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GALOIS REPRESENTATIONS, DIFFERENTIAL EQUATIONS, AND q -DIFFERENCE EQUATIONS : SKETCH OF A p -ADIC UNIFICATION

by

Yves André

Abstract. — This is a broad introduction to the following, more technical, paper [AdV]. We explain how [AdV] relates to two major themes of J.-P. Ramis' work, which eventually become unified in the p -adic world.

Résumé (Représentations galoisiennes, équations différentielles et aux q -différences: esquisse d'une unification p -adique)

Ce texte est une introduction développée à l'article suivant, plus technique [AdV]. Nous expliquons comment [AdV] est lié à deux thèmes majeurs de l'œuvre de J.-P. Ramis, et comment ceux-ci trouvent leur unification en passant au monde p -adique.

Introduction

Two remarkable analogies have played an important role in Jean-Pierre Ramis' work:

- the analogy between linear complex differential equations and coverings in characteristic p (reported in D. Bertrand's contribution to this volume),
- the analogy between linear differential equations and q -difference equations (reported in J. Sauloy's contribution).

Our aim is to explain the analogs of these analogies in the p -adic world. We will see that once transposed into that context, these analogies become much more precise, and eventually lead to some equivalences of categories!

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Key words and phrases. — Differential equations, q -difference equations, coverings, wild singularities, local Galois representation, overconvergence, p -adic local monodromy.

1. A mysterious analogy: linear complex differential equations and coverings in characteristic p , tame and wild

1.1. A dictionary. — This analogy grew out of discussions between J.-P. Ramis and M. Raynaud during the “Nuit de la Musique 1993”⁽¹⁾. Let us recall it in the form of a “dictionary”:

Differential side	Characteristic- p side
$X = \overline{X} \setminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \setminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ ($\overline{X}$ complete)
differential module / X	unramified Galois covering of X
singular point (in S)	branch point (in S)
regular singular point	tame branch point, <i>i.e.</i> , the ramification index at s is prime to p
irregular singular point	wild branch point
local differential Galois group at $s \in S$	inertia group at $s \in S$
(global) differential Galois group G (a linear alg. group / $\mathbb{C}$)	covering group G (a finite group)
<i>torus</i> in G	<i>p-Sylow subgroup</i> of G
$L(G)$: normal subgroup generated by all tori	$p(G)$: normal subgroup generated by all p -Sylow's
monodromy map $\mu : \pi_1(X) \rightarrow G/L(G)$	monodromy map $\mu : \pi_1^{(p')}(X) \rightarrow G/p(G)$
μ has Zariski-dense image (Ramis condition for the existence of a diff. module on X with diff. Galois group G , all singularities $s \in S$ being regular but one)	μ is surjective (Harbater condition for the existence of an unramified G -covering of X , all branch points $s \in S$ being tame but one).

Comment. — In the right-hand column, $\overline{\mathbf{F}}_p$ denotes a fixed algebraic closure of the field $\mathbf{F}_p$ with p -elements, and $\pi_1^{(p')}(X)$ denotes the profinite group which classifies unramified coverings of X of degree prime to p , *i.e.*, the prime-to- p quotient of Grothendieck's algebraic fundamental group $\pi_1(X)$ of X . According to Grothendieck,

⁽¹⁾Older sources, in the ℓ -adic context, will be evoked in the next section.

$\pi_1^{(p')}(X)$ is a free prime-to- p profinite group on $2g + |S| - 1$ generators (g denotes the genus of $\overline{X}$ and S is assumed to be non-empty).⁽²⁾

1.2. ℓ -adic linearized variant ($\ell \neq p$). — There is a somewhat older and more standard version of this dictionary (*cf. e.g.* the end of [K]) in which objects in the right-hand column are replaced by more linear ones (in fact $\mathbb{Z}_\ell$ -linear⁽³⁾ ones, for some fixed (but arbitrary) prime number $\ell \neq p$). It consists essentially in considering at once the whole tower of unramified coverings of X of degree a power of ℓ . In that way, finite groups are replaced, in the right-hand column, by ℓ -adic Lie groups, or even by algebraic groups over $\mathbb{Q}_\ell$ (by taking a suitable algebraic envelope).

