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STATISTICAL PROPERTIES OF UNIMODAL MAPS: SMOOTH FAMILIES WITH NEGATIVE SCHWARZIAN DERIVATIVE

by

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Abstract. — We prove that there is a residual set of families of smooth or analytic unimodal maps with quadratic critical point and negative Schwarzian derivative such that almost every non-regular parameter is Collet-Eckmann with subexponential recurrence of the critical orbit. Those conditions lead to a detailed and robust statistical description of the dynamics. This proves the Palis conjecture in this setting.

1. Introduction

'The main strategy of the study of all mathematical models is, according to Poincaré, the consideration of each model as a point of a space of different but similar admissible systems' (V. Arnold in [**Ar**]). One of the main concerns of dynamical systems is to establish properties valid for typical systems. Since the space of such systems is usually infinite dimensional, there are of course many concepts of 'typical'. According to [**Ar**] again, 'The most physical genericity notion is defined by Kolmogorov (1954), who suggested to call a property of dynamical systems exceptional, if it holds only on Lebesgue measure zero set of values of the parameters in every (topologically) generic family of systems, depending on sufficiently many parameters'.

In the last decade Palis [**Pa**] described a general program for (dissipative) dynamical systems in any dimension. He conjectured that a typical dynamical system has a finite number of attractors described by physical measures, the union of their basins has full Lebesgue measure, and those physical measures are stochastically stable. Typical was to be interpreted in the Kolmogorov sense: full measure in generic families. Our aim here is to give a proof of this conjecture for an important class of one-dimensional dynamical systems.

Here we consider unimodal maps, that is, continuous maps from an interval to itself which have a unique turning point. More specifically, we consider S-unimodal maps, that is, we assume that the map is C^3 with negative Schwarzian derivative and that the critical point is non-degenerate.

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1.1. The quadratic family. — The basic model for unimodal maps is the quadratic family, $q_a(x) = a - x^2$, where $-1/4 \leqslant a \leqslant 2$ is a parameter. Despite its simple appearance, the dynamics of those maps presents many remarkable phenomena. Restricting to the probabilistic point of view, its richness first became apparent with the work of Jakobson [**J**], where it was shown that a positive measure set of parameters corresponds to quadratic maps with stochastic behavior. More precisely, those parameters possess an absolutely continuous invariant measure (the physical measure) with positive Lyapunov exponent. On the other hand, it was later shown by Lyubich [**L2**] and Graczyk-Swiatek [**GS**] that regular parameters (with a periodic hyperbolic attractor) are (open and) dense. So at least two kinds of very distinct observable behavior are present on the quadratic family, and they alternate in a complicate way.

Besides regular and stochastic behavior, different behavior was shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Those pathologies were shown to be non-observable in [**L3**] and [**MN**]. Finally in [**L4**] it was proved that almost every real quadratic map is either regular or stochastic.

Among stochastic maps, a specific class grabbed lots of attention in the 90's: Collet-Eckmann maps. They are characterized by a positive Lyapunov exponent for the critical value, and gradually they were shown to have 'best possible' near hyperbolic properties: exponential decay of correlations, validity of central limit and large deviations theorems, good spectral properties and zeta functions ([**KN**], [**Y**]). Let us call attention to the robustness of the statistical description, with a good understanding of stochastic perturbations: strong stochastic stability ([**BV**]), rates of convergence to equilibrium ([**BBM**]).

In [**AM1**] the regular or stochastic dichotomy was extended by showing that almost every stochastic map is actually Collet-Eckmann and has polynomial recurrence of its critical point, in particular implying the validity of the above mentioned results.

The position of the quadratic family in the borderline of real and complex dynamics made it a meeting point of many different techniques: most of the deeper results depend on this interaction. It gradually became clear however that studying the quadratic family allows one to obtain results on more general unimodal maps.

1.2. Universality. — Starting with the works of Milnor-Thurston, and also through the discoveries of Feigenbaum and Coullet-Tresser, the quadratic family was shown to be a prototype for other families of unimodal maps which presents universal combinatorial and geometric features. More recently, the result of density of hyperbolicity among unimodal maps was obtained in [**K**] exploiting the validity of this result for quadratic maps.

In [**ALM**], a general method was developed to transfer information from the quadratic family to real analytic families of unimodal maps. It was shown that the decomposition of spaces of analytic unimodal maps according to combinatorial behavior is essentially a codimension-one lamination.

Thinking of two analytic families as transversals to this lamination, one may try to compare the parameter space of both families via the holonomy map. A straightforward application of this method allows one to conclude that the bifurcation pattern of a general analytic family is locally the same as in the quadratic family from the topological point of view (outside of countably many 'bad parameters').

The 'holonomy' method was then successfully applied to extend the regular or stochastic dichotomy from the quadratic family to a general analytic family. The probabilistic point of view presents new difficulties however. First, the statistical properties of two topologically conjugate maps need not correspond by the (generally not absolutely continuous) conjugacy. Fortunately many properties are preserved, in particular the criteria used by Lyubich in his result.

The second difficulty is that the holonomy map is usually not absolutely continuous, so typical combinatorics for the quadratic family may not be typical for other families: it has to be shown that the class of regular or stochastic maps is still typical after application of the holonomy map.

1.3. Results and outline of the proof. — Let us call a k-parameter family good if almost every non-regular parameter is Collet-Eckmann (and satisfies some additional technical conditions). Our goal will be to prove that good families are generic. This question naturally makes sense in different spaces of unimodal maps (corresponding to different degrees of smoothness). We only deal with the last steps of this problem (going from the quadratic family to analytic and then smooth categories), basing ourselves on the building blocks [**L3**], [**L4**], [**ALM**], and [**AM1**].

We start by describing how the holonomy method of [**ALM**] can be applied to generalize the results of [**AM1**] to general analytic families (to put together those two papers we need to do a non-trivial strengthening of [**AM1**]). As a consequence we conclude that essentially all analytic families are good.

To get to the smooth setting (at least C^3, since we are assuming negative Schwarzian derivative), our strategy is different: we show a certain robustness of good families, which together with their denseness (due to the analytic case) will yield genericity. Our main tool is one of the nice properties of Collet-Eckmann maps: persistence of the Collet-Eckmann condition under generic unfolding (a result of [**T1**]). By means of some general argument, we reduce the global result to this local one.

Let us mention that the results of this paper are still valid without the negative Schwarzian derivative assumption (also allowing one to get to C^2 smoothness), see [**A**], [**AM4**]. The techniques are very different however, since we replace the global holonomy method we use here by a local holonomy analysis based on a "macroscopic" version of the infinitesimal perturbation method of [**ALM**]. For analytic maps this also allowed us to obtain better asymptotic estimates which have interesting consequences, for instance pathological measure-theoretical behavior of the lamination by combinatorial classes (see [**AM2**]).

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2. General definitions

2.1. Notation. — Let $I = [-1, 1]$ and let B^k be the closed unit ball in $\mathbb{R}^k$ (we will use the notation I for the *dynamical* interval, while B^1 will be reserved for the one-dimensional *parameter* space). We will consider B^k endowed with the Lebesgue measure normalized so that $|B^k| = 1$. Let $C^r(I)$ denote the space of C^r maps $f : I \to \mathbb{R}$.

By a *unimodal map* we will mean a smooth (at least C^2) symmetric (even) map $f : I \to I$ with a unique critical point at 0 such that $f(-1) = -1$, $Df(-1) \geqslant 1$, and if $Df(-1) = 1$ then $D^2f(-1) < 0$. If f is C^3, we define the Schwarzian derivative on $I \setminus \{0\}$ as

$$Sf = \frac{D^3f}{Df} - \frac{3}{2}\left(\frac{D^2f}{Df}\right)^2.$$

For $a > 0$, let $\Omega_a \subset \mathbb{C}$ denote an a neighborhood I.

Let $\mathcal{A}_a$ denote the space of holomorphic maps on Ω_a which have a continuous extension to $\partial\Omega_a$, satisfying $\phi(z) = \phi(-z)$, $\phi(-1) = \phi(1) = -1$ and $\phi'(0) = 0$.

Notice that $\mathcal{A}_a$ is a closed affine subspace of the Banach space of bounded holomorphic maps of Ω_a. We endow it with the induced metric and affine structure.

We define $\mathcal{A}_a^{\mathbb{R}} \subset \mathcal{A}_a$ the space of maps which are real symmetric.

2.2. More on unimodal maps. — A C^3 unimodal map such that $Sf < 0$ on $I \setminus \{0\}$ and such that its critical point is non-degenerate (that is, $D^2f \neq 0$) will be called a S-*unimodal map*.

We say that x is a periodic orbit (of period n) for f if $f^n(x) = x$ and $n \geqslant 1$ is minimal with this property. In this case we define $Df^n(x)$ as the multiplier of x. Notice that this definition depends only on the orbit of x. We say that x is hyperbolic if $|Df^n(x)| \neq 1$.

A unimodal map is called *regular* (or hyperbolic) if all periodic orbits are hyperbolic and the iterates of the critical point converge to an attracting periodic orbit. This condition is C^2-open, moreover a S-unimodal map is regular if and only if it has a hyperbolic periodic attractor (see [**MvS**]).

A k-*parameter family* of unimodal maps is a map $F : B^k \times I \to I$ such that for $p \in B^k$, $f_p(x) = F(p, x)$ is a unimodal map. Such a family is said to be C^n or analytic, according to F being C^n or analytic. We introduce the natural topology in spaces of smooth families (C^n with $n = 2, \ldots, \infty$), but do not introduce any topology in the space of analytic families (however, we will refer from time to time to induced C^n topologies).

An analytic family of S-unimodal maps F will be called *non-trivial* if there exists a regular parameter. Notice that this condition is C^3-open.

A unimodal map f is called *Collet-Eckmann* (CE) if there exists constants $C > 0$, $\lambda > 1$ such that for every $n > 0$,

$$|Df^n(f(0))| > C\lambda^n.$$

This means that the map is strongly hyperbolic along the critical orbit. It is also useful to study the hyperbolicity of backward iterates of the critical point, so we say that f is *Backwards Collet-Eckmann* (BCE) if there exists $C > 0$, $\lambda > 1$ such that for any $n > 0$ and any x with $f^n(x) = 0$, we have

$$|Df^n(x)| > C\lambda^n.$$

By a result of Nowicki (see [**MvS**]), for S-unimodal maps CE implies BCE, so we will mostly discuss the Collet-Eckmann condition (except for the last section where we consider C^2 unimodal maps as well).

Very often it is useful to estimate how fast is the recurrence of the critical orbit. We will be mainly interested in two kinds of control: *Polynomial Recurrence* (P) if there exists $\alpha > 0$ such that

$$|f^n(0)| > n^{-\alpha}$$

for big enough n and *Subexponential Recurrence* (SE) if for all $\alpha > 0$,

$$|f^n(0)| > e^{-\alpha n}$$

for n big enough.

We will say that f is *Weakly Regular* (WR) if

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \sum_{\substack{1\leqslant k\leqslant n \\ f^k(0)\in(-\delta,\delta)}} \ln |Df(f^k(0))| = 0.$$

This condition is used in proofs of stochastic stability for C^2 maps, see [**T2**].

We will consider spaces of S-unimodal maps: we define $\mathcal{U}^r \subset C^r(I)$ the set of S-unimodal maps. Spaces of analytic unimodal maps are now easily defined: $\mathcal{U}_a = \mathcal{U}^3 \cap \mathcal{A}_a^{\mathbb{R}}$.

2.3. The quadratic family. — The quadratic family is the most studied family of unimodal maps. It is usually parametrized by

$$q_t(x) = t - x^2,$$

so that for $-1/4 \leqslant t \leqslant 2$, there exists a unique symmetric interval $I_t = [-\beta_t, \beta_t]$ such that $q_t(I_t) \subset I_t$ and $q_t(-\beta_t) = -\beta_t$, so q_t can be seen as a unimodal map of I_t (which depends on t). Moreover $Sq_t(x) < 0$ if $x \neq 0$.

By an affine reparametrization of the parameter t and of each interval I_t, we obtain a canonical one-parameter family of S-unimodal maps in the interval I, which we denote p_t, $t \in B^1$, which will be called the quadratic family as well.

2.4. Quasisymmetric maps. — Let $\gamma \geqslant 1$ be given. We say that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is *quasisymmetric* (qs) if there exists a constant $k > 1$ such that for all $x \in \mathbb{R}$ and any $h > 0$

$$\frac{1}{k} \leqslant \frac{f(x+h) - f(x)}{f(x) - f(x-h)} \leqslant k.$$

A homeomorphism h is quasisymmetric if and only if it admits a real-symmetric extension to a quasiconformal map $\widetilde{h} : \mathbb{C} \to \mathbb{C}$ (Ahlfors-Beurling). We will say that h is γ-qs (or that γ is a qs constant for h) if the dilatation of $\widetilde{h}$ is bounded by γ. This definition of the quasisymmetric constant is convenient since the composition of quasisymmetric maps g and f is readily seen to be quasisymmetric and the qs constant of $g \circ f$ is bounded by the product of the qs constants of g and f.

