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MONODROMY PRESERVING DEFORMATION AND DIFFERENTIAL GALOIS GROUP I

by

Hiroshi Umemura

For J.-P. Ramis on the occasion of his 60th birthday

Abstract. — In 1914, J. Drach interpreted in terms of his infinite dimensional differential Galois theory R. Fuchs' work on the monodromy preserving deformation and the sixth Painlevé equation. This note of Drach contains a quite original idea but it is difficult to understand. We analyze his note by our infinite dimensional differential Galois theory. We get non-trivial examples of which we can calculate our Galois group.

***Résumé* (Déformation isomonodromique et groupe de Galois différentiel).** — En 1914, J. Drach interpréta le travail de R. Fuchs sur les déformations isomonodromiques et la sixième équation de Painlevé en termes de sa théorie de Galois de dimension infinie. La note de Drach contient une idée très originale mais difficile à comprendre. Nous analysons sa note en appliquant notre théorie de Galois différentielle de dimension infinie. Cela nous donne des exemples non triviaux dont nous pouvons calculer notre groupe de Galois.

1. Introduction

Today, there are a variety of ways of defining the Painlevé equations. Most of them are unimaginable from the original definition.

(1) Historically the origin of the Painlevé equations goes back to the pursuit of special functions defined by algebraic differential equations of the second order. Around 1900 Painlevé succeeded in classifying algebraic differential equations $y'' = F(t, y, y')$ without movable singular points, where F is a rational function of t, y and y' and t is the independent variable so that $y' = dy/dt$ and $y'' = d^2y/dt^2$. The property of being free from the movable singularities is nowadays called the Painlevé property. After he classified the equations satisfying the condition, Painlevé then threw away those

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equations that he could integrate by the so far known functions and thus he arrived at the list of the six Painlevé equations. This is the first definition of the Painlevé equations. It is, however, very lucky that he could discover the Painlevé equations in this manner.

(2) In 1907, R. Fuchs discovered that the sixth Painlevé equation describes a monodromy preserving deformation of a second order ordinary linear equation $y'' = p(x)y$. Later R. Garnier generalized this for the other Painlevé equations.

(3) In our former work [**5**], we showed that we can recover the second Painlevé equation form a rational surface with a rational double point. We can regard this as an algebro-geometric definition of the second Painlevé equation.

(4) Masatoshi Noumi and Yasuhiko Yamada interpreted theory of Painlevé equations form the view point of Kač-Moody Lie algebra. They not only uniformly reviewed the theory of τ function of the Painlevé equations but also generalized the Painlevé equations in a natural frame work.

(5) There is another definition due to J. Drach [**1**] in 1914. He asserts the equivalence of the following two conditions for a function $\lambda(t)$.

(i) $\lambda(t)$ satisfies the sixth Painlevé equation.

(ii) The dimension of the Galois group of a non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}$$

is finite.

In the second condition, the Galois group of general algebraic differential equation is involved. Namely the second condition depends on his infinite dimensional differential Galois theory, which has been an object of discussion since he proposed it in his thesis in 1898.

In this note, we apply our infinite dimensional Galois theory of differential equations [**7**] to study the result of J. Drach. We prove that the first condition (i) implies the second (ii).

Theorem 1.1. — *Let $\lambda(t)$ be a function of t satisfying the sixth Painlevé equation. Let $K = \mathbf{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies*

$$\frac{dy}{dt} = \frac{y(y-1)(t-\lambda)}{t(t-1)(y-\lambda)}.$$

Then the Galois group $\operatorname{InfGal}(L/K)$ *is at most of dimension* 3.

Remark 1.2. — We can expect that generically the dimension of $\operatorname{InfGal}(L/K)$ is 3. Yet inequality $\dim \operatorname{InfGal}(L/K) < 3$ may occur. So it is important to determine the solutions λ of the sixth Painlevé equation and the corresponding $\operatorname{InfGal}(L/K)$ such that $\dim \operatorname{InfGal}(L/K) < 3$.

For the first Painlevé equation, we can prove a more precise result. However, this still relies on a statement about constant fields, called Proposition 5.3 below, which will be proven in [**8**].

***Theorem 1.3* (assuming Proposition 5.3 in § 5).** — *Let $\lambda(t)$ be a function of t satisfying the first Painlevé equation $\lambda'' = 6\lambda^2 + t$. Let $K = \mathbf{C}(t, \lambda(t), \lambda'(t))$ which is a differential field with derivation d/dt. Let $L = K(y)$ be a differential field extension of K such that y is transcendental over K and such that y satisfies q*

$$\frac{dy}{dt} = \frac{1}{2}\frac{1}{y - \lambda(t)}$$

Then the Galois group

$$\mathrm{InfGal}(L/K) \simeq \widehat{\mathrm{SL}}_{2L^{\natural}}.$$

Remarks 1.4. — As the proof of the Theorems shows, it is difficult to imagine how to deduce the condition (i) from (ii).

The assertion of Drach should be properly understood otherwise we would have counter examples. In fact, the second condition (ii) is closed under the specialization of the function $\lambda(t)$, whereas the first (i) is not so. Hence the first condition (i) should be replaced by

(i)' The function $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI} or a degeneration of P_{VI}.

Why are the Theorems interesting? Because the Galois group, which is a formal group of infinite dimension in general, is very difficult to calculate. We have only two types of examples where we can calculate the Galois group. (1) If L/K is a strongly normal extension in the sense of Kolchin which is his generalization of classical Galois extension so that the Galois group $G := \mathrm{Gal}(L/K)$ of the extension is an algebraic group, then $\mathrm{InfGal}(L/K) = \widehat{G}$ and (2) for differential field extension $L = K(y)/K$ such that y is a solution of a Riccati equation with coefficients in K, $\mathrm{InfGal}(L/K)$ is a formal subgroup of $\widehat{\mathrm{SL}}_2$ (*cf.* Theorem (5.16), [**7**]).

