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INVARIANT MEASURES FOR TYPICAL QUADRATIC MAPS

by

Marco Martens & Tomasz Nowicki

Abstract. — A sufficient geometrical condition for the existence of absolutely continuous invariant probability measures for S−unimodal maps will be discussed. The Lebesgue typical existence of Sinai-Bowen-Ruelle-measures in the quadratic family will be a consequence.

1. Introduction

A general belief, or hope, in the theory of dynamical systems is that typical dynamical systems have well-understood behavior. This belief has two forms, depending on the meaning of the word "typical". It could refer to the topological generic situation or to the Lebesgue typical situation in parameter space. In this work *typical* will refer to Lebesgue typical and the behavior of a Lebesgue typical quadratic map on the interval will be discussed.

The quadratic family is formed by the maps $q_t : [-1,1] \to [-1,1]$ with $t \in [0,1]$ and

$$q_t(x) = -2tx^\alpha + 2t - 1,$$

with the critical exponent $\alpha = 2$. The maps in this family can be classified as follows. The maps in

$$\mathcal{P} = \{t \in [0,1] \mid q_t \text{ has a periodic attractor}\}$$

have a unique periodic orbit whose basin of attraction is an open and dense set. Moreover this basin has full Lebesgue measure. In particular the invariant measure on the periodic attractor is the SBR-measure for the map. Recall that a measure μ on $[-1,1]$ is called an S(inai)-B(owen)-R(uelle)-measure for q_t if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{q_t^k(x)} = \mu,$$

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for typical $x \in [-1,1]$.

The maps in

$$\mathcal{R} = \{t \in [0,1] \mid q_t \text{ is infinitely renormalizable}\}$$

have a unique invariant minimal Cantor set which attracts both generic and typical orbits. This Cantor set is uniquely ergodic and has zero Lebesgue measure, [**BL2**], [**G**], [**M1**]. The unique invariant measure on this Cantor set is the SBR-measure for the system. The maps in

$$\mathcal{I} = [0,1] \setminus \{\mathcal{P} \cup \mathcal{R}\}$$

have a periodic interval whose orbit is the limit set of generic orbits. The orbit of this periodic interval absorbs also the orbit of typical points. These maps are ergodic with respect to the Lebesgue measure, [**BL1**], [**GJ**], [**K**], [**M1**]. In the quadratic family, $\alpha = 2$, the limit set of typical points is actually also the orbit of this periodic interval, [**L1**]. However, in families with α big enough there are maps in $\mathcal{I}$ whose typical limit set is not this periodic interval, [**BKNS**].

Before discussing the behavior of typical quadratic maps let us include the behavior of generic quadratic maps .

Theorem 1.1 **([GS], [L3]).** — *Hyperbolicity is dense in the quadratic family, e.g.* $\overline{\mathcal{P}} = [0,1]$.

We will continue to specify the behavior of a typical map in $\mathcal{I}$. The dynamics of maps in

$$\mathcal{M} = \{t \in \mathcal{I} \mid q_t \text{ has an absolutely continuous invariant probability measure}\}$$

is well-understood. The measure is unique and its support is the orbit of the above periodic interval. Moreover it has positive Lyapunov exponent, [**K**], [**Ld**]. Starting in [**NU**], where it was shown that $q_1 \in \mathcal{M}$, more and more maps q_t were shown to have such a measure ([**B**], [**R**], [**Mi**]). Finally it was shown in [**Ja**] that $\mathcal{M}$ has positive measure.

Main Theorem 1.2 **(joint with Lyubich).** — *A typical quadratic map has a unique SBR-measure. More specifically*

(1) *for* $t \in \mathcal{P}$ *the support of the SBR-measure is the periodic attractor,*
(2) *for* $t \in \mathcal{R}$ *the SBR-measure is supported on a Cantor set,*
(3) *for* $t \in \mathcal{M}$ *the SBR-measure is an absolutely continuous measure supported on the orbit of a periodic interval,*
(4) *the set* $\mathcal{P} \cup \mathcal{R} \cup \mathcal{M} \subset [0,1]$ *has full Lebesgue measure.*

The general belief that typical dyamical systems have well understood behavior has been precisely formulated in the Palis-Conjecture [**P**]. Now, by Theorem 1.2, this Conjecture has been proved for the quadratic family.

Johnson constructed unimodal maps in $\mathcal{I}$ (with arbitrary critical exponent) which do not have an absolutely continuous invariant measure, **[Jo]**. More careful combinatorial Johnson-Examples were made without SBR-measure **[HK]**. The same work shows the existence of maps in $\mathcal{I} \setminus \mathcal{M}$ which have an SBR-measure but this measure is not absolutely continuous. The complications which occur in $\mathcal{I} \setminus \mathcal{M}$ are thoroughly studied in **[Br]**.

