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## ON BOUNDS FOR THE CONCENTRATION FUNCTION. 1

by

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**Abstract.** — We give an upper bound for the concentration function of a sum of independent identically distributed integral valued random variables in terms of a lower bound for their tail, under the necessary extra condition that the random variables are not essentially supported in a proper arithmetic progression.

### 1. Introduction

Let  $X_1, \dots, X_k, \dots$  be independent real random variables and  $S_n = \sum_{k=1}^n X_k$ . It is well known that, in general, the distribution of  $S_n$  spreads out as  $n$  grows. When all the  $X_k$ 's are square-integrable, the relation  $\sigma^2(S_n) = \sum_{k=1}^n \sigma^2(X_k)$  is a way to express this fact. In the general case, Doeblin and Lévy [2] were the first to measure this phenomenon in terms of concentration functions. The concentration function of a real random variable  $X$  is defined by

$$Q(X; \lambda) = \sup_t P\{t < X \leq t + \lambda\} \text{ for } \lambda \geq 0.$$

The results of Doeblin and Lévy have been successively improved by Kolmogorov [6], Rogozin [12] and Kesten [5]. Let us quote a corollary to Kesten's result, for the case when the  $X_k$ 's are identically distributed.

**Theorem (Kesten [5], Corollary 1, p. 134).** — *There exists an absolute constant  $C$  such that for any set of independent identically distributed random variables  $X_1, \dots, X_n$*

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and any  $0 < \lambda \leq 2L$  we have

$$(1.1) \quad Q(S_n; L) \leq C \frac{L}{\lambda} \frac{Q(X_1; L)}{\sqrt{n(1 - Q(X_1; \lambda))}}.$$

Let us consider the case when the  $X_k$ 's follow a Cauchy law  $\mathcal{C}(1)$ , where the Cauchy law  $\mathcal{C}(a)$  with parameter  $a > 0$  has density  $a/(\pi(t^2 + a^2))$ . One readily sees that for  $L = 1$  and  $0 < \lambda \leq 2$ , the right hand side of (1.1) has order of magnitude  $(\lambda\sqrt{n})^{-1}$  and is never  $o(1/\sqrt{n})$ . However, the random variable  $S_n$  follows the law  $\mathcal{C}(n)$ , and so

$$Q(S_n; 1) = \frac{2}{\pi} \arctan\left(\frac{1}{2n}\right) = \frac{1}{n\pi}(1 + o(1)).$$

The dispersion (in the standard sense) of  $S_n$  is due to the dispersion of the  $X_k$ 's themselves; but the dispersion of the  $X_k$ 's is not reflected in a small concentration  $Q(X_1; \lambda)$  for *small*  $\lambda$ 's, but indeed for *large*  $\lambda$ 's: the law of  $X_1$  has a large tail, as can be seen from the fact that  $X_1$  is not integrable.

A connection between the moments of the  $X_k$ 's and the concentration of their sums has been provided by Esséen [3], who proves that the integrability of  $|X_1|^r$  for some  $0 < r \leq 2$  implies the *lower bound*

$$Q(S_n; L) \geq K(r)L(L + (n\mu_r)^{1/r})^{-1},$$

where  $\mu_r = \inf_a E(|X_1 - a|^r)$  and  $K(r)$  is an explicitly given expression that only depends on  $r$ .

We aim at giving an *upper bound* for the concentration function of  $S_n$  in terms of the *tail* of the distribution of the  $X_k$ 's. There is however a difficulty that will be better seen on discrete random variables. Let us consider an integer  $q > 1$  and two integral valued random variables  $X_1$  and  $X'_1$  such that

$$\begin{aligned} P\{X_1 = 0\} &= P\{X'_1 = 0\} = 1/2, \\ P\{X'_1 = \ell\} &= \begin{cases} P\{X_1 = \ell/q\} \neq 0 & \text{when } q \text{ divides } \ell, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We clearly have  $Q(X_1; 1) = Q(X'_1; 1) = 1/2$  and the tail of the distribution of  $X'_1$  is heavier than that of  $X_1$ . However, if we consider two sets  $X_1, \dots, X_n$  and  $X'_1, \dots, X'_n$  of  $n$  independent identically distributed random variables, their sums  $S_n$  and  $S'_n$  are such that  $Q(S_n; 1) = Q(S'_n; 1)$ ; we have indeed  $P\{S_n = N\} = P\{S'_n = qN\}$  and so

$$Q(S_n; 1) = \max_N P\{S_n = N\} = \max_M P\{S'_n = M\} = Q(S'_n; 1).$$

We give in this paper an upper bound for the concentration function of a sum of independent identically distributed integral valued random variables in terms of the measure of their tail, under the assumption that the support of the random variables is not essentially contained in a proper arithmetic progression.

