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WALKS IN RIGID ENVIRONMENTS: SYMMETRY AND DYNAMICS

by

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Abstract. — We study dynamical systems generated by a motion of a particle in an array of scatterers distributed in a lattice. Such deterministic cellular automata are called Lorentz-type lattice gases or walks in rigid environments. It is shown that these models can be completely solved in the one-dimensional case. The corresponding regimes of motion can serve as the simple dynamical examples of diffusion, sub- and super-diffusion.

1. Introduction

Deterministic (dynamical systems) or stochastic (random processes) models are the ones which were used traditionally to model real phenomena and processes. The theory of these two types of models, purely deterministic and purely stochastic ones, is very rich and therefore the intuition on evolution of such systems is well developed. The intuition means a right expectation of what should happen in the course of evolution of some concrete system even though the rigorous mathematical analysis is usually lacking.

Such intuition is based on some explicitly solvable simple (but nontrivial) and visible examples, i.e., on the comprehensive mathematical analysis of the corresponding models. These fundamental models in the theory of stochastic processes include sequences of identically distributed independent random variables (Bernoulli shifts), a random walk, etc. In dynamical systems such fundamental models include a rotation of a circle, an algebraic toral automorphism, some billiard models, etc. Certainly, this class of completely solvable models is growing, and our intuition is essentially growing with it. I cannot resist to mention the quadratic family which now finally belongs to this class as well [**14**].

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However, dynamics of many (and actually of a majority) of real systems is neither purely stochastic nor purely deterministic but it rather has both these components. Certainly, it is the well known fact and traditional attempts to account for that is to study e.g., small random perturbations of dynamical systems or to add a small deterministic flow term (advection) to a diffusion process. Such small perturbations, while being very important to study, do not address the question on the behavior of hybrid systems with (nonsmall) deterministic and stochastic features in their evolution. In fact, in applications almost always models were chosen as stochastic ones (instead of hybrid ones) with the standard argument that each real phenomenon or process has infinitely many features neglected by any model and therefore it is, in fact, a random process.

There are large areas like e.g., operations research, logistics, etc., which still completely belong to the probability theory while already the first applications of the dynamical systems methods allowed to achieve very encouraging results by essentially increasing production rates of certain production lines [**2**].

Another class of hybrid systems goes back to the classical Lorentz gas. Recall that in the Lorentz gas (light) point, noninteracting between themselves, particles move by inertia in an array of immovable scatterers and collide with scatterers elastically. It is a dynamical system which can be reduced to Sinai billiard. This system has been comprehensively studied and until now it is the only one nontrivial system for which time irreversible macroscopic dynamics (governed by the diffusion equation) has been rigorously derived from the time reversible microscopic dynamics (governed by Newton equations). It is transparent that this result has been obtained only for periodic configurations of scatterers (under the condition that a free path of the point particle is bounded, see details in [**6**]).

The very interesting mathematically and important problem for various applications is to study this system in case when the scatterers are distributed randomly. It seems, at the first sight, that this problem should follow from the one with periodic distribution of scatterers because of some additional "self-averaging" generated by a random distribution of scatterers. Indeed, it seems that such "self-averaging" should just improve stochastic properties of the corresponding dynamical system with periodically placed scatterers. However, this idea is totally wrong. In fact, in the Lorentz gas with randomly distributed scatterers we encounter a hybrid system, which has both deterministic and stochastic features. (Certainly, the Lorentz gas with randomly distributed scatterers can be described as purely deterministic (dynamical) system. However, it does not make this system to be deterministic, as well as the representation of a stationary random process a shift in the space of its realizations does not transform this stochastic process into a deterministic one.)

If an interesting and important system does not allow a comprehensive analysis then it is natural to consider some simpler model which retains (some) principal features of this system. Such simplified Lorentz gas model has been introduced in [**18**]. In this model scatterers (usually of two different types, e.g., left and right mirrors aligned along the diagonals of the square lattice) are randomly distributed on vertices of the square lattice. The point particle moves with unit speed along the bonds of this lattice and get reflected by the scatterers. These systems were naturally called Lorentz Lattice Gases (LLG). It is worthwhile to mention that this model is the generalization of another classical model in nonequilibrium statistical mechanics, which is called the (Ehrenfests') Wind-Tree model. In the Wind-Tree model a (light) point particle moves in an array of randomly distributed scatterers, which are identical rhombuses with parallel diagonals. The particle moves parallel to one of the diagonals of rhombuses and therefore after (elastic) reflection from the boundary of some scatterer, its velocity becomes parallel to another diagonal of the rhombuses and so on.

The Lorentz Lattice Gases belong to the class of systems which can be naturally called Deterministic Walks in Random Environments (DWRE). Indeed the dynamics of these systems is generated by deterministic motion of the particle, where both the free motion and reflections from the boundary of scatterers are deterministic, while distribution of scatterers is random.

It occurred that the Lorentz Lattice Gases were studied (without using this name) in lots of applications, e.g., in material science, superconductivity, chemical kinetics, information transmission and especially in the theoretical computer science. All these studies were exclusively numerical and these systems were included in the class of systems which are conventionally called "complex systems" (and are often discussed in the journal with the same name).

In fact, in many applications there were considered so called flipping LLG, where the moving particle has impact on an environment as well. Formally dynamics of such models is defined by the rule that after reflection of the moving particle from a scatterer this scatterer instantly changes its type. Therefore in flipping LLG there is also a dynamics of an environment formed by the configuration of scatterers. Hence for such models it makes sense to consider dynamics of many particles moving along the bonds of a lattice rather than of a single one. Indeed, even though the moving particles do not interact directly they do, in fact, interact via changing the environment to each other. It allows to account for an "information exchange" between particles (signals, etc.) and environment (neurons, etc.), see. e.g., [**1, 7, 9, 10, 12**].

From the mathematical point of view all these models are dynamical systems. In fact, they belong to the class of deterministic cellular automata. However, this formal observation does not help much in studying these systems. In fact, it occurred that the much more productive approach is to consider all these models as Deterministic Walks in Random Environments. (To make clear distinction with purely stochastic models of this kind we mention that in the last ones a scatterer after colliding with particle "flips a coin" to decide whether it should change its type.)