If $X \subset \mathbb{R}$ and $h : X \to \mathbb{R}$ has a γ-quasisymmetric extension to $\mathbb{R}$ we will also say that h is γ-qs.

3. Statement of the results

3.1. A dichotomy for generic families of S-unimodal maps. — We would like to classify the typical behavior in generic families of unimodal maps. This classification should reveal refined information on the stochastic description of the dynamics of those typical parameters.

We will therefore consider a smooth enough family of unimodal maps F. The techniques of the present paper will need the fact that F is a family of S-unimodal maps. This includes two main restrictions: the negative Schwarzian derivative and the quadratic critical point. The first one is serious, since this condition is not dense, but can be removed with more refined techniques (see [**A**]). The second one (which is not present in the usual definition of S-unimodal map, but is rather a convention in this paper) is no serious loss of generality, since quadratic critical point is certainly typical among unimodal maps.

Remark 3.1. — Families of unimodal maps with a fixed critical exponent different from 2 have also been subject of much study. This theory has many similarities, but also some important differences and new features, and is not nearly as complete as the case of criticality 2. It is however widely expected that the Palis conjecture (and indeed our Theorems A, B and C) still holds in this setting.

We first consider the analytic case.

Theorem A. — *Let F be a non-trivial k-parameter analytic family of S-unimodal maps. Then for almost every non-regular parameter $p \in B^k$, f_p satisfies the Collet-Eckmann and Polynomial Recurrence conditions.*

Notice that the set of non-trivial analytic families is indeed generic in any meaningful sense: its complement has "infinite codimension", see Proposition 4.3. Moreover, if an analytic family is non-trivial, it is possible to verify the non-triviality in finite time (with an infinite precision computer [(1)]).

Our second result about non-trivial analytic families is the robustness of a slightly weaker dichotomy under C^2 perturbations of the family.

Theorem B. — *Let F be a non-trivial k-parameter analytic family of S-unimodal maps. Let $F^{(n)}$ be a sequence of C^2 families such that $F^{(n)} \to F$ in the C^2 topology. For each n, let X_n be the set of parameters $p \in B^k$ where $F^{(n)}$ is either regular or has only repelling periodic orbits and satisfies simultaneously the Backwards Collet-Eckmann, Collet-Eckmann, Subexponential Recurrence and Weak Regularity conditions. Then $|X_n| \to 1$. In particular, almost every parameter of F is Weakly Regular.*

As a consequence, we can use a Baire argument to conclude that the dichotomy is still valid among topologically generic smooth families (that is, belonging to some residual set), obtaining the following corollary of Theorems A and B.

***Theorem C* (Smooth Dichotomy)**. — *In topologically generic k-parameter C^r, $r = 3, 4, \dots, \infty$ families of S-unimodal maps, almost every non-regular parameter satisfies the Backwards Collet-Eckmann, Collet-Eckmann, Subexponential Recurrence and Weak Regularity conditions.*

It is good to recall that both types of behavior described by the dichotomy are indeed observable for open sets of families of unimodal maps ([**J**], [**BC**]).

Remark 3.2. — The space of S-unimodal maps is easy to describe and easier to work with but has some disadvantages. One of them is that it is not an intrinsic condition, in particular it is not invariant by analytic change of coordinates. A more natural class to work with is the space of quasiquadratic unimodal maps as defined by [**ALM**]. A unimodal map f is called quasiquadratic if there exists a C^3-neighborhood of f where all maps are topologically conjugate to some quadratic map. The results of this paper are still valid in spaces of quasiquadratic unimodal maps (which includes S-unimodal maps). The proofs are unchanged, since the results we need from [**ALM**] are stated and proved for quasiquadratic maps. We remark further that the description of quasiquadratic unimodal maps can be used to describe all unimodal maps: it is proved

(1) Since regular parameters form an open set (non-empty if the family is non-trivial), and any regular parameter one can be also checked in finite time (by locating the attracting hyperbolic periodic orbit).

in [**A**], [**AM4**] that (Kolmogorov) typical (analytic or smooth) unimodal maps have either a quasiquadratic renormalization or a quasiquadratic unimodal restriction.

3.2. Ergodic consequences. — The importance of the above dichotomy is the fact that each of the two possibilities has very well defined stochastic properties. We quickly recall those (we assume that maps are S-unimodal).

Regular maps have a periodic attractor whose basin is big both topologically (open and dense set) as in the measure-theoretical sense (full measure). Moreover the attractor and its basin are stable under C^1 perturbations. The dynamics of such maps can be described in deterministic terms.

Maps satisfying CE and SE have non-deterministic dynamics. They can be however described through their stochastic properties, and it turns out that such maps have the main good properties usually found in hyperbolic maps. First, there is a physical measure, that is an invariant probability which describes asymptotic behavior of orbits: for almost every x and for every continuous $\phi : I \to \mathbb{R}$,

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} \phi(f^k(x)) = \int \phi d\mu.$$

This physical measure has a positive Lyapunov exponent and is indeed absolutely continuous and supported on a cycle of intervals, so the asymptotic behavior is non-deterministic. The convergence to the asymptotic stochastic model is exponential, see the results on decay of correlations and convergence to equilibrium ([**KN**], [**Y**]). Those properties are beautifully related to a spectral gap of a transfer operator and to zeta functions, see [**KN**]. Notice finally that exponential decay of correlations is actually equivalent to the Collet-Eckmann condition (see [**NS**]).

While the dynamics is highly unstable under deterministic perturbations (nearby maps can be regular for instance), the stochastic description given by the physical measure μ is robust under stochastic perturbations: the perturbed system has a stationary measure which is close to μ in the sense of the L^1 distance between their densities ([**BV**]). For studies of decay of correlations for the perturbed systems, see [**BBM**].

4. Analytic families

4.1. Hybrid classes and holonomy maps. — Two S-unimodal maps $f, \widetilde{f}$ are said to be *hybrid equivalent* if they are topologically conjugate and, in case they are regular, their attracting periodic orbits have the same multiplier.

The set of all maps which are hybrid equivalent to some f is called the *hybrid class* of f. The partition of S-unimodal maps into hybrid classes is thus a refinement of the partition in topological conjugacy classes.

It follows from a result of Guckenheimer (see [**MvS**]) that any S-unimodal map f is topologically conjugate to some quadratic map. It turns out that if f has a hyperbolic attractor, we can select the quadratic map with a hyperbolic attractor with the same multiplier[(2)]. In particular, each hybrid class intersects the quadratic family in at least one point.

The problem of uniqueness is much harder. The following result is due to Lyubich [**L2**] and Graczyk-Swiatek [**GS**], and is a consequence of (the proof of) the equivalent rigidity result for quadratic maps:

Theorem 4.1. — *Let h be a topological conjugacy between two analytic S-unimodal maps f and $\widetilde{f}$ which have all periodic orbits repelling. Then h is quasisymmetric.*

Remark 4.1. — Although we won't use it here, a similar theorem still holds for maps with non-repelling periodic orbits: if f and $\widetilde{f}$ are two topologically conjugate S-unimodal maps and have non-repelling periodic orbits then we can select a topological conjugacy which is quasisymmetric (the choice of the topological conjugacy is not unique). This result is considerably easier than the case where all periodic orbits are repelling, and does not use analyticity.

This rigidity result has a remarkable consequence for quadratic maps: each hybrid class intersects the quadratic family at a unique parameter. Thus, any S-unimodal map f is hybrid equivalent to a unique quadratic map $\chi(f)$. The map χ is called the *straightening* [(3)].

Lemma 4.2. — *Let f be an analytic S-unimodal map. Then $\chi(f)$ is regular/CE/P if and only if f also satisfies the corresponding property.*

Proof. — The property of being regular is clearly invariant under hybrid equivalence, so we only have to analyze invariance of the conditions CE and P.

By [**NP2**], the Collet-Eckmann condition is topologically invariant, so it is preserved under hybrid equivalence.

To check invariance of polynomial recurrence of the critical orbit, first assume that f has some non-repelling periodic point p. In this case, the the orbit of p must attract the critical point. In particular, the critical point is either non-recurrent (in

[(2)] This follows for instance from Milnor-Thurston kneading theory and the fact that the quadratic family is a full family. Another way to see this is to notice that in each "hyperbolic window" of quadratic maps (a maximal parameter interval (a, b) such that p_t is hyperbolic for $t \in (a, b)$), the multiplier of the hyperbolic attractor induces a homeomorphism from (a, b) to $(-1, 1)$ (this is a consequence for instance of the work of Douady-Hubbard on the complex quadratic family).

[(3)] We should point out that there is also a notion of hybrid class in complex dynamics. In that context, the fact that each hybrid class (of quadratic-like maps with connected Julia set) contains exactly one quadratic polynomial is a consequence of the Straightening Theorem of Douady-Hubbard. Our definition of hybrid class is motivated precisely by the possibility of defining an analogous straightening map (whose existence is proved by quite different methods).

which case both f and $\chi(f)$ satisfy P in a trivial way) or periodic (in which case f and $\chi(f)$ do not satisfy P also in a trivial way).

If f has all periodic orbits repelling, by Theorem 4.1, the conjugacy between f and $\chi(f)$ is quasisymmetric, and in particular Hölder. It is easy to see that P is invariant by Hölder conjugacy. □

Remark 4.2. — By [**NP1**], two S-unimodal Collet-Eckmann maps which are topologically conjugate are Hölder conjugate, so using [**NP2**] we see that the joint conditions CE and P are topologically invariant. This joint invariance of CE and P is all that will be used in the further arguments. Notice that [**NP1**] and [**NP2**] do not assume analyticity, and are more elementary than Theorem 4.1.

4.2. Hybrid laminations. — It is natural to study the hybrid class of some map f. This is what is done in Theorem A of [**ALM**] in the analytic setting, where it is shown that in $\mathcal{U}_a$, every hybrid class is a codimension-one analytic submanifold. Moreover, different hybrid class fit together in some nice structure, called *hybrid lamination.*

Remark 4.3. — It is not known if the hybrid lamination is really a lamination everywhere. In [**ALM**], it is shown that the hybrid lamination is a lamination (in the usual sense) "almost everywhere" (more precisely, if restricted to an open set containing the complement of countably many classes corresponding to existence of neutral periodic orbits), which is enough for our purposes.

A k-parameter analytic family of S-unimodal maps can be thought as an analytic map from B^k to some $\mathcal{U}_a$. As a consequence, the structure of the hybrid lamination implies that non-trivial analytic families are indeed quite frequent.

***Lemma 4.3* (Most analytic families are non-trivial)**. — *If a k-parameter analytic family of S-unimodal maps is not contained in some non-regular hybrid class then it is non-trivial. In particular, non-trivial analytic families are dense in the space of C^n families of S-unimodal maps, $n = 3, \dots, \infty$.*

Proof. — Let us consider an analytic family of S-unimodal maps F. By the theory of Milnor-Thurston, see [**MvS**], either all parameters have the same non-periodic *kneading sequence*, or there exists a parameter with periodic critical point. In the latter case, the family is of course non-trivial, so let us consider the former case. Two S-unimodal maps with the same kneading sequence are either topologically conjugate, or one of them possess a neutral periodic orbit (see Corollary, Chapter 2, page 157 of [**MvS**]), and it follows that the other is necessarily regular. Thus, if the family F does not have regular parameters, all maps are non-regular and topologically conjugate, that is, F is contained in a non-regular hybrid class.

For the denseness result, given a C^r family F, approximate it by an analytic family $\widetilde{F}$. If such an analytic family is contained in a hybrid class, we can perturb it further in order to intersect two hybrid classes, since each hybrid class is a codimension-one submanifold. □

Let us consider the case where F is a one-parameter analytic family of S-unimodal maps, that is, an analytic curve in some $\mathcal{U}_a$. A consequence of the nice structure of the hybrid lamination is the following result:

Lemma 4.4 **(see the proof of Theorem C of [ALM]).** — *If F is a one-parameter analytic family of S-unimodal maps which is not contained in some hybrid class then there is an open set of parameters, with countable complement, where F is transverse to the hybrid lamination.*

Define the map χ_F on B^1 by $\chi_F(t) = \chi(f_t)$. In [**ALM**] the map χ_F is considered as the holonomy map from F to the quadratic family along the hybrid lamination in some $\mathcal{U}_a$. Using this interpretation, they obtain the following result:

Theorem 4.5 **(Theorem C of [ALM]).** — *Let F be a one-parameter family of unimodal maps which is not contained in some hybrid class. Then there is an open set $U \subset B^1$ with countable complement such that the straightening χ_F is quasisymmetric in any compact interval $J \subset U$.*

4.3. Dichotomy in the quadratic family.

— The main result of [**AM1**] is that almost every parameter in the quadratic family is either regular or Collet-Eckmann with a polynomial recurrence of the critical orbit. To obtain the same result for a non-trivial analytic family using Theorem 4.5, we will need a stronger estimate, since quasisymmetric maps are not in general absolutely continuous.