In this work we will formulate a geometrical condition on maps in $\mathcal{I}$ sufficient for the existence of absolutely continuous invariant probability measures. The geometric condition is formulated in terms of a decreasing sequence of *central intervals* $U_n = (-u_n, u_n)$, $n \geq 1$, which are defined for all unimodal maps with recurrent critical orbit. The domain $D_n \subset U_n$ of the first return map $R_n : D_n \to U_n$, $n \geq 1$ is a countable collection of intervals. The central component of D_n is U_{n+1}. The first return map R_n is said to have a *central return* when

$$R_n(0) \in U_{n+1}.$$

The sufficient geometrical condition for the existence of absolutely continuous measures is stated in terms of *scaling factors*

$$\sigma_n = \frac{u_{n+1}}{u_n}, \quad n \geq 1.$$

These scaling factors describe the small scale geometrical properties of the system but they are also strongly related to distortion questions. The main consequence of the distortion Theory developed in **[M1]** are the a priori bounds on the distortion of each R_n. The renormalization Theory developed in **[L1]** and **[LM]** achieved much stronger results: if a quadratic unimodal map has only finitely many central returns then the scaling factors tend exponentially to zero.

The scaling factors are related to small scale geometry, distortion but also expansion. The technical step in this work is to show that small scaling factors imply strong expansion along the critical orbit. In **[NS]** it was shown that enough expansion along the critical orbit causes the existence of an absolutely continuous invariant probability measure. In particular, if

$$\sum_{n \geq 1} \left| Dq_t^n(q_t(0)) \right|^{-1/\alpha} < \infty$$

then q_t has an absolutely continuous invariant probability measure.

Main Theorem 1.3. — *Let f be an $S-$unimodal map with critical exponent $\alpha > 1$. If f has summable scaling factors, that is*

$$\sum_{n \geq 1} \sigma_n^{1/\alpha} < \infty,$$

then it has an absolutely continuous invariant probability measure.

Corollary 1.4. — *If a quadratic map has only finitely many central returns then it has an absolutely continuous invariant probability measure.*

The (Johnson-)Examples in **[Jo]** have infinitely many (cascades of) central returns. The corollary states that the only quadratic unimodal maps in $\mathcal{I}$ which do not have an absolutely continuous invariant measure are Johnson-Examples. The families $\{q_t\}$ with α big enough have maps in $\mathcal{I}$ which do not have an absolutely continuous invariant probability measure and which are also not Johnson-Examples, **[BKNS]**.

In **[L2]**, Lyubich studies the parameter space of the (holomorphic) quadratic family. A new proof showing that $\mathcal{I}$ has positive Lebesgue measure is given (compare with the Jacobson-Theorem **[Ja]**). Moreover it is shown that for almost every parameter in $\mathcal{I}$ the corresponding quadratic map has only finitely many central returns. This, together with Theorem 1.3, implies Theorem 1.2.

Conjecture 1.5. — *A typical map in the family $\{q_t\}$, with critical exponent $\alpha > 1$, has a unique SBR-measure. More specifically*

(1) *for $t \in \mathcal{P}$ the support of the SBR-measure is the periodic attractor,*
(2) *for $t \in \mathcal{R}$ the SBR-measure is supported on a Cantor set,*
(3) *for $t \in \mathcal{M}$ the SBR-measure is an absolutely continuous measure supported on the orbit of a periodic interval,*
(4) *the set $\mathcal{P} \cup \mathcal{R} \cup \mathcal{M} \subset [0,1]$ has full Lebesgue measure.*

An appendix is added to collect the standard notions and Lemmas in interval dynamics.

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2. Central Intervals

Throughout the following sections we will fix an $S-$unimodal map $f : [-1,1] \to [-1,1]$ with critical exponent $\alpha > 1$ and without periodic attractors. Furthermore assume that the critical orbit is recurrent.

The set of nice points is

$$\mathcal{N} = \{x \in [-1,1] \mid \forall i \geq 0 \ f^i(x) \notin (-|x|,|x|)\}$$

This set is closed and not empty. For example the fixed point of f in $(0,1)$ is in $\mathcal{N}$.

For $x \in \mathcal{N}$ let $D_x \subset U_x = (-|x|,|x|)$ be the set of points whose orbit returns to U_x. The first return map to U_x is denoted by

$$R_x : D_x \longrightarrow U_x.$$

The next Lemma is a straightforward consequence of the fact that the boundary of each U_x is formed by nice points.