Since we can prove only one direction of the assertion of Drach, our result is not satisfactory in the sense that it does not give us a new definition of the Painlevé equation. It offers us, however, highly non-trivial examples of differential field extensions of which we can calculate our Galois group.

The author would like to acknowledge his indebtedness to Daniel Bertrand. Without his constant interest in the subject and valuable discussions with him, this work would not have been done. It is a pleasure to thank B. Malgrange who kindly permitted us to add his letter to D. Bertrand as an appendix to this note.

2. Review of R. Fuchs' paper

R. Fuchs studied a monodromy preserving deformation of a linear differential equation $d^2y/dx^2 = p(x)y$. Namely he considered a system of linear equations

$$(1) \qquad \begin{cases} \dfrac{\partial^2 y_i}{\partial x^2} = py_i, \\ \dfrac{\partial y_i}{\partial t} = By_i - A\dfrac{\partial y_i}{\partial x}, \end{cases} \quad \text{for } i = 1, 2,$$

where

$$p = \frac{a}{x^2} + \frac{b}{(x-1)^2} + \frac{c}{(x-t)^2} + \frac{e}{(x-\lambda)^2} + \cdots$$

and we assume that λ is not a function of x but it is a function of t, *i.e.*, $\partial\lambda/\partial x = 0$. y_1 and y_2 are linearly independent solutions. The integrability of the system (1) implies

$$A(x, t) = \frac{x(x-1)(t-\lambda)}{t(t-1)(x-\lambda)} \quad \text{and} \quad B(x, t) = \frac{1}{2}\frac{\partial A}{\partial x}$$

and $\lambda(t)$ satisfies the sixth Painlevé equation P_{VI}.

Where does the non-linear differential equation

$$\frac{dy}{dt} = \frac{y(y-1)(y-\lambda)}{t(t-1)(t-\lambda)}$$

in Theorem 1.1 come from?

Lemma 2.1. — *We may assume that the Wronskian*

$$W_r = \begin{vmatrix} y_1 & y_2 \\ \partial y_1/\partial x & \partial y_2/\partial x \end{vmatrix} = 1.$$

Proof. — It is an exercise to check $\partial W_r/\partial t = \partial W_r/\partial x = 0$ so that W_r is a constant. It is sufficient to replace y_i by $(1/\sqrt{W_r})y_i$ for $i = 1, 2$. □

From now on we write T for t, W for x so that we consider the system

$$(2) \qquad \begin{cases} \dfrac{\partial^2 y_i}{\partial W^2} = py_i, \\ \dfrac{\partial y_i}{\partial T} = B(W, T)y_i - A(W, T)\dfrac{\partial y_i}{\partial W}, \end{cases} \quad \text{for } i = 1, 2,$$

Lemma 2.2. — *If we set $y = y_2/y_1$, then we have*

$$\begin{cases} \dfrac{\partial y}{\partial W} = \dfrac{1}{y_1^2}, \\ \dfrac{\partial y}{\partial T} = -A\dfrac{1}{y_1^2}. \end{cases}$$

Proof. — This is a consequence of Lemma 2.1. □

We are working in the partial differential field

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle = \mathbf{C}(W,T,\lambda(T),\lambda'(T),\dots)(y_1,y_2,\partial y_1/\partial T,\partial y_2/\partial T) \tag{3}$$

with derivations $\{\partial/\partial W,\ \partial/\partial T\}$. The differential field extension

$$\mathbf{C}(W,T)\langle\lambda(T)\rangle(y_1,y_2,\partial y_1/\partial T,\partial y_2/\partial T)/\mathbf{C}(W,T)\langle\lambda(T)\rangle$$

is defined by the adjunction of the solutions y_i, y_2 of the system (2) of linear equations.

Now we introduce differential operators

$$\begin{cases} D_t = \dfrac{\partial}{\partial T} + A\,\dfrac{\partial}{\partial W}, \\ D_w = \quad y_1^2\,\dfrac{\partial}{\partial W}. \end{cases} \tag{4}$$

Then $[D_t, D_w] = 0$ so that the field (3) is a differential field with derivations $\{D_t,\ D_w\}$. If we regard the the field (3) as a partial differential field with derivations $\{D_t,\ D_w\}$, then it involves non-linear differential equations.

Lemma 2.3. — $D_t W = A(W,\ T)$.

Proof. — This follows from the definition of the operator D_t. □

Lemma 2.4. — $\dfrac{\partial y}{\partial T} + A\dfrac{\partial y}{\partial W} = 0$ *so that* $D_t y = 0$.

Proof. — This is a consequence of Lemma 2.2. □

Lemma 2.4 shows that y is a first integral of a non-linear ordinary differential equation $dY/dT = A(Y,\ T)$.

It follows from the definition of the operator(4) $D_w(W) = y_1^2$ and hence y_1 is algebraic of degree at most 2 over $\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle$. Here $\langle\,\rangle$ should be interpreted in the partial differential field (3) with derivations $\{D_t,\ D_w\}$. Since $y_2 = yy_1$,

$$(\mathbf{C}(W,\ T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W,\ \partial y_2/\partial W : \mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle) \leqslant 2. \tag{5}$$

3. Infinite dimensional differential Galois theory

Let us now explain our differential Galois theory of infinite dimension using a particular example. Namely we consider a differential field extension $L = \mathbf{C}(T)\langle\lambda\rangle(W)$ over $K = \mathbf{C}(T)\langle\lambda\rangle$ with derivation D_t such that

(1) $D_t(T) = 1$,
(2) W is transcendental over the field K.
(3) $D_t W = \dfrac{W(W-1)(T-\lambda)}{T(T-1)(W-\lambda)}$ and
(4) $\lambda(T)$ is an arbitrary function of T.

