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SERGEI BRODSKY

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ON GROUPS GENERATED BY A PAIR OF ELEMENTS WITH SMALL THIRD OR FOURTH POWER

by

Sergei Brodsky

Abstract. — The paper is devoted to an investigation of two-generated groups such that the $m-$th power of the generating pair contains less than 2^m elements . It is proved, in particular, that if the cube of the generating pair contains less than 7 elements or its fourth power contains less than 11 elements, then the group is solvable. Otherwise, it is not necessarily solvable. The proofs use computer calculations.

1. Introduction

Let G be a group. A finite subset M of G is called *a set with small $m-$th power* (m is some integer) if $|M^m| < |M|^m$ (here $M^m = \{a_1 \dots a_m | a_1, \dots, a_m \in M\}$ and $| \, . \, |$ denotes the cardinality of the set). The structure of the groups in which each $p-$element subset has a small $m-$th power (for some small p and m), as well as the structure of the set of all special elements (1), was investigated in papers [1-5,7], among others. Notice that the notion of identification pattern, which is introduced in the present paper, is close to the notion "type of square" which was introduced in [3], but we will not discuss the relationship between these concepts.

In this paper we are interested in the structure of groups generated by a two-element set $M = \{a, b\}$ with a small third and fourth power. The proofs are based on pure combinatorial considerations, and are ultimately reduced to enumerating a list of very concrete groups, unfortunately; the total number of cases which appear here is so large that we need to use a computer. All computer calculations were developed by the author on an IBM PC using self-made programs which were written in the frame-work of the mathematical package MATLAB-386 (2). These programs provide a simplification of finite group presentations using Tietze transformations, a calculation

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(1)The element $a \in G$ is called *special* if the set $\{a, b\}$ has the small $m-$th power for some fixed integer m and each $b \in G$.

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of a commutator subgroups in the case of a finite index, and also recognition of groups of some types. The methods of programming are in some interest. Since their description would lead us too far from the topic of the present paper, the topic could be a subject of a separate publication. The results of the mentioned calculations are given in the Appendix.

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Let us formulate a general combinatorial assertion which will be needed below. Let A be a finite set, θ an equivalence relation on A, and $R \subseteq A \times A$. We say that the equivalence relation θ *is generated by* R, and write $\theta = \mathrm{eq}(R)$ if θ is the least equivalence relation contaning R. The relation θ will be called *independent* if θ is the minimal generating relation for its closure $\mathrm{eq}(R)$. The following lemma can be easily proved using induction on $|R|$.

Lemma 1. — *Let θ be an equivalence relation on the set A generated by a relation $R \subseteq A \times A$. Then $|A/\theta| \geq |A| - |R|$. If, in addition, R is independent, then $|A/\theta| = |A| - |R|$.*

2. Identification graphs and their properties

Let G be a group generated by two elements a and b: $G = \mathrm{gp}(a, b)$. We fix a and b as signature constants and regard the group G as the quotient-group of the free group $F = \langle a, b\rangle$. The natural epimorphism $\Phi_G : F \to G$ defines an equivalence relation on the group F which will be denoted by the symbol θ_G. We define $H(G)$ as the normal closure of the element ab^{-1} in G: $H(G) = (ab^{-1})^G$, and set $u_i = a^i ba^{-i-1}$ for each $i \in \mathbb{Z}$, so $H = \mathrm{gp}(u_i | i \in \mathbb{Z})$. For each element, or a subset P of $H(G)$, we let $P^{(s)}$ denote the element (the subset) $a^s Pa^{-s}$; it is clear that $P^{(s)}$ can be obtained from P by adding s to all indices of the u-symbols. We also apply the same notation to elements and subsets of the Cartesian square $H_G \times H_G$: $(P, Q)^{(s)} = (P^{(s)}, Q^{(s)})$. Since $|\{a, b\}^m| = |\{a, b\}^m a^{-m}|$, the condition $|\{a, b\}^m| = n \leq 2^m$ $(m \geq 2)$ is equivalent to the condition $|H_m(G)| = n$ where $H_m(G) = \{a, b\}a^{-m}$. One can see that $H_m(G)$ consists of values in G of all strictly increasing positive words in symbols $u_0, \ldots, u_{m-1}$:

$$H_m(G) = \{u_{i_1} \ldots u_{i_k} \mid 0 \leq i_1 < \cdots < i_k \leq m-1,\ 0 \leq k < m\} \subseteq H(G).$$

We denote by U_m the set of all strictly increasing positive words in symbols $u_0, \ldots, u_{m-1}$ itself, so that $H_m(F) = \mathrm{gp}(U_m)$ and $H_m(G) = \mathrm{gp}(\Phi_G(U_m))$.

For $S, T \in U_m$ we say that the pair (S, T) is an *irreducible m-pair* if exactly one of the words S, T begins with u_0 and exactly one of them ends with u_{m-1}. If the irreducible m-pair e has the form (u_0P, Qu_{m-1}) we say that it is *positive*, otherwise e has the form (u_0Pu_{m-1}, Q) and in this case we say that e is *negative*. In both cases we define $i(e) = P$ and $t(e) = Q$. The set of all positive irreducible m-pairs is denoted by I_m^+ and the set of all negative irreducible pairs is denoted as I_m^-.

For given $R \in U_m$, let $\tilde{R}$ be the word in symbols a and b which freely equals R; it is clear that $\tilde{R}$ is a positive word of length m. We say that an irreducible m-pair

(S,T) is *degenerate* if there exists some irreducible $(m-1)$-pair $(P,Q) \in \theta_G$ such that one of the words $\tilde{P}, \tilde{Q}$ is a subword of one of the words $\tilde{S}, \tilde{T}$. The following lemma is obvious.

Lemma 2. — *Let* $\theta_0 = \theta_G \cap (U_{m-1} \times U_{m-1})$ *and let* (S,T) *be a degenerate irreducible* m*-pair. Then* $(S,T) \in \theta$ *if and only if* $(S,T) \in \mathrm{eq}(\theta_0 \cup \theta_0^{(1)} \cup \theta_0 u_{m-1} \cup u_0\theta_0^{(1)})$.

Let us now define the *positive identification* m*-graph* $\Gamma_m^+(G)$ of G as the oriented graph with the set of vertices $H_{m-2}^{(1)}$ and the set of edges $E_m^+(G) = (\Phi_G \times \Phi_G)(I_m^+ \cap \theta_G)$, and the *negative identification* m*-graph* $\Gamma_m^-(G)$ of G as the graph with the same set of vertices and the set of edges $E_m^-(G) = (\Phi_G \times \Phi_G)(I_m^- \cap \theta_G)$. The incidence relations in both these graphs are given by the following rule: if $e \in E_m^+ \cup E_m^-$ and $e = (\Phi_G \times \Phi_G)(e_0)$, where e_0 is some irreducible m-pair, then the initial vertex of e is $\Phi_G(i(e_0))$ and the terminal vertex of e is $\Phi_G(t(e_0))$.

The correctness of the last definition, as well as the validity of the following lemma, can be easily verified.

Lemma 3. — *Let* $G = \mathrm{gp}(a,b)$ *and* $m \geq 2$. *Then each vertex of the positive* m*-identification graph* $\Gamma_m^+(G)$, *and each vertex of the negative* m*-identification graph* $\Gamma_m^-(G)$, *has at most one incoming edge and at most one outgoing edge.*

For $e \in E_m^+(G) \cup E_m^-(G)$, we call e a *degenerate* edge if and only if the set $(\Phi_G \times \Phi_G)^{-1}(e)$ contains some degenerate irreducible pair. Lastly, let $\mathrm{def}_m(G)$ denote the total number of nondegenerate edges in the set $E_m^+(G) \cup E_m^-(G)$.

Lemma 4. — *Let* $G = \mathrm{gp}(a,b)$ *and* $m \geq 2$. *Then*

$$\mathrm{def}_m(G) \geq -2^m - |H_m(G)| + 4|H_{m-1}(G)|.$$

Proof. — Let $d = 2^{m-1} - |H_{m-1}(G)|$. Then, by Lemma 1, the trace θ_0 of the equivalence relation θ_G on the set U_{m-1} is generated by some relation R_0 of cardinality d. Since $U_m \times U_m = (U_{m-1} \times U_{m-1}) \cup (U_{m-1}^{(1)} \times U_{m-1}^{(1)}) \cup (U_{m-1}u_{m-1} \times U_{m-1}u_{m-1}) \cup (u_0U_{m-1}^{(1)} \times u_0U_{m-1}^{(1)})$, the trace θ of the equivalence relation θ_G on the set U_m can be represented as the union of their traces $\theta_0, \theta_1, \theta_2, \theta_3$ on the sets $U_{m-1}, U_{m-1}^{(1)}, U_{m-1}u_{m-1}, u_0U_{m-1}^{(1)}$, respectively, and the relation $(I_G^+ \cup I_G^-) \cap \theta_G$. Each of the equivalence relations θ_k $(k = 1,2,3,4)$ is generated by a d-element relation $(R_0, R_0^{(1)}, R_0u_{m-1}, u_0R_0^{(1)}$, respectively). The union R of last the four relations contains no more than $4d$ elements. By Lemma 2, the difference $(I_G^+ \cup I_G^-) \cap \theta_G \setminus \mathrm{eq}(R)$ is contained in the set of all nondegenerate irreducible m-pairs from θ. Now let us define R_1 as the set which contains one $\Phi_G \times \Phi_G$ pre-image of each nondegenerate edge from $E_m^+(G) \cup E_m^-(G)$. Then $\theta_0 = \mathrm{eq}(R \cup R_1)$, and it only remains to apply Lemma 1.