Let us say that a set $X \subset B^1$ has *total qs-probability* if the image of $B^1 \setminus X$ by any quasisymmetric map $h : B^1 \to B^1$ has zero Lebesgue measure.

By an improvement of the proofs in [**AM1**] (see appendix), it is possible to obtain the following result:

Theorem 4.6. — *The set of quadratic maps which are either regular or simultaneously CE and P has total qs-probability.*

Remark 4.4. — In [**AM1**] a better result than polynomial recurrence is obtained in the quadratic family. Namely it is shown that the asymptotic exponent of the recurrence

$$\limsup_{n\to\infty} \frac{-\ln|f^n(0)|}{\ln n}$$

is exactly 1 for almost every non-regular map. However, for a set of total qs-probability, we are only able to show that the asymptotic exponent is bounded.

4.4. Proof of Theorem A. — Let F be a non-trivial analytic family. If all parameters are regular, there is nothing to prove, so assume that there is a non-regular parameter.

First assume F is one-parameter. By Theorems 4.6 and 4.5, for almost every $t \in B^1$, $\chi_F(t)$ is either regular or satisfies CE and P. By Lemma 4.2, this implies that f_t is either regular or CE and P.

Assume now that F is a k-parameter family. Let $p \in B^k$ be a regular parameter. Let $L : B^1 \to B^k$ be an affine map such $p \in L(B^1)$. Let F^L be the one-parameter family defined by $f_t^L = f_{L(t)}$. Then F^L is a non-trivial one-parameter analytic family and hence for almost every t, f_t^L is either regular or CE and P. The result follows by application of Fubini's Theorem.

5. Robustness of the dichotomy

To obtain the robustness claimed on Theorem B our approach will be to exploit an important result of Tsujii, whose core is a strong generalization of Benedicks-Carleson result and techniques. This result establishes that the CE and SE conditions are infinitesimally persistent in one-parameter families unfolding generically: they are density points of CE and SE parameters. The connection with our robustness result, which has a global nature, is done using some general argument.

5.1. Tsujii's theorem. — Let F be a C^2 k-parameter family of unimodal maps. Assume that p_0 is a parameter such that f_{p_0} satisfies CE, BCE, SE, has a quadratic critical point and all periodic orbits repelling. Tsujii's Theorem considers the case where F is a generic unfolding at p_0. For one-parameter families, generic unfolding means precisely

$$\sum_{j=0}^{\infty} \frac{v(f_{p_0}^j(0))}{Df_{p_0}^j(f_{p_0}(0))} \neq 0, \quad \text{where} \quad v = \frac{d}{dp} f_p \bigg|_{p=p_0}. \tag{5.1}$$

This transversality condition will be called *Tsujii transversality.*

If F is a one-parameter family, we will say that (F, p_0) satisfies the *Tsujii conditions* if all above requirements are satisfied.

The following is an immediate consequence of the main theorem of Tsujii in [**T1**].

Theorem 5.1. — *Let F be a C^2 one-parameter family of unimodal maps. Assume (F, t_0) satisfies the Tsujii conditions. Then t_0 is a density point of parameters t for which (F, t) satisfies the Tsujii conditions and for which f_t is WR.*

5.2. A higher dimensional version. — In order to pass from one-parameter to k-parameters, we will need the following easy proposition. Let us say that $p \in B^k$ is a density point of a set X along a line l through p if p is a density point of $l \cap X$ in l (endowed with the linear Lebesgue measure).

Proposition 5.2. — *If $p \in B^k$ is a density point of X along almost every line, then p is a density point of X in B^k.*

Proof. — Let E be the characteristic function of X. For each line l through p, let $A_l : \mathbb{R} \to l$ be an isometric parametrization of l taking 0 into p. Let P^{k-1} be the space of such lines with the natural probability measure (obtained by identification with the $k-1$ dimensional projective space). Let

$$\rho_\varepsilon(l) = \int_{-1}^{1} |r| E(A_l(\varepsilon r)) dr.$$

Assuming that p is a density point of X along almost every l we have, for almost every l

$$\lim_{\varepsilon \to 0} \rho_\varepsilon(l) = 1.$$

Using polar coordinates, the relative measure of X in an ε ball around p is given by

$$\int_{P^{k-1}} \rho_\varepsilon(l) dl.$$

By the Lebesgue Convergence Theorem,

$$\lim_{\varepsilon \to 0} \int \rho_\varepsilon(l) dl = \int \lim_{\varepsilon \to 0} \rho_\varepsilon(l) dl = 1.$$

This shows that p is a density point of X. □

We say that a k-parameter F satisfies the Tsujii transversality at p_0 if there exists a line through p_0 along which the one-parameter Tsujii transversality condition is satisfied. In other words, there exists an affine map $L : B^1 \to B^k$ such that $L(t_0) = p_0$ for some $t_0 \in \operatorname{int} B^1$ and such that the induced one-parameter family F^L defined by $f_t^L = f_{L(t)}$ is Tsujii transverse at the parameter t_0.

By linearity of (5.1) with respect to v, if (F, p_0) is Tsujii transverse then all lines passing through p_0 are Tsujii transverse except the lines parallel to a certain codimension-one space of $\mathbb{R}^k$.

Lemma 5.3. — *Let F be a C^2 k-parameter family of unimodal maps. Assume (F, p_0) satisfies the Tsujii conditions. Then p_0 is a density point of parameters p for which (F, p) satisfies the Tsujii conditions and for which f_p is WR.*

Proof. — If F is Tsujii transverse at p_0 then it is Tsujii transverse along almost every line through p_0. Along such a line it is a density point of parameters satisfying the Tsujii conditions and WR. The result follows from Proposition 5.2. □

5.2.1. Tsujii transversality and hybrid lamination. — Let us take a closer look at the Tsujii transversality for an analytic F. Let $f_p = f$.

Assuming the summability condition,

$$\sum_{k=0}^{\infty} \frac{1}{|Df^k(f(0))|} < \infty \tag{5.2}$$

(in particular if f is CE), let

$$\nu_f(v) = \sum_{k=0}^{\infty} \frac{v(f^k(0))}{Df^k(f(0))} = v(0) + \sum_{k=1}^{\infty} \frac{v(f^k(0))}{Df^k(f(0))}$$

be a functional defined on continuous vector fields v on the interval.

Lemma 5.4. — *If f satisfies the summability condition then there exists an even polynomial vector field v, with $v(-1) = v(1) = 0$ and such that $\nu_f(v) \neq 0$.*

Proof. — Let $S = \sum |Df^k(f(0))|^{-1}$. Let ε be so small that

$$\sum_{\substack{k>0 \\ f^k(0)\in(-\varepsilon,\varepsilon)}} \frac{1}{|Df^k(f(0))|} < 1/3.$$

Let v be an even polynomial vector field satisfying $v(-1) = v(1) = 0$,

$$\begin{aligned} |v(x)| &< 2, &&\text{for} \quad x \in I, \\ v(x) &> 1, &&\text{for} \quad x \in (-\varepsilon/2, \varepsilon/2), \\ |v(x)| &< \frac{1}{10S}, &&\text{for} \quad x \in I \setminus (-\varepsilon, \varepsilon). \end{aligned}$$

Then $\nu_f(v) > 1 - 2/3 - 1/10 > 0$. □

Lemma 5.5. — *The kernel of ν_f intersected with $T\mathcal{A}_a^{\mathbb{R}}$ is the tangent space to the hybrid class of f.*

Proof. — By the previous lemma, ν_f is non-trivial over $T\mathcal{A}_a^{\mathbb{R}}$, so the above intersection is a closed codimension-one subspace of $T\mathcal{A}_a^{\mathbb{R}}$. So it is enough to show that if v is tangent then $\nu_f(v) = 0$. Assuming that v is tangent, consider an analytic family f_t contained in the hybrid class of f, such that $f_0 = f$ and

$$\frac{d}{dt} f_t \bigg|_{t=0} = v.$$

It is remarked in [**ALM**] that

$$\alpha_{n+1} = Df^n(f(0)) \sum_{k=0}^{n} \frac{v(f^k(0))}{Df^k(f(0))} = Df^n(f(0))\nu_f(v)$$

is precisely

$$\frac{d}{dt} f_t^{n+1}(0) \bigg|_{t=0}.$$

Moreover, $t \mapsto f_t^{n+1}(0)$ are holomorphic functions of the complex parameter t, taking values in Ω_a, and whose domain is some definite neighborhood of 0. It follows by Cauchy estimates on the derivative that this sequence is bounded independently of n. By the summability condition (5.2), $|Df^n(f(0))| \to \infty$, so we have necessarily $\nu_f(v) = 0$. □

Remark 5.1. — It is shown in [**ALM**] that the sequence α_n is not only bounded (for tangent vector fields v), but that the vector field defined on the orbit of the critical value by $w(f^k(0)) = \alpha_k$, $k > 0$, extends to a quasiconformal vector field on $\mathbb{C}$.

So Tsujii transversality can be interpreted for such a map (satisfying the summability condition (5.2)) as transversality of the family to the hybrid class of f_p.

Since for maps with negative Schwarzian derivative CE implies the BCE and that all periodic orbits are repelling, we can conclude from Theorem A, Lemma 4.4 and this discussion the following result:

Lemma 5.6. — *If F is a non-trivial k-parameter analytic family of S-unimodal maps then almost every parameter is regular or satisfies the Tsujii conditions.*

5.3. Estimates of density in perturbed families. — Let K be the space of C^2 k-parameter families of unimodal maps (without, naturally, the hypothesis of negative Schwarzian derivative).

Let $X \subset K \times B^k$ be the set of (F, p) such that either f_p is regular or satisfies the Tsujii conditions and WR. For $F \in K$, let $X_F = \{p \in B^k | (F, p) \in X\}$.

Let $Y \subset B^k$ be measurable with $|Y| > 0$. We define the density of X along F on Y as

$$d(F, Y) = \frac{|Y \cap X_F|}{|Y|}.$$

Instead of defining the classical infinitesimal density:

$$\liminf_{\varepsilon \to 0} d(F, B_\varepsilon(p))$$

we will need to consider the stability of the density with respect to perturbations of F. With this in mind we introduce two parameters. Let

$$D^-(F, p) = \liminf_{\substack{\widehat{F} \to F \\ \widehat{f}_p = f_p}} \liminf_{\varepsilon \to 0} d(\widehat{F}, B_\varepsilon(p)),$$

$$D^+(F, p) = \liminf_{\varepsilon \to 0} \liminf_{\widehat{F} \to F} d(\widehat{F}, B_\varepsilon(p)).$$

Remark 5.2. — Notice that in the definition of $D^-(F, p)$ we only consider families through a fixed map, while in the definition of $D^+(F, p)$ we do not make this restriction.

Theorem A and Tsujii's result give a direct way to estimate D^-:

Lemma 5.7. — *Let F be a non-trivial analytic family of S-unimodal maps. Then for almost every $p \in B^k$, $D^-(F,p) = 1$.*

Proof. — Indeed, by Lemma 5.6, almost every parameter is either regular or satisfies the Tsujii conditions. Since the set of regular maps is C^2 open, $D^-(F,p) = 1$ at any regular parameter p.

Let us show that this still holds for parameters p satisfying the Tsujii conditions. Since Tsujii transversality *through a fixed CE map* is clearly an open condition, if $\widehat{F}$ is any C^2 family near F with $\widehat{f}_p = f_p$ then $(\widehat{F}, p)$ also satisfies the Tsujii conditions. By Lemma 5.3,

$$\lim_{\varepsilon \to 0} d(\widehat{F}, B_\varepsilon(p)) = 1.$$

Thus $D^-(F,p) = 1$. □

However, for measure estimates in perturbed families, $D^+(F,p)$ is more relevant. We proceed to discuss the effect of the interchange of limits in the definitions of $D^-(F,p)$ and $D^+(F,p)$.

Lemma 5.8. — *In this setting,*

$$D^+(F,p) \geqslant D^-(F,p).$$

Proof. — The idea is to construct, arbitrarily near F, a family $\widetilde{F}$ with $\widetilde{f}_p = f_p$ and

$$\lim_{j \to \infty} d(\widetilde{F}, B_{\varepsilon_j}(p)) = D^+(F,p),$$

for some sequence $\varepsilon_j \to 0$, which implies $D^+(F,p) \geqslant D^-(F,p)$. To construct $\widetilde{F}$, we will interpolate F with a certain sequence $F^{(n)}$ which realizes the limit in the definition of $D^+(F,p)$.