Differential side	Characteristic- p side
$X = \overline{X} \setminus S$ affine curve / $\mathbb{C}$ ($\overline{X}$ complete)	$X = \overline{X} \setminus S$ affine curve / $k \subset \overline{\mathbb{F}}_p$ ($\overline{X}$ complete)
differential module M on X	lisse ℓ -adic sheaf $\mathcal{L}$ on X (ℓ -adic continuous representation of $\pi_1(X)$)
differential Galois group (an algebraic group / $\mathbb{C}$)	monodromy group (image of $\pi_1(X)$ or its Zariski closure, an algebraic group / $\mathbb{Q}_\ell$)
local differential Galois group	image of inertia group $\mathcal{I}$ (or its Zariski closure)
de Rham cohomology groups $H_{\text{dR}}^i(X, M)$	étale cohomology groups $H_{\text{ét}}^i(X, \mathcal{L})$
$\chi(M) = \sum (-1)^i \dim H_{\text{dR}}^i(X, M)$	$\chi(\mathcal{L}) = \sum (-1)^i \dim H_{\text{ét}}^i(X, \mathcal{L})$
<i>Deligne-Malgrange irregularity</i> $\text{irr}(M, s)$ at s	<i>Swan conductor</i> $\text{sw}(M, s)$ at $s \in S$
Deligne's formula for $\chi(M)$ in terms of $\text{rk } M$ and irregularities	Grothendieck's formula for $\chi(\mathcal{L})$ in terms of $\text{rk } M$ and Swan conductors.

⁽²⁾Referee's remark. Earlier presentations of the Ramis-Raynaud dictionary can be found in M. van der Put's Bourbaki talk: Recent work on differential Galois theory (Exposé 849, Astérisque 252 (1998), 341-367), as well as in van der Put and Singer's book *Galois Theory of Linear Differential Equations*, Springer-Verlag (2003).

⁽³⁾Recall that the ring of ℓ -adic integers $\mathbb{Z}_\ell$ is the limit of the system $\cdots \rightarrow \mathbb{Z}/\ell^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/\ell^n\mathbb{Z} \rightarrow \cdots \rightarrow \mathbb{Z}/\ell\mathbb{Z} = \mathbf{F}_\ell$, so that any ℓ -adic integer can be expressed as a series $\sum_{n=0}^{\infty} a_n \ell^n$ where $a_n \in \{0, 1, \dots, \ell-1\}$. The field of fractions of $\mathbb{Z}_\ell$ is $\mathbb{Q}_\ell = \mathbb{Z}_\ell[\frac{1}{\ell}]$. In the sequel, we denote by $\overline{\mathbb{Q}}_\ell$ a fixed algebraic closure of $\mathbb{Q}_\ell$.

1.3. The ℓ -adic local monodromy theorem ($\ell \neq p$). — Let us recall the structure of the absolute Galois groups which play a role in the “characteristic- p side”. We now assume that $k = \mathbf{F}_{p^n} \subset \bar{k} = \bar{\mathbf{F}}_p$ is the field with p^n elements. Then,

$$G_k := \mathrm{Gal}(\bar{k}/k) = \hat{\mathbb{Z}} = \prod_{\ell'} \mathbb{Z}_{\ell'} \quad \text{and}$$

$G_{k((x))} := \mathrm{Gal}(k((x))^{\mathrm{sep}}/k((x)))$ can be unscrewed via two exact sequences:

$$1 \longrightarrow \mathcal{I} \longrightarrow G_{k((x))} \longrightarrow G_k \longrightarrow 1$$

and

$$1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{I} \longrightarrow \mathbb{Z}_{\ell} \times \prod_{\ell' \neq p, \ell} \mathbb{Z}_{\ell'} \longrightarrow 1$$

where $\mathcal{I} = G_{\bar{k}((x))}$ is the *inertia group*, and $\mathcal{P}$ is a pro- p -group called the *wild inertia group*.