In the studies of DWRE the important role is played both by the structure of a lattice where particles move (which could be e.g., the square, triangular, cubic, random, etc. lattice) and by the types of scatterers considered (e.g., there are 4^4 types of scatterers in a square lattice). It is not surprising, of course, because a lattice defines a configuration space and the types of scatterers (together with a lattice) define the dynamics (equations of motion).

The great majority of papers on DWRE are numerical. There are as well quite a few mathematical results on dynamics of DWRE. They usually use some specific features of the given model, which allow sometimes to come up with complete solution. For instance, it is possible to reduce a (purely deterministic) problem to a (purely probabilistic) percolation problem on some graph [**4**]. (It is worthwhile to mention that such graph is defined not only by the lattice but by the types of scatterers as well.) Sometimes it was possible to completely solve the problem by constructing some peculiar class of solutions and by proving that no other solutions exist (see e.g., [**5**]). However, in most cases the results were rather counterintuitive. Actually, in almost all cases when dealing with the hybrid (neither purely deterministic, nor purely stochastic) systems the authors confessed that they obtained results different from what they expected.

This situation clearly calls for some kind of a general view at these systems, especially the one which would allow to integrate the studies of DWRE in fixed and in evolving (e.g., flipping) environments. The corresponding approach has been developed in [**3**] where these two classes of DWRE, were integrated into one class of dynamical systems called Walks in Rigid Environments (WRE). (Observe that R in DWRE refers to "random," while in WRE it refers to "rigid").

WRE is also a dynamical system generated by motion of point particles in some graph (e.g., in a lattice). For the sake of simplicity we will consider here only one-particle systems. Some scatterers are randomly distributed along the vertices of this graph. (Again for the sake of simplicity we assume that the scatterers are distributed independently, even though one may assume that they interact via some potential.)

The crucial feature of WRE is the new parameter r which is called a rigidity of an environment. The rigidity determines how many times the particle must collide with the given scatterer in order to change its type. In the other words, the scatterer at a given site changes its type at the moment after the rth visit of the moving particle to this site. It is easy to see that the LLGs with fixed environment correspond to the case $r = \infty$, while the LLGs with flipping environment correspond to the case $r = 1$. Thus the two studied so far classes of LLG form, in fact, two extreme sub-classes of WRE.

Besides the introduction of Walks in Rigid Environments allowed to move rigorous studies of LLG to another level and to address the central problem of the theory of such systems which is the diffusion problem. Until [**3**] the mathematical papers on Deterministic Walks in Random Environments usually addressed the problem whether a typical path of a particle is bounded or unbounded. However, the most important question which one can ask about evolution of a system generated by a motion of some object (particle, signal, etc.) is where this object is going to be at a sufficiently large moment of time t. The quantity of interest is the mean square displacement $Ez^2(t)$ (or, in other words, the expectation, taken with respect to the distribution of environments, of a (squared) position $z(t)$ of the particle at time t). One distinguishes diffusive, subdiffusive and superdiffusive behavior which correspond to the linear, slower than linear and to faster than linear growth of $Ez^2(t)$ respectively.

It has been shown in [**3**] that the asymptotic behavior of the particles' position is determined by an interplay between the symmetries of the lattice and symmetries of scatterers.

The present paper deals with WRE where the problem of the particle's diffusion can be solved completely. We give the examples of all three situations, i.e., diffusion, sub- and super-diffusion. Moreover, in these examples it was possible to completely "separate" stochastic and deterministic elements of the evolution of these models.

Qualitatively the situation is the following one. Stochastic evolution of the system takes place when the particle visits some site of the lattice at the first time, while between two consecutive visits to the new (nonvisited before) sites the particle undergoes a deterministic evolution. This deterministic evolution is completely defined by the types of the scatterers allowed in the model under study and by their symmetries. It is exactly this deterministic evolution defines the speed of growth of visited (exited) domain.

Such separation of the evolution into random events and intermediate deterministic motion allowed to describe in one-dimensional case all three types of behavior in the same way.

It occurred that the evolution of the particle can be broken into the qualitatively similar stages. Each such stage is characterized by deterministic motion of the particle in some box of a random size. In cases of diffusion and of subdiffusion the sizes of these boxes are growing in time, while in case of super-diffusion the sizes of these boxes fluctuate and the boxes are moving along the lattice in one direction, which is defined by the initial distribution of scatterers near the origin.

Actually the analysis of all these three models is rather straight-forward and they could be used in the first courses of dynamical systems and/or random processes as completely solvable models which are neither purely deterministic nor purely stochastic to develop intuition on systems with such mixed type of behavior.

The structure of the paper is the following. In Sect. 2 we give the necessary definitions and formulate the results. The proofs are given in Sect. 3. The last Sect. 4 contains some concluding remarks.

2. Definitions and main results

Consider an one-dimensional regular lattice which, without any loss of generality, could be identified with the set of integers $\mathbb{Z}$. We assume that at each site $z \in \mathbb{Z}$ there is a scatterer of some type. A particle moves with the unit speed along the lattice $\mathbb{Z}$, i.e., $v(t) = 1$ or $v(t) = -1$ at each moment of time t. Denote by $z(t)$ position of the particle at time t. Then the position of the particle at the next moment of time is determined by $v(t)$ and by the type of scatterer located at the site $z(t)$. Certainly it is enough to consider a discrete time. To distinguish between two moments of time when the particle reached some site of the lattice but had not yet reflected by a scatterer at this site and the one when it was just reflected by a scatterer we will denote these moments by t and t_+ respectively. Hence $v(t)$ is the velocity with which the particle approaches a site $z(t)$ and $v(t_+)$ is the velocity with which the particle leaves this site.

It is clear that in dimension one there are 2^2 possible scatterers (or local scattering rules), which we will denote by BS, FS, LS and RS. Here BS is the backward scatterer, which changes the velocity of the particle to the opposite one. In other words, if BS is located at a site $z(t) \in \mathbb{Z}$ then $v(t_+) = -v(t)$. FS is the trivial, or forward scatterer which does not change the velocity of the particle, i.e., $v(t_+) = v(t)$ if at the site $z(t)$ was the forward scatterer. The last two types of scatterers, LS and RS, which we will refer to as the left and the right scatterer respectively, are the semi-transparent ones. Namely LS (RS) sends all scattered particles to the left (right), i.e., if a LS (RS) is located at a site $z(t) \in \mathbb{Z}$ then $z(t+1) = z(t) - 1 (z(t+1) = z(t) + 1)$.

