Astérisque

PATRICK POLO
JACQUES TILOUINE

Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $\mathbb{Z}_p$ for representations with p-small weights

Astérisque, tome 280 (2002), p. 97-135
<http://www.numdam.org/item?id=AST_2002__280__97_0>

© Société mathématique de France, 2002, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Astérisque
280, 2002, p. 97–135

BERNSTEIN-GELFAND-GELFAND COMPLEXES AND COHOMOLOGY OF NILPOTENT GROUPS OVER $\mathbb{Z}_{(p)}$ FOR REPRESENTATIONS WITH p-SMALL WEIGHTS

by

Patrick Polo & Jacques Tilouine

Abstract. — Given a connected reductive group defined and split over $\mathbb{Z}_{(p)}$, we study Bernstein-Gelfand-Gelfand complexes over $\mathbb{Z}_{(p)}$ and prove a $\mathbb{Z}_{(p)}$-analogue of Kostant's theorem computing the $\mathfrak{n}$-homology of the Weyl module $V(\lambda)$, when λ belongs to the closure of the fundamental p-alcove.

***Résumé* (Complexes de Bernstein-Gelfand-Gelfand et cohomologie de groupes nilpotents sur $\mathbb{Z}_{(p)}$ pour les représentations de poids p-petits)**

Étant donné un groupe réductif connexe défini et déployé sur $\mathbb{Z}$, nous étudions certains complexes de Bernstein-Gelfand-Gelfand sur $\mathbb{Z}_{(p)}$ et établissons un analogue sur $\mathbb{Z}_{(p)}$ d'un théorème de Kostant, en calculant la $\mathfrak{n}$-homologie du module de Weyl $V(\lambda)$ lorsque λ appartient à l'adhérence de la p-alcôve fondamentale.

Introduction

Let G be a connected reductive linear algebraic group defined and split over $\mathbb{Z}$, let T be a maximal torus, W the Weyl group, R the root system, $R^\vee$ the set of coroots, R^+ a set of positive roots, and ρ the half-sum of the elements of R^+. Let $X = X(T)$ be the character group of T and let X^+ be the set of those $\lambda \in X$ such that $\langle \lambda, \alpha^\vee \rangle \geqslant 0$ for all $\alpha \in R^+$.

For any $\lambda \in X^+$, let $V_{\mathbb{Z}}(\lambda)$ be the Weyl module for G over $\mathbb{Z}$ with highest weight λ (see 1.3) and, for any commutative ring A, let $V_A(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} A$.

Let p be a prime integer and let

$$\overline{C}_p := \{\nu \in X \mid 0 \leqslant \langle \nu + \rho, \beta^\vee \rangle \leqslant p, \quad \forall \beta \in R^+\},$$

the closure of the fundamental p-alcove.

The aim of this paper is to prove that several results about $V_{\mathbb{Q}}(\lambda)$, due to Kostant [**33**], Bernstein-Gelfand-Gelfand [**3**], Lepowsky [**37**], Rocha [**46**], and Pickel [**43**], hold

2000 ***Mathematics Subject Classification.*** — 17B50, 17B56, 20G30, 20J06, 57T15.
Key words and phrases. — BGG complexes, (co)homology of nilpotent groups, Weyl modules.

© Astérisque 280, SMF 2002

true over $\mathbb{Z}_{(p)}$ when $\lambda \in X^+ \cap \overline{C}_p$: this is the precise meaning of the notion of p-smallness mentioned in the title.

In more details, let B be the Borel subgroup corresponding to R^+, let P be a standard parabolic subgroup containing B, let P^- be the opposed parabolic subgroup containing T, let U_P^- be its unipotent radical, and let $L = P \cap P^-$, a Levi subgroup. Let R_L be the root system of L, let $R_L^+ = R_L \cap R^+$, and

$$X_L^+ := \{\xi \in X \mid \langle \xi, \alpha^\vee \rangle \geqslant 0, \quad \forall \alpha \in R_L^+\}.$$

For any $\xi \in X_L^+$ and any commutative ring A, let $V_A^L(\xi)$ be the Weyl module for L over A with highest weight ξ.

Let $\mathfrak{g}, \mathfrak{p}, \mathfrak{u}_P^-$ be the Lie algebras ove. $\mathbb{Z}$ of G, P, U_P^-, respectively, and let $U(\mathfrak{g})$ and $U(\mathfrak{p})$ be the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{p}$. For $\xi \in X_L^+$, consider the generalized Verma module

$$M_{\mathfrak{p}}^{\mathbb{Z}}(\xi) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_{\mathbb{Z}}^L(\xi).$$

For any commutative ring A, let $M_{\mathfrak{p}}^A(\xi) = M_{\mathfrak{p}}^{\mathbb{Z}}(\xi) \otimes_{\mathbb{Z}} A$.

Let $N = |R^+|$ and, for $i = 0, 1, \dots, N$, let $W(i) := \{w \in W \mid \ell(w) = i\}$, where ℓ denotes the length function on W relative to B. Further, let

$$W^L = \{w \in W \mid wX^+ \subseteq X_L^+\} \quad \text{and} \quad W^L(i) := W^L \cap W(i).$$

After several recollections in Section 1, we prove in Section 2 the following Theorem (under certain restrictions on G and p, see 2.8).

Theorem A. — *Let $\lambda \in X^+ \cap \overline{C}_p$. There exists an exact sequence of $U(\mathfrak{g})$-modules:*

$$0 \longrightarrow D_N(\lambda) \longrightarrow \cdots \longrightarrow D_0(\lambda) \longrightarrow V_{\mathbb{Z}_{(p)}}(\lambda) \longrightarrow 0,$$

where each $D_i(\lambda)$ admits a finite filtration of $U(\mathfrak{g})$-submodules with associated graded

$$\operatorname{gr} D_i(\lambda) \cong \bigoplus_{w \in W^L(i)} M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(w(\lambda+\rho)-\rho).$$

That is, following the terminology introduced in [**46**], $V_{\mathbb{Z}_{(p)}}(\lambda)$ admits a weak generalized Bernstein-Gelfand-Gelfand resolution. From this, one obtains immediately the following (see 2.1 and 2.9).

***Theorem B* (Kostant's theorem over $\mathbb{Z}_{(p)}$).** — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each i, there is an isomorphism of L-modules:*

$$H_i(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V_{\mathbb{Z}_{(p)}}^L(w(\lambda+\rho)-\rho).$$

Let $\Gamma := U_P^-(\mathbb{Z})$ be the group of $\mathbb{Z}$-points of U_P^-, it is a finitely generated, torsion free, nilpotent group. By a result of Pickel [**43**], there is a natural isomorphism $H_*(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda)) \cong H_*(\Gamma, V_{\mathbb{Q}}(\lambda))$. In Section 3, we prove a slightly weaker version of this result over $\mathbb{Z}_{(p)}$ when λ is p-small (see 3.8).

Theorem C. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each $n \geqslant 0$, $H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda))$ has a natural $L(\mathbb{Z})$-module filtration such that*

$$\operatorname{gr} H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(n)} V^L_{\mathbb{Z}_{(p)}}(w(\lambda+\rho)-\rho).$$

The proof of this result has two parts. In the first, we develop certain general results valid for any finitely generated, torsion free, nilpotent group Γ. In particular, using a beautiful theorem of Hartley [**22**], we obtain in an algebraic manner a spectral sequence relating the homology of a certain graded, torsion-free, Lie ring $\operatorname{gr}_{\mathrm{isol}} \Gamma$ associated with Γ to the homology of Γ itself, the coefficients being a Γ-module with a "nilpotent" filtration and its associated graded (see Theorem 3.5). This gives a purely algebraic, homological version (with coefficients) of a cohomological spectral sequence obtained, using methods of algebraic topology, by Cenkl and Porter [**9**]. In fact, our methods also have a cohomological counterpart. This will be developped in a subsequent paper [**44**].

In the second part of the proof, we first show that in our case where $\Gamma = U_P^-(\mathbb{Z})$, one has $\operatorname{gr}_{\mathrm{isol}} \Gamma \cong \mathfrak{u}_P^-$, and then deduce from the truth of Kostant's theorem over $\mathbb{Z}_{(p)}$ that the spectral sequence mentioned above degenerates at E_1.

Next, in Section 4, we obtain a result à la Bernstein-Gelfand-Gelfand concerning now the distribution algebras $\operatorname{Dist}(G)$ and $\operatorname{Dist}(P)$. In this case, there exists a standard *complex* (not a resolution!)

$$\mathcal{S}_\bullet(G,P,\lambda) = \operatorname{Dist}(G) \otimes_{\operatorname{Dist}(P)} (\Lambda^\bullet(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)).$$

For $\xi \in X_L^+$, consider the generalized Verma module (for $\operatorname{Dist}(G)$ and $\operatorname{Dist}(P)$)

$$\mathcal{M}_P^{\mathbb{Z}}(\xi) := \operatorname{Dist}(G) \otimes_{\operatorname{Dist}(P)} V_{\mathbb{Z}}^L(\xi),$$

and, for any commutative ring A, set $\mathcal{S}_\bullet^A(G,P,\lambda) = \mathcal{S}_\bullet(G,P,\lambda) \otimes_{\mathbb{Z}} A$ and $\mathcal{M}_P^A(\xi) = \mathcal{M}_P^{\mathbb{Z}}(\xi) \otimes_{\mathbb{Z}} A$.

Under the assumption that $\mathfrak{u}_P^-$ is abelian, we obtain, by using an idea borrowed from [**16**, § VI.5] plus arguments from Section 2, the following result (see 4.3). Let $\mathcal{D}G$ denote the derived subgroup of G.

Theorem D. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion and that $\mathfrak{u}_P^-$ is abelian. Let $\lambda \in X^+ \cap \overline{C}_p$. Then the standard complex $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ contains as a direct summand a subcomplex $\mathcal{C}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ such that, for every $i \geqslant 0$,*

$$\mathcal{C}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda) \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w(\lambda+\rho)-\rho).$$

Presumably, the hypothesis that $\mathfrak{u}_P^-$ be abelian can be removed, but the proof would then require considerably more work. Since the abelian case is sufficient for the applications in the companion paper by A. Mokrane and J. Tilouine [**39**], we content ourselves with this result. We hope to come back to the general case later.

To conclude this introduction, let us mention that the results of this text are used in [**39**] in the case where G is the group of symplectic similitudes. In this case, $\mathcal{D}G$ is simply-connected and $\mathbb{Z}R$ is a direct summand of $X(T)$. When P is the Siegel parabolic, Theorem D occurs in [**39**, § 5.4] as an important step to establish a modulo p analogue of the Bernstein-Gelfand-Gelfand complex of [**16**, Chap.VI, Th. 5.5], while Theorem C (in its cohomological form) is used in [**39**, § 8.3] to study mod. p versions of Pink's theorem on higher direct images of automorphic bundles.

The notations of [**39**] follow those of [**16**] and are therefore different from the ones used in the present paper, which are standard in the theory of reductive groups. A dictionary is provided in the final section of this text.

Acknowledgements. — We thank the referee for his/her thoughtful reading of the manuscript and for several valuable suggestions (including references [**4**] and [**29**]). We also thank Jens Carsten Jantzen for providing us with a copy of his article [**29**].

1. Notation and preliminaries

1.1. Let G be a connected reductive linear algebraic group, defined and split over $\mathbb{Z}$. Let T be a maximal torus, W the Weyl group, R the root system and $R^\vee$ the set of coroots. Fix a set Δ of simple roots, let R^+ and R^- be the corresponding sets of positive and negative roots, and let B, B^- denote the associated Borel subgroups and U, U^- their unipotent radicals. (For all this, see, for example, [**11**] or [**28**, § II.1]).

Let $X = X(T)$ (resp. $X^\vee = X^\vee(T)$) be the group of characters (resp. cocharacters) of T, and denote by $\langle\ ,\ \rangle$ the natural pairing between them. Elements of X will be called weights, in accordance with the terminology in Lie theory. Let $\leqslant$ denote the partial order on X defined by the positive cone $\mathbb{N}R^+$, that is, $\mu \leqslant \lambda$ if and only if $\lambda - \mu \in \mathbb{N}R^+$. Let $\mathbb{Z}R \subset X$ be the root lattice and let ρ be the half-sum of the positive roots; it belongs to $X \otimes \mathbb{Z}[1/2]$. Define, as usual, the dot action of W on X by

$$w \cdot \lambda = w(\lambda + \rho) - \rho,$$

for $\lambda \in X, w \in W$. It is easy to see that $w\rho - \rho \in \mathbb{Z}R$: applying w to the equality $2\rho = \sum_{\beta \in R^+} \beta$ and substracting, one obtains the well-known formula

$$(*) \qquad \rho - w\rho = \sum_{\beta \in R^+, w^{-1}\beta \in R^-} \beta.$$

Therefore, denoting by $N(w)$ the term on the right hand-side of $(*)$, one may also define the dot action by the formula

$$w \cdot \lambda = w\lambda - N(w),$$

from which it is clear that $w \cdot \lambda$ does indeed belong to X.

Let X^+ be the set of dominant weights:

$$X^+ := \{\lambda \in X \mid \forall \alpha \in R^+, \quad \langle \lambda, \alpha^\vee \rangle \geqslant 0\},$$

where $\alpha^\vee$ denotes the coroot associated with α.

1.2. Enveloping and distribution algebras.— Let $\mathfrak{g} = \mathrm{Lie}(G)$ (resp. $\mathfrak{t} = \mathrm{Lie}(T)$) be the Lie algebra of G (resp. T); they are finite free $\mathbb{Z}$-modules. Let $U(\mathfrak{g})$ denote the enveloping algebra of $\mathfrak{g}$ over $\mathbb{Z}$, and let $\mathrm{Dist}(G)$ denote the algebra of distributions of G (see [**28**, Chap. I.7]). If G is semi-simple and simply-connected, $\mathrm{Dist}(G)$ coincides with the Kostant $\mathbb{Z}$-form of $U(\mathfrak{g})$ ([**34**]), see [**28**, § II.1.12] or [**5**, VIII, §§ 12.6–8]. We shall denote it by $\mathcal{U}_{\mathbb{Z}}(\mathfrak{g})$ or simply $\mathcal{U}(\mathfrak{g})$; sometimes it will also be convenient to denote it by $\mathcal{U}_{\mathbb{Z}}(G)$.

Similarly, if H is a closed subgroup of G defined over $\mathbb{Z}$, we shall denote $\mathrm{Dist}(H)$ also by $\mathcal{U}_{\mathbb{Z}}(H)$.

By an H-module we shall mean a rational H-module, that is, a $\mathbb{Z}[H]$-comodule. More generally, for any commutative ring A, an H_A-module means an A-module with a structure of $A[H]$-comodule. If V is an H-module, then, as is well-known, V is also an $\mathcal{U}_{\mathbb{Z}}(H)$-module and a fortiori an $U(\mathrm{Lie}(H))$-module.

If M is a T-module, it is the direct sum of its weight spaces M_λ, for $\lambda \in X$, see, for example, [**28**, § I.2.11].

For future use, let us record here the following

Proposition. — *Let P be a standard parabolic subgroup of G, let V be a finite dimensional $P_{\mathbb{Q}}$-module and let M be a $\mathbb{Z}$-lattice in V. Then M is a P-submodule if and only if it is an $\mathcal{U}_{\mathbb{Z}}(P)$-submodule.*

Proof. — Without loss of generality we may assume that P contains B. Let P^- be the opposed standard parabolic subgroup and let U_P^- be its unipotent radical. By the Bruhat decomposition, the multiplication map induces an isomorphism of $U_P^- \times B$ onto an open subset of P, see, for example, [**28**, § II.1.10]. This implies that the arguments in [**28**, II.8.1] are valid for P, and the proposition then follows from [**28**, I.10.13].

1.3. Weyl modules.— For $\lambda \in X^+$, let $V_{\mathbb{Q}}(\lambda)$ denote the irreducible $G_{\mathbb{Q}}$-module with highest weight λ, and let $V_{\mathbb{Z}}(\lambda)$ be the corresponding Weyl module over $\mathbb{Z}$; that is,

$$V_{\mathbb{Z}}(\lambda) := \mathcal{U}_{\mathbb{Z}}(G)v_\lambda$$

is the $\mathcal{U}_{\mathbb{Z}}(G)$-submodule generated by a fixed vector $v_\lambda \neq 0$ of weight λ. It is a G-module by Proposition 1.2 above. Of course, up to isomorphism, $V_{\mathbb{Z}}(\lambda)$ does not depend on the choice of v_λ. For future use, let us also record the following (obvious) lemma.

Lemma. — *Let M be a $\mathbb{Z}$-free, G-module and $v \in M$ an element fixed by U and of weight λ. Then the submodule $\mathcal{U}_{\mathbb{Z}}(G)v$ is isomorphic to $V_{\mathbb{Z}}(\lambda)$.*

Proof. — The $\mathcal{U}_{\mathbb{Q}}(G)$-submodule of $M \otimes \mathbb{Q}$ generated by v is isomorphic to $V_{\mathbb{Q}}(\lambda)$.

1.4. Contravariant duals.— Let us fix an anti-involution τ of G which is the identity on T and exchanges B and B^- (see [**28**, II.1.16]). Then τ induces anti-involutions on $\mathcal{U}_{\mathbb{Z}}(G)$, on $\mathfrak{g}$ and on $U_{\mathbb{Z}}(\mathfrak{g})$, which we denote by the same letter τ.

For any ring A and G_A-module V, let us denote by V^τ the A-dual $\mathrm{Hom}_A(V, A)$, regarded as a G_A-module via τ. It may be called the "contravariant dual" of V, as for $V = V_{\mathbb{Z}}(\lambda)$ this is closely related to the so-called "contravariant form" on $V_{\mathbb{Z}}(\lambda)$; see [**28**, II.8.17] and the discussion in the next subsection 1.5.

Note that if V is a free A-module, the weights of T in V and V^τ are the same. In particular, the irreducible $G_{\mathbb{Q}}$-modules $V_{\mathbb{Q}}(\lambda)$ and $V_{\mathbb{Q}}(\lambda)^\tau$ are isomorphic.

1.5. Admissible lattices.— For use in the companion article by Mokrane and Tilouine [**39**] and also in the next subsection, let us discuss some properties of admissible lattices. Of course, this is fairly well-known to representation theorists, but we spell out the details for the convenience of readers with a different background.

As noted above, we may identify $V_{\mathbb{Q}}(\lambda) = V_{\mathbb{Q}}(\lambda)^\tau$. Under this identification, $V_{\mathbb{Q}}(\lambda)$ becomes equipped with a non-degenerate, G-invariant bilinear form $\langle\ ,\ \rangle$ such that

$$(*) \qquad \langle gv, v'\rangle = \langle v, \tau(g)v'\rangle \quad \text{and} \quad \langle Xv, v'\rangle = \langle v, \tau(X)v'\rangle,$$

for $v, v' \in V_{\mathbb{Q}}(\lambda)$, $g \in G$, $X \in \mathcal{U}_{\mathbb{Z}}(G)$. (This is the contravariant form mentioned in the previous subsection).

