$$d_{q^{p^{sn}}} y(x) = \frac{a_n(x) - 1}{\left(q^{p^{sn}} - 1\right)x} y(x)$$

with an action of the Frobenius, which "tends" to the differential equation $\dfrac{dy}{dx}(x) = g(x)y(x)$, having a strong Frobenius structure.

Proposition 7.3. — *For a q-difference equation $y(qx) = a(x)y(x)$, with $a(x) \in (\mathcal{E}^\dagger)^*$, it is equivalent to have a strong Frobenius structure or a confluent weak Frobenius structure.*

Proof. — Let us suppose that $y(qx) = a(x)y(x)$ has a Frobenius structure. Then the sequence of iterated difference equations

$$y(q^{p^{sn}}x) = a_n(x)y(x), \quad \text{with } a_n(x) = \prod_{i=0}^{p^{sn}-1} a(q^i x).$$

satisfies condition 1. Since $y(qx) = a(x)y(x)$ is solvable, the equation $y(q^{p^{sn}}x) = a_n(x)y(x)$ is solvable for any $n \geqslant 1$, hence

$$\left| \frac{a_n(x) - 1}{(q^{p^{sn}} - 1)\, x} \right|_{\mathcal{E}^\dagger} \leqslant 1.$$

This proves that $a_n(x) \to 1$ when $n \to \infty$. Moreover it follows from (6.9) that $y(qx) = a(x)y(x)$ admits a solution $y(x)$ such that $y(x)^{p^h} \in \mathcal{E}^\dagger$, for some $h \in \mathbb{Z}_{\geqslant 0}$. Since

$$\frac{d_{q^{p^{sn}}}\left(y(x)^{p^h}\right)}{y(x)^{p^h}} = \frac{a_n(x)^{p^h} - 1}{(q^{p^{sn}} - 1)x} = \frac{a_n(x) - 1}{(q^{p^{sn}} - 1)x} \prod_{\substack{\zeta^{p^h}=1 \\ \zeta \neq 1}} (a_n(x) - \zeta)\,.$$

we obtain

$$\begin{aligned} \frac{\frac{d}{dx}\left(y(x)^{p^h}\right)}{y(x)^{p^h}} &= \lim_{n\to\infty} \frac{d_{q^{p^{sn}}}\left(y(x)^{p^h}\right)}{y(x)^{p^h}} \\ &= \lim_{n\to\infty} \left(\frac{a_n(x) - 1}{(q^{p^{sn}} - 1)x} \prod_{\substack{\zeta^{p^h}=1 \\ \zeta \neq 1}} (a_n(x) - \zeta) \right) \\ &= p^h \lim_{n\to\infty} \frac{a_n(x) - 1}{(q^{p^{sn}} - 1)x}, \end{aligned}$$

which proves that $(a_n(x) - 1)/(q^{p^{sn}} - 1)x$ has a limit $g(x)$ in $\mathcal{E}^\dagger$. The existence of the sequence $u_n(x)$ and its limit, as well as the strong Frobenius structure of $\frac{dy}{dx}(x) = g(x)y(x)$, is a consequence of the fact that $\frac{dy}{dx}(x) = g(x)y(x)$ and $y(x) = a(x)y(x)$ have the same solution in some finite extension of $\mathcal{E}^\dagger$.

Suppose we have a sequence of equations satisfying 1. and 2. Then clearly $\frac{dy}{dx}(x) = g(x)y(x)$ has a strong Frobenius structure, hence it is solvable.

We claim that every equation $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable. We set

$$d^k_{q^{p^{sn}}} y_n(x) = h^k_n(x)y_n(x) \quad \text{and} \quad \frac{d^k y}{dx^k}(x) = g_k(x)y(x).$$

Then for any $n \gg 1$ we have $|h^1_n(x)|_{\mathcal{E}^\dagger} = |g_1(x)|_{\mathcal{E}^\dagger}$ and $|q^{p^{sn}} - 1| < |\pi|$, which means that $|h^k_n(x)/[k]^!_{q^{p^{sn}}}|_{\mathcal{E}^\dagger} = |g_k(x)/k!|_{\mathcal{E}^\dagger}$. It follows from lemma 3.3 that $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable for any $n \gg 1$. We deduce that $y_n(q^{p^{sn}}x) = a_n(x)y_n(x)$ is solvable for any $n \geqslant 0$ from the inequality (19), describing the action of the Frobenius on the generic radius of convergence.

For any $n \geqslant 0$ consider the decomposition of $a_n(x)$ (*cf.* (5.1), Step 0):

$$a_n(x) = \frac{\lambda_n}{x^{N_n}}\, l_n(x)\, m_n(x),$$

with $\lambda_n \in K$, $N_n \in \mathbb{Z}$, $l_n(x) \in (1+x\mathcal{B})^*$ and $m_n(x) \in (1+\frac{1}{x}\mathcal{H}^\dagger)^*$, and the analogous decomposition of $u_n(x)$:

$$u_n(x) = \frac{\lambda'_n}{x^{N'_n}}\, l'_n(x)\, m'_n(x).$$

By performing a gauge transformation we can assume that $N'_n = 0$ for any n. Moreover since $y_n(q^{p^{s n}}x) = a_n(x)y_n(x)$ is solvable for any $n \geqslant 0$, we necessarily have $N_n = 0$ and $\lambda_n \in q^{\mathbb{Z}_p}$ for any $n \geqslant 0$ (*cf.* the proof of (5.1)). Moreover condition 1. and the uniqueness of the above decompositions imply that

$$\begin{cases} \lambda_n^{\tau^s} = \lambda_{n-1}. \\ \dfrac{l'_n(q^{p^{s(n-1)}}x)}{l'_n(x)} l_n(x)^\phi = l_{n-1}(x). \\ \dfrac{m'_n(q^{p^{s(n-1)}}x)}{m'_n(x)} m_n(x)^\phi = m_{n-1}(x). \end{cases} \qquad \text{for any } n \geqslant 1.$$

The first equality means that $y(q^{p^{s(n-1)}}x) = \lambda_n^{\tau^s} y(x)$ and $y(q^{p^{s(n-1)}}x) = \lambda_{n-1}y(x)$ are $\mathcal{E}^\dagger$-equivalent. Since $q^{\tau^s} = q$ and $\lambda_n \in q^{\mathbb{Z}_p}$, λ_n is τ^s-invariant. Hence we have $\lambda_n = \lambda_{n-1} = \lambda_0$. The unicity of the decomposition above shows that λ_0 is the exponent of the limit differential equation $\frac{dy}{dx}(x) = g(x)y(x)$ and hence that $\lambda_o = q^{l_0}$, with $(p^n-1)l_0 \in \mathbb{Z}$, for some positive integer n. We conclude from theorem 6.4 that $y(qx) = a(x)y(x)$ has a strong Frobenius structure. □

8. The group of isomorphism classes of q-difference equations of rank 1 admitting a Frobenius structure

We know that a solvable differential equation (*resp.* a differential equation with Frobenius structure) of rank one $y' = gy$ defined over $\mathcal{E}^\dagger$ has a solution of the form $x^\alpha u(x)$ (*resp.* $v(x)$), where $\alpha \in \mathbb{Z}_p$ and $u(x)$ (*resp.* $v(x)$) is an element of some finite unramified extension of $\mathcal{E}^{\dagger(9)}$ (*cf.* [**Cr87**], [**T98b**], [**Co01**]). The same is true for q-difference equations:

Proposition 8.1. — *Let $y(qx) = a(x)y(x)$, with $a(x) \in \mathcal{E}^\dagger$, be a solvable q-difference equation (resp. a q-difference equation with Frobenius structure). Then $y(qx) = a(x)y(x)$ has a solution of the form $x^\alpha u(x)$ (resp. $v(x)$), where $\alpha \in \mathbb{Z}_p$ and $u(x)$ (resp. $v(x)$) is an element of a finite unramified extension of $\mathcal{E}^\dagger$.*

(9) Following [**S68**], we call unramified extension of valued fields an extension of valued fields such that the ramification index is equal to 1 and the extension of the residue fields is separable.

Proof. — We know from (5.1), (6.7) and (6.9) that $y(qx) = a(x)y(x)$ has a solution of the form $x^\alpha u(x)$, where $\alpha \in \mathbb{Z}_p$ and $u(x)$ is an element of an extension of $\mathcal{E}^\dagger$, solution of a q-difference equation with Frobenius structure. Moreover we know that the Frobenius structure of $y(qx) = a(x)y(x)$ forces $\alpha \in \mathbb{Z}/(p^s - 1)$ for some positive integer s, which amounts to saying that $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$. It follows from (7.3) that $u(x)$ is solution of a differential equation over $\mathcal{E}^\dagger$ with strong Frobenius structure, hence that $u(x)$ (*resp.* $v = x^\alpha u$) is an element of a finite unramified extension of $\mathcal{E}^\dagger$. □

The $\mathcal{E}^\dagger$-equivalence classes of differential equations of rank one $y'(x) = g(x)y(x)$ form a group with respect to addition of the coefficient $g(x)$, and equivalence classes of equations with Frobenius structure form a subgroup, which we denote by $d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$. Note that, for such equations, the space $K.y \in (\mathcal{E}^\dagger)'$ defines a character of the absolute Galois group $G_{k((x))}$, which depends only on the class of the equation, and this provides a homomorphism

$$d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \longrightarrow X_K(G_{k((x))})$$

to the group of K-rational characters of $G_{k((x))}$. If the residue field k is algebraically closed, this is a bijection; the inverse map associates to the character χ the χ-eigenspace in any big enough finite unramified extension $(\mathcal{E}^\dagger)'$ of $\mathcal{E}^\dagger$ (this is an $\mathcal{E}^\dagger$-line), endowed with the natural derivation.

In the same vein, the $\mathcal{E}^\dagger$-equivalence classes of q-difference equations of rank one $y(qx) = a(x)y(x)$ form a group with respect to multiplication of the coefficient $a(x)$. Equations with strong Frobenius structure form a subgroup, which we denote by $\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$. Similarly, the space $K.y \in (\mathcal{E}^\dagger)'$ defines a character of the absolute Galois group $G_{k((x))}$, which depends only on the class of the equation, and this provides a homomorphism

$$\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \longrightarrow X_K(G_{k((x))})$$

to the character group of $G_{k((x))}$. If the residue field k is algebraically closed, this is a bijection; the inverse map associates to the character χ the χ^{-1}-eigenspace in any big enough finite unramified extension $(\mathcal{E}^\dagger)'$ of $\mathcal{E}^\dagger$ (this is an $\mathcal{E}^\dagger$-line), endowed with the natural σ_q-action.

On the other hand, proposition 7.3 associates by "confluence" to any element of $\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$ an element of $d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$, and it is easy to see that they correspond to the same character of $G_{k((x))}$. One thus arrives at the following

Theorem 8.2. — *Let us assume that k is algebraically closed. There are canonical group isomorphisms*

$$\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger} \xrightarrow{\cong} X_K(G_{k((x))}) \xrightarrow{\cong} d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger},$$

the composite being given by "confluence".

Remark 8.3. — The group of tame characters of $G_{k((x))}$ is canonically isomorphic to $(\mathbb{Z}_p \cap \mathbb{Q})/\mathbb{Z}$. For any $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$, the corresponding object of $d\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$ (*resp.* $\sigma_q\text{-}eq^{(\phi)}_{\mathcal{E}^\dagger}$) is represented by $d/dx + \alpha/x$ (*resp.* $d_q + [\alpha]_q/x$) and depends only on the class of α mod $\mathbb{Z}$.

APPENDICES TO PART I

9. Frobenius structure of $d_q y(x) = \pi_q y(x)$

9.1. q-analog of the Dwork exponential. — Recall that the "Dwork exponential" $\theta(x) = \exp(\pi x - \pi x^p)$ expresses the Frobenius structure of the differential equation $y' = \pi y$, and has radius of convergence > 1.

In the same way one can define an element π_q such that $|\pi_q|$ is the radius of convergence of $e_q(x)$[(10)]. Consider the q-difference equation satisfied by $e_q(\pi_q x)$:

$$y(qx) = a(x)y(x) \quad \text{with } a(x) = (1 + (q-1)\pi_q x). \tag{23}$$

It is defined over $\mathcal{E}^\dagger_{1/x}$ and it has a Frobenius structure (*cf.* proposition 6.7). Equation (23) can be iterated in the following way:

$$y(q^{p^s}x) = a_{p^s}(x)y(x) \quad \text{with } a_{p^s}(x) = a(x)a(qx)\cdots a(q^{p^s-1}x).$$

Then the series $e_q(\pi_q x)^\phi = e_q(\pi_q^{\tau^s} x^{p^s})$ is solution of the *q-difference* equation

$$y(qx) = a_{p^s}(x)^\phi y(x). \tag{24}$$

The fact that (23) has a Frobenius structure means that (23) and (24) are $\mathcal{E}^\dagger_{1/x}$-equivalent, *i.e.*, that

$$\frac{e_q(\pi_q x)}{e_q(\pi_q x)^\phi} = \frac{e_q(\pi_q x)}{e_q(\pi_q^{\tau^s} x^{p^s})} \in \left(\mathcal{E}^\dagger_{1/x}\right)^*.$$

---

(10) The choice of such an element is not canonical. A possible choice would be the following: let n be the smallest positive integer such that $|q^{p^n} - 1| < |\pi|$; then one can choose π_q such that

$$\pi_q^{p^n} = (-[p]_q)^{p^{n-1}} (-[p]_{q^p})^{p^{n-2}} \cdots \left(-[p]_{q^{p^{n-1}}}\right) \pi.$$

In other words, we have chosen $\pi_{q^{p^n}} = \pi$ and we have set recursively $\pi^p_{q^{p^i}} = -[p]_{q^{p^i}} \pi_{q^{p^{i+1}}}$, for any $i = 0, \ldots, n-1$. This is a good choice, in fact if $|q^{p^n} - 1| < |\pi|$ then $|[p]_{q^{p^n}}| = |p|$ and hence

$$\begin{aligned} |\pi_q| &= \textstyle\prod_{i=0}^{n-1} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}} |\pi|^{1/p^n} \\ &= \textstyle\prod_{i=0}^{n-1} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}} |p\pi|^{1/p^{n+1}} \\ &= \textstyle\prod_{i=0}^{n} \left|[p]_{q^{p^i}}\right|^{1/p^{i+1}} |\pi|^{1/p^{n+1}}. \end{aligned}$$

This proves that $|\pi_q|$ is equal to the radius of convergence of $e_q(x)$ (*cf.* (2.1)).

Since τ is a Frobenius automorphism such that τ^s fixes q, the definition of π_q implies that $|\pi_q^\tau - \pi_q| < |\pi_q|$. Therefore the existence of a strong Frobenius structure for (23) is equivalent to the overconvergence of

$$\frac{e_q(\pi_q x)}{e_q(\pi_q x^{p^s})}.$$

A reasonable analog of Dwork's exponential $\theta(x)$ (from the viewpoint of strong Frobenius structures[11]) would be the following series

$$\frac{e_q(\pi_q x)}{e_q(\pi_q x^p)}.$$

Unfortunately we are not able to prove that this series is overconvergent, unless $|1-q| < |\pi|$. In fact if $|1-q| < |\pi|$ its overconvergence is a immediate consequence of the overconvergence of the Dwork's exponential $\theta(x)$ and (2.5).