The inequality which was obtained in Lemma 4 provides us with good necessary conditions for a group to be generated by a pair with a small power. However, we need a more detailed version of this result which also includes some sufficient conditions.

Lemma 5. — *Let $G = \mathrm{gp}(a, b)$ and $H_{m-1}(G) \geq 2^{m-1} - 1$ $(m \geq 2)$. Then*

$$\mathrm{def}_m(G) = -2^m - |H_m(G)| + 4|H_{m-1}(G)|.$$

Proof. — Let $H_{m-1}(G) = 2^{m-1}$. Preserving the notations which were introduced in the Proof of Lemma 4, we have here that $R = \varnothing$ and R_1 coinsides with $E_m^+(G) \cup E_m^-(G)$. Lemma 3 asures us that the last relation is independent. By Lemma 1, the inequality of Lemma 4 becomes an exact equality.

Let now $H_{m-1}(G) = 2^{m-1} - 1$. In this case R consists of four pairs, and one can verify that it is independent. Repeating the previous argument, and bearing in mind that the definition of a nondegenerate edge provides the independence of the united relation R_1 we again have an exact equality - instead of the inequality - in Lemma 4.

The fact that the quotient group $G/H(G)$ is cyclic reduces the investigation of the group $G(\Gamma)$ to an investigation of the group $H(G)$. The following lemma shows that in nontrivial situations this group is finitely generated.

Lemma 6. — *Let $|H_m(G)| < 2^m$. Then $H(G) = \mathrm{gp}(u_0, \ldots, u_{m-2})$.*

Proof. — If $m = 1$ then $u_0 = 1$ and $H = 1$. Hence, we may assume that $m \geq 2$. Without loss of generality, we may also assume that $|H_{m-1}(G)| = 2^{m-1}$. By Lemma 4, $\mathrm{def}_m(G) \geq 1$, and thus there exists an irreducible m-pair (S, T) such that G satisfies the equality $S = T$ - implying that G also satisfies the equality $S^{(i)} = T^{(i)}$ for each $i \in \mathbb{Z}$. Therefore, for each $i \in \mathbb{Z}$, $u_i \in \mathrm{gp}(u_{i-m+1}, \ldots, u_{i-1})$ and $u_i \in \mathrm{gp}(u_{i+1}, \ldots, u_{i+m-1})$. Now, using induction on i, one can prove that for each $i \in \mathbb{Z}$, $u_i \in \mathrm{gp}(u_0, \ldots, u_{m-2})$.

It should be noted that in the case $m = 2$ Lemma 6 asserts that the group H is cyclic. (In fact, this assertion is obvious and well known).

3. Identification patterns and their universal groups

Let us consider a finite sequence $\Gamma = \langle E_2^+, E_2^-, \ldots, E_m^+, E_m^- \rangle$ such that the set E_k^+ of its *positive k-edges* and the set of E_k^- of its *negative k-edges* consist of positive and negative irreducible k-pairs, respectively $(2 \leq k \leq m)$. For each $e \in E_k^+ \cup E_k^-$, we define the *initial* vertex of e as $i(e)$ and the *terminal* vertex of e as $t(e)$; so for each $2 \leq k \leq m$ we obtain two oriented graphs with the set of vertices U_{k-2}: the *positive k-graph of* Γ which will be denoted by $(\Gamma)_k^+$, and the *negative k-graph of* Γ which will be denoted by $(\Gamma)_k^-$. We write $e = (w_1, w_2)_k^+$ (or $e = (w_1, w_2)_k^-$) if e is a positive (or a negative) k-edge with the initial vertex w_1 and the terminal vertex w_2. If we need to describe any such sequence in a concrete situation, we do this by enumerating of its edges. Further, we consider the sequence of groups $\{H_k(\Gamma) | 2 \leq k \leq m\}$ which are defined in the set of generators $\{u_i | i \in \mathbb{Z}\}$ by the sets of relations $\bigcup\{\mathcal{R}_k(\Gamma)^{(s)} | s \in \mathbb{Z}\}$, where $\mathcal{R}_k(\Gamma) = \{u_0 i(e) = t(e) u_{p-1}^{\varepsilon(e)} | e \in E_p^+ \cup E_p^-,\ 2 \leq p \leq k\}$, $\varepsilon(e) = 1$ for $e \in E_p^+$ and $\varepsilon(e) = -1$ for $e \in E_p^-$. For each of these groups, the natural epimorphism $\Phi_{\Gamma,k} : U_k \to H_k$ defines the equivalence relation on the group U_k which is denoted by the symbol $\theta_{\Gamma,k}$. Let us denote the quotient-graphs $(\Gamma)^+_k/\theta_{\Gamma,k-2}$ and $(\Gamma)^i_k/\theta_{\Gamma,k-2}$ by the symbols $[\Gamma]^+_k$ and $[\Gamma]^-_k$, respectively.

As above, we say that an irreducible k-pair (S,T) is *degenerate* (in respect to Γ) if there exists some irreducible $(k-1)$-pair $(P,Q) \in \theta_{\Gamma,k}$ such that one of the words $\tilde{P}, \tilde{Q}$ is a subword of one of the words $\tilde{S}, \tilde{T}$.

Finally, we call the sequence Γ to be an *identification pattern* if, for each $3 \le k \le m$, the set $E^+_k \cup E^-_k$ consists of nondegenerate pairs, and each of the graphs $[\Gamma]^+_k, [\Gamma]^-_k$ has the property that each of its vertices has at most one incoming and at most one outgoing edge.

For a given identification pattern Γ, we let the symbol $\mathcal{G}(\Gamma)$ denote the class of all groups $G = \mathrm{gp}(a,b)$, such that for each $2 \le k \le m$, $E^+_k(G) \supseteq (\Phi_G \times \Phi_G)(E^+_k)$ and $E^-_k(G) \supseteq (\Phi_G \times \Phi_G)(E^-_k)$. Let us now define the *universal group* $G(\Gamma)$ of the identification pattern Γ as the infinite cyclic extension of the group $H(\Gamma) = H_m(\Gamma)$ with the naturally defined extending automorphism: $G(\Gamma) = \langle a \rangle \land H(\Gamma)$, $au_ia^{-1} = u_{i+1}$. It is easy to see that for each identification pattern $\Gamma, G(\Gamma) \in \mathcal{G}(\Gamma)$ and $\mathcal{G}(\Gamma)$ consists of all quotient-groups of $G(\Gamma)$. The group $H(\Gamma)$ itself we call the *universal kernel* of Γ.

Example 1. — Let $\Gamma = \langle (1,1)^+_3, (u_1u_2, u_1u_2)^+_4 \rangle$. Then the universal kernel of Γ has the following presentation: $H(\Gamma) = \langle u_0, u_1 \mid u_0u_1u_0 = u_1u_0u_1 \rangle$; and the inner automorphism, afforded by a, acts in the following way: $au_0a^{-1} = u_1, au_1a^{-1} = u_0$. Using the Reidemeister-Schreier method (see, for instance, [8,9]), we see that the group H is the infinite cyclic extension of the free group $K = \langle v_0, v_1 \rangle$ with the extending automorphism defined by the equalities $u_1v_0u_1^{-1} = v_1$, $u_1v_1u_1^{-1} = v_0^{-1}v_1$ $(u_0 = v_0u_1)$. Direct calculations show that $|H_3(G(\Gamma))| = 7$ and $|H_4(G(\Gamma))| = 11$.

Example 2. — Let $\Gamma = \langle (1,1)^+_4, (u_1u_2, u_1u_2)^+_4 \rangle$. Then

$$H(\Gamma) = \langle u_0, u_1, u_2 \mid u_0u_1u_2 = u_1u_2u_0 = u_2u_0u_1 \rangle,$$

$au_0a^{-1} = u^1, au_1a^{-1} = u_2$ and $au_2a^{-1} = u_0$. Using Tietze transformations (see, for instance, [8,9]), we have $H(\Gamma) = \langle v_0, v_1, v_2 \mid v_0v_2 = v_2v_0, v_1v_2 = v_2v_1 \rangle$, where $u_0 = v_0, u_1 = v_0^{-1}v_1, u_2 = v_1^{-1}v_2$. That is, $H(\Gamma)$ is a direct product of the free group $\langle v_0, v_1 \rangle$ of rank two and the infinite cyclic group $\langle v_2 \rangle$. In this case we have $|H_3(G(\Gamma))| = 8$ and $|H_4(G(\Gamma))| = 14$.

In an informal way the above examples show that there exist arbitrarily large groups generated by a pair of elements with small third and fourth powers. In precise terms we have the following two theorems:

Theorem 1. — *For each countable (finite) group L, there exists a (finite) group $G = \mathrm{gp}(a,b)$ such that $|\{a,b\}^3| = 7$, $|\{a,b\}^4| = 11$, and the group L is the homomorphic image of a subgroup of G.*

Theorem 2. — *For each countable (finite) group L, there exists a (finite) group $G = \mathrm{gp}(a,b)$ such that $|\{a,b\}^3| = 8$, $|\{a,b\}^4| = 14$, and the group L is the homomorphic image of a subgroup of G.*

Proof. — In order to prove the infinite versions of these theorems, it is enough to note that each of the groups $G(\Gamma)$ in the above examples contains the free group of rank two, and also to realize that each countable group can be embedded into a two-generated group ([6]).