Let us fix, once for all, a non-zero vector $v_\lambda \in V_{\mathbb{Q}}(\lambda)_\lambda$. The identification $V_{\mathbb{Q}}(\lambda) = V_{\mathbb{Q}}(\lambda)^\tau$ may be chosen so that $\langle v_\lambda, v_\lambda\rangle = 1$.

Recall that a $\mathbb{Z}$-lattice $\mathcal{L} \subset V_{\mathbb{Q}}(\lambda)$ is called an admissible lattice if it is stable under $\mathcal{U}_{\mathbb{Z}}(G)$. By Proposition 1.2, this implies that $\mathcal{L}$ is a G-module and is therefore the direct sum of its T-weight spaces.

Let $\mathcal{E}(\lambda)$ denote the set of admissible lattices $\mathcal{L} \subset V_{\mathbb{Q}}(\lambda)$ such that $\mathcal{L} \cap V_{\mathbb{Q}}(\lambda)_\lambda = \mathbb{Z}v_\lambda$. Clearly, $V_{\mathbb{Z}}(\lambda) := \mathcal{U}_{\mathbb{Z}}(G)v_\lambda$ is the unique minimal element of $\mathcal{E}(\lambda)$.

For any $\mathcal{L} \in \mathcal{E}(\lambda)$, the dual G-module $\mathcal{L}^\tau$ identifies with

$$\{x \in V_{\mathbb{Q}}(\lambda) \mid \langle x, \mathcal{L}\rangle \subseteq \mathbb{Z}\}.$$

It follows from $(*)$ that $\mathcal{L}^\tau$ is an admissible lattice, and since $\langle v_\lambda, v_\lambda\rangle = 1$ it belongs to $\mathcal{E}(\lambda)$. Therefore, $\mathcal{L}^\tau \supseteq V_{\mathbb{Z}}(\lambda)$ and hence $\mathcal{L} \subseteq V_{\mathbb{Z}}(\lambda)^\tau$. Let us record this as the next

Lemma. — *The set of admissible lattices $\mathcal{L} \subset V_{\mathbb{Q}}(\lambda)$ such that $\mathcal{L} \cap V_{\mathbb{Q}}(\lambda)_\lambda = \mathbb{Z}v_\lambda$ contains a unique minimal element, $V_{\mathbb{Z}}(\lambda)$, and a unique maximal element, $V_{\mathbb{Z}}(\lambda)^\tau$.*

The above minimal and maximal lattices are denoted by $V(\lambda)_{\min}$ and $V(\lambda)_{\max}$ in [**39**] and in Section 5 below.

1.6. Weyl modules and induced modules.— Let us recall the definition of the induction functor $\mathrm{Ind}_{B^-}^G$. For any B^--module M,

$$\mathrm{Ind}_{B^-}^G(M) := (\mathbb{Z}[G] \otimes M)^{B^-},$$

where $\mathbb{Z}[G]$ is regarded as a $G\times B^-$-module via $((g,b)\phi)\,(g') = \phi(g^{-1}g'b)$, for $g, g' \in G$, $b \in B^-$ and where the invariants are taken with respect to the diagonal action of B^-; it is a left exact functor, see [**28**, § I.3.3]. As in [**28**, § II.2.1], we shall denote simply by $H^i(\ \ )$ the right derived functors $R^i\operatorname{Ind}_{B^-}^G(\ \ )$.

Let $\lambda \in X$; it may be regarded in a natural manner as a character of either B^- or B. Moreover, since τ is the identity on T, one has $\lambda(\tau(b)) = \lambda(b)$ for any $b \in B^-$.

For any ring A, let us denote by A_λ the free A-module of rank one on which B^- acts via the character λ. Then,

$$H^0(A_\lambda) \cong \{\phi \in A[G] \mid \phi(gb) = \lambda(b^{-1})\phi(g),\ \ \forall\, g \in G, b \in B^-\}.$$

Proposition. — *Let $\lambda \in X^+$.*

a) $H^0(\mathbb{Z}_\lambda) \cong V_{\mathbb{Z}}(\lambda)^\tau$.

b) *If k is a field, $H^0(k_\lambda) \cong H^0(\mathbb{Z}_\lambda) \otimes k \cong V_k(\lambda)^\tau$. Thus, in particular, $V_k(\lambda)$ is irreducible if and only if $H^0(k_\lambda)$ is so.*

Proof. — First, by flat base change ([**28**, I.3.5]), one has $H^0(\mathbb{Z}_\lambda) \otimes \mathbb{Q} \cong H^0(\mathbb{Q}_\lambda)$. Moreover, $H^0(\mathbb{Q}_\lambda) \cong V_{\mathbb{Q}}(\lambda)$, by the theorem of Borel-Weil-Bott (see, for example, [**28**, II.5.6]).

Further, since $\mathbb{Z}[G]$ is a free $\mathbb{Z}$-module (being a subring of $\mathbb{Z}[U] \otimes \mathbb{Z}[B^-]$), so is $H^0(\mathbb{Z}_\lambda)$. Therefore, $H^0(\mathbb{Z}_\lambda)$ may be identified with a G-submodule of $V_{\mathbb{Q}}(\lambda)$, and the identification may be chosen so that $H^0(\mathbb{Z}_\lambda) \cap V_{\mathbb{Q}}(\lambda)_\lambda = \mathbb{Z}v_\lambda$, *i.e.*, so that $H^0(\mathbb{Z}_\lambda)$ belongs to $\mathcal{E}(\lambda)$.

Now, there is a natural G-module map $\phi : V_{\mathbb{Z}}(\lambda)^\tau \to H^0(\mathbb{Z}_\lambda)$ defined by

$$x \longmapsto \big(g \mapsto \langle x, \tau(g^{-1})v_\lambda\rangle\big)\,.$$

Moreover, since $V_{\mathbb{Z}}(\lambda)$ is generated by v_λ as a G-module, ϕ is injective. Since $V_{\mathbb{Z}}(\lambda)^\tau$ is the largest element of $\mathcal{E}(\lambda)$, this implies that ϕ induces an isomorphism $V_{\mathbb{Z}}(\lambda)^\tau \cong H^0(\mathbb{Z}_\lambda)$. This proves assertion a).

Let us prove assertion b). For each $i \geqslant 0$, there is an exact sequence

$$0 \longrightarrow H^i(\mathbb{Z}_\lambda) \otimes k \longrightarrow H^i(k_\lambda) \longrightarrow \operatorname{Tor}^{\mathbb{Z}}(H^{i+1}(\mathbb{Z}_\lambda), k) \longrightarrow 0,$$

see [**28**, I.4.18]. Next, by Kempf's vanishing theorem ([**28**, II.4.6]), one has $H^i(\mathbb{Z}_\lambda) = 0$ for $i \geqslant 1$. The first isomorphism of assertion b) follows. Finally, the second is a consequence of assertion a) and the natural isomorphisms

$$\operatorname{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}(\lambda), \mathbb{Z}) \otimes k \cong \operatorname{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}(\lambda), k) \cong \operatorname{Hom}_k(V_k(\lambda), k).$$

This completes the proof of the proposition.

1.7. Parabolic subgroups and unipotent radicals.— Now, let P be a standard parabolic subgroup of G containing B, let L be the Levi subgroup of P containing T, and let P^- be the standard parabolic subgroup opposed to P, that is, P^- is the unique parabolic subgroup containing B^- such that $P^- \cap P = L$.

Let U_P^- (resp. U_P) denote the unipotent radical of P^- (resp. P) and let $\mathfrak{u}_P^- = \mathrm{Lie}(U_P^-)$, $\mathfrak{u}_P = \mathrm{Lie}(U_P)$ and $\mathfrak{p} = \mathrm{Lie}(P)$. Then $\mathfrak{u}_P^-$, $\mathfrak{u}_P$ and $\mathfrak{p}$ are free $\mathbb{Z}$-modules and $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}_P^-$. Thus, in particular, $\mathfrak{g}/\mathfrak{p}$ is a free $\mathbb{Z}$-module.

Further, if V is a P-module then, by standard arguments, the homology groups

$$H_i(\mathfrak{u}_P^-, V) := \mathrm{Tor}_i^{U(\mathfrak{u}_P^-)}(\mathbb{Z}, V)$$

carry a natural structure of L-modules. For example, they can be computed as the homology of the standard Chevalley-Eilenberg complex $\Lambda^\bullet(\mathfrak{u}_P^-) \otimes V$, which carries a natural action of L.

For any commutative ring A, we set $V_A(\lambda) := V_{\mathbb{Z}}(\lambda) \otimes A$ and $\mathfrak{g}_A := \mathfrak{g} \otimes A$. The enveloping algebra of $\mathfrak{g}_A$ identifies with $U_{\mathbb{Z}}(\mathfrak{g}) \otimes A$ and is denoted by $U_A(\mathfrak{g})$. One defines similarly $U_A(\mathfrak{u}_P^-)$ and $\mathcal{U}_A(\mathfrak{g})$, etc...

Since $U_{\mathbb{Z}}(\mathfrak{u}_P^-)$ is a free $\mathbb{Z}$-module, one has, for every $i \geqslant 0$,

$$\mathrm{Tor}_i^{U_A(\mathfrak{u}_P^-)}(A, V_A(\lambda)) \cong \mathrm{Tor}_i^{U_{\mathbb{Z}}(\mathfrak{u}_P^-)}(\mathbb{Z}, V_A(\lambda)).$$

We shall denote these groups simply by $H_i(\mathfrak{u}_P^-, V_A(\lambda))$; as noted above they are L_A-modules.

Our goal in Section 2 is to show that celebrated results of Kostant ([**33**, Cor. 8.1]) and Bernstein-Gelfand-Gelfand ([**3**, Th. 9.9]), which describe respectively, for any $\lambda \in X^+$, the L-module structure of $H_\bullet(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda))$ and a minimal $U_{\mathbb{Q}}(\mathfrak{u}_P^-)$-resolution of $V_{\mathbb{Q}}(\lambda)$, hold true when $\mathbb{Q}$ is replaced by $\mathbb{Z}_{(p)}$, for any prime integer p such that

$$p \geqslant \langle \lambda + \rho, \alpha^\vee \rangle, \quad \forall \alpha \in R^+.$$

1.8. Weyl modules for a Levi subgroup.— We need to introduce more notation. Let W_L and R_L denote the Weyl group and root system of L, and let $R_L^\pm := R_L \cap R^\pm$. Let X_L^+ denote the set of L-dominant weights:

$$X_L^+ := \{\lambda \in X \mid \forall \alpha \in R_L^+, \quad \langle \lambda, \alpha^\vee \rangle \geqslant 0\}.$$

Let $W^L := \{w \in W \mid wX^+ \subseteq X_L^+\}$. It is well-known, and easy to check, that W^L is also equal to $\{w \in W \mid w^{-1}R_L^+ \subseteq R^+\}$.

Let ℓ and $\leqslant$ denote the length function and Bruhat-Chevalley order on W associated with the set Δ of simple roots. Then, for $i \geqslant 0$, set

$$W(i) := \{w \in W \mid \ell(w) = i\} \quad \text{and} \quad W^L(i) := W^L \cap W(i).$$

For any $\xi \in X_L^+$, let $V_{\mathbb{Q}}^L(\xi)$ denote the irreducible $L_{\mathbb{Q}}$-module with highest weight ξ and let $V_{\mathbb{Z}}^L(\xi)$ be the corresponding Weyl module for L. Observe that $V_{\mathbb{Q}}^L(\xi)$ (and then $V_{\mathbb{Z}}^L(\xi)$) identifies with the $L_{\mathbb{Q}}$-submodule of $V_{\mathbb{Q}}(\xi)$ (resp. L-submodule of $V_{\mathbb{Z}}(\xi)$) generated by v_ξ.

More generally, one has the following

Lemma. — *Let M be a P-module which is $\mathbb{Z}$-free and let $v \in M$ be a non-zero element of weight ξ. Assume that v is U-invariant (this is the case, for instance, if ξ is a maximal weight of M). Then the $\mathcal{U}_{\mathbb{Z}}(P)$-submodule of M generated by v is isomorphic to $V_{\mathbb{Z}}^L(\xi)$.*

Proof. — Recall that $\mathcal{U}_{\mathbb{Z}}(P) \cong \mathcal{U}_{\mathbb{Z}}(L) \otimes \mathcal{U}_{\mathbb{Z}}(U_P)$ (see [**28**, § II.1.12]). Since v is fixed by U, it is annihilated by the augmentation ideal of $\mathcal{U}_{\mathbb{Z}}(U_P)$. Therefore, $\mathcal{U}_{\mathbb{Z}}(P)v = \mathcal{U}_{\mathbb{Z}}(L)v$ and, since M is $\mathbb{Z}$-free, the result follows from Lemma 1.3.

1.9. The fundamental p-alcove.— In this subsection and the next one, let p be a prime integer. The notion of p-smallness mentioned in the title of this article is defined as follows. We shall say that $\lambda \in X$ is p-small if it satisfies the condition:

$$(\dagger) \qquad \langle \lambda + \rho, \alpha^\vee \rangle \leqslant p, \quad \forall\, \alpha \in R.$$

An equivalent definition of p-smallness is as follows. Let W_p denote the affine Weyl group with respect to p. Recall that W_p is the subgroup of automorphisms of $X(T) \otimes \mathbb{R}$ generated by the reflections $s_{\beta,np}$, for $\beta \in R^+$, $n \in \mathbb{Z}$, where, for $\lambda \in X(T) \otimes \mathbb{R}$,

$$s_{\beta,np}(\lambda) = \lambda - (\langle \lambda, \beta^\vee \rangle - np)\beta,$$

and that W_p is the semi-direct product of W and the group $p\mathbb{Z}R$ acting by translations. We consider the dot action of W_p on $X(T) \otimes \mathbb{R}$, defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

The fundamental p-alcove C_p is defined by

$$C_p := \{\lambda \in X(T) \otimes \mathbb{R} \mid 0 < \langle \lambda + \rho, \beta^\vee \rangle < p, \ \ \forall\, \beta \in R^+\}.$$

Its closure

$$\overline{C}_p := \{\lambda \in X(T) \otimes \mathbb{R} \mid 0 \leqslant \langle \lambda + \rho, \beta^\vee \rangle \leqslant p, \ \ \forall\, \beta \in R^+\}$$

is a fundamental domain for the dot action of W_p on $X(T) \otimes \mathbb{R}$ (for all this, see for example [**28**, § II.6.1]).

Then, for $\lambda \in X^+$, the condition of p-smallness is equivalent to the requirement that λ belongs to $\overline{C}_p$. Thus, an arbitrary $\lambda \in X$ is p-small if and only if it belongs to $W \cdot \overline{C}_p$.

Let ρ_L be the half-sum of the elements of R_L^+. Note that $\langle \rho_L, \alpha^\vee \rangle = 1$ for any $\alpha \in \Delta \cap R_L$ and hence $\rho - \rho_L$ vanishes on R_L. Therefore, if a weight $\xi \in X_L^+$ is p-small, it is a fortiori p-small for L.

The fact that $V_{\mathbb{F}_p}(\lambda)$ is irreducible when λ is p-small is of course very well-known to representation-theorists; for the convenience of readers with a different background, we record this here as the next

Lemma. — *Let $\lambda \in X^+$ and $\xi \in X_L^+$. If λ (resp. ξ) is p-small, $V_{\mathbb{F}_p}(\lambda)$ (resp. $V_{\mathbb{F}_p}^L(\xi)$) is irreducible and self-dual for the contravariant duality.*

Proof. — The first assertion is a consequence of [**28**, II.8.3], combined with Proposition 1.6. Further, since irreducible $G_{\mathbb{F}_p}$-modules are determined by their highest weight, the second assertion follows from the first.

Corollary. — *If $\lambda \in X^+ \cap \overline{C}_p$ then, for any $\Lambda \in \mathcal{E}(\lambda)$, one has*

$$V_{\mathbb{Z}_{(p)}}(\lambda) = \Lambda \otimes \mathbb{Z}_{(p)} = V_{\mathbb{Z}_{(p)}}(\lambda)^\tau.$$

Proof. — By the previous lemma, one has $V_{\mathbb{F}_p}(\lambda) = V_{\mathbb{F}_p}(\lambda)^\tau$. The result then follows by Nakayama's lemma.

1.10. A vanishing result.— Let us record the following

Lemma. — *For all $\lambda, \mu \in X^+$, one has $\operatorname{Ext}^1_G(V_{\mathbb{F}_p}(\lambda), V_{\mathbb{F}_p}(\mu)^\tau) = 0$ and also*

$$\operatorname{Ext}^1_G(V_{\mathbb{Z}}(\lambda), V_{\mathbb{Z}}(\mu)^\tau) = 0 = \operatorname{Ext}^1_G(V_{\mathbb{Z}_{(p)}}(\lambda), V_{\mathbb{Z}_{(p)}}(\mu)^\tau).$$

Proof. — Since $V_{\mathbb{F}_p}(\mu)^\tau \cong H^0(\mu)$, by Proposition 1.6, the assertion over $\mathbb{F}_p$ is a consequence of [**28**, Prop. II.4.13]. The assertions over $\mathbb{Z}$ or $\mathbb{Z}_{(p)}$ then follow from a theorem of universal coefficients [**28**, Prop. I.4.18].

Corollary. — *Suppose that $\lambda, \mu \in X^+ \cap \overline{C}_p$. Then*

$$\operatorname{Ext}^1_G(V_{\mathbb{F}_p}(\lambda), V_{\mathbb{F}_p}(\mu)) = 0 = \operatorname{Ext}^1_G(V_{\mathbb{Z}_{(p)}}(\lambda), V_{\mathbb{Z}_{(p)}}(\mu)).$$

Proof. — By the results in 1.9, $V_{\mathbb{F}_p}(\mu)$ and $V_{\mathbb{Z}_{(p)}}(\mu)$ are self-dual. Thus, the corollary follows from the previous lemma.

1.11. We shall need later the following lemma. Recall that U_P denotes the unipotent radical of P and that one has $P = L \ltimes U_P$.

Lemma. — *Let M be a P-module, finite free over $\mathbb{Z}_{(p)}$. Assume that each weight ν of M satisfies $\langle \nu + \rho, \alpha^\vee \rangle \leqslant p$, for any $\alpha \in R_L$.*

a) *There exists a sequence of P-submodules $0 = M_0 \subset \cdots \subset M_r = M$ such that*

$$M_i/M_{i-1} \cong V^L_{\mathbb{Z}_{(p)}}(\xi_i), \text{ where } \xi_i \in X^+_L \text{ and } \xi_j \leqslant \xi_i \text{ if } j \geqslant i.$$

The set $\{\xi_1, \ldots, \xi_r\}$ is uniquely determined by M; in fact the $V^L_{\mathbb{Q}}(\xi_i)$ are the irreducible composition factors of the $L_{\mathbb{Q}}$-module $M_{\mathbb{Q}}$.

b) *Moreover, there is an isomorphism of L-modules $M_{|L} \cong \bigoplus_{i=1}^r V^L_{\mathbb{Z}_{(p)}}(\xi_i)$. In particular, if U_P acts trivially on M, then $M \cong \bigoplus_{i=1}^r V^L_{\mathbb{Z}_{(p)}}(\xi_i)$.*

Proof. — Let us prove assertion a) by induction on the rank of M, following [**15**, Lemma 11.5.3]. There is nothing to prove if $M = 0$. If $M \neq 0$, let ξ_1 be a maximal weight of M, let $v \in M$ be a primitive element of weight ξ_1 and denote by N the $\mathcal{U}_{\mathbb{Z}_{(p)}}(P)$-submodule generated by v. Then $N \cong V^L_{\mathbb{Z}_{(p)}}(\xi_1)$, by Lemma 1.8. By assumption, $\xi_1 \in \overline{C}_p$ and hence $N_{\mathbb{F}_p} := N \otimes \mathbb{F}_p$ is irreducible.