This reflects the fact that in contrast to the char. 0 case, the algebraic closure of $\bar{k}((x))$ contains many more elements than just Puiseux series. For instance, roots z of the Artin-Schreier equation $z^{-p} - z^{-1} = x^{-1}$ cannot be expressed as Puiseux series.

Correspondingly, one has a tower of Galois extensions

$$k((x)) \subset \bar{k}((x)) \stackrel{\text{tame}}{\subset} \bigcup_{p \nmid n} \bar{k}((x^{1/n})) \stackrel{\text{wild}}{\subset} (k((x)))^{\mathrm{sep}},$$

with respective Galois groups $G_k, \mathcal{I}/\mathcal{P}$ and $\mathcal{P}$.

Theorem 1.1 (Grothendieck [G]). — *Every ℓ -adic representation of $G_{k((x))}$ is quasi-unipotent, i.e., a suitable open subgroup of $\mathcal{I}$ acts (through its quotient in $\mathbb{Z}_{\ell}$) by unipotent matrices.*

This can also be formulated, in the “Tannakian vein”, as an equivalence of $\otimes$ -categories

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_{\ell}}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{continuous } \overline{\mathbb{Q}}_{\ell}\text{-reps. of } \mathcal{I} \text{ which extend to reps. of } G_F\}$$

where $\mathcal{I}$ appears in the left-hand side as a constant group-scheme (and representations are understood in the scheme-theoretic sense), and in the right-hand side as a profinite topological group.

2. The p -adic analog of this analogy. An equivalence of categories.

2.1. Some motivation. Frobenius and overconvergence. — At least two aspects of the above dictionary 1.2 are unsatisfactory: the arbitrariness of the auxiliary prime number ℓ , and the very different natures of the cohomologies occurring in the left-hand (De Rham) and right-hand (étale) columns.

Both drawbacks would disappear, and the analogy would become much closer, if one could replace ℓ by p , and étale cohomology by some appropriate cohomology of De Rham type.

It turns out that this is indeed possible, provided one lifts the geometric situation from characteristic p to characteristic 0 (p -adic lifting), and replaces ℓ -adic sheaves by some analytic differential modules. The basic ideas here are due to Dwork⁽⁴⁾.

i) The relevant “lifting” of X is constructed as follows: one fixes a p -adic field K (a finite extension of $\mathbb{Q}_p$) with residue field $k = \mathbf{F}_{p^n}$, and one fixes a smooth complete curve $\overline{X}/K$ whose reduction modulo p is our given complete curve $\overline{X}/k$. Removing the open disks of radius 1 in $\overline{X}$ which reduce mod. p to the points in S , one gets a p -adic analytic curve $\mathcal{X}$ which lifts X .

As was pointed out by Dwork, one should actually remove “infinitesimally more”: the relevant space is the limit $\mathcal{X}^\dagger$ of $\overline{X}$ deprived from disks of radius $1 - \varepsilon$ around the singularities, when $\varepsilon \rightarrow 1$ ($\mathcal{X}^\dagger$ is a pro-ringed space⁽⁵⁾).

ii) According to Dwork again, the relevant p -adic differential modules should have two features:

- 1) *Frobenius structure*: after change of variable $x \mapsto x^{p^n}$, the new differential module is isomorphic to the old one,
- 2) *Overconvergence*: the differential module (and its Frobenius structure) should be also defined in some annulus inside each singular disk; in other words, should be defined over $\mathcal{X}^\dagger$.

Examples

a) $\overline{X} = \mathbb{P}_k^1$, $S = \{0\}$, $\mathcal{X}$ = outer unit disk $|x| \geq 1$. Let π be Dwork’s constant, *i.e.*, a fixed root of the equation $\pi^{p-1} = -p$ in $\overline{\mathbb{Q}_p}$. Then the differential equation $y' = -\pi/x^2 y$ has the required properties (overconvergence of Frobenius means that $y(x^p)/y(x) = e^{(\pi/x^p) - (\pi/x)}$ has a p -adic radius of convergence > 1 in $1/x$).

b) Let us consider the differential module $x^{-\alpha} e^{1/x} \mathcal{O}$ (endowed with the derivation where xd/dx) where $\mathcal{O}$ is some ring of analytic functions away from the origin.