**Theorem 1.** — Let  $\frac{\log 4}{\log 3} < \sigma < 2$ ,  $\varepsilon > 0$ ,  $A \geq 1$  and  $a > 0$  be given real numbers. Let  $n$  be a positive integer and  $X_1, \dots, X_n$  a set of independent identically distributed

*integral valued random variables such that*

$$(1.2) \quad \max_{q \geq 2} \max_{s \bmod q} \sum_{\ell \equiv s \pmod{q}} P\{X_1 = \ell\} \leq 1 - \varepsilon ,$$

$$(1.3) \quad \forall L \geq A : Q(X_1; L) \leq 1 - aL^{-\sigma} .$$

*Then we have*

$$(1.4) \quad Q(S_n; 1) \leq cn^{-1/\sigma} ,$$

*where  $c$  depends on  $\sigma, \varepsilon, A$  and  $a$  at most.*

The main aim of this paper being to illustrate the use of inverse additive results to probability theory, we kept the statement and proof of our main result as simple as possible. We have thus restricted our attention to integral valued random variables, have not considered the general case when  $0 < \sigma < 2$ , and have not made explicit the dependence of  $c$  on the parameters  $\varepsilon, A$  and  $a$ . Let us simply notice here that Theorem 1 is valid under the condition  $1 < \sigma < 2$ : this depends on the fact that, under iterated applications of Lemma 3, the constant  $3^k$  that arise may be improved to  $(4 - \varepsilon)^k$ , an observation which is basically due to Lev. However, when  $\sigma < 1$ , new phenomena enter the matter (generalized arithmetic progressions); we shall soon return to this topic.

The statement of Theorem 1 becomes false if condition (1.2) is suppressed. Of course, if the constant  $c$  in (1.4) is allowed to depend on the law of  $X_1$ , then condition (1.2) is no longer necessary.

The proof of this theorem may be summarized as follows. The concentration  $Q(S_n; 1)$  is majorized by the mean value of the modulus of the characteristic function of  $S_n$ ; this latter is the  $n$ -th power of that of  $X_1$ , which we call  $\varphi$ , so that the problem reduces to the study of the large values of  $\varphi$ . Here we use two ideas that have been introduced by Freiman, Moskvina and Yudin in [4] in the context of local limit theorems. The first one, which can be seen as a consequence of Bochner's theorem, is that  $\varphi(t_1 + t_2)$  is large as soon as both  $\varphi(t_1)$  and  $\varphi(t_2)$  are large. The second one comes from the structure theory of set addition: either the set  $E$  of the arguments of the large values of  $\varphi$  is small, or it has a structure. In the first case,  $\varphi$  cannot be too large, and so we get (1.4). It remains to exclude the second case; were it to occur, then, as we shall see, either  $E$  would contain the vertices of a regular polygon, which would violate (1.2), or it would contain a large interval around 0, which would contradict (1.3).

Problems of estimating the measure of the set of large values of the characteristic function have also been studied by Arak and Zaitsev [1]. This gave them the possibility to solve a famous problem of Kolmogorov on the estimation of the approximation of the  $n$ -th convolution of any probability distribution by that of an infinitely divisible law.

As a warm up, and in order to introduce some tools and techniques, we devote the second paragraph to prove a special case of the Doeblin-Lévy-Kolmogorov-Rogozin-Kesten (DLKRK) inequality which stems from the same ideas and follows [10], [11].

The interested reader will find questions of a similar flavour in the classical monographs by Petrov [9] and the more recent one by Ledoux and Talagrand [7].