The proofs of the finite versions can be obtained by the method of the proof of Theorem 1 in [10],which asserts that a semidirect product of a residually finite group and a finitely generated residually finite group is residually finite.

Let us prove in details Theorem 1. Let $G_0 = G(\Gamma), H = H(\Gamma)$, and K be the groups from Example 1. First we embed the group L into some two-generated finite group P (we can take, for instance, $P = S_n$ for the relevant permutation group S_n). Consider the two-generator free group $\tilde{F}$ of the variety generated by P and the relevant verbal subgroup M of K, so that $\tilde{F} = K/M$ and $|\tilde{F}| < \infty$ (see, for instance, [12]). The subgroup M is a normal divisor of the group H, the quotient-group H/M is the semidirect product of an infinite cyclic group, and the group $\tilde{F}$: $H/M = \langle u_1 \rangle \leftthreetimes \tilde{F}$. Since the group $\tilde{F}$ is finite, the extending automorphism of this semidirect product has finite order, say l, and hence we may consider the semidirect product $P_1 = \langle u_1 \mid u_1^l = 1 \rangle \leftthreetimes \tilde{F}$ which is a finite group. We then obtain the following chain of epimorphisms and embeddings:

$$H \twoheadrightarrow P_1 \hookleftarrow \tilde{F} \twoheadrightarrow P \hookleftarrow L.$$

Repeating these considerations, with the usage of P_1 instead of P, H instead of K, and G_0 instead of H, we can extend the above chain to the chain

$$G_0 \twoheadrightarrow P_2 \hookleftarrow \tilde{F}_1 \twoheadrightarrow P_1 \hookleftarrow \tilde{F} \twoheadrightarrow P \hookleftarrow L,$$

where all groups besides G are finite. Now, using Theorem 1 from [10], we may assert that the group G_0 is residually finite. Therefore, it is possible to insert into the last chain the finite group G which satisfies the conditions of Theorem 1:

$$G_0 \twoheadrightarrow G \twoheadrightarrow P_2 \hookleftarrow \tilde{F}_1 \twoheadrightarrow P_1 \hookleftarrow \tilde{F} \twoheadrightarrow P \hookleftarrow L.$$

The proof of the finite version of Theorem 2 is obtained by similar considerations.

Theorems 1 and 2 show that if we want to obtain any definite information about the groups generated by a pair with a small third or fourth power, we need to impose stronger restrictions on the cardinalities of these powers than those used in the above theorems. Noting that in the case where $|H_2(G)| < 4$ the group H is cyclic, we have to investigate only the following situations:

(a) $|H_2(G)| = 4$ and $|H_3(G)| \leq 7$;
(b) $|H_3(G)| = 7$ and $|H_4(G)| \leq 11$;
(c) $|H_3(G)| = 8$ and $|H_4(G)| \leq 14$.

Using lemmas 2,3 and 5, one can easily verify the following three lemmas.

Lemma 7. — *Let $G = \mathrm{gp}(a, b)$ and $|H_2(G)| = 4$. Then $|H_3(G)| < 7$ if and only if G is a quotient of the universal group $G(\Gamma)$ for some identification pattern Γ with two 3-edges (and no other edges).*

***Lemma* 8.** — *Let $G = \mathrm{gp}(a,b)$ and $|H_3(G)| = 7$. Then $|H_4(G)| < 11$ if and only if G is a quotient of the universal group $G(\Gamma)$ for some identification pattern Γ with one 3-edge and two 4-edges (and no other edges).*

***Lemma* 9.** — *Let $G = \mathrm{gp}(a,b)$ and $|H_3(G)| = 8$. Then $|H_4(G)| \leq 16 - k$ if and only if G is a quotient of the universal group $G(\Gamma)$ for some identification pattern Γ with k 4-edges (and no other edges).*

The conditions of lemmas 7 - 9 provide the diagonality of the relations $\theta_{1,\Gamma}, \theta_{2,\Gamma}$, and therefore the graphs $[\Gamma]_3^+$, $[\Gamma]_3^-$, $[\Gamma]_4^+$, $[\Gamma]_4^-$ coinside with the graphs $(\Gamma)_3^+$, $(\Gamma)_3^-$, $(\Gamma)_4^+$, $(\Gamma)_4^-$ respectively. We see now that the problem of describing groups which satisfy the conditions (a)-(c) above is reduced to enumerating the relevant graphs with the sets of vetices $U_1^{(1)}$ and $U_2^{(1)}$ and calculating the relevant universal kernels. The major part of this enumeration can be eliminated by using the considerations below.

For a word $P \in U_k$, we define the *k-complementary* word $\alpha_k(P)$ as the word from U_k such that the set of all $u-$symbols which occur in $\alpha_k(P)$ is the complement in $\{u_0, \dots u_{k-1}\}$ of the set of all u-symbols which occur in P. If $P = u_{i_1}u_{i_2}\dots u_{i_l}$, we define the *k-opposite* word $\beta_k(P) = u_{k-1-i_l}\dots u_{k-1-i_2}u_{k-1-i_1}$. Extending these mappings componentwise onto the Cartesian square $U_k \times U_k$, we obtain two sign preserving involutions on the set of all irreducible k-pairs which we denote by the same symbols α_k and β_k. It follows from the definitions that these involutions commute, and hence they define an action of the Klein four-group $\mathcal{K}$ on the set of all irreducible k-pairs. Furthermore, for $g \in \mathcal{K}$ and any identification pattern $\Gamma = \langle E_2^+, E_2^-, \dots, E_m^+, E_m^-\rangle$, we define $g(\Gamma) = \langle g(E_2^+), g(E_2^-), \dots, g(E_m^+), g(E_m^-)\rangle$ and so we obtain the action of $\mathcal{K}$ on the set of all identification patterns. We say that two identification patterns are $\mathcal{K}$ *– equivalent* if they belong to the same orbit of this action.

***Lemma* 10.** — *If identification patterns Γ_1 and Γ_2 are $\mathcal{K}$-equivalent, then $H(\Gamma_1) \cong H(\Gamma_2)$ and $G(\Gamma_1) \cong G(\Gamma_2)$.*

Proof. — In order to prove this lemma it is enough to note that the map α is the restriction of the automorphism of the free group $F = \langle a, b\rangle$ defined by the rule $a \mapsto b, b \mapsto a$, and the map β is the restriction of the composition of the automorphism of F defined by the rule $a \mapsto a^{-1}, b \mapsto b^{-1}$ and the group inversion $g \mapsto g^{-1}$.

4. Main results

Now we turn directly to the problem of calculating the universal kernel for a given identification pattern Γ. By Lemma 6, $H(\Gamma)$ is finitely generated, but it is not necessarily finitely presented. Let us denote by the symbol $H^{[n]}(\Gamma)$ the group which is defined in the set of generators $\{u_i | 0 \leq i \leq n-1\}$ by the set of all relations from the union $\bigcup\{\mathcal{R}_m(\Gamma)^{(s)} | s \in \mathbb{Z}\}$ which contain only the symbols $u_0, \dots, u_{n-1}$; we call this group the *n-particular kernel* of Γ. The group $H(\Gamma)$ is the direct limit of the family of groups $\{H^{[n]}(\Gamma) | n > 0\}$; if we have, for some n, $H^{[n]}(\Gamma) \cong H^{[n+1]}(\Gamma)$, and the group $H^{[n]}(\Gamma)$ is hopfian, then we may conclude that $H(\Gamma) = H^{[n]}(\Gamma)$. The lists

1 and 2 of the universal kernels in the Appendix are obtained using this argument: for each type of identification patterns which appears in lemmas 7-9, we enumerate up to $\mathcal{K}$-equivalence identification patterns of the given type, and calculate $H^{[5]}(\Gamma)$ and $H^{[6]}(\Gamma)$ (taking into account only the patterns for which the order of $H^{[n]}(\Gamma)$ is large enough). It is shown by the calculations that the first of above conditions holds for each identification pattern of those types. On the other hand, all of the groups in these lists are finite, except the first one in List 1 and the second one in List 2; yet these two groups are finite extensions of residaully finite groups and so they are residually finite themselves. Therefore, all groups in the lists 1 and 2 are hopfian, so these lists present the exact description of the needed universal kernels. List 3 is obtained in the similar way using $H^{[7]}(\Gamma)$ and $H^{[8]}(\Gamma)$. For a few identificational patterns of this type it turns out that $H^{[7]}(\Gamma) \neq H^{[8]}(\Gamma)$. In this case we also calculate $H^{[9]}(\Gamma)$, and have $H^{[8]}(\Gamma) = H^{[9]}(\Gamma)$. Again, all of these groups are residually finite, and therefore they are hopfian. In order to prove this assertion, we can apply the same line of argument, or, in some cases, Malčev's theorem, which is mentioned in the proof of Theorem 1.

Summarizing the information which is contained in the mentioned lists, and bearing in mind lemmas 7-9, we obtain the following theorems:

Theorem 3. — *Let $G = gp(a,b)$, $|\{a,b\}^2| = 4$ and $|\{a,b\}^3| < 7$. Then the normal subgroup $H = (ab^{-1})^G$ of the group G, generated by the element ab^{-1}, is isomorphic to one of the following groups:*
a) cyclic group of order 5;
b) direct product of two cyclic groups of the same order p ($2 \leq p \leq \infty$);
c) dihedral group of order greater than 2;
d) quaternion group.
All these possibilities are realizable.