Remark 3.1. — The extension $L = \mathbf{C}(T)\langle\lambda\rangle(W)/K = \mathbf{C}(T)\langle\lambda\rangle$ of the previous section is a special case of the extension introduced above. In the previous section, we assumed moreover that λ is a solution of the sixth Painlevé equation and that the fields L and K are differential subfields of

$$\mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W).$$

We start from the differential field extension $L = K(W)/K$ with derivation D_t. We define its Galois group. We consider the universal Taylor morphism $i : L \to L^\natural[[\tau]]$. Namely we set for an element $a \in L^\natural[[\tau]]$

$$i(a) = \sum_{n=0}^{\infty} \frac{1}{n!} D_t^n(a)\tau^n.$$

Here $L^\natural$ is the abstract field structure of the differential field L. Namely we forget in the differential field L the derivation D_t. The map i introduced above is a morphism of rings compatible with the derivations D_t and $\partial/\partial\tau$.

Consider now on $L^\natural$, the derivation $\partial/\partial W$, which we denote by $(\partial/\partial W)^\natural$ to avoid confusions. The differential field endowed with $(\partial/\partial W)^\natural$ will be denoted by $L^\sharp$. So we have in the power series ring $L^\sharp[[\tau]]$ two mutually commutative derivations $\partial/\partial\tau$ and $(\partial/\partial W)^\natural$. The latter operates as a derivation of coefficients of a power series.

The quotient field of $L^\sharp[[\tau]]$ is the field $L^\sharp[[\tau]][\tau^{-1}]$ of Laurent series that is the differential field with derivations $\partial/\partial\tau$ and $(\partial/\partial W)^\natural$. In this partial differential field $L^\sharp[[\tau]][\tau^{-1}]$, let $\mathcal{L}$ be the partial differential subfield generated by $i(L)$ and $L^\natural$ and we define $\mathcal{K}$ as the partial differential subfield generated by $i(K)$ and $L^\natural$.

Remark 3.2. — The $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$ of $K^\natural$-derivations of $L^\natural$ is 1-dimensional, and so it is spanned by any non zero element of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$. Hence we have

$$\mathrm{Der}(L^\natural/K^\natural) = L^\natural(\partial/\partial W)^\natural$$

Therefore the construction of $\mathcal{L}$ and $\mathcal{K}$ is independent of the choice of a generator of the $L^\natural$-vector space $\mathrm{Der}(L^\natural/K^\natural)$.

Now considering again the Taylor expansion of the coefficients of a Laurent series, we have a differential algebra morphism $L^\sharp[[\tau]][\tau^{-1}] \to L^\natural[[\xi]][[\tau]][\tau^{-1}]$, where ξ is the variable appearing when we expand the coefficients of our Laurent series.

$$L^\sharp \longrightarrow L^\natural[[\xi]], \quad a \longmapsto \sum_{n=0}^{\infty} \frac{1}{n!}\left((\partial/\partial W)^\natural\right)^n(a)\xi^n.$$

So now $\mathcal{L}$ and $\mathcal{K}$ are differential subfields of $L^\natural[[\xi]][[\tau]][\tau^{-1}]$ with derivations $\{\partial/\partial\xi, \partial/\partial\tau\}$.

Now we consider the functor of infinitesimal deformations of $\mathcal{L}/\mathcal{K}$ in

$$L^\natural[[\xi]][[\tau]][\tau^{-1}]$$

that is a principal homogeneous space of a formal group $\mathrm{InfGal}(L/K)$ of infinite dimension in general. This is the definition of our Galois group. To be more precise, we consider the category $\mathrm{Alg}(L^\natural)$ of commutative $L^\natural$-algebras. We define the functor $F : \mathrm{Alg}(L^\natural) \to (\mathrm{Sets})$ by setting for $A \in \mathrm{Alg}(L^\natural)$

$$F(A) := \{\varphi \in \mathcal{L} \to A[[\xi]][[\tau]][\tau^{-1}] \mid \varphi \text{ is a differential algebra morphism satisfying the following two conditions below}\}$$

(1) φ induces the identity map on $\mathcal{K}$.

(2) Let $N(A)$ be the ideal of the algebra A consisting of all the nilpotent elements of A. So we have a canonical morphism

$$r : A[[\xi]][[\tau]][\tau^{-1}] \longrightarrow A/N(A)[[\xi]][[\tau]][\tau^{-1}]$$

of reducing the coefficients of Laurent series modulo the ideal $N(A)$. Let $j : \mathcal{L} \to A[[\xi]][[\tau]][\tau^{-1}]$ be the composite of the inclusions

$$\mathcal{L} \subset L^\natural[[\xi]][[\tau]][\tau^{-1}] \subset A[[\xi]][[\tau]][\tau^{-1}].$$

Using this notation, the condition that we require is $r \circ \varphi = r \circ j$.

Intuitively φ is an infinitesimal deformation of the inclusion map j. Let

$$\mathcal{W}(\xi, \tau) \in L^\natural[[\xi]][[\tau]][\tau^{-1}]$$

be the image of $W \in L$ by the canonical map

$$L \longrightarrow L^\natural[[\xi]][[\tau]][\tau^{-1}].$$

Let $\varphi \in F(A)$. Then there exists a power series

$$\psi(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots \in A[[\xi]]$$

such that

$$\varphi(W) = \mathcal{W}(\psi(\xi), \tau)$$

and such that $\psi(\xi)$ is congruent to ξ modulo $N(A)$. More precisely

$$a_0,\ a_1 - 1,\ a_2,\ a_3,\ \cdots \in N(A).$$

The infinitesimal deformation φ is determined by the power series $\psi(\xi)$ because the $\{D_t, (\partial/\partial W)^\natural\}$-differential field $\mathcal{L}$ is generated over $\mathcal{K}$ by $\mathcal{W}(\xi, \tau)$. The set

$$G(A) = \{\psi(\xi) = a_0 + a_1\xi +_2 \xi^2 + \cdots \in A[[\xi]] \mid a_0,\ a_1 - 1,\ a_2,\ a_3,\ \cdots \in N(A).\}$$

of formal power series congruent to the identity ξ modulo $N(A)$ forms a group by the composite of power series. The group functor G plays the role of the Lie pseudo-group of all the coordinate transformations of 1-variable. We can show that

$$H(A) = \{\psi \mid \mathcal{W}(\psi(\xi), \tau) \text{ defines an element of } F(A)\}$$

forms a subgroup of $G(A)$. The subgroup functor H is defined by a set of algebraic differential equations. We baptized such a functor a Lie-Ritt functor in [**7**]. In the classical language H is a Lie pseudo-group and $H(A)$ operates on $F(A)$ functorially. The group functor H is the Galois group $\mathrm{InfGal}(L/K)$ of L/K.