## 2. A DLKRR inequality for discrete random variables

**Theorem 2 (DLKRR).** — *Let  $X_1, \dots, X_n$  be independent identically distributed integral valued random variables, and let  $S_n$  be their sum and  $p = \max_N P\{X_1 = N\}$ . For every integer  $N$ , we have*

$$P\{S_n = N\} \leq 40 \frac{p}{\sqrt{n(1-p)}}.$$

Let us start by giving some notation that will be used in this paragraph and the next. We let

$$\begin{aligned} p_\ell &= P\{X_1 = \ell\} \text{ for any } \ell \in \mathbb{Z}, \\ \varphi(t) &= \sum_{\ell \in \mathbb{Z}} p_\ell \exp(2\pi i t \ell) \text{ for } t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \\ E(\theta) &= \{t \in \mathbb{T} : |\varphi(t)| \geq \cos \theta\} \text{ for } 0 \leq \theta \leq \pi/2, \\ \theta^* &\text{ be such that } \cos \theta^* = \min |\varphi(t)| \text{ and } 0 \leq \theta^* \leq \pi/2. \end{aligned}$$

The proof of Theorem 2 will be based on the following two results, for the first of which we give a sketch of a proof.

**Lemma 1 (cf. [4]).** — *For  $\theta_1 \geq 0$ ,  $\theta_2 \geq 0$  and  $\theta_1 + \theta_2 \leq \frac{\pi}{2}$ , we have*

$$E(\theta_1) + E(\theta_2) \subset E(\theta_1 + \theta_2).$$

*Proof.* — For  $j = 1, 2$ , we consider  $t_j$  in  $E(\theta_j)$ , and let  $\alpha_j = \arg^t \varphi(t_j)$  and  $\lambda_j = \sqrt{1 - |\varphi(t_{3-j})|^2} e^{-i\alpha_j}$ . We use the Cauchy inequality to get an upper bound on

$$|\lambda_1 \varphi(t_1) + \lambda_2 \varphi(-t_2)|^2 = \left| \sum_{\ell} \sqrt{p_\ell} (\lambda_1 \sqrt{p_\ell} e^{2\pi i \ell t_1} + \lambda_2 \sqrt{p_\ell} e^{-2\pi i \ell t_2}) \right|^2.$$

□

**Lemma 2 (Macbeath-Kneser Theorem, cf. [8], p. 13-14).** — *Let  $E_1$  and  $E_2$  be two non-empty closed sets in  $\mathbb{T}$ . We have*

$$|E_1 + E_2| \geq \min(1, |E_1| + |E_2|),$$

where  $|A|$  represents the Haar measure of  $A$  in  $\mathbb{T}$ .

*Proof of Theorem 2.* — We may of course assume that  $p$  is strictly less than 1 and so  $\theta^*$  is strictly positive. Our first task is to show that

$$(2.1) \quad |E(\theta)| \leq 12 \frac{\theta p}{\sqrt{1-p}}, \text{ for } \theta \in ]0, \theta^*/2[,$$

and

$$(2.2) \quad |E(\theta)| \leq p / \cos^2 \theta, \text{ for } \theta \in ]0, \frac{\pi}{2}].$$

By the definition of  $\theta^*$ , we have

$$\cos^2 \theta^* \leq \int_{\mathbb{T}} |\varphi(t)|^2 dt = \sum_{\ell} p_{\ell}^2 \leq p \sum_{\ell} p_{\ell} = p ,$$

whence

$$(2.3) \quad \theta^* > |\sin \theta^*| = \sqrt{1 - \cos^2 \theta^*} \geq \sqrt{1 - p} .$$

Now let  $\theta$  be in  $]0, \theta^*/2[$ , and let  $M = [\theta^*/\theta]$ ; we have  $M \geq 2$ . We may write

$$\begin{aligned} p &\geq \int_{\mathbb{T}} |\varphi(t)|^2 dt \\ &\geq \sum_{m=1}^M \int_{\cos(m\theta) \leq |\varphi(t)| < \cos((m-1)\theta)} |\varphi(t)|^2 dt \\ &\geq \sum_{m=1}^{[M/2]} |E(m\theta) \setminus E((m-1)\theta)| \cos^2 m\theta , \end{aligned}$$

and by appealing to Lemma 2, we get

$$\begin{aligned} p &\geq |E(\theta)| \sum_{m=1}^{[M/2]} \cos^2 m\theta \\ &\geq |E(\theta)| \cdot [M/2] \cdot \frac{1}{2} \\ &\geq \frac{M}{6} |E(\theta)| \\ &\geq \frac{\theta^*}{12\theta} |E(\theta)| . \end{aligned}$$