Lemma 2.1 **([M1])**. — *For every $x \in \mathcal{N}$ there exists a collection of pairwise disjoint intervals $\mathcal{U}_x$ with*

(1) *$I \subset U_x$ for all $I \in \mathcal{U}_x$,*
(2) *$\bigcup_{I\in\mathcal{U}_x} I = D_x$,*
(3) *if $I \in \mathcal{U}_x$ and $0 \notin I$ then $R_x : I \to U_x$ is monotone and onto,*
(4) *if $I \in \mathcal{U}_x$ and $0 \in I$ then $R_x : I \to U_x$ is 2 to 1 onto the image. Moreover $R_x(\partial I) = \{x\}$ or $\{-x\}$.*

Define the function $\psi : \mathcal{N} \to \mathcal{N}$ by

$$\psi(x) = \partial V_x \cap (0, 1),$$

where $V_x \in \mathcal{U}_x$ is the *central* interval: $0 \in V_x$. Say $R_x|_{V_x} = f^{q_x}$ and observe that

$$\{f(\psi(x)), f^2(\psi(x)), \ldots, f^{q_x}(\psi(x)) = x\} \cap U_x = \varnothing$$

which follows from the fact that $R_x : D_x \to U_x$ is the first return map. In particular $\psi(x) \in \mathcal{N}$ and we can consider the first return map to V_x. It will also satisfy Lemma 2.1.

Choose $u_1 \in \mathcal{N}$ and consider the sequence $u_n = \psi(u_{n-1})$ with $n \geq 1$. Use the simplified notation $\mathcal{U}_n$ for $\mathcal{U}_{u_n}$ and denote the first return maps by

$$R_n : D_n \longrightarrow U_n$$

instead of $R_{u_n} : D_{u_n} \to U_{u_n}$. All these first return maps have the properties stated in Lemma 2.1. Observe that $|U_n| = 2u_n$.

Let $\sigma_n = u_{n+1}/u_n$, $n \geq 1$. We call σ_n the *scaling factor* of level n. We will assume that

$$\sigma_n \longrightarrow 0.$$

However, the main Proposition 3.1, can also be proved by using only an a priori bound on the scaling factors. The assumption $\sigma_n \to 0$ will make the exposition less cumbersome.

Lemma 2.2. — *If $I \in \mathcal{U}_n$ and $R_n|_I = f^t$ then there exists an interval $J \supset f(I)$ such that*

$$f^{t-1} : J \longrightarrow U_{n-1}$$

is monotone onto. In particular all the maps $f^{t-1} : f(I) \to U_n$, $I \in \mathcal{U}_n$ have uniformly bounded distortion. Moreover these maps will be essentially linear when $n \to \infty$.

Proof. — Let $I \in \mathcal{U}_n$ with $R_n|I = f^t$ and let $J \supset f(I)$ be the maximal interval on which f^{t-1} is monotone. The maximality implies the existence of $i < t-1$ such that $0 \in \partial f^i(J)$. Observe that $f^i(f(I)) \cap U_n = \varnothing$, the first return happens after $t - 1 > i$ steps. So u_n (or $-u_n$) $\in f^i(J)$. We observed before that the orbit of $f(u_n)$ never enters U_{n-1}, $f^{t-1}(J) \supset U_{n-1}$. □

Lemma 2.3. — *For $\varepsilon > 0$ there exists $n_0 \geq 1$ such that the hyperbolic length of any $I \in \mathcal{U}_n$ is small,*

$$\text{hyp}(I, U_n) \leq \varepsilon \text{ and also } \frac{|f(I)|}{|f(U_n)|} \leq \varepsilon,$$

whenever $n \geq n_0$.

Proof. — Let $I \in \mathcal{U}_n$, say $R_n|I = f^t|I$. The previous Lemma states the existence of an interval $J \supset f(I)$ such that $f^{t-1} : (J, f(I)) \to (U_{n-1}, U_n)$ is monotone onto. For n large we see that U_n is a very small middle interval in U_{n-1}, it has a very small hyperbolic length. Because the map f has negative Schwarzian derivative we get that $f(I) \subset J$ has a very small hyperbolic length. Observe that $f^{-1}(J) \subset U_n$. Otherwise the orbit of u_n would pass through U_{n-1}. This is impossible: we saw before that the orbit of $\psi(x) = u_n$ does not cross $U_x = U_{n-1}$. The Lemma will be proved by pulling back the pair $(J, f(I))$ one step more. □

3. Derivatives along Recurrent Orbits

Let $\rho_n = \min\{1/\sigma_{n-1}, 1/\sigma_n\}$. In this section we will prove

Proposition 3.1. — *There exist $n_0 \geq 1$, $\theta < 1$ and $C > 0$ with the following property. If $n \geq n_0$, $x \in U_{n+1}$ and $R_n^i(x) \notin U_{n+1}$ for $i \leq s$ then*

$$|Df^T(f(x))| \geq C \cdot \rho_n \cdot \theta^{-(s-1)},$$

where $R_n^s(x) = f^T(x)$.

In **[VT]** a similar estimate in the case $s = 1$ was obtained for circle homeomorphisms with irrational rotation number of bounded type. The proof of Proposition 3.1 will use the following Lemmas and notation. Fix $x \in U_{n+1}$ according to the Proposition, say $R_n^i(x) = f^{t_i}(x)$ with $i \leq s$.

Lemma 3.2. — *For each $i \leq s$ there exists an interval $S_i \ni f(x)$ such that*

$$f^{t_i - 1} : S_i \longrightarrow U_n$$

is monotone and onto.