9.2. q-analog of Artin-Hasse exponential series. — Another reasonable q-analog of the Dwork exponential, from the viewpoint of confluent weak Frobenius structures, is the modified series

$$\theta_q(x) = \frac{e_q(\pi_q x)}{e_{q^p}(\pi_{q^p} x^p)}.$$

If $|q-1| < |\pi|$, the overconvegence of $\theta_q(x)$ immediately follows from the overconvergence of $\theta(x)$ and (2.5). To prove the overconvergence of $\theta_q(x)$ under more general hypotheses, one could try to construct an analog of the Artin-Hasse series, but our proof, which is quite similar to the one in the differential case, works only under restrictive assumptions. Anyway we are going to sketch it.

Proposition 9.1. *Suppose that there exists $\widetilde{q} \in \mathbb{Q}_p$ such that $\widetilde{q}^p = q$. Let*

$$E_q(x) = \prod_{i \geqslant 0} e_{q^{p^i}}\left(\frac{x^{p^i}}{[p^i]_q}\right) = e_q(x) e_{q^p}\left(\frac{x^p}{[p]_q}\right) e_{q^{p^2}}\left(\frac{x^{p^2}}{[p^2]_{q^p}[p]_q}\right)\cdots$$

Then $E_q(x)$ is analytic and bounded by 1 on the disk $D(0, 1^-)$.

Sketch of the proof

Step 1. – We deduce from the formula

$$e_{\widetilde{q}}([p]_{\widetilde{q}} x) = e_q(x) e_q(\widetilde{q} x) \cdots e_q(\widetilde{q}^{p-1} x)$$

that

$$\frac{E_q(x) E_q(\widetilde{q} x) \cdots E_q(\widetilde{q}^{p-1} x)}{E_q(x^p)} = e_{\widetilde{q}}([p]_{\widetilde{q}} x) \in 1 + [p]_{\widetilde{q}} x \mathcal{O}_K[\![x]\!].$$

The hypothesis $\widetilde{q} \in \mathbb{Q}_p$ is used here to prove that $[p]_q^n / [n]_{\widetilde{q}}^!$ is a p-adic integer. Such an estimate seems to be difficult when $|\widetilde{q} - 1| \geqslant |\pi|$.

[11] not from the viewpoint of confluent weak Frobenius structures, *cf.* §9.2, the next appendix, and 12.12.

Step 2 (q-analog of the Dieudonné's theorem). One proves that for any $f(x) \in 1 + xK[\![x]\!]$ we have

$$f(x) \in 1 + \mathcal{O}_K[\![x]\!] \iff \frac{f(x)f(\widetilde{q}x)\cdots f(\widetilde{q}^{p-1}x)}{f^\tau(x^p)} \in 1 + [p]_{\widetilde{q}} x\mathcal{O}_K[\![x]\!].$$

One concludes the proof remembering that $q^\tau = q$, since $q \in \mathbb{Q}_p$, and hence that $E_q^\tau(x^p) = E_q(x^p)$. □

Corollary 9.2. — *The series*

$$e_q(\pi x)e_{q^p}(-\pi x^p) = \frac{e_q(\pi x)}{e_{q^{\ p}}(\pi x^p)}$$

is overconvergent.

To prove the corollary one follows the proof in the differential case. One has to take into account that $q \in \mathbb{Q}_p$ implies $|[n]_q| = n$ for any integer n. Moreover $|p\pi - [p]_q\pi| < |\pi|$ implies that $e_{q^p}(\pi^p x^p/[p]_q)\, e_{q^p}(-\pi x^p)^{-1}$ is overconvergent (*cf.* corollary 2.2).

10. p-adic q-exponential and Koblitz' Gamma function

In this appendix, we leave our local framework and outline some *global aspects* of q-difference equations of rank one with overconvergent Frobenius structure. More precisely, we sketch a q-analog of part of Dwork's paper [**D83**], in which he related Morita's p-adic Gamma function Γ_p to the Frobenius structure of his exponential modules.

We shall recover in this way a function $\Gamma_{p,q}$ which is a p-adic analog of Jackson's Γ_q function as well as a q-analog of Morita's Γ_p function, and which had been previously introduced by N. Koblitz [**Ko80**][**Ko82b**] using Morita's approach[(12)].

10.1. Dwork's operator ψ. — In this appendix, the singular disk is the unit disk at infinity, so that the relevant ring of overconvergent functions is $\mathcal{H}^\dagger_{1/x} = \cup_{\varepsilon>0}\mathcal{A}([0, 1+\varepsilon[)$. We work over $K = \mathbb{Q}_p(\pi)$ and with $\tau = \mathrm{id}$ for simplicity.

Following Dwork, let us introduce the operator ψ defined by

$$\psi\Big(\sum a_n x^n\Big) = \sum a_{pn} x^n.$$

This is a left inverse of the Frobenius operator induced by $\phi : x \mapsto x^p$. It acts on $\mathcal{H}^\dagger$ and intertwines $\delta_q = xd_q$ and xd_{q^p} up to multiplication by $[p]_q$:

$$[p]_q \delta_{q^p}\psi = \psi\delta_q.$$

More generally, for any $a \in \mathbb{Z}_p$, ψ sends the space $x^a\mathcal{H}^\dagger$ to $x^b\mathcal{H}^\dagger$ where b is the so-called *successor* of $a \in \mathbb{Z}_p$, *i.e.*, the unique p-adic integer b such that $pb - a \in \mathbb{Z} \cap [0, p[$.

(12) In fact, one of us defined $\Gamma_{p,q}$ using q-exponential modules - à la Dwork -, and was told afterwards by F. Sullivan that this function had been defined earlier - à la Morita - by Koblitz.

10.2. q-Exponential modules. — The simplest examples of Dwork's "exponential modules" are of the form $x^a e^{\pi x}\mathcal{H}^\dagger_{1/x}$. They lack strong Frobenius structure (except when a is a rational number) but have a weak Frobenius structure: if b denotes the successor of a, $x^a e^{\pi x}\mathcal{H}^\dagger_{1/x}$ is isomorphic to the Frobenius inverse image of $x^b e^{\pi x}\mathcal{H}^\dagger_{1/x}$ (due to the overconvergence of Dwork's exponential).

A similar phenomenon occurs in the q-difference case. Proposition 2.1 suggests to replace $e^{\pi x}$ by $e_q(\pi_q x)$ for some constant π_q with the following property: defining the sequence $(\pi_q^{p^n})$ by $\pi_{q^{p^{n+1}}} = -\pi^p_{q^{p^n}}/[p]_{q^{p^n}}$, we assume that $|\pi_{q^{p^n}}| = |\pi|$ for n big enough[13]. Then proposition 2.1 shows that $e_q(\pi_q x)$ has radius of convergence 1.

However, we shall have to assume that $|q-1| < p^{-1/p-1}$ in order to ensure that the series

$$\theta_q(x) = \frac{e_q(\pi_q x)}{e_{q^p}(\pi_{q^p} x^p)}$$

is overconvergent. The equation $\pi_q^p = -[p]_q \pi_{q^p}$ is irrelevant at this point but will be used in the sequel.

We set

$$f_{q,a} = x^a e_q(\pi_q x),$$

and consider the $\mathcal{H}^\dagger_{1/x}$-module $f_{q,a}\mathcal{H}^\dagger_{1/x}$ equipped with the natural action of $\delta_q = xd_q$. A simple computation shows that $\delta_q f_{q,a} = (q^a \pi_q x + [a]_q) f_{q,a}$. Let us write the classes modulo $\operatorname{Im}\delta_q$ in braces. So,

$$\{f_{q,a+1}\} = -q^{-a}\pi_q^{-1}[a]_q\{f_{q,a}\} \text{ in } \frac{f_{q,a}\mathcal{H}^\dagger_{1/x}}{\delta_q\left(f_{q,a}\mathcal{H}^\dagger_{1/x}\right)}.$$

from which one deduces that this cokernel has dimension 1 over K and is generated by $\{f_{q,a}\}$.

If b is the successor of a, then

$$\frac{f_{q,a}(x)}{f_{q^p,b}(x^p)} = \frac{e_q(\pi_q x)}{e_{q^p}(\pi_q x^p)}$$

is overconvergent, which expresses the fact that $f_{q,a}\mathcal{H}^\dagger_{1/x}$ is isomorphic to the Frobenius inverse image of $f_{q^p,b}\mathcal{H}^\dagger_{1/x}$. The same argument as in the differential case shows that ψ extends to a mapping $f_{q,a}\mathcal{H}^\dagger_{1/x} \to f_{q^p,b}\mathcal{H}^\dagger_{1/x}$ by the following formula

$$\psi(f_{q,a}.f) = f_{q^p,b}\psi(x^{a-pb}\theta_q(x)f).$$

[13] It is easy to see that such a π_q exists at least after replacing K by a finite extension. On the other hand, there does not seem to be a "uniform", canonical choice for such π_q, and one can actually show that the sequence $(\pi_{q^{p^n}})$ never converges.

10.3. $\Gamma_{p,q}$**.** — Since ψ intertwines δ_q and δ_{q^p} up to a factor $[p]_q$, it passes to the "cohomology":

$$\psi : \frac{f_{q,a}\mathcal{H}^\dagger_{1/x}}{\delta_q\big(f_{q,a}\mathcal{H}^\dagger_{1/x}\big)} \longrightarrow \frac{f_{q^p,b}\mathcal{H}^\dagger_{1/x}}{\delta_{q^p}\big(f_{q^p,b}\mathcal{H}^\dagger_{1/x}\big)}$$

and one can define an element $\Gamma_{p,q}(a) \in K^*$ by

$$\psi\{f_{q,a}\} = \pi_q^{pb-a}\Gamma_{p,q}(a)\{f_{q^p,b}\}.$$

One then computes:

$\psi\{f_{q,0}\} = \Gamma_{p,q}(0)\{f_{q^p,0}\} = \{f_{q^p,0}\psi(\theta_q(x))\} = \{f_{q^p,0}\}$.

If a is a unit, b is a successor of $a+1$, and one has

$$\begin{aligned}\Gamma_{p,q}(a+1)&\{f_{q^p,b}\}\\ &= \pi_q^{a-pb+1}\psi\{f_{q,a+1}\} = -q^{-a}\pi_q^{a-pb}[a]_q\psi\{f_{q,a}\} = -q^{-a}[a]_q\Gamma_{p,q}(a)\{f_{q^p,b}\}\\ &= [-a]_q\Gamma_{p,q}(a)\{f_{q^p,b}\}.\end{aligned}$$

If a is divisible by p, then $a = pb$ and $b+1$ is a successor of $a+1$, whence:

$$\begin{aligned}\Gamma_{p,q}(a+1)&\{f_{q^p,b+1}\}\\ &= \pi_q^{1-p}\psi\{f_{q,a+1}] = -\pi_q^{-p}q^{-a}[a]_q\psi\{f_{q,a}\} = -\pi_q^{-p}q^{-a}[a]_q\Gamma_{p,q}(a)\{f_{q^p,b}\}\\ &= -\Gamma_{p,q}(a)\{f_{q^p,b+1}\}.\end{aligned}$$

where

$$-\frac{\pi_q^{-p}\pi_{q^p}[a]_q}{[b]_{q^p}} = -\pi_q^{-p}\pi_{q^p}[p]_q = 1$$

by assumption.

Therefore, as a function of $a \in \mathbb{Z}_p$, $\Gamma_{p,q}(a)$ satisfies the functional equations of Koblitz' function of the same name:

$$\Gamma_{p,q}(0) = 1, \quad \frac{\Gamma_{p,q}(\alpha+1)}{\Gamma_{p,q}(a)} = \begin{cases}[-a]_q & \text{if } a \text{ is a unit,}\\ -1 & \text{if } |\alpha|_p < 1.\end{cases}$$

In order to check that $\Gamma_{p,q}(a)$ is Koblitz' function, it remains to prove its continuity. In fact, we shall prove the so-called "Boyarsky principle" for q-exponential modules $f_{q,a}\mathcal{H}^\dagger_{1/x}$, *i.e.*, the analyticity of $\Gamma_{p,q}$ on each disk $D(-k,|p|^+)$, $k = 0,1,\dots,p-1$.

Let us expand $\theta_q(x) = \sum \varepsilon_n x^n$. For every $a \in \mathbb{Z}_p \cap D(-k,1^-)$, one has

$$\begin{aligned}\psi\{f_{a,q}\} &= \pi_q^k\Gamma_{p,q}(a)\{f_{q^p,b}\} = \{f_{q^p,b}\psi(x^{-k}\theta_q(x))\} = \{\textstyle\sum \varepsilon_{pn+k}f_{q^p,b+n}\}\\ &= \{\textstyle\sum \varepsilon_{pn+k}(-1)^n q^{-npb-n(n-1)p/2}\pi_{q^p}^{-n}[b]_{q^p}[b+1]_{q^p}\dots[b+n-1]_{q^p}f_{q^p,b}\}\end{aligned}$$

hence

$$\begin{aligned}\Gamma_{p,q}(a) = \sum \varepsilon_{pn+k}(-1)^n q^{(-n(a+k)-n(n-1)p)/2}\pi_{q^p}^{-n-k}\\ \cdot\left[\frac{a+k}{p}\right]_{q^p}\left[\frac{a+k}{p}+1\right]_{q^p}\dots\left[\frac{a+k}{p}+n-1\right]_{q^p}\end{aligned}$$

which is analytic in $D(-k, |p|^+)$, since $|1-q| < |\pi|$, as follows from the expansion

$$[c]_{q^p} = -1 + \sum_{m \geqslant 0} \binom{c}{m} (q^p - 1)^{m-1}.$$

It would be interesting to extend this approach "à la Boyarsky" to other q-difference modules with p-adic parameters, notably to Koblitz' p-adic hypergeometric q-difference equations [**Ko82b**].

PART II
HIGHER RANK

11. Preliminaries: unramified extensions of $\mathcal{E}^\dagger$

In this part, it will be essential to deal not only with $\mathcal{E}^\dagger_x$ but also with its unramified extensions as well. Whereas any such extension is of the form $\mathcal{E}^\dagger_{x'}$ for some new variable x', the q-difference operator d_q and Frobenius fail to act on x' as simply as they do on x. Thus the relatively down-to-earth methods of part I do not apply to the case of $\mathcal{E}^\dagger_{x'}$. In this preliminary section, we give some tools to handle this issue.

11.1. Topologies. — Let K be a complete non-archimedean field of characteristic 0, with residue field k of characteristic $p > 0$. We keep the notation of part I for rings of analytic functions. For any interval I, the ring $\mathcal{B}(I)$ of bounded analytic functions on the annulus $\mathcal{C}(I)$ is endowed with the topology given by the sup-norm $|\ |_I$, for which it is complete. It is also endowed with a coarser topology (strictly coarser if I is not closed): the Frechet topology defined by the norms $|\ |_J$ for all closed $J \subset I$. Its Frechet completion is $\mathcal{A}(I)$.