This last inequality and (2.3) imply (2.1). Inequality (2.2) simply follows from

$$p \geq \int_{\mathbb{T}} |\varphi(t)|^2 dt \geq |E(\theta)| \cdot \cos^2 \theta .$$

For every integer  $N$ , we have

$$\begin{aligned} P\{S_n = N\} &= \int_{\mathbb{T}} \varphi^n(t) \exp(-2\pi i N t) dt \\ &\leq \int_{\mathbb{T}} |\varphi(t)|^n dt . \end{aligned}$$

By change of variable, and then integration by parts, we have

$$\begin{aligned}
\int_{\mathbb{T}} |\varphi(t)|^n dt &= \int_0^{\pi/2} \cos^n \theta d|E(\theta)| \\
&= \int_0^{\pi/2} n \cos^{n-1} \theta \sin \theta \cdot |E(\theta)| d\theta .
\end{aligned}$$

We now use (2.1), getting

$$\begin{aligned}
\int_0^{\theta^*/2} n \cos^{n-1} \theta \sin \theta |E(\theta)| d\theta \\
\leq \frac{12p}{\sqrt{1-p}} \int_0^{\pi/2} \cos^n \theta d\theta \\
\leq \frac{12p\sqrt{\pi/2}}{\sqrt{(1-p)}\sqrt{n}} .
\end{aligned}$$

On the remaining interval  $[\theta^*/2, \pi/2]$ , we use (2.2), which leads, for  $n \geq 3$ , to

$$\begin{aligned}
\int_{\theta^*/2}^{\pi/2} n \cos^{n-1} \theta \sin \theta \frac{p}{\cos^2 \theta} d\theta &\leq \frac{n}{n-2} p \int_{\theta^*/2}^{\pi/2} (n-2) \cos^{n-3} \theta \sin \theta d\theta \\
&= \frac{np}{n-2} \cos^{n-2} \left( \frac{\theta^*}{2} \right) \\
&\leq \frac{np}{n-2} \cos^{n-2} \left( \frac{\sqrt{(1-p)}}{2} \right) .
\end{aligned}$$

From the inequality  $x \cos^n x \leq \frac{1}{\sqrt{n}}$ , we get

$$\begin{aligned}
\int_{\theta^*/2}^{\pi/2} n \cos^{n-1} \theta \sin \theta |E(\theta)| d\theta \\
\leq \frac{2np}{(n-2)\sqrt{(n-2)(1-p)}} \\
\leq \frac{12p}{\sqrt{n(1-p)}} .
\end{aligned}$$

We have completed the proof of Theorem 2 when  $n \geq 3$ . For  $n \leq 2$ , we simply have to notice that  $P\{S_n = N\} \leq p \leq \frac{40p}{\sqrt{n(1-p)}}$ .  $\square$

### 3. Proof of Theorem 1

The letter  $c$  (with or without indices) always represents a constant that depends on  $\sigma, \varepsilon, A$  and  $a$  at most. The value may change from one occurrence to the next.

The key ingredient which will play the rôle of Lemma 2 is the following result, which will be applied with  $k = 2^{2/\sigma}$ .

**Lemma 3 (Freiman - Moskvina - Yudin [4]).** — *Let  $2 < k < 3$ ; there exists a positive real number  $\mu = \mu(k)$  such that for any closed set  $F$  in  $\mathbb{T}$ , symmetric with respect to the origin and satisfying:*

$$|F| \leq \mu \quad \text{and} \quad |2F| < k|F| ,$$

*there exists a positive integer  $q$  such that*

$$\bigcup_{r=0}^{q-1} \left[ \frac{r}{q} - \frac{|F|}{q}, \frac{r}{q} + \frac{|F|}{q} \right] \subset 3F .$$

We give two further results connecting the values of the characteristic function with the concentration of the associated distribution, the second one being of an arithmetical nature.