Proof. — Lemma 2.2 applied to $U_{n+1} \in \mathcal{U}_n$ states the existence of S_1. The proof will proceed by induction. Assume that $S_i \ni f(x)$ exists. Then $f^{t_i-1} : S_i \to U_n$ monotone and onto. Moreover $f^{t_i-1}(f(x)) \in I_{i+1} \in \mathcal{U}_n$. Because $f^{t_i-1}(f(x)) \notin U_{n+1}$ we have that $I_{i+1} \neq U_{n+1}$ and $R_n : I_{i+1} \to U_n$ is monotone and onto. Now let $S_{i+1} = f^{-(t_i-1)}(I_{i+1}) \cap S_i$. □

Observe that $f^{t_{i-1}-1}(S_i) = I_i \in \mathcal{U}_n$.

Lemma 3.3. — *There exist $n_0 \geq 1$ and $K < \infty$ with the following property. If the distortion of*

$$f^{t_i - 1} : S_i \longrightarrow U_n \text{ with } n \geq n_0$$

is bigger than K then

$$I_i \subset \Big(-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n \Big).$$

Proof. — Lemma 2.2 states that $f^{t_1 - 1} : S_1 \to U_n$ has a monotone extension up to U_{n-1}, the map is essentially linear for big enough n. Hence $i \geq 2$. Consider the following decomposition

$$f^{t_i - 1} | S_i = f^{t_i - t_{i-1} - 1} | f(I_i) \circ f | I_i \circ f^{t_{i-1} - 1} | S_i.$$

The factor $f^{t_{i-1}-1}|S_i$ has a monotone extension up to U_n. In particular, for big enough n, it is essentially linear. This is because the image I_i has a small hyperbolic length within U_n (Lemma 2.3). The factor $f^{t_i - t_{i-1} - 1}|f(I_i)$ has a monotone extension up to U_{n-1} (Lemma 2.2), it is also essentially linear. The distortion of $f^{t_i - 1}|S_i$ can only be caused by the factor $f|I_i$ and only if I_i is very close to 0. There is some n_0 such that $I_i \subset (-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n)$, whenever $n \geq n_0$. Here we used Lemma 2.3 which states that I_i has also very small hyperbolic length within U_n. □

Lemma 3.4. — *For any $\theta < 1$ there exist $n_0 \geq 1$ and $C < \infty$ such that*

$$|S_i| \leq C \cdot u_{n+1}^{\alpha} \cdot \theta^{i-1},$$

whenever $n \geq n_0$ and $i \geq 2$.

Proof. — Observe that $f(0) \in S_1 \supset S_2 \supset \cdots \supset S_i$ and $f(U_{n+1}) \subset S_1$, $i \geq 2$. Let $L_j \subset S_1$ be the connected component of $S_1 - S_j$ with $L_j \subset f(U_{n+1})$, $2 \leq j \leq i$. For n big enough we get from the proof of Lemma 3.2 and from Lemma 2.3 that the hyperbolic length of S_j within S_{j-1} is very small, $2 \leq j \leq i$. It is easily seen that this implies

$$\begin{aligned} |S_i| &\leq C \cdot \theta^{i-1} \cdot |L_i| \\ &\leq C \cdot \theta^{i-1} \cdot |f(U_{n+1})| \\ &\leq C \cdot \theta^{i-1} \cdot u_{n+1}^{\alpha}. \end{aligned}$$

□

Lemma 3.5. — *There exist $n_0 \geq 1$ and $C > 0$ such that the following holds for $n \geq n_0$. If $|R_n(U_{n+1})| = |f^{t_1}(U_{n+1})| \geq \frac{1}{10} u_n$ then*

$$|Df^{t_1 - 1}_{|S_1}| \geq C \cdot \frac{u_n}{u_{n+1}^{\alpha}}.$$

If $|R_n(U_{n+1})| < \frac{1}{10} u_n$ then

$$|Df^{t_1 - 1}_{|S_1}| \geq C \cdot \frac{u_{n-1}}{u_n^{\alpha}}.$$

Proof. — Consider the map $f^{t_1-1} : S_1 \to U_n$. From Lemma 2.2 we know that this map has a monotone extension up to U_{n-1}. The map is essentially linear because $u_{n-1} \gg u_n$ whenever n is big enough. The derivative $|Df^{t_1-1}_{|S_1}|$ is essentially constant and can be estimated by

$$|Df^{t_1-1}_{|S_1}| \geq C\frac{|R_n(U_{n+1})|}{|f(U_{n+1})|} \geq C\frac{u_n}{u^{\alpha}_{n+1}}.$$