On the other hand, since M is free over $\mathbb{Z}_{(p)}$, one obtains an exact sequence of P-modules

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}_{(p)}}(M/N, \mathbb{F}_p) \longrightarrow N_{\mathbb{F}_p} \xrightarrow{\phi} M_{\mathbb{F}_p},$$

and $\phi(v) \neq 0$, as v is a primitive element. Since $N_{\mathbb{F}_p}$ is irreducible, ϕ is injective. Thus, $\mathrm{Tor}_1^{\mathbb{Z}_{(p)}}(M/N, \mathbb{F}_p) = 0$ and this implies that M/N is free over $\mathbb{Z}_{(p)}$. Since M/N has smaller rank than M, the first part of assertion a) follows by the inductive hypothesis. The second part is then clear.

Finally, the first part of assertion b) follows from Corollary 1.10, applied to L, and the last part is clear.

2. Kostant's theorem over $\mathbb{Z}_{(p)}$

2.1. Our goal in this section is to prove the following theorem. Recall from 1.9 the definition of $\overline{C}_p$, the closure of the fundamental p-alcove.

Theorem. — *Let $\lambda \in X^+$ and let p be a prime integer such that $\lambda \in \overline{C}_p$. Then, for each i, there is an isomorphism of L-modules*

$$H_i\big(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)\big) \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

By standard arguments, it suffices to prove the theorem in the case where G is semi-simple; one can further assume that G is simply-connected and, then, that the root system R is irreducible. Similarly, the result for SL_n is easily derived from the result for GL_n (for technical reasons, the latter is easier to handle, see below).

Therefore, while in 2.2–2.8 G still denotes an arbitrary connected reductive linear algebraic group, defined and split over $\mathbb{Z}$, we shall assume in subsection 2.9, where we prove Theorem 2.1, that G is either GL_n or almost simple and simply-connected of type $\neq A$.

Remark. — The hypothesis $\lambda \in X^+ \cap \overline{C}_p$ implies that

$$(\dagger) \qquad p \geqslant \langle \lambda + \rho, \alpha^\vee \rangle \geqslant \langle \rho, \alpha^\vee \rangle, \quad \forall\, \alpha \in R^+.$$

Recall also that it is customary, in representation theory, to introduce the so-called Coxeter number of G, defined by

$$h := 1 + \mathrm{Max}\{\langle \rho, \alpha^\vee \rangle,\ \alpha \in R^+\}.$$

Therefore, the condition $(\dagger)$ above implies that $p \geqslant h - 1$, and reduces to this inequality when $\lambda = 0$.

2.2. Standard resolutions for $U(\mathfrak{g})$.— Recall first the standard Koszul resolution of the trivial module:

$$\cdots \longrightarrow U(\mathfrak{g}) \otimes \Lambda^2(\mathfrak{g}) \xrightarrow{d_2} U(\mathfrak{g}) \otimes \mathfrak{g} \xrightarrow{d_1} U(\mathfrak{g}) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where each differential d_k is defined by the formula

$$\begin{aligned} d_k(u \otimes x_1 \wedge \cdots \wedge x_k) := &\sum_{i=1}^{k} (-1)^{i-1}\, ux_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_k \\ &+ \sum_{1 \leqslant i < j \leqslant k} (-1)^{i+j}\, u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \widehat{x_j} \wedge \cdots \wedge x_k. \end{aligned}$$

Let $\pi_{\mathfrak{p}}$ denote the natural projection $\Lambda^\bullet(\mathfrak{g}) \to \Lambda^\bullet(\mathfrak{g}/\mathfrak{p})$; it is a morphism of P-modules. Then, there is a surjective morphism of $U(\mathfrak{g})$-modules:

$$\begin{aligned} \phi_{\mathfrak{p}} : U(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \Lambda^\bullet(\mathfrak{g}/\mathfrak{p}) \\ u \otimes x &\longmapsto u \otimes_{U(\mathfrak{p})} \pi_{\mathfrak{p}}(x). \end{aligned}$$

It is well-known, and easy to check, that each d_k induces a map $d_k^{\mathfrak{p}}$ such that $\phi_{\mathfrak{p}} \circ d_k = d_k^{\mathfrak{p}} \circ \phi_{\mathfrak{p}}$. Thus, one obtains a complex of $U(\mathfrak{g})$-modules

$$\cdots \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \Lambda^2(\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_2^{\mathfrak{p}}} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_1^{\mathfrak{p}}} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

which is still exact, for it is easily seen that the proof of [**3**, Th.9.1] is valid over $\mathbb{Z}$. This complex is called the standard resolution of the trivial module $\mathbb{Z}$ relative to $U(\mathfrak{g})$ and $U(\mathfrak{p})$. We shall denote it by $S_\bullet(\mathfrak{g}, \mathfrak{p}, \mathbb{Z})$ or simply $S_\bullet(\mathfrak{g}, \mathfrak{p})$.

Let V be a $\mathbb{Z}$-free $U(\mathfrak{g})$-module. Then $S_\bullet(\mathfrak{g}, \mathfrak{p}) \otimes V$, with the diagonal action of $\mathfrak{g}$, is an $U(\mathfrak{g})$-resolution of V by modules which are free over $U(\mathfrak{u}_P^-)$.

Further, recall the "tensor identity" [**19**, Prop. 1.7] : for any $U(\mathfrak{p})$-module E, there is a natural isomorphism of $U(\mathfrak{g})$-modules

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E) \otimes V \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (E \otimes V_{|\mathfrak{p}}),$$

where $V_{|\mathfrak{p}}$ denotes V regarded as an $U(\mathfrak{p})$-module. Applying these isomorphisms to the terms of the resolution $S_\bullet(\mathfrak{g}, \mathfrak{p}) \otimes V$, one obtains an $U(\mathfrak{g})$-resolution

$$\begin{aligned} \cdots \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\Lambda^2(\mathfrak{g}/\mathfrak{p}) \otimes V|_{\mathfrak{p}}) \xrightarrow{d_2} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\mathfrak{g}/\mathfrak{p} \otimes V|_{\mathfrak{p}}) \\ \xrightarrow{d_1} U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V|_{\mathfrak{p}} \xrightarrow{\varepsilon} V \longrightarrow 0, \end{aligned}$$

where the differentials d_k are now given by

$$d_k(1\otimes\overline{x}_1\wedge\cdots\wedge\overline{x}_k\otimes v):=\sum_{i=1}^{k}(-1)^{i-1}\,x_i\otimes\overline{x}_1\wedge\cdots\wedge\widehat{\overline{x}_i}\wedge\cdots\wedge\overline{x}_k\otimes v$$
$$+\sum_{1\leqslant i<j\leqslant k}(-1)^{i+j}\,1\otimes\pi_{\mathfrak{p}}([x_i,x_j])\wedge\overline{x}_1\wedge\cdots\wedge\widehat{\overline{x}_i}\wedge\cdots\widehat{\overline{x}_j}\wedge\cdots\wedge\overline{x}_k\otimes v$$
$$+\sum_{i=1}^{k}(-1)^{i}\,1\otimes\overline{x}_1\wedge\cdots\wedge\widehat{\overline{x}_i}\wedge\cdots\wedge\overline{x}_k\otimes x_iv,$$

for $x_1,\ldots,x_k\in\mathfrak{g}$ and $v\in V$ (we have denoted $\pi_{\mathfrak{p}}(x_i)$ by $\overline{x}_i$). We shall call it the standard resolution of V relative to the pair $(U(\mathfrak{g}),U(\mathfrak{p}))$, and denote it by $S_\bullet(\mathfrak{g},\mathfrak{p},V)$. When $V=V_{\mathbb{Z}}(\lambda)$, we shall denote it by $S_\bullet(\mathfrak{g},\mathfrak{p},\lambda)$.

2.3. Let p be a prime integer and recall the notation of 1.9.

Lemma. — *Let $\lambda\in X^+\cap\overline{C}_p$. Then all weights of $V_{\mathbb{Z}}(\lambda)\otimes\Lambda(\mathfrak{g}/\mathfrak{p})$ are p-small.*

Proof. — As T-module, $\Lambda(\mathfrak{g}/\mathfrak{p})$ identifies with $\Lambda(\mathfrak{u}_P^-)$ and hence is a submodule of $\Lambda(\mathfrak{u}^-)$, where $\mathfrak{u}^-$ is the Lie algebra of U^-.

By a result of Kostant ([**33**, Lemma 5.9]), there is a T-isomorphism

$$\rho\otimes\Lambda(\mathfrak{u}^-)\cong V_{\mathbb{Z}}(\rho).$$

Therefore, if ν is a weight of $V_{\mathbb{Z}}(\lambda)\otimes\Lambda(\mathfrak{g}/\mathfrak{p})$, then $\nu+\rho$ is a weight of $V_{\mathbb{Z}}(\lambda)\otimes V_{\mathbb{Z}}(\rho)$. This implies that $\langle\nu+\rho,\alpha^\vee\rangle\leqslant p$, for all $\alpha\in R$.

Indeed, let μ be the dominant W-conjugate of $\nu+\rho$, it is also a weight of $V_{\mathbb{Z}}(\lambda)\otimes V_{\mathbb{Z}}(\rho)$. Clearly, it suffices to prove that $\langle\mu,\alpha^\vee\rangle\leqslant p$, for all $\alpha\in R^+$. Further, since μ is dominant, it suffices to prove that $\langle\mu,\gamma^\vee\rangle\leqslant p$ when $\gamma^\vee$ is a maximal coroot. But it is well-known that a maximal coroot is a dominant coweight, i.e. satisfies $\langle\beta,\gamma^\vee\rangle\geqslant 0$ for all $\beta\in R^+$, see e.g. [**5**, VI,§ 1, Prop.8]. Finally, since $\mu=\lambda+\rho-\theta$ with $\theta\in\mathbb{N}R^+$, it follows that

$$\langle\mu,\gamma^\vee\rangle\leqslant\langle\lambda+\rho,\gamma^\vee\rangle\leqslant p.$$

This proves the lemma.

2.4. Verma modules and filtrations.— For any $\xi\in X_L^+$, define the generalized Verma module (for $U(\mathfrak{g})$ and $U(\mathfrak{p})$)

$$M_{\mathfrak{p}}(\xi):=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}V_{\mathbb{Z}}^L(\xi).$$

For any commutative ring A, set $M_{\mathfrak{p}}^A(\xi):=M_{\mathfrak{p}}(\xi)\otimes_{\mathbb{Z}}A$ and observe that it identifies with $U_A(\mathfrak{g})\otimes_{U_A(\mathfrak{p})}V_A^L(\xi)$.

For $\lambda\in X^+$, we set also

$$S_\bullet^A(\mathfrak{g},\mathfrak{p},\lambda):=S_\bullet(\mathfrak{g},\mathfrak{p},\lambda)\otimes_{\mathbb{Z}}A.$$

Let us assume that $\lambda \in X^+ \cap \overline{C}_p$. Then, by Lemma 2.3, all weights of $V_{\mathbb{Z}}(\lambda) \otimes \Lambda^\bullet(\mathfrak{g}/\mathfrak{p})$ are p-small. Therefore, by Lemma 1.11, there exists, for each i, a P-module filtration

$$0 = F_0 \subset \cdots \subset F_r = \Lambda^i(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}_{(p)}}(\lambda)$$

such that each F_j/F_{j-1} is isomorphic to $V^L_{\mathbb{Z}_{(p)}}(\xi^i_j)$, for some $\xi^i_j \in X^+_L$ (not necessarily distinct). Let us denote by $\Omega^i_{\mathfrak{p}}(\lambda)$ the multiset of those ξ^i_j (each $\xi \in X^+_L$ occuring as many times as $V^L_{\mathbb{Z}_{(p)}}(\xi)$ occurs in the filtration).

Moreover, as $U(\mathfrak{g})$ is free over $U(\mathfrak{p})$, the functor $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} -$ is exact. Therefore, one obtains the

Lemma. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then each $S^{\mathbb{Z}_{(p)}}_i(\mathfrak{g}, \mathfrak{p}, \lambda)$ admits a finite filtration by $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-modules such that the successive quotients are the $M^{\mathbb{Z}_{(p)}}_{\mathfrak{p}}(\xi)$, for $\xi \in \Omega^i_{\mathfrak{p}}(\lambda)$.*

2.5. A conjugacy result in $\mathfrak{g}^*$.— We will need in the next subsection the following lemma. It is proved in [**29**, Lemma 6.6] under the assumption that $\mathfrak{g}_{\overline{\mathbb{F}}_p} \cong \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ as G-modules, and in [**31**, Lemma 3.3] under the assumption that G is almost simple and distinct from SO_{2n+1} if $p = 2$. Let $\mathfrak{u}$ be the Lie algebra over $\mathbb{Z}$ of U and let $\mathfrak{u}_{\overline{\mathbb{F}}_p} = \mathfrak{u} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$.

Lemma. — *Each $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ is conjugate under G to an element χ' such that $\chi'(\mathfrak{u}_{\overline{\mathbb{F}}_p}) = 0$.*

Proof. — Let $\mathcal{B}$ denote the variety of Borel subgroups of G, let Z be the closed subvariety of $\mathcal{B} \times \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ consisting of pairs (B', χ) such that χ vanishes on the derived subalgebra of $\operatorname{Lie} B'$, and let π denote the projection $Z \longrightarrow \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$. Then, the lemma is equivalent to the surjectivity of π.

But, $\mathcal{B}$ being projective, $\pi(Z)$ is a closed subvariety and, since $\dim Z = \dim \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$, the surjectivity of π will follow if we show that the set of those $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ such that $\pi^{-1}(\chi)$ is finite, is not empty. But this follows from an argument of Steinberg [**50**, Lemma 3.2] (one may also consult [**25**, Prop. 4.1]). Namely, for each $\beta \in R$, let X_β be a generator of $\mathfrak{g}_\beta$. We claim that if $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ satisfies $\chi(\mathfrak{b}_{\overline{\mathbb{F}}_p}) = 0$ and $\chi(X_{-\alpha}) \neq 0$, for every $\alpha \in \Delta$, then $\pi^{-1}(\chi) = \{B\}$.

Indeed, let B' be a Borel subgroup such that χ vanishes on $\mathfrak{u}'$, the derived subalgebra of $\operatorname{Lie} B'$. Then $B' = g(B)$ for some $g \in G$ and, using the Bruhat decomposition, one may write $g = u n_w b$ for some $w \in W$, $b \in B$ and $u \in U \cap n_w^{-1}(U)$. If $w \neq 1$, there exists a simple root $\alpha \in \Delta$ such that $w^{-1}\alpha \in R^-$. Let $\beta = -w^{-1}\alpha$, then $n_w X_\beta = c X_{-\alpha}$ for some non-zero $c \in \overline{\mathbb{F}}_p$. Set $x = b^{-1}c^{-1}X_\beta$. Then $x \in \mathfrak{u}_{\overline{\mathbb{F}}_p}$ and, by hypothesis,

$$0 = \chi(gx) = \chi(uX_{-\alpha}).$$

But $uX_{-\alpha} - X_{-\alpha}$ belongs to $\mathfrak{b}_{\overline{\mathbb{F}}_p}$ and hence the assumptions on χ imply that $\chi(uX_{-\alpha}) = \chi(X_{-\alpha}) \neq 0$, a contradiction. This contradiction shows that $w = 1$, whence $g \in B$ and $B' = B$. This completes the proof of the lemma.

2.6. The Harish-Chandra homomorphism

2.6.1. Let $\mathfrak{u}^- = \operatorname{Lie} U^-$ and let A be a commutative ring. By the PBW theorem, one has

$$U_A(\mathfrak{g}) = U_A(\mathfrak{t}) \oplus \left(\mathfrak{u}^- U_A(\mathfrak{g}) + U_A(\mathfrak{g})\mathfrak{u}\right).$$

Let δ_A denote the A-linear projection from $U_A(\mathfrak{g})$ to $U_A(\mathfrak{t})$ defined by this decomposition.

Let $U_A(\mathfrak{g})^G \subset U_A(\mathfrak{g})^T$ be the subrings of G-invariant and T-invariant elements for the adjoint action. Observe that, since elements of $U_A(\mathfrak{g})^T$ have weight zero,

$$U_A(\mathfrak{g})^T \subseteq U_A(\mathfrak{t}) \oplus \mathfrak{u}^- U_A(\mathfrak{g})\mathfrak{u}.$$

The restriction of δ_A to $U_A(\mathfrak{g})^T$ is a ring homomorphism; indeed one sees easily that the arguments in the proof of [**13**, Lemme 7.4.2] or [**31**, Lemma 5.1] carry over in our case. Let θ_A denote the restriction of δ_A to $U_A(\mathfrak{g})^G$.

Lemma. — *$\theta_{\overline{\mathbb{F}}_p} : U_{\overline{\mathbb{F}}_p}(\mathfrak{g})^G \to U_{\overline{\mathbb{F}}_p}(\mathfrak{t})$ is injective.*

Proof. — Taking into account Lemma 2.5, the proof is exactly the same as the one of [**29**, Lemma 9.1]. For the convenience of the reader, we record it briefly. Let $U = U_{\overline{\mathbb{F}}_p}(\mathfrak{g})$, let $x \mapsto x^{[p]}$ denotes the p-th power map of $\mathfrak{g}_{\overline{\mathbb{F}}_p}$ and, for $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$, let U_χ denote the quotient of U by the two-sided ideal generated by the elements $x^p - x^{[p]} - \chi(x)$, for $x \in \mathfrak{g}_{\overline{\mathbb{F}}_p}$.

Let $u \in U^G$ with $\theta_{\overline{\mathbb{F}}_p}(u) = 0$. Then, $u \in \mathfrak{u}^- U \mathfrak{u}$ and, being G-invariant, $u = g(u)$ belongs to $g(\mathfrak{u}^-)Ug(\mathfrak{u})$, for every $g \in G$. Let L be a simple U-module. By Lemma 2.5 and, say, [**29**, 2.4], L is a $U_{g\chi}$-module, for some $g \in G$ and $\chi \in \mathfrak{g}^*_{\overline{\mathbb{F}}_p}$ such that $\chi(\mathfrak{u}) = 0$. Then, one deduces from [**29**, § 6.7] or [**17**, Prop. 1.5] that L is generated by a vector v annihilated by $g(\mathfrak{u})$ (in [**17**], it is assumed that G is semi-simple and simply-connected but this hypothesis is not used in the proof of Prop. 1.5). Thus, $uv = g(u)v = 0$ and hence $uL = 0$. Therefore, u annihilates every simple U-module, that is, belongs to every maximal left ideal of U. Hence, $1 + u$ is a unit in U; but the only units in U are the non-zero scalars, and it follows that $u = 0$. (The last part of the argument is due to Curtis [**10**]).