Let $\varepsilon_j \to 0$ be a sequence such that

$$\lim_{j \to \infty} \liminf_{\widehat{F} \to F} d(\widehat{F}, B_{\varepsilon_j}(p)) = D^+(F,p).$$

Passing to a subsequence, we may assume that

$$\lim_{j \to \infty} \frac{\varepsilon_{j+1}}{\varepsilon_j} = 0.$$

Let $K_j \subset B_{\varepsilon_j}(p) \smallsetminus \overline{B_{\varepsilon_{j+1}}(p)}$ be compact sets such that

$$\lim_{j \to \infty} \frac{|\operatorname{int} K_j|}{|B_{\varepsilon_j}(p)|} = 1. \tag{5.3}$$

Let $\phi_j : \mathbb{R}^k \to \mathbb{R}$ be a C^∞ function supported in $B_{\varepsilon_j}(p) \smallsetminus \overline{B_{\varepsilon_{j+1}}(p)}$ such that $\phi_j | K_j = 1$.

For a sequence $F^{(n)} \to F$, let us define $\widetilde{F} : B^k \times I \to I$ by

$$\widetilde{f}_q = f_q + \sum_{j=1}^{\infty} \phi_j(q)(f_q^{(j)} - f_q).$$

It is easy to see that for every $\delta > 0$ there exists a sequence $\delta_n > 0$, $n \geqslant 1$, such that, if $\|F^{(n)} - F\|_{C^2} < \delta_n$ then $\|\widetilde{F} - F\|_{C^2} < \delta$ (and in particular $\widetilde{F}$ is C^2). In other words, if $F^{(n)} \to F$ sufficiently fast then $\widetilde{F}$ is C^2 and close to F in the C^2 topology.

Notice that $\widetilde{F}$ interpolates F and the sequence $F^{(n)}$ in such a way that inside each $B_{\varepsilon_n}(p)$, $\widetilde{f}_p = f_p^{(n)}$ for p in int K_n. Thus,

$$X_{\widetilde{F}} \cap K_n = X_{F^{(n)}} \cap K_n. \tag{5.4}$$

Fix $\delta > 0$ and select $F^{(n)}$ such that

$$\lim_{n\to\infty} d(F^{(n)}, B_{\varepsilon_n}(p)) = D^+(F^{(0)}, p) \tag{5.5}$$

and moreover $\|F^{(n)} - F\|_{C^2} < \delta_n$, so that $\|\widetilde{F} - F\|_{C^2} < \delta$. By (5.3), (5.4), and (5.5),

$$\liminf_{\varepsilon\to 0} d(\widetilde{F}, B_\varepsilon(p)) \leqslant \lim_{n\to\infty} d(\widetilde{F}, B_{\varepsilon_n}) = \lim_{n\to\infty} d(F^{(n)}, B_{\varepsilon_n}) = D^+(F, p).$$

Making $\delta \to 0$, $\widetilde{F}$ converges to F and we obtain $D^+(F, p) \geqslant D^-(F, p)$. □

5.4. Proof of Theorem B. — Let F be a non-trivial analytic family of S-unimodal maps. Then almost every parameter satisfies $D^-(F, p) = 1$. Hence, for almost every p we have $D^+(F, p) = 1$.

Fix $\varepsilon > 0$. Let $p \in B^k$ be such that $D^+(F, p) = 1$. By definition of D^+ there exists a sequence of balls $U^n(p)$ centered at p and converging to p, and neighborhoods $\mathcal{V}^n(p) \subset K$ of F such that if $\widetilde{F} \in \mathcal{V}^n(p)$ then

$$d(\widetilde{F}, U^n(p)) > 1 - \varepsilon/2.$$

By Vitali's Lemma, there exist sequences p_j, n_j such that $U^{n_j}(p_j)$ are disjoint and $|\cup U^{n_j}(p_j)| = 1$. Let m be such that $\cup_{j=1}^m U^{n_j}(p_j) > 1 - \varepsilon/2$. Let $\mathcal{V} = \cap_{j=1}^m \mathcal{V}^{n_j}(p_j)$. Then if $\widetilde{F} \in \mathcal{V}$, $d(\widetilde{F}, B^k) \geqslant 1 - \varepsilon$. If $F^{(n)} \to F$ in the C^2 topology then $F^{(n)} \in \mathcal{V}$ for n large enough and the set of parameters for $F^{(n)}$ which are either regular or satisfy the Tsujii conditions and Weak Regularity have measure at least $1 - \varepsilon$, as required.

Moreover, considering the sequence $F^{(n)} \equiv F$, we conclude that almost every parameter for F is Weakly Regular, hence the last claim of Theorem B.

5.5. Proof of Theorem C (Smooth Dichotomy). — By Proposition 4.3 non-trivial analytic families are dense among C^n families of S-unimodal maps, $n = 3, \dots, \infty$. Theorem B implies that for all ε the set D_ε of C^n families of S-unimodal maps for which the set of bad parameters (not regular or BCE, CE, SE and WR) has measure less then ε, contains a neighborhood of all non-trivial analytic families, that is, an open and dense set. Therefore $\cap D_{1/2^n}$ is a residual set. Clearly any family in $\cap D_{1/2^n}$ satisfies the stated dichotomy.

Appendix
Quasisymmetric robustness of Collet-Eckmann and polynomial recurrence

The aim of this Appendix is to sketch a proof of Theorem 4.6. This proof is similar in strategy to the one of the main results of [**AM1**], however non-trivial modifications are needed. To avoid too much intersection, this will be a concise exposition concentrated mainly on the new steps needed for this improvement: the reader can find a full proof of this result in [**AM3**].

A.1. Quasisymmetric maps

A.1.1. Quasisymmetric reparametrization. — Let now H be an arbitrary but fixed $\widehat{\gamma}$-quasisymmetric map from B^1 to the parameter space of the quadratic family. To prove Theorem 4.6, it will be enough to show that almost every $t \in B^1$ correspond under H to a parameter of the quadratic family which is either regular or satisfies the Collet-Eckmann and Polynomial Recurrence conditions.

From now on, all mentions to parameter space will (unless explicitly stated otherwise) refer to the above reparametrization.

A.1.2. Quasisymmetric capacities. — The γ-capacity of a set $X \subset \mathbb{R}$ in an interval I is defined as follows:

$$p_\gamma(X|I) = \sup \frac{|h(X \cap I)|}{|h(I)|}$$

where the supremum is taken over all γ-qs maps $h : \mathbb{R} \to \mathbb{R}$.

Notice that if I^j are disjoint subintervals of I and $X \subset \cup I^j$ then

$$p_\gamma(X|I) \leqslant p_\gamma(\cup_j I^j|I) \sup_j p_\gamma(X|I^j).$$

A.2. Sequence of first return maps. — The statistical analysis of [**AM1**] concerns mainly the following objects: we are given a unimodal map (which we will assume finitely renormalizable and with a recurrent critical point) $f : I \to I$ and a sequence of nested intervals $I_n \subset I$. The inductive relation between the I_n is as follows: the domain of the first return map R_n to I_n consists of countably many intervals $\{I_n^j\}_{j\in\mathbb{Z}}$, with the convention that $0 \in I_n^0$ (the central component), and we let $I_n^0 = I_{n+1}$.

The special sequence of intervals I_n that we consider is called the principal nest, see [**L2**]. Since we assume f to be finitely renormalizable, there exists a smallest symmetric interval $T \subset I$ which is periodic (say, of period m). For the principal nest, $I_1 = [-p, p]$, where p is the orientation reversing fixed point of $f^m : T \to T$. A level n of the principal nest is called central if $R_n(0) \in I_{n+1}$. Let us say that f is a *simple map* if its principal nest has at most finitely many central levels.

Each non-central branch of R_n is a diffeomorphism onto I_n. Let us introduce some convenient notation related to the iteration of the non-central branches of R_n. Let Ω

be the set of finite sequences of non-zero integers (the empty sequence is included), an element of Ω is denoted $\underline{d} = (j_1, \dots, j_m)$. If $\underline{d} \in \Omega$ has length $|\underline{d}| = m$, we denote $R_n^{\underline{d}}$ the branch of $R_n^{|\underline{d}|}$ with combinatorics $\underline{d}$, that is, the domain of $R_n^{\underline{d}}$ is the set

$$I_n^{\underline{d}} = \{x \in I | R_n^{k-1}(x) \in I_n^{j_k}, 1 \leqslant k \leqslant m\}.$$

We let $C_n^{\underline{d}} = (R_n^{\underline{d}})^{-1}(I_{n+1})$.

Let us denote by L_n the first landing map from I_n to I_{n+1}. This map relates easily to R_n using the above description: the domain of L_n is $\cup C_n^{\underline{d}}$, and $L_n|C_n^{\underline{d}} = R_n^{\underline{d}}$. The reader should think of L_n as a high iterate of R_n. This leads to the following inductive relation between return maps: $R_{n+1} = L_n \circ R_n | I_{n+1}$.

The return time of a point x belonging to an interval I_n^j is denoted by $r_n(x)$ (or $r_n(j)$, since it does not depend on $x \in I_n^j$), that is, $R_n|I_n^j = f^{r_n(j)}$. The landing time is denoted by $l_n(x) \equiv l_n(j)$. The combinatorics at level n of a point x is denoted $\underline{d}^{(n)}(x)$, so that $x \in C_n^{\underline{d}^{(n)}(x)}$. Let $j^{(n)}(x)$ be such that $x \in I_n^{j^{(n)}(x)}$. We let $\tau_n = j^{(n)}(R_n(0))$, so that $R_n(0) \in I_n^{\tau_n}$. The return time of the critical point is denoted $v_n = r_n(0)$. Let $s_n = |\underline{d}^{(n)}(R_n(0))|$.

Notice that $I_{n+1} = R_{n-1}^{-1}(C_{n-1}^{\underline{d}})$ for some $\underline{d}$. The interval $\widetilde{I}_{n+1} = R_{n-1}^{-1}(I_{n-1}^{\underline{d}}) \subset I_n$ is a big neighborhood of I_{n+1} which will be useful later. This choice of neighborhood is particularly good for simple maps, and it turns out that in this case $\widetilde{I}_{n+1}$ is still much smaller than I_n for big n.

A.2.1. Phase-parameter relation. — The starting point of [**AM1**] are two theorems of Lyubich describing the (unreparametrized) parameter space of the quadratic family: infinitely renormalizable maps have zero Lebesgue measure [**L4**] and almost every finitely renormalizable non-regular map is simple [**L3**]. We will need the following remark of [**ALM**]: Lyubich's proof actually allows one to conclude that the set of regular or simple maps has full measure after any quasisymmetric reparametrization.

In view of those results, Theorem 4.6 is reduced to proving that the set of parameters which are Collet-Eckmann and polynomially recurrent have full measure (*after reparametrization by H*) among simple maps. From now on we exclude non-simple maps from measure-theoretic considerations, and we will use "with total probability" to refer to a set of parameters with full measure (*after reparametrization by H*) among simple maps.

To estimate the probability in the parameter corresponding to a certain behavior of the n-th stage of the principal nest, we make use of the Phase-Parameter Lemmas of [**AM1**]. They describe how the partition of the phase space induced by return and landing maps R_n and L_n induce parameter partitions of certain parameter windows J_n.

The topological part of the phase-parameter relation is described in the following:

Theorem A.1. — *For each non-renormalizable quadratic map f with a recurrent critical point, there exists a sequence of parameter intervals $\{J_n\}$ such that:*

(1) *J_n is the maximal interval containing f such that for all $g \in J_n$, there exists a continuation $I_{n+1}[g]$ of I_{n+1} with the "same combinatorics" in the following sense. There exists a continuous family of homeomorphisms $h_n[g] : I \to I$, $g \in J_n$ which is equivariant with respect to the actions of $g|(I \setminus I_{n+1}[g])$ and $f|(I \setminus I_{n+1})$, so that if $x \in I \setminus I_{n+1}[f]$ then $g \circ h_n[g](x) = h_n[g] \circ f(x)$.*

(2) *There exists a homeomorphism $\Xi_n : I_n \to J_n$ such that $\Xi_n(C^{\underline{d}}_n)$ is the set of all $g \in J_n$ such that $R_n[g](0) \in h_n[g](C^{\underline{d}}_n)$.*

This result follows immediately from the Topological Phase-Parameter relation for the unreparametrized quadratic family (Theorem 2.2 of [**AM1**]), since the reparametrization is a homeomorphism.