Now we will define the dynamics of our system. In order to do it we introduce an integer r, $1 \leqslant r \leqslant \infty$, which we will refer to as a *rigidity* of an environment. Let $\widehat{S}$ be a space of all possible scatterers on a lattice under consideration. (Recall that in this paper we discuss only WRE in one-dimensional lattice $\mathbb{Z}$, i.e., $\widehat{S} = \{BS, FS, LS, RS\}$.)

WRE is defined by three objects:

(1) A subspace $S \subset \widehat{S}$ of scatterers, which we will call a space of allowed scatterers.
(2) An integer $r > 0$ (rigidity).
(3) A function $e : S \to S$.

Let $S_r = S \times \{0, 1, \ldots, r-1\}$ and $\pi : S_r \to S$ is the natural projection. Denote a function $a : S_r \to S_r$ as

$$a(S, i) = \begin{cases} (S, i+1), & \text{if } 0 \leqslant i < r-1 \\ (e(s), 0), & \text{if } i = r-1, \end{cases} \tag{1}$$

where $s \in S$. We will call i an index of the corresponding scatterer.

We will denote by $s(z)$ a type of scatterer which is located at the site $z \in \mathbb{Z}$. The type of scatterer at z may change in the course of dynamics (if $r < \infty$). By $(s(z))_t$ we denote the type of a scatterer located at a site $z \in \mathbb{Z}$ at a moment of time t. The notation $s(z(t))$ will be referred to a type of scatterer located at a moment t at the site where the particle sits at this moment.

The configuration space of our system $W = S_r^{\mathbb{Z}} \times \mathbb{Z}$, where $S_r^{\mathbb{Z}}$ is a configuration of scatterers (together with a number of visits occurred to a site $z \in \mathbb{Z}$ while a scatterer of some fixed type was located there) and the second factor $\mathbb{Z}$ corresponds to the position of the particle. The phase space $\Omega = W \times \{-1, 1\}$.

Now we are able to write the equations governing the dynamics

$$
\begin{aligned}
v(t+1) &= g(v(t), s(z(t))), \\
z(t+1) &= z(t) + v(t+1), \\
((s(z))_{t+1}, i) &= ((s(z))_t, i) \qquad \text{if } z \neq z(t) \\
((s(z(t), i) &= a(s(z(t)), i) \qquad \text{if } z = z(t).
\end{aligned}
\tag{2}
$$

The function $g(v(t), s(z(t)))$ in (2) is completely defined by the type of scatterer $s(z(t))$. (The formal expressions for an abstract scatterer are rather cumbersome. It would become simple though when we consider concrete models of WRE.)

We will introduce two such models. In the first model we will take semi-transparent scatterers LS and RS as the set S of admissible scatterers. The second model corresponds to $S = \{BS, FS\}$.

We describe now the dynamics of these two models informally (but precisely and in more visible way than it is formally defined by the relations (2)).

Each of the models under consideration deals with two types of scatterers. The particle moves with unit velocity along the lattice $\mathbb{Z}$. At each integer moment of time t it comes to some vertex $z(t) \in \mathbb{Z}$ and gets scattered by the scatterer located at this moment at $z(t)$. (A function $g(\cdot, \cdot)$ is immediately specified by the type of this scatterer.) If the particle was scattered r consecutive times by this scatterer located at $z(t)$ (i.e., if particle returned to this site with this very scatterer r times) then this scatterer gets changed to another type.

Now we need to specify initial conditions for our dynamical system. Without any loss of generality we can always assume that the particle starts at the origin with the initial velocity $v(0) = 1$. We take Bernoulli measure on space of scatterers' initial configurations, i.e., the types of scatterers at different sites are chosen independently and have the same distributions.

Two models under consideration have quite different symmetry properties. The only nontrivial symmetry of the lattice $\mathbb{Z}$ is the reflection with respect to the origin. (Indeed the probability distributions on initial configurations of scatterers are translationally invariant.) Observe now that LS and RS do respect this symmetry, while BS and FS do not. It is the key point why dynamical properties of these models are quite different as we will see later.

It is easy to see that an orbit of any WRE is completely defined by the initial configurations of scatterers. We will use sometimes the same notation ω to denote an orbit of a dynamical system and the corresponding configuration of scatterers. Another remark is that initially (at $t = 0$) all scatterers have indices zero.

For the sake of brevity we will refer to the model with $S = \{LS, RS\}$ as to the model with oriented scatterers (OS-model) and to the model with $S = \{BS, FS\}$ as to NOS-model (the model with non-oriented scatterers).

We start with the formulation of the results on the qualitative behavior of the OS- and NOS-models and then turn to their quantitative behavior.

The first simple remark is that the dynamics of both models is trivial (and similar) in case when the environment does not change in time ($r = \infty$). Indeed the particle will with probability one oscillate between two closest to the origin BS (for the NOS-model) with positive and non-positive coordinate respectively, or between the closest to the origin LS with positive coordinate and the closest to the origin RS with non-positive coordinate in the OS-model.

It is the characteristic feature of hybrid systems (intermediate ones between purely deterministic and purely stochastic) that an exceptional set of orbits of measure zero can often be completely characterized. For instance, if $r = \infty$ this set consists of initial configurations of scatterers where all scatterers with positive coordinates are RS (for the OS-model) or FS (for the NOS-model) or/and all scatterers with non-positive coordinates are LS (for the OS-model) or FS (for the NOS-model). The dynamics of the OS-model is characterized qualitatively by the following statement.

Theorem 1. — *In the OS-model for any value of rigidity $r < \infty$ the particle will almost surely visit each site of the lattice $\mathbb{Z}$ infinitely many times. Moreover, for almost every point $\omega \in \Omega$ of the phase space there exists a sequence of moments of time τ_i, $i = 0, 1, \ldots$, $\tau_0 = 0$, $\tau_i < \tau_{i+1}$, $\tau_i \to \infty$ as $i \to \infty$ and a corresponding sequence of closed intervals $B_i(\omega) = [a_i(\omega), b_i(\omega)] \subset \mathbb{Z}$, $i = 1, 2, \ldots$, $a_i(\omega) \leqslant 0$, $b_i(\omega) > 0$, $B_i(\omega) \subset B_{i+1}(\omega), B_i(\omega) \to (-\infty, \infty)$, as $i \to \infty$, such that within a time interval $\tau_{i-1} < t < \tau_i$, $i = 1, 2, \ldots$, the particle stays inside the interval $B_i(\omega)$ and visits the origin $z = 0$ $2r$ times.*

Thus Theorem 1 shows that in the OS-model for any finite value of rigidity the particle will oscillate about origin with an increasing amplitude. We will say that a point $\omega \in \Omega$ has a positive (negative) tail of scatterers of some type if there exists $z_+ > 0$ ($z_- \leqslant 0$) such that all scatterers at the sites $z \geqslant z_+$ ($z \leqslant z_-$) are of one and the same type.