Let us first consider the complex case. The corresponding “period” is

$$\int_{\gamma} x^{-\alpha} e^{1/x} \frac{dx}{x} = -\frac{2\pi i}{\Gamma(1-\alpha)} \sim \Gamma(\alpha). \quad (6)$$

According to Ramis’ precise Gevrey theory, an optimal choice for $\mathcal{O}$ is

$$\mathbb{C}[[1/x]]_{-1,1^-} := \left\{ \sum a_n x^{-n} \mid \exists \kappa > 0, \exists r \in]0, 1[; |a_n| \leq \kappa r^n / n! \right\}. \quad (7)$$

⁽⁴⁾We refer to [R] for a nice introduction to this circle of ideas.

⁽⁵⁾Working with $\mathcal{X}$ instead of $\mathcal{X}^\dagger$ would provide unwanted infinite-dimensional cohomology spaces in general.

⁽⁶⁾Here π is the usual one! The symbol $\sim$ means equality up to some factor in $\overline{\mathbb{Q}^*}$. The chosen loop comes from $-\infty$ and returns to $-\infty$ after turning once counterclockwise around the origin.

⁽⁷⁾ $(-1, 1^-)$ is a characteristic index for which $\dim H^1 = 1$.

Let us now turn to the p -adic case. Overconvergence is satisfied after rescaling $1/x \mapsto \pi/x$. However strange this condition may look at first sight, it is nothing else than a Gevrey condition: indeed, in the p -adic case,

$$\left\{ \sum a_n x^{-n} \mid \exists \kappa > 0, \exists r \in]0, 1[; |a_n| \leq \kappa r^n / |n|! \right\}$$

is precisely the ring of *analytic functions* on a disk of radius $> |\pi|$, which gives the ring of analytic functions on $\mathcal{X}^\dagger$ after rescaling $1/x \mapsto \pi/x$. On the other hand, one can evaluate Frobenius, and it turns out that its eigenvalues in some appropriate sense are, up to some algebraic factor, special values $\sim \Gamma_p(\alpha)$ of Morita's p -adic gamma function.

This is a general phenomenon, and the new version of the dictionary, with p -adic right-hand column, now looks as follows:

Differential side	Characteristic- p side
$X = \overline{X} \setminus S$ affine curve / $\mathbb{C}$	$X = \overline{X} \setminus S$ affine curve / $k \subset \overline{\mathbf{F}}_p$ overconvergent lifting $\mathcal{X}^\dagger$ of X over a p -adic field
differential module M on X	differential module $\mathcal{M}^\dagger$ on $\mathcal{X}^\dagger$ (admitting a Frobenius structure)
$H_{\text{dR}}^i(X, M)$	$H_{\text{dR}}^i(X^\dagger, \mathcal{M}^\dagger)$
<i>periods</i>	<i>eigenvalues of Frobenius.</i>

Remark. — At the referee's suggestion, let us mention that there exists a completely different approach to the Ramis-Raynaud analogies, also involving p -adic differential equations, namely the theory of iterative differential Galois groups developed by H. Matzat and M. van der Put ([**MvP1**, **MvP2**]). This variant of differential Galois theory applies to function fields of characteristic p , and relies on the notion of an iterative derivation (Hasse, F.-K. Schmidt). The differential Galois groups attached to iterative differential modules are linear algebraic groups (not necessarily finite) over the field of (iterative) constants k , and one has a Galois correspondence. In special cases, Matzat and van der Put establish an iterative differential analog of the Abhyankar conjecture.

Iterative differential modules can be lifted to global p -adic differential equations of a very special kind, but the relationship with the above theory remains unclear.

2.2. The p -adic local monodromy theorem (Crew's conjecture). — Let again K be a finite extension of $\mathbb{Q}_p$, with residue field $k = \mathbf{F}_{p^n}$.