4. Proof of Theorem 1.1

We prove Theorem 1.1. Namely, we consider a differential field extension $L = \mathbf{C}(T)\langle\lambda(T)\rangle(W)$ over $K = \mathbf{C}(T)\langle\lambda(T)\rangle$ with derivation D_t satisfying the following conditions.

(1) D_t is a derivation over $\mathbf{C}$. Or equivalently $D_t(\mathbf{C}) = 0$.
(2) T is a variable over $\mathbf{C}$ and $D_t(T) = 1$.
(3) $\lambda(T)$ is a function of T satisfying the sixth Painlevé equation.
(4) W is transcendental over K.
(5) $D_t W = \dfrac{W(W-1)(T-\lambda)}{T(T-1)(W-\lambda)}$.

We have to show that the Galois group $\mathrm{InfGal}(L/K)$ is at most of dimension 3. We know that since λ satisfies the sixth Painlevé equation, we are in the framework of monodromy preserving deformation of R. Fuchs treated in § 2 and § 3. The differential fields L and K with derivation D_t are differential subfields of the partial differential field

$$\mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W).$$

with derivations D_t, D_w. We will show in [**8**] the following general fact. We prove, however, a part of the assertion that allows us to show Theorem 1.1.

Proposition 4.1. — *Consider in general an ordinary differential field extension L/K of an ordinary differential field K with derivation D_t such that L is finitely generated over K as an abstract field. Then, the dimension of the formal group* $\mathrm{InfGal}(L/K)$ *is equal to the transcendence degree* $\mathrm{tr.d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$ *of the abstract field extension $\mathcal{L}^\natural/\mathcal{K}^\natural$.*

Proof. — We give a sketch of a proof. Let us content ourselves to prove that the dimension of the Lie algebra Lie $(\mathrm{InfGal}(L/K))$ is equal to the transcendence degree $\mathrm{tr.d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$. We construct $\mathcal{L}$ and $\mathcal{K}$ by the general procedure of [**7**]. Namely we consider the universal Taylor morphism

$$i : L \longrightarrow L^\natural[[\tau]].$$

Let $y_1, y_2, \ldots, y_n$ be a transcendence base of the abstract field extension $L^\natural/K^\natural$. Let

$$\partial/\partial y_i : K(y_1, y_2, \ldots, y_n) \longrightarrow K(y_1, y_2, \ldots, y_n)$$

be the derivation over $K^\natural$ of the rational function field

$$K(y_1, y_2, \ldots, y_n)^\natural$$

for $1 \leqslant i \leqslant n$. The derivation $\partial/\partial y_i$ can be extended to a unique derivation over $K^\natural$ of $L^\natural$ that we denote also by $\partial/\partial y_i$ for $1 \leqslant i \leqslant n$. The partial differential field $L^\natural$

endowed with derivations $\partial/\partial y_i$ $(1 \leqslant i \leqslant n)$ will be denoted by $L^\sharp$. $\mathcal{L}$ is the partial differential field generated by the image $i(L)$ and $L^\sharp$ in $L^\sharp[[\tau]][\tau^{-1}]]$ with derivations

$$(\partial/\partial y_1)^\natural, (\partial/\partial y_2)^\natural, \ldots, (\partial/\partial y_n)^\natural,\ d/d\tau, \tag{6}$$

where

$$(\partial/\partial y_1)^\natural, (\partial/\partial y_2)^\natural, \ldots, (\partial/\partial y_n)^\natural$$

are derivations of coefficients of Laurent series so that derivations (6) of $\mathcal{L}$ commute. The partial differential field $\mathcal{K}$ is defined in a similar way. Namely it is the partial differential subfield of $L^\sharp[[\tau]][\tau^{-1}]$ generated by $L^\sharp$ and $i(K)$. So $\mathcal{L}$ and $\mathcal{K}$ are partial differential fields of $L^\sharp[[\tau]][\tau^{-1}]]$ with derivations

$$(\partial/\partial y_1)^\natural, (\partial/\partial y_2)^\natural, \ldots, (\partial/\partial y_n)^\natural,\ d/d\tau.$$

The Galois group $\mathrm{InfGal}(L/K)$ is a formal group or a Lie-Ritt functor on the category of $\mathrm{Alg}(L^\natural)$ of $L^\natural$-commutative algebras. We consider the restriction

$$\mathrm{InfGal}(L/K) \otimes_{L^\natural} \mathcal{L}^\natural$$

of the functor

$$\mathrm{InfGal}(L/K)$$

on the subcategory $\mathrm{Alg}(\mathcal{L}^\natural)$ of $\mathcal{L}^\natural$ commutative algebras. Another ingredient is the functor

$$\mathfrak{F}_{L/K} : \mathrm{Alg}(\mathcal{L}^\natural) \longrightarrow (\mathrm{Sets}),$$

where (Sets) is the category of sets. Now we consider the universal Taylor morphism

$$\mathcal{L} \longrightarrow \mathcal{L}^\natural[[\boldsymbol{t},\ \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]],$$

where $\mathbf{t}$ and $\boldsymbol{w}_i$ are introduced by the derivations $\partial/\partial\tau$ and $\partial/\partial w_i$, which we identify with $\partial/\partial y_i$, for $1 \leqslant i \leqslant n$. So the universal Taylor morphism

$$\mathcal{L} \longrightarrow \mathcal{L}^\natural[[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]]$$

maps an element $a \in \mathcal{L}$ to

$$\sum_{m=(m_0,m_1,\ldots,m_n)\in\mathbf{N}^{n+1}} \frac{1}{m!}\frac{\partial^{m_0}}{\partial\tau^{m_0}}\frac{\partial^{m_1}}{\partial w_1^{m_1}}\frac{\partial^{m_2}}{\partial w_1^{m_2}}\cdots\frac{\partial^{m_n}}{\partial w_1^{m_n}}(a)\,\boldsymbol{t}^{m_0}\boldsymbol{w}_1^{m_1}\mathbf{w_2}^{m_2}\cdots\boldsymbol{w}_n^{m_n}.$$