Remark. — In [**31**, 9.4.d)], it is mistakenly asserted that $\theta_{\overline{\mathbb{F}}_p}$ is not injective in the case where $G = \mathrm{SO}(2n+1)$ and $p = 2$; but in fact the element q considered in [**31**, 9.1] is not G-invariant.

2.6.2. Note that $U_{\overline{\mathbb{F}}_p}(\mathfrak{t}) = S_{\overline{\mathbb{F}}_p}(\mathfrak{t})$ identifies with $\mathcal{P}(\mathfrak{t}^*_{\overline{\mathbb{F}}_p})$, the algebra of regular functions on

$$\mathfrak{t}^*_{\overline{\mathbb{F}}_p} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}, \overline{\mathbb{F}}_p) \cong X(T) \otimes \overline{\mathbb{F}}_p.$$

The dot action of W on $U_{\overline{\mathbb{F}}_p}(\mathfrak{t})$ is defined, therefore, by $(w \cdot P)(\lambda) = P(w^{-1} \cdot \lambda)$, for $w \in W$, $P \in U_{\overline{\mathbb{F}}_p}(\mathfrak{t})$, $\lambda \in \mathfrak{t}^*_{\overline{\mathbb{F}}_p}$. For typographical reasons, let us denote by $U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}$ the subalgebra of invariants for this action. Then, as in [**31**, Lemma 5.2] or [**29**, 9.5], one obtains that $\theta_{\overline{\mathbb{F}}_p}(U_{\overline{\mathbb{F}}_p}(\mathfrak{g})^G) \subseteq U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}$. Moreover, under certain assumptions on G and p, this inclusion is an equality. Recall that a prime p is called *good* for R if it satisfies the following: for every $\gamma^\vee \in R^\vee$ expressed in terms of the simple coroots as

$$\gamma^\vee = \sum_{\alpha \in \Delta} n_\alpha(\gamma^\vee)\alpha^\vee,$$

one has $p > n_\alpha(\gamma^\vee)$ for all α. Then, one has the following mod. p analogue of Harish-Chandra's isomorphism. Let $\mathcal{D}G$ denote the derived subgroup of G, see [**28**, II.1.18].

Theorem ([**29**]). — *Assume that $\mathcal{D}G$ is simply-connected, that p is good for R, and that $X(T)/\mathbb{Z}R$ has no p-torsion. Then $\theta_{\overline{\mathbb{F}}_p}$ induces an isomorphim of algebras*

$$U_{\overline{\mathbb{F}}_p}(\mathfrak{g})^G \cong U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}.$$

Proof. — Under the stated assumptions, this is proved in [**29**, § 9.6]. For the convenience of the reader, let us outline the steps of the proof. Firstly, it is proved in [**29**, § 9.6] that it suffices to prove that the natural map

$$U_{\mathbb{Z}_{(p)}}(\mathfrak{t})^{W\bullet} \otimes \overline{\mathbb{F}}_p \longrightarrow U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^{W\bullet}$$

is surjective. Secondly, since $\mathcal{D}G$ is simply-connected, $\{\alpha^\vee,\ \alpha \in \Delta\}$ is part of a basis of $X^\vee(T)$; see [**28**, II.1.18] or [**48**, Prop. 8.1.8.(iii)], and it follows that the previous map is surjective if and only if the analogous map $U_{\mathbb{Z}_{(p)}}(\mathfrak{t})^W \otimes \overline{\mathbb{F}}_p \to U_{\overline{\mathbb{F}}_p}(\mathfrak{t})^W$ is so. Finally, this surjectivity result follows, under the assumption that p is good and does not divide $|X(T)/\mathbb{Z}R|$, from [**12**], Cor. of Th. 2 (applied to the lattice $M = X^\vee(T) \cong \mathrm{Lie}\, T$ and the root system $R^\vee$).

Remark. — The theorem is proved by completely different methods in [**31**] in the case where G is almost simple and $p \neq 2$ if $G = \mathrm{SO}(2n+1)$; these methods can be extended to the case where G is reductive under the assumption that $p \neq 2$ if $\alpha^\vee/2 \in X^\vee(T)$, for some $\alpha \in R$. However, the version of the theorem given above is sufficient for our purposes.

2.6.3. Central characters. — For any $\mu \in X(T)$, its differential $d\mu$ induces an A-linear map $\mathfrak{t}_A \to A$ and hence an A-algebra morphism $U_A(\mathfrak{t}) \to A$, still denoted by $d\mu$. Thus, μ gives rise to an A-algebra morphism $\chi_{\mu,A} := d\mu \circ \theta_A$, from $U_A(\mathfrak{g})^G$ to A.

For any morphism of commutative rings $f : A \to B$, it is easily seen that the following diagram is commutative:

$$\begin{array}{ccccc} U_A(\mathfrak{g})^G & \xrightarrow{\theta_A} & U_A(\mathfrak{t}) & \xrightarrow{d\mu} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ U_B(\mathfrak{g})^G & \xrightarrow{\theta_B} & U_B(\mathfrak{t}) & \xrightarrow{d\mu} & B. \end{array}$$

Thus, one has $\chi_{\mu,B} \circ f = f \circ \chi_{\mu,A}$.

Recall that $U_A(\mathfrak{g})^G \subseteq U_A(\mathfrak{t}) \oplus \mathfrak{u}^- U_A(\mathfrak{g})\mathfrak{u}$. Thus, if M is a $U_A(\mathfrak{g})$-module generated by an element v of weight μ annihilated by $\mathfrak{u}$, then $U_A(\mathfrak{g})^G$ acts on M by the character $\chi_{\mu,A}$ (see [**13**, Prop.7.4.4]).

Let π denote the morphism $\mathbb{Z}_{(p)} \to \overline{\mathbb{F}}_p$, let $\chi_{\mu,p} := \chi_{\mu,\mathbb{Z}_{(p)}}$ and $\overline{\chi}_{\mu,p} := \pi \circ \chi_{\mu,p} = \chi_{\mu,\overline{\mathbb{F}}_p}$, and set $J_{\mu,p} := \operatorname{Ker} \chi_{\mu,p}$. Then, one deduces immediately from the previous theorem the following

Corollary. — *Keep the hypotheses of the previous theorem. Let $\lambda, \mu \in X(T)$. If $\chi_{\lambda,\overline{\mathbb{F}}_p} = \chi_{\mu,\overline{\mathbb{F}}_p}$, there exists $w \in W$ such that $\mu - w \cdot \lambda \in pX(T)$.*

2.7. Decomposition w.r.t. central characters mod. p.— Let $\lambda \in X^+$ and let p be a prime integer such that $\lambda \in \overline{C}_p$. Recall the multisets $\Omega^i_{\mathfrak{p}}(\lambda)$ from 2.4 and let $\Omega^\bullet_{\mathfrak{p}}(\lambda)$ denote their disjoint union.

By Lemma 2.4, each $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ admits a finite $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-filtration, whose quotients are the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$, where ξ runs through $\Omega^i_{\mathfrak{p}}(\lambda)$. It follows that $S_\bullet^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ is annihilated by the ideal

$$I := \prod_{\xi \in \Omega^\bullet_{\mathfrak{p}}(\lambda)} J_{\xi,p}$$

(each ξ being counted with its multiplicity).

The following lemma is straightforward.

Lemma. — *Let A be a commutative ring and let $P_1, \ldots, P_r$ be ideals of A such that $P_1 \cdots P_r = 0$ and $P_i + P_j = A$ if $j \neq i$. Then, for any A-module M, one has*

$$M = \bigoplus_{i=1}^{r} M^{P_i}, \quad \text{where } M^{P_i} = \{m \in M \mid P_i m = 0\}.$$

Further, the assignment $M \mapsto M^{P_i}$ is an exact functor.

We shall apply the lemma to $A := U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G / I$. Note that A is a finite $\mathbb{Z}_{(p)}$-module. Moreover, it is easily seen that the maximal ideals of A are the $pA + J_{\xi,p} = \operatorname{Ker} \overline{\chi}_{\xi,p}$, for $\xi \in \Omega^\bullet_{\mathfrak{p}}(\lambda)$. (By abuse of notation, we still denote by $J_{\xi,p}$ the image of $J_{\xi,p}$ in A).

Let $\overline{\chi}_1, \dots, \overline{\chi}_r$ be the distinct algebra homomorphisms $A \to \overline{\mathbb{F}}_p$, numbered so that $\overline{\chi}_1 = \overline{\chi}_{\lambda,p}$, and, for $i = 1, \dots, r$, let

$$P_i := \prod_{\substack{\xi \in \Omega_{\mathfrak{p}}^{\bullet}(\lambda) \\ \overline{\chi}_{\xi,p} = \overline{\chi}_i}} J_{\xi,p}.$$

Clearly, $P_1 \cdots P_r = 0$ and $pA + P_i + P_j = A$ if $j \neq i$. Since A is a finite $\mathbb{Z}_{(p)}$-module, the latter implies, by Nakayama lemma, that $P_i + P_j = A$ if $j \neq i$.

Then, one deduces from the previous lemma that $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ is the direct sum of the $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-submodules corresponding to the characters $\overline{\chi}_1, \dots, \overline{\chi}_r$, that is,

$$(*) \qquad S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda) = \bigoplus_{i=1}^{r} S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)^{P_i}.$$

Moreover, since the differentials in the complex $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ are $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-equivariant, each $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)^{P_i}$ is a direct summand *subcomplex*. In particular, since $\overline{\chi}_1 = \overline{\chi}_{\lambda,p}$, this is true for

$$S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}} := S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)^{P_1}.$$

Further, since $M \mapsto M_{\overline{\chi}_{\lambda,p}}$ is an exact functor and since

$$M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)_{\overline{\chi}_{\lambda,p}} = \begin{cases} M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi) & \text{if } \overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}\,; \\ 0 & \text{otherwise,} \end{cases}$$

one obtains, as in [**3**, Lemma 9.7], the following

Corollary. — *$S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)$ contains the subcomplex $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ as a direct summand. Moreover, for $i \geqslant 0$, each $S_{i}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ has a filtration whose quotients are the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$, for those $\xi \in \Omega_{\mathfrak{p}}^{i}(\lambda)$ (counted with multiplicities) such that $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$.*

2.8. The main step towards the description of $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ is the following proposition.

Proposition. — *Assume that $\mathcal{D}G$ is simply-connected and $X(T)/\mathbb{Z}R$ has no p-torsion. Let $\lambda \in X^+ \cap \overline{C}_p$ and $\xi \in \Omega_{\mathfrak{p}}^{\bullet}(\lambda)$. If $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$, then $\xi = w \cdot \lambda$ for some $w \in W^L$.*

Proof. — Let ξ be as in the proposition. Observe that, by 2.1(†), the assumption $X^+ \cap \overline{C}_p \neq \varnothing$ implies that p is good for R. Therefore, the hypotheses of Theorem 2.6.2 are satisfied. Thus, by Corollary 2.6.3, $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$ implies that there exist $y \in W$ and $\nu \in X(T)$ such that $y \cdot \xi = \lambda + p\nu$. Moreover, since $y \cdot \xi$ is a weight of $\Lambda(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)$, then $y \cdot \xi - \lambda \in \mathbb{Z}R$ and hence $p\nu \in \mathbb{Z}R \cap pX(T)$. Since $X(T)/\mathbb{Z}R$ has no p-torsion, it follows that $\nu \in \mathbb{Z}R$ and hence $\xi \in W_p \cdot \lambda$.

Now, let $w \in W$ such that $w^{-1}(\xi + \rho)$ is dominant and let $\xi^+ := w^{-1} \cdot \xi$. Then, by Lemma 2.3, $\xi^+ \in \overline{C}_p$. But $\xi^+ \in W_p \cdot \lambda$; since $\overline{C}_p$ is a fundamental domain for the dot action of W_p, it follows that $\xi^+ = \lambda$, and hence $\xi = w \cdot \lambda$.

Further, since $\xi \in \Omega_{\mathfrak{p}}^{\bullet}(\lambda) \subseteq X_L^+$, for any $\alpha \in R_L^+$ one has $\langle w \cdot \lambda, \alpha^\vee \rangle \geqslant 0$ and hence

$$\langle \lambda + \rho, w^{-1}\alpha^\vee \rangle \geqslant \langle \rho, \alpha^\vee \rangle > 0.$$

This implies that $w \in W^L$. The proposition is proved.

Remark. — In the first version of this paper, the previous proposition was stated under the assumption that G is either GL_n or almost simple and simply connected of type $\neq A$ and the proof relied on [**31**, Th. 1] in the second case and on results of Carter and Lusztig ([**8**], proof of Theorems 3.8 and 4.1) in the first case. We are indebted to the referee for pointing out that the result could be stated and proved in a uniform manner by using the version of Harish-Chandra's isomorphism given in [**29**, §9].

We can now prove the following analogue of [**3**, Th. 9.9] and [**37**, Th. 3.10], [**46**, Th. 7.11].

Theorem. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion, and that $\lambda \in X^+ \cap \overline{C}_p$. Then $S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ is an $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})$-resolution of $V_{\mathbb{Z}_{(p)}}(\lambda)$ and each $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ with $i \geqslant 0$ has a filtration whose quotients are exactly the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(w \cdot \lambda)$, for $w \in W^L(i)$, each occuring once.*

Proof. — By Corollary 2.7 and the previous proposition, each $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ with $i \geqslant 0$ has a filtration whose quotients are the $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$, for those $\xi \in \Omega_{\mathfrak{p}}^i(\lambda)$ (counted with multiplicities) such that $\xi = w \cdot \lambda$ for some $w \in W^L$.

Conversely, for $w \in W^L$, Kostant has showed that $V_{\mathbb{Q}}^L(w \cdot \lambda)$ occurs with multiplicity one in $\Lambda^{\bullet}(\mathfrak{g}/\mathfrak{p}) \otimes V_{\mathbb{Z}}(\lambda)$, in degree equal to $\ell(w)$, see [Ko1], Lemma 5.12 and end of proof of Th. 5.14. This completes the proof of the theorem.

2.9. Proof of theorem 2.1.— In this subsection, we assume that G is either GL_n or almost simple and simply-connected of type $\neq A$. As observed in 2.1, this assumption entails no loss of generality in the proof of Kostant's theorem over $\mathbb{Z}_{(p)}$. Keep the notation of 2.7-2.8. Note that $\mathbb{Z}R$ is a direct summand of $X(T)$ if $G = \mathrm{GL}_n$, while if G is almost simple of type $\neq A$, the assumption $X^+ \cap \overline{C}_p \neq \varnothing$ implies that $X(T)/\mathbb{Z}R$ has no p-torsion. Therefore, the hypotheses of Theorem 2.8 are satisfied.

Observe next that, as $U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-)$-module, any $M_{\mathfrak{p}}^{\mathbb{Z}_{(p)}}(\xi)$ is isomorphic to $U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-) \otimes V_{\mathbb{Z}_{(p)}}^L(\xi)$, hence free. Thus, by Theorem 2.8, $S_i^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}$ is a free $U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-)$-module, for each $i \geqslant 0$.

Therefore, $H_{\bullet}(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda))$ is the homology of the complex

$$C_{\bullet} := \mathbb{Z}_{(p)} \otimes_{U_{\mathbb{Z}_{(p)}}(\mathfrak{u}_P^-)} S_{\bullet}^{\mathbb{Z}_{(p)}}(\mathfrak{g}, \mathfrak{p}, \lambda)_{\overline{\chi}_{\lambda,p}}.$$

Further, by Theorem 2.8, again, for $i \geqslant 0$ each C_i has an L-module filtration whose successive quotients are the $V_{\mathbb{Z}_{(p)}}^L(w \cdot \lambda)$, for $w \in W^L(i)$.

By Corollary 1.10, applied to L, one obtains that these filtrations split, that is, for each $i \geqslant 0$ one has isomorphisms of L-modules

$$C_i \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

Further, we claim that the differentials $d_i : C_i \to C_{i-1}$ are zero. Indeed, one has $H_i(C_\bullet) \otimes \mathbb{Q} \cong H_i(\mathfrak{u}^-_P, V_{\mathbb{Q}}(\lambda))$ and, by Kostant's theorem ([**33**, Cor. 8.1] or [**3**], Cor. of Th. 9.9), the latter is isomorphic to $C_i \otimes \mathbb{Q}$. It follows, for a reason of dimension, that $d_i \otimes 1 = 0$. Since C_{i-1} is torsion-free, this implies that $d_i = 0$.

Thus, we have obtained, for each $i \geqslant 0$, an isomorphism of L-modules

$$H_i(\mathfrak{u}^-_P, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

This completes the proof of Theorem 2.1.

2.10. Analogue in cohomology.— Recall the anti-involution τ from 1.4; it exchanges P^- and P and stabilizes L. Let $\lambda \in X^+ \cap \overline{C}_p$. Since $H_\bullet(\mathfrak{u}^-_P, V)$ is a free $\mathbb{Z}_{(p)}$-module, one obtains, by standard arguments, an isomorphism of L-modules

$$H_\bullet(\mathfrak{u}^-_P, V_{\mathbb{Z}_{(p)}}(\lambda))^\tau \cong H^\bullet(\mathfrak{u}_P, V_{\mathbb{Z}_{(p)}}(\lambda)^\tau).$$

Further, since $V_{\mathbb{Z}_{(p)}}(\lambda) = V_{\mathbb{Z}_{(p)}}(\lambda)^\tau$ and $V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda) = V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda)^\tau$, for $w \in W^L$, by Corollary 1.9, applied to G and L, one obtains the

Corollary. — *Let $\lambda \in X^+ \cap \overline{C}_p$. For each $i \geqslant 0$, there is an isomorphism of $L_{\mathbb{Z}_{(p)}}$-modules*

$$H^i(\mathfrak{u}_P, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w \in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

3. Cohomology of the groups $U^-_P(\mathbb{Z})$

3.1. Let us recall several definitions and facts about finitely generated, torsion free, nilpotent groups. Let Γ be such a group, say of class c. Let $\mathcal{F}$ be a finite series

$$\Gamma = F^1\Gamma \supset F^2\Gamma \supset \cdots \supset F^{d+1}\Gamma = \{1\}$$

of normal subgroups of Γ. Following the terminology in Passman's book [**42**, p.85], let us say that $\mathcal{F}$ is an N-series if $(F^i\Gamma, F^j\Gamma) \subseteq F^{i+j}\Gamma$ for all i, j. Since every subgroup of Γ is finitely generated (see [**21**, Lemma 1.9] or [**42**, Chap. 3, Lemma 4.2]), each $F^i\Gamma/F^{i+1}\Gamma$ is then a finitely generated abelian group.