**Lemma 4.** — *Let  $\alpha$  and  $L^{-1}$  be two positive real numbers. If  $|\varphi(t)| \geq 1 - \alpha$  for all  $t$  in  $[-\frac{1}{2L}, \frac{1}{2L}]$ , then  $Q(X_1; L) \geq 1 - 6\alpha$ .*

*Proof.* — Let us write  $Q = Q(L)$ , and define  $x_0$  the supremum of

$$\{x : P\{X_1 \leq x\} \leq (1 - Q)/2\} .$$

Since  $x \mapsto P\{X_1 \leq x\}$  is continuous from the right, we have  $P\{X_1 \leq x_0\} \geq (1 - Q)/2$ . On the other hand, for  $x < x_0$ , we have

$$P\{X_1 \geq x + L\} \geq 1 - P\{x < X_1 \leq x + L\} - P\{X_1 \leq x\} \geq (1 - Q)/2 ,$$

and, since  $P\{X_1 \geq x + L\}$  is continuous from the left, we have  $P\{X_1 \geq x_0 + L\} \geq (1 - Q)/2$ .

Let us define

$$\varphi_-(t) = \sum_{k \leq x_0} p_k \exp(2\pi i k t), \quad \varphi_+(t) = \sum_{\ell \geq x_0 + L} p_\ell \exp(2\pi i \ell t)$$

and  $\varphi_0(t) = \sum_{x_0 < m < x_0 + L} p_m \exp(2\pi i m t).$



Let us denote by  $I$  the interval  $[-\frac{1}{2L}, \frac{1}{2L}]$ . By the Schwarz inequality, we get

$$\begin{aligned}
 \left( L \int_I |\varphi_-(t) + \varphi_+(t)| dt \right)^2 &\leq L \int_I |\varphi_-(t) + \varphi_+(t)|^2 dt \\
 &\leq \varphi_-^2(0) + \varphi_+^2(0) + 2L \sum_{k \leq x_0} \sum_{\ell \geq x_0+L} p_k p_\ell \left| \int_I \cos(2\pi(k-\ell)t) dt \right| \\
 &\leq \varphi_-^2(0) + \varphi_+^2(0) + \frac{2}{\pi} \varphi_-(0) \varphi_+(0) \\
 &\leq \left( (\varphi_-(0) + \varphi_+(0))^2 - \left(2 - \frac{2}{\pi}\right) \varphi_-(0) \varphi_+(0) \right) \\
 &= (\varphi_-(0) + \varphi_+(0))^2 \left( 1 - \left(2 - \frac{2}{\pi}\right) \frac{\varphi_-(0) \varphi_+(0)}{(\varphi_-(0) + \varphi_+(0))^2} \right) \\
 &\leq (\varphi_-(0) + \varphi_+(0))^2 \left( 1 - \left(1 - \frac{1}{\pi}\right) \frac{\varphi_-(0) \varphi_+(0)}{(\varphi_-(0) + \varphi_+(0))^2} \right)^2,
 \end{aligned}$$

and so we get

$$L \int_I |\varphi_-(t) + \varphi_+(t)| dt \leq \varphi_-(0) + \varphi_+(0) - \frac{2}{3} \frac{\varphi_-(0) \varphi_+(0)}{\varphi_-(0) + \varphi_+(0)}.$$

This leads to

$$1 - \alpha \leq L \int_I |\varphi(t)| dt \leq \varphi_0(0) + \varphi_-(0) + \varphi_+(0) - \frac{2}{3} \frac{\varphi_-(0) \varphi_+(0)}{\varphi_-(0) + \varphi_+(0)}.$$

Since the minimum of  $\frac{xy}{x+y}$  under the conditions  $\frac{1-Q}{2} \leq x, y \leq \frac{1+Q}{2}$  and  $1 - Q \leq x + y \leq 1$  is  $\frac{1-Q}{4}$ , we have

$$1 - \alpha \leq 1 - \frac{2}{3} \left( \frac{1-Q}{4} \right)$$

which implies

$$1 - Q \leq 6\alpha.$$

□

**Lemma 5.** — *Let  $q \geq 1$  be an integer. We have*

$$\frac{1}{q} \sum_{r=0}^{q-1} \left| \varphi\left(\frac{r}{q}\right) \right|^2 \leq \max_s \sum_{\ell \equiv s \pmod{q}} p_\ell.$$