Similarly, the topology of the Robba ring $\mathcal{R}$ is the finest for which the injections $\mathcal{A}(]1-\varepsilon, 1[) \hookrightarrow \mathcal{R}$, are continuous, *cf. e.g.*, [**Cr98**, 4], and the induced topology on $\mathcal{E}^\dagger$ is coarser than the topology defined by the p-adic norm $|\ | = |\ |_{\mathcal{E}^\dagger}$ [(14)].

The truncation $\gamma_{x>0}$ in positive degrees is continuous, both as an operator $\mathcal{R} \to x\mathcal{A}([0,1[)$ and as an operator $\mathcal{E}^\dagger \to x\mathcal{B}([0,1[)$.

If K is discretely valued and J is closed, $\mathcal{A}(J)$ is a Banach space, and the Banach norm coincides with the sup-norm on $\mathcal{C}(J)$ if the endpoints of J lie in $\sqrt{|K^*|}$ ([**Cr98**, 4.2]).

11.2. Unramified extensions and absolute values. — We assume in the rest of this section that k is perfect and that K is discretely valued. Then $\mathcal{E}^\dagger_{K,x}$ is the field of fractions of the henselian ring $\mathcal{O}_{\mathcal{E}^\dagger_{K,x}}$ with residue field $k((x))$. Any finite unramified

[(14)] In [**Ke**], $\mathcal{R}$ is denoted by $\Gamma^{k((x))}_{\mathrm{an,con}}$ and $\mathcal{E}^\dagger$ by $\Gamma^{k((x))}_{\mathrm{con}}[1/p]$.

extension of $\mathcal{E}^\dagger_{K,x}$ is of the form $\mathcal{E}^\dagger_{K',x'}$, for a finite unramified extension K'/K and a new variable x' algebraic over $\mathcal{E}^\dagger_{K,x}$ [**Ma95**].

Among the finite étale extensions of the Robba ring $\mathcal{R}_{K,x}$, those of the form $\mathcal{R}_{K',x'} = \mathcal{R}_{K,x} \otimes_{\mathcal{E}^\dagger_{K,x}} \mathcal{E}^\dagger_{K',x'}$ play a distinguished role in the local theory of p-adic differential equations (as was first emphasized in the work of R. Crew [**Cr98**]). We shall see that they play a similar role in the local theory of p-adic q-difference equations. The most suitable way to deal with all these extensions is to embed them (up to isomorphism) in a fixed canonical "big Robba ring" $\widetilde{\mathcal{R}}$, as was done by K. Kedlaya [**Ke**][(15)] (the field of constants of $\widetilde{\mathcal{R}}$ is $\widetilde{K} := K \otimes_{W(k)} W(\overline{k})$).

For any element $y \in \mathcal{R}_{K,x}$ and any $r \in]0,1[$, one defines $|y|_{|x|=r}$ (also written, abusively, $|y|_r$) to be the sup-norm of y on the circle $|x| = r$ if $y \in \mathcal{A}([r,1[)$, ∞ otherwise. Similarly for $\mathcal{R}_{K',x'}$. Note that if $y \in \mathcal{E}^\dagger_{K,x}$, $|y|_r = |y|$ for any r close enough to 1.

The drawback of this notion is its dependence on the choice of x'. Kedlaya has shown how to bypass it by introducing (partially defined) canonical absolute values $|\,|_{r,\mathrm{can}}$ on $\widetilde{\mathcal{R}}$[(16)]. He proves that for any $y \in \mathcal{R}_{K',x'} \subset \widetilde{\mathcal{R}}$ and any r close enough to 1, $|y|_{r,\mathrm{can}}$ coincides with the naive absolute value $|y|_{|x'|=r}$ [**Ke**, 3.7].

On the other hand, let τ denote a Frobenius endomorphism of K. It extends uniquely to $\widetilde{K}$. Let us fix a positive integer s and consider the endomorphism $\phi_s = \phi$ of $\mathcal{R}_{K,x}$ given by $\phi(\sum a_n x^n) = \sum \tau^s(a_n) x^{p^s n}$. Then ϕ extends canonically to a τ^s-semilinear endomorphism of $\widetilde{\mathcal{R}}$, and one has the formula $|y^\phi|_{r^{1/p^s},\mathrm{can}} = |y|_{r,\mathrm{can}}$ [**Ke**, following prop. 3.11]. Hence for r close enough to 1,

$$|y^\phi|_{|x'|=r^{1/p^s}} = |y|_{|x'|=r}.$$

11.3. Good coordinates. — Because $\mathcal{E}^\dagger_x$ is henselian, ϕ lifts uniquely to any finite unramified extension of $\mathcal{E}^\dagger_x$. By definition $\phi(x) = x^p$. For a finite unramified extension of $\mathcal{E}^\dagger_x$ with tame residual extension of degree say n, one can take $x' = x^{1/n}$ so that again $\phi(x') = (x')^p$. However, in the case of a wild totally ramified residual extension (say of degree p^m), it is not possible to choose x' such that $\phi(x') = (x')^p$, although $\phi(x') \equiv (x')^p \bmod p$.

In order to get some control on the Frobenius action in the course of computations, it is important to choose x' carefully. For this purpose, one can use so-called Katz-Gabber extensions. According to Katz-Gabber [**Ka86**, 1.4.2], one can choose x' in such a way that x' is algebraic of degree p^m over $k(x)$ and such that the extension $k(x,x')/k(x)$ defines (via x) a finite covering $\overline{f}_k : \overline{C}_k \to \mathbf{P}^1_k$ unramified above $\mathbf{P}^1 \smallsetminus 0$ and totally ramified at $x' = x = 0$. We denote by $f_k : C_k \to \mathbf{P}^1 \smallsetminus 0$ the restriction

(15) denoted by $\Gamma^{\overline{k((x))}}_{an,con}$ or $\Gamma^{alg}_{an,con}$ in *loc. cit.*

(16) Those attached to his partial valuations $w_{\frac{\log r^{-1}}{\log p}}$.

of $\overline{f}_k$ above $\mathbf{P}^1 \smallsetminus 0$. Extending k if necessary, we assume that one of points, say P_k, of $\overline{C}_k$ with $x' = \infty$ is k-rational.

Let $\overline{f} : \overline{C} \to \mathbf{P}^1_{\mathcal{O}_K}$ be a lifting of this finite covering, with $\overline{C}$ projective and flat over $\mathcal{O}_K$[17]. We denote again by the letter x (*resp.* x') a lifting of the first (*resp.* second) coordinate to $\overline{C}$ (we refer to such a coordinate x' as a "*good coordinate*" for this finite extension of $\mathcal{E}^\dagger_x$).

We denote by $f : C \to \mathbf{P}^1 \smallsetminus 0$ the restriction of $\overline{f}$ above $\mathbf{P}^1 \smallsetminus 0$. The finite covering $f_K : \overline{C}_K \to \mathbf{P}^1_K \smallsetminus 0$ is unramified above the disk $D([1, \infty])$.

Let $\mathcal{O}^\dagger(C)$ denote the (p-adic) weak completion of the affine algebra $\mathcal{O}(C)$. Via f^*, this an étale extension of $\mathcal{O}_{\mathcal{H}^\dagger_x}$. It is known that $\mathcal{O}^\dagger(C)$ is henselian (*cf. e.g.*, [**E02**, th. 3]), from which it follows that the endomorphism ϕ of $\mathcal{E}^\dagger_{x'}$ preserves $\mathcal{O}^\dagger(C)$. On the other hand, $\mathcal{O}_K[[1/x]]$ and $\mathcal{O}_K[[1/x']]$ are both ϕ-equivariantly isomorphic to the completion of $\mathcal{O}(C)$ at P_k.

We now fix an element $q \in K^*$ (not a root of unity) satisfying $|q| = 1$, and fixed under τ^s. The homothety $x \mapsto qx$ extends uniquely to an automorphism of each of the topological K-algebras $\mathcal{E}^\dagger_x, \mathcal{E}^\dagger_{x'}, \mathcal{R}_x, \mathcal{R}_{x'}, \widetilde{\mathcal{R}}$, and also of $\mathcal{O}^\dagger(C)$ (the latter ring being henselian). We denote all these extensions somewhat abusively by the symbol σ_q. Of course, in general $\sigma_q(x')$ is not proportional to x'.

The assumption $\tau^s(q) = q$ ensures that σ_q and ϕ generate a twisted polynomial ring $K[\sigma_q, \phi]$ of endomorphisms of any of the previous rings:

$$\sigma_q \, \phi = \phi \, \sigma_q^{p^s} = \phi \, \sigma_{q^{p^s}} .$$

12. q-difference modules and Frobenius structures

12.1. σ-modules. — Let R be a commutative integral $\mathbb{Q}$-algebra and let σ be an injective endomorphism of R.

Definition 12.1. — A *σ-module*, or *σ-difference module* over R is a free R-module M of finite rank equipped with an R-linear *isomorphism*

$$\Sigma : \sigma_* M := M \otimes_{R,\sigma} R \longrightarrow M.$$

The σ-modules over R form a category in an obvious way[18], which is linear over the fixed ring of R under σ.

In terms of a basis $(m_i)_{i=1,\dots,\mu}$ of M, the datum of Σ is equivalent to the datum of an invertible matrix, which it is convenient to define as the *inverse* $A \in GL_\mu(R)$ of the matrix of Σ. With this convention, the σ-difference matrix system

$$\sigma(Y) = AY$$

is equivalent to condition that $\sum_i Y_{ij} m_i$ is fixed by Σ.

---

[17] For a conceptual proof of the existence of such a lifting, *cf. e.g.*, [**Cr98**, 8.3]; here, it suffices to lift to characteristic 0 an equation relating x and x'.

[18] and even a monoïdal symmetric rigid category

If (m'_j) is another basis, with $m'_j = \sum (H^{-1})_{ij} m_i$, then the inverse A' of the matrix of Σ in this new basis is $H^\sigma A H^{-1}$.

Remark 12.2. — σ-modules can also be understood in terms of non-commutative connections, *cf.* [**A01**].

In the sequel, we consider the case where R is $\mathcal{E}^\dagger_x$ or $\mathcal{R}_x$, or (if k is perfect and K is discretely valued) one of their finite extensions $\mathcal{E}^\dagger_{x'}$, $\mathcal{R}_{x'}$. In such cases, we consider σ-modules as topological modules. When $\sigma = \sigma_q$, the notion of σ-module amounts to that of q-difference module over R, our main object of study. When $\sigma = \phi$, we denote the isomorphism defining the structure of ϕ-module by

$$\Phi : \phi_* M := M \otimes_{R,\phi} R \longrightarrow M.$$

This example is studied at length in [**Ke**].

Remark 12.3. — If K is discretely valued, $\mathcal{E}^\dagger$ is a field and $\mathcal{R}$ is a *Bézout ring*: every finitely generated ideal is principal (as was remarked by several authors (*cf. e.g.*, [**Cr98**, 4.9]), this follows from Lazard's theory of principal parts). Any Bézout ring R is integrally closed and coherent, and any finitely generated R-module is a direct sum of its torsion submodule and of a free module, *cf. e.g.*, [**Cr98**, 4.9]. Thus, in that case, one could replace "free" by "locally free" in the definition of σ-modules.

If R is Bézout, and the subring of σ_q-constants is a field K, then the category of σ_q-modules is tannakian over K.

12.2. Strong Frobenius structure. — We assume that q is fixed under τ^s. We shall be interested in situations where the given module M is at the same time a σ_q-module and a ϕ-module.

Definition 12.4. — A *strong Frobenius structure* on a q-difference module (M, Σ_q) over R is the datum of a structure of ϕ-module Φ on M, Σ_q and Φ being subject to the following "integrability condition":

$$\Sigma_q \, \Phi = \Phi \, (\Sigma_q)^{p^s},$$

to be understood in the sense that the following diagram commutes

$$\begin{array}{ccc} \phi_* \sigma_{q^{p^s},*} M = \sigma_{q,*} \phi_* M & \xrightarrow{\phi_* \Sigma_{q^{p^s}} = \phi_* \Sigma_q^{p^s}} & \phi_* M \\ \Big\downarrow{\scriptstyle \sigma_{q,*}\Phi} & & \Big\downarrow{\scriptstyle \Phi} \\ \sigma_{q,*} M & \xrightarrow{\Sigma_q} & M. \end{array}$$

We also say that (M, Σ_q, Φ) is a σ_q-ϕ-module over R.

If Σ_q and Φ are represented by the matrices A_q^{-1} and F^{-1} respectively in a given basis (m_i) of M, the integrability condition translates into

$$F^{\sigma_q} A_q = A_{q^{p^s}}^{\phi} F. \quad \text{where } A_{q^{p^s}} := A_q^{\sigma_{q^{p^s}}} A_q^{\sigma_{q^{p^s-1}}} \cdots A_q.$$

Remark 12.5. — Let us consider the q-difference system $Y^{\sigma_q} = AY$ attached to $(M, \Sigma_q, (m_i))$. Then $Y^{\phi} = FY$ satisfies the q^{1/p^s}-difference system with matrix A_q^{ϕ}, which can be turned, after p^s iterations, into the q-difference system $(Y^{\phi})^{\sigma_q} = A_{q^{p^s}} Y^{\phi}$.

We denote by $\sigma_q\text{-}Mod_R^{\phi}$ (*resp.* $\sigma_q\text{-}Mod_R^{(\phi)}$) the category of σ_q modules over R endowed with a specified (*resp.* an unspecified) strong Frobenius structure. If R is Bézout and if the subring of σ_q-constants is a field K, then $\sigma_q\text{-}Mod_R^{(\phi)}$ is tannakian over K while $\sigma_q\text{-}Mod_R^{\phi}$ is tannakian over the field K_s of elements of K fixed under τ^s.

For any ring homomorphism $R \to R'$ compatible with σ_q, τ[19], and for any replacement of ϕ by some power ϕ^i, there is an obvious functor between the corresponding categories of q-difference modules with strong Frobenius structures.

Example 12.6 (**q-Exponential**). — In the case of a q-difference module of rank $\mu = 1$, identified, after the choice of a basis, with a difference equation $y(qx) = a(x)y(x)$ with $a(x) \in R^*$, the strong Frobenius structure is unique if it exists and is given by y^{ϕ}/y provided it belongs to R (otherwise, there is no strong Frobenius structure).

This is the case of the q-difference equation satisfied by $e_q(\pi_q x)$ as in appendix 9.

Example 12.7 (**q-Logarithm**). — The equation $d_q \ell_q(x) = 1/x$, or equivalently

$$\ell_q(qx) = \ell_q(x) + q - 1$$

has the obvious solution ("q-logarithm")

$$\ell_q(x) = \frac{q-1}{\log q} \log x.$$

One has

$$\ell_q(x)^{\phi} = p^s \ell_q(x).$$

Consider the linear system of order 2

$$\sigma_q Y(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} Y(x), \tag{25}$$

having $\begin{pmatrix} q-1 \\ \ell_q(x) \end{pmatrix}$ as solution. Then $\begin{pmatrix} q-1 \\ \ell_q(x)^{\phi} \end{pmatrix} = \begin{pmatrix} q-1 \\ p^s \ell_q(x) \end{pmatrix}$ is a solution of

$$\sigma_q Y(x) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{p^s} Y(x), \tag{26}$$

[19] satisfying the usual relation $\sigma_q \phi = \phi \sigma_q^{p^s}$

Since

$$\begin{pmatrix} q-1 \\ p^s \ell_q(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^s \end{pmatrix} \begin{pmatrix} q-1 \\ \ell_q(x) \end{pmatrix}$$

we conclude that (25) and (26) are $\mathcal{E}^\dagger$-equivalent.