In words, the sequence J_n in Theorem A.1 denotes the maximal interval containing f where we can consider a continuation of I_n (recall that the boundary of I_n is preperiodic), and such that the first return map to this continuation does not change combinatorics, so that its domain changes continuously. When we change the map g inside the interval J_n, the critical value of $R_n[g]$ varies inside the interval $I_n[g]$ "properly", that is, moves from one boundary point to the other. In doing so, it goes through the partition induced by the $C^{\underline{d}}_n$ in a well behaved ("monotonic") way: it goes through each member of the partition exactly once, and thus defines a partition in the parameter interval J_n, corresponding topologically to the partition in the phase interval I_n. Theorem A.1 thus establishes that the "diagonal" motion of the critical value and the "horizontal" motion of the partition of the phase space are "transversal". This is indeed how the proof of Lyubich goes (using complex analysis). This result can also be established using the Milnor-Thurston's combinatorial theory of unimodal maps together with the monotonicity property of the quadratic family.

The next component of the phase-parameter relation is a quantitative estimate on the regularity of the phase-parameter homeomorphisms Ξ_n. While the topological part is based on a very general transversality argument, the quantitative part depends on the delicate geometric estimates of Lyubich.

We let $J^{\tau_n}_n = \Xi_n(I^{\tau_n}_n)$. The correspondence Ξ_n is uniquely defined if restricted to $K_n = I_n \setminus \cup C^{\underline{d}}_n$. More importantly, it is quasisymmetric if restricted to certain subsets of K_n. To make this precise, let $K^\tau_n = K_n \cap I^{\tau_n}_n$ (forgetting information outside $I^{\tau_n}_n$) and $\widetilde{K}_n = I_n \setminus (\cup I^j_n \cup \widetilde{I}_{n+1})$ (forgetting information inside each I^j_n and also inside $\widetilde{I}_{n+1}$).

Theorem A.2. — *Let f be a simple map. Then, for all $\gamma = (1+\delta)\widehat{\gamma} > \widehat{\gamma}$, there exist $n_0 > 0$ such that for all $n > n_0$,*

PhPa1: $\Xi_n|K^\tau_n$ *is γ-qs;*
PhPa2: $\Xi_n|\widetilde{K}_n$ *is γ-qs;*
PhPh1: $h_n[g]|K_n$ *is $1+\delta$-qs for all $g \in J^{\tau_n}_n$;*
PhPh2: $h_n[g]|\widetilde{K}_n$ *is $1+\delta$-qs for all $g \in J_n$.*

This theorem is a straightforward consequence of the Phase-Parameter relation for the unreparametrized quadratic family (Theorem 2.3 of [**AM1**]). While in [**AM1**] the quasisymmetric constants in PhPa1 and PhPa2 could be taken arbitrarily close to 1 (the unreparametrized case corresponds to taking $H = \mathrm{id}$, that is, $\widehat{\gamma} = 1$) for deeper levels of the principal nest, this does not hold here due to introduction of reparametrization, which multiply all phase-parameter constants by $\widehat{\gamma}$ (notice that PhPh1 and PhPh2 are estimates which do not depend on reparametrization, so we can still choose constants close to 1). This will be the source of many difficulties addressed in this Appendix.

A.3. The statistical argument. — For the remaining of this Appendix we fix some constant $\gamma > \widehat{\gamma}$, and we will start our consideration with levels of the principal nest where the reparametrized phase-parameter relation is already γ-qs. We will also need some very large constants $\widetilde{b} < b$ which depend only on γ (the relation can be computed explicitly following the proof, in particular, $\widetilde{b}$ should be at least so big that $\widetilde{b}^{-1}$ is a lower bound on the Hölder constant of γ-qs maps). We let $a = b^{-1}$ and $\widetilde{a} = \widetilde{b}^{-1}$.

From now on we will always estimate the γ-capacity of bad sets in the phase space. To conclude results for the parameter we will use the following variation of the Borel-Cantelli Lemma (this is Lemma 3.1 of [**AM1**]).

Lemma A.3. — *Let $X \subset \mathbb{R}$ be a measurable set such that for each $x \in X$ there is a sequence $D_n(x)$ of nested intervals converging to x such that for all $x_1, x_2 \in X$ and any n, $D_n(x_1)$ is either equal or disjoint to $D_n(x_2)$. Let Q_n be measurable subsets of $\mathbb{R}$ and $q_n(x) = |Q_n \cap D_n(x)|/|D_n(x)|$. Let Y be the set of x in X which belong to finitely many Q_n. If $\sum q_n(x)$ is finite for almost any $x \in X$ then $|Y| = |X|$.*

In practice, the D_n will be the parameter windows defined before (either J_n or $J_n^{\tau_n}$), and Q_n will be certain subsets of J_n or $J_n^{\tau_n}$ corresponding (under the phase-parameter map) to branches of the return map (in the case of J_n) or landings (in the case of $J_n^{\tau_n}$), whose behavior we want to avoid. We will then show that such bad events have summable γ-capacity in the phase space, which will yield the conclusion for Lebesgue measure of the parameter using PhPa1 (for landings) or PhPa2 (for returns).

A.3.1. A simple application: torrential decay of geometry. — We will now illustrate the use of Lemma A.3 and the phase-parameter relation with an estimate on the decay of geometry. More precisely, we will consider the *scaling factor*

$$c_n = \frac{|I_{n+1}|}{|I_n|}.$$

The scaling factor is a particularly important parameter in the subsequent analysis: all statistical estimates that follow will be related to c_n.

One initial information on the scaling factors is provided by the following result of Lyubich:

***Theorem A.4* (see [L1])**. — *If f is simple than there exists $C > 0$, $\lambda < 1$ such that $c_n < C\lambda^n$.*

We will now show that, with total probability, the decay of c_n is much faster than exponential. To express this decay, let us consider the tower function defined by recursion $T(1) = 2$, $T(n+1) = 2^{T(n)}$. We will show that, with total probability, the c_n decrease torrentially to 0, that is, there exists $k > 0$ such that $c_n^{-1} > T(n-k)$ for n big enough. More precisely, we will show that c_{n+1}^{-1} behaves as an exponential of (a bounded power of) c_n^{-1}.

This very fast decay implies that the landing map to I_{n+1} is essentially a very high iterate of the return map to I_n (since it takes a long time to hit a very small interval). This very high iteration time will allow us to conclude that the characteristics (say, return time) of each level tend to be better behaved than in the previous one due to fast convergence to some average (some kind of Law of Large Numbers). The fact that we must deal with qs-capacity instead of Lebesgue measure will essentially reflect in the presence of errors terms (whose size depend on $\widehat{\gamma}$) in certain exponents in the above description.

In order to estimate c_n, we first consider the related quantity $s_n = |\underline{d}^{(n)}(R_n(0))|$, which denotes the number of times the critical orbit visits I_n before hitting I_{n+1}.

If the critical orbit behaved as a sequence of random points (uniformly distributed with respect to Lebesgue), the expectation of this first hitting time should be c_n^{-1}. More relevant for us, the distribution of the first hitting time (for the random model) should be concentrated about c_n^{-1}: with large probability (say, less than 2^{-n}), the first hitting time is in some "neighborhood" of c_n^{-1} (say, $[4^{-n}c_n^{-1}, 4^n c_n^{-1}]$). The corresponding statement for our actual dynamical system is that the distribution of $|\underline{d}^{(n)}(x)|$, with respect to Lebesgue measure on $x \in I_n$ is concentrated around c_n^{-1}, which can be easily checked by the reader: the estimates are not significantly affected in the non-random case.

However, due to the nature of the phase-parameter relation, we must estimate the distribution of $|\underline{d}^{(n)}(x)|$ in terms of capacities. This will affect drastically the estimates. To understand why, keep in mind that γ-qs maps are only Hölder (with some constant bounded from below by $\widetilde{b}^{-1}$), so they can potentially distort the logarithm of the ratio between I_{n+1} and I_n by such a constant. Aside from this problem, the information we need can be computed quite easily and is summarized below.

Lemma A.5. — *With total probability, for all n sufficiently big we have*

(1) $$p_{2\gamma}(|\underline{d}^{(n)}(x)| \leqslant k | I_n) < k c_n^{\tilde{a}},$$

(2) $$p_{2\gamma}(|\underline{d}^{(n)}(x)| \geqslant k | I_n) < e^{-k c_n^{\tilde{b}}}.$$

We also have

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \leqslant k|I_n^{\tau_n}) < kc_n^{\tilde{a}}, \tag{3}$$

$$p_{2\gamma}(|\underline{d}^{(n)}(x)| \geqslant k|I_n^{\tau_n}) < e^{-kc_n^{\tilde{b}}}. \tag{4}$$

This lemma corresponds to Lemma 4.2 of [**AM1**].

The phase-parameter lemmas (specially PhPa1) allow us to transfer the last pair of estimates to the parameter space: for n sufficiently big, (Lebesgue) most parameters in $J_n^{\tau_n}$ satisfy

$$c_n^{-\tilde{a}/2} < s_n < c_n^{-2\tilde{b}}.$$

Here 'most' means that the complement has probability bounded by $c_n^{\tilde{a}/3}$. But c_n (and thus $c_n^{\tilde{a}/3}$) decays exponentially for every simple map (by Theorem A.4). So $\sum c_n^{\tilde{a}/3} < \infty$ and we are able to apply Lemma A.3 to obtain the following:

Lemma A.6. — *With total probability, for n sufficiently big we have*

$$c_n^{-\tilde{a}/2} < s_n < c_n^{-2\tilde{b}}.$$

This lemma corresponds to Lemma 4.3 of [**AM1**].

Remark A.1. — This result implies easily torrential decay of c_n: $\ln c_{n+1}^{-1}$ can be easily bounded from below by Ks_n for some universal $K > 0$, and thus for big n,

$$c_{n+1}^{-1} \geqslant e^{c_n^{-\tilde{a}/3}}.$$

A.4. Derivatives. — We proceed to estimate derivatives of branches of the return map. All lemmas in this section can be proved using the same argument as in [**AM1**].

The first step is to exclude the possibility of a 'too recurrent' or 'too low' return. It is analogous to Lemma 4.8 of [**AM1**], being a simple application of PhPa2.

Lemma A.7. — *With total probability, the distance between $R_n(0)$ and $\partial I_n \cup \{0\}$ is at least $|I_n|n^{-\tilde{b}}$. In particular $R_n(0) \notin \tilde{I}_{n+1}$ for all n large enough.*

Recall that the *distortion* of a diffeomorphism ϕ on an interval T is defined by

$$\operatorname{Dist}(\phi|T) = \frac{\sup_T |D\phi|}{\inf_T |D\phi|}.$$

Lemma A.7 allows us to start estimating the distortion of iterates of f. The following estimate corresponds to Lemma 4.9 of [**AM1**]. It is based on the fact that the distortion of branches of return maps is due to the position of the branch with respect to the critical point. Using PhPa1, we are able to give polynomial lower bounds on the distance between the critical point with respect to non-central branches, which are valid with total probability.

Lemma A.8. — *With total probability, for n big enough and $j \neq 0$*

$$\operatorname{Dist}(f|I_n^j) \leqslant n^{\tilde{b}}.$$

The following estimate is analogous to Lemma 4.10 of [**AM1**]. It is based on the previous one and the observation that return branches are torrentially expansive in average (from the decay of geometry).

Lemma A.9. — *With total probability, for n big enough and for all $\underline{d} \in \Omega$*

$$\operatorname{Dist}(R_n^{\underline{d}}) \leqslant n^{\tilde{b}}.$$

In particular, for n big enough, $|DR_n(x)| > 2$ if $x \in \cup_{j \neq 0} I_n^j$.

Lemma A.9 gives estimates of derivatives under iterates of R_n. To obtain estimates of derivatives under iterates of f, we will need the following very general result of Guckenheimer which shows that quadratic maps are hyperbolic away from critical points and parabolic points (this actually generalizes to very general one-dimensional systems by a result of Mañé), see [**MvS**]. We state just a consequence adapted to our particular setting.

Theorem A.10. — *Let f be a quadratic map without non-repelling periodic orbits (in particular if f is a simple map). For every $\varepsilon > 0$, there exists $C > 0$, $\lambda > 1$ such that if $|f^k(x)| > \varepsilon$ for $0 \leqslant k \leqslant m$ then $Df^{m+1}(x) > C\lambda^m$.*

With this information we are now able to give a lower bound on the derivative of iterates of f. The next lemma is identical to Lemma 4.11 of [**AM1**], and is based on the idea that full returns to sufficiently deep levels cause expansion (from the previous lemma), while the dynamics outside a definite neighborhood of the critical point is hyperbolic (by Theorem A.10).