Proof. — By Lemma 7, the group G satisfies the conditions of the Theorem if and only if its subgroup H is a homomorphic image of some group in List 1 in the Appendix. Taking into account that groups number 3,7 and 9 are all isomorphic to the quaternion group, we see that all homomorphic images of the groups 1,3,4,6,7,8 and 9, which have at least four elements, is one of the groups described in items *a),c),d)*. The groups 2 and 5 are free abelian of rank two, and it is easy to verify that their free generators are conjugated by the element a. Thus images of these generators are conjugated in each quotient-group of the universal group $G(\Gamma)$. Therefore, these quotient-groups satisfy the condition *b)* for H. A similar situation holds also for group 1 in List 1: it is the free product of two groups of order two which are conjugated by a. It follows that each normal subgroup P of this group is a-invariant (that is P is normal in the group $G(\Gamma)$) and hence for each dihedral group $\hat{H}$ there exists homomorphic image $\hat{G}$ of $G(\Gamma)$ with $H(\hat{G}) \cong \hat{H}$.

Since the condition of $|H| < 4$ implies the cyclicity of H (Lemma 6) for all groups satisfying *b),c),d)* we have that $|H| = 4$. Group 4 satisfies the condition *a)* which may be checked directly.

Corollary 1. — *Let $G = gp(a, b)$, $|\{a, b\}^3| < 7$. Then the group G is solvable of derived length not greater than three.*

Theorem 4. — *Let $G = gp(a, b)$, $|\{a, b\}^3| = 7$ and $|\{a, b\}^4| < 11$. Then the normal subgroup $H = (ab^{-1})^G$ of the group G, generated by the element ab^{-1}, is isomorphic to one of the following groups:*

a) cyclic group of order 7;
b) direct product of two cyclic groups of the same order p ($3 \le p \le \infty$);
c) direct product of cyclic groups of orders 2 and 4;
d) quaternion group;
e) nonabelian semidirect product of cyclic group of even finite or infinite order with a cyclic group of order 3;
f) nonabelian semidirect product of cyclic group of order 3 with a Klein four-group;
g) group defined by presentation $\langle x, y \mid x^2 = y^2, (xy)^2 = 1\rangle$ (extension of cyclic group of order 4 by group of order 2);
h) special linear group $\mathbf{SL}(2, 3)$;

All these possibilities are realizable.

Proof. — At first let us remark that there exist identification patterns with one 3-edge and two 4-edges such that $H(\Gamma)$ is the cyclic group of order 7 (we may take as the example $\Gamma = \langle (u_1, 1)_3^-, (1, u_1)_4^-, (u_1, u_2)_4^- \rangle$) and so there exists a group G satisfying the conditions of the Theorem such that H satisfies the condition *a)*. If H is not isomorphic to the cyclic group of order 7 then, by Lemma 8, it must by a homomorphic image of some group in List 2. Taking into account that the group 2 has the presentation $\langle v, w \mid w^3 = 1, v^{-1}wv = w^{-1}\rangle$, that the groups 12 and 19 are isomorphic to the quaternion group and that the group 5 is isomorphic to $\mathbf{SL}(2, 3)$ (the last isomorphism can be defined by the rule $x \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right], y \mapsto \left[\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right]$), we see that each group in List 2 satisfies one of the conditions *b)-h)*. In order to complete the proof, it remains to make the following observations: it is possible to apply to groups 2,14 and 18 considerations similar to those which were applied to the groups 1,2 and 5 in the proof of Theorem 3; the unique quotient-group of $\mathbf{SL}(2, 3)$ which has order greater than 7 satisfies condition *f)*; and the unique quotient-group of the group 1 which has order greater than 7, satisfies condition *c)*.

Corollary 2. — *Let $G = gp(a, b)$, $|\{a, b\}^3| = 7$ and $|\{a, b\}^4| < 11$. Then the group G is solvable of derived length not greater than four.*

Theorem 5. — *Let $G = gp(a, b)$, $|\{a, b\}^3| = 8$ and $|\{a, b\}^4| < 14$. Then either the normal subgroup $H = (ab^{-1})^G$ of the group G generated by the element ab^{-1} is solvable of derived length not greater than three, or it is a central extension of a cyclic group of order not greater two by the alternating group A_5.*

Proof. — By Lemma 9, our group is a homomorphic image of some group in List 3. All these groups except group 17 are solvable of derived length not greater than three. Group 17 is presented in this list in the following way: $H = \langle x, y \mid xyx = yxy, xyx^{-1}yx = y^2\rangle$, but in generators $v = xy$ and $w = xyx$ it has the presentation

$\langle v, w \mid v^3 = w^2, (vw^{-1})^5 w^2 = 1\rangle$. The quotient-group of this group by the central cyclic subgroup generated by the element w^2 is isomorphic (as was proved in ([11]) to the alternating group A_5. Moreover, let us define homomorphism $\varphi : H \to A_5$ by the rule $v \mapsto (135)$ and $w \mapsto (12)(34)$. A computation using the Reidemeister-Schreier method shows that φ is an epimorphism, and its kernel is isomorphic to a cyclic group of order two.

Remark. — Using List 3, one can make a full classification of groups which satisfy the conditions of Theorem 5 as it has done in theorems 3 and 4, but it seems to be too extensive for our liking.

Corollary 3. — *Let $G = gp(a, b)$, $|\{a, b\}^3| = 8$ and $|\{a, b\}^4| < 14$. Then either the group G is solvable of derived length not greater than four, or it has an invariant series $1 \trianglelefteq N \triangleleft H \trianglelefteq G$ such that N is a cyclic central subgroup of H, $H/N \cong A_5$ and G/H is cyclic of order not greater two.*

Theorem 6. — *Let $G = gp(a, b)$, $|\{a, b\}^3| = 8$ and $|\{a, b\}^4| < 13$. Then the normal subgroup $H = (ab^{-1})^G$ of the group G generated by the element ab^{-1} is solvable of derived length not greater than three.*

Proof. — If G satisfies the conditions of the Theorem, but H does not satisfy its conclusion, then, by Lemma 9 and properties of groups in lists 3,4, it must be a quotient-group of some identification pattern with four 3-edges such that all of them are contained in the union of $\mathcal{K}$-orbit of the identification pattern number 17 in List 3. This union consists of edges $(1,1)_4^+$, $(u_1, u_1u_2)_4^+$, $(u_1u_2, u_2)_4^+$, $(1, u_2)_4^+$, $(u_1, 1)_4^+$, $(u_1u_2, u_1u_2)_4^+$ and it is easy to see that it is impossible to construct any four-element identification pattern of these edges.

Corollary 4. — *Let $G = gp(a, b)$, $|\{a, b\}^3| = 8$ and $|\{a, b\}^4| < 13$. Then the group G is solvable of derived length not greater than four.*

Corollary 5. — *Let $G = gp(a, b)$ and $|\{a, b\}^4| < 11$. Then the group G is solvable of derived length not greater than four.*

5. Appendix

Below are given results of mechanical computations of the universal kernels for the identification patterns which appear in the proofs of the theorems of the last section. These tables use the notation abel$(m_1, \ldots, m_k)$ for the direct product of k cyclic groups of orders $m_1, \ldots, m_k$ $(2 \leq m_i \leq \infty)$.

List 1
Universal kernels of the identification patterns with two 3-edges, only for $|H(\Gamma)| > 4$.

N	Γ	$H(\Gamma)$
1	$\langle (1,1)_3^+, (1,1)_3^- \rangle$	$H = \langle x, y \mid x^2, y^2 \rangle, H' = \text{abel}(1)$;
2	$\langle (1,1)_3^+, (u_1, u_1)_3^+ \rangle$	$H = \text{abel}(\infty, \infty)$;
3	$\langle (1,1)_3^+, (u_1, u_1)_3^- \rangle$	$H = \langle x, y \mid yxyx^{-1}, xyxy^{-1} \rangle$,
		$H/H' = \text{abel}(2,2), H' = \text{abel}(2)$;
4	$\langle (1,1)_3^-, (1, u_1)_3^+ \rangle$	$H = \text{abel}(5)$;
5	$\langle (1,1)_3^-, (u_1, u_1)_3^- \rangle$	$H = \text{abel}(\infty, \infty)$;
6	$\langle (1, u_1)_3^+, (1, u_1)_3^- \rangle$	$H = \text{abel}(2,2)$;
7	$\langle (1, u_1)_3^+, (u_1, 1)_3^+ \rangle$	$H = \langle x, y \mid y^2x^{-2}, yxyx^{-1}, yxy^{-1}x \rangle$,
		$H/H' = \text{abel}(2,2), H' = \text{abel}(2)$;
8	$\langle (1, u_1)_3^+, (u_1, 1)_3^- \rangle$	$H = \text{abel}(2,2)$;
9	$\langle (1, u_1)_3^-, (u_1, 1)_3^- \rangle$	$H = \langle x, y \mid yxyx^{-1}, y^2x^2, xy^{-1}xy \rangle$,
		$H/H' = \text{abel}(2,2), H' = \text{abel}(2)$

List 2
Universal kernels of the identification patterns with one 3-edge and two 4-edges, only for $|H(\Gamma)| > 7$.