Let us denote temporarily by $r(\mathcal{F})$ the rank of $\bigoplus_{i=1}^d F^i\Gamma/F^{i+1}\Gamma$. This rank is in fact an invariant of Γ. Indeed, $\mathcal{F}$ can be refined to a sequence of normal subgroups

$$\Gamma = H^1 \supset H^2 \supset \cdots \supset H^{n+1} = \{1\}$$

such that each H^i/H^{i+1} is cyclic, and for any such refinement the number of infinite cyclic quotients equals $r(\mathcal{F})$. But, for any subnormal series $\Gamma = S^1 \rhd S^2 \rhd \cdots \rhd S^{m+1} = \{1\}$ such that each quotient S^i/S^{i+1} is cyclic, the number of infinite cyclic quotients is an invariant called the rank, or Hirsch number, of Γ and denoted by $h(\Gamma)$; see the discussion before Lemma 10.2.10 in [**42**] or [**51**, Chap. 2, Th. 3.20]. Together, these arguments show that $r(\mathcal{F}) = h(\Gamma)$.

If $\mathcal{F}$ is an N-series, the associated graded abelian group

$$\operatorname{gr}_{\mathcal{F}} \Gamma := \bigoplus_{i\geqslant 1} F^i\Gamma/F^{i+1}\Gamma$$

has a natural structure of Lie algebra over $\mathbb{Z}$ (see, for example, [**36**, Chap. I, Th. 2.1]).

Further, $\mathcal{F}$ is called an N_0-series if it is an N-series and each $F^i\Gamma/F^{i+1}\Gamma$ is torsion-free. Such series exist, see [**30**, Th. 2.2] or [**42**, Chap. 11, Lemma 1.8], and in this case $\operatorname{gr}_{\mathcal{F}}\Gamma$ is a free $\mathbb{Z}$-module of rank $h(\Gamma)$.

Let $\{C^i(\Gamma)\}_{i\geqslant 1}$ denote the lower central series; as is well-known, it is the fastest descending N-series. We shall denote the corresponding graded Lie algebra simply by $\operatorname{gr}\Gamma$. Further, for each i, set

$$C^{(i)}(\Gamma) := \{x \in \Gamma \mid x^n \in C^i(\Gamma) \text{ for some } n > 0\}.$$

By [**42**, Lemma 11.1.8] (see also [**21**, § 4]), $\{C^{(i)}(\Gamma)\}_{i\geqslant 1}$ is an N_0-series. It is clearly the fastest descending N_0-series. Following [**21**, § 4], we will call it the isolated lower central series. We will denote by $\operatorname{gr}_{\mathrm{isol}}\Gamma$ the associated Lie algebra over $\mathbb{Z}$

$$\operatorname{gr}_{\mathrm{isol}}\Gamma := \bigoplus_{i\geqslant 1} C^{(i)}(\Gamma)/C^{(i+1)}(\Gamma);$$

it is a free $\mathbb{Z}$-module of rank $h(\Gamma)$. Clearly, there is an isomorphism of graded Lie algebras $\operatorname{gr}\Gamma\otimes\mathbb{Q} \cong \operatorname{gr}_{\mathrm{isol}}\Gamma\otimes\mathbb{Q}$.

Let I denote the augmentation ideal of the group ring $\mathbb{Z}\Gamma$ and, for $n \geqslant 0$, let $I^{(n)}$ denote the isolator of I^n, that is,

$$I^{(n)} := \{x \in \mathbb{Z}\Gamma \mid mx \in I^n \text{ for some } m > 0\}.$$

Equivalently, if $I_{\mathbb{Q}}$ denotes the augmentation ideal of $\mathbb{Q}\Gamma$, then $I^{(n)} = \mathbb{Z}\Gamma \cap I^n_{\mathbb{Q}}$.

Let us consider the graded rings

$$\operatorname{gr}_{\mathrm{isol}}\mathbb{Z}\Gamma := \bigoplus_{n\geqslant 0} I^{(n)}/I^{(n+1)} \quad\text{and}\quad \operatorname{gr}\mathbb{Q}\Gamma := \bigoplus_{n\geqslant 0} I^n_{\mathbb{Q}}/I^{n+1}_{\mathbb{Q}}.$$

The former is a subring of the latter and, by a result of Quillen ([**45**]), there is an isomorphism of graded Hopf algebras $U_{\mathbb{Q}}(\operatorname{gr}\Gamma\otimes\mathbb{Q}) \cong \operatorname{gr}\mathbb{Q}\Gamma$. Further, one has the following more precise result of Hartley :

Theorem ([**23**, Th. 2.3.3′]). — *There is an isomorphism of graded Hopf algebras*

$$U_{\mathbb{Z}}(\operatorname{gr}_{\mathrm{isol}}\Gamma) \cong \operatorname{gr}_{\mathrm{isol}}\mathbb{Z}\Gamma.$$

3.2. Let A be a finitely generated subring of $\mathbb{Q}$ (thus, $A = \mathbb{Z}[1/m]$ for some m and A is a PID). Let $\mathfrak{u}$ be a nilpotent Lie algebra of class c over A, which is a finite free A-module, say of rank r. Let $\mathfrak{u}_{\mathbb{Q}} = \mathfrak{u} \otimes_A \mathbb{Q}$, then $U_{\mathbb{Q}}(\mathfrak{u}_{\mathbb{Q}}) \cong U_A(\mathfrak{u}) \otimes_A \mathbb{Q}$; we shall denote it by $U_{\mathbb{Q}}(\mathfrak{u})$. By the PBW theorem, $U_A(\mathfrak{u})$ is a subalgebra of $U_{\mathbb{Q}}(\mathfrak{u})$.

Let $\mathcal{F}$ be a finite sequence

$$\mathfrak{u} = F^1\mathfrak{u} \supset F^2\mathfrak{u} \supset \cdots \supset F^{d+1}\mathfrak{u} = \{0\}$$

of Lie ideals of $\mathfrak{u}$. As in the previous paragraph, let us say that $\mathcal{F}$ is an N-series if $[F^i\mathfrak{u}, F^j\mathfrak{u}] \subseteq F^{i+j}\mathfrak{u}$, and is an N_0-series if further each $F^i\mathfrak{u}/F^{i+1}\mathfrak{u}$ (which is a finitely generated module over the PID A) is torsion free, and hence a free A-module.

Let $\{C^i(\mathfrak{u})\}_{i\geqslant 1}$ denote the lower central series of $\mathfrak{u}$ and define the isolated lower central series $\{C^{(i)}(\mathfrak{u})\}_{i\geqslant 1}$ by

$$C^{(i)}(\mathfrak{u}) := \{x \in \mathfrak{u} \mid nx \in C^i(\mathfrak{u}) \text{ for some } n > 0\}.$$

This is, clearly, the fastest descending N_0-series of $\mathfrak{u}$. Consider the graded Lie algebras

$$\operatorname{gr}_{\text{isol}} \mathfrak{u} := \bigoplus_{i\geqslant 1} C^{(i)}(\mathfrak{u})/C^{(i+1)}(\mathfrak{u}) \quad \text{and} \quad \operatorname{gr} \mathfrak{u}_{\mathbb{Q}} := \bigoplus_{i\geqslant 1} C^i(\mathfrak{u}_{\mathbb{Q}})/C^{i+1}(\mathfrak{u}_{\mathbb{Q}}).$$

Then $\operatorname{gr}_{\text{isol}} \mathfrak{u}$ is a free A-module of rank r and there is an isomorphism of graded Lie algebras $(\operatorname{gr}_{\text{isol}} \mathfrak{u}) \otimes_A \mathbb{Q} \cong \operatorname{gr} \mathfrak{u}_{\mathbb{Q}}$.

Let $J_{\mathbb{Q}}$ denote the augmentation ideal of $U_{\mathbb{Q}}(\mathfrak{u})$. Then the graded algebra

$$\operatorname{gr} U_{\mathbb{Q}}(\mathfrak{u}) := \bigoplus_{n\geqslant 0} J_{\mathbb{Q}}^n/J_{\mathbb{Q}}^{n+1}$$

is a primitively generated, graded Hopf algebra; it is isomorphic to $U_{\mathbb{Q}}(\operatorname{gr} \mathfrak{u}_{\mathbb{Q}})$, by [**32**] or [**52**, Prop. 1]. In fact, as in the case of group rings, a little more is true. For $n \geqslant 1$, let $J^{(n)} = U_A(\mathfrak{u}) \cap J_{\mathbb{Q}}^n$. Then the graded ring

$$\operatorname{gr}_{\text{isol}} U_A(\mathfrak{u}) := \bigoplus_{n\geqslant 0} J^{(n)}/J^{(n+1)}$$

identifies with a subring of $\operatorname{gr} U_{\mathbb{Q}}(\mathfrak{u})$. Further, one deduces from the proof of [**52**, Prop. 1] the following result. Let $X_1, \ldots, X_r$ be an A-basis of $\mathfrak{u}$ compatible with the filtration $\{C^{(i)}(\mathfrak{u})\}_{i=1}^c$, *i.e.*, such that for $s = 1, \ldots, c$, the X_j with $j > r - \dim C^s(\mathfrak{u}_{\mathbb{Q}})$ form an A-basis of $C^{(s)}(\mathfrak{u})$, and, for each i, let $\mu(i)$ be the largest integer k such that $X_i \in C^{(k)}(\mathfrak{u})$.

Proposition

a) *The ordered monomials* $X_1^{n_1} \cdots X_r^{n_r}$ *with* $\sum_{i=1}^r n_i\, \mu(i) \geqslant n$ *form an* A*-basis of* $J^{(n)}$, *for any* $n \geqslant 0$.

b) *There is an isomorphism of graded Hopf algebras* $U_A(\operatorname{gr}_{\text{isol}} \mathfrak{u}) \cong \operatorname{gr}_{\text{isol}} U_A(\mathfrak{u})$.

3.3. Let Γ be, as in 3.1, a finitely generated, torsion free, nilpotent group of class c and let $\Gamma = H^1 \supset \cdots \supset H^{r+1} = \{1\}$ be a refinement of the isolated lower central series such that each H^i/H^{i+1} is an infinite cyclic group, generated by the image of an element g_i of H^i. Then, $r = h(\Gamma)$ and $\{g_1, \ldots, g_r\}$ is called a system of canonical parameters (or canonical basis) of Γ; it induces a bijection $\mathbb{Z}^r \cong \Gamma$, given by $(n_1, \ldots, n_r) \mapsto g_1^{n_1} \cdots g_r^{n_r}$; we will denote the R.H.S. simply by $g(n_1, \ldots, n_r)$. Let $\{e_1, \ldots, e_r\}$ be the standard basis of $\mathbb{Z}^r$; then $g(e_i) = g_i$.

Let $\mathcal{P}_{r,r}$ denote the subring of the polynomial ring $\mathbb{Q}[\xi_1, \ldots, \xi_r, \eta_1, \ldots, \eta_r]$ consisting of those polynomials which take integral values on $\mathbb{Z}^r \times \mathbb{Z}^r$. By a result of Ph. Hall [**21**, Th. 6.5], there exist polynomials $P_1, \ldots, P_r \in \mathcal{P}_{r,r}$ such that

$$(\star) \qquad g(x_1, \ldots, x_r)\, g(y_1, \ldots, y_r) = g(P_1(x,y), \ldots, P_r(x,y)),$$

for any $x, y \in \mathbb{Z}^r$.

Therefore, there exists an algebraic unipotent group scheme U, defined over a finitely generated subring A of the rationals, and whose underlying scheme is affine space $\mathbb{A}^r_A$, such that Γ identifies with the subgroup $\mathbb{Z}^r$ of $U(A) = A^r$.

Remark. — If Γ is of class c, one may take $A = \mathbb{Z}[1/c!]$; this can be deduced, for example, from the Campbell-Hausdorff formula.

Let $k \in \{1, \ldots, r\}$. Since $P_k(x, 0) = x_k$ and $P_k(0, y) = y_k$ for every $x, y \in \mathbb{Z}^r$, the part of degree $\leqslant 1$ of P_k is $\xi_k + \eta_k$ and its part of degree 2, call it b_k, is bilinear in the ξ_i and the η_j. Thus, one has

$$P_k(\xi, \eta) = \xi_k + \eta_k + \sum_{i,j=1}^{r} b_k(e_i, e_j)\, \xi_i \eta_j + \text{terms of degree} > 2.$$

Let $\mathfrak{m}$ denote the ideal $(\xi_1, \ldots, \xi_r)$ of $A[U] = A[\xi_1, \ldots, \xi_r]$, let

$$\mathfrak{u} := \operatorname{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, A)$$

be the Lie algebra of U over A, and let $\{v_1, \ldots, v_r\}$ be the A-basis of $\mathfrak{u}$ dual to the basis $\{\overline{\xi}_1, \ldots, \overline{\xi}_r\}$. Then, the Lie brackets are given by

$$(1) \qquad [v_i, v_j] = \sum_{k=1}^{r} (b_k(e_i, e_j) - b_k(e_j, e_i))\, v_k,$$

see, for example, [**35**, § 1] or [**9**, § 1].

Proposition. — *There is an isomorphism of graded Lie algebras over A*

$$\operatorname{gr}_{\mathrm{isol}} \Gamma \otimes_{\mathbb{Z}} A \cong \operatorname{gr}_{\mathrm{isol}} \mathfrak{u},$$

under which each $\overline{g}_i$ corresponds to $\overline{v}_i$.

Proof. — First, for each i, let $\nu(i)$ denote the largest integer n such that $g_i \in C^{(n)}(\Gamma)$. Denote by $\overline{g}_i$ the image of g_i in $\mathrm{gr}_{\mathrm{isol}}^{\nu(i)}\Gamma$; then $\{\overline{g}_1, \dots, \overline{g}_r\}$ is a $\mathbb{Z}$-basis of $\mathrm{gr}_{\mathrm{isol}}\Gamma$.

For $k = 1, \dots, r$, let $Q_k := P_k - \xi_k - \eta_k$ be the part of P_k of degree > 1. Recall that, for $x_1, \dots, x_r \in \mathbb{Z}$, $g(\sum_{i=1}^r x_i e_i)$ denotes the element $g_1^{x_1} \cdots g_r^{x_r}$ of Γ.

Let $i, j \in \{1, \dots, r\}$ be arbitrary with $i < j$. Then, for every $x, y \in \mathbb{Z}^r$, one has $g(xe_i)g(ye_j) = g(xe_i + ye_j)$ and hence $Q_k(xe_i, ye_j) = 0 = b_k(xe_i, ye_j)$ for any k. In particular, $b_k(e_i, e_j) = 0$.

On the other hand, since $g_j^x \in C^{(\nu(j))}(\Gamma)$ and $g_i^y \in C^{(\nu(i))}(\Gamma)$ one has,

$$g_j^x\, g_i^y \equiv g_i^y\, g_j^x\, g\Big(\sum_{\substack{k \\ \nu(k)=\nu(i)+\nu(j)}} Q_k(x,y)e_k\Big) \quad \text{mod. } C^{(\nu(i)+\nu(j)+1)}(\Gamma).$$

Further, since the commutator induces a bilinear map on $\mathrm{gr}_{\mathrm{isol}}\Gamma$, one has, when $\nu(k) = \nu(i) + \nu(j)$,

$$Q_k(xe_j, ye_i) = xyQ_k(e_j, e_i) = xyb_k(e_j, e_i).$$

Then, an easy computation shows that

$$g_i^x\, g_j^y\, g_i^{-x}\, g_j^{-y} \equiv g\Big(\sum_{\substack{k \\ \nu(k)=\nu(i)+\nu(j)}} -xyb_k(e_j, e_i)\, e_k\Big) \quad \text{mod. } C^{(\nu(i)+\nu(j)+1)}(\Gamma).$$

Using the fact that $b_k(e_i, e_j) = 0$, one deduces that the Lie bracket on $\mathrm{gr}_{\mathrm{isol}}\Gamma$ is given by

$$[\overline{g}_i, \overline{g}_j] = \sum_{\substack{k \\ \nu(k)=\nu(i)+\nu(j)}} \big(b_k(e_i, e_j) - b_k(e_j, e_i)\big)\, \overline{g}_k. \tag{2}$$

The proposition is then a consequence of the following claim.

Claim. — *For $\ell = 1, \dots, c$, $C^{(\ell)}(\mathfrak{u})$ is the A-span of those v_k such that $\nu(k) \geqslant \ell$.*

Indeed, using (1), the claim implies that $\mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}$ is the Lie algebra having an A-basis $\{\overline{v}_1, \dots, \overline{v}_r\}$ and brackets given by

$$[\overline{v}_i, \overline{v}_j] = \sum_{\substack{k \\ \nu(k)=\nu(i)+\nu(j)}} \big(b_k(e_i, e_j) - b_k(e_j, e_i)\big)\, \overline{v}_k. \tag{3}$$

Comparing with (2), one obtains that $\mathrm{gr}_{\mathrm{isol}}\Gamma \otimes_{\mathbb{Z}} A \cong \mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}$.

Let us now prove the claim by induction on $r + \ell$. Recall that c denotes the class of Γ. By induction, we may reduce to the case where $C^{(c)}(\Gamma) = \mathbb{Z}g_r$.

Since $\mathrm{gr}_{\mathrm{isol}}\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathrm{gr}\,\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ is generated in degree 1, there exist $s < t < r$ such that $\nu(t) = c - 1$ and $[\overline{g}_s, \overline{g}_t] = n\overline{g}_r$, for some non-zero integer n. Then, $(g_s, g_t) = g_r^n$ and hence, by the previous calculations, one has $b_r(e_t, e_s) = -n$, while $b_r(e_s, e_t) = 0$. Therefore, by (1), $[v_s, v_t] = nv_r$.

For any $k < r$, the image of v_k in $\mathfrak{u}/Av_r$ belongs to $C^{(\nu(k))}(\mathfrak{u}/Av_r)$, by induction hypothesis. Thus, there exist a positive integer m_k and $a_k \in A$ such that

$$m_k v_k - a_k v_r \in C^{\nu(k)}(\mathfrak{u}). \tag{4}$$

Applying this to $k = t$ and using the fact that v_r is central, one obtains that

$$m_t\, n\, v_r = [v_s, m_t v_t - a_t v_r]$$

belongs to $C^c(\mathfrak{u})$, and hence $v_r \in C^{(c)}(\mathfrak{u})$. In turn, this implies, by (4), that $v_k \in C^{(\nu(k))}(\mathfrak{u})$, for each $k < r$. This proves the claim and completes the proof of the proposition.

3.4. Filtered Noetherian rings with the AR-property.— Let us recall several results about the homology of filtered Noetherian rings with the Artin-Rees property. Some basic references for this material are [**47**], [**6**], [**20**]; see also [**40**, Chap. I] and [**14**, § 1]. (Note, however, that in [**20**] the assertions in lines 8-12 of 2.8 and assertion (ii) of Theorem 3.3 are not correct; it is not difficult to provide counter-examples).

Let S be a left Noetherian ring. A sequence $\mathcal{I} := \{I_1, I_2, \ldots\}$ of two-sided ideals is said to be *admissible* if $I_1 \supseteq I_2 \supseteq \cdots$ and $I_j I_k \subseteq I_{j+k}$ for $j, k \geqslant 0$ (where one sets $I_0 = S$). Given such a sequence, let

$$\operatorname{gr} S := \bigoplus_{n\geqslant 0} I_n/I_{n+1} \quad \text{and} \quad \widehat{S} := \operatorname*{proj.lim.}_{n\geqslant 0} S/I_n$$

be the associated graded ring and completion, respectively.