The differential Galois group is obviously the additive group $\mathbb{G}_a$, and one has a canonical fully faithful $\otimes$-functor

$$Rep_K \mathbb{G}_a \longrightarrow \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}_x}$$

which sends the standard two-dimensional representation to the q-difference module attached to ℓ_q.

***Example 12.8* (q-Difference modules arising from Galois representations)**

Let us assume that k is perfect and K is discretely valued. Let $k'((x'))$ be a finite Galois extension of $k((x))$, and let $\mathcal{E}^\dagger_{K',x'}$ be the corresponding finite unramified Galois extension of $\mathcal{E}^\dagger_{K,x}$. Then $\mathcal{E}^\dagger_{K',x'}$ has a canonical structure of σ_q-ϕ-module over $\mathcal{E}^\dagger_{K,x}$, given by the canonical extension of σ_q and ϕ to $\mathcal{E}^\dagger_{K',x'}$.

We denote by $d\text{-}Mod^\phi_R$ (*resp.* $d\text{-}Mod^{(\phi)}_R$) the category of differential modules (free) over R endowed with a specified (*resp.* an unspecified) strong Frobenius structure. If R is Bézout and if the subring of σ_q-constants is a field K, then $d\text{-}Mod^{(\phi)}_R$ is tannakian over K while $d\text{-}Mod^\phi_R$ is tannakian over the field K_s of elements of K fixed under τ^s.

Let $G_{k((x))}$ be the absolute Galois group of $k((x))$. There are well-known $\otimes$-functors

$$D^\phi_d : Rep_{K_s} G_{k((x))} \longrightarrow d\text{-}Mod^\phi_{\mathcal{R}_x},$$
$$D^{(\phi)}_d : Rep_K G_{k((x))} \longrightarrow d\text{-}Mod^{(\phi)}_{\mathcal{R}_x},$$

given by

$$V \longmapsto (V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$$

for suitable $\mathcal{R}_{K',x'}$ (depending on V), *cf.* [**T98b**] (here $G_{k((x))}$ is considered as a constant profinite group-scheme, and representations are understood in the algebraic sense, *i.e.*, as representations of a group scheme; in particular, representations of $G_{k((x))}$ in this sense have finite image).

This carries over to the q-difference case. For any $V \in Rep_K G_{k((x))}$ of dimension μ, there is a finite unramified extension $\mathcal{E}^\dagger_{K',x'}$ (say in $\widetilde{\mathcal{R}}$) such that $(V \otimes_K \mathcal{E}^\dagger_{K',x'})^{G_{k((x))}}$ (*resp.* $(V \otimes_K \mathcal{R}_{K',x'})^{G_{k((x))}}$) is a $\mathcal{E}^\dagger_{K,x}$-module (*resp.* $\mathcal{R}_{K,x}$-module) M of rank μ. This module inherits a natural q-difference structure and strong Frobenius structure from $\mathcal{E}^\dagger_{K',x'}$ (*resp.* $\mathcal{R}_{K',x'}$). If $K = K'$ (*e.g.*, if k is algebraically closed), one recovers V from M by the following recipe "à la Fontaine": $V \cong (M \otimes \mathcal{E}^\dagger_{K',x'})^{\Sigma_q}$ (*resp.* $\cong V(M) := (M \otimes \mathcal{R}^\dagger_{K',x'})^{\Sigma_q}$).

Proposition 12.9. — *There is a canonical fully faithful K_s-linear $\otimes$-functor*

$$D^\phi_{\sigma_q} : Rep_{K_s} G_{k((x))} \longrightarrow \sigma_q\text{-}Mod^\phi_{\mathcal{R}_x}.$$

and a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\phi)}_{\sigma_q} : Rep_K G_{k((x))} \longrightarrow \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto (V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$$

for suitable $\mathcal{R}_{K',x'}$ (depending on V).

The fact that the functor is full is seen as usual using internal $\underline{Hom}$: it reduces to the fact that $D^{(\phi)}_{\sigma_q}(V^{G_{k((x))}})$ is the largest trivial subobject of $D^{(\phi)}_{\sigma_q}(V)$.

Remark 12.10. — Combining the last two examples, one finds a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\phi)}_{\sigma_q} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto D^{(\phi)}_{\sigma_q}(V) := (V \otimes \mathcal{R}_{K',x'}[\log x])^{G_{k((x))} \times \mathbb{G}_a(K)}$$

for suitable $\mathcal{R}_{K',x'}$.

12.3. Confluent weak Frobenius structure. — Let us remark that if M is a $q^{p^{(i+1)s}}$-difference module, then $\phi_* M$ has a natural structure of $q^{p^i s}$-difference module (in a given basis they are "given by" matrices $A_{q^{p^{is}}}$ and $A^{\phi}_{q^{p^{is}}}$ respectively).

On the other hand, $q^{p^{is}} \to 1$ when $i \to \infty$, and the phenomenon of confluence occurs in this natural way, in the p-adic setting.

Combining these two remarks, this suggests to introduce another type of Frobenius structure, which seems to have no counterpart in the differential case: a sequence of Frobenius predecessors ($q^{p^{is}}$-difference modules) which for $i \to \infty$ converge to a strong Frobenius structure on the limit differential module. More precisely:

Definition 12.11. — A *confluent weak Frobenius structure* on a q-difference module (M, Σ_q) over R is a sequence $(M_i = (M, \Sigma_{q^{p^{is}}}))_{i \geqslant 0}$ of $q^{p^{is}}$-difference structures on M, and isomorphisms

$$\Phi_i : \phi_* M_{i+1} \longrightarrow M_i$$

of $q^{p^{is}}$-difference modules, such that

1) the operators $\Delta_i = \frac{1}{(q^{p^{is}}-1)}(\Sigma_{q^{p^{is}}} - \mathrm{Id})$ converge to a derivation Δ_∞ on M,
2) Φ_i converges to a strong Frobenius Φ_∞ for the differential module (M, Δ_∞).

In terms of associated matrices, this amounts to the data of a sequence of matrices F_i, $A_{q^{p^{is}}} \in GL_\mu(R)$, $i \geqslant 0$, related by the following relations:

$$F_i^{\sigma_{q^{p^{is}}}} A_{q^{p^{is}}} = A^{\phi}_{q^{p^{(i+1)s}}} F_i.$$

and such that the sequences (F_i) and $(G_i := (q^{p^i}-1)^{-1}(A_{q^{p^i}}-I))$ both converge in $M_\mu(R)$ to some limits F_∞ and G_∞ respectively (F_i represents the "quotient" $Y_{i+1}^{\phi}Y_i^{-1}$ where Y_i is a fundamental solution of the system $\sigma_{q^{p^{is}}}Y_i = A_{q^{p^{is}}}Y_i$).

We denote by $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}\,\phi}$ the category of σ_q modules over R endowed with a specified confluent weak Frobenius structure.

We denote by $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}(\phi)}$ the analogous category where one leaves the morphisms Φ_i unspecified: objects consist of a sequence $(M, \Sigma_{q^{p^{is}}}))_{i\geqslant 0}$ of $q^{p^{is}}$-difference structures on M converging to a differential module structure on M, these data being part of an unspecified confluent weak Frobenius structure.

If R is Bézout and if the subring of σ_q-constants is a field K, then $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}(\phi)}$ is tannakian over K while $\sigma_q\text{-}\,Mod_R^{\mathrm{conf}\,\phi}$ is tannakian over K_s.

One has a canonical $\otimes$-functors "limit differential module":

$$\mathrm{Lim}_\infty^{\phi} : \sigma_q\text{-}\,Mod_R^{\mathrm{conf}\,\phi} \longrightarrow d\text{-}\,Mod_R^{\phi}, \qquad (M, \Sigma_q, (M_i, \Phi_i)) \longmapsto (M_\infty, \Delta_\infty, \Phi),$$
$$\mathrm{Lim}_\infty^{(\phi)} : \sigma_q\text{-}\,Mod_R^{\mathrm{conf}(\phi)} \longrightarrow d\text{-}\,Mod_R^{(\phi)}, \qquad (M, \Sigma_q, (M_i,)) \longmapsto (M_\infty, \Delta_\infty).$$

***Example 12.12* (q-Exponential).** We assume that $\pi^\tau = \pi$ and that $|q-1| < p^{-1/p-1}$. Let us consider the q-difference module ($M = \mathcal{R}, \Sigma_q$) of rank $\mu = 1$ attached to q-difference system $d_q y = \pi y$ (with solution $e_q(\pi x)$); explicitly,

$$\Sigma_q(1) = (1 + (q-1)\pi x)^{-1}.$$

Then corollary 2.5 shows that $(\Sigma_{q^{p^{is}}})$ together with

$$(\Phi_i, \Phi_i(1) = e_{q^{p^i}}(\pi x^{p^i})/e_{q^{p^{(i+1)}}}(\pi x^{p^{i+1}}))$$

form a confluent weak Frobenius structure with limit differential module $(\mathcal{R}, \Delta_\infty = x\frac{d}{dx} + \pi x)$.

***Example 12.13* (q-Logarithm).** The sequence of matrices $A_{q^{p^{is}}} = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $F_i = I$, defines a confluent weak Frobenius structure for the q-logarithm (in this case $Y_i = \left(\begin{smallmatrix} (q-1)[p^{is}]_q & 0 \\ (q-1)\frac{[p^{is}]_q}{p^{is}} \log x & 1 \end{smallmatrix}\right)$). One has a canonical fully faithful $\otimes$-functor

$$Rep_K\, \mathbb{G}_a \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R},x}^{(\mathrm{conf}\,\phi)}$$

which sends the standard two-dimensional representation to the q-difference module attached to ℓ_q.

***Example 12.14* (q-Difference modules which arise from Galois representations)**

Let us assume that k is perfect and K is discretely valued. Let $k'((x'))$ be a finite Galois extension of $k((x))$, let $\mathcal{E}^{\dagger}_{K',x'}$ be the corresponding finite unramified Galois extension of $\mathcal{E}^{\dagger}_{K,x}$, and let $\mathcal{R}_{K',x'}$ be the corresponding finite étale extension of $\mathcal{R}_{K,x}$.

Remark 12.15. $(\mathcal{E}^{\dagger}_{K',x'},\sigma_q)$ has a canonical confluent weak Frobenius structure, given by $\Sigma_{q^{p^{is}}} = \sigma_q^{p^{is}}$ and $\Phi_i = \phi$. One has $\delta_{q^{p^{is}}} = \frac{1}{(q^{p^{is}}-1)}(\sigma_q^{p^{is}} - \mathrm{Id}) \to xd/dx$ on $\mathcal{E}^{\dagger}_{K,x}$, and we are about to see that the same holds on $\mathcal{E}^{\dagger}_{K',x'}$.

We may assume that $K' = K$. Let ϖ be a uniformizer of $\mathcal{O}_K$, and let us write $ord_{\varpi}(q^{p^{is}} - 1) = n_i$ ($n_i \to \infty$ with i). Then $\sigma_q^{p^{is}}$ is identity on the henselian ring $\mathcal{O}_{\mathcal{E}^{\dagger}_x}/\varpi^{n_i}$, hence also on its étale extension $\mathcal{O}_{\mathcal{E}^{\dagger}_{x'}}/\varpi^{n_i}$. This means that all $\delta_{q^{p^{is}}}$ extend to $\mathcal{O}_K$-linear endomorphisms of $\mathcal{O}_{\mathcal{E}^{\dagger}_{x'}}$. The formula $\delta_{q^{p^{is}}}(ab) - a\delta_{q^{p^{is}}}(b) - \delta_{q^{p^{is}}}(a)b = (q^{p^{is}} - 1)\delta_{q^{p^{is}}}(a)\delta_{q^{p^{is}}}(b)$ shows that $\delta_{q^{p^{is}}}$ induces a derivation of $\mathcal{O}_{\mathcal{E}^{\dagger}_{x'}}/\varpi^{n_i}$. Since its restriction to $\mathcal{O}_{\mathcal{E}^{\dagger}_x}/\varpi^{n_i}$ is xd/dx, it is xd/dx. Therefore $\delta_{q^{p^{is}}} \to xd/dx$ on $\mathcal{E}^{\dagger}_{x'}$.

Remark 12.16. For r close enough to 1, one has $|\sigma_q(x')|_{|x'|=r'} = |\sigma_q(x')|_{r',\mathrm{can}} = r'$ for every $r' \in [r,1[$, hence $\delta_q(\mathcal{A}_{x'}([r,r']) \subset (\mathcal{A}_{x'}([r,r'])$. Arguing as above, one shows that $\delta_{q^{p^{is}}} \to xd/dx$ on $\mathcal{A}_{x'}([r,r'])$. Similarly, $\delta_{q^{p^{is}}} \to xd/dx$ on $\mathcal{R}_{K',x'}$, and $\mathcal{R}_{K',x'}$ has a canonical confluent weak Frobenius structure, given by $\Sigma_{q^{p^{is}}} = \sigma_q^{p^{is}}$ and $\Phi_i = \phi$.

One can then play the game of (12.8) with confluent weak Frobenius structures instead of strong Frobenius structures, and get:

Proposition 12.17. *There is a canonical fully faithful K_s-linear $\otimes$-functor*

$$D^{\mathrm{conf}\,\phi}_{\sigma_q} : Rep_{K_s} G_{k((x))} \longrightarrow \sigma_q\text{-}Mod^{\mathrm{conf}\,\phi}_{\mathcal{R}_x},$$

and a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\mathrm{conf}\,\phi)}_{\sigma_q} : Rep_K G_{k((x))} \longrightarrow \sigma_q\text{-}Mod^{(\mathrm{conf}\,\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto (V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$$

for suitable $\mathcal{R}_{K',x'}$ (depending on V).

Remark 12.18. Combining the last two examples, one finds a canonical fully faithful K-linear $\otimes$-functor

$$D^{(\mathrm{conf}\,\phi)}_{\sigma_q} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}Mod^{(\mathrm{conf}\,\phi)}_{\mathcal{R}_x}$$

given by

$$V \longmapsto D^{(\mathrm{conf}\,\phi)}_{\sigma_q}(V) := (V \otimes \mathcal{R}_{K',x'}[\log x])^{G_{k((x))} \times \mathbb{G}_a(K)}$$

for suitable $\mathcal{R}_{K',x'}$, as well as a canonical fully faithful K_s-linear $\otimes$-functor

$$D^{\mathrm{conf}\,\phi}_{\sigma_q} : Rep_{K_s}(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}Mod^{\mathrm{conf}\,\phi}_{\mathcal{R}_x}.$$

Composing the former with the limit functor $\mathrm{Lim}_{\infty}{}^{(\phi)}$, one gets the $\otimes$-functor

$$D^{(\phi)}_{d} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow d\text{-}Mod^{(\phi)}_{\mathcal{R}_x}.$$

This is easily checked on the regular representation of a finite Galois quotient G of $G_{k((x))}$, for which $(V \otimes \mathcal{R}_{K',x'})^{G_{k((x))}}$ is nothing but the Galois extension $\mathcal{R}'/\mathcal{R}$ in $\widetilde{\mathcal{R}}$ with Galois group G.