Corollary 1. — *The exceptional set of measure zero in* Theorem 1 *consists of such points $\omega \in \Omega$, where the corresponding configurations of scatterers contains a positive tail of RS or/and a negative tail of LS.*

Denote by $z_{\max}(t)$ and $z_{\min}(t)$ the sites with the maximal and the minimal coordinates respectively visited by the particle to a moment t. The next theorem describes quantitative features of the dynamics of the OS-model. Namely, it says that the size of the region visited by the particle to a moment t grows diffusively.

Theorem 2. — *In OS-model $Ez^2_{\max}(t)$, $Ez^2_{\min}(t)$ and $Ez^2(t)$ grow linearly in t.*

In the NOS-model the scatterers are invariant with respect to reflections. This is the reason why the dynamics of this model is quite different from the one of the OS-model.

Theorem 3. — *In the NOS the particle visits almost surely all sites of the lattice $\mathbb{Z}$ infinitely many times if the rigidity r is an even number. Besides for almost every $\omega \in \Omega$ there exist sequences of moments of time τ_i, $i = 0, 1, 2, \ldots$, and of closed intervals $B_i(\omega)$, $i = 1, 2, \ldots$, with properties analogous to the ones in* Theorem 2. *If the rigidity r is an odd number then for all $\omega \in \Omega$, the particle visits all sites in $[0, \infty)$, $[-1, \infty)$, or $(-\infty, 1]$ and only these sites. Besides the particle visits each of these sites no more than $3r$ times. Moreover in this case there exist sequences of moments of time $\widehat{\tau}_i(\omega)$, $i = 0, 1, 2, \ldots$, and of closed intervals $\widehat{B}_i(\omega) = [\widehat{a}_i(\omega), \widehat{b}_i(\omega)]$, $i = 1, 2, \ldots$ such that $\widehat{\tau}_0(\omega) = 0$, $\widehat{\tau}_i(\omega) < \widehat{\tau}_{i+1}(\omega)$, $\widehat{a}_i(\omega) < \widehat{a}_{i+1}(\omega) < \widehat{b}_i(\omega)$, $\widehat{a}_i(\omega) \to \infty$ as $i \to \infty$ or $\widehat{b}_i(\omega) \to -\infty$ as $i \to \infty$ and the particle stays inside $\widehat{B}_i(\omega)$, within the time interval $[\widehat{\tau}_i(\omega), \widehat{\tau}_{i+1}(\omega)]$.*

We recall that the particle always starts at the origin with positive velocity. This explains why in Theorem 3 the semi-interval $(-\infty, 0)$ does not show up.

By comparison of Theorems 1 and 3 one can immediately see that a parity of rigidity does not play any role in the OS-model while in the NOS-model it completely defines its qualitative behavior.

Remark. — Observe that in case of odd rigidity Theorem 3 refers to the behavior of all (rather than of almost all) orbits.

Corollary 2. — *Let in the NOS-model the rigidity be even. Then the exceptional set of measure zero orbits in* Theorem 3 *corresponds to the configuration of scatterers with a positive tail of FS or/and with a negative tail of FS.*

The next statement immediately follows from Theorem 3.

Corollary 3. — *Let the rigidity r be an odd number. Then in the NOS-model the particle will for all $\omega \in \Omega$ propagate in one direction with a random velocity.*

Indeed the particle at any moment of time is confined to some segment (box) $B_i(\omega)$ where it goes back and forth. These boxes move in one direction and the particle eventually propagates with them. At each first visit to any site of $\mathbb{Z}$ the particle can be scattered backward or forward according to a random initial distribution of scatterers. Therefore the particle propagates with a random speed.

The next theorem gives the quantitative description of the dynamics of NOS-model.

Theorem 4. — *In NOS-model $Ez^2(t)$ grows as const t^2 if r is an odd number. Otherwise, if r is an even number, $Ez^2(t)$ grows as const* $\log t$.

Because of the deterministic evolution of WRE it is possible to give much more detailed description of the motion of the particle within random boxes in each of the models under study. On the other hand transition from one box where the particle gets confined for some time to the next such box is a random event.

We describe now a geometric nature of typical orbits in the OS- and NOS-models. At first we introduce some notions and notations.

We will denote by Ω_1 and Ω_2 the phase spaces of the OS-model and of the NOS-model respectively. It is convenient to introduce the reduced phase spaces $\widehat{\Omega}_1 = \{LS, RS\}^{\mathbb{Z}}$ and $\widehat{\Omega}_2 = \{BS, FS\}^{\mathbb{Z}}$. Thus $\widehat{\Omega}_1$ and $\widehat{\Omega}_2$ refer just to a type of scatterer at any site of $\mathbb{Z}$, without taking into account how many times the particle has already been reflected by this scatterer. Dynamics of the OS-model (NOS-model) we will define by $f_1 : \Omega_1 \to \Omega_1$ ($f_2 : \Omega_2 \to \Omega_2$). Let $\pi_1' : \Omega_1 \to \widehat{\Omega}_1$, $\pi_2' : \Omega_2 \to \widehat{\Omega}_2$, $\pi_1'' : \Omega_1 \to \{-1, 1\}$, $\pi_2'' : \Omega_2 \to \{-1, 1\}$ are the natural projections.

For $x, y \in \widehat{\Omega}_1$ ($x, y \in \widehat{\Omega}_2$) we define the distance $d(x, y)$ as $d(x, y) = 2^{-n}$ if $x_i = y_i$ for $|i| < n$ and $x_i \neq y_i$ for $i = n$ or $i = -n$, i.e., if configurations of scatterers restricted to $(-n, n)$ coincide for x and y.