The lifting process associates to X over the finite field k the p -adic pro-space $\mathcal{X}^\dagger$ over K . In this process, localization around a point $s \in S$ becomes localization on an “infinitely thin annulus”.

One is thus led to consider the so-called *Robba ring*, i.e., the ring of K -analytic functions⁽⁸⁾ on arbitrarily thin annuli $\mathcal{A}_{|1-\varepsilon, 1|} : 1-\varepsilon < |x| < 1$ (x denotes a local coordinate in the singular unit disk above s).

$$\mathcal{R} = \mathcal{R}_K = \bigcup_{\varepsilon} \{K\text{-analytic functions on } \mathcal{A}_{|1-\varepsilon, 1|}\}.$$

The subring $\mathcal{E}^\dagger$ of *bounded* functions also plays an important role, because it turns out to be an *henselian field* with residue field $k((x))$; in other words, giving a finite separable extension of $k((x))$ amounts to giving a finite unramified extension $\mathcal{E}^{\dagger'}$ of $\mathcal{E}^\dagger$ (and this provides a finite étale extension $\mathcal{R}'$ of $\mathcal{R}$).

Theorem 2.1. — *Every differential module over $\mathcal{R}$ which admits a Frobenius structure is quasi-unipotent, i.e., has a basis of solutions in $\mathcal{R}'[\log x]$, where $\mathcal{R}'$ is the finite étale extension of $\mathcal{R}$ attached to a finite separable extension of $k((x))$.*

This is the p -adic (deeper) analog of Grothendieck's ℓ -adic local monodromy theorem 1.1, as the following “Tannakian formulation” puts in evidence:

$$\text{Rep}_{\overline{\mathbb{Q}_p}}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{differential modules}/\mathcal{R}_{\overline{\mathbb{Q}_p}} \text{ admitting a Frobenius structure}\}.$$

Example. — The 1-dimension representation of $\mathcal{I} = G_{k((x))}^-$ attached to Artin-Schreier equation $z^{-p} - z^{-1} = x^{-1}$ corresponds to the differential module $y' = -\frac{\pi}{x^2}y$.

2.3. Hasse-Arf filtrations. — There are several approaches to Thm. 2.1. My own approach [A2] is based on the notion of *Hasse-Arf filtration* in a Tannakian category $\mathcal{T}$ ⁽⁹⁾.

Data. — For every object $M \in \mathcal{T}$, a separated decreasing filtration $(F^{>\lambda}M)_{\lambda \geq 0}$ functorial and exact in M , satisfying

$$F^{>\lambda}(\mathbf{1}) = 0, \quad F^{>\lambda}(M) = F^{>\lambda}(N) = 0 \quad \implies \quad F^{>\lambda}(M \otimes N) = F^{>\lambda}(M^\vee) = 0.$$

One shows that every M then admits a canonical finite decomposition $M = \oplus \text{gr}^{\lambda_i} M$, where $\text{gr}^{\lambda_i} M$ is “of pure slope λ_i ”. This allows to attach to M its Newton polygon $NP(M)$ following the usual recipe. The “height” of $NP(M)$ is denoted by $h(M)$.

We say that the functorial filtration $(F^{>\lambda})$ is a *Hasse-Arf filtration* if $\forall M$, $h(M) \in \mathbb{N}$ (equivalently, if all Newton polygons have integral vertices).

Examples

1) The oldest example comes from arithmetic. $\mathcal{T} = \text{Rep } G_{k((x))}$ ⁽¹⁰⁾. The classical theory of ramification provides a non-decreasing sequence of normal subgroups $G^\mu \subset G_{k((x))}$ (the higher ramification groups), and one defines a filtration as follows:

$$F^{>\lambda}M = 0 \iff G^\mu \text{ acts trivially on } M \quad \forall \mu > \lambda.$$

⁽⁸⁾We refer to [R'] for a nice introduction to the theory of p -adic analytic functions.

⁽⁹⁾See also [A4] for a more detailed introduction to this topic.

⁽¹⁰⁾One could replace $k((x))$ by any local field.