Let S-filt denote the category of $\mathbb{N}$-filtered left S-modules: objects are left S-modules M equipped with a decreasing filtration $M = F^0M \supseteq F^1M \supseteq \cdots$ such that $I_nF^kM \subseteq F^{n+k}M$, and a morphism $f : M \to N$ between two such objects is an S-morphism which preserves the filtrations. Then f induces a morphism of $\operatorname{gr} S$-modules $\operatorname{gr} f : \operatorname{gr} M \to \operatorname{gr} N$ and this defines a functor gr from S-filt to the category of $\mathbb{N}$-graded $\operatorname{gr} S$-modules. Further, f is called *strict* if one has $f(M) \cap F^kN = f(F^kM)$ for any k.

An object M of S-filt is called *separated* if $\bigcap_{n\geqslant 0} F^nM = \{0\}$, and *discrete* if $F^nM = \{0\}$ for some $n \geqslant 0$.

The category S-filt is equipped with shift functors s^n, for $n \geqslant 0$, defined as follows. If M is an object of S-filt, $s^nM = M$ as S-module but $F^p(s^nM) = F^{p-n}M$ for $p \geqslant 0$, with the convention that $F^kM = M$ if $k < 0$. If M is an $\mathbb{N}$-graded S-module, the shifted module s^nM is defined in an analogous manner.

An object L of S-filt is called *filt-free* if it a direct sum of shifted modules $s^{d(\lambda)}S$, for λ running in some index set Λ. Then, $\operatorname{gr} L \cong \bigoplus_{\lambda\in\Lambda} s^{d(\lambda)} \operatorname{gr} S$.

Let M be an object of S-filt. Then a *strict filt-free resolution* of M is an S-module resolution

$$\cdots \longrightarrow L_1 \xrightarrow{f_1} L_0 \xrightarrow{f_0} M \longrightarrow 0 \tag{$\mathcal{E}$}$$

such that every L_n is filt-free and every f_n is a strict morphism in S-filt. By [**47**, Lemmas 1,2], the associated graded complex $(\operatorname{gr}\mathcal{E})$ is then a free $\operatorname{gr}S$-resolution of $\operatorname{gr}M$ and, conversely, if S is complete with respect to $\mathcal{I}$, any free $\operatorname{gr}S$-resolution of $\operatorname{gr}M$ can be obtained in this manner.

Let us consider also the category filt-S of $\mathbb{N}$-filtered *right* S-modules. All notions introduced previously for S-filt have, of course, their right-handed analogues. Now, if N (resp. M) is an object of filt-S (resp. S-filt), the abelian group $N\otimes_S M$ has a natural $\mathbb{N}$-filtration, defined by

$$F^n(N\otimes_S M) := \operatorname{Im}\Big(\sum\nolimits_{p+q=n} F^pN\otimes_S F^qM \longrightarrow N\otimes_S M\Big).$$

Moreover, it is easily seen that if either of N or M is a filt-free object, then the natural map $\operatorname{gr}N\otimes_{\operatorname{gr}S}\operatorname{gr}M\to\operatorname{gr}(N\otimes_S M)$ is an isomorphism.

Therefore, if one considers a strict filt-free resolution $L_\bullet$ of, say, M, the filtration on $N\otimes_S L_\bullet$ induces a natural spectral sequence with E_1-term (in cohomological notation)

$$E_1^{p,-q} = H^{p-q}(\operatorname{gr}N\otimes_S\operatorname{gr}L_\bullet)_p = \operatorname{Tor}^{\operatorname{gr}S}_{q-p}(\operatorname{gr}N,\operatorname{gr}M)_p.$$

Moreover, certain finiteness conditions ensure that this spectral sequence converges finitely to $\operatorname{Tor}^S_*(N,M)$. Firstly, by [**47**, Lemma 2.(g)] or [**20**, Th. 2.9], one has the following

Proposition (C). — *Assume that S is complete with respect to the filtration $\mathcal{I}$ and that $\operatorname{gr}S$ is left Noetherian. Let M,N be objects of S-filt and filt-S, respectively, such that M is separated and $\operatorname{gr}M$ finitely generated over $\operatorname{gr}S$, while N is discrete. Then the spectral sequence above converges finitely to $\operatorname{Tor}^S_*(N,M)$.*

Proof. — By the references cited above, any resolution of $\operatorname{gr}M$ by free $\operatorname{gr}S$-modules can be lifted to a strict filt-free resolution of M. Since $\operatorname{gr}M$ is finitely generated over $\operatorname{gr}S$, which is left Noetherian, one deduces that M admits a strict filt-free resolution $L_\bullet\to M\to 0$ such that each L_n is finitely generated. As N is assumed to be discrete, the filtration on $N\otimes_S L_\bullet$ is then discrete (and exhaustive) in each degree, and the proposition follows.

Secondly, the assumption that S be complete can be relaxed if one assumes that the sequence $\mathcal{I}=\{I=I_1\supseteq I_2\supseteq\cdots\}$ has the left Artin-Rees property, *i.e.*, that $\mathcal{I}$ satisfies the following : for any finitely generated left S-module M, any submodule $N\subseteq M$ and any $n\geqslant 0$, there exists $n'\geqslant n$ such that $N\cap I_{n'}M\subseteq I_nN$.

For any left S-module M, let us denote by $\widehat{M}$ its completion with respect to the filtration $\{I_nM\}$; it is an $\widehat{S}$-module and there is a natural morphism of $\widehat{S}$-modules $\tau_M:\widehat{S}\otimes_S M\to\widehat{M}$. As observed in [**6**, Prop. 3], one has the following proposition, which is proved exactly as in the commutative I-adic case (see [**2**, Chap. 10]).

Proposition (AR). — *Assume that S is left Noetherian and that $\mathcal{I}$ satisfies the left AR-property. Then, τ_M is an isomorphism for any finitely generated left S-module M and, therefore,*

a) *$\widehat{S}$ is flat as right S-module,*

b) *for each n, $\widehat{S}I_n = \operatorname{Ker}(\widehat{S} \to S/I_n)$ is a two-sided ideal and hence $\{\widehat{S}I_n\}$ is an admissible sequence in $\widehat{S}$,*

c) *the associated graded $\operatorname{gr}\widehat{S}$ is isomorphic to $\operatorname{gr} S$.*

Thus, in particular, if $P_\bullet \to S/I \to 0$ is a resolution of S/I by free S-modules, then $\widehat{S} \otimes_S P_\bullet$ is a free $\widehat{S}$-resolution of

$$\widehat{S} \otimes_S (S/I) = \widehat{S/I} = S/I.$$

Thus, for any right $\widehat{S}$-module N, there is a natural isomorphism

$$\operatorname{Tor}_\bullet^{\widehat{S}}(N, S/I) \cong \operatorname{Tor}_\bullet^S(N, S/I).$$

This is the case, in particular, if N is a right S-module with a discrete filtration. Therefore, one obtains the following theorem, which is essentially contained in [**20**, Th. 3.3′.(i)].

Theorem 3.4.1. — *Let S be a left Noetherian ring, $\mathcal{I}$ an admissible sequence of ideals. Suppose that $\mathcal{I}$ satisfies the left AR property and that $\operatorname{gr} S$ is left Noetherian. Let N be a right S-module with a discrete filtration. Then there is a finitely convergent spectral sequence*

$$E_1^{p,-q} = \operatorname{Tor}_{q-p}^{\operatorname{gr} S}(\operatorname{gr} N, S/I)_p \Longrightarrow \operatorname{Tor}_{q-p}^{\widehat{S}}(N, S/I) \cong \operatorname{Tor}_{q-p}^{S}(N, S/I).$$

For future use, let us derive the following equivariant version of the theorem. Let Λ be a group of automorphisms of S preserving the sequence $\mathcal{I}$. Let $S\Lambda$ denote the smash product $S\#\mathbb{Z}\Lambda$, that is, $S\Lambda = S \otimes_{\mathbb{Z}} \mathbb{Z}\Lambda$ as $(S, \mathbb{Z}\Lambda)$-bimodule, the multiplication being defined by

$$(s \otimes \lambda)(s' \otimes \lambda') = s\lambda(s') \otimes \lambda\lambda'.$$

Similarly, denote by $\widehat{S}\Lambda$ the smash product $\widehat{S}\#\mathbb{Z}\Lambda$. Observe that an $S\Lambda$-module is the same thing as an S-module M equipped with an action of Λ such that $\lambda s m = \lambda(s)\lambda m$, for $m \in M$, $s \in S$, $\lambda \in \Lambda$.

For every $n \geqslant 0$, let I'_n (resp. $\widehat{I'_n}$) denote the left ideal of $S\Lambda$ (resp. $\widehat{S}\Lambda$) generated by I_n; they are two-sided ideals and form an admissible sequence in $S\Lambda$ (resp. $\widehat{S}\Lambda$). In both cases, the associated graded is isomorphic to $(\operatorname{gr} S)\Lambda := (\operatorname{gr} S)\#\Lambda$.

Theorem 3.4.2. — *With notation as above, let N be a discrete object of $S\Lambda$-filt. There is a finitely convergent spectral sequence of Λ-modules*

$$E_1^{p,-q} = \operatorname{Tor}_{q-p}^{\operatorname{gr} S}(\operatorname{gr} N, S/I)_p \Longrightarrow \operatorname{Tor}_{q-p}^{S}(N, S/I).$$

Proof. — First, $I' := (S\Lambda)I$ is a two-sided ideal of $S\Lambda$, and $S\Lambda \otimes_S (S/I) \cong S\Lambda/I'$. Then, by standard arguments, it suffices to prove that: *i*) $\widehat{S}\Lambda$ is flat as right $S\Lambda$-module, and: *ii*) $\widehat{S}\Lambda \otimes_S (S/I) \cong S\Lambda/I'$.

But $\widehat{S}\Lambda$ is isomorphic to $\widehat{S} \otimes_S S\Lambda$ as $(\widehat{S}, S\Lambda)$-bimodule, and to $S\Lambda \otimes_S \widehat{S}$ as $(S\Lambda, \widehat{S})$-bimodule. This implies *i*) and *ii*).

3.5. Let us return to the finitely generated, torsion free, nilpotent group Γ and the associated unipotent algebraic group U_A. Recall the notation of subsections 3.1–3.3.

It is known that $\mathbb{Z}\Gamma$ and $U_A(\mathfrak{u})$ are left and right Noetherian and have the left and right AR-property with respect to the filtration by the powers of the augmentation ideal, see, for example, [**42**, Th. 2.7 & § 11.2], [**41**] and [**6**, Th. 1].

Further, by [**22**, Cor. 3.5], one has $I^{(cn)} \subseteq I^n$, where c is the class of Γ (and also the class of $\mathfrak{u}$), and a similar argument, using Proposition 3.2.a) shows that $J^{(cn)} \subseteq J^n$. From this one deduces easily that the sequences $\{I^{(n)}\}$ and $\{J^{(n)}\}$ also have the left and right AR-property. In the sequel, we equip $\mathbb{Z}\Gamma$ and $U_A(\mathfrak{u})$ with these sequences, which we call $\mathcal{I}$ and $\mathcal{J}$ respectively. By Theorem 3.1 and Proposition 3.2, the associated graded rings are left and right Noetherian.

Let V be an U_A-module. Then V is in a natural manner a representation of the Lie algebra $\mathfrak{u}$ and of the abstract group Γ. Let $\mathcal{F}$ be a finite sequence $V = F^0V \supset \cdots \supset F^{s+1}V = \{0\}$ of U_A-submodules. Let us say that $\mathcal{F}$ is an *admissible filtration* of V if it is an $\mathcal{I}$ (resp. $\mathcal{J}$) filtration of V regarded as $\mathbb{Z}\Gamma$ (resp. $U_A(\mathfrak{u})$) module, *i.e.*, if for any $i, n \geqslant 0$, both $I^{(n)}(F^iV)$ and $J^{(n)}(F^iV)$ are contained in $F^{i+n}V$.

Lemma. — *If V is an U_A-module which is finite free over A, it admits an admissible filtration.*

Proof. — By the theorem of Lie-Kolchin applied to $V_{\mathbb{Q}}$, one obtains that V^U, the submodule of invariants, is non-zero. Since

$$V^U = \{x \in V \mid \Delta_V(x) = x \otimes \varepsilon\},$$

where Δ_V is the coaction defining the comodule structure and ε is the augmentation of $A[U]$, and since $V \otimes_A A[U]$ is a free A-module, one sees that V/V^U is torsion-free, hence a free A-module.

Therefore, if one sets $F_0V = 0$ and defines inductively F_kV as the inverse image in V of the U-invariants in $V/F_{k-1}V$, the sequence $\{F_kV\}$ is increasing strictly, as long as $F_kV \neq V$, and each V/F_kV, if non-zero, is a finite free A-module. Since V is a Noetherian A-module, $F_NV = V$ for some N. Setting $F^iV = F_{N-i}V$, it is easily seen that, for any $i, n \geqslant 0$, both $I^n(F^iV)$ and $J^n(F^iV)$ are contained in $F^{i+n}V$. Further, since every $F^iV/F^{i+n}V$ is torsion-free, one obtains that $\{F^iV\}_{i=0}^N$ is an admissible filtration of V.

Then, one deduces from the results of 3.4 the following theorem. There are, obviously, equivariant versions; we leave their formulation to the reader.

Theorem. — *Let V be an U_A-module which is finite free over A and let $\mathcal{F}$ be any admissible filtration on V. Then there are two finitely convergent spectral sequences:*

i) $$E_1^{p,-q} = H_{q-p}(\mathrm{gr}_{\mathrm{isol}}\,\Gamma, \mathrm{gr}_{\mathcal{F}}\,V)_p \Longrightarrow H_{q-p}(\Gamma, V),$$

ii) $$E_1^{p,-q} = H_{q-p}(\mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}, \mathrm{gr}_{\mathcal{F}}\,V)_p \Longrightarrow H_{q-p}(\mathfrak{u}, V).$$

3.6. Finally, let us return to the setting of Sections 1 and 2. The unipotent group U_P^- is defined over $\mathbb{Z}$. Let $\Gamma := U_P^-(\mathbb{Z})$; it is, clearly, a torsion-free nilpotent group.

For each $\beta \in R$, let U_β be the corresponding root subgroup, let X_β be a generator of $\mathfrak{g}_\beta = \mathrm{Lie}\,U_\beta$, and let θ_β be the isomorphism $\mathbb{G}_a \to U_\beta$ such that $d\theta_\beta(1) = X_\beta$. Set $I := \Delta \setminus R_L^+$ and let $f_I : \mathbb{Z}R \to \mathbb{Z}$ be the additive function which coincides on the basis Δ with the negative of the characteristic function of I. That is,

$$f_I(\alpha) = \begin{cases} -1 & \text{if } \alpha \in I; \\ 0 & \text{if } \alpha \in \Delta \cap R_L^+. \end{cases}$$

Choose a numbering $\alpha_1, \ldots, \alpha_r$ of the elements of $R^- \setminus R_L^-$ such that $f_I(\alpha_i) \leqslant f_I(\alpha_j)$ if $i \leqslant j$. The multiplication map induces an isomorphism of $\mathbb{Z}$-schemes

$$U_{\alpha_1} \times \cdots \times U_{\alpha_r} \xrightarrow{\cong} U_P^-.$$

Moreover, it follows from the commutation formulas in [**49**, Lemma 15] or [**7**, 3.2.3–3.2.5] that, for any $s = 1, \ldots, r$, $U_{\alpha_s} \cdots U_{\alpha_r}$ is a closed, normal subgroup of U_P^-. One deduces that the $g_i := \theta_{\alpha_i}(1)$ generate Γ and, moreover, form a system of canonical parameters, that U_P^- is the algebraic group associated in 3.3 to Γ, and that the basis $\{v_1, \ldots, v_r\}$ of $\mathfrak{u}_P^-$ identifies with $\{X_{\alpha_1}, \ldots, X_{\alpha_r}\}$.

Lemma. — *One has* $\mathfrak{u}_P^- \cong \mathrm{gr}_{\mathrm{isol}}\,\mathfrak{u}_P^-$.

Proof. — Since T acts on $\mathfrak{u}_P^-$ by Lie algebra automorphisms, $\mathfrak{u}_P^-$ has a structure of graded Lie algebra given by the function f_I. That is, if one sets, for $i \geqslant 1$,

$$\mathfrak{u}_P^-(i) := \bigoplus_{\substack{\alpha \in R^- \\ f_I(\alpha) = i}} \mathfrak{g}_\alpha,$$

then

$$\mathfrak{u}_P^- = \bigoplus_{i \geqslant 1} \mathfrak{u}_P^-(i) \quad \text{and} \quad [\mathfrak{u}_P^-(i), \mathfrak{u}_P^-(j)] \subseteq \mathfrak{u}_P^-(i+j).$$

Therefore, the lemma will follow if we show that $C^{(i)}(\mathfrak{u}_P^-) = \mathfrak{u}_P^-(\geqslant i)$, where $\mathfrak{u}_P^-(\geqslant i)$ is defined in the obvious manner. Clearly, $C^i(\mathfrak{u}_P^-) \subseteq \mathfrak{u}_P^-(\geqslant i)$ and, since $\mathfrak{u}_P^-/\mathfrak{u}_P^-(\geqslant i)$ is torsion-free, one obtains that $C^{(i)}(\mathfrak{u}_P^-) \subseteq \mathfrak{u}_P^-(\geqslant i)$.

The converse inclusion $\mathfrak{u}_P^-(\geqslant i) \subseteq C^{(i)}(\mathfrak{u}_P^-)$ follows from an argument in the proof of [**4**, Prop. 4.7.(iii)]. For the convenience of the reader, let us recall here this short argument. Using induction on i, it suffices to prove that for any $\beta \in R^-$ such that $f_I(\beta) = i \geqslant 2$, there exists $\alpha \in R^+$ such that $f_I(-\alpha) = 1$ and $\langle \beta, \alpha^\vee \rangle < 0$, since then $\beta + \alpha \in R^- \setminus R_L^-$ and $[X_{-\alpha}, X_{\beta+\alpha}] = mX_\beta$ for some non-zero integer m.

As f_I is constant on orbits of $W_{\Delta\setminus I}$, we may assume that β belongs to X_L^+. Then, since $\beta \in R^-$ whilst dominant roots are positive, there exists $\alpha \in I$ such that $\langle \beta, \alpha^\vee \rangle < 0$. This completes the proof of the lemma.

Remark. — Our original proof of the inclusion $\mathfrak{u}_P^-(\geqslant i) \subseteq C^{(i)}(\mathfrak{u}_P^-)$ relied on the fact that, by [**33**, Cor. 8.1], $H_1(\mathfrak{u}_{P,\mathbb{Q}}^-, \mathbb{Q}) \cong \bigoplus_{\alpha\in I} V_{\mathbb{Q}}^L(-\alpha)$. We are indebted to the referee for pointing out the simple, direct argument in [**4**].