Here we used that $|R_n(U_{n+1})| \geq \frac{1}{10}u_n$. Now consider the other case: $|R_n(U_{n+1})| < \frac{1}{10}u_n$. Let $K \supset f(U_{n+1})$ be the interval which is mapped monotonically onto U_{n-1}: $f^{t_1-1} : K \to U_{n-1}$. Observe that $f^{-1}(K) \subset U_n$. This follows from the fact that the orbit of $f(u_n)$ never hits U_{n-1}, which was observed in section 2. Let $K' \subset K$ be such that $f^{t_1-1} : K' \to (-\frac{3}{4} \cdot u_{n-1}, \frac{3}{4} \cdot u_{n-1})$ is monotone and onto. This map has uniform bounded distortion because it has a monotone extension up to U_{n-1}. Let $K'' = f^{-1}(K') \subset U_n$. The derivative $|Df^{t_1-1}_{|S_1}|$ can be estimated by

$$|Df^{t_1-1}_{|S_1}| \geq C\frac{|f^{t_1}(K'')|}{|f(K'')|} \geq C\frac{u_{n-1}}{u^{\alpha}_{n}}.$$

□

Proof of Proposition 3.1. — Assume first that $s = 1$. This is an application of the previous Lemma 3.5. If $|R_n(U_{n+1})| \geq \frac{1}{10}u_n$ then

$$|Df^{t_1}(f(x))| \geq C \cdot \frac{u_n}{u^{\alpha}_{n+1}} \cdot u^{\alpha-1}_{n+1} = C \cdot \frac{u_n}{u_{n+1}} \geq C \cdot \rho_n,$$

where we used that $R_n(x) \notin U_{n+1}$.

In the other case when $|R_n(U_{n+1})| < \frac{1}{10}u_n$, we have

$$|Df^{t_1}(f(x))| \geq C \cdot \frac{u_{n-1}}{u^{\alpha}_{n}} \cdot u^{\alpha-1}_{n} = C \cdot \frac{u_{n-1}}{u_n} \geq C \cdot \rho_n,$$

where we used that in this case $f^{t_1-1}(x) \in R_n(U_{n+1})$ which is close to the boundary of U_n. The case with $s = 1$ is finished.

Now assume that $s \geq 2$. The proof will be split in two cases. Let $n \geq n_0 \geq 1$ be big enough such that Lemma 3.3 and 3.4 can be applied. Let $K < \infty$ be the constant from Lemma 3.3 and $\theta < 1$ the constant from Lemma 3.4.

Case I ($f^T(x) = R^s_n(x) \in (-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n)$). — Let $H \subset S_s$ be such that $f^{T-1} = f^{t_s-1} : H \to (-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n)$ is onto. This map has a monotone extension up to U_n.

Hence it has a uniformly bounded distortion,

$$\begin{aligned}|Df^T(f(x))| &\geq C \cdot \frac{|f^{T-1}(H)|}{|H|} \cdot u_{n+1}^{\alpha-1} \\ &\geq C \cdot \frac{u_n}{|S_s|} \cdot u_{n+1}^{\alpha-1} \\ &\geq C \cdot \frac{u_n}{\theta^{s-1} \cdot u_{n+1}^{\alpha}} \cdot u_{n+1}^{\alpha-1} \\ &\geq C \cdot \rho_n \cdot \theta^{-(s-1)}.\end{aligned}$$

Case II ($f^T(x) = R_n^s(x) \notin (-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n)$). — If the distortion of $f^{T-1} : S_s \to U_n$ is bounded by K then we can give the same argument as in case I:

$$\begin{aligned}|Df^T(f(x))| &\geq C \cdot \frac{|U_n|}{|S_s|} \cdot u_{n+1}^{\alpha-1} \\ &\geq C \cdot \frac{u_n}{\theta^{s-1} \cdot u_{n+1}^{\alpha}} \cdot u_{n+1}^{\alpha-1} \\ &\geq C \cdot \rho_n \cdot \theta^{-(s-1)}.\end{aligned}$$

Now let us assume that this distortion is bigger than K. Apply Lemma 3.3, which states $I_s \subset (-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n)$. Then

$$\begin{aligned}|Df^T(f(0))| &= |Df^{t_{s-1}}(f(0))| \cdot |Df^{T-t_{s-1}}(f^{t_{s-1}}(f(0)))| \\ &\geq C \cdot \rho_n \cdot \theta^{-(s-2)} \cdot |Df^{T-t_{s-1}}(f^{t_{s-1}}(f(0)))|.\end{aligned}$$

For $s-1 \geq 1$ we get this estimate from case I: $R_n^{s-1}(0) \in I_s \subset (-\frac{3}{4} \cdot u_n, \frac{3}{4} \cdot u_n)$. When $s-1 = 1$ it follows from the Proof of Proposition 3.1 for $s = 1$.

The last factor can be estimated by using the fact that $f^{T-t_{s-1}-1} : f(I_s) \to U_n$ has a monotone extension up to U_{n-1}, see Lemma 2.2. It is essentially linear and its derivative can be estimated

$$|Df^{T-t_{s-1}-1}|_{f(I_s)}| \geq C \cdot \frac{|U_n|}{|f(I_s)|} \geq C \frac{|U_n|}{\varepsilon \cdot |f(U_n)|},$$

where $\varepsilon > 0$ is given by Lemma 2.3. By taking $n_0 \geq 1$ big enough we can assume that $\varepsilon > 0$ is arbitrarily small.