Lemma 5. — *In the OS-model for almost every point $\omega \in \Omega_1$ there exists an infinite sequence of moments of time $\tau_k = \tau_k(\omega)$, $k = 1, 2, \ldots$, $\tau_k \to \infty$ as $k \to \infty$, such that*

(i) $\pi_1''(f_1^{\tau_k}(\omega)) = 1$

(ii) $(\pi_1'(f_1^{\tau_k}(\omega)))_i = (RS, 0)$ *if* $0 \leqslant i \leqslant k$

(iii) $(\pi_1'(f_1^{\tau_k}(\omega)))_i = (LS, 0)$ *if* $-k \leqslant i < 0$.

In other words, Lemma 5 states that a typical orbit of the OS-model returns into the smaller and smaller neighborhoods of the orbit, for which at $t = 0$ at all positive sites of $\mathbb{Z}$ were right scatterers with zero indices while at all nonpositive sites of the lattice were left scatterers with zero indices.

Lemma 6. — *Let the rigidity r be an even number. Then in the NOS-model for almost every point $\omega \in \Omega_2$ there exists an infinite sequence of moments of time $\tau_k = \tau_k(\omega)$, $k = 1, 2, \ldots, \tau_k < \tau_{k+1}$, $\tau_k \to \infty$ as $k \to \infty$, such that*

(i) $\pi_2''(f_2^{\tau_k}(\omega)) = 1$

(ii) $\pi_1''(f_2^{\tau_k}(\omega))_i = (BS, 0)$ *for* $|i| < k$.

Lemma 7. — *Consider the NOS-model. Let the rigidity r be an odd number. Then for any point $\omega \in \Omega_2$ and any $z \in \mathbb{Z}$ there exist a moment of time $\tau = \tau(\omega, z)$ such that:*

(i) *The type of scatterer at z never changes after the moment $\tau(\omega, z)$.*

(ii) *The type of scatterer located at z after the moment $\tau(\omega, z)$ is BS if at the next site of $\mathbb{Z}$ in the direction of propagation, it was initially FS, or FS if the next site of $\mathbb{Z}$ in the direction of propagation was initially occupied by BS.*

Lemma 7 states that the initial configuration of scatterers gets flipped (each FS becomes BS and vice versa) and shifted on one site in the direction opposite to the eventual direction of the particle's propagation.

3. Proofs

We have already mentioned that a WRE in a fixed environment ($r = \infty$) $\mathbb{Z}$ is always trivial, i.e., the particle will be moving forever back and forth in a segment between two closest BS (in NOS-model) or LS with a positive coordinate and RS with a negative coordinate (in OS-model).

Denote by $\eta(z,t)$ a number of visits of the particle to a site $z \in \mathbb{Z}$, which occurred between the last moment of time $\tau = \tau(z,t)$, $0 < \tau(z,t) < t$, when a scatterer at z flipped and t. It is easy to see that $\eta(z,t)$ equals the index of the scatterer at the site z at time t. Thus, to make the scatterer at the site z flip requires another $r - \eta(z,t)$ visit of the particle to this site.

Proof of Theorem 1. — Recall that we always assume that the particle starts at the origin $z = 0$ with velocity $v = 1$. Therefore the initial segment of any orbit is the motion of the particle with the unit speed until it will get to the closest to the origin site $z = b_1 > 0$ with LS. At such site z (at the moment $t = b_1$) the particle will turn, i.e., its velocity becomes $v = -1$.

Now (if $r > 1$) for some time the particle will be confined between the sites $z = b_1 - 1$ and $z = b_1$. Indeed, both indices $\eta(b_1 - 1, b_1)$ and $\eta(b_1, b_1)$ equal 1. Then (if $r > 1$) there is RS at the site $b_1 - 1$ and at the moment $t = b_1 + 1$ the particle gets reflected back and hits again LS at b_1 at the moment $b_1 + 2$ and so on.

It is easy to see that those oscillations between $z = b_1 - 1$ and $z = b_1$ will be over at the moment $t = b_1 + 2r - 1$. In fact at the moment $b_1 + 2r$ the particle will be in the site $z = b_1 - 2$ and $\eta(b_1, (b_1 + 2r - 1)) = \eta(b_1 - 1, b_1 + 2r - 1) = 0$, i.e., the scatterers at the sites $b_1 - 1$ and b_1 changed their type. (If $r = 1$ then the particle makes no oscillations in its way between $z = b_1 - 1$ and $z = b_1$. Instead the particle after the reflection at $z = b_1$ goes back and passes at the next step the site $z = b_1 - 1$.)

Therefore at the moment $t = b_1 + (2r-1)(b_1 - 1) + 1$ the particle will return back to the origin $z = 0$. If there was at $t = 0$ RS at the origin then at the moment $t = 2rb_1$ it will start its travel from $z = 0$ into the positive semiaxis. In this case we set $a_1 = 0$. Otherwise, the particle starts to move from the origin into the negative semiaxis. The same consideration can obviously apply to this piece of trajectory, where one only needs to change LS into RS and vice versa and to change $v = 1$ into $v = -1$. Denote the site where the particle meets its first RS by $z = a_1 < 0$.

Observe now that at the moment of time when the particle goes through the origin with the velocity $v = -1$ (i.e., at this moment there is LS at the origin) all scatterers at the sites $z = 1, 2, \ldots, b_1$ are RS. Analogously, when the particle will cross the origin next time with the velocity $v = 1$ at all sites $z = 0, -1, -2, \dots, a_1 + 1, a_1$ there will be LS.

Therefore, at the moment of time $t = 2r(b_1 - a_1)$ the particle will be again (as at $t = 0$) at the origin with the velocity $v = 1$. Besides at this moment all sites $z = 1, 2, \dots, b_1$ will be occupied again by RS with the indices equal zero. Therefore, now the particle will travel into the positive semiaxis $\mathbb{Z}_+$ at least $b_1 + 1$ consecutive steps, i.e., it will penetrate into $\mathbb{Z}_+$ farther than at its first excursion to $\mathbb{Z}_+$ when it was backscattered by LS at the site $\mathbb{Z} = b_1$. Denote the closest (at this moment of time) to the origin (positive) site with LS by $z = b_2 > b_1$ and the closest to $z = 0$ negative site by $z = a_2 < a_1$. Then the same arguments as before are applied.