N	Γ	$H(\Gamma)$
1	$\langle (1,1)_3^+, (u_2, u_1)_4^+,$	$H = \langle x, y \mid y^2x^{-2}, yxyx \rangle$,
	$(u_1u_2, 1)_4^- \rangle$	$H/H' = \text{abel}(2,4); H' = \text{abel}(2)$;
2	$\langle (1,1)_3^+, (u_2, u_1)_4^+,$	$H = \langle x, y \mid y^2x^{-2}, yxyx^{-1}y^{-1}x^{-1} \rangle$,
	$(u_1u_2, u_1u_2)_4^+ \rangle$	$H/H' = \text{abel}(\infty), H' = \text{abel}(3)$;
3	$\langle (1,1)_3^+, (u_2, u_1)_4^+,$	$H = \langle x, y \mid y^2x^{-2}, yxyxy^{-1}x^{-1}, y^2x^2 \rangle$,
	$(u_1u_2, u_1u_2)_4^- \rangle$	$H/H' = \text{abel}(4); H' = \text{abel}(3)$;
4	$\langle (1,1)_3^+, (u_2, u_1u_2)_4^+,$	$H = \langle x, y \mid xyxy^{-1}xy^{-1}, y^2, x^3 \rangle$,
	$(u_1u_2, 1)_4^- \rangle$	$H/H' = \text{abel}(3); H' = \text{abel}(2,2)$;

List 2
(continuation)

N	Γ	$H(\Gamma)$
5	$\langle (1,1)_3^+, (u_2, u_1u_2)_4^+, (u_1u_2, u_1)_4^+ \rangle$	$H = \langle x, y \mid xyxy^{-2}, yxyx^{-2} \rangle, H/H' = \text{abel}(3); H'/H'' = \text{abel}(2,2); H'' = \text{abel}(2);$
6	$\langle (1,1)_3^+, (u_1u_2, 1)_4^-, (u_1u_2, u_1)_4^+ \rangle$	$H = \langle x, y \mid xyxy^{-1}xy^{-1}, y^2, x^3 \rangle, H/H' = \text{abel}(3); H' = \text{abel}(2,2);$
7	$\langle (1,1)_3^-, (1, u_1u_2)_4^-, (u_1, u_2)_4^+ \rangle$	$H = \text{abel}(2,4);$
8	$\langle (1, u_1)_3^+, (u_1, 1)_4^-, (u_2, 1)_4^+ \rangle$	$H = \text{abel}(3,3);$
9	$\langle (1, u_1)_3^+, (u_1, 1)_4^-, (u_1u_2, u_1)_4^+ \rangle$	$H = \text{abel}(3,3);$
10	$\langle (1, u_1)_3^+, (u_1, 1)_4^-, (u_1u_2, u_2)_4^- \rangle$	$H = \text{abel}(3,3);$
11	$\langle (1, u_1)_3^+, (u_2, 1)_4^+, (u_1u_2, u_1)_4^+ \rangle$	$H = \text{abel}(3,3);$
12	$\langle (1, u_1)_3^-, (1,1)_4^+, (u_1u_2, u_1u_2)_4^+ \rangle$	$H = \langle x, y \mid xyx^{-1}y, yxy^{-1}x, x^{-2}y^2 \rangle, H/H' = \text{abel}(2,2); H' = \text{abel}(2);$
13	$\langle (1, u_1)_3^-, (1,1)_4^-, (u_1, u_2)_4^+ \rangle$	$H = \text{abel}(3,3);$
14	$\langle (1, u_1)_3^-, (1,1)_4^-, (u_1u_2, u_1u_2)_4^- \rangle$	$H = \text{abel}(\infty, \infty);$
15	$\langle (1, u_1)_3^-, (u_1, u_2)_4^+, (u_1u_2, 1)_4^- \rangle$	$H = \langle x, y \mid y^3, x^3, xyxy \rangle, H/H' = \text{abel}(3); H' = \text{abel}(2,2);$
16	$\langle (u_1, 1)_3^-, (1, u_1u_2)_4^-, (u_1, u_1)_4^+ \rangle$	$H = \text{abel}(3,3);$
17	$\langle (u_1, 1)_3^-, (1, u_1u_2)_4^-, (u_2, u_1)_4^+ \rangle$	$H = \langle x, y \mid y^3, x^3, xy^{-1}xy^{-1} \rangle, H/H' = \text{abel}(3); H' = \text{abel}(2,2);$
18	$\langle (u_1, 1)_3^-, (u_1, u_1)_4^+, (u_2, u_2)_4^+ \rangle$	$H = \text{abel}(\infty, \infty);$
19	$\langle (u_1, 1)_3^-, (u_1, u_1)_4^-, (u_2, u_2)_4^- \rangle$	$H = \langle x, y \mid xyxy^{-1}, x^{-1}yxy \rangle, H/H' = \text{abel}(2,2); H' = \text{abel}(2)$

List 3
Universal kernels of the identification patterns with three 4-edges, only for $|H(\Gamma)| > 7$.

N	Γ	$H(\Gamma)$
1	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(1,1)_4^-\rangle$	$H=\mathrm{abel}(2,2,2)$;
2	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(1,u_1u_2)_4^-\rangle$	$H=\mathrm{abel}(3,\infty)$;
3	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1,1)_4^-\rangle$	$H=\mathrm{abel}(9)$;
4	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1,u_1)_4^-\rangle$	$H=\mathrm{abel}(2,2,2)$;
5	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1,u_2)_4^-\rangle$	$H=\mathrm{abel}(14)$;
6	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_2,1)_4^-\rangle$	$H=\mathrm{abel}(9)$;
7	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_2,u_1)_4^-\rangle$	$H=\mathrm{abel}(14)$;
8	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_2,u_2)_4^+\rangle$	$H=\mathrm{abel}(\infty,\infty,\infty)$;
9	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_2,u_2)_4^-\rangle$	$H=\mathrm{abel}(2,2,2)$;
10	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1u_2,u_1)_4^-\rangle$	$H=\mathrm{abel}(9)$;
11	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1u_2,u_2)_4^-\rangle$	$H=\mathrm{abel}(9)$;
12	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1u_2,u_1u_2)_4^+\rangle$	$H=\mathrm{abel}(\infty,\infty,\infty)$;
13	$\langle(1,1)_4^+,(u_1,u_1)_4^+,(u_1u_2,u_1u_2)_4^-\rangle$	$H=\mathrm{abel}(2,2,2)$;
14	$\langle(1,1)_4^+,(u_1,u_2)_4^+,(1,u_1u_2)_4^-\rangle$	$H=\mathrm{abel}(\infty)$;
15	$\langle(1,1)_4^+,(u_1,u_2)_4^+,(u_2,u_1)_4^+\rangle$	$H=\langle x,y \mid y^3,x^{-1}yxy\rangle$, $H/H'=\mathrm{abel}(\infty), H'=\mathrm{abel}(3)$;

List 3
(continuation)

N	Γ	$H(\Gamma)$
16	$\langle (1,1)_4^+, (u_1,u_2)_4^+, (u_1u_2,u_1u_2)_4^+ \rangle$	$H = \text{abel}(\infty)$;
17	$\langle (1,1)_4^+, (u_1,u_1u_2)_4^+, (u_1u_2,u_2)_4^+ \rangle$	$H = \langle x,y \mid xyxy^{-1}x{-}1y{-}1, xyx^{-1}yxy^{-2} \rangle$, $H/H' = 1$;
18	$\langle (1,1)_4^+, (u_2,u_1)_4^+, (1,u_1u_2)_4^- \rangle$	$H = \langle x,y \mid yx^{-1}yx^{-1}, x^2yx^{-1}y^{-2} \rangle$, $H/H' = \text{abel}(\infty), H' = \text{abel}(2,2)$;
19	$\langle (1,1)_4^+, (u_2,u_1)_4^+, (u_1u_2,u_1u_2)_4^+ \rangle$	$H = \langle x,y \mid yxyx^{-1}y^{-1}x^{-1}, y^3x^{-1}y^{-3}x \rangle$, $H/H' = \text{abel}(\infty)$, $H'/H'' = \text{abel}(2,2), H'' = \text{abel}(2)$;
20	$\langle (1,1)_4^+, (u_1u_2,u_1u_2)_4^+, (1,1)_4^- \rangle$	$H = \langle x,y,z \mid x^2, y^2, z^2, zxyzyx, xyzxzy, yzxyxz \rangle$, $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2)$;
21	$\langle (1,1)_4^+, (u_1u_2,u_1u_2)_4^+, (1,u_1u_2)_4^- \rangle$	$H = \langle x,y \mid y^2x^{-1}y^2x^{-1}y^{-1}x^{-1}, x^3y^{-3} \rangle$, $H/H' = \text{abel}(3,\infty), H' = \text{abel}(\infty,\infty)$;
22	$\langle (1,1)_4^+, (u_1u_2,u_1u_2)_4^+, (u_1,1)_4^- \rangle$	$H = \text{abel}(9)$;
23	$\langle (1,1)_4^+, (u_1u_2,u_1u_2)_4^+, (u_1,u_1)_4^- \rangle$	$H = \langle x,y,z \mid zxzx^{-1}, yzy^{-1}z, xyxy^{-1}, zxz^{-1}x, yzyz^{-1}, xyx^{-1}y \rangle$, $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2)$;
24	$\langle (1,1)_4^+, (u_1u_2,u_1u_2)_4^+, (u_1,u_2)_4^- \rangle$	$H = \text{abel}(14)$;
25	$\langle (1,u_1)_4^+, (u_1,1)_4^+, (1,u_1)_4^- \rangle$	$H = \text{abel}(2,2,2)$;
26	$\langle (1,u_1)_4^+, (u_1,1)_4^+, (u_1,1)_4^- \rangle$	$H = \text{abel}(2,2,2)$;
27	$\langle (1,u_1)_4^+, (u_1,1)_4^+, (u_2,u_1u_2)_4^+ \rangle$	$H = \text{abel}(2,2,2)$;
28	$\langle (1,u_1)_4^+, (u_1,1)_4^+, (u_2,u_1u_2)_4^- \rangle$	$H = \text{abel}(2,2,2)$;
29	$\langle (1,u_1)_4^+, (u_1,1)_4^+, (u_1u_2,u_2)_4^+ \rangle$	$H = \text{abel}(2,2,2)$;