Recall the integers $\nu(i)$ introduced in the proof of Proposition 3.3. From this proposition and the previous lemma (and their proofs), one deduces the following

Corollary. — *There is an isomorphism of graded Hopf algebras* $\operatorname{gr}_{\mathrm{isol}} \mathbb{Z}\Gamma \cong U(\mathfrak{u}_P^-)$, *under which each* $\overline{g_i - 1}$ *corresponds to* X_{α_i}. *Further, for* $i = 1, \dots, r$, *one has* $\nu(i) = f_I(\alpha_i)$.

3.7. For any $\lambda \in X^+$, set

$$V_{\mathbb{Z}}(\lambda)(i) := \bigoplus_{\substack{\mu\in X \\ f_I(\mu-\lambda)=i}} V_{\mathbb{Z}}(\lambda)_\mu,$$

where the subscript μ denotes the μ-weight space. Then, each $V_{\mathbb{Z}}(\lambda)(i)$ is an L-submodule and there is an isomorphism of L-modules

$$V_{\mathbb{Z}}(\lambda) \cong \bigoplus_{i\geqslant 0} V_{\mathbb{Z}}(\lambda)(i).$$

Set $F^k V_{\mathbb{Z}}(\lambda) := \bigoplus_{i\geqslant k} V_{\mathbb{Z}}(\lambda)(i)$; this defines a filtration $\mathcal{F}$ of $V_{\mathbb{Z}}(\lambda)$ by P^--submodules, such that the associated graded is isomorphic to $V_{\mathbb{Z}}(\lambda)$ as L-module.

Proposition. — *One has* $I^{(n)} F^k V_{\mathbb{Z}}(\lambda) \subseteq F^{n+k} V_{\mathbb{Z}}(\lambda)$, *and* $\operatorname{gr}_{\mathcal{F}} V_{\mathbb{Z}}(\lambda) \cong V_{\mathbb{Z}}(\lambda)$ *as representations of* $\operatorname{gr}_{\mathrm{isol}} \mathbb{Z}\Gamma \cong U_{\mathbb{Z}}(\mathfrak{u}_P^-)$.

Proof. — Following [**22**], set, for $i = 1, \dots, r$ and $n \geqslant 0$,

$$u_i^{(n)} := g_i^{-[(n+1)/2]} (g_i - 1)^n,$$

where $[x]$ denotes the greatest integer not greater than x, and observe that $u_i^{(n)} \equiv (g_i - 1)^n$ modulo I^n. Further, for $\boldsymbol{j} \in \mathbb{N}^r$, set

$$u(\boldsymbol{j}) := u_1^{(j_1)} \cdots u_r^{(j_r)} \quad \text{and} \quad \nu(\boldsymbol{j}) = \sum_i j_i\, \nu(i).$$

Then, by [**22**, Theorem 3.2 (i) and Lemma 3.1], the elements $u(\boldsymbol{j})$ satisfying $\nu(\boldsymbol{j}) \geqslant n$ form a $\mathbb{Z}$-basis of $I^{(n)}$, for every $n \geqslant 0$.

From this one deduces that, in order to prove the proposition, it suffices to prove that, for any $v \in F^k V_{\mathbb{Z}}(\lambda)$ and $i = 1, \dots, r$, one has

$$(*) \qquad (g_i - 1)v - X_{\alpha_i} v \in F^{k+\nu(i)+1} V_{\mathbb{Z}}(\lambda).$$

The distribution algebra $\operatorname{Dist}(U_P^-)$ has a $\mathbb{Z}$-basis formed by the ordered products

$$X_{\alpha_1}^{(m_1)} \cdots X_{\alpha_r}^{(m_r)}, \quad \text{for } (m_1, \dots, m_r) \in \mathbb{N}^r,$$

where the elements $X_\beta^{(m)}$ satisfy $X_\beta^m = m!\,X_\beta^{(m)}$ for every $m \geqslant 0$. Further, the structure of $\mathbb{Z}[G]$-comodule on $V_{\mathbb{Z}}(\lambda)$ is such that, for any ring Ω, any $t \in \Omega$ and $v \in V_\Omega(\lambda)$, and any root α, one has

$$\theta_\alpha(t)v = \sum_{m\geqslant 0} t^m X_\alpha^{(m)} v,$$

where the R.H.S. is in fact a finite sum. Since $g_i = \theta_{\alpha_i}(1)$ and since each $X_{\alpha_i}^{(m)}$ has weight $m\alpha_i$ for the adjoint action of T, this immediately implies formula $(*)$. The proposition is proved.

3.8. We can now prove Theorem C of the Introduction. The discrete group $\Lambda = L(\mathbb{Z})$ normalizes $\Gamma = U_P^-$ and, hence, preserves the isolated powers of the augmentation ideal of $\mathbb{Z}\Gamma$. Therefore, by the equivariant version of Theorem 3.5 *i*), combined with Proposition 3.7, there is a finitely convergent spectral sequence of $L(\mathbb{Z})$-modules

$$(1) \qquad H_*(\mathfrak{u}_P^-, V_{\mathbb{Z}}(\lambda)) \cong H_*(\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma, V_{\mathbb{Z}}(\lambda)) \Longrightarrow H_*(\Gamma, V_{\mathbb{Z}}(\lambda)).$$

It is, clearly, compatible with flat base change. Thus, for any prime integer p, one has a finitely convergent spectral sequence

$$(2) \qquad H_*(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong H_*(\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda)) \Longrightarrow H_*(\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda)).$$

Moreover, it is not difficult to check, by standard arguments, that the natural structure of $L(\mathbb{Z})$-module on $H_*(\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda))$ considered in Theorem 3.4.2 is the restriction to $L(\mathbb{Z})$ of the natural structure of L-module on $H_*(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda))$. Therefore, if λ is p-small then, by Theorem 2.1, one obtains an isomorphism of $L(\mathbb{Z})$-modules

$$H_i(\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong H_i(\mathfrak{u}_P^-, V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w\in W^L(i)} V^L_{\mathbb{Z}_{(p)}}(w\cdot\lambda),$$

for every $i \geqslant 0$. In particular, $H_*(\mathrm{gr}_{\mathrm{isol}}\, \mathbb{Z}\Gamma, V_{\mathbb{Z}_{(p)}}(\lambda))$ is a free $\mathbb{Z}_{(p)}$-module.

Finally, it is well-known that $\mathfrak{u}_P^- \otimes \mathbb{Q}$ is isomorphic to the Malcev-Jennings Lie algebra of Γ; this follows, for example, from the proof of [**35**, Lemma 1.9]. Therefore, by a result of Pickel [**43**, Th. 10], there is an isomorphism of graded vector spaces

$$H_\bullet(\mathfrak{u}_P^-, V_{\mathbb{Q}}(\lambda)) \cong H_\bullet(\Gamma, V_{\mathbb{Q}}(\lambda)).$$

This implies that the abutment of the spectral sequence in (2) has the same rank over $\mathbb{Z}_{(p)}$ as the E_1-term. Since the latter is a free $\mathbb{Z}_{(p)}$-module, one deduces that the spectral sequence degenerates at E_1. Therefore, we have obtained the following

Theorem. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then, for each $n \geqslant 0$, $H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda))$ has a finite, natural $L(\mathbb{Z})$-module filtration such that*

$$\mathrm{gr}\, H_n(U_P^-(\mathbb{Z}), V_{\mathbb{Z}_{(p)}}(\lambda)) \cong \bigoplus_{w\in W^L(n)} V^L_{\mathbb{Z}_{(p)}}(w\cdot\lambda).$$

By the universal coefficient theorem, one then obtains a similar result over $\mathbb{F}_p$. Finally, by an argument similar to the one in 2.10, one obtains the following analogue in cohomology.

Corollary. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then, for each $n \geqslant 0$, $H^n(U_P(\mathbb{Z}), V_{\mathbb{F}_p}(\lambda))$ has a finite, natural $L(\mathbb{Z})$-module filtration such that*

$$\operatorname{gr} H^n(U_P(\mathbb{Z}), V_{\mathbb{F}_p}(\lambda)) \cong \bigoplus_{w \in W^L(n)} V^L_{\mathbb{F}_p}(w \cdot \lambda).$$

3.9. Let us derive in this subsection a corollary about the p-Lie algebra associated with the p-lower central series of Γ. (This result will not be used in the sequel).

Let $\mathcal{F}$ be a decreasing sequence $\Gamma = F^1\Gamma \supseteq F^2\Gamma \supseteq \cdots$ of normal subgroups of Γ. It is called an N_p-sequence if it is an N-sequence and $x \in F^i\Gamma$ implies that $x^p \in F^{pi}\Gamma$. In this case, $\operatorname{gr}_{\mathcal{F}} \Gamma$ is a graded p-Lie algebra, see [**36**, Chap. I, Cor. 6.8] or [**5**, Chap. II, § 5, Ex. 10].

For our purposes, it is convenient to define the p-lower central series $\{F^n_p\Gamma\}_{n\geqslant 1}$ as follows. Denoting by $I_{\mathbb{F}_p}$ the augmentation ideal of $\mathbb{F}_p\Gamma$, set

$$F^n_p\Gamma := \{x \in \Gamma \mid x - 1 \in I^n_{\mathbb{F}_p}\}.$$

This is an N_p-sequence (see [**42**, Lemma 3.3.1]), and we denote the associated graded p-Lie algebra by $\operatorname{gr}^\bullet_p \Gamma$.

The n-th term $F^n_p\Gamma$ of the p-lower central series is sometimes defined as the subgroup of Γ generated by all elements x^{p^s} satisfying $p^s\omega(x) \geqslant n$, where $\omega(x)$ denotes the largest integer i such that $x \in C^i(\Gamma)$. That the two definitions agree is due to Lazard [**36**, Chap. I, Th. 5.6 & 6.10] and Quillen [**45**], see also [**42**, § 11.1].

Let us denote by $\mathcal{L}ie_{\mathbb{F}_p}$ the category of Lie algebras over $\mathbb{F}_p$, by p-$\mathcal{L}ie_{\mathbb{F}_p}$ the subcategory of p-Lie algebras, and by gr-$\mathcal{L}ie_{\mathbb{F}_p}$ and p-gr-$\mathcal{L}ie_{\mathbb{F}_p}$, respectively, the subcategories of graded and graded p-Lie algebras over $\mathbb{F}_p$. The forgetful functor p-$\mathcal{L}ie_{\mathbb{F}_p} \to \mathcal{L}ie_{\mathbb{F}_p}$ has a left adjoint, denoted by p-$\mathcal{L}$; it takes gr-$\mathcal{L}ie_{\mathbb{F}_p}$ to p-gr-$\mathcal{L}ie_{\mathbb{F}_p}$.

Corollary. — *Let Γ be a finitely generated, torsion-free, nilpotent group, say of class c. Suppose that $\bigoplus_{i=1}^c C^{(i)}(\Gamma)/C^i(\Gamma)$ has no p-torsion. Then, there is an isomorphism of graded p-Lie algebras*

$$\operatorname{gr}^\bullet_p \Gamma \cong p\text{-}\mathcal{L}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p).$$

Proof. — The hypothesis implies easily that $\operatorname{gr}\Gamma \otimes \mathbb{F}_p \cong \operatorname{gr}_{\mathrm{isol}} \Gamma \otimes \mathbb{F}_p$. Moreover, it follows from the proof of [**22**, Th. 3.2 (i)] that every $I^{(n)}/I^n$ has no p-torsion. This implies that, inside $\mathbb{F}_p\Gamma$, one has the identifications $I^{(n)} \otimes \mathbb{F}_p = I^n \otimes \mathbb{F}_p = I^n_{\mathbb{F}_p}$. One deduces from this, coupled with Theorem 3.1, the isomorphisms

$$\operatorname{gr}\mathbb{F}_p\Gamma \cong (\operatorname{gr}_{\mathrm{isol}} \mathbb{Z}\Gamma) \otimes \mathbb{F}_p \cong U_{\mathbb{Z}}(\operatorname{gr}_{\mathrm{isol}} \Gamma) \otimes \mathbb{F}_p \cong U_{\mathbb{F}_p}(\operatorname{gr}_{\mathrm{isol}} \Gamma \otimes \mathbb{F}_p) \cong U_{\mathbb{F}_p}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p).$$

On the other hand, by Quillen [**45**], $\operatorname{gr}\mathbb{F}_p\Gamma$ is isomorphic as graded Hopf algebra to $U^{res}_{\mathbb{F}_p}(\operatorname{gr}^\bullet_p \Gamma)$, the restricted enveloping algebra of the p-Lie algebra $\operatorname{gr}^\bullet_p \Gamma$.

Recall that $U^{res}_{\mathbb{F}_p}$, the restricted enveloping algebra functor, is left adjoint to the forgetful functor $\mathcal{A}s_{\mathbb{F}_p} \to p\text{-}\mathcal{L}ie_{\mathbb{F}_p}$, where $\mathcal{A}s_{\mathbb{F}_p}$ denotes the category of associative $\mathbb{F}_p$-algebras (with unit), while the usual enveloping algebra functor is left adjoint to the forgetful functor $\mathcal{A}s_{\mathbb{F}_p} \to \mathcal{L}ie_{\mathbb{F}_p}$. Thus, since the adjoint of a composite is the composite of the adjoints, one has $U_{\mathbb{F}_p}(L) \cong U^{res}_{\mathbb{F}_p}(p\text{-}\mathcal{L}(L))$, for any $\mathbb{F}_p$-Lie algebra L.

Therefore, one obtains an isomorphism of graded Hopf algebras

$$U^{res}_{\mathbb{F}_p}(p\text{-}\mathcal{L}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p)) \cong U^{res}_{\mathbb{F}_p}(\operatorname{gr}^\bullet_p \Gamma).$$

Taking primitive elements, this gives, by the theorem of Milnor-Moore [**38**, Th. 6.11], an isomorphism of graded p-Lie algebras $p\text{-}\mathcal{L}(\operatorname{gr}\Gamma \otimes \mathbb{F}_p) \cong \operatorname{gr}^\bullet_p \Gamma$. The corollary is proved.

Remark. — It is easy to see that the torsion primes in $\bigoplus_{i=1}^c C^{(i)}(\Gamma)/C^i(\Gamma)$ and in $\operatorname{gr}\Gamma$ are the same. Presumably, it should not be difficult to extract from the proof of Proposition 3.3 that the torsion primes in $\operatorname{gr}\mathfrak{u}$ are also the same.

4. Standard and BGG complexes for distribution algebras

4.1. As in subsection 2.2, there is defined a complex

$$\cdots \longrightarrow \mathcal{U}(G) \otimes_{\mathcal{U}(P)} \Lambda^2(\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_2^{\mathfrak{p}}} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} (\mathfrak{g}/\mathfrak{p}) \xrightarrow{d_1^{\mathfrak{p}}} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

the differentials being defined by the same formula as in 2.2. Note, however, that this complex is *not* exact. We shall denote it by $\mathcal{S}_\bullet(G,P)$.

More generally, let V be a G-module and let $V_{|P}$ denote V regarded as an $\mathcal{U}(P)$-module. Then one obtains, as in 2.2, a complex of $\mathcal{U}(G)$-modules

$$\cdots \longrightarrow \mathcal{U}(G) \otimes_{\mathcal{U}(P)} (\Lambda^2(\mathfrak{g}/\mathfrak{p}) \otimes V|_P) \xrightarrow{d_2} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} (\mathfrak{g}/\mathfrak{p} \otimes V|_P)$$
$$\xrightarrow{d_1} \mathcal{U}(G) \otimes_{\mathcal{U}(P)} V|_P \xrightarrow{\varepsilon} V \longrightarrow 0.$$

We shall call it the standard complex of V relative to the pair $(\mathcal{U}(G),\mathcal{U}(P))$, and denote it by $\mathcal{S}_\bullet(G,P,V)$. When $V = V_{\mathbb{Z}}(\lambda)$, we shall denote it simply by $\mathcal{S}_\bullet(G,P,\lambda)$.

Further, as in 2.4, let us define, for any $\xi \in X_L^+$, the generalized Verma module (for $\mathcal{U}(G)$ and $\mathcal{U}(P)$)

$$\mathcal{M}_P(\xi) := \mathcal{U}(G) \otimes_{\mathcal{U}(P)} V^L_{\mathbb{Z}}(\xi).$$

Set $\mathcal{M}_P^{\mathbb{Z}_{(p)}}(\xi) := \mathcal{M}_P(\xi) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) := \mathcal{S}_\bullet(G,P,\lambda) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, for any $\lambda \in X^+$.

4.2. For the rest of this section, let us fix $\lambda \in X^+$ and a prime integer p such that $\lambda \in \overline{C}_p$. Recall from 2.4 that $\Omega^i_{\mathfrak{p}}(\lambda)$ denotes the multiset of those $\xi \in X_L^+$ such that $V^L_{\mathbb{Q}}(\xi)$ is a composition factor of $\Lambda^i(\mathfrak{g}/\mathfrak{p})_{\mathbb{Q}}$.

Since $\mathcal{U}(G)$ is free over $\mathcal{U}(P)$ (see, for example, [**28**, § II.1.12]), one obtains exactly as in 2.4 the following

Lemma. — *Let $\lambda \in X^+ \cap \overline{C}_p$. Then each $\mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ admits a finite filtration by $\mathcal{U}_{\mathbb{Z}_{(p)}}(G)$-modules such that the successive quotients are the $\mathcal{M}_P^{\mathbb{Z}_{(p)}}(\xi)$, for $\xi \in \Omega_{\mathfrak{p}}^i(\lambda)$.*

Next, since $U_{\mathbb{Z}_{(p)}}(\mathfrak{g}) \subset \mathcal{U}_{\mathbb{Z}_{(p)}}(G) \subset U_{\mathbb{Q}}(\mathfrak{g})$, one deduces that $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G$ is contained in the center of $\mathcal{U}_{\mathbb{Z}_{(p)}}(G)$. Recall the characters $\chi_{\mu,p}$ and $\overline{\chi}_{\mu,p} = \pi \circ \chi_{\mu,p}$, where π is the morphism $\mathbb{Z}_{(p)} \to \overline{\mathbb{F}}_p$, introduced in 2.6.3. If M is a $\mathcal{U}_{\mathbb{Z}_{(p)}}(G)$-module generated by an element of weight μ annihilated by $\mathfrak{u}$, then $U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G$ acts on M by the character $\chi_{\mu,p}$ (see 2.6.3).