It is easy to see that in the same way we can construct segments $B_i = [a_i, b_i]$, $i = 1, 2, \dots$, with the properties satisfying to Theorem 1. Obviously these intervals as well as the corresponding intervals of time $[\tau_{i-1}, \tau_i]$, $i = 1, 2 \dots$, when the particle is confined within B_i are completely defined by the initial distribution of scatterers ω, i.e., $\tau_i = \tau_i(\omega)$ and $a_i = a_i(\omega)$, $b_i = b_i(\omega)$. □

Proof of Corollary 1. — It follows from the proof of Theorem 1 that the only case when there is no infinite sequence of closed intervals $B_{i+1}(\omega) \supset B_i(\omega)$ occurs when $b_k(\omega) = \infty$ or $a_k(\omega) = -\infty$ for some integer $k > 0$. But it means that the configuration of scatterers ω has a positive tail (where $z_+ = b_k(\omega)$) or it has a negative tail (where $z_- = a_k(\omega)$). □

Proof of Theorem 2. — It follows from the proof of Theorem 1 that for almost every initial configuration of scatterers there exists a sequence $\tau_i(\omega)$ such that within the interval $[\tau_i(\omega), \tau_{i+1}(\omega)]$, $i = 0, 1, 2, \dots$, the particle moves (starting at the origin) inside the interval $B_{i+1}(\omega) = [a_{i+1}(\omega), b_{i+1}(\omega)]$. Besides it follows from the proof of Theorem 1 that the length of the interval $[\tau_i(\omega), \tau_{i+1}(\omega)]$ equals

$$\Delta\tau_{i+1} = 2r(b_{i+1}(\omega) - a_{i+1}(\omega)).$$

Moreover we know exactly how the particle moves in this interval. Indeed, the particle visits within the interval $\Delta\tau_{i+1}$ each site in $B_{i+1}(\omega)$ exactly $2r$ times.

Hence to prove Theorem 2 we need to evaluate expected length of an interval $B_i(\omega) = [a_i(\omega), b_i(\omega)]$. Let us note first that now it would be more convenient to use the probabilistic approach and language. Indeed, $b_1(\omega), b_{i+1}(\omega) - b_i(\omega)$ and $-a_1(\omega), a_i(\omega) - a_{i+1}(\omega)$, $i = 1, 2, \dots$, are sequences of independent identically distributed random variables. These random variables are the ones we need to analyze because the proof of Theorem 1 provided us with the complete description of the deterministic motion of the particle inside the (random) intervals $B_i(\omega)$.

Let q $((1-q))$ be a probability that LS (RS) is located at any given site of the lattice $\mathbb{Z}$. Recall that according to our assumptions the scatterers were placed independently at the different sites. Therefore both $b_{i+1}(\omega) - b_i(\omega)$ and $a_i(\omega) - a_{i+1}(\omega)$ have the geometric probability distribution, i.e., for any $i = 0, 1, 2, \dots,$

$$\text{Prob}\{b_{i+1}(\omega) - b_i(\omega) = k\} = (1-q)^{k-1}q \tag{3}$$

and

$$\text{Prob}\{a_i(\omega) - a_{i+1}(\omega) = k\} = q^{k-1}(1-q) \tag{4}$$

where $k \geqslant 1$ is an integer and $a_0(\omega) = b_0(\omega) = 0$.

Denote by $f(t,z)$ probability that the particle will visit a site $z > 0$ at the first time at some moment t. Then one can write the following recurrence equation

$$\begin{aligned} f(z,t) &= (1-q)f(z-1,t-1) \\ &\quad + (1-q)q\sum_{k=1}^{\infty} q^{k-1} f(z-1, t-1-[b(t-1)-a(t-1)+k]2r) \end{aligned} \tag{5}$$

where $b(t)$ and $a(t)$ are the maximal, and the minimal coordinates of sites visited by the particle to the moment t. It is easy to see that $b(t-1)$ in our case equals $z-1$ and therefore we can rewrite (5) as

$$\begin{aligned} f(z,t) &= (1-q)f(z-1,t-1) \\ &\quad + (1-q)q\sum_{k=1}^{\infty} q^{k-1} f(z-1, t-[z-a(t-1)+k]2r) \end{aligned} \tag{6}$$

Similar equations can be written for $z \leqslant 0$. Certainly $b(t) = b(t,\omega)$ and $a(t) = a(t,\omega)$, i.e., both these quantities depend upon the initial configuration of scatterers ω.

Let $m_+(t,\omega)$ be a number of LS located between the origin and $b(t-1)$ in a configuration ω. Then there are two possibilities. Either a number $m_-(t,\omega)$ of RS located between 0 (including the origin itself) and $a(t-1)$ equals $m_+(t,\omega)$ or $m_+(t,\omega) - m_-(t,\omega) = 1$. Because we assumed that $z > 0$ the second possibility holds.

It follows from Theorem 1 that at each moment of time τ the particle almost surely is confined in some segment $B(\tau,\omega) = [a(\tau,\omega), b(\tau,\omega)]$.

We will now compute the expected values of $a(\tau)$ and $b(\tau)$. It is enough to do it for $b(\tau)$ because the procedure of computing $a(\tau)$ is completely similar. One can write

$$b(\tau,\omega) = b_1(\omega) + (b_2(\omega) - b_1(\omega)) + \dots + (b_{m_+(\tau,\omega)}(\omega) - b_{m_+(\tau,\omega)-1}(\omega)). \tag{7}$$

The probability distributions of the terms in this sum are given by (3). Therefore we just need to find the expected value of $m_+(\tau,\omega)$.

It follows from Theorem 1 and Corollary 1 that for almost every orbit ω of OS-model there exist infinite sequences of moments of time $\tau_i^+(\omega)$ $(\tau_i^-(\omega))$, $i = 1, 2, \dots,$ such that at the moment τ_k^+ (τ_k^-) the orbit visits at the first time the right (left) end $b_k(\omega)$ $(a_k(\omega))$ of the interval $B_k(\omega)$.

We restrict the consideration to the set of orbits $\Omega_1' \subset \Omega_1$ of measure one described in Corollary 1. Then for any configuration $\omega \in \Omega_1'$ and for any moment of time $\tau > 0$ one can write the following identity

$$\tau = 2r \sum_{i=1}^{m_+(\tau,\omega)-1} (b_i(\omega) - a_i(\omega)) + \gamma(\tau, \omega), \tag{8}$$

where $\gamma(\tau, \omega)$ is the length of the interval of time between the moment when the particle returned to the origin with $v = 1$ after visiting $2r$ times all sites in the interval $B_{m_+(\tau,\omega)-1}(\omega)$ and the moment τ.