List 3
(continuation)

N	Γ	$H(\Gamma)$
30	$\langle (1,u_1)_4^+, (u_1,1)_4^+, (u_1u_2,u_2)_4^- \rangle$	$H = \text{abel}(2,2,2)$;
31	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (1,1)_4^- \rangle$	$H = \text{abel}(9)$;
32	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (u_1,u_1)_4^- \rangle$	$H = \text{abel}(9)$;
33	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (u_2,1)_4^+ \rangle$	$H = \langle x,y \mid yx^{-2}yx, y^{-2}xyx \rangle$, $H/H' = \text{abel}(3)$, $H'/H'' = \text{abel}(2,2), H'' = \text{abel}(2)$;
34	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (u_2,u_2)_4^- \rangle$	$H = \text{abel}(9)$;
35	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (u_2,u_1u_2)_4^+ \rangle$	$H = \text{abel}(9)$;
36	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (u_1u_2,1)_4^- \rangle$	$H = \langle x,y \mid yx^{-1}yxyx^{-1}, x^2, y^3 \rangle$, $H/H' = \text{abel}(3), H' = \text{abel}(2,2)$;
37	$\langle (1,u_1)_4^+, (u_1,u_2)_4^+, (u_1u_2,u_1u_2)_4^- \rangle$	$H = \text{abel}(9)$;
38	$\langle (1,u_1)_4^+, (u_2,1)_4^+, (u_1u_2,1)_4^- \rangle$	$H = \langle x,y \mid x^3, yxyx, y^3 \rangle$, $H/H' = \text{abel}(3), H' = \text{abel}(2,2)$;
39	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (1,1)_4^- \rangle$	$H = \text{abel}(9)$;
40	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (1,u_1)_4^- \rangle$	$H = \text{abel}(2,2,2)$;
41	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (1,u_2)_4^- \rangle$	$H = \text{abel}(11)$;
42	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_1,1)_4^- \rangle$	$H = \text{abel}(2,2,2)$;
43	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_1,u_1)_4^- \rangle$	$H = \text{abel}(9)$;
44	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_1,u_2)_4^- \rangle$	$H = \text{abel}(17)$;

List 3
(continuation)

N	Γ	$H(\Gamma)$
45	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_2,1)_4^- \rangle$	$H = \mathrm{abel}(3,3)$;
46	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_2,u_1)_4^- \rangle$	$H = \mathrm{abel}(13)$;
47	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_1u_2,1)_4^+ \rangle$	$H = \mathrm{abel}(11)$;
48	$\langle (1,u_1)_4^+, (u_2,u_1u_2)_4^+, (u_1u_2,u_2)_4^+ \rangle$	$H = \mathrm{abel}(2,2,2)$;
49	$\langle (1,u_1)_4^+, (u_1u_2,1)_4^+, (1,u_2)_4^- \rangle$	$H = \mathrm{abel}(11)$;
50	$\langle (1,u_1)_4^+, (u_1u_2,1)_4^+, (u_1,u_1u_2)_4^- \rangle$	$H = \mathrm{abel}(11)$;
51	$\langle (1,u_1)_4^+, (u_1u_2,u_2)_4^+, (1,u_1)_4^- \rangle$	$H = \mathrm{abel}(2,2,2)$;
52	$\langle (1,u_1)_4^+, (u_1u_2,u_2)_4^+, (u_1,1)_4^- \rangle$	$H = \mathrm{abel}(2,2,2)$;
53	$\langle (1,u_1)_4^+, (u_1u_2,u_2)_4^+, (u_2,u_1u_2)_4^- \rangle$	$H = \mathrm{abel}(2,2,2)$;
54	$\langle (1,u_1)_4^+, (u_1u_2,u_2)_4^+,$	$H = \mathrm{abel}(2,2,2)$;
55	$\langle (1,u_2)_4^+, (u_1,u_1u_2)_4^+, (1,u_2)_4^- \rangle$	$H = \mathrm{abel}(2,2,2)$;
56	$\langle (1,u_2)_4^+, (u_1,u_1u_2)_4^+, (u_1,1)_4^- \rangle$	$H = \mathrm{abel}(11)$;
57	$\langle (1,u_2)_4^+, (u_1,u_1u_2)_4^+, (u_1,u_2)_4^- \rangle$	$H = \mathrm{abel}(19)$;
58	$\langle (1,u_2)_4^+, (u_1,u_1u_2)_4^+, (u_2,1)_4^+ \rangle$	$H = \mathrm{abel}(2,2,2)$;
59	$\langle (1,u_2)_4^+, (u_1,u_1u_2)_4^+, (u_2,1)_4^- \rangle$	$H = \mathrm{abel}(2,2,2)$;

List 3
(continuation)

N	Γ	$H(\Gamma)$
60	$\langle (1,u_2)_4^+, (u_1,u_1u_2)_4^+, (u_2,u_1)_4^- \rangle$	$H = \text{abel}(11);$
61	$\langle (1,u_1u_2)_4^+, (u_1,u_1)_4^+, (1,u_1u_2)_4^- \rangle$	$H = \langle x,y \mid x^2, yxyx, y^5 \rangle,$ $H/H' = \text{abel}(2), H' = \text{abel}(5);$
62	$\langle (1,u_1u_2)_4^+, (u_1,u_1)_4^+, (u_1,u_1)_4^- \rangle$	$H = \langle x,y \mid y^2, x^2, xyxyxyxyxy \rangle,$ $H/H' = \text{abel}(2), H' = \text{abel}(5);$
63	$\langle (1,u_1u_2)_4^+, (u_1,u_1)_4^+, (u_1u_2,1)_4^+ \rangle$	$H = \langle x,y \mid y^2, yx^{-1}yx^{-1}, x^4yx^{-1}y \rangle,$ $H/H' = \text{abel}(2), H' = \text{abel}(5);$
64	$\langle (1,u_1u_2)_4^+, (u_1,u_2)_4^+, (1,u_1u_2)_4^- \rangle$	$H = \text{abel}(2,2,2);$
65	$\langle (1,u_1u_2)_4^+, (u_1,u_2)_4^+, (u_1,u_2)_4^- \rangle$	$H = \text{abel}(2,2,2);$
66	$\langle (1,u_1u_2)_4^+, (u_1,u_2)_4^+, (u_2,u_1)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
67	$\langle (1,u_1u_2)_4^+, (u_1,u_2)_4^+, (u_2,u_1)_4^- \rangle$	$H = \text{abel}(2,2,2);$
68	$\langle (1,u_1u_2)_4^+, (u_1,u_2)_4^+, (u_1u_2,1)_4^+ \rangle$	$H = \langle x,y,z \mid zxzx^{-1}, yzyz^{-1}, x^2y^{-2},$ $yxyx^{-1}, yxy^{-1}x, z^2y^{-2}, zxz^{-1}x, yzy^{-1}z \rangle,$ $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2);$
69	$\langle (1,u_1u_2)_4^+, (u_1,u_2)_4^+, (u_1u_2,1)_4^- \rangle$	$H = \text{abel}(2,2,2);$
70	$\langle (1,u_1u_2)_4^+, (u_2,u_1)_4^+, (1,u_1u_2)_4^- \rangle$	$H = \text{abel}(2,2,2);$
71	$\langle (1,u_1u_2)_4^+, (u_2,u_1)_4^+, (u_1,u_2)_4^- \rangle$	$H = \text{abel}(2,2,2);$
72	$\langle (1,u_1u_2)_4^+, (u_2,u_1)_4^+, (u_2,u_1)_4^- \rangle$	$H = \text{abel}(2,2,2);$
73	$\langle (1,u_1u_2)_4^+, (u_2,u_1)_4^+, (u_1u_2,1)_4^+ \rangle$	$H = \langle x,y,z \mid x^2z^2, z^2y^2, z^{-1}yzy,$ $z^{-1}xzx, yx^{-1}yx, x^{-1}zxz, xzy^{-1}xzy^{-1} \rangle,$ $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2);$

List 3
(continuation)