In this context, $h(M)$ is called the Swan conductor of M and is denoted by $\text{sw}(M)$. Its integrality is a classical theorem by Hasse and Arf.

2) $\mathcal{T} = \{\text{differential modules } M/\mathbb{C}[[x]]\}$. Every object is endowed with the Turrittin-Levelt slope filtration. In this context, $h(M)$ is the irregularity $\text{irr}(M)$, and its integrality follows from the definition.

3) $\mathcal{T} = \{\text{differential modules } M/\mathcal{R} \text{ admitting a Frobenius structure}\}$. Looking at the growth of solutions toward the *outer boundary* of $\mathcal{A}_{]1-\varepsilon, 1[}$, Christol and Mebkhout have defined the (analytic) *filtration by p -adic slopes* of M . In this context, $h(M)$ is called the p -adic irregularity of M and is denoted by $\text{irr}_p(M)$. Christol and Mebkhout have shown that $\text{irr}_p(M)$ is always an integer [CM].

It turns out that, despite their very different natures, examples 1) and 3) correspond to each other via the equivalence of categories 2.1:

Theorem 2.2 (Matsuda, Tsuzuki; Crew [C]). — *The canonical $\otimes$ -equivalence:*

$$\text{Rep}_{\overline{\mathbb{Q}}_p}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{\text{differential modules } / \mathcal{R}_{\overline{\mathbb{Q}}_p} \text{ admitting a Frobenius structure}\}$$

is compatible with the canonical filtrations (Hasse-Arf on the L.H.S., by p -adic slopes on the R.H.S.).

This can be summarized by the slogan

$$\boxed{\text{sw} = \text{irr}_p}$$

3. Another analogy:

linear differential equations and q -difference equations; confluence

3.1. The q -world. — Let us provisionally abandon p in favor of q . The q -calculus has a long history (Euler, Gauss, Jacobi, Heine, ..., Ramanujan, ...⁽¹¹⁾). It is based on the replacement of ordinary integers n by their q -analogs

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

The usual derivation d/dx is then replaced by the q -derivation

$$d_q : f(x) \mapsto \frac{f(x) - f(qx)}{(1 - q)x}$$

which sends x^n to $[n]_q x^{n-1}$ (and the q -exponential $e_q^x = \sum x^n / [n]_q!$ to itself).

Differential equations are thus replaced by q -difference equations. The phenomenon of *confluence* arises when $q \rightarrow 1$: then $n_q \rightsquigarrow n$, $d_q \rightsquigarrow d/dx$, and q -difference equations tend to differential equations under appropriate convergence conditions. Conversely, for q close to 1, q -difference equations may be considered as deformations of differential equations.

⁽¹¹⁾See some highlights in [E].

3.2. Non-commutative connections and q -deformations. — Let R be some “ring of functions” stable under the dilatation $\sigma_q : f(x) \mapsto f(qx)$.

Recall that a more intrinsic version of differential equations is provided by differential modules, or even better, by connections. Similarly, a more intrinsic version of q -differences is provided by q -difference modules: R -module M (projective of finite rank) + σ_q -linear automorphism.

This setting has one drawback: it does not allow to obtain the limit differential module in the case of confluence ($q \rightarrow 1$). However, one can also present differential modules as *connections*:

$$\nabla : M \longrightarrow \Omega_q^1 \otimes M, \quad (\nabla(rm) = r\nabla(m) + dr \otimes m),$$

$\Omega_q^1 =$ *non-commutative* bimodule of rank one: $f\omega = \omega \cdot \sigma_q(f)$, $d : R \rightarrow \Omega_q^1$, $f \mapsto \omega \cdot d_q(f)$.

This gives rise to a unified theory of Galois differential groups in the differential and q -difference cases [A1], [A3], and a relevant setting for the algebraic study of confluence.

3.3. Analytic theory. — Here, for many reasons (*e.g.*, to avoid difficult problems of small divisors), one assumes $|q| \neq 1$.

The analytic theory of q -difference equations has been initiated by Adams, Birkhoff *etc.*, and was revived by J.-P. Ramis in the early 90’s. The analogy with differential equations is often straightforward at the formal/combinatorial level, but rather subtle at the analytic level, especially when wild phenomena or confluence are involved, *cf.* [S].