Observe that $|Df(f^{T-1}(f(0)))| \geq C \cdot u_n^{\alpha-1}$. This is because $|f^{T-1}(f(0))| \geq \frac{3}{4} \cdot u_n$. We can finish the estimate for $|Df^T(f(0))|$ by observing that

$$\begin{aligned}|Df^{T-t_{s-1}}(f^{t_{s-1}}(f(0)))| &= |Df^{T-t_{s-1}-1}(f^{t_{s-1}}(f(0)))| \cdot |Df(f^{T-1}(f(0)))| \\ &\geq C \cdot \frac{|U_n|}{\varepsilon \cdot |f(U_n)|} \cdot u_n^{\alpha-1} \\ &\geq C \cdot \frac{u_n}{\varepsilon \cdot u_n^{\alpha}} \cdot u_n^{\alpha-1} \geq C \cdot \frac{1}{\varepsilon} \geq \frac{1}{\theta}.\end{aligned}$$

□

4. Telemann Decomposition of the Critical Orbit

In this section we will prove Theorem 1.3. Let f be a unimodal map such that

$$\sum_{n\geq 1} \sigma_n^{1/\alpha} < \infty.$$

The existence of an absolutely continuous invariant probability measure follows from [NS] in where it was shown that the Summability Condition on Derivatives

$$\sum_{k\geq 1} |Df^k(f(0))|^{-1/\alpha} < \infty$$

is sufficient for the existence of absolutely continuous invariant probability measures.

In the sequel we will prove that the summability of scaling factors implies the Summability Condition of derivatives. Choose $n_0 \geq 1$ big enough such that Proposition 3.1 can be applied and moreover

$$a = \sum_{\substack{n\geq n_0\\ s\geq 0}} \frac{\theta^{s/\alpha}}{(C\cdot\rho_n)^{1/\alpha}} = \frac{1}{1-\theta^{1/\alpha}} \sum_{n\geq n_0} \frac{1}{(C\cdot\rho_n)^{1/\alpha}} < 1,$$

where C and θ are the constants from Proposition 3.1.

Fix $k \geq 1$. The estimate for $|Df^k(f(0))|$ is based on the *Telemann decomposition* of the critical orbit up to time k. Consider the orbit of $f(0)$ up to time $k-1$. Let $m \geq 0$ be such that U_{n_0+m} is the smallest central interval which is crossed by this piece of the orbit:

$$n_0 + m = \max\{j \geq 0 \mid \exists 0 < i \leq k,\ f^i(0) \in U_j\}$$

and the last moment of crossing is denoted by

$$k_m = \max\{1 \leq j \leq k \mid f^j(0) \in U_{n_0+m}\}.$$

The moments $k_m \leq k_{m-1} \leq \ldots k_1 \leq k_0$ are such that k_i is the last moment that the orbit hits U_{n_0+i}: if $\{k_i < j \leq k \mid f^j(0) \in U_{n_0+i-1}\} = \varnothing$ then $k_{i-1} = k_i$ otherwise

$$k_{i-1} = \max\{k_i < j \leq k \mid f^j(0) \in U_{n_0+i-1}\} \text{ with } 1 \leq i \leq m.$$

Let $r(k) = k - k_0$ and if $k_{i-1} \neq k_i$ then

$$s_{i-1}(k) = \#\{k_i < j \leq k_{i-1} \mid f^j(0) \in U_{n_0+i-1}\}, \quad 1 \leq i \leq m,$$

the number of returns trough U_{n_0+i-1}.

The chain-rule applied to $Df^k(f(0))$ gives

$$Df^k(f(0)) = Df^{r(k)}(f^{k_0}(f(0))) \cdot Df^{k_m}(f(0)) \cdot \prod_{i=0}^{m-1} Df^{k_i - k_{i+1}}(f^{k_{i+1}}(f(0))).$$

The first factor can be estimated by using

Proposition 4.1 **([G], [Ma]).** — *Given $n_0 \geq 1$ there exist constants $C > 0$ and $\lambda > 1$ such that*

$$|Df^r(x)| \geq C\lambda^r,$$

whenever $f^i(x) \notin U_{n_0}$ with $i \leq r$.