Lemma A.11. — *With total probability, if n is sufficiently big and if $x \in I_n^j$, $j \neq 0$, and $R_n|I_n^j = f^r$, then for $1 \leqslant k \leqslant r$, $|Df^k(x)| > |x| c_{n-1}^3$.*

A.5. How to deal with hyperbolicity. — Keeping in mind that our analysis of the statistical properties of the dynamics of f is made in terms of the induced return maps R_n, we see that in order to estimate the hyperbolicity along the critical orbit (to obtain the Collet-Eckmann condition) we must have a convenient way to quantify the hyperbolicity of (for instance) non-central return branches. To do so, for $j \neq 0$, we define the quantity

$$\lambda_n(j) = \inf_{x \in I_n^j} \frac{\ln |DR_n(x)|}{r_n(j)}.$$

We let $\lambda_n = \inf_{j \neq 0} \lambda_n(j)$.

To analyze the behavior of λ_n, we start with the general information provided by Theorem A.10. Coupled with exponential upper bounds on distortion for returns (which competes with torrential expansion of each non-central branch from the decay of c_n), the hyperbolicity of f in the complement of I_{n+1} immediately implies the following estimate (identical to Lemma 7.9 of [**AM1**]).

Lemma A.12. — *With total probability, for all n sufficiently big, $\lambda_n > 0$.*

The "minimum hyperbolicity" $\liminf \lambda_n$ of the parameters we will obtain will in fact be positive, as it follows from one of the properties of Collet-Eckmann parameters (uniform hyperbolicity on periodic orbits), together with our estimates on distortion.

Our strategy however is not to show that the minimum hyperbolicity is positive, but that the typical value of $\lambda_n(j)$ stays big as n grows (and is in fact bigger than $\lambda_{n_0}/2$ for $n > n_0$ big). In this sense, it is convenient to think of $\lambda_n(j)$ as a random variable whose distribution we are interested in.

There is an inductive relation between the random variables $\lambda_n(j)$ for different values of n: this is related to the fact that if $R_n(I^j_{n+1}) \subset C^{\underline{d}}_n$, $\underline{d} = (j_1, \ldots, j_m)$, we have $R_{n+1}|I^j_{n+1} = L_n|C^{\underline{d}}_n \circ R_n|I^j_{n+1}$. The hyperbolicity of the "landing part" $L_n|C^{\underline{d}}_n$ is essentially a weighted sum

$$\frac{\sum_{i=1}^m \lambda_n(j_i) r_n(j_i)}{\sum_{i=1}^m r_n(j_i)}. \tag{A.1}$$

So if the "return part" $R_n|I^j_{n+1}$ does not carry a big weight on the computation of $\lambda_{n+1}(j)$ (outside a set of branches with small γ-qs capacity), we can think of $\lambda_{n+1}(j)$ as distributed according to the weighted sum (A.1). This turns out to be the case as the return part does not affect much the denominator (time) and does not have a bad effect on the numerator (derivative). Indeed, in the next section we will see that the return time of $R_n|I^j_{n+1}$ (given by v_n) is much smaller (of order c^{-1}_{n-1}) than the total return time $R_{n+1}|I^j_{n+1}$ (of order c^{-1}_n). Moreover, if I^j_{n+1} is outside a small neighborhood of 0, $|DR_n|I^j_{n+1}|$ is bigger than 1.

Since we also have to estimate the hyperbolicity of truncated branches (as the Collet-Eckmann condition is a condition along the full critical orbit, and not only at full returns), it will not be enough to just obtain that the distribution of $\lambda_n(j)$ is concentrated around some value bigger than $\lambda_{n_0}/2$. In order to state exactly what kind of hyperbolicity estimate we need, it is convenient to introduce a certain class of branches: good returns.

We define the set of good returns $G(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0, n \in \mathbb{N}$, $n \geqslant n_0$ as the set of all j such that

G1: (hyperbolic return)

$$\lambda_n(j) \geqslant \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2},$$

G2: (hyperbolicity in truncated return) for $c^{-3/(n-1)}_{n-1} \leqslant k \leqslant r_n(j)$ we have

$$\inf_{I^j_n} \frac{\ln |Df^k|}{k} \geqslant \lambda_{n_0} \frac{1 + 2^{n_0 - n + 1/2}}{2} - c^{2/(n-1)}_{n-1}.$$

Of course we still have to show that the set of returns which fail to be good has small γ-qs capacity. In order to do so, we will construct explicitly a class of branches whose complement has small γ-qs capacity and then show that this class of branches is contained in good branches (see Lemma A.20). Before doing so, we must first estimate the distribution of return times, since they have an important role in the computation of $\lambda_n(j)$.

A.6. Distribution of return and landing times. — To estimate the distribution of return and landing times, it is convenient to also think of $r_n(j)$ and $l_n(j)$ as "random variables" which are related by some simple rules: if $\underline{d} = (j_1, \ldots, j_m)$ then $l_n(\underline{d}) = \sum_{i=1}^{m} r_n(j_i)$ and $r_{n+1}(j) = v_n + l_n(\underline{d})$ where $R_n(I_{n+1}^j) \subset C_n^{\underline{d}}$. In particular, since the distribution of $|\underline{d}^{(n)}|$ is concentrated around c_n^{-1} which is torrentially big, the random variable l_n behaves like a very large sum of random variables distributed as r_n. On the other hand, r_{n+1} should have distribution approximately like l_n itself, once we show that v_n does not make an important contribution.

The main tool to do the actual analysis is to prove first a Large Deviation Estimate for r_n using only the torrential decay of c_n, and then show that such estimate leads to much more precise control of the subsequent levels.

Since the transition between different levels introduces some distortion (although torrentially small), we are forced to deal with a sequence of quasisymmetric constants in our estimates: instead of just estimating γ-qs capacities for some fixed γ, we must consider a sequence $\gamma_n = \gamma(n+1)/n$ and $\widetilde{\gamma}_n = \gamma(2n+3)/(2n+1)$. The basic idea is that control of the distribution of r_n with respect to γ_n-capacities will provide control of the distribution of l_n with respect to $\widetilde{\gamma}_n$ capacities which in turn will allow to estimate the distribution of r_{n+1} with respect to γ_{n+1} capacities. Notice that $\inf \gamma_n = \inf \widetilde{\gamma}_n = \gamma$. (This ideas are introduced in § 5 of [**AM1**].)

Although very technical, this part is very similar to the analysis made on (the several lemmas of) § 6 of [**AM1**] (differing only by change of constants), so we will only state the final estimate which summarizes the results of that section and provide a short outline of the argument.

Lemma A.13. — *With total probability, for all n sufficiently large we have*

(1) $p_{\widetilde{\gamma}_n}(l_n(x) < c_n^{-s} | I_n) < c_n^{\tilde{a}^2 - s} < c_n^{a-s}$, *with* $s > 0$,
(2) $p_{\widetilde{\gamma}_n}(l_n(x) < c_n^{-s} | I_n^{\tau_n}) < c_n^{a-s}$, *with* $s > 0$,
(3) $p_{\widetilde{\gamma}_n}(l_n(x) > c_n^{-s} | I_n) < e^{-c_n^{b-s}}$, *with* $s > b$,
(4) $p_{\widetilde{\gamma}_n}(l_n(x) > c_n^{-s} | I_n^{\tau_n}) < e^{-c_n^{b-s}}$, *with* $s > b$,
(5) $p_{\gamma_n}(r_n(x) < c_{n-1}^{-s} | I_n) < c_{n-1}^{\tilde{a}^2 - s} < c_{n-1}^{a-s}$, *with* $s > 0$,
(6) $p_{\gamma_n}(r_n(x) > c_{n-1}^{-s} | I_n) < e^{-c_{n-1}^{\sqrt{b}-s}} < e^{-c_{n-1}^{b-s}}$ *with* $s > b$.
(7) $c_{n-1}^{-a} < r_n(\tau_n) < c_{n-1}^{-b}$.
(8) $c_{n-1}^{-a} < v_n < c_{n-1}^{-b}$.
(9) $c_{n-1}^{-a} < \ln(c_n^{-1}) < c_{n-1}^{-b}$.

A.6.1. Outline of the proof of Lemma A.13. — The estimates from below are relatively easy. Estimates (1) and (2) follow directly from $l_n(\underline{d}) \geqslant |\underline{d}|$ and Lemma A.5. Estimate (5) follows from (1) using the relation between r_{n+1} and l_n. The estimate from below in (8) follows from (2) and PhPa1, and the estimate from below in (7) follows from (5) and PhPa2. The estimate from below on (9) was computed on Remark A.1.

The estimates from above are much more delicate. In what follows we will ignore the difference between I_n and $I_n^{\tau_n}$, since it is not substantial for the argument. The key estimate is (6), which says that the tail $p_{\gamma_n}(r_n(x) > k)$ decays exponentially fast (in k) with some specific rate (polynomial in c_{n-1}). On the other hand, decay with *some* rate is easy: f is hyperbolic outside I_{n+1} (see Theorem A.10), so there exists some (small) $\alpha_n > 0$ with $p_{\gamma_n}(r_n(x) > k\alpha_n^{-1}) < e^{-k}$ for $k \geqslant 1$. This exponential decay implies that it is very unlikely that a large sequence $\underline{d} = (j_1, \dots, j_m)$ will have a landing time $l_n(\underline{d}) = \sum_{i=1}^m r_n(j_i)$ much bigger than $m\alpha_n^{-1}$.

From this relation between r_n and l_n, we see that there exists some β_n with $p_{\tilde{\gamma}_n}(l_n(x) > k\beta_n^{-1}) < e^{-k}$, and moreover we can estimate β_n in terms of α_n and the size of a typical $\underline{d}^{(n)}$ (which is given by a polynomial on c_n^{-1}): β_n^{-1} is bounded by a polynomial (this polynomial error is related to γ) on $\alpha_n^{-1}c_n^{-1}$. From the relation between l_n and r_{n+1} we obtain an estimate on α_{n+1} in terms of v_n and β_n, which we can rewrite in terms of v_n, c_n and α_n: $\alpha_{n+1}^{-1} - v_n$ is bounded by some polynomial on $\alpha_n^{-1}c_n^{-1}$.

Since $p_{\tilde{\gamma}_n}(l_n(x) > \beta_n^{-1}c_n^{-1})$ is summable (by definition of β_n), it follows that $v_{n+1} - v_n$ is bounded by a polynomial on $\alpha_n^{-1}c_n^{-1}$ with total probability (use PhPa1), in particular, for n big we can bound v_{n+1} with a polynomial on $\alpha_n^{-1}c_n^{-1}$.

In particular, if $\alpha_n^{-1} > c_n^{-1}$, α_{n+1}^{-1} is bounded by a *polynomial* in α_n^{-1}. Although initially we did not have any control on the value of α_n, we know that c_{n+1}^{-1} behaves as an *exponential* on c_n^{-1} (torrential growth), so eventually it catches up with α_n^{-1}: for n big, $c_n^{-1} > \alpha_n^{-1}$.

So for n big α_n^{-1} can be bounded exclusively by a polynomial on c_{n-1}^{-1} as stated in (6). This automatically implies the estimate from above in (7) using PhPa2. Since β_n^{-1} and v_{n+1} are bounded by a polynomial on $\alpha_n^{-1}c_n^{-1}$ we obtain (3) and (4) and the estimate from above in (8).

Since f^{v_n} expands I_{n+1} to an interval of size at least $2^{-n}|I_n|$, and the derivative of f is bounded by 4, we have $2^n c_n^{-1} < 4^{v_n}$, so the estimate from above on (9) follows from the estimate from above in (8).

A.7. Constructing hyperbolic branches. — In this section we show by an inductive process that the great majority of branches are reasonably hyperbolic (good branches). In order to do that, in the following subsection, we define some classes of branches with 'very good' distribution of times and which are not too close to the critical point. The definition of 'very good' distribution of times has an inductive component: they are composition of many 'very good' branches of the previous level. The fact that most branches are 'very good' is related to the validity of some kind of Law of Large Numbers estimate. The inductive definition will guarantee that the 'very good' distribution of times holds in all scales and allows us to preserve hyperbolicity from one step to the other: very good branches are good.

Remark A.2. — The several classes of branches that we will define do not correspond exactly to the same classes in [**AM1**], although classes with the same name have essentially the same function in the proof. There are some non-trivial steps to make this adaptation work, since the previous proof uses strongly small quasisymmetric constants. This will lead to consideration of extra classes below (bad returns and fast landings).

Remark A.3. — This section contains the main modifications with respect to [**AM1**] (precisely the introduction of bad returns and fast landings). The role of those modifications is explained in Remark A.4.

A.7.1. Standard landings. — Let us define the set of standard landings at time n, $LS(n) \subset \Omega$ as the set of all $\underline{d} = (j_1, \dots, j_m)$ satisfying the following:

LS1: (m is not too small or large) $c_n^{-a/2} < m < c_n^{-2b}$,

LS2: (No very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.