Indeed, it follows from the proof of Theorem 1 that any orbit $\omega \in \Omega_1'$ has the following structure. First, it visits $2r$ times all sites in the interval $B_1(\omega)$ and occurs at the origin after that with the positive velocity, then it visits $2r$ times all sites in the interval $B_2(\omega)$ and returns to $z = 0$ with $v = 1$ and so on.

Therefore we have

$$E\gamma(\tau, \omega) \leqslant 2rE(b_{m_+(\tau,\omega)} - a_{m_+(\tau,\omega)}) \tag{9}$$

Hence, we need to find $Eb_{m_+(\tau,\omega)}$ and $Ea_{m_+(\tau,\omega)}$. By making use of (3), (4) it is easy to compute

$$\begin{aligned} Eb_{m_+(\tau,\omega)} &= \frac{1}{q} Em_+(\tau, \omega) \\ Ea_{m_+(\tau,\omega)} &= \frac{1}{1-q} Em_+(\tau, \omega). \end{aligned} \tag{10}$$

Indeed, (3) and (4) imply that

$$\begin{aligned} Eb_1(\omega) = E(b_{i+1}(\omega) - b_i(\omega)) &= \frac{1}{q}, \\ E(-a_1(\omega)) = E(a_i(\omega) - a_{i+1}(\omega)) &= \frac{1}{1-q}, \end{aligned} \tag{11}$$

where $i = 1, 2, \ldots$.

Recall now, that $b_1(\omega)$, $(b_{i+1}(\omega) - b_i(\omega))$, and $-a_1(\omega)$, $(a_i(\omega) - a_{i+1}(\omega))$, $i = 1, 2, \ldots$, are two sequences of independent identically distributed random variables.

It follows from (8) that

$$\begin{aligned} 2r \sum_{i=1}^{m_+(\tau,\omega)-1} (m_+(\tau, \omega) - i) \left[(b_i(\omega) - b_{i-1}(\omega)) + (a_{i-1}(\omega) - a_i(\omega))\right] \leqslant \tau \\ \leqslant 2r \sum_{i=1}^{m_+(\tau,\omega)} (m_+(\tau, \omega) - i + 1) \left[(b_i(\omega) - b_{i-1}(\omega)) + (a_{i-1}(\omega) - a_i(\omega))\right] \end{aligned} \tag{12}$$

where again $a_0(\omega) = b_0(\omega) = 0$.

The relations (10)–(12) imply

$$\left(\frac{1}{q}+\frac{1}{1-q}\right)\frac{Em_+(\tau,\omega)(Em_+(\tau,\omega)-1)}{2} \leqslant \tau \leqslant \frac{Em_+(\tau,\omega)(Em_+(\tau,\omega)+1)}{2}\left(\frac{1}{q}+\frac{1}{1-q}\right). \tag{13}$$

Therefore there exist such positive constants C_1 and C_2 that

$$C_1 t^{1/2} \leqslant Em_+(t,\omega) \leqslant C_2 t^{1/2} \tag{14}$$

for sufficiently large t.

It follows from (14) that there exist positive constants C_1', C_2', C_1'', C_2'' such that for sufficiently large t one has

$$\begin{aligned} C_1' t &\leqslant E z^2_{\max}(t) \leqslant C_2' t,\\ C_1'' t &\leqslant E z^2_{\min}(t) \leqslant C_2'' t. \end{aligned} \tag{15}$$

It remains to prove that $Ez^2(t)$ has the same asymptotics. This fact immediately follows from (15) and Theorem 1. Indeed, it has been shown in the proof of Theorem 1 that within the interval of time $\tau_i(\omega) \leqslant t \leqslant \tau_{i+1}(\omega)$, $i = 0, 1, 2, \dots$ the particle for any $\omega \in \Omega_1'$ spends the same amount of time (equal $2r$) at each site of the interval $B_i(\omega)$.

Therefore, position of the particle is uniformly distributed within $B_{i+1}(\omega)$ in the time interval $[\tau_i(\omega), \tau_{i+1}(\omega)]$, and the last statement of Theorem 2 follows. □

Proof of Theorem 3. — Theorem 3 follows from Theorem 1 and Theorem 2 in [**3**]. Therefore we just outline the proof.

Let us consider the NOS-model and assume first that the rigidity r is an odd number. Then the particle will travel from the origin till the closest to $z=0$ site $\widehat{b}_1 = \widehat{b}_1(\omega) > 0$ where in the configuration ω there is a back-scatterer BS. At $z=\widehat{b}_1$ the particle will turn back and travel now in the negative direction until it reaches the closest to $z=\widehat{b}_1$ site $z=\widehat{a}_1(\omega)$ with FS. Observe that if the rigidity $r=1$ then $\widehat{a}_1(\omega) = \widehat{b}_1(\omega) - 1$, unless $\widehat{b}_1(\omega) = 1$ and a FS is located at the origin in the configuration ω. In this case the scatterer at the origin becomes BS after the particle pass $z=0$ in the negative direction. Therefore, it is enough for $r=1$ to consider only such cases when there was a BS at the origin at $t=0$.

We return now to the general case of an odd rigidity. According to the dynamics the particle will move back and forth in the segment $\widehat{B}_1(\omega) = [\widehat{a}_1(\omega), \widehat{b}_1(\omega)]$ until it hits the BS located at the site $z=b_1$ at the $\left(\frac{r+1}{2}\right)$th time. Denote this moment by $\widehat{\tau}_1(\omega)$. Observe that to this moment of time the particle will visit all internal sites of $B_1(\omega)$ exactly r times. Recall that initially at all these sites were located forward scatterers. Therefore to $t=\widehat{\tau}_1$ all of those got substituted by BS.