*Proof.* — We have

$$\varphi\left(\frac{r}{q}\right) = \sum_{\ell} p_\ell \exp\left(2\pi i \ell \frac{r}{q}\right) = \sum_{s=0}^{q-1} \exp\left(2\pi i s \frac{r}{q}\right) P_s,$$

where  $P_s = \sum_{\ell \equiv s \pmod q} p_\ell$ , so that

$$\begin{aligned} \frac{1}{q} \sum_{r=0}^{q-1} \left| \varphi\left(\frac{r}{q}\right) \right|^2 &= \frac{1}{q} \sum_{r=0}^{q-1} \sum_{s=0}^{q-1} \sum_{t=0}^{q-1} \exp\left(2\pi i \frac{r(s-t)}{q}\right) P_s P_t \\ &= \sum_{s=0}^{q-1} \sum_{t=0}^{q-1} \left\{ \frac{1}{q} \sum_{r=0}^{q-1} \exp\left(2\pi i \frac{r(s-t)}{q}\right) \right\} P_s P_t \\ &= \sum_{s=0}^{q-1} P_s P_s \\ &\leq \left( \max_{s \pmod q} P_s \right) \sum_{s=0}^{q-1} P_s, \end{aligned}$$

but

$$\sum_{s=0}^{q-1} P_s = \sum_{s=0}^{q-1} \sum_{\ell \equiv s \pmod q} p_\ell = \sum_{\ell} p_\ell = 1,$$

and Lemma 5 is proved.  $\square$

We keep the notation that we introduced at the beginning of section 2, and further let

$$(3.1) \quad \theta_1 = \min \left( \theta^*/4, \frac{\mu\sqrt{1-p}}{12p}, \sqrt{\varepsilon}/2, \frac{\sqrt{1-p}}{12pA} \right),$$

so that (2.1) implies that  $|E(\theta)| \leq \mu$ , for  $\theta \leq \theta_1$ .

Let us assume that there exists  $\theta_0 \leq \theta_1$  such that  $|2E(\theta_0)| < k|E(\theta_0)|$ . According to Lemma 3, one of the following possibilities holds true:

$$(3.2) \quad \exists q \geq 2 : \left\{ \frac{0}{q}, \frac{1}{q}, \dots, \frac{q-1}{q} \right\} \subset E(3\theta_0)$$

$$(3.3) \quad \left[ -|E(\theta_0)|, |E(\theta_0)| \right] \subset E(3\theta_0).$$

Let us first assume that (3.2) is satisfied. We use Lemma 5, which leads to

$$\max_s \sum_{\ell \equiv s \pmod q} p_\ell \geq \cos^2 3\theta_0,$$

and by condition (1.2) of Theorem 1, we obtain

$$\cos^2 3\theta_0 \leq 1 - \varepsilon,$$

whence

$$\theta_0 \geq \sqrt{\varepsilon/3},$$

in contradiction with the inequalities  $\theta_0 \leq \theta_1 \leq \sqrt{\varepsilon}/2$ .

Hence, condition (3.3) is satisfied. Lemma 4 then leads us to

$$Q(X_1; |E(\theta_0)|^{-1}) \geq 1 - 6(1 - \cos(3\theta_0))$$

and by condition (1.3) of Theorem 1,

$$1 - 6(1 - \cos 3\theta_0) \leq 1 - a|E(\theta_0)|^\sigma ,$$

whence

$$a|E(\theta_0)|^\sigma \leq 6(1 - \cos(3\theta_0))$$

or

$$(3.4) \quad |E(\theta_0)| \leq c\theta_0^{2/\sigma} .$$

Let us summarize what we have proved so far, remembering that Lemma 1 implies  $2E(\theta) \subset E(2\theta)$ . For  $\theta \leq \theta_1$ , we have

$$\text{either } k|E(\theta)| \leq |E(2\theta)| \quad \text{or} \quad |E(\theta)| \leq c\theta^{2/\sigma} .$$

Our next step is to prove that for  $\theta \leq \theta_1$ , we have

$$(3.5) \quad |E(\theta)| \leq k \max \left( c, \frac{|E(\theta_1)|}{\theta_1^{2/\sigma}} \right) \theta^{2/\sigma} .$$