The other factors can be estimated by Proposition 3.1. The decomposition was set up to make Proposition 3.1 applicable to the factors:

$$\begin{aligned}|Df^k(f(0))|^{-1/\alpha} &\leq \Big(C\lambda^{r(k)} \cdot \prod_{\substack{i \leq m-1 \\ k_i \neq k_{i+1}}} C \cdot \rho_{n_0+i} \cdot \theta^{-(s_i(k)-1)}\Big)^{-1/\alpha} \\ &\leq C\Big(\frac{1}{\lambda^{1/\alpha}}\Big)^{r(k)} \cdot \prod_{\substack{i \leq m-1 \\ k_i \neq k_{i+1}}} \frac{(\theta^{1/\alpha})^{s_i(k)-1}}{(C \cdot \rho_{n_0+i})^{1/\alpha}}\end{aligned}$$

Lemma 4.2. — *Let s_i, r and s'_i, r' correspond to the Teleman decomposition of respectively k and k'. If $k \neq k'$ then $r \neq r'$ or $s_i \neq s'_i$ for some $i \geq 0$.*

Proof. — Assume that $r = r'$ and $s_i = s'_i$ for all $i \geq 1$. We have to show that $k = k'$. Observe that $f^{k_m}(0) = R^{s_m}_{n_0+m}(0)$ but also $f^{k'_m}(0) = R^{s'_m}_{n_0+m}(0) = R^{s_m}_{n_0+m}(0)$. So $k_m = k'_m$. Now repeat this argument to show that $k_i = k'_i$ for $0 \leq i \leq m$. In particular we get $k_0 = k'_0$. So

$$k' = k'_0 + r' = k_0 + r = k. \qquad \square$$

Proof of the Summability Condition for Derivatives. — The number $a < 1$ was defined in the beginning of this section. Let $\beta = 1/\alpha$.

$$\begin{aligned}\sum_{k \geq 0} |Df^k(f(0))|^{-1/\alpha} &= \sum_{r \geq 0} \sum_{\substack{k \geq 0 \\ r(k)=r}} |Df^k(f(0))|^{-1/\alpha} \\ &\leq \sum_{r \geq 0} C\Big(\frac{1}{\lambda^\beta}\Big)^r \cdot \sum_{\substack{k \geq 0 \\ r(k)=r}} \prod_{\substack{0 \leq i \leq m-1 \\ k_i \neq k_{i+1}}} \frac{(\theta^\beta)^{s_i(k)-1}}{(C \cdot \rho_{n_0+i})^\beta}.\end{aligned}$$

Now observe that for each r there are no two products appearing in the second sum which are formed by the same factors, according to Lemma 4.2. If we expand a^n we see that the sum of all possible different products can be estimated by $1+a+a^2+a^3+\ldots$. Hence

$$\begin{aligned}\sum_{k \geq 0} |Df^k(f(0))|^{-1/\alpha} &\leq \sum_{r \geq 0} C\Big(\frac{1}{\lambda^\beta}\Big)^r \cdot (1 + a + a^2 + a^3 + \ldots) \\ &\leq \frac{1}{1-a} \sum_{r \geq 0} C\Big(\frac{1}{\lambda^\beta}\Big)^r \\ &\leq C \cdot \frac{1}{1-a} \cdot \frac{1}{1 - 1/\lambda^\beta} < \infty.\end{aligned}$$

5. Appendix

In this appendix some basic notions of interval dynamics are collected. The details can be found in [**MS**].

The hyperbolic length of an interval $I \subset T \subset [-1,1]$ within T is defined to be

$$\text{hyp}(I,T) = \ln \frac{|L \cup I| \cdot |R \cup I|}{|L| \cdot |R|},$$

where $L, R \subset T$ are the connected components of $T \setminus I$ and $|J|$ stands for the length of the interval $J \subset [-1,1]$.

The Schwarzian derivative of a C^3 map $f : [-1,1] \to [-1,1]$ is

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2} \cdot \frac{D^2 f(x)}{Df(x)}.$$

where $D^i f(x)$ stands for the i^{th} derivative of f in $x \in [-1,1]$.

Expansion-Lemma 5.1. — *If $f : [-1,1] \to [-1,1]$ has $Sf(x) \le 0$ for all $x \in [-1,1]$ and $f^n|T$ is monotone then*

$$\text{hyp}(f^n(I), f^n(T)) \ge \text{hyp}(I,T),$$

where $I \subset T$.

Koebe-Lemma 5.2. — *For each $\tau > 0$ there exists $1 \le K(\tau) < \infty$ with the following property. Let $f^n : T \to f^n(T)$ be monotone and $Sf(x) \le 0$ for all $x \in [-1,1]$. If $I \subset T$ is an interval such that both component $L, R \subset T \setminus I$ satisfy*

$$\frac{|f^n(L)|}{|f^n(T)|}, \frac{|f^n(R)|}{|f^n(T)|} \ge \tau$$

then $f^n|I$ has bounded distortion

$$\frac{|Df^n(x)|}{|Df^n(y)|} \le K(\tau),$$

for all $x, y \in I$. Moreover $K(\tau) \to 1$ when $\tau \to \infty$.

An S-unimodal map is a C^3 endomorphism $f : [-1,1] \to [-1,1]$ such that

(1) $f(\pm 1) = -1$,
(2) $Df(x) > 0$ for $x < 0$,
(3) $Df(x) < 0$ for $x > 0$,
(4) $Df(0) = 0$ and up to a C^1 change of coordinates f equals locally $x \to |x|^\alpha$ with $\alpha > 1$. The point $x = 0$ is called the critical point and $\alpha > 1$ is called the critical exponent of f.
(5) $Sf(x) < 0$, $x \neq 0$.