12.4. Solvability at the outer boundary. — We generalize definition 3.1 to the higher rank case. As in § 1.4, let t_r be a generic point of absolute value $r \in]|q-1|, 1[$ in some complete extension Ω of K. For any $\rho \in]|q-1|, r]$, let $\mathcal{A}_\Omega(t_r, \rho)$ be the ring of analytic functions in the Ω-disk $|x - t_r| < \rho$. This is a q-difference ring in a canonical way, and the canonical embedding $\mathcal{A}_K([r, 1[) \hookrightarrow \mathcal{A}_\Omega(t_r, \rho)$ is compatible with σ_q.

Let r' be in $]|q-1|, 1[$.

Definition 12.19. — A q-difference module M over $\mathcal{A}_K([r', 1[)$ is solvable (at the outer boundary) if there is a function

$$r \in]r', 1[\longmapsto \rho(r) \in]|q-1|, r]$$

such that $\lim_{r\to 1} \rho(r) = 1$ and $M \otimes_{\mathcal{A}_K([r,1[)} \mathcal{A}_\Omega(t_r, \rho(r))$ has a basis of elements fixed under Σ_q.

Since any q-difference module M over $\mathcal{E}^\dagger_x$ or $\mathcal{R}_x$ is "defined over $\mathcal{A}_K([r, 1[)$" for some r close enough to 1, this provides a definition of solvability (at the outer boundary) for such modules. We say that a q-difference module M over some finite unramified extension $\mathcal{E}^\dagger_{x'}$ (*resp.* $\mathcal{R}_{x'}$) is solvable if the underlying q-difference module over $\mathcal{E}^\dagger_x$ (*resp.* $\mathcal{R}_x$) is.

Let us choose a basis of M and denote by A_q the inverse of the matrix of Σ_q in this basis. Let us define a sequence of matrices

$$G_{q,0} = I, \quad G_{q,1} = \frac{1}{q-1}(A_q - I), \quad G_{q,m+1} = x \cdot d_q(G_{q,m}) + G_{q,m}^{\sigma_q}(G_{q,1} - m_q \cdot q^{-m}.I).$$

The q-difference system $Y^{\sigma_q} = A_q Y$ then gives rise to the sequence of systems

$$x^m d_q^m Y = G_{q,m} Y.$$

For the solution around t_r normalized by $Y(t_r)$, one has the twisted Taylor expansion (*cf.* (1.2), [**DV02**, 4], [**DV03**, 3])

$$Y(x) = \sum_{m \geqslant 0} t_r^{-m} \frac{G_{q,m}(t_r)}{[m]_q^!} (x, t_r)_{q,m}.$$

From this formula, denoting by $|\pi_q| < 1$ the limit of $|[m]_q^!|^{1/m}$, the following lemma follows immediately.

Lemma 12.20. — *M is solvable if and only if*

$$\limsup_{r\to 1} \limsup_m |G_{q,m}|_r^{1/m} \leqslant |\pi_q|.$$

Proposition 12.21. — *Any q-difference module M over $\mathcal{R}_{K',x'}$ with a strong Frobenius structure is solvable.*

Indeed, the strong Frobenius structure M induces a strong Frobenius structure on the underlying q-difference module over $\mathcal{R}_{K,x}$, hence we may assume that $\mathcal{R}_{K',x'} = \mathcal{R}_{K,x}$. Dwork's well-known argument applies (if M is defined over $\mathcal{A}_K([r^p, 1[)$ and

$M \otimes_{\mathcal{A}_K([r^p,1[)} \mathcal{A}_\Omega(t_{r^p},\rho^p)$ has a basis of elements fixed under Σ_q, and if r and ρ are close enough to 1, then $\phi_* M$ is defined over $\mathcal{A}_K([r,1[)$ and $\phi_* M \otimes_{\mathcal{A}_K([r,1[)} \mathcal{A}_\Omega(t_r,\rho)$ has a basis of elements fixed under Σ_q).

Proposition 12.22. — *For any solvable q-difference module over $\mathcal{R}_{K',x'}$ (resp. $\mathcal{E}^\dagger_{K',x'}$), $\lim_{i\to\infty}(\Sigma_q)^{p^i} = I$ in $GL(M)$.*

Proof. — Let us first consider the case of $\mathcal{R}_{K',x'}$. This is a problem about the underlying q-difference module over $\mathcal{R}_{K,x}$, hence we may assume that $\mathcal{R}_{K',x'} = \mathcal{R}_{K,x}$. Let us take a basis of M, and consider matrices A_q and $G_{q,m}$ as before, and the sequence $A_{q^n} = A^{\sigma_q^{n-1}} \cdots A^{\sigma_q} A$. We have to show that $\lim_{i\to\infty} A_{q^{p^i}} = I$ in $GL_\mu(\mathcal{R})$. The relations between the iterates of σ_q and of d_q are given by the formula (*cf.* [**DV02**, 1.1.11], [**DV03**, 1.2])

$$\sigma_q^n = \sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m q^{m(m-1)/2} x^m d_q^m .$$

It implies the following relation

$$A_{q^n} = \sum_{m=0}^{n} \binom{n}{m}_q (q-1)^m q^{m(m-1)/2} G_{q,m} .$$

Let us now take $n = p^i$, and let i tend to ∞. Let us cut the previous sum into three pieces $I + \sum_{m<p^{i/2}} + \sum_{m\geqslant p^{i/2}}$ and write, for any fixed r close enough to 1,

$$|A_{q^n}|_r \leqslant \max\left(1,\ \sup_{0<m<p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r ,\ \sup_{m\geqslant p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r\right).$$

The quantity $\sup_{0<m<p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r$ tends to 0 because the factors $\binom{p^i}{m}_q$ tend uniformly to 0. The quantity $\sup_{m\geqslant p^{i/2}} \left|\binom{p^i}{m}_q\right| |q-1|^m |G_{q,m}|_r$ tends to 0 because the factors $|G_{q,m}|_r$ tend uniformly to 0 due to the solvability condition.

Let us now assume that M is defined over $\mathcal{E}^\dagger_{K,x}$. In order to get $\lim_{i\to\infty} A_{q^{p^i}} = I$ in $GL_\mu(\mathcal{E}^\dagger)$, one needs some uniformity in r in the previous estimates. This is provided by the effective bound à la Dwork-Robba [**DV03**, 5]: for $\rho(r)$ as in (12.19),

$$|G_{q,m}|_r \leqslant \left(\sup_{m^{(1)}<\cdots<m^{(\mu-1)}\leqslant m} \left|\frac{[m]_q^!}{\prod m_q^{(i)}}\right|\right)\left(\sup_{n<\mu} |G_{q,n}|_r\right)\left(\frac{r}{\rho(r)}\right)^m .$$

which is valid for any r close enough to 1, and gives

$$|G_{q,m}| \leqslant \left(\sup_{m^{(1)}<\cdots<m^{(\mu-1)}\leqslant m} \left|\frac{[m]_q^!}{\prod m_q^{(i)}}\right|\right)\left(\sup_{n<\mu} |G_{q,n}|\right),$$

at the limit $r = 1$ by solvability (at the outer boundary). □

13. "Unit-root" q-difference modules

In this section, we study unit-root q-difference modules over $\mathcal{R}$, that is σ_q-ϕ-modules M over $\mathcal{R}$ for which there exists a $\mathcal{O}_{\mathcal{E}^\dagger}$-lattice $\mathcal{M}$ in M such that Φ induces an isomorphism $\phi_*\mathcal{M} \cong \mathcal{M}$.

We prove the q-analog of Tsuzuki's theorem (in the differential case): after passing to a finite separable extension of $k((x))$ and to the corresponding finite étale extension of $\mathcal{R}$, M admits a basis of vectors fixed by Σ_q and Φ simultaneously.

We follow Christol's approach [**C01**] of that theorem, which is more analytic than Tsuzuki's proof. The fact that what follows looks more involved than [**C01**] is not due to pecularities of the q-difference theory, but to the fact that we had to fill two gaps in Christol's paper[(20)].

13.1. Overconvergence of solutions. — We start with a complete non-archimedean field of characteristic 0, with residue field k of characteristic $p > 0$. We don't assume K discretely valued nor k perfect, but we assume that K contains the p-th roots of unity (so that Dwork's constant π belongs to K), and we fix $q \in K$ such that $|q-1| < |\pi|$.

Let α and η be real numbers such that $0 < \alpha < 1 < \eta$. Define

$$\beta = p\frac{\log \alpha}{\log \alpha + \log |p|},$$

so that

$$\alpha^{1-\log r/p\log \eta} p^{-\log r/p \log \eta}$$

takes the value 1 at $r = \eta^\beta$, and

$$\alpha < \pi \iff \beta > 1.$$

Notice that $\beta < p$.

Lemma 13.1. — *Let $Y \in GL_\mu(\mathcal{B}([0,1[))$ be such that*

$$|Y - I|_1 \leqslant \alpha \quad \textit{and} \quad G := x \cdot d_q Y \cdot Y^{-1} \in M_\mu(\mathcal{O}_{\mathcal{B}([0,\eta[)}).$$

Then there exists $Y' \in GL_\mu(\mathcal{B}([0,1[))$ such that

$$|Y' - I|_1 \leqslant \alpha \quad \textit{and} \quad G' := x \cdot d_q Y' \cdot {Y'}^{-1} \in M_\mu(\mathcal{O}_{\mathcal{B}([0,\eta^\beta[)})$$

and such that

$$Y'(x^p) \cdot Y^{-1} \in GL_\mu(\mathcal{B}([0,\eta[)).$$

(20) The first gap lies in [**C01**, prop. 13] where two different computations are made in $\mathcal{B}([0,1[)$ and $\mathcal{E}^\dagger_{1/x}$ respectively, and are subsequently compared in the "intersection $\mathcal{B}([0,1[) \cap \mathcal{E}^\dagger_{1/x}$", which is meaningless. The second gap occurs at the end of the proof (th. 17), where one is supposed to redo the argument of prop. 13 over a finite unramified extension $\mathcal{E}^\dagger_{1/x'}$ of $\mathcal{E}^\dagger_{1/x}$. But the argument of prop. 13 makes strong use of the explicit form $x \to x^p$ of Frobenius, and breaks down for $\mathcal{E}^\dagger_{1/x'}$.

Proof. — (The proof is entirely parallel to that given in [**C01**, Lemma 1] in the differential case.) There are three steps.

Step 1. — It is straightforward to check that the matrix G_m defined inductively by

$$G_0 = I, \quad G_{m+1} = x \cdot d_q(G_m) + G_m(qx)(G(x) - m_q \cdot q^{-m} \cdot I)$$

satisfies

$$x^m d_q^m Y = G_m Y.$$

From these formulas, taking into account the fact that the operator $d_q^m/[m]_q^!$ does not increase the sup-norm on $\mathcal{B}([0,1[)$ (*cf.* [**DV02**, 4.2.1], [**DV03**, 2.1]), one derives the following estimates:

$$|G_m|_1 \leqslant \left| [m]_q^! \cdot \frac{d_q^m}{[m]_q^!}(Y - I) \right| \leqslant \left| [m]_q^! \right| \cdot \alpha, \quad |G_m|_\eta \leqslant 1.$$

Since $\log r \mapsto \log |G_m|_r$ is a convex function, one obtains by interpolation

$$|G_m|_r \leqslant \left(\left| [m]_q^! \right| \cdot \alpha \right)^{1 - \log r/\log \eta}$$

for every r between 1 and η. In particular, $|G_m|_r < 1$ and $\lim_m |G_m|_r = 0$ if $r < \eta$.

Step 2. — We shall see that

$$Y' := \frac{1}{p} \sum_{y^p = x} Y(y)$$

satisfies the conditions of the lemma. Since the m-th Taylor coefficient of Y' is nothing but the pm-th Taylor coefficient of Y, it is clear that $|Y' - I|_1 \leqslant \alpha$.

Moreover, considering the q-analog of the Taylor expansion, one finds

$$\begin{aligned} Y'(x^p)Y(x)^{-1} &= \frac{1}{p} \sum_{\zeta^p = 1} Y(\zeta x)Y(x)^{-1} \\ &= I + \sum_1^\infty G_m(x) \left(\frac{1}{p} \sum_{\zeta^p = 1} \frac{(\zeta - 1) \dots (\zeta - q^{m-1})}{[m]_q^!} \right). \end{aligned}$$

Since $|q - 1| < |\pi|$, the coefficient of G_m is in $\mathcal{O}_K$, and it easily follows that $H := Y'(x^p) \cdot Y^{-1} \in GL_\mu(\mathcal{B}([0, \eta[))$.

Step 3. — One has

$$p_q G'(x^p) = (x \cdot d_q)(Y(x^p))Y(x^p)^{-1} = x d_q H \cdot H^{-1}(qx) + HGH^{-1}(qx)$$

which lies in $GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta[)})$, hence $G' \in M_\mu(\mathcal{B}([0, \eta^p[))$ and $|G'|_{\eta^p} \leqslant |p_q|^{-1} = |p|^{-1}$.

On the other hand, $|G'|_1 = |d_q Y' . (Y')^{-1}|_1 \leqslant \alpha$. By log-convex interpolation, one finds

$$|G'|_r \leqslant \alpha^{1 - \log r/p \log \eta} p^{-\log r/p \log \eta}$$

for every r between 1 and η^p. In particular, $|G'|_{\eta^\beta} \leqslant 1$ by the choice of β, whence the lemma. □

Assume moreover that $\alpha < |\pi|$ and let n be a non-negative integer such that $\eta^{\beta''} \cdot |\pi| > 1$. Then the real number $\eta' := (\eta^{\beta''} \cdot |\pi|)^{p^{-n}}$ lies in the interval $]1, \eta[$.

Proposition 13.2. – *Under the assumption of the lemma, $Y \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$.*

Proof (cf. [**C01**, Prop. 3]*).* – One has $|x^{-m}(G_m/[m]_q^!)|_\eta \leqslant |1/[m]_q^!|\eta^{-m} \leqslant (|\pi|\eta)^{-m}$. If $n = 0$, this is $\leqslant (\eta')^{-m}$, and by the q-analog of the Taylor expansion of Y, one concludes that $Y \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$.

One then argues by induction on n. By the previous lemma, there is Y' which satisfies our assumptions with n replaced by $n-1$ and η replaced by η^β. By induction, $Y' \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,(\eta')^{p'}[)})$. Hence $Y'(x^{p'}) \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$, and since $H \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta[)}) \subset GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$, one concludes that $Y = H^{-1}Y'(x^{p'}) \in GL_\mu(\mathcal{O}_{\mathcal{B}([0,\eta'[)})$ as well. □

Corollary 13.3. – *Let $Y \in GL_\mu(\mathcal{B}(]1,\infty])$ be such that $|Y - I| < |\pi|$ and $Y^{\sigma_q}Y^{-1} \in M_\mu(\mathcal{H}_x^\dagger)$. Then $Y \in GL_\mu(\mathcal{H}_x^\dagger)$.*

Indeed, the assumption implies $(xd_qY) \cdot Y^{-1} \in M_\mu(\mathcal{H}_x^\dagger)$. In fact, since actually $Y \in GL_\mu(\mathcal{O}_{\mathcal{B}(]1,\infty])})$, $(xd_qY) \cdot Y^{-1} \in M_\mu(\mathcal{O}_{\mathcal{H}_x^\dagger})$. After change of variable $x \mapsto 1/x$, and for η close enough to 1, one is in the situation of the previous proposition.