N	Γ	$H(\Gamma)$
74	$\langle (1,u_1u_2)_4^+,(u_2,u_1)_4^+,$ $(u_1u_2,1)_4^-\rangle$	$H=\text{abel}(2,2,2);$
75	$\langle (1,u_1u_2)_4^+,(u_1u_2,1)_4^+,$ $(1,u_1u_2)_4^-\rangle$	$H=\langle x,y,z \mid x^2,z^2,yzyz,yzxzyzxz,yxyx\rangle,$ $H/H'=\text{abel}(2,2,2), H'=\text{abel}(\infty,\infty);$
76	$\langle (1,u_1u_2)_4^+,(u_1u_2,1)_4^+,$ $(u_1,u_1)_4^-\rangle$	$H=\langle x,y \mid x^2,y^2,xyxyxyxyxy\rangle,$ $H/H'=\text{abel}(2), H'=\text{abel}(5);$
77	$\langle (1,u_1u_2)_4^+,(u_1u_2,1)_4^+,$ $(u_1,u_2)_4^-\rangle$	$H=\text{abel}(2,2,4);$
78	$\langle (1,u_1u_2)_4^+,(u_1u_2,1)_4^+,$ $(u_1u_2,1)_4^-\rangle$	$H=\langle x,y,z \mid x^2,y^2,xzxz,zyzy\rangle,$ $H/H'=\text{abel}(2,2,2), H'=\text{abel}(\infty,\infty);$
79	$\langle (u_1,u_1)_4^+,(u_2,u_2)_4^+,$ $(1,1)_4^-\rangle$	$H=\langle x,y,z \mid zx^{-1}zx,y^{-1}xyx,$ $z^{-1}yzy,zy^{-1}zy,z^{-1}xzx,xyx^{-1}y\rangle,$ $H/H'=\text{abel}(2,2,2), H'=\text{abel}(2);$
80	$\langle (u_1,u_1)_4^+,(u_2,u_2)_4^+,$ $(1,u_1u_2)_4^-\rangle$	$H=\text{abel}(3,\infty);$
81	$\langle (u_1,u_1)_4^+,(u_2,u_2)_4^+,$ $(u_1,1)_4^-\rangle$	$H=\text{abel}(9);$
82	$\langle (u_1,u_1)_4^+,(u_2,u_2)_4^+,$ $(u_1,u_1)_4^-\rangle$	$H=\langle x,y,z \mid x^2,y^2,xzxz,zxyxz^{-1}xyx,$ $yxzyxz^{-1},zxyz^{-1}xy,z^{-1}yzy\rangle,$ $H/H'=\text{abel}(2,2,2), H'=\text{abel}(2);$
83	$\langle (u_1,u_1)_4^+,(u_2,u_2)_4^+,$ $(u_1,u_2)_4^-\rangle$	$H=\text{abel}(14);$
84	$\langle (u_1,u_2)_4^+,(u_2,u_1)_4^+,$ $(1,1)_4^-\rangle$	$H=\langle x,y \mid yx^{-1}yx,x^2y^{-1}x^2y,y^3\rangle,$ $H/H'=\text{abel}(4), H'=\text{abel}(3);$
85	$\langle (u_1,u_2)_4^+,(u_2,u_1)_4^+,$ $(1,u_1u_2)_4^-\rangle$	$H=\text{abel}(2,2,\infty);$
86	$\langle (u_1,u_2)_4^+,(u_2,u_1)_4^+,$ $(u_1,u_2)_4^-\rangle$	$H=\langle x,y,z \mid z^2,xz^{-1}xz^{-1},xzxz,y^2,xyxy^{-1}\rangle,$ $H/H'=\text{abel}(2,2,2), H'=\text{abel}(\infty,\infty);$
87	$\langle (u_1,u_2)_4^+,(u_2,u_1)_4^+,$ $(u_1u_2,1)_4^-\rangle$	$H=\langle x,y,z \mid y^2,zyxz^{-1}yx^{-1},xyzxyz,$ $z^2yx^{-1}yx^{-1},yzyx^{-2}z,z^{-1}x^{-1}zx\rangle,$ $H/H'=\text{abel}(2,2,4), H'=\text{abel}(2);$

List 3
(continuation)

N	Γ	$H(\Gamma)$
88	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(1,1)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
89	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(1,u_1)_4^+ \rangle$	$H = \text{abel}(9);$
90	$\langle (1,1)_4^-, (u_1,u_1)_4^-, \rangle$ $(1,u_1u_2)_4^+$	$H = \text{abel}(2,7);$
91	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_1,u_1)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
92	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_1,u_2)_4^+ \rangle$	$H = \text{abel}(3,\infty);$
93	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_2,1)_4^+ \rangle$	$H = \text{abel}(9);$
94	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_2,u_2)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
95	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_2,u_2)_4^- \rangle$	$H = \text{abel}(\infty,\infty,\infty);$
96	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_2,u_1u_2)_4^+ \rangle$	$H = \text{abel}(9);$
97	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_1u_2,1)_4^+ \rangle$	$H = \text{abel}(14);$
98	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_1u_2,u_1)_4^+ \rangle$	$H = \text{abel}(9);$
99	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_1u_2,u_1u_2)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
100	$\langle (1,1)_4^-, (u_1,u_1)_4^-,$ $(u_1u_2,u_1u_2)_4^- \rangle$	$H = \text{abel}(\infty,\infty,\infty);$
101	$\langle (1,1)_4^-, (u_1u_2,u_1u_2)_4^-,$ $(1,1)_4^+ \rangle$	$H = \langle x,y,z \mid x^2,y^2,z^2,zxyzyx,$ $xyzxzy, yzxyxz, zxyzyx \rangle,$ $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2);$

List 3
(continuation)

N	Γ	$H(\Gamma)$
102	$\langle (1,1)_4^-, (u_1u_2, u_1u_2)_4^-, (1, u_1)_4^+\rangle$	$H = \text{abel}(9)$;
103	$\langle (1,1)_4^-, (u_1u_2, u_1u_2)_4^-, (1, u_1u_2)_4^+\rangle$	$H = \text{abel}(14)$;
104	$\langle (1,1)_4^-, (u_1u_2, u_1u_2)_4^-, (u_1, u_1)_4^+\rangle$	$H = \langle x,y,z \mid y^{-1}zyz, y^{-1}xyx, xz^{-1}xz, z^{-1}yzy, yxyx^{-1}, zx^{-1}zx\rangle$, $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2)$;
105	$\langle (1,1)_4^-, (u_1u_2, u_1u_2)_4^-, (u_1, u_2)_4^+\rangle$	$H = \text{abel}(3,\infty)$;
106	$\langle (1,u_1)_4^-, (u_1, 1)_4^-, (1, u_1)_4^+\rangle$	$H = \text{abel}(2,2,2)$;
107	$\langle (1,u_1)_4^-, (u_1, 1)_4^-, (u_1, 1)_4^+\rangle$	$H = \text{abel}(2,2,2)$;
108	$\langle (1,u_1)_4^-, (u_1, 1)_4^-, (u_2, u_1u_2)_4^+\rangle$	$H = \text{abel}(2,2,2)$;
109	$\langle (1,u_1)_4^-, (u_1, 1)_4^-, (u_2, u_1u_2)_4^-\rangle$	$H = \text{abel}(2,2,2)$;
110	$\langle (1,u_1)_4^-, (u_1, 1)_4^-, (u_1u_2, u_2)_4^+\rangle$	$H = \text{abel}(2,2,2)$;
111	$\langle (1,u_1)_4^-, (u_1, 1)_4^-, (u_1u_2, u_2)_4^-\rangle$	$H = \text{abel}(2,2,2)$;
112	$\langle (1,u_1)_4^-, (u_2, u_1u_2)_4^-, (1, u_1)_4^+\rangle$	$H = \text{abel}(2,2,2)$;
113	$\langle (1,u_1)_4^-, (u_2, u_1u_2)_4^-, (1, u_1u_2)_4^+\rangle$	$H = \text{abel}(11)$;
114	$\langle (1,u_1)_4^-, (u_2, u_1u_2)_4^-, (u_1, 1)_4^+\rangle$	$H = \text{abel}(2,2,2)$;
115	$\langle (1,u_1)_4^-, (u_2, u_1u_2)_4^-, (u_2, 1)_4^+\rangle$	$H = \text{abel}(11)$;
116	$\langle (1,u_1)_4^-, (u_2, u_1u_2)_4^-, (u_1u_2, 1)_4^+\rangle$	$H = \text{abel}(19)$;

List 3
(continuation)