For every $\mathcal{L}^\natural$-algebra A, we have

$$\mathrm{Hom}_{\text{differential algebra}}(\mathcal{L},\ A[[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]]) \simeq \mathrm{Hom}_{\text{algebra}}(\mathcal{L}^\natural,\ A). \tag{7}$$

We set for an $\mathcal{L}^\natural$-algebra A

$$\begin{aligned}\mathfrak{F}_{L/K}(A) := \{f : \mathcal{L} \to A[[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]] \mid f \text{ is a partial differential algebra}\\ \text{morphism that is an infinitesimal deformation over } \mathcal{K} \text{ of the universal}\\ \text{Taylor morphism}\}\end{aligned}$$

We know that the functor $\mathrm{InfGal}(L/K) \otimes_{L^\natural} \mathcal{L}^\natural$ operates on the functor $\mathfrak{F}_{L/K}$ in such a way that

$$(\mathrm{InfGal}(L/K) \otimes_{L^\natural} \mathcal{L}^\natural,\ \mathfrak{F}_{L/K})$$

is a principal homogeneous space. See Theorem (5.11), [**7**]. So we have to show that the tangent space of the functor $\mathfrak{F}_{L/K}$ at the point $\mathfrak{F}_{L/K}(\mathcal{L}^\natural)$ is equal to $\mathrm{tr.\,d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$. In fact by (7),

$$\begin{aligned}\mathfrak{F}_{L/K}(\mathcal{L}^\natural)[\varepsilon]) = \{f : \mathcal{L} \to \mathcal{L}^\natural[\varepsilon][[\boldsymbol{t}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n]] \mid f \text{ is a partial differential}\\ \text{algebra morphism that is an infinitesimal } \mathcal{K}\text{-deformation of}\\ \text{the universal Taylor morphism}\}\end{aligned}$$

coincides with

$$\mathrm{Hom}_{\mathcal{K}^\natural}(\mathcal{L}^\natural,\ \mathcal{L}^\natural[\varepsilon]).$$

The latter is nothing but

$$\mathrm{Der}_{\mathcal{K}^\natural}(\mathcal{L}^\natural)$$

of which the dimension is $\mathrm{tr.\,d.}[\mathcal{L}^\natural : \mathcal{K}^\natural]$ □

To prove Theorem 1.1, by Proposition 4.1, we have to show $\mathrm{tr.\,d.}[\mathcal{L} : \mathcal{K}] \leqslant 3$. The following fact is well-known. See [**6**], Lemma (1.1) for example.

Lemma 4.2. — *Let F be an ordinary differential field and A a differential F-algebra. Let C_K and C_A be the rings of constants of F and A respectively. Then F and C_A are linearly disjoint over C_F.*

Let us denote

$$\mathbf{C}(W,\, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W,\, \partial y_2/\partial W)$$

by M, which is a partial differential field with derivations $\{D_t, D_w\}$. M is an over field of L/K so that we have

$$L^\natural[[\tau]][\tau^{-1}] \subset M^\natural[[\tau]][\tau^{-1}].$$

Let us consider subfields $M^\natural\mathcal{L}$ and $M^\natural\mathcal{K}$ in $M^\natural[[\tau]][\tau^{-1}]$. They are closed under the derivation $d/d\tau$.

Lemma 4.3. — *In $M^\natural[[\tau]][\tau^{-1}]$, (i)$M^\natural$ and $\mathcal{L}$ are linearly disjoint over $L^\natural$ and (ii)$M^\natural$ and $\mathcal{K}$ are linearly disjoint over $L^\natural$.*

Proof. — We prove the first assertion, the second being proved similarly. Let us consider the composite field $M^\natural\mathcal{L}$ and $\mathcal{L}$ in $M^\natural[[\tau]][\tau^{-1}]$. They are partial differential fields. We consider only the derivation $d/d\tau$. If we notice that the ring of $d/d\tau$-constants of $M^\natural\mathcal{L}$ is $M^\natural$ and that the field of $d/d\tau$-constants of $\mathcal{L}$ is $L^\natural$, then the assertion (i) follows from Lemma 4.2. □

It follows from Lemma 4.3 that

$$\operatorname{tr.d.}[\mathcal{L} : \mathcal{K}] = \operatorname{tr.d.}[M^\natural \otimes_{L^\natural} \mathcal{L} : M^\natural \otimes_{L^\natural} \mathcal{K}] \tag{8}$$

$$= \operatorname{tr.d.}[M^\natural \mathcal{L} : M^\natural \mathcal{K}]. \tag{9}$$

$\mathcal{L}$ is the partial differential field in $L^\natural[[\tau]][\tau^{-1}]$ generated by $i(L)$ and $L^\natural$ with respect to the derivations $(\partial/\partial W)^\natural$ and $d/d\tau$. Hence $M^\natural \mathcal{L}$ is the differential field in $M^\natural[[\tau]][\tau^{-1}]$ generated by $i(L)$ and $M^\natural$ with respect to the derivations $(\partial/\partial W)^\natural$ and $d/d\tau$. Since $\partial/\partial W = y_1^2 D_w$ in M by (4) and since $y_1^2 \in M$, $M^\natural \mathcal{L}$ is the partial differential field in

$$M^\natural[[\tau]][\tau^{-1}]$$

generated by $i(L)$ and $M^\natural$ with respect to the derivations $D_w^\natural$ and $d/d\tau$. Let us denote by abuse of notation by

$$i : M \longrightarrow M^\natural[[\tau]] \subset M^\natural[[\tau]][\tau^{-1}]$$

the universal Taylor morphism of D_t-differential field M. Since D_t commutes with D_w,

$$\begin{aligned} i(D_w(a)) &= \sum_{n=0}^{\infty} \frac{1}{n!} D_t^n(D_w(a))\tau^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} D_w(D_t^n(a))\tau^n \\ &= D_w^\natural(i(a)) \end{aligned}$$

for every element $a \in M$. Hence

$$i(D_w(W)) = D_w^\natural(i(W))$$

is in $M^\natural \mathcal{L}$. Since $D_t y = 0$,

$$i(y) = y^\natural \in M^\natural \subset M^\natural \mathcal{L} \subset M^\natural[[\tau]][\tau^{-1}].$$