13.2. q-analog of Tsuzuki's theorem: the case when F is close to I

We assume that K *is endowed with a Frobenius automorphism τ and that k is algebraically closed.* We fix a positive integer s and consider the Frobenius $\phi : \sum a_n x^n \mapsto \sum \tau^s(a_n) x^{p^s n}$ on $\mathcal{E}_x^\dagger$. We *assume that $q^{\tau^s} = q$.*

Lemma 13.4. – *For any $F_0 \in GL_\mu(\mathcal{O}_K)$ such that $|F_0 - I| < 1$, there exists $H_0 \in GL_\mu(\mathcal{O}_K)$ such that $H_0^{\tau^s} F_0 H_0^{-1} = I$ and $|H_0 - I| = |H_0^{-1} - I| \leqslant |F_0 - I|$.*

For $s = 1$, this is part of [**C01**, Lemma 12]. The same proof works for any $s > 0$.

Lemma 13.5. – *For any $F \in GL_\mu(\mathcal{E}_x^\dagger)$ such that $|F - I| < 1$, there exists $H \in GL_\mu(\mathcal{O}_K[[x]]) \subset GL_\mu(\mathcal{O}_{\mathcal{E}_x^\dagger})$ such that $H^\phi F H^{-1} \in GL_\mu(\mathcal{H}_x^\dagger))$ and $|H - I| = |H^{-1} - I| \leqslant |F - I|$.*

Proof. – For $s = 1$, this is part of [**C01**, Lemma 8]. The proof for any $s > 0$ is similar. One starts by noticing that the endomorphism $id - \phi$ of $xK[[x]]$ has an inverse: $\psi = \sum_1^\infty \phi^n$, and that this inverse stabilizes the subring $\gamma_{x>0}\mathcal{E}$. Let $\mathcal{U}$ be the closed subspace of $M_\mu(\mathcal{O}_K[[x]])$ consisting of matrices H satisfying $|H - I| = |H^{-1} - I| \leqslant |F - I|$, and let us consider the following endomorphism f of $\mathcal{U}$:

$$f(H) = H + (\psi \circ \gamma_{x>0})(H^\phi F H^{-1})$$

which can also be written as

$$f(H) = I + (\psi \circ \gamma_{x>0})(H^\phi F H^{-1} + H - H^\phi).$$

Let us check that f is a contraction; one has

$$\begin{aligned} f(H) - f(H') &= (\psi \circ \gamma_{x>0})(H^{\phi}(FH^{-1} - I) - (H')^{\phi}(F(H')^{-1} - I) + H - H') \\ &= (\psi \circ \gamma_{x>0})\big((H^{\phi} - (H')^{\phi})(FH^{-1} - I)\big) \\ &\qquad - \big((H')^{\phi}F(H')^{-1}(H - H')H^{-1} + H - H'\big), \end{aligned}$$

and it is clear that the norm of both terms in the difference is $< |H - H'|$. Thus f has a fixed point H, and since ψ is invertible, $\gamma_{x>0}(H^{\phi}FH^{-1}) = 0$ as wanted. □

Proposition 13.6. — *Let $F \in GL_{\mu}(\mathcal{E}_x^{\dagger})$ satisfy $|F - I| < |\pi|$, and let A_q, $A_{q^{p^s}} \in GL_{\mu}(\mathcal{R}_x)$ be such that*

$$F^{\sigma_q}A_q = A_{q^{p^s}}^{\phi}F, \quad A_{q^{p^s}} = A_q^{\sigma_{q^{p^s-1}}} \cdots A_q.$$

Then $A_q \in GL_{\mu}(\mathcal{O}_{\mathcal{E}_x^{\dagger}})$ and there exists $Y \in GL_{\mu}(\mathcal{O}\mathcal{E}_x^{\dagger})$ such that

$$Y^{\sigma_q} = A_qY, \quad Y^{\phi} = FY.$$

Proof. — Using the two previous lemmas, one reduces to the case when $F \in GL_{\mu}(\mathcal{O}_{\mathcal{H}_x^{\dagger}})$ and $F(0) = I$ (still with $|F - I| < |\pi|$). Let us define a sequence of matrices $Y_m \in GL_{\mu}(\mathcal{O}_{\mathcal{H}_x^{\dagger}})$ by

$$Y_m = F^{-1}(F^{\phi})^{-1} \cdots (F^{\phi^m})^{-1}.$$

As above, let us set, for any positive integer n, $A_{q^n} = A_q^{\sigma_{q^{n-1}}} \cdots A_q$. From the relation $A_{q^{p^s}}^{\phi} = F^{\sigma_q}A_qF^{-1}$, one derives by induction on n the following relation in $GL_{\mu}(\mathcal{R}_x)$:

$$A_{q^{np^s}}^{\phi} = F^{\sigma_{q^n}}A_{q^n}F^{-1}.$$

Applying this to the powers of p^s, one then computes

$$(Y_m^{\sigma_q})^{-1}A_qY_m = (F^{\phi^m})^{\sigma_q} \cdots (F^{\phi})^{\sigma_q}F^{\sigma_q}A_qF^{-1}(F^{\phi})^{-1} \cdots (F^{\phi^m})^{-1} = A_{q^{p^{ms}}}^{\phi^m}.$$

We now use the fact (12.21, 12.22) that the sequence $A_{q^{p^{ms}}}$, hence also $A_{q^{p^{ms}}}^{\phi^m}$, tends to I in $GL_{\mu}(\mathcal{R}_x)$, and derive that $A_q \in GL_{\mu}(\mathcal{H}_x^{\dagger})$. Indeed, since $\gamma_{x>0}Y_m = \gamma_{x>0}Y_m^{\sigma_q} = 0$, $\gamma_{x>0}A_q = Y_m^{\sigma_q}(\gamma_{x>0}(A_{q^{p^{ms}}}^{\phi^m}))Y_m^{-1}$; one has $\gamma_{x>0}(A_{q^{p^{ms}}}^{\phi^m}) \to \gamma_{x>0}I = 0$, whence $\gamma_{x>0}A_q = 0$, and it follows that $A_q \in GL_{\mu}(\mathcal{H}_x^{\dagger})$.

By (12.21), (12.22) again, $|A_{q^{p^{ms}}}^{\phi^m} - I|_{\mathcal{E}^{\dagger}} \to 0$, hence $(Y_m^{\sigma_q})^{-1}A_qY_m$ tends to I also in $GL_{\mu}(\mathcal{B}(]1, \infty])$. But Y_m has a limit Y in $GL_{\mu}(\mathcal{O}_{\mathcal{B}(]1,\infty]})$, hence $Y^{\sigma_q}Y^{-1} = A_q \in GL_{\mu}(\mathcal{B}(]1, \infty])$. Applying (13.3), one concludes that $Y \in GL_{\mu}(\mathcal{H}_x^{\dagger})$, and that $Y^{\sigma_q}Y^{-1} = A_q$. Since $Y \in GL_{\mu}(\mathcal{O}_{\mathcal{B}(]1,\infty]})$, the entries of A_q and Y actually lie in $\mathcal{O}_{\mathcal{H}_x^{\dagger}}$.

On the other hand, it is obvious that $Y^{\phi} = FY$ in $GL_{\mu}(\mathcal{O}_{B(]1,\infty])})$, hence also in $GL_{\mu}(\mathcal{H}_x^{\dagger})$. □

Corollary 13.7. — *Let (M, Σ_q) be a q-difference module over $\mathcal{R}_x$ with a strong Frobenius structure Φ. Let us assume that there is a basis of M in which the Frobenius*

matrix F has entries in $\mathcal{E}^\dagger$ and satisfies $|F-I|<|\pi|$. Then there is a basis of M which is fixed under both Σ_q and Φ.

13.3. Going up in finite unramified extensions of $\mathcal{E}^\dagger$. — We now assume in addition that K *is discretely valued* (and again that k is algebraically closed). Hence K is a finite extension of the field of fractions of the Witt ring $W(k)$.

We are now looking for generalizations of the previous results when $\mathcal{E}^\dagger_{K,x}$ is replaced by a finite unramified extension $\mathcal{E}^\dagger_{K,x'}$ (x' being a good coordinate as in § 11.3). The difficulty lies of course in the fact that one cannot assume that $\sigma_q(x')$ is proportional to x', nor that $\phi(x')=(x')^p$.

We use again the notations $f: C \xrightarrow{x} \mathbf{P}^1_{\mathcal{O}_K} \smallsetminus 0$, etc... of § 11.3. Let d be the degree of f. Let us first generalize corollary 13.3.

Proposition 13.8. — *Let $Y' \in GL_\mu(B_{x'}(]1,\infty])$ be such that $|Y'-I|<|\pi|$ and $(Y')^{\sigma_q}(Y')^{-1} \in M_\mu(\mathcal{O}^\dagger(C)_K)$. Then $Y' \in GL_\mu(\mathcal{O}^\dagger(C)_K)$.*

Proof. — We first notice that it is sufficient to prove that $Y' \in GL_\mu(\mathcal{O}^\dagger(C)[1/h])$ for some $h \in \mathcal{H}^\dagger_x$ of norm 1: the fact that $A'_q := (Y')^{\sigma_q}(Y')^{-1}$ has no pole at $h=0$ will imply that Y' has no pole as well at $h=0$. After such a localization, we may assume that $\mathcal{O}^\dagger(C)[1/h]$ is a free $\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h]$-module of rank d.

In the second place, we notice that $A'_q \in M_\mu(\mathcal{O}^\dagger(C))$. Let us endow the free $\mathcal{O}^\dagger(C)[1/h]$-module of rank μ, say $M' = \oplus_{i=1}^{i=\mu} \mathcal{O}^\dagger(C)[1/h]m'_i$, with the σ_q-linear endomorphism Σ_q defined by the matrix $(A'_q)^{-1}$ in the basis (m_i). Via f, M' gives rise to an "underlying" q-difference module M over $\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h]$. We endow it with the basis (m_j) constructed via (m'_i) and a fixed basis of $\mathcal{O}^\dagger(C)[1/h]$ over $\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h]$. In this basis, the q-difference system has a solution $Y \in GL_{\mu d}(\mathcal{B}_x(]1,\infty]))$ with $Y(0)=I$. Let ϖ be a uniformizing parameter for $\mathcal{O}_K$, and let n be such that $|\pi|=|\varpi|^{n-1}$. The condition $|Y'-I|<|\pi|$ translates into: the image of the vectors m'_i in M'/ϖ^n are fixed under Σ_q. Since $\sigma_q \equiv \mathrm{id} \bmod \varpi^n$, this implies that the image of the vectors m_j in M/ϖ^n are fixed under Σ_q. Hence $|Y-I|<|\pi|$, and by (13.3), $Y \in GL_{\mu d}(\mathcal{O}_{\mathcal{H}^\dagger_x}[1/h])$, which implies that $Y' \in GL_\mu(\mathcal{O}^\dagger(C)[1/h])$. □

Let us now generalize lemma 13.5:

Lemma 13.9. — *For any $F' \in GL_\mu(\mathcal{E}^\dagger_{x'})$ such that $|F'-I|<1$, there exists $H \in GL_\mu(\mathcal{O}_K[[x']]) \subset GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger_{x'}})$ such that $H^\phi F' H^{-1} \in GL_\mu(\mathcal{O}^\dagger(C)_K)$ and $|H-I| = |H^{-1}-I| \leqslant |F'-I|$.*

Proof. — One can even require that $H^\phi F' H^{-1} \in GL_\mu(\mathcal{H}^\dagger_{x'})$. The proof is almost the same as in (13.5), except that one has to deal with two Frobenius endomorphisms at the same time: ϕ, and ϕ' which raises to x' to the power p^s. One introduces the

inverse $\psi' = \sum_1^\infty (\phi')^n$ of $id - \phi'$, and the space $\mathcal{U}'$ analogous to $\mathcal{U}$ with x' in place of x. One checks that the formula

$$f(H) = H + (\psi \circ \gamma_{(x')^{>0}})(H^\phi F' H^{-1})$$

(with ϕ, not ϕ'!) defines an endomorphism of $\mathcal{U}'$. One checks as in (13.5) that f is a contraction by writing $f(H)$ in the form $f(H) = I + (\psi \circ \gamma_{(x')^{>0}})(H^\phi (F' H^{-1} - I) + H + (H^\phi - H^{\phi'}))$ and using the fact that $|H^\phi - (H')^\phi - H^{\phi'} + (H')^{\phi'}| < |H - H'|$. □

At last, let us generalize proposition 13.6:

Proposition 13.10. — *Let $F' \in GL_\mu(\mathcal{E}^\dagger_{x'})$ satisfy $|F' - I| < |\pi|$, and let A'_q, $A'_{q^{p^s}} \in GL_\mu(\mathcal{R}_{x'})$ be such that*

$$(F')^{\sigma_q} A'_q = (A'_{q^{p^s}})^\phi F', \quad A'_{q^{p^s}} = (A'_q)^{\sigma_{q^{p^{s-1}}}} \cdots A'_q.$$

Then $A'_q \in GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger_{x'}})$ there exists $Y' \in GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger_{x'}})$ such that

$$(Y')^{\sigma_q} = A'_q Y', \quad (Y')^\phi = F' Y'.$$

Proof. — Using (13.8) and (13.9) in the place of (13.3) and (13.5) respectively, the argument is the same as in (13.6), except that one cannot directly apply the truncation operator $\gamma_{x'^{>0}}$ since σ_q does not commute with $\gamma_{x'^{>0}}$. We are in the situation where F', hence Y'_m, is in $GL_\mu(\mathcal{O}^\dagger(C)_K)$, and where $((Y'_m)^{\sigma_q})^{-1} A'_q Y'_m = (A'_{q^{p^{ms}}})^{\phi^m} \to I$ in $GL_\mu(\mathcal{R}_{x'})$, and we have to derive that $A'_q \in GL_\mu(\mathcal{O}^\dagger(C)_K)$.

We first notice that $\mathcal{O}^\dagger_K$ is a free $\mathcal{H}^\dagger_x$-module of rank d (it comes from a locally free $\mathcal{A}([r,\infty])$-module of rank d ($r < 1$), which is automatically free since $\mathcal{A}([r,\infty])$ is principal).