It is easy to see that at the moment $t=\widehat{\tau}_1(\omega)$ the BS located at the site $z=\widehat{b}_1(\omega)$ has the index $(r+1)/2$ while the BS at $z=\widehat{b}_1(\omega)$ has the index zero. Therefore the particle will move now for $(r+1)$ moments of time between $z = \widehat{b}_1(\omega)$ and $z = \widehat{b}_1(\omega) - 1$. Finally, at the moment $\tau_1^* = \widehat{\tau}_1(\omega) + (r+1)$ the particle will pass the site $z = \widehat{b}_1(\omega)$ with positive velocity and travel until the closest site $z = \widehat{b}_2(\omega)$ with a backward scatterer. At the moment $t = \tau_1^*$ the BS located at $z = \widehat{b}_1(\omega) - 1$ will have the index equal $(r+1)/2$.

Therefore, after the moment $t = \tau_1^*$ the particle will move back and forth between the sites $z = \widehat{b}_1(\omega) - 1$ and $z = \widehat{b}_2(\omega)$ until the moment $t = \tau_2^*(\omega)$, when it will pass the site $z = \widehat{b}_2(\omega)$ with the positive velocity. We denote $\widehat{a}_2(\omega) = \widehat{b}_1(\omega) - 1$.

In the same manner one can construct intervals $\widehat{B}_i(\omega) = [\widehat{a}_i(\omega), \widehat{b}_i(\omega)]$ and the corresponding sequence of times $\widehat{\tau}_i(\omega)$, $i = 1, 2, \ldots$.

Let now the rigidity r is an even number. Then again the first segment of any orbit will travel till the closest to $z = 0$ site $z = b_1(\omega)$ with a backward scatterer. Then the particle will travel from $z = b_1(\omega)$ in the negative direction till the closest site $z = a_1(\omega) \leqslant 0$ with BS. After it reaches $z = a_1(\omega)$ the particle continues to move back and forth within the segment $B_1(\omega) = [a_1(\omega), b_1(\omega)]$.

The crucial difference with the case of odd rigidity is that at all internal sites of $B_1(\omega)$ will appear BS (with index 0) at the moment $\widetilde{\tau}_1(\omega)$, when the particle will return to the origin at the rth time. At $t = \widetilde{\tau}_1(\omega)$ the indices of BSs at $z = a_1(\omega)$ and $z = b_1(\omega)$ equal $r/2$.

Therefore it will take now a very long time for the particle to get out of the segment $B_1(\omega)$. Indeed, all scatterers located in the internal sites of this segment must change their type before that back to FS.

At the moment $\tau_1'(\omega)$ when it happens the particle will start again to move back and forth in $B_1(\omega)$ from its left end $a_1(\omega)$ till its right end $b_1(\omega)$. It is easy to see that at the rth visit of the particle to the origin $z = 0$ in the process of these consecutive trespassing of $B_1(\omega)$ at all internal sites of $B_1(\omega)$ will be BS with the index 0 while at $z = a_1(\omega)$ and $z = b_1(\omega)$ will be FS with the index 0.

Therefore after the next repetition of the same process of turning all BS at the internal sites of $B_1(\omega)$ into FS the particle will get out of $B_1(\omega)$ and will become confined to some interval $B_2(\omega) = [a_2(\omega), b_2(\omega)]$, where the similar process will take place. Here $z = a_2(\omega)$ ($z = b_2(\omega)$) is the closest to $a_1(\omega)$ ($b_1(\omega)$) site with negative (positive) coordinate where there is a BS. In the same way one can construct a sequence of closed segments $B_i(\omega)$, $i = 1, 2, \ldots$, with the required properties. The corresponding sequence $\tau_i(\omega)$, $i = 1, 2, \ldots$, is naturally defined by the condition that the particle remains confined to $B_i(\omega)$ until the moment $t = \tau_i(\omega) + 1$, when it leaves this segment at the first time. □

Proof of Corollary 2. — Consider the NOS-model with an even rigidity. Then it follows from the proof of Theorem 3 that the particle will visit the origin infinitely many times unless $a_i(\omega) = -\infty$ or/and $b_j(\omega) = \infty$ for some positive integers i, j. □

Proof of Theorem 4. — The case of an odd rigidity r has been considered in [**3**].

Let the rigidity r is an even number. Consider any site $z \in \mathbb{Z}$, $z > 0$. Denote by τ_z the moment of time when the particle visits the site z at the first time in such state that there is a forward scatterer at z. In other words $\tau_z = \tau_z(\omega)$ is the moment of the first visit of the particle to the site z if there was a FS at $t = 0$ at this site, or it is the moment of the first visit of the particle to the site z after a BS at this site has been changed to a FS.

It has been shown in [**3**] that the expectation of the random variable $\tau_z(\omega)$ equals

$$E\tau_z = 2r + 1 + (z-1)[(1-q) + r(1 + 4q + qz)] + q(3^{z-1} - z) \tag{16}$$

The analogous formula holds for $z \geqslant 0$. The statement of Theorem 4 for even rigidity immediately follows from Theorem 3 and (16). □

Lemma 5 is the immediate corollary of Theorem 1.

Lemma 6 is the immediate corollary of Theorem 3.

Lemma 7 follows from the proof of Theorem 3.

4. Concluding remarks

One may get the impression that the phenomena discussed and results obtained in this paper are essentially restricted to the one-dimensional case. It is, certainly, the simplest possible situation, when one studies walks in $\mathbb{Z}$ and, perhaps, it is not feasible to hope that the same type of comprehensive analysis would be possible for deterministic walks on some sufficiently general class of graphs.

However, various regimes of anomalous diffusion were observed in computer experiments with WRE (see e.g., [**8**]). For instance, the phenomenon of propagation in a random environment has been proven to exist [**13**] in the triangular lattice as well. It is worthwhile to mention that this propagation reminds very much the famous gliders in the Conway's Game of Life [**9**]. Observe though that the glider is just a particular solution to this dynamical system, while propagation in WRE takes place for any orbit of a certain deterministic walk in the triangular lattice. Moreover, this propagation occurs with random velocity, while in the Game of Life gliders always move with one and the same velocity. This and other features of WRE are currently explored in the theoretical computer sciences (see e.g., [**11**]).

We believe that the rigorous theory of Walks in Rigid Environments could be developed much farther. Although these dynamical systems demonstrate various features of stochastic (chaotic) behavior, their behavior is quite different from the one which we encounter in familiar classes of chaotic dynamical systems. For instance, these systems are nonexpansive [**5**].

On the other hand WRE provide clearer models than probabilistic models of various types of random walks and they do not require detailed assumptions about probability distributions involved on contrary to the purely probabilistic models (see e.g., [**15, 16, 17**]).

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