So let

$$\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle$$

denote as in § 2 the partial differential field generated by λ, y over $\mathbf{C}(T)$ with respect to the derivations $\{D_t, D_w\}$, then

$$i(\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle) \subset M^\natural \mathcal{L}.$$

Hence

$$M^\natural i(\mathbf{C}(T)\langle\lambda\rangle\langle y\rangle\langle W\rangle) = M^\natural \mathcal{L}.$$

It follows from (5) and from Lemma 4.2

$$\operatorname{tr.d.}[M^\natural \mathcal{L} : M^\natural \mathcal{K}] = \operatorname{tr.d.}[M^\natural i(M) : M^\natural i(K)] \tag{10}$$

$$= \operatorname{tr.d.}[M^\natural \otimes_{C_M} M : M^\natural \otimes_{C_K} K]. \tag{11}$$

Here C_M, C_K denote respectively the field of D_t-constants of M and K.

Lemma 4.4. $\operatorname{tr.d.}[M^\natural \otimes_{C_M} M : M^\natural \otimes_{C_K} K] \leqslant 3.$

Proof. — Let us recall

$$M = \mathbf{C}(W, T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W), \qquad K = \mathbf{C}(T)\langle\lambda\rangle,$$

with

$$W_r = y_1\frac{\partial y_2}{\partial W} - y_2\frac{\partial y_1}{\partial W} = 1$$

and $y = y_2/y_1 \in C_M$. Therefore $\operatorname{tr.d.}[M : KC_M] \leqslant 3$. □

Now Theorem 1.1 follows from (8), (9), (10), (11) and Lemma 4.4.

5. Framework of proving Theorem 1.3

Now we show a framework of proving Theorem 1.3. We consider the system (12) below according to Garnier [**2**], p. 49.

$$\left\{\begin{aligned} \frac{\partial^2 y_i}{\partial W^2} &= p y_i, \\ \frac{\partial y_i}{\partial T} &= -\frac{1}{2}\frac{\partial B}{\partial W}\, y_i + B\,\frac{\partial y_i}{\partial W}, \end{aligned}\right. \qquad \text{for } i = 1,\, 2, \tag{12}$$

where

$$\begin{aligned} p(W) = c_4[W^4 - \lambda^4 + T(W^2 - \lambda^2)] + 2c_3[2(W^3 - \lambda^3) + T(W - \lambda)] \\ + c_2(W^2 - \lambda^2) + 2c_1(W - \lambda) + \frac{3}{4(W-\lambda)^2} - \frac{\lambda'}{W - \lambda} + \lambda'^2 \end{aligned}$$

and

$$B(W) = \frac{1}{2}\,\frac{1}{W - \lambda},$$

and y_1, y_2 are linearly independent solutions of the first equation of (12), λ being a function of T, $\lambda' = d\lambda/dT$ and $\lambda'' = d^2\lambda/dT^2$. After Garnier, the integrability condition of (12) implies that $\lambda(T)$ satisfies

$$\lambda'' = c_4(2\lambda^3 + \lambda T) + c_3(6\lambda^2 + t) + c_2\lambda + c_1, \tag{13}$$

where c_1, c_2, c_3, c_4 are constants.

Remark 5.1. — The ordinary differential equation of the second order (13) contains both the second Painlevé equation

$$\lambda'' = 2\lambda^3 + \lambda T + c_1,$$

when $c_4 = 1$, $c_3 = 0\, c_2 = 0$, and the first Painlevé equation

$$\lambda'' = 6\lambda^2 + T.$$

when $c_4 = 0$, $c_3 = 1$, $c_2 = c_1 = 0$.

Using the notation of this section, we consider the partial differential fields

$$M = \mathbf{C}(W,T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W) \tag{14}$$

with respect to the derivations $\partial/\partial W$, $\partial/\partial T$. We introduce mutually commutative differential operators on M

$$\begin{cases} D_t &= \dfrac{\partial}{\partial T} + \dfrac{1}{2}\dfrac{1}{W-\lambda(T)}\dfrac{\partial}{\partial W}, \\ D_w &= y_1^2 \dfrac{\partial}{\partial W}. \end{cases}$$

Remark 5.2. — It follows from the definition of differential operator D_t that

$$D_t T = 1, \qquad D_t^n \lambda(T) = \frac{\partial^n \lambda(T)}{\partial T^n} \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence the first Painlevé equation

$$\lambda'' = 6\lambda^2 + T$$

is equivalent to

$$D_t^2 \lambda = 6\lambda^2 + T$$

in M.

Drach's idea for the sixth Painlevé equation suggests the equivalence of the following two conditions on a function $\lambda(T)$, where T is an independent variable. We use the same notation as above but logically they have nothing to do with the variables W, T and derivations D_t, D_w for a moment.

(1) $\lambda(T)$ satisfies the first Painlevé equation (13) with respect to the derivation D_t with

$$D_t(\mathbf{C}) = 0, \ D_t(T) = 1.$$

Namely $D_t^2\lambda = 6\lambda^2 + T$.

(2) $\mathrm{InfGal}(L/K)$ is finite dimensional, where $L = K(W)$, $K = \mathbf{C}(T)\langle\lambda\rangle$ such that D_t is a differential operator on L satisfying

$$D_t(\mathbf{C}) = 0, \qquad D_t T = 1, \qquad D_t W = \frac{1}{2}\frac{1}{W-\lambda},$$

the element W being transcendental over K.