Let us endow the free $\mathcal{O}^\dagger(C)_K$-module of rank μ, say $M' = \oplus_{i=1}^{i=\mu} \mathcal{O}^\dagger(C)_K m'_i$, with a ϕ-linear endomorphism Φ defined by the matrix $(F')^{-1}$ in the basis (m_i), and endow $M' \otimes \mathcal{E}^\dagger_{x'}$ with the σ_q-linear endomorphism Σ_q defined by the matrix $(A'_q)^{-1}$ in the same basis. Via f, M' gives rise to an "underlying" ϕ-difference module M over $\mathcal{H}^\dagger_x$, and $M \otimes \mathcal{E}^\dagger_x$ becomes a σ_q-ϕ-module over $\mathcal{E}^\dagger_x$. We endow M with the basis (m_j) constructed via (m'_i) and a fixed basis of $\mathcal{O}^\dagger_K$ over $\mathcal{H}^\dagger_x$, and we denote by A_q and F respectively the inverse matrices of Σ_q and Φ in this basis. Then the argument of (13.6) applies and shows that $A_q \in GL_\mu(\mathcal{H}^\dagger_x)$, which implies that $A'_q \in GL_\mu(\mathcal{O}^\dagger(C)_K)$. □

13.4. q-analog of Tsuzuki's theorem: the general case. — In this section K is a complete discrete valuation field of characteristic 0, with *algebraically closed residue field* k of characteristic p. We assume that $\pi \in K$, that $|q-1| < |\pi|$. K is endowed with a Frobenius τ, and we assume that, for a given $s > 0$, $\tau^s(q) = q$.

As before, ϕ is the τ^s-linear endomorphism of $\mathcal{E}^\dagger$ (or $\mathcal{R}$) which sends x to x^{p^s}.

Theorem 13.11. — *Let (M, Σ_q) be a q-difference module over $\mathcal{R}_x$ with a strong Frobenius structure Φ. Let us assume that there is a $\mathcal{O}_{\mathcal{E}_x^\dagger}$-lattice $\mathcal{M}$ in M such that Φ induces an isomorphism*

$$\phi_*\mathcal{M} \xrightarrow{\cong} \mathcal{M}.$$

Then there is a finite unramified extension $\mathcal{E}_{x'}^\dagger$ of $\mathcal{E}_x^\dagger$ and a basis of $\mathcal{M} \otimes_{\mathcal{O}_{\mathcal{E}_x^\dagger}} \mathcal{O}_{\mathcal{E}_{x'}^\dagger}$ which is fixed under both Σ_q and Φ.

In particular, $(M \otimes_{\mathcal{R}_x} \mathcal{R}_{x'}, \Sigma_q)$ is a trivial q-difference module.

This follows from the previous proposition and the following lemma of N. Tsuzuki [**T98b**, 5.2.2] (*cf.* also [**C01**, Lemma 16]):

Lemma 13.12. — *For any $F \in GL_\mu(\mathcal{O}_{\mathcal{E}_x^\dagger})$, there exists a finite unramified extension $\mathcal{E}_{x'}^\dagger$ and $H \in GL_\mu(\mathcal{O}_{\mathcal{E}_{x'}^\dagger})$ such that $|H^\phi F H^{-1} - I| < |\pi|$.*

This is proven in [**T98b**, 5.1.1], [**C01**, lemme 16] for $s = 1$, but the proof works for any $s > 0$.

14. Quasi-unipotence

In this section K is assumed to be complete, discretely valued, of characteristic 0, with residue field k perfect of characteristic $p > 0$. K is endowed with a Frobenius τ. As before, ϕ is the τ^s-linear endomorphism of $\mathcal{E}^\dagger$ (or $\mathcal{R}$) which sends x to x^{p^s}.

We shall prove the quasi-unipotence of q-difference modules over $\mathcal{R}$ which admit a strong Frobenius structure, using Kedlaya's structure theorem for ϕ-modules over $\mathcal{R}$.

Another path toward quasi-unipotence seems possible, through the structure theorem of [**A02**] for tannakian categories with a Hasse-Arf filtration. However, this would require a q-analog of Christol-Mebkhout theory of exponents and analytic slopes.

14.1. Kedlaya's filtration. —

We shall rely on the following theorem [**Ke**, 6.10]:

Theorem 14.1. — *Let (M, Φ) be a ϕ-module over $\mathcal{R}$. Then there exists a unique finite ascending filtration (M_j) by saturated ϕ-modules such that*

i) *the quotients M_j/M_{j+1} have a single Φ-slope s_j,*

ii) *$j \mapsto s_j$ is increasing,*

iii) *each quotient M_j/M_{j+1} comes from a unique ϕ-module (N_j, Φ) over $\mathcal{E}^\dagger$, and $N_j \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ has a unique slope in the sense of Dieudonné-Manin, which is s_j.*

Here, $\mathcal{E}$ is the usual notation for the p-adic completion of $\mathcal{E}^\dagger$. "M_j saturated" means as usual that M/M_j is torsion-free over $\mathcal{R}$, which implies that it is free, since it is also finitely presented and $\mathcal{R}$ is a Bézout ring. Notice that $M_1 = N_1 \otimes_{\mathcal{E}^\dagger} \mathcal{R}$.

The definition of Φ-slopes[21] involves the "big Robba ring" $\widetilde{\mathcal{R}}$ (*cf.* § 11.2), whose precise definition will not matter here. A ϕ-module L over $\mathcal{R}$ is said to have a single slope s if there exists a basis of eigenvectors for Φ in $L \otimes_{\mathcal{R}} \widetilde{\mathcal{R}}$ with eigenvalues in some finite extension of $\mathcal{O}_K$ of valuation equal to s.

We shall also use the following result:

Proposition 14.2. — *Let (M, Φ) be a ϕ-module over $\mathcal{R}$. Then up to replacing $k((x))$ by a finite separable extension (and $\mathcal{R}$ by the corresponding étale extension) and K by a finite extension,*

i) *there exists a basis $\underline{m}$ of M in which the inverse matrix F of Φ can be written as $F = D^{-1}U$, where D is a diagonal matrix with diagonal entries D_i in K and of non-decreasing valuations, and with $U \in GL_\mu(\mathcal{O}_{\mathcal{E}^\dagger})$, $|U - I| < 1$,*

ii) *the Φ-slopes of $N := \oplus \mathcal{E}^\dagger m_i$ coincide with the slopes of $N \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ in the sense of Dieudonné-Manin,*

iii) *N contains N_1 (in the notation of § 14.1),*

iv) *the valuations of D_i are equal for $i < \operatorname{rk} N$, and are strictly smaller than the valuations of D_j for $j > \operatorname{rk} N$,*

v) *N is the direct sum of N_1 and $N_1' := \oplus_{i > \operatorname{rk} N} \mathcal{E}^\dagger m_i$,*

vi) *moreover, one may choose $\underline{m}$ in such a way that $N = \oplus_{i \leqslant \operatorname{rk} N} \mathcal{E}^\dagger m_i$, so that U may be written in block form as $U = \left(\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right)$, where $X \in GL_{\operatorname{rk} N}(\mathcal{O}_{\mathcal{E}^\dagger})$, $Z \in GL_{\mu - \operatorname{rk} N}(\mathcal{O}_{\mathcal{E}^\dagger})$, and Y has entries in $\mathcal{O}_{\mathcal{E}}^\dagger$.*

Proof. — Multiplying F by some constant and replacing K by a finite extension, one may assume that the Φ-slopes belong to the value group of $\mathcal{O}$.

For i) and ii), see [**Ke**, 6.9], where it is shown that the Dieudonné-Manin slopes of $N \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ are the valuations of the D_i's.

The construction and characterization of N_1 in [**Ke**, 6.10] also shows iii). On the other hand, setting $\widetilde{\mathcal{E}} = K \otimes_{\mathbb{Z}_p} W(\overline{k((x))})$, it is shown in [**Ke**, 5.9] that Φ can be put into diagonal form D by some change of basis via a matrix $V \in GL_\mu(\mathcal{O}_{\widetilde{\mathcal{E}}})$ (applied to $\underline{m}$), with $|V - I| < 1$. Points iv), v) and v) follow easily. □

14.2. The local monodromy theorem. — We assume that $|q-1| < |\pi|$ and that $\tau^s(q) = q$.

Theorem 14.3. — *Any q-difference module (M, Σ_q) over $\mathcal{R}$ which admits a strong Frobenius structure is quasi-unipotent: after replacing $k((x))$ by a finite separable extension (and $\mathcal{R}$ by the corresponding étale extension), it admits a filtration by saturated q-difference submodules, with trivial quotients*[22].

[21] called special slopes in [**Ke**]: we call them Φ-slopes here in order to prevent any confusion with the (analytic) slopes à la Christol-Mebkhout, whose definition does not make use of any Frobenius structure, and which will appear in the sequel.

[22] as q-difference modules, *i.e.*, they admit a basis which is fixed under Σ_q.

Proof. — Multiplying F by some constant and replacing K by a finite extension, one may assume that the Φ-slopes belong to the value group of $\mathcal{O}$, and that the least Φ-slope is 0.

We may freely replace $k((x))$ by a finite separable extension and extend the discretely valued field K, so that we may assume that we are in the situation of the previous proposition, that $\pi \in K$ and that k is algebraically closed.

We shall prove that M_1 is stable under Σ_q. Applying (13.11), one deduces that it is trivialized, as a q-difference module, in some finite extension of $\mathcal{R}$ coming from a finite separable extension of $k((x))$. The same will be true for the M_j by induction, whence the quasi-unipotence of (M, Σ_q).

We rely of course on the integrability condition $F^{\sigma_q} A_q = A_{q^{p^s}}^{\phi^s} F$, in the basis provided by (14.2) vi). We write $A_q, A_{q^{p^s}}$ in block form

$$A_q = \begin{pmatrix} P_q & Q_q \\ R_q & S_q \end{pmatrix}, \quad A_{q^{p^s}} = \begin{pmatrix} P_{q^{p^s}} & Q_{q^{p^s}} \\ R_{q^{p^s}} & S_{q^{p^s}} \end{pmatrix}.$$

We have to show that $R_q = 0$.

The lower left corner of the integrability condition gives rise to the equation

$$R_q = (Z^{\sigma_q})^{-1} D'' R_{q^{p^s}}^{\phi} (D')^{-1} X,$$

where D' (*resp.* D'') is the diagonal matrix with diagonal entries D_i, $i \leqslant \operatorname{rk} M_1$ (*resp.* $i > \operatorname{rk} M_1$).

Let us fix r sufficiently close to 1 (so that our q-difference module with Frobenius structure is defined over $\mathcal{A}([r, 1[)$). Then the canonical absolute values $|\ |_{r^{1/p^s},\mathrm{can}}$ are defined for each of the matrices entering the last displayed equation. Moreover

$$|(Z^{\sigma_q})^{-1} D''|_{r^{1/p^s},\mathrm{can}} \cdot |(D')^{-1} X|_{r^{1/p^s},\mathrm{can}} \leqslant 1,$$

and $|R_{q^{p^s}}^{\phi}|_{r^{1/p^s},\mathrm{can}} = |R_{q^{p^s}}|_{r,\mathrm{can}}$. Thus $|R_q|_{r^{1/p^s},\mathrm{can}} \leqslant |R_{q^{p^s}}|_{r,\mathrm{can}}$. Now we may replace ϕ by a large power, *i.e.*, s is replaced by a large multiple. When $s \to \infty$, we know that $A_{q^{p^s}} \to I$ in $\mathcal{R}$ (by (12.21), (12.22)), hence $|R_{q^{p^s}}|_{r,\mathrm{can}} \to 0$. We conclude that $R_q = 0$. □

14.3. Logarithmic variant

Theorem 14.4. — *For any q-difference module (M, Σ_q) over $\mathcal{R}_x$ which admits a strong Frobenius structure, there is a finite étale extension $\mathcal{R}_{x'}/\mathcal{R}_x$ coming from a finite separable extension $k'((x'))/k((x))$, such that $M \otimes_{\mathcal{R}_x} \mathcal{R}_{x'}[\log x]$ has a basis fixed by Σ_q.*

This follows from the previous theorem and the following lemma:

Lemma 14.5. — *$d_q : \mathcal{R}_{x'}[\log x] \to \mathcal{R}_{x'}[\log x]$ is surjective.*

Proof. — It is easy straightforward that $\delta_q = x d_q = (q-1)^{-1}(\sigma_q - id)$ induces a K-linear isomorphism $\mathcal{R}_x/K \to \mathcal{R}_x/K$. Besides, note that $\delta_q(x^n log^k x)$ is a polynomial of degree $\leqslant k$ in $\log x$, with coefficients in $K[x, 1/x]$. From there, using the twisted

Leibniz rule, it is not difficult to conclude by induction on the degree of log that $\delta_q : \mathcal{R}_x[\log x] \to \mathcal{R}_x[\log x]$ is surjective.

Let us now turn to the etale extension $\mathcal{R}_{K',x'}[\log x]$. We may assume that $K = K'$. Again, by induction on the degree of log, one reduces the statement to the surjectivity of $\delta_q : \mathcal{R}_{x'}/K \to \mathcal{R}_{x'}/K$.

Recall (*cf.* (12.16)) that if I is a closed interval contained in $]0,1[$ and close enough to the point 1, one has $\delta_q(\mathcal{A}_{x'}(I)) \subset (\mathcal{A}_{x'}(I))$, and that $\delta_{q^{p^{is}}} \to xd/dx$ when $i \to \infty$.

One has

$$\delta_q(f) = g \Longleftrightarrow \delta_{q^{p^{is}}} f = \psi_i(g) := ([p^{is}]_q)^{-1}(g^{\sigma_{q^{p^{is}}-1}} + \cdots + g)\,.$$

It is easy to see that for given $g \in \mathcal{A}_{x'}(I)$, $\psi_i(g)$ forms a Cauchy sequence (for $j < i$, write $\psi_i(g)$ as $([p^{js}]_q)([p^{is}]_q)^{-1} \sum_{m=0}^{m=p^{(i-j)s}-1} \psi_j(g^{\sigma_{(q^{p^{js}})^m}})$ and compare each of the $p^{(i-j)s}$ terms to $\psi_j(g)$). Hence it converges in the Banach space $\mathcal{A}_{x'}(I)$.

On the other hand, although $x'd/dx' : \mathcal{A}_{x'}(I)/K \to \mathcal{A}_{x'}(I)/K$ is *not* surjective, every $h \in \mathcal{A}_{x'}(I)/K$ has a $(x'd/dx')$-preimage in $\mathcal{A}_{x'}(I')/K$, for any closed interval I' contained in the interior of I. For I close enough to 1, the same is true for xd/dx, which is a multiple of $x'd/dx'$ by a unit in $\mathcal{A}_{x'}(I)$: there exists $f \in \mathcal{A}_{x'}(I')/K$ such that $xd/dx(f) = h$. In the case $h = \lim \psi_i(g)$, one finds $\delta_q(f) = g$ in $\mathcal{A}_{x'}(I)/K$. Whence the result. □

As in the differential case (*cf.* [**A02**]), theorem 14.4 can be expressed in the following "tannakian way":

Theorem 14.6. – *Let us assume that k is algebraically closed. Then the canonical $\otimes$-functor*

$$D^{(\phi)}_{\sigma_q} : Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}_x}$$

is an equivalence of categories. In fact, there is a canonical quasi-inverse $V^{(\phi)}_{\sigma_q}$ and a canonical isomorphism $D^{(\phi)}_{\sigma_q} \circ V^{(\phi)}_{\sigma_q} \cong \mathrm{Id}$.