The orbit of the critical point $x = 0$ is called the critical orbit. The critical orbit is said to be recurrent if it accumulates at the critical point.

Usage of constants. — Every uniform constant, that is a constant which is independent of the actual map, appearing in estimates will be denoted by C. Constants which play a specific role in the statement of Lemmas will usually have a specific name.

References

[BKNS] H. Bruin, G. Keller, T. Nowicki, S. van Strien, *Wild Attractors Exist*, Ann. of Math. **143** (1996), 97-130.

[B] L.A. Bunimovich, *A Transformation of the Circle*, Math. Notes **8** no. 2 (1970), 587-592.

[Br] H. Bruin, *Invariant Measures of Interval Maps*, Thesis (1994), Technical University Delft, Delft.

[BL1] A.M. Blokh, M. Lyubich, *Attractors of Maps on the interval*, Func. Anal. and Appl. **21** (1987), 32-46.

[BL2] A.M. Blokh, M. Lyubich, *Measure of Solenoidal Attractors of Unimodal Maps of the Segment*, Math. Notes **48** no. 5 (1990), 1085-1090.

[G] J. Guckenheimer, *Limit Sets of S−unimodal Maps with Entropy Zero*, Comm. Math. Phys. **110** (1987), 655-659.

[GJ] J. Guckenheimer, S. Johnson, *Distortion of S−unimodal Maps*, Ann. of Math. **132** (1990), 71-130.

[GS] J. Graczyk, G. Swiatek, *Hyperbolicity in the Real Quadratic Family*, Report no. PM 192 PennState (1995).

[HK] F. Hofbauer, G. Keller, *Quadratic Maps without Asymptotic Measure*, Comm. Math. Phys. **127** (1990), 319-337.

[Ja] M.V. Jakobson, *Absolutely Continuous Invariant Measures for one-parameter Families of One-dimensional Maps*, Comm. Math. Phys. **81** (1981), 39-88.

[Jo] S. Johnson, *Singular Measures without restrictive interval*, Comm. Math. Phys. **110** (1987), 185-190.

[K] G. Keller, *Exponents, Attractors and Hopf Decomposition for Interval Maps*, Erg. Th. & Dyn. Sys. **10** (1990), 717-744.

[L1] M. Lyubich, *Combinatorics, Geometry and Attractors of Quasi-Quadratic Maps*, Ann. of Math **140** (1994), 347-404.

[L2] M. Lyubich, *Geometry of Quadratic Polynomials: Moduli, Rigidity, and Local Connectivity*, IMS at Stony Brook preprint 1993/9.

[L3] M. Lyubich, *Dynamics of Quadratic Polynomials, II Rigidity*, IMS at Stony Brook preprint 1995/14.

[L4] M. Lyubich, *Dynamics of Quadratic Polynomials, III Parapuzzles*, IMS at Stony Brook preprint 1996/5 and this volume.

[LM] M. Lyubich, J.W. Milnor, *The Fibonacci Unimodal Map*, J.A.M.S. **6** (1993), 425-457.

[Ld] F. Ledrappier, *Some Properties of Absolutely Continuous Invariant Measures of an Interval*, Erg. Th. & Dyn. Sys. **1** (1981), 77-93.

[Ma] R. Mañé, *Hyperbolicity, Sinks, and Measure in One-dimensional Dynamics*, Comm. Math. Phys. **100** (1985), 495-524 and Erratum, Comm. Math. Phys. **112** (1987), 721-724.

[M1] M. Martens, *Interval Dynamics*, Thesis (1990), Technical University Delft, Delft, or **[M2]**.

[M2] M. Martens, *Distortion Results and Invariant Cantor Sets of Unimodal Maps*, Erg. Th. & Dyn. Sys. **14** (1994), 331-349.

[Mi] M. Misiurewicz, *Absolutely Continuous Measures for certain maps of an Interval*, Publ. Math. I.H.E.S. **53** (1981), 17-51.

[MS] W. de Melo and S. van Strien, *One-dimensional Dynamics*, Springer-Verlag, 1993.

[NS] T. Nowicki, S. van Strien, *Absolutely Continuous Measures under a Summability Condition*, Invent. Math. **93** (1988), 619-635.

[NU] J. von Neumann, S.M. Ulam, *On Combinatorics of Stochastic and Deterministic Processes*, Bull. A.M.S. **53** (1947), 1120.

[P] J. Palis, *A Global View on Dynamics: Conjectures and Results*, this volume.

[R] D. Ruelle, *Applications conservant une mesure absolument continue par rapport à dx sur* $[0,1]$, Comm. Math. Phys. **55** (1977), 47-51.

[VT] J.J.P. Veerman, F.M. Tangerman, *Scalings in Circle Maps (I)*, Comm. Math. Phys. **134** (1990), 89-107.

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