To prove Theorem 1.3, we argue as in § 3. Let us assume the condition (i) above that λ satisfies the first Painlevé equation

$$D_t^2 \lambda = 6\lambda^2 + T.$$

We start from the differential fields L, K in Condition (ii). Then we know by Garnier by virtue of Condition (i) that we are in the situation of the beginning of this section. Namely $K = \mathbf{C}(T)\langle T\rangle$, $L = K(W)$ are partial differential subfields of

$$M = \mathbf{C}(W,T)\langle\lambda\rangle(y_1, y_2, \partial y_1/\partial W, \partial y_2/\partial W)$$

with derivations D_t, D_w. We construct as in §3, partial differential field $\mathcal{L}$, $\mathcal{K}$. We have to show that

$$\text{tr. d.}[\mathcal{L} : \mathcal{K}] = 3. \tag{15}$$

This implies that the dimension of the Lie algebra of the formal group $\operatorname{InfGal}(L/K) = 3$. Then Lie's classification of Lie algebras operating on a manifold of dimension 1 shows

$$\text{Lie algebra of } \operatorname{InfGal}(L/K) \simeq sl_2$$

and hence

$$\operatorname{InfGal}(L/K) \simeq \widehat{\mathrm{SL}}_{2L^\natural}.$$

To prove (15), we have to show

$$\text{tr. d.}[M : K] = 4 \tag{16}$$

and

$$\text{tr. d.}[C_M : C_K] = 1, \tag{17}$$

where $K = \mathbf{C}(T)\langle\lambda\rangle$ and C_M, C_K are respectively D_t constants of M, K.

We will give a proof of (16) and (17) in [**8**]. The equality (16) is proved in a standard way. The proof of equality (17) is reduced to the following assertion.

Proposition 5.3. — *The field of constants of the differential field*

$$\mathbf{C}(T, \lambda, \lambda', W, y_1, \partial y_1/\partial W)$$

with derivation D_t coincides with $\mathbf{C}$.

The proof of the Proposition 5.3 is as much involved as that of the irreducibility Theorem of the first Painlevé equation. See also Remark 1.2.

6. Questions

Malgrange [**3**] proposes a differential Galois theory of infinite dimension. The relation between his theory and ours remains to be clarified. The following natural questions arises.

Question 6.1. — *Interprete Drach's paper by Malgrange's theory.*

Looking at the first version of this note, Malgrange proved, in the frame work of his differential Galois theory, an analogue of the inequality in Theorem 1.1 for the first and sixth Painlevé equations. So far as the inequality is concerned, his marvelous proof is geometric and more accessible than our algebraic argument. We notice here a difference. Namely his proof depends on the monodromy preserving deformation of a linear differential equation for a 2×2 matrix introduced by Jimbo and Miwa. On the other hand, Noumi and Yamada [**4**] introduced a more symmetric monodromy preserving deformation associated with $\widehat{\mathfrak{so}}(8)$ to define the sixth Painlevé equation. So it is natural to ask

Question 6.2. — *Can one use the Noumi-Yamada system associated with $\widehat{\mathfrak{so}}(8)$ to prove Theorem 1.1?*

We naively believe that our extensions L/K offer us totally new examples of which we can determine the Galois group $\mathrm{InfGal}(L/K)$. You may doubt this belief.

Question 6.3. — *Are the extensions L/K studied in this note subextensions of strongly normal extensions? To be more affirmative, prove that they are not so.*

7. Appendix: Extract from a letter of B. Malgrange to D. Bertrand

Grenoble, le 16.10.03

[...] Voici les détails dont je te parlais avant-hier.

7.1. Sorite général. — On considère un système linéaire 2×2,

$$dF = \Omega F, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ et } \Omega = (\omega_{ij}),\ 1 \leqslant i, j \leqslant 2.$$

Les ω_{ij} sont holomorphes sur un ouvert de $\mathbf{C}^n$ (par exemple ; ou aussi polynomiales en $x_1, \ldots, x_n$, etc. ; dans chaque cas, on a un résultat analogue). Ici, n est $\geqslant 2$ et, à part ça, quelconque. Ω doit vérifier la condition d'intégrabilité $d\Omega = \Omega \wedge \Omega$.

Posons $f_2/f_1 = w$.

w vérifie $dw + \alpha + \beta w + \gamma \frac{w^2}{2}$, avec $\alpha = -\omega_{21}, \beta = \omega_{11} - \omega_{22}, \gamma = 2\omega_{12}$; la condition d'intégrabilité s'écrit ici $d\pi = \pi \wedge \partial\pi/\partial w$, $\pi = dw + \alpha + \beta w + \gamma w^2/2$; explicitement, $d\alpha = \alpha \wedge \beta$, $d\beta = \alpha \wedge \gamma$, $d\gamma = \beta \wedge \gamma$.

On a évidemment une structure projective transverse associée au feuilletage $\pi = 0$ dans l'espace (x, w) ; d'où *par restriction à* $w = 0$ une structure projective transverse associée au feuilletage $\omega_{21} (= -\alpha) = 0$ dans l'espace des x (structure projective à prendre dans le sens de mon article[1] sur Galois différentiel = des équations différentielles qui définissent localement à un difféomorphisme près une structure projective au sens usuel).

7.2. Application à Painlevé. — On applique ça aux équations de Painlevé qui proviennent de systèmes linéaires du type précédent ; ici $n = 2$, et on écrit (x, t) au lieu de (x_1, x_2).

[1] B. Malgrange, *Le groupoïde de Galois d'un feuilletage*, Monogr. Ens. math., 38, 2001, 465-501.