This quasi-inverse is given by $V^{(\phi)}_{\sigma_q}(M) := (M \otimes_{\mathcal{R}} \mathcal{R}'[\log x])^{\Sigma_q}$ for $\mathcal{R}'$ big enough (in $\widetilde{\mathcal{R}}$), and the isomorphism $M \xrightarrow{\cong} D^{(\phi)}_{\sigma_q} V^{(\phi)}_{\sigma_q}(M)$ is induced by the canonical isomorphism

$$(M \otimes_{\mathcal{R}} \mathcal{R}'[\log x])^{\Sigma_q} \otimes_K \mathcal{R}'[\log x] \xrightarrow{\cong} M \otimes_{\mathcal{R}} \mathcal{R}'[\log x].$$

Proof. – We know that this functor is fully faithful, and its essential surjectivity is ensured by (14.4). The rest is formal and left to the reader. □

Remark 14.7. – This functor actually comes from a fully faithful $\otimes$-functor

$$Rep_K(G_{k((x))} \times \mathbb{G}_a) \longrightarrow \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{E}^\dagger_x},$$

but the latter is *not* essentially surjective. For instance

$$\mathrm{Ext}_{\sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}}}(\mathcal{R},\mathcal{R}) = \mathrm{Ext}_{Rep\,\mathbb{G}_a}(K,K) = K,$$

while $\mathrm{Ext}_{\sigma_q\text{-}Mod^{(\phi)}_{\mathcal{E}^\dagger}}(\mathcal{E}^\dagger,\mathcal{E}^\dagger)$ is a K-space of infinite dimension.

Corollary 14.8. — *There is a canonical K-linear fully faithful $\otimes$-functor*

$$\sigma_q\text{-}\,Mod_{\mathcal{R}}^{(\phi)} \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}}^{\mathrm{conf}(\phi)}$$

given by $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$[(23)].

Is it an equivalence of categories? This is likely. One way to tackle the question would be to prove directly the essential surjectivity of $D_{\sigma_q}^{(\mathrm{conf}\,\phi)}$ along the above lines; the major technical problem in this direction is to control the variation of Kedlaya's filtration attached to the confluent sequence of $q^{p^{is}}$-difference modules when $i \to \infty$.

15. Applications

15.1. Confluence. — We assume again that k is algebraically closed. We assume that $\pi \in K$, that $|q-1| < |\pi|$. K is endowed with a Frobenius τ, and we assume that, for a given $s > 0$, $\tau^s(q) = q$.

As before, ϕ is the τ^s-linear endomorphism of $\mathcal{R}$ which sends x to x^{p^s}.

We call the composed $\otimes$-functor

$$\mathrm{Conf} : \sigma_q\text{-}\,Mod_{\mathcal{R}}^{(\phi)} \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}}^{\mathrm{conf}(\phi)} \longrightarrow d\text{-}\,Mod_{\mathcal{R}}^{(\phi)},$$

where the first functor is $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_{\sigma_q}^{(\phi)}$ and the second is $\mathrm{Lim}_{\infty}^{(\phi)}$, "*the p-adic confluence functor*".

Theorem 15.1. — *One has* $\mathrm{Conf} \circ D_{\sigma_q}^{(\phi)} = D_d^{(\phi)}$. *In particular,* Conf *is an equivalence.*

Proof. — This is clear from remark 12.18, and from the fact that $D_{\sigma_q}^{(\phi)}$ and $D_d^{(\phi)}$ are equivalences, by the quasi-unipotence theorems. □

Remark 15.2. — It is clear that the restriction of these functors to rank one objects gives rise to the group isomorphisms of (8.2).

15.2. Canonical q-deformation. — We call the composed $\otimes$-functor

$$q\text{-Def} = \mathrm{D}_{\sigma_q}^{(\phi)} \circ \mathrm{V}_{\mathrm{d}}^{(\phi)} : d\text{-}\,Mod_{\mathcal{R}}^{(\phi)} \longrightarrow \sigma_q\text{-}\,Mod_{\mathcal{R}}^{(\phi)}$$

"*the p-adic q-deformation functor*".

Theorem 15.3. — q-Def *is a quasi-inverse of* Conf. *Moreover, it is canonically isomorphic to* $\phi_* \circ (q^{p^s}\text{-Def})$.

Proof. — By the quasi-unipotence theorems, it is clear that q-Def is an equivalence. In order to prove the first assertion, it is thus enough to see that q-Def is left quasi-inverse to Conf, which follows immediately from the fact that $\mathrm{Conf} \circ D_{\sigma_q}^{(\phi)} = D_d^{(\phi)}$.

For the second assertion, one remarks that q-Def(M) (*resp.* q^{p^s}-Def(M)) is the first (*resp.* the second) term in the sequence of modules defined by $D_{\sigma_q}^{\mathrm{conf}(\phi)} \circ V_d^{(\phi)}(M)$. □

(23) using tacitly the canonical isomorphism $D_{\sigma_q}^{(\phi)} \circ V_{\sigma_q}^{(\phi)} \cong \mathrm{Id}$.

15.3. Analytic slopes and exponents. — Let r' be in $]|q-1|,1[$, and λ be a non negative real number. By analogy with the differential case (*cf.* [**CM02**]), we introduce the following

Definition 15.4. — A q-difference module M over $\mathcal{A}_K([r',1[)$ has (analytic) *slopes* $\leqslant \lambda$ if there is a function

$$r \in]r',1[\longmapsto \rho(r) \in]r^{1+\lambda}, r]$$

such that $M \otimes_{\mathcal{A}_K([r,1[)} \mathcal{A}_\Omega(t_r, \rho(r))$ has a basis of elements fixed under Σ_q.

It is clear that this implies solvability in the sense of (12.19).

Since any q-difference module M over $\mathcal{R}_x$ is "defined over $\mathcal{A}_K([r,1[)$" for some r close enough to 1, this provides a definition of "having slopes $\leqslant \lambda$" for such modules. This property is stable by passage to subquotients, dual, tensor products and extensions. One checks exactly as in the differential case that for n prime to p, the base-change by $x \mapsto x^n$ of M is a $q^{1/n}$-difference module of slopes $\leqslant n\lambda$.

It is likely that the Christol-Mebkhout theory of slope filtrations carries over to the q-difference case, but we haven't checked all details.

We now assume for simplicity that the residue field k is algebraically closed.

***Example 15.5* (q-difference equations of rank 1 with Frobenius structure)**

We have seen in (7.3) that any q-difference equation over $\mathcal{R}$ with Frobenius structure has a non-zero solution in some finite unramified extension $\mathcal{E}^\dagger_{x'}$ of $\mathcal{E}^\dagger_x$, and that $y'/y \in \mathcal{E}^\dagger_x$. Thus the group isomorphism $\sigma\text{-}cq^{(\phi)}_{\mathcal{E}^\dagger} \xrightarrow{\cong} d\text{-}cq^{(\phi)}_{\mathcal{E}^\dagger}$ from (8.2) preserves the property of "having slopes $\leqslant \lambda$".

In particular, q-difference equations of rank 1 with Frobenius structure and slope 0 (*i.e.*, slope $\leqslant 0$) correspond to tame characters of $G_{k((x))}$ (via (8.2)), hence are of the form $d_q - [\alpha]_q/x$ with $\alpha \in \mathbb{Z}_p \cap \mathbb{Q}$.

Theorem 15.6. — *Any $M \in \sigma_q\text{-}Mod^{(\phi)}_{\mathcal{R}}$ with slope 0 has a filtration with graded pieces of rank one.*

Proof. — By quasi-unipotence, we may assume, after taking subquotients, that M corresponds to a representation V of $G_{k((x))}$ with finite image G (which is the semi-direct product of a cyclic group of order prime to p by a p-group, *cf.* [**S68**]). Representations of G then correspond to q-difference modules in the tannakian category generated by M, which have slope 0. By base-change by $x \mapsto x^n$ for suitable n prime to p, we may assume that G is a p-group, *i.e.*, that with are in the purely wild case. By the previous example, there is no one-dimensional representation of G which corresponds to a q-difference module of slope 0. Since G is a p-group, we conclude that it is trivial. □

It follows from that M is an iterated extension of q-difference modules of type $\mathcal{R}[d_q]/(d_q + [\alpha_i]_q)$, $i = 1, \ldots, \mu = \operatorname{rk} M$, with $\alpha_i \in \mathbb{Z}_p \cap \mathbb{Q}$ well defined mod. $\mathbb{Z}$ and up

to permutation. Let us call these p-adic numbers (mod. $\mathbb{Z}$ and up to permutation) the *exponents* of M. Taking into account remark 8.3, we have:

Proposition 15.7. — *The functors* Conf *and* q-Def *preserve objects of (analytic) slope* 0 *and their exponents.*

References

[A01] Y. ANDRÉ – "Différentielles non commutatives et théorie de Galois différentielle ou aux différences", *Ann. Sci. École Norm. Sup.* **5** (2001), p. 1–55.

[A02] ———, "Filtrations de type Hasse-Arf et monodromie p-adique", *Invent. Math.* **148** (2002), no. 2, p. 285–317.

[A] ———, "Galois representations, differential equations and q-difference equations: sketch of a p-adic unification", this volume.

[CC96] B. CHIARELLOTTO & G. CHRISTOL – "On overconvergent isocrystals and F-isocrystals of rank one", *Compositio Math.* **100** (1996), p. 77–99.

[C81a] G. CHRISTOL – "Décomposition des matrices en facteurs singuliers. Applications aux équations différentielles", in *Groupe d'étude sur l'analyse ultramétrique. 7'-8' années (Paris, 1979/1981)*, vol. 5, Secrétariat Math., Paris, 1981.

[C81b] ———, "Systèmes différentiels linéaires p-adiques, structure de Frobenius faible", *Bull. Soc. math. France* **109** (1981), no. 1, p. 83–122.

[C86] ———, "Fonctions et éléments algébriques", *Pacific J. Math.* **125** (1986), p. 1–37.

[C01] ———, "About a Tsuzuki theorem", in *p-adic functional analysis (Ioannina, 2000)*, Lecture Notes in Pure and Appl. Math., vol. 222, Dekker, New York, 2001, p. 63–74.

[CD94] G. CHRISTOL & B. DWORK – "Modules différentiels sur des couronnes", *Ann. Inst. Fourier (Grenoble)* **44** (1994), no. 3, p. 663–701.

[CM02] G. CHRISTOL & Z. MEBKHOUT – "Équations différentielles p-adiques et coefficients p-adiques sur les courbes", in *Cohomologies p-adiques et applications arithmétiques. II*, Astérisque, vol. 279, Société Mathématique de France, 2002, p. 125–183.

[Co01] P. COLMEZ – "Les conjectures de monodromie p-adiques", in *Sém. Bourbaki 2001/2002*, Astérisque, vol. 290, Société Mathématique de France, 2003, exp. n° 897, p. 53–101.

[Cr87] R. CREW – "F-isocrystals and p-adic representations", in *Algebraic geometry, Bowdoin, 1985 , Part 2 (Brunswick, Maine, 1985)*, Proc. Sympos. Pure Math., vol. 46, American Mathematical Society, Providence, RI, 1987, p. 111–138.

[Cr98] ———, "Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve", *Ann. Sci. École Norm. Sup.* **31** (1998), p. 717–763.

[DV02] L. DI VIZIO – "Arithmetic theory of q-difference equations: the q-analog of Grothendieck-Katz's conjecture on p-curvatures", *Invent. Math.* **150** (2002), no. 3, p. 517–578.

[DV03] ———, "Introduction to p-adic q-difference equations (weak Frobenius structure and transfer theorems)", in *Geometric Aspects of Dwork's Theory*, vol. 2, de Gruyter, 2004, p. 615–675.

[D83] B. DWORK – "On the Boyarsky principle", *Amer. J. Math.* **285** (1983), p. 115–156.

[DR77] B. DWORK & P. ROBBA – "On ordinary linear p-adic differential equations", *Trans. Amer. Math. Soc.* **231** (1977), no. 1, p. 1–46.

[E02] J.-Y. ÉTESSE – "Relèvement de schémas et algèbres de Monsky-Washnitzer: théorèmes d'équivalence et de pleine fidélité", *Rend. Sem. Mat. Univ. Padova* **107** (2002), p. 111–138.

[HL46] G.H. HARDY & J.E. LITTLEWOOD – "Notes on the theory of series XXIV. A curious power-series", *Math. Proc. Cambridge Philos. Soc.* **42** (1946), p. 85–90.

[Ka86] N. KATZ – "Local-to-global extensions of representations of fundamental groups", *Ann. Inst. Fourier (Grenoble)* **36** (1986), p. 69–106.

[Ke] K. KEDLAYA – "A p-adic monodromy theorem", to appear in *Ann. of Math.*

[Ko80] N. KOBLITZ – "q-extension of the p-adic gamma function", *Trans. Amer. Math. Soc.* **260** (1980), no. 2, p. 449–457.

[Ko82a] ———, "q-extension of the p-adic gamma function. II", *Trans. Amer. Math. Soc.* **273** (1982), no. 1, p. 111–129.

[Ko82b] ———, "p-adic analog of Heine's hypergeometric q-series", *Pacific J. Math.* **102** (1982), no. 2, p. 373–383.

[Ma95] S. MATSUDA – "Local indices of p-adic differential operators corresponding to Artin-Schreier-Witt coverings", *Duke Math. J.* **77** (1995), p. 607–625.

[Mo77] E. MOTZKIN – "La décomposition d'un élément analytique en facteurs singuliers", *Ann. Inst. Fourier (Grenoble)* **27** (1977), p. 67–82.

[Q] C. QUESNE – "Disentangling q-exponentials: A general approach", arXiv: `math-ph/0310038`.

[Ra98] L. RAMERO – "On a class of étale analytic sheaves", *J. Algebraic Geom.* **7** (1998), no. 3, p. 405–504.

[R85] P. ROBBA – "Indice d'un opérateur différentiel p-adique IV. Cas des systèmes. Mesure de l'irrégularité dans un disque", *Ann. Inst. Fourier (Grenoble)* **35** (1985), p. 13–55.

[RC94] P. ROBBA & G. CHRISTOL – *Équations différentielles p-adiques. Applications aux sommes exponentielles*, Actualités Mathématiques, Hermann, Paris, 1994.

[Ro00] A. ROBERT – *A course in p-adic analysis*, Graduate Texts in Math., vol. 198, Springer-Verlag, New York, 2000.

[S68] J.-P. SERRE – *Corps locaux*, Hermann, Paris, 1968.

[T98a] N. TSUZUKI – "The local index and the Swan conductor", *Compositio Math.* **111** (1998), p. 245–288.

[T98b] ———, "Finite local monodromy of overconvergent unit-root F-isocrystals on a curve", *Amer. J. Math.* **120** (1998), p. 1165–1190.

Y. ANDRÉ, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris (France)
E-mail : `yves.andre@ens.fr`

L. DI VIZIO, Laboratoire Émile Picard, Université Paul Sabatier, U.F.R. M.I.G., 118, route de Narbonne, 31062 Toulouse cedex 4 (France) • *E-mail :* `divizio@picard.ups-tlse.fr`