LS3: (Short times are sparse in large enough initial segments) For $c_{n-1}^{-2b} \leqslant k \leqslant m$

$$\#\{1 \leqslant i \leqslant k,\, r_n(j_i) < c_{n-1}^{-a/2}\} < (6 \cdot 2^n) c_{n-1}^{a/2} k.$$

We also define the set of fast landings at time n, $LF(n) \subset \Omega$ by the following conditions

LF1: (m is small) $m < c_n^{-a/2}$.

LS2: (No very large times) $r_n(j_i) < c_{n-1}^{-3b}$ for all i.

It is easy to convince oneself that most landings are standard. Indeed, the distribution of $|\underline{d}^{(n)}(x)|$ is concentrated around c_n^{-1} as requested by LS1. Moreover, branches with very large times (larger than c_{n-1}^{-3b}) are so few that even a long sequence $(j_1, \dots, j_m)$ with $m < c_{n-1}^{-2b}$ is not likely to contain such an event, as required by LS2. Finally, the Law of Large Numbers indicates that a long sequence $(j_1, \dots, j_m)$ will seldom contain a proportion of short times much bigger than their frequency as given by Lemma A.13, as required by LS3.

Since fast landings are not standard, they must be few. However, they correspond to most of the branches which are not standard. The reason for this comes from the requirements of LS1, which imposes two conditions (an upper and a lower bound on m). The upper bound condition is much more rarely violated (by one exponential order of magnitude) than the lower bound (just check Lemma A.5). Fast landings essentially capture the violations of the lower bound (LF1).

The actual estimates for the frequency of standard and fast landings are provided below. They can be obtained from the estimates of distribution of return times (contained in Lemma A.13) following the general lines of Lemma 7.1 of [**AM1**]. This step is purely dynamical (no further parameter exclusion is made).

Lemma A.14. — *With total probability, for all n sufficiently big,*

(1) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n)|I_n) < c_n^{a/3}/2$,
(2) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n)|I_n) < c_n^{n^2}/2$,
(3) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n)|I_n^{\tau_n}) < c_n^{a/3}/2$,
(4) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LS(n) \cup LF(n)|I_n^{\tau_n}) < c_n^{n^2}/2$.

A.7.2. Very good returns, bad returns and excellent landings. — Define the set of very good returns, $VG(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0 \leqslant n \in \mathbb{N}$ and the set of bad returns, $B(n_0, n) \subset \mathbb{Z} \setminus \{0\}$, $n_0 \leqslant n \in \mathbb{N}$, by induction as follows. We let $VG(n_0, n_0) = \mathbb{Z} \setminus \{0\}$, $B(n_0, n_0) = \varnothing$ and supposing $VG(n_0, n)$ and $B(n_0, n)$ defined, define the set of excellent landings $LE(n_0, n) \subset LS(n)$ satisfying the following extra assumptions.

LE1: (Not very good moments are sparse in large enough initial segments) For all $c_{n-1}^{-2b} < k \leqslant m$

$$\#\{1 \leqslant i \leqslant k,\ j_i \notin VG(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{a^2} k,$$

LE2: (Bad moments are sparse in large enough initial segments) For all $c_n^{-1/n} < k \leqslant m$

$$\#\{1 \leqslant i \leqslant k,\ j_i \notin B(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{n} k,$$

We define $VG(n_0, n+1)$ as the set of j such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \in LE(n_0, n)$ and the extra condition:

VG: (distant from 0) The distance of I_{n+1}^j to 0 is bigger than $c_n^{n^2}|I_{n+1}|$.

And we define $B(n_0, n+1)$ as the set of $j \notin VG(n_0, n+1)$ such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ with $\underline{d} \notin LF(n)$.

Very good returns are designed to carry hyperbolicity from level to level: since they are composed of many very good returns of the previous level (LE1), and are not too close to 0 (VG), they should keep most of the hyperbolicity of level n_0 (given by $\lambda_{n_0} > 0$). For this to work, we must control the distribution of return times of the previous level inside a very good branch. The risky situation is the presence of not very good branches which have a large return time: those are contained in the bad branches defined above. It turns out that they can not spoil the hyperbolicity because they are too few (LE2). This basic idea will be carried out in detail through a series of lemmas.

Very good and bad returns can be estimated in an inductive fashion analogously to the estimate of Lemmas 7.2 and 7.3 of [**AM1**]: initially all branches are very good and no branches are bad, and as n grows the Law of Large Numbers indicates that conditions LE1 and LE2 should be rarely violated so that very good branches should continue to be frequent and bad branches rare. This estimate is again purely dynamical.

Lemma A.15. — *With total probability, for all n_0 sufficiently big,*

(1) $p_{\gamma_n}(j^{(n)}(x) \notin VG(n_0, n)|I_n) < c_{n-1}^{a^2}$,

(2) $p_{\gamma_n}(j^{(n)}(x) \in B(n_0, n)|I_n) < 2c_{n-1}^{2n}$,
(3) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n)|I_n) < c_n^{2a/5}$,
(4) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n) \cup LF(n)|I_n) < c_n^{bn}$,
(5) $p_{\tilde{\gamma}_n}(\underline{d}^{(n)}(x) \notin LE(n_0, n)|I_n^{\tau_n}) < c_n^{2a/5}$.

This translates immediately using PhPa2 to a parameter estimate analogous to Lemma 7.4 of [**AM1**]:

Lemma A.16. — *With total probability, for all n_0 big enough, for all n big enough (depending on n_0), $\tau_n \in VG(n_0, n)$.*

Before going on we will need two simple estimates: one is for the return time of very good branches and another is for the return time of branches which are neither very good or bad. The first of those estimates is analogous to Lemma 7.5 of [**AM1**], and follows directly from the definitions of very good and bad branches.

Lemma A.17. — *With total probability, for all n_0 big enough and for all $n \geqslant n_0$, if $j \in VG(n_0, n+1)$ then*

$$m < r_{n+1}(j) < mc_{n-1}^{-4b},$$

where, as usual, m is such that $R_n(I_{n+1}^j) = C_n^{\underline{d}}$ and $\underline{d} = (j_1, \dots, j_m)$.

Lemma A.18. — *With total probability for all n_0 sufficiently big, if $n > n_0$, if $j \notin VG(n_0, n) \cup B(n_0, n)$ then $r_n(j) < c_{n-1}^{-a/2} c_{n-2}^{-4b}$.*

Proof. — Indeed, if $j \notin VG(n_0, n) \cup B(n_0, n)$ then $R_{n-1}(I_n^j) \subset C_{n-1}^{\underline{d}}$ with $\underline{d} \in LF(n-1)$. By definition of fast landing, $l_{n-1}(\underline{d}) < c_{n-1}^{-a/2} c_{n-2}^{-3b}$, so

$$r_n(j) = v_{n-1} + l_{n-1}(\underline{d}) < c_{n-1}^{-a/2} c_{n-2}^{-3b} + c_{n-2}^{-b}. \qquad \square$$

At this stage we have most of the tools to show that almost every parameter is "Collet-Eckmann at first returns", that is, $|Df^{k_n}(f(0))|$ is exponentially big for the sequence k_n of first landings of $f(0)$ in I_n. To obtain the full Collet-Eckmann condition (exponential growth for all k), we will need to analyze truncations of branches or landings, that is, we will consider iterates of the type $f^k|I_n^j$ (or $f^k|C_n^{\underline{d}}$) for k less then the return time $r_n(j)$ (or $l_n(\underline{d})$).

We now show that very good branches are well behaved when truncated at a reasonably big time. Here "well behaved" means "spending most of the time in very good branches of the previous level". So if we are able to control the hyperbolicity of very good branches in some level we will have a good possibility of controlling truncated very good branches in the next level. This lemma corresponds to Lemma 7.6 of [**AM1**], but the proof must be modified, with the use of bad returns and fast landings.

Lemma A.19. — *With total probability, for all n_0 big enough and for all $n \geqslant n_0$, the following holds.*

Let $j \in VG(n_0, n+1)$, as usual let $R_n(I^j_{n+1}) \subset C^{\underline{d}}_n$ and $\underline{d} = (j_1, \dots, j_m)$. Let m_k be biggest possible with

$$v_n + \sum_{j=1}^{m_k} r_n(j_i) \leqslant k$$

(the amount of full returns to level n before time k) and let

$$\beta_k = \sum_{\substack{1 \leqslant i \leqslant m_k, \\ j_i \in VG(n_0, n)}} r_n(j_i).$$

(the total time spent in full returns to level n which are very good before time k) Then $1 - \beta_k/k < c_{n-1}^{a^2/3}$ if $k > c_n^{-2/n}$.

Proof. — Let us estimate first the time i_k which is not spent on non-critical full returns:

$$i_k = k - \sum_{j=1}^{m_k} r_n(j_i).$$

This corresponds exactly to v_n plus some incomplete part of the return $j_{m_{k+1}}$. This part can be bounded by $c_{n-1}^{-b} + c_{n-1}^{-3b}$ (use the estimate of v_n and LS2 to estimate the incomplete part).

Using LS2 we conclude now that

$$m_k > (k - c_{n-1}^{-b}) c_{n-1}^{3b} > c_n^{-1/n}$$

so m_k is not too small.

Let us now estimate the contribution h_k from bad full returns j_i. The number of such returns must be less than $c_{n-1}^{n/2} m_k$ by LE2 and the estimate on m_k. By LS2 their total time is at most $c_{n-1}^{(n/2)-3b} m_k < m_k$.

The non very good full returns on the other hand can be estimated by LE1 (given the estimate on m_k), they are at most $c_{n-1}^{a^2} m_k$. So we can estimate the total time l_k of non very good or bad full returns (with time less then $c_{n-1}^{-a/2} c_{n-2}^{-4b}$ by Lemma A.18) by

$$c_{n-1}^{a^2} c_{n-1}^{-a/2} c_{n-2}^{-4b} m_k,$$

while β_k can be estimated from below by

$$(1 - c_{n-1}^{a/4}) c_{n-1}^{-a/2} m_k.$$

It is easy to see then that $i_k/\beta_k \ll c_{n-1}^{a/5}$, $h_k/\beta_k \ll c_{n-1}^{a/5}$. We also have

$$\frac{l_k}{\beta_k} < 2 c_{n-1}^{a^2/2}.$$

So $(i_k + h_k + l_k)/\beta_k$ is less then $c_{n-1}^{a^2/3}$. Since $i_k + h_k + l_k + \beta_k = k$ we have $1 - \beta_k/k < (i_k + h_k + l_k)/\beta_k$. ☐

Remark A.4. — This lemma illustrates the main reason why the original argument of [**AM1**] must be changed in order to deal with big quasisymmetric constants. Indeed, in [**AM1**], we do not need to split the branches which are not very good in bad branches and otherwise (fast). The reason is that in [**AM1**] the distribution of $r_n(j)$ is concentrated in a much narrower window around c_{n-1}^{-1} (say, $(c_{n-1}^{-1+2\varepsilon}, c_{n-1}^{-1-2\varepsilon})$). In particular, in a large sequence $(j_1, \ldots, j_k)$ (which should be thought as an initial segment of an excellent landing), we can estimate the proportion of the total return time due to very good branches essentially by considering the proportion of very good branches in the sequence.

In this Appendix, the distribution of $r_n(j)$ is located in a much larger window $(c_{n-1}^{-a}, c_{n-1}^{-b})$. The risky situation is to have a large sequence $(j_1, \ldots, j_k)$ with a large proportion of very good branches, but whose return time is near the bottom of the window (c_{n-1}^{-a}), while the not very good branches in the sequence have all return time near the top (c_{n-1}^{-b}). In this case, the proportion of the total time due to very good branches could be very small.

The solution given in this Appendix is based on the idea that the not very good branches *with large time* (bad branches) are really very few: most of the not very good branches are indeed fast. Paying attention to this asymmetry, we can indeed prove that in such a sequence $(j_1, \ldots, j_k)$, most of the total time is due to very good branches.

This argument (most branches with atypical time are fast) is based implicitly in the following asymmetry which appeared already in our first statistical estimate, Lemma A.5, when we showed that the distribution of $|\underline{d}^{(n)}(x)|$ is concentrated around c_n^{-1}: there is a big difference (one extra exponential) in the estimates on the upper tail (γ-qs capacity of $\{|\underline{d}^{(n)}(x)| > c_n^{-k\tilde{b}}\}$) and the lower tail ($\gamma$-qs capacity of $\{|\underline{d}^{(n)}(x) < c_n^{-k\tilde{a}}\}$).

(Essentially the same problem, with the same solution, appears in Lemma A.22.)