Cas de P_1 (cf. Jimbo-Miwa[(2)], Appendix C). On considère le système

$$\frac{\partial F}{\partial x} = AF\ ,\ \frac{\partial F}{\partial t} = BF,\ \text{avec}$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x^2 + \begin{pmatrix} 0 & y \\ 4 & 0 \end{pmatrix} x + \begin{pmatrix} -z & y^2 + t^2/2 \\ -4y & z \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & y \\ 2 & 0 \end{pmatrix}.$$

Les conditions d'intégrabilité s'écrivent ici $dy/dt = z$, $dz/dt = 6y^2 + t$, *i.e.*, y satisfait à P_1 (y et z sont supposés a priori *dépendre de t seul*). Alors, modulo ces équations, on peut appliquer ce qui précède ; on a $\Omega = Adx + Bdt$, $\omega_{12} = 4(x-y)dx + 2dt$, *i.e.*, modulo que y satisfait à P_1, on a une structure projective transverse[(3)] (*cf.* Umemura[(4)], th. 1.3).

Cas de P_6

$$A = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}\ , B = -\frac{A_t}{x-t}$$

avec les notations A_0, A_1, A_t données dans Miwa-Jimbo (et reprises, avec un léger changement de notations dans Boalch[(5)], p. 29-32).

On prend ici $\omega_{12} = 0$; en explicitant, on trouve $\dfrac{y-x}{x(x-1)}dx = \dfrac{y-t}{t(t-1)}dt$, avec y fonction de t vérifiant P_6 (*cf.* Umemura, th. 1.1). Détails ci-dessous.

Bien sûr, cela s'appliquerait à tous les sytèmes 2×2 vérifiant la condition d'intégrabilité.

Tout cela est immédiat. La question difficile est de savoir s'il y a ou non des réductions supplémentaires (probablement pas pour P_1, ni pour P_6 dans le cas générique).

B. Malgrange

Détail du calcul sur P_6 (cf. Jimbo-Miwa ou Boalch, loc. cit.). On a

$$A_0 = \begin{pmatrix} z_0 + \theta_0 & -uz_0 \\ (z_0 + \theta_0)/u & -z_0 \end{pmatrix},\ A_1 = \begin{pmatrix} z_1 + \theta_1 & -vz_1 \\ (z_1 + \theta_1)/v & -z_1 \end{pmatrix},\ A_t = \begin{pmatrix} z_t + \theta_t & -wz_t \\ (z_t + \theta_t)/w & -z_t \end{pmatrix}.$$

(Boalch écrit z_1, z_3, z_2 au lieu de z_0, z_1, z_t.)

Conditions : $A_0 + A_1 + A_t = -\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ (les θ et k sont fixés), *i.e.*,

$$z_0 + z_1 + z_t = k_2\ ,\ uz_0 + vz_1 + wz_t = 0\ ,\ (z_0 + \theta_0)/u + (z_1 + \theta_1)/v + (z_t + \theta_t)/w = 0$$

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(2) M. Jimbo, T. Miwa, *Monodromy preserving deformations of LDEs with rational coefficients II*, Physica 2D, 1981, 407-448.

(3) Il faut prendre cette assertion au sens de mon article, avec toutefois un grain de sel (interpréter le fait que y satisfait à P_1 au moyen d'un système de 2 feuilletages emboîtés, en regardant des transversales à l'un dans des transversales de l'autre. Il y a un sorite à écrire que je n'ai pas vérifié en détail).

(4) H. Umemura, *Monodromy preserving deformation and differential Galois group I*, ce volume.

(5) P. Boalch, *The Klein solution to Painlevé's sixth equation*, math.AG/0308221.

(et alors, on a : $k_1 + k_2 + \theta_0 + \theta_1 + \theta_t = 0$).

Ici, $a_{12} = -\dfrac{uz_0}{x} - \dfrac{vz_1}{x-1} - \dfrac{wz_t}{x-t}$, $b_{12} = \dfrac{wz_t}{x-t}$, et l'équation est

$$\Big(\frac{uz_0}{x} + \frac{vz_1}{x-1} + \frac{wz_t}{x-t}\Big)dx = \frac{wz_t}{x-t}\,dt.$$

Éliminons les variables inutiles avec les conditions précédentes, auxquelles on peut rajouter (*cf.* Boalch) $(1+t)uz_0 + tvz_1 + wz_t = 1$. On trouve

$$\frac{-x + tuz_1}{x(x-1)}\,dx = wz_t\,dt.$$

Posant $y = tuz_1$, la dernière condition d'intégrabilité est que *y satisfasse à* P_6 (les autres conditions ont déjà été données implicitement, en écrivant que les classes de conjugaison des A sont fixées, et que leur somme est constante).

On trouve, d'autre part, $wz_t = \dfrac{y-t}{t(t-1)}$ (*loc. cit.*), d'où le résultat final

$$\frac{y-x}{x(x-1)}\,dx = \frac{y-t}{t(t-1)}\,dt,$$

annoncé plus haut, pour $\omega_{12} = 0$ (avec y satisfaisant à P_6).

References

[1] J. DRACH – "Sur les équations différentielles du premier ordre et du premier degré", *C. R. Acad. Sci. Paris* (1914), p. 926–929.

[2] M.R. GARNIER – "Sur les équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes", *Ann. Sci. École Norm. Sup.* (1912), p. 1–126.

[3] B. MALGRANGE – "Le groupoïde de Galois d'un feuilletage", in *Essays on geometry and related topics. Vol. 1, 2*, Monogr. Enseign. Math., vol. 38, l'Enseignement Mathématique, Genève, 2001, p. 465–501.

[4] M. NOUMI & Y. YAMADA – "A new Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}(8)}$", arXiv: `math-ph/0203029`, 2002.

[5] M.-H. SAITO & H. UMEMURA – "Painlevé equations and deformations of rational surfaces with rational double points", in *Proc. of Nagoya international workshop on Physics and Combinatorics*, World Scientific, Singapore, 2001, p. 320–365.

[6] H. UMEMURA – "Galois theory of algebraic and differential equations", *Nagoya Math. J.* **144** (1996), p. 1–58.

[7] ______, "Differential Galois theory of infinite dimension", *Nagoya Math. J.* **144** (1996), p. 59–135.

[8] ______, "Monodromy preserving deformation and differential Galois group II", in preparation.

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