Let $I = \prod_{\xi \in \Omega_{\mathfrak{p}}^\bullet(\lambda)} \operatorname{Ker} \chi_{\xi,p}$ (each ξ being counted with its multiplicity). It follows from the previous lemma that $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ is a module over the ring $A := U_{\mathbb{Z}_{(p)}}(\mathfrak{g})^G/I$, which is a finite $\mathbb{Z}_{(p)}$-module. Let $\overline{\chi}_1, \dots, \overline{\chi}_r$ be the distinct algebra homomorphisms $A \to \overline{\mathbb{F}}_p$, numbered so that $\overline{\chi}_1 = \overline{\chi}_{\lambda,p}$, and, for $j = 1, \dots, r$, set

$$\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_j} := \Big\{ x \in \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) \mid \prod_{\substack{\xi \in \Omega_{\mathfrak{p}}^\bullet(\lambda) \\ \overline{\chi}_{\xi,p} = \overline{\chi}_j}} (\operatorname{Ker} \chi_{\xi,p}) x = 0 \Big\}.$$

These are, clearly, subcomplexes of $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)$. Then, exactly as in 2.7(*), one obtains the

Corollary. — *One has $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) = \bigoplus_{j=1}^r \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_j}$, a direct sum of complexes.*

4.3. Our aim in this section is to prove the following theorem.

Theorem. — *Assume that $\mathcal{D}G$ is simply-connected, that $X(T)/\mathbb{Z}R$ has no p-torsion and that $\mathfrak{u}_P^-$ is abelian. Let $\lambda \in X^+ \cap \overline{C}_p$. Consider the direct summand subcomplex $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}}$ defined in 4.2. Then, for each $i \geqslant 0$, one has*

$$\mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}} \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w \cdot \lambda).$$

As in [**16**, VI.5], we treat first the case $\lambda = 0$ and then derive from it the general case.

4.4. The case $\lambda = 0$.— Since $\mathfrak{u}_P^-$ is abelian, the differentials in the standard Koszul complex computing $H_\bullet(\mathfrak{u}_P^-)$ are all zero and hence $H_\bullet(\mathfrak{u}_P^-) \cong \Lambda^\bullet(\mathfrak{u}_P^-)$. Therefore, by a result of Kostant [**33**, § 8.2], the composition factors of $\Lambda^i(\mathfrak{g}/\mathfrak{p})_{\mathbb{Q}}$ are exactly the $V_{\mathbb{Q}}^L(w \cdot 0)$, for $w \in W^L(i)$, each occuring with multiplicity one.

Moreover, as easily seen, the assumption that $\mathfrak{u}_P^-$ is abelian is equivalent to the fact that if $\alpha, \beta \in R^+ \setminus R_P^+$ then $\alpha + \beta \notin R$, which is also equivalent to the fact that U_P acts trivially on $\mathfrak{g}/\mathfrak{p}$. Therefore, by Corollary 1.10.b), each $\Lambda^i(\mathfrak{g}/\mathfrak{p})_{\mathbb{Z}_{(p)}}$ is the direct sum of the Weyl modules $V_{\mathbb{Z}_{(p)}}^L(w \cdot 0)$, for $w \in W^L(i)$. It follows that

$$(*) \qquad \mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P) \cong \bigoplus_{w \in W^L(i)} \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w \cdot 0),$$

and, therefore, $\mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P) = \mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P)_{\overline{\chi}_{\lambda,p}}$ in this case. This proves the sought-for result when $\lambda = 0$ and $p \geqslant h-1$. (Note that no further assumption on G and p is needed in this case).

4.5. The general case.— Now, let $\lambda \in X^+ \cap \overline{C}_p$ be arbitrary. First, since $\mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) = \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P) \otimes V(\lambda)$, one deduces from 4.4 (∗) and the tensor identity ([**19**, Prop. 1.7]) that, for $i \geqslant 0$,

$$(1) \qquad \mathcal{S}_i^{\mathbb{Z}_{(p)}}(G,P,\lambda) \cong \bigoplus_{w\in W^L(i)} \mathcal{U}_{\mathbb{Z}_{(p)}}(G) \otimes_{\mathcal{U}_{\mathbb{Z}_{(p)}}(P)} \left(V^L_{\mathbb{Z}_{(p)}}(w\rho-\rho)\otimes V_{\mathbb{Z}_{(p)}}(\lambda)\right).$$

Let $\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ denote the summand corresponding to w on the R.H.S. Then

$$(2) \qquad \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda) = \bigoplus_{w\in W^L} \mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda),$$

each $\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)$ occuring in degree $\ell(w)$.

Therefore, by 4.2, one obtains that

$$(3) \qquad \mathcal{S}_\bullet^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}} \cong \bigoplus_{w\in W^L} \mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}}.$$

Lemma. — *Assume further that $\mathcal{D}G$ is simply-connected and that $X(T)/\mathbb{Z}R$ has no p-torsion. Then, for every $w \in W^L$,*

$$\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}} \cong \mathcal{M}_P^{\mathbb{Z}_{(p)}}(w\cdot\lambda).$$

Proof. — First, exactly as in 2.7, one obtains that each $\mathcal{S}_w^{\mathbb{Z}_{(p)}}(G,P,\lambda)_{\overline{\chi}_{\lambda,p}}$ has a filtration whose quotients are the $\mathcal{M}_P^{\mathbb{Z}_{(p)}}(\xi)$ for those ξ such that $V^L_{\mathbb{Q}}(\xi)$ is a composition factor of the $L_{\mathbb{Q}}$-module $V^L_{\mathbb{Q}}(w\cdot 0)\otimes V_{\mathbb{Q}}(\lambda)$ (counted with multiplicities) and such that $\overline{\chi}_{\xi,p} = \overline{\chi}_{\lambda,p}$.

Moreover, under the assumptions of the lemma, one obtains, exactly as in the proof of Proposition 2.8, that any such ξ has the form $y\cdot\lambda$, for some $y \in W^L$.

But then the assumption that $V^L_{\mathbb{Q}}(y\cdot\lambda)$ occurs as a composition factor of $V^L_{\mathbb{Q}}(w\cdot 0)\otimes V_{\mathbb{Q}}(\lambda)$ implies that $y = w$ and that the multiplicity is one. This may be deduced, for example, from [**27**, Satz 2.25]. For the convenience of the reader, let us record a proof. Firstly, it is well-known that any composition factor of the $L_{\mathbb{Q}}$-module $V^L_{\mathbb{Q}}(w\cdot 0)\otimes V_{\mathbb{Q}}(\lambda)$ has the form $V^L_{\mathbb{Q}}(w\cdot 0+\nu)$, for some weight ν of $V_{\mathbb{Q}}(\lambda)$ and occurs with a multiplicity at most equal to $\dim V_{\mathbb{Q}}(\lambda)_\nu$, see, for example, [**24**, § 24, Ex. 12] or, better, the proof of Cor. 4.7 in [**1**]. Secondly, for such a ν, suppose that $w\cdot 0+\nu = y\cdot\lambda$, for some $y \in W$. Then,

$$y^{-1}w\rho - \rho = \lambda - y^{-1}\nu.$$

Let θ denote this weight. Since $y^{-1}w\rho$ (resp. $y^{-1}\nu$) is a weight of $V_{\mathbb{Q}}(\rho)$ (resp. $V_{\mathbb{Q}}(\lambda)$), one has $\theta \in -\mathbb{N}R^+$ (resp. $\theta \in \mathbb{N}R^+$) and, therefore, $\theta = 0$. Thus, since the stabilizer of ρ in W is trivial, $y = w$. Finally, $\nu = w\lambda$ has multiplicity one in $V_{\mathbb{Q}}(\lambda)$. This completes the proof of the lemma and, therefore, of Theorem 4.3.

5. Dictionary

Let $G = \mathrm{GSp}(2g)_{\mathbb{Z}}$ be the split reductive Chevalley group over $\mathbb{Z}$ defined by ${}^tXJX = \nu \cdot J$ where J is given by $g \times g$-blocks

$$J = \begin{pmatrix} 0_g & & & \iddots \\ & & 1 & \\ & \iddots & & \\ -1 & & 0_g & \end{pmatrix}.$$

Let $B = TN$, resp. $Q = MU$, be the Levi decomposition of the upper triangular subgroup of G, resp. of the Siegel parabolic, *i.e.*, the maximal parabolic associated to α, the longest simple root for (G, B, T), so $M = L_I$ where $I = \Delta\backslash\{\alpha\}$. Note that $\mathcal{D}G = \mathrm{Sp}(2g)$ is simply-connected and that the unipotent radical of Q is abelian.

The group of characters X of T is identified to the sublattice

$$\{(a_g, \cdots, a_1; c) \in \mathbb{Z}^g \times \mathbb{Z} \mid c \equiv a_g + \cdots + a_1 \text{ mod. } 2\}$$

of $\mathbb{Z}^{g+1}$ in the following manner. The character $(a_g, \cdots, a_1; c)$ is defined by

$$\mathrm{diag}(t_g, \ldots, t_1, x \cdot t_1^{-1}, \ldots, x \cdot t_g^{-1}) \longmapsto t_g^{a_g} \cdot \cdots \cdot t_1^{a_1} \cdot x^{(c-a_1-\cdots-a_g)/2}.$$

The weight lattice $P(R)$ coincides with X, and the root lattice $\mathbb{Z}R$ is the intersection of X with the hyperplane $\{c = 0\}$. In particular, $X/\mathbb{Z}R$ is torsion free. The cone $X^+ \subset X$ of dominant weights of G is given by the conditions $a_g \geqslant \cdots \geqslant a_1 \geqslant 0$. The half-sum ρ of the positive roots of G is then $\rho = (g, \ldots, 1; 0)$.

If $(\varepsilon_g, \ldots, \varepsilon_1)$ is the canonical basis of $\mathbb{Z}^g$, the highest coroot $\gamma^\vee$ of G is $\varepsilon_g + \varepsilon_{g-1}$. The condition $\langle \lambda + \rho, \gamma^\vee \rangle \leqslant p$ reads, therefore,

$$a_g + a_{g-1} + g + (g-1) \leqslant p.$$

For a character $\phi = (a_g, \ldots, a_1; c)$ we define its degree as $|\phi| = \sum_{i=1}^g a_i$; the dual character $\widehat{\phi} = (a_g, \ldots, a_1; -c)$ of ϕ has the same degree. Note that $|\rho| = g(g+1)/2$. So,

$$\langle \lambda + \rho, \gamma^\vee \rangle \leqslant |\lambda + \rho|$$

with equality for $g \leqslant 2$.

Let $\mathbf{V} = \langle e_g, \ldots, e_1, e_1^*, \ldots, e_g^* \rangle$ be the standard $\mathbb{Z}$-lattice on which G acts; given two vectors $v, w \in \mathbf{V}$, we write $\langle v, w \rangle_J = {}^tvJw$ for their symplectic product. Then Q is the stabilizer of the standard lagrangian lattice $\mathbf{W} = \langle e_g, \ldots, e_1 \rangle$; we have $\mathbf{V} = \mathbf{W} \oplus \mathbf{W}^*$; $M = L_I$ is the stabilizer of the decomposition $(\mathbf{W}, \mathbf{W}^*)$; one has $M \cong \mathrm{GL}(g) \times \mathbb{G}_m$. Let $B_M = B \cap M$ be the standard Borel of M.

Recall from 1.5 the definition of admissible lattices and, for $\lambda \in X^+$, the $\mathbb{Z}$-lattices $V(\lambda)_{\min}$ and $V(\lambda)_{\max}$.

Let $\lambda \in X^+$ and $n = |\lambda|$; for any (i, j) with $1 \leqslant i < j \leqslant n$, let $\phi_{i,j} : \mathbf{V}^{\otimes n} \to \mathbf{V}^{\otimes(n-2)}$ denote the contraction given by

$$v_1 \otimes \cdots \otimes v_n \longmapsto \langle v_i, v_j \rangle_J \; v_1 \otimes \cdots \otimes \widehat{v}_i \otimes \cdots \otimes \widehat{v}_j \otimes \cdots \otimes v_n,$$

and let $\mathbf{V}^{\langle n\rangle}$ be the submodule of $\mathbf{V}^{\otimes n}$ defined as intersection of the kernels of the $\phi_{i,j}$. By applying the Young symmetrizer $c_\lambda = a_\lambda \cdot b_\lambda$ (see [**18**], 15.3 and 17.3) to $\mathbf{V}^{\langle n\rangle}$, one obtains an admissible $\mathbb{Z}$-lattice $V(\lambda)_{\mathrm{Young}}$ in $V_{\mathbb{Q}}(\lambda)$.

Then, by Corollary 1.9, one has the

Corollary. — *For any p-small weight $\lambda \in X^+$, one has canonically*

$$V(\lambda)_{\min} \otimes \mathbb{Z}_{(p)} = V(\lambda)_{\mathrm{Young}} \otimes \mathbb{Z}_{(p)} = V(\lambda)_{\max} \otimes \mathbb{Z}_{(p)}.$$

Moreover, a similar result holds for a weight $\mu \in X_M^+$ of M, provided it is p-small for M.

References

[1] H. H. Andersen, A new proof of old character formulas, pp.193-207 in: *Invariant theory*, Contemporary Math., Vol. 88 (R. Fossum, W. Haboush, M. Hochster, V. Lakshmibai eds.), Amer. Math. Soc., 1989.

[2] M. Atiyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.

[3] I. N. Bernstein, I. M. Gelfand, S. I. Gelfand, Differential operators on the base affine space and a study of **g**-modules, in *Lie Groups and their Representations*, ed. I. M. Gelfand, Proc. Conf. Budapest 1971, Adam Hilger Publ., 1975.

[4] A. Borel, J. Tits, Homomorphismes "abstraits" de groupes algébriques simples, *Annals Math.* 97 (1973), 499-571.

[5] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap.VII–VIII, Hermann, 1975.

[6] K. S. Brown, E. Dror, The Artin-Rees property and homology, *Israel J. Math.*, 22 (1975), 93-109.

[7] F. Bruhat, J. Tits, Groupes réductifs sur un corps local II, *Publ. Math. IHES*, 60 (1984), 5-184.

[8] R. Carter, G. Lusztig, On the modular representations of the general linear and symmetric groups, *Math. Z.*, 136 (1974), 193-242.

[9] B. Cenkl, R. Porter, Nilmanifolds and associated Lie algebras over the integers, *Pacific J. Math.*, 193 (2000), 5-29.

[10] C. W. Curtis, Noncommutative extensions of Hilbert rings, *Proc. Amer. Math. Soc.*, 4 (1953), 945-955.

[11] M. Demazure, Schémas en groupes réductifs, *Bull. Soc. math. France*, 93 (1965), 369-413.

[12] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, *Invent. math.*, 21 (1973), 287-301.

[13] J. Dixmier, *Algèbres enveloppantes*, Gauthier-Villars, 1974.

[14] S. Donkin, On the Noetherian property in endomorphism rings of certain comodules, *J. Algebra*, 70 (1981), 394-419.

[15] S. Donkin, *Rational representations of algebraic groups*, Lecture Notes in Maths. 1140, Springer-Verlag, 1985.

[16] G. Faltings, C. L. Chai, *Degeneration of abelian varieties*, Springer-Verlag, 1990.

[17] E. M. Friedlander, B. J. Parshall, Modular representation theory of Lie algebras, *Amer. J. Math.*, 110 (1988), 1055-1094.

[18] W. Fulton, J. Harris, *Representation theory (a first course)*, Springer-Verlag, 1994.

[19] H. Garland, J. Lepowsky, Lie algebra homology and the MacDonald-Kač formulas, *Invent. math.*, 34 (1976), 37-76.

[20] L. Grünenfelder, On the homology of filtered and graded rings, *J. Pure Appl. Algebra*, 14 (1979), 21-37.

[21] Ph. Hall, Nilpotent groups, Canadian Math. Congress, Edmonton 1957, reprinted pp.417-462 in *Collected works of Philip Hall*, Oxford Univ. Press, 1988.

[22] B. Hartley, Powers of the augmentation ideal in group rings of infinite nilpotent groups, *J. London Math. Soc.* (2), 25 (1982), 43-61.

[23] B. Hartley, Topics in the theory of nilpotent groups, pp. 61-120 in *Group Theory. Essays for Philip Hall*, Academic Press, 1984.

[24] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.

[25] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Amer. Math. Soc., 1995.

[26] J. C. Jantzen, Darstellungen halbeinfacher Gruppen und kontravariante Formen, *J. Reine Ang. Math.*, 290 (1977), 117-141.

[27] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Maths. 750, Springer-Verlag, 1979.

[28] J. C. Jantzen, *Representations of algebraic groups*, Academic Press, 1987.

[29] J. C. Jantzen, Representations of Lie algebras in prime characteristic, pp.185-235 in: *Representation theories and algebraic geometry* (Montréal 1997), A.Broer ed., Kluwer, 1998.

[30] S. A. Jennings, The group ring of a class of infinite nilpotent groups, *Can. J. Math.*, 7 (1955), 169-187.

[31] V. Kac, B. Weisfeiler, Coadjoint action of a semi-simple algebraic group and the center of the enveloping algebra in characteristic p, *Indag. Math.*, 38 (1976), 136-151.

[32] M. A. Knus, On the enveloping algebra and the descending central series of a Lie algebra, *J. Algebra*, 12 (1969), 335-338.

[33] B. Kostant, Lie algebra cohomology and the generalized Borel-Weil theorem, *Ann. Math.*, 74 (1961), 329-387.

[34] B. Kostant, Groups over $\mathbb{Z}$, pp.90-98 in: *Algebraic groups and their discontinuous subgroups*, Proc. Symp. Pure Math. IX (A. Borel, G.D. Mostow eds.), Amer. Math. Soc., 1966.

[35] L. A. Lambe, S. B. Priddy, Cohomology of nilmanifolds and torsion-free nilpotent groups, *Trans. A.M.S.*, 273 (1982), 39-55.

[36] M. Lazard, Sur les groupes nilpotents et les anneaux de Lie, *Ann. Éc. Norm. Sup.* (3ème série), 71 (1954), 101-190.

[37] J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, *J. Algebra*, 49 (1977), 496-511.

[38] J. W. Milnor, J. C. Moore, On the structure of Hopf algebras, *Ann. Math.*, 81 (1965), 211-264.

[39] A. Mokrane, J. Tilouine, Cohomology of Siegel varieties with p-adic integral coefficients and applications, *this volume.*

[40] C. Năstăsescu, F. Van Oystaeyen, *Graded and filtered rings and modules*, Lecture Notes in Maths. 758, Springer-Verlag, 1979.

[41] Y. Nouazé, P. Gabriel, Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente, *J. Algebra*, 6 (1967), 77-99.

[42] D. S. Passman, *The algebraic structure of group rings*, Wiley, 1977.

[43] P. F. Pickel, Rational cohomology of nilpotent groups and Lie algebras, *Comm. Algebra*, 6 (1978), 409-419.

[44] P. Polo, paper in preparation.

[45] D. Quillen, On the associated graded ring of a group ring, *J. Algebra*, 10 (1968), 411-418.

[46] A. Rocha-Caridi, Splitting criteria for $\mathfrak{g}$-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite dimensional, irreducible $\mathfrak{g}$-module, *Trans. A.M.S.*, 262 (1980), 335-366.

[47] G. Sjödin, On filtered modules and their associated graded modules, *Math. Scand.*, 33 (1973), 229-249.

[48] T. A. Springer, *Linear algebraic groups* (2nd edition), Birkhäuser, 1998.

[49] R. Steinberg, *Lectures on Chevalley groups*, Yale University lecture notes, 1968.

[50] R. Steinberg, Regular elements of semi-simple algebraic groups, *Publ. Math. IHES*, 25 (1965), 49-80.

[51] M. Suzuki, *Group Theory* I, Springer-Verlag, 1982.

[52] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. math. France*, 98 (1970), 81-116.