4. The p -adic analog of this analogy. Another equivalence of categories [AdV]

4.1. Frobenius structure. — We fix $n > 0$, and a prime p . Recall that a Frobenius structure on a differential module is an isomorphism between M and its “pull-back” by the change of variable $\phi : x \mapsto x^{p^n}$.⁽¹²⁾

In the q -difference case, one has the relation

$$\boxed{\sigma_q \phi = \phi \sigma_{q^{p^n}}},$$

so that the pull-back of a σ_q -module M is a priori a $\sigma_{q^{1/p^n}}$ -module. In order to make a σ_q -module out of it, it suffices however to iterate p^n times the action of $\sigma_{q^{1/p^n}}$. We denote by $\phi^! M$ this new σ_q -module M . A Frobenius structure may then be defined to be an isomorphism between $\phi^! M$ and M .⁽¹³⁾

⁽¹²⁾Actually, one also has to twist the coefficients by some power of the so-called Frobenius automorphism of K , but we neglect this fact here.

⁽¹³⁾In [AdV], another notion of Frobenius structure is also considered.

4.2. p -adic confluence and canonical q -deformation. — We come back to the p -adic situation: K is p -adic field with residue field $k \subset \mathbf{F}_{p^n}$. We fix $q \in K$, $q \neq$ root of unity. We are interested in “confluence” (note that here $|q-1| < 1 \rightarrow |q| = 1$, in contrast to the usual postulate in the complex case). Actually, it simplifies matters to assume

$$\boxed{|1 - q| < p^{-\frac{1}{p^n-1}}}.$$

Let M be a σ_q -module over the Robba ring $\mathcal{R} = \mathcal{R}_K$.

Theorem 4.1. — *There is a canonical “functor of confluence”*

$$\begin{aligned} \{q\text{-difference modules } / \mathcal{R}_{\overline{K}} \text{ admitting a Frobenius structure}\} \\ \longrightarrow \{\text{differential modules } / \mathcal{R}_{\overline{K}} \text{ admitting a Frobenius structure}\} \end{aligned}$$

which is an equivalence of tannakian categories.

Its construction uses *quasi-unipotence* (q -analog of Crew’s conjecture), cf. [AdV]. In fact for any q -difference module over $\mathcal{R}_K$ admitting a Frobenius structure, there is a canonical sequence of $q^{p^{i_n}}$ -difference structures on the same underlying $\mathcal{R}$ -module (with $i \rightarrow \infty$, so that $q^{p^{i_n}} \rightarrow 1$) which converges to a differential structure.

4.3. Another Hasse-Arf filtration? — This subsection is tentative. By looking at the growth of solutions toward the *outer boundary* of $\mathcal{A}_{|1-\varepsilon, 1[}$, it seems (not all details have been checked) that one can define, à la Christol and Mebkhout, a *filtration by p -adic slopes* on M , whence a notion of p -adic q -irregularity $q\text{-irr}_p(M)$, and that one has the following q -analog of 2.2:

Conjecture 4.2. — *The canonical $\otimes$ -equivalence:*

$$\text{Rep}_{\overline{\mathbb{Q}_p}}(\mathcal{I} \times \mathbb{G}_a) \xrightarrow{\sim} \{q\text{-difference modules } / \mathcal{R}_{\overline{\mathbb{Q}_p}} \text{ admitting a Frobenius structure}\}$$

is compatible with the canonical filtrations (Hasse-Arf on the L.H.S., by p -adic slopes on the R.H.S.).

This can be summarized by the slogan

$$\boxed{\text{sw} = q\text{-irr}_p}.$$

Remark. — In this context, it is interesting to notice that the formal slope filtration for complex q -difference modules with $|q| \neq 1$ (cf. [S]) is *not* a Hasse-Arf filtration (in contrast both to the differential case and to the p -adic case), since negative slopes may occur, for instance. Once again, we see that the q -analogy is much tighter in the p -adic case.

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