Now we conclude that very good (that is, most) branches are good, justifying our previous hints.

Lemma A.20. — *With total probability, for n_0 big enough and for all $n > n_0$, $VG(n_0, n) \subset G(n_0, n)$.*

The proof is the same as for Lemma 7.10 of [**AM1**], the two main features of very good branches exploited here are their good distribution of return times and the condition VG which allows us to avoid drastic losses of derivative due to starting very close to the critical point. The argument is by induction: first, all very good branches of level n_0 satisfy condition G1 of a good branch, that is, a full return is very hyperbolic (this follows from the definition of λ_{n_0}). Then, supposing that all very good branches of level n satisfy G1, we conclude that very good branches of level $n+1$ have enough hyperbolic branches in its composition (even if truncated) to satisfy both conditions G1 and G2.

A.7.3. Cool landings. — As we hinted in the last section, very good branches play the role of building blocks of hyperbolicity. We must now show that the critical point spends most of its time in very good branches. To do so, we will define a class of landings which are composed by many very good branches, but which are controlled to an ever greater detail than excellent landings. Their design will allow to estimate their hyperbolicity if truncated outside a relatively small initial segment.

We define the set of cool landings $LC(n_0, n) \subset \Omega$, $n_0, n \in \mathbb{N}$, $n \geqslant n_0$ as the set of all $\underline{d} = (j_1, \dots, j_m)$ in $LE(n_0, n)$ satisfying

LC1: (Starts very good) $j_i \in VG(n_0, n)$, $1 \leqslant i \leqslant c_{n-1}^{-a^2/2}$.

LC2: (Not very good moments are sparse in large enough initial segments) For all $c_{n-1}^{-a^2/4} < k \leqslant m$

$$\#\{1 \leqslant i \leqslant k,\, r_n(j_i) < c_{n-1}^{-a/2}\} < (6 \cdot 2^n) c_{n-1}^{a/3} k,$$

LC3: (Bad moments are sparse in large enough initial segments) For $c_{n-1}^{-n/3} \leqslant k \leqslant m$

$$\#\{1 \leqslant i \leqslant k,\, j_i \in B(n_0, n)\} < (6 \cdot 2^n) c_{n-1}^{n/6} k,$$

LC4: (Starts with no bad moments) $j_i \notin B(n_0, n)$, $1 \leqslant i \leqslant c_{n-1}^{-n/2}$.

As in Lemma 7.7 of [**AM1**], cool landings are frequent and we get the following parameter estimate analogous to Lemma 7.8 of [**AM1**]. The ideas of this estimate are quite similar to the case of standard landings.

Lemma A.21. — *With total probability, for all n_0 big enough, for all n big enough we have $R_n(0) \in LC(n_0, n)$.*

Let us now show that cool landings inherit hyperbolicity from very good returns. This result corresponds to Lemma 7.11 of [**AM1**], but the proof of this fact needs adjustments for big quasisymmetric constants, so we provide it here.

Lemma A.22. — *With total probability, if n_0 is sufficiently big, for all n sufficiently big, if $\underline{d} \in LC(n_0, n)$ then for all $c_{n-1}^{-4/(n-1)} < k \leqslant l_n(\underline{d})$,*

$$\inf_{C_n^{\underline{d}}} \frac{\ln |Df^k|}{k} \geqslant \frac{\lambda_{n_0}}{2}.$$

Proof. — Fix such $\underline{d} \in LC(n_0, n)$, and let $\underline{d} = (j_1, \dots, j_m)$.

Let

$$a_k = \inf_{C_n^{\underline{d}}} \frac{\ln |Df^k|}{k}.$$

Analogously to Lemma A.19, we define m_k as the number of full returns before k, that is, the biggest integer such that

$$\sum_{i=1}^{m_k} r_n(j_i) \leqslant k.$$

We define

$$\beta_k = \sum_{\substack{1\leqslant i\leqslant m_k,\\ j_i\in VG(n_0,n+1)}} r_n(j_i),$$

(counting the time up to k spent in complete very good returns) and

$$i_k = k - \sum_{i=1}^{m_k} r_n(j_i).$$

(counting the time in the incomplete return at k).

Let us then consider two cases: small m_k ($m_k < c_{n-1}^{-a^2/2}$) and otherwise.

Case 1 ($m_k < c_{n-1}^{-a^2/2}$). The idea of the first case is that all full returns are very good by LC1, and the incomplete time is also part of a very good return.

Since full very good returns are very hyperbolic by G1 and very good returns are good, we just have to worry about possibly losing hyperbolicity in the incomplete time. To control this, we introduce the queue (or tail) $q_k = \inf_{C_n^d} \ln |Df^{i_k} \circ f^{k-i_k}|$. We have $-q_k < -\ln(c_{n-1}^{1/3} c_{n-1}^3)$ by VG and Lemma A.11. Let us split again in two cases: i_k big or otherwise.

Subcase 1a ($i_k > c_{n-1}^{-4/(n-1)}$). If the incomplete time is big, we can use G2 to estimate the hyperbolicity of the incomplete time (which is part of a very good return). The reader can easily check the estimate in this case.

Subcase 1b ($i_k < c_{n-1}^{-4/(n-1)}$). If the incomplete time is not big, we can not use G2 to estimate q_k, but in this case i_k is much less than k: since $k > c_{n-1}^{-4/(n-1)}$, at least one return was completed ($m_k \geqslant 1$), and since it must be very good we conclude that $k > c_{n-1}^{-a/2}$ by LS1, so

$$a_k > \lambda_{n_0} \frac{(1+2^{n_0-n})}{2} \cdot \frac{k-i_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}.$$

Case 2 ($m_k > c_{n-1}^{-a^2/2}$). For an incomplete time we still have $-q_k < -\ln(c_n c_{n-1}^3)$, so $-q_k/k < c_{n-1}^{a^2/3}$.

Arguing as in Lemma A.19, we split $k - \beta_k - i_k$ (time of full returns which are not very good) in part relative to bad returns h_k and in part relative to returns that are not very good or bad (which must be fast) l_k. Using LC3 and LC4 to bound the number of bad returns and LS2 to bound their time, we get

$$h_k < c_{n-1}^{-3b} c_{n-1}^{n/7} m_k,$$

and using LC1 and LC2 we have

$$l_k < c_{n-1}^{-a/2} c_{n-2}^{-4b} (6 \cdot 2^n) c_{n-1}^{a^2} m_k,$$

By LC1 and LC2 again, using LS1 to estimate the time of a very good return by $c_{n-1}^{-a/2}$, we have that $\beta_k > c_{n-1}^{-a/2} m_k/2$, thus we get

$$\frac{h_k + l_k}{\beta_k} < c_{n-1}^{a^2/2}, \tag{A.2}$$

which is very small.

On the other hand, $\beta_k > c_{n-1}^{-a/2} c_{n-1}^{-a^2/2}/2$ by hypothesis on m_k. Let us split in three cases according to the behavior of i_k.

Subcase 2a (i_k not very good or bad). In this case, $i_k < c_{n-1}^{-a/2} c_{n-2}^{-4b}$, so i_k/β_k is very small, and we actually have $1 - \beta_k/k < c_{n-1}^{a^2/10}$. Since very good returns are good and even not very good returns have derivative at least 1,

$$a_k > \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2} \cdot \frac{\beta_k}{k} - \frac{-q_k}{k} > \frac{\lambda_{n_0}}{2}. \tag{A.3}$$

Subcase 2b (i_k very good). If i_k is very good and $i_k > c_{n-1}^{-4/(n-1)}$, we can reason as in Subcase 1a that $G2$ can be used for the estimate of q_k so that we have

$$a_k > \lambda_{n_0} \frac{1 + 2^{n_0 - n}}{2} \cdot \frac{\beta_k}{k} + \frac{i_k}{k} \cdot \frac{\lambda_{n_0}}{2} < \frac{\lambda_{n_0}}{2}$$

by (A.2).

If $i_k \leqslant c_{n-1}^{-4/(n-1)}$, then i_k/β_k is very small and so $1 - \beta_k/k < c_{n-1}^{a^2/10}$, and we obtain (as in Subcase 2a) estimate (A.3).

Subcase 2c (i_k bad). If i_k is bad, by LC4 we have that $m_k > c_{n-1}^{-a/2}$, but $i_k < c_{n-1}^{-3b}$ by LS2, so i_k/β_k is very small again and we have $1 - \beta_k/k < c_{n-1}^{a^2/10}$, so estimate (A.3) applies and we are done. □

A.8. Collet-Eckmann. — Since the critical point always falls in cool landings (see Lemma A.21), the Collet-Eckmann condition follows easily from Lemma A.22 (which guarantees gain of derivative after large truncations), together with Lemma A.11, which controls loss of derivative at small truncations. This argument is identical to the one in § 8.1 of [**AM1**], but we reproduce it here for the convenience of the reader.

Let

$$a_k = \frac{\ln |Df^k(f(0)))}{k}$$

and $e_n = a_{v_n - 1}$.

It is easy to see that if n_0 is big enough such that both Lemmas A.21 and A.22 we obtain for n big enough that

$$e_{n+1} \geqslant e_n \frac{v_n - 1}{v_{n+1} - 1} + \frac{\lambda_{n_0}}{2} \cdot \frac{v_{n+1} - v_n}{v_{n+1} - 1}$$

and so

$$\liminf_{n \to \infty} e_n \geqslant \frac{\lambda_{n_0}}{2}. \tag{A.4}$$

Let now $v_n - 1 < k < v_{n+1} - 1$. Define $q_k = \ln |Df^{k - v_n}(f^{v_n}(0))|$.

Assume first that $k < v_n + c_{n-1}^{-4/(n-1)}$. From LC1 we know that τ_n is very good, so by LS1 we have $r_n(\tau_n) > c_{n-1}^{-a/2}$, so k is in the middle of this branch (that is, $v_n \leqslant k \leqslant v_n + r_n(\tau_n) - 1$). Using that $|R_n(0)| > |I_n|/2^n$ (by Lemma A.7), we get by Lemma A.11 that $-q_k < -\ln(2^{-n} c_{n-1} c_{n-1}^5)$. We then get from $v_n > c_{n-1}^{-a}$ that

$$a_k \geqslant e_n \frac{v_n - 1}{k} - \frac{-q_k}{k} > \left(1 - \frac{1}{2^n}\right) e_n - \frac{1}{2^n}. \tag{A.5}$$

If $k > v_n + c_{n-1}^{-4/(n-1)}$ using Lemma A.22 we get

$$a_k \geqslant e_n \frac{v_n - 1}{k} + \frac{\lambda_{n_0}}{2} \cdot \frac{k - v_n + 1}{k}. \tag{A.6}$$

Estimates (A.4), (A.5), and (A.6) imply that $\liminf a_k \geqslant \lambda_{n_0}/2$ and so f is Collet-Eckmann.

A.9. Recurrence. — To show that the critical point is polynomially recurrent, we can follow the same lines from [**AM1**]. First we look at the essentially Markov process $R_n|(I_n \setminus I_{n+1})$, which shows that with total probability, most (in the γ-qs sense) points in I_n approach 0 with a polynomial rate (the exponent must be chosen according to γ) until the first time they fall in I_{n+1}. More precisely, we show (after transferring to the parameter) the following estimate (analogous to Corollary 8.3 of [**AM1**]).

Lemma A.23. — *With total probability, for n big enough and for $1 \leqslant i \leqslant s_n$,*

$$\frac{\ln |R_n^i(0)|}{\ln(c_{n-1})} < b^2 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

To obtain the polynomial recurrence for f we relate the return times in terms of R_n to return times in terms of f. In other words, letting k_i be such that $R_n^i(0) = f^{k_i}(0)$, we must relate k_i and i. It is enough to do the estimate for a cool landing and we obtain the following estimate (as in Corollary 8.5 of [**AM1**]).

Lemma A.24. — *With total probability, for n big enough and for $1 \leqslant i \leqslant s_n$,*

$$\frac{\ln(k_i)}{\ln(c_{n-1}^{-1})} > a/3 \left(1 + \frac{\ln(i)}{\ln(c_{n-1}^{-1})}\right).$$

Let now $v_n \leqslant k < v_{n+1}$. If $|f^k(0)| < k^{-3b^3}$ we have $f^k(0) \in I_n$ and so $k = k_i$ for some i. It follows from Lemmas A.23 and A.24 that

$$|f^{k_i}(0)| > k_i^{-3b^3}.$$

This concludes the proof of polynomial recurrence. We notice that polynomial lower bounds are easily obtained: considering $|R_n(0)| = |f^{v_n}(0)| < c_{n-1}$ and using $v_n < c_{n-1}^{-b}$ we get

$$\limsup_{n\to\infty} \frac{-\ln |f^n(0)|}{\ln n} \geqslant a.$$

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