Indeed, by induction, we have for any  $\ell \geq 0$ :

$$|E(\theta_1/2^\ell)| \leq \max \left( c, \frac{|E(\theta_1)|}{\theta_1^{2/\sigma}} \right) (\theta_1/2^\ell)^{2/\sigma} ,$$

either by using

$$|E(\theta_1/2^{\ell+1})| \leq \frac{1}{k} |E(\theta_1/2^\ell)| = \frac{1}{2^{2/\sigma}} |E(\theta_1/2^\ell)| ,$$

or directly

$$|E(\theta_1/2^{\ell+1})| \leq c(\theta_1/2^{\ell+1})^{2/\sigma}$$

when the previous inequality does not hold. Now, for  $\theta \leq \theta_1$ , we choose  $\ell$  such that  $\theta_1/2^{\ell+1} < \theta \leq \theta_1/2^\ell$ , and notice that  $E(\theta) \subset E(\theta_1/2^\ell)$ , so that we have

$$\begin{aligned} |E(\theta)| &\leq \max \left( c, \frac{|E(\theta_1)|}{\theta_1^{2/\sigma}} \right) (\theta_1/2^\ell)^{2/\sigma} \\ &\leq k \max \left( c, \frac{|E(\theta_1)|}{\theta_1^{2/\sigma}} \right) (\theta_1/2^{\ell+1})^{2/\sigma} , \end{aligned}$$

whence (3.5) follows.

As in section 2, we write

$$\begin{aligned} P\{S_n = N\} &\leq \int_{\mathbb{T}} |\varphi(t)|^n dt \\ &\leq \int_0^{\frac{\pi}{2}} n \cos^{n-1} \theta \sin \theta |E(\theta)| d\theta , \end{aligned}$$

and we majorize  $|E(\theta)|$  by (3.5) on the set  $[0, \theta_1]$ , and by (2.2) on  $[\theta_1, \frac{\pi}{2}]$ . We get, for  $n \geq 3$

$$\begin{aligned} P\{S_n = N\} &\leq k \max \left( c, \frac{|E(\theta_1)|}{\theta_1^{2/\sigma}} \right) \int_0^{\theta_1} n \cos^{n-1} \theta \sin \theta \cdot \theta^{2/\sigma} d\theta \\ &\quad + \frac{n}{n-2} \cos^{n-2} \theta_1 . \end{aligned}$$

We now find upper bounds for the terms containing  $\theta_1$ , as well as for the integral. We have  $\theta_1 \leq \pi/8$  so that  $\theta_1^{2/\sigma} \leq (\pi/8)^{2/\sigma}$ , and

$$|E(\theta_1)| \leq \frac{1}{\cos^2 \theta_1} \leq \frac{1}{\cos^2 \pi/8} .$$

By (2.3), we have  $\theta^* \geq \sqrt{1-p} \geq \sqrt{\varepsilon}$ , so that  $\theta_1$ , defined in (3.1), depends at most on  $\mu$  (and so on  $\sigma$ ),  $\varepsilon$ ,  $A$ ,  $a$ , so that  $\cos^{n-2} \theta_1 \leq c/n^2$ .

We finally have, for  $1 \leq \ell \leq 2$

$$\begin{aligned} \int_0^{\theta_1} n \cos^{n-1} \theta \sin \theta \cdot \theta^\ell d\theta &\leq 2 \int_0^{\pi/8} \theta^{\ell-1} \cos^n \theta d\theta \\ &\leq 2 \int_0^{\pi/8} \theta^{\ell-1} \exp \left( -\frac{\theta^2 n}{2} \right) d\theta \\ &\leq 2n^{-\ell/2} \int_0^\infty (\theta\sqrt{n})^{\ell-1} \exp \left( -\frac{\theta^2 n}{2} \right) d(\theta\sqrt{n}) \\ &\leq cn^{-\ell/2} , \end{aligned}$$

where  $c$  is an absolute constant.

We have thus obtained

$$P\{S_n = N\} \leq cn^{-1/\sigma} ,$$

where  $c$  depends at most on  $\sigma, \varepsilon, A$  and  $a$ . □

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