N	Γ	$H(\Gamma)$
117	$\langle(1,u_1)_4^-,(u_1u_2,u_2)_4^-,(1,u_1)_4^+\rangle$	$H=\text{abel}(2,2,2);$
118	$\langle(1,u_1)_4^-,(u_1u_2,u_2)_4^-,(u_1,1)_4^+\rangle$	$H=\text{abel}(2,2,2);$
119	$\langle(1,u_1)_4^-,(u_1u_2,u_2)_4^-,(u_2,u_1u_2)_4^+\rangle$	$H=\text{abel}(2,2,2);$
120	$\langle(1,u_1)_4^-,(u_1u_2,u_2)_4^-,(u_1u_2,u_2)_4^+\rangle$	$H=\text{abel}(2,2,2);$
121	$\langle(1,u_1u_2)_4^-,(u_1,1)_4^-,(1,1)_4^+\rangle$	$H=\text{abel}(9);$
122	$\langle(1,u_1u_2)_4^-,(u_1,1)_4^-,(u_1,u_1)_4^+\rangle$	$H=\text{abel}(9);$
123	$\langle(1,u_1u_2)_4^-,(u_1,1)_4^-,(u_2,u_2)_4^+\rangle$	$H=\langle x,y\mid y^{-2}xyxy^{-1}x,y^2x^2y^{-1}x\rangle,$ $H/H'=\text{abel}(9),H'=\text{abel}(7);$
124	$\langle(1,u_1u_2)_4^-,(u_1,1)_4^-,(u_1u_2,u_1)_4^-\rangle$	$H=\langle x,y\mid x^3,yxyxy^{-1}x^{-1},y^3\rangle,$ $H/H'=\text{abel}(3),H'=\text{abel}(7);$
125	$\langle(1,u_1u_2)_4^-,(u_1,1)_4^-,(u_1u_2,u_2)_4^-\rangle$	$H=\text{abel}(9);$
126	$\langle(1,u_1u_2)_4^-,(u_1,1)_4^-,(u_1u_2,u_1u_2)_4^+\rangle$	$H=\text{abel}(9);$
127	$\langle(1,u_1u_2)_4^-,(u_1,u_1)_4^-,(1,u_1u_2)_4^+\rangle$	$H=\langle x,y\mid y^2,yx^{-1}yx^{-1},xyxy,x^5\rangle,$ $H/H'=\text{abel}(2),H'=\text{abel}(5);$
128	$\langle(1,u_1u_2)_4^-,(u_1,u_1)_4^-,(u_1,u_1)_4^+\rangle$	$H=\langle x,y\mid xyxy,x^2,y^3x^{-1}y^{-2}x,xy^{-1}xy^{-1}\rangle,$ $H/H'=\text{abel}(2),H'=\text{abel}(5);$
129	$\langle(1,u_1u_2)_4^-,(u_1,u_1)_4^-,(u_1,u_2)_4^+\rangle$	$H=\text{abel}(\infty);$
130	$\langle(1,u_1u_2)_4^-,(u_1,u_1)_4^-,(u_2,u_2)_4^-\rangle$	$H=\text{abel}(\infty);$
131	$\langle(1,u_1u_2)_4^-,(u_1,u_1)_4^-,(u_1u_2,1)_4^-\rangle$	$H=\langle x,y\mid x^{-1}yxy,y^5\rangle,$ $H/H'=\text{abel}(\infty),H'=\text{abel}(5);$

List 3
(continuation)

N	Γ	$H(\Gamma)$
132	$\langle (1, u_1u_2)_4^-, (u_1, u_2)_4^-,$ $(1, u_1u_2)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
133	$\langle (1, u_1u_2)_4^-, (u_1, u_2)_4^-,$ $(u_1, u_2)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
134	$\langle (1, u_1u_2)_4^-, (u_1, u_2)_4^-,$ $(u_2, u_1)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
135	$\langle (1, u_1u_2)_4^-, (u_1, u_2)_4^-,$ $(u_2, u_1)_4^- \rangle$	$H = \langle x, y, z \mid z^2x^2, zx^{-1}zx, z^{-2}y^2,$ $zyz^{-1}y, z^{-1}yxz^{-1}yx^{-1}, y^{-1}zyz, y^{-1}xyx^{-1} \rangle,$ $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2)$
136	$\langle (1, u_1u_2)_4^-, (u_1, u_2)_4^-,$ $(u_1u_2, 1)_4^+ \rangle$	$H = \text{abel}(2,2,2);$
137	$\langle (1, u_1u_2)_4^-, (u_1, u_2)_4^-,$ $(u_1u_2, 1)_4^- \rangle$	$H = \langle x, y, z \mid y^2x^2, y^{-1}zyz, z^2x^2,$ $xy^{-1}xy, xyx^{-1}y, xzx^{-1}z, yz^{-1}yz, xzxz^{-1} \rangle,$ $H/H' = \text{abel}(2,2,2), H' = \text{abel}(2);$
138	$\langle (1, u_1u_2)_4^-, (u_1u_2, 1)_4^-,$ $(1, u_1u_2)_4^+ \rangle$	$H = \langle x, y, z \mid x^2, xyxy, zyz^{-1}y, z^2 \rangle,$ $H/H' = \text{abel}(2,2,2), H' = \text{abel}(\infty, \infty);$
139	$\langle (1, u_1u_2)_4^-, (u_1u_2, 1)_4^-,$ $(u_1, u_2)_4^+ \rangle$	$H = \text{abel}(2,2,\infty);$
140	$\langle (1, u_1u_2)_4^-, (u_1u_2, 1)_4^-,$ $(u_2, u_1)_4^+ \rangle$	$H = \langle x, y, z \mid z^2, xy^{-1}xy^{-1}, y^2x^{-1}zx^{-1}z,$ $yzyzx^{-2}, xz^{-1}xyzy, x^2y^2 \rangle,$ $H/H' = \text{abel}(2,2,4), H' = \text{abel}(2);$
141	$\langle (u_1, 1)_4^-, (u_2, u_1)_4^-,$ $(1, u_2)_4^+ \rangle$	$H = \text{abel}(11);$
142	$\langle (u_1, 1)_4^-, (u_2, u_1)_4^-,$ $(u_1, u_1u_2)_4^+ \rangle$	$H = \text{abel}(11);$
143	$\langle (u_1, 1)_4^-, (u_2, u_1)_4^-,$ $(u_1u_2, u_2)_4^- \rangle$	$H = \text{abel}(11);$
144	$\langle (u_1, 1)_4^-, (u_1u_2, u_1)_4^-,$ $(u_2, u_2)_4^+ \rangle$	$H = \langle x, y \mid x^3, y^3, xyxy^{-1}x^{-1}y \rangle,$ $H/H' = \text{abel}(3), H' = \text{abel}(7);$
145	$\langle (u_1, 1)_4^-, (u_1u_2, u_2)_4^-,$ $(1, 1)_4^+ \rangle$	$H = \text{abel}(9);$

List 3
(continuation)

N	Γ	$H(\Gamma)$
146	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (1,u_1)_4^+\rangle$	$H = \mathrm{abel}(2,2,2);$
147	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (1,u_2)_4^+\rangle$	$H = \mathrm{abel}(11);$
148	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (1,u_1u_2)_4^+\rangle$	$H = \mathrm{abel}(17);$
149	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (u_1,1)_4^+\rangle$	$H = \mathrm{abel}(2,2,2);$
150	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (u_1,u_1)_4^+\rangle$	$H = \mathrm{abel}(9);$
151	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (u_2,1)_4^+\rangle$	$H = \mathrm{abel}(3,3);$
152	$\langle (u_1,1)_4^-, (u_1u_2,u_2)_4^-, (u_1u_2,1)_4^+\rangle$	$H = \mathrm{abel}(13);$
153	$\langle (u_1,u_1)_4^-, (u_2,u_2)_4^-, (1,1)_4^+\rangle$	$H = \langle x,y,z \mid zxzx^{-1}, xyxy^{-1}, yzyz^{-1}, zyzy^{-1}, xzxz^{-1}, yxyx^{-1}\rangle,$ $H/H' = \mathrm{abel}(2,2,2), H' = \mathrm{abel}(2);$
154	$\langle (u_1,u_1)_4^-, (u_2,u_2)_4^-, (1,u_1)_4^+\rangle$	$H = \mathrm{abel}(9);$
155	$\langle (u_1,u_1)_4^-, (u_2,u_2)_4^-, (1,u_1u_2)_4^+\rangle$	$H = \mathrm{abel}(14);$
156	$\langle (u_1,u_1)_4^-, (u_2,u_2)_4^-, (u_1,u_1)_4^+\rangle$	$H = \langle x,y,z \mid x^2, z^2, xy^{-1}xy^{-1}, y^{-1}z^{-1}yz, xy^{-1}z^{-1}x^{-1}yz\rangle,$ $H/H' = \mathrm{abel}(2,2,2), H' = \mathrm{abel}(2);$
157	$\langle (u_1,u_1)_4^-, (u_2,u_2)_4^-, (u_1,u_2)_4^+\rangle$	$H = \langle x,y \mid xy^{-2}xyx^{-2}y, y^{-3}x^3\rangle,$ $H/H' = \mathrm{abel}(3,\infty), H' = \mathrm{abel}(\infty,\infty);$
158	$\langle (u_1,u_1)_4^-, (u_2,u_2)_4^-, (u_1u_2,1)_4^-\rangle$	$H = \langle x,y \mid x^2y^{-1}x^{-2}y, y^{-2}xyx^{-1}yx\rangle,$ $H/H' = \mathrm{abel}(\infty),$ $H'/H'' = \mathrm{abel}(2,6), H'' = \mathrm{abel}(2);$
159	$\langle (u_1,u_1)_4^-, (u_1u_2,1)_4^-, (1,u_1u_2)_4^+\rangle$	$H = \langle x,y \mid x^2, y^2, xyxyxyxyxy\rangle,$ $H/H' = \mathrm{abel}(2); H' = \mathrm{abel}(5);$

List 3
(continuation)

N	Γ	$H(\Gamma)$
160	$\langle (u_1,u_1)_4^-, (u_1u_2,1)_4^-, (u_1,u_1)_4^+\rangle$	$H=\langle x,y \mid x^2,y^2,xyxyxyxyxy\rangle$, $H/H'=\text{abel}(2); H'=\text{abel}(5)$;
161	$\langle (u_1,u_1)_4^-, (u_1u_2,1)_4^-, (u_1,u_2)_4^+\rangle$	$H=\langle x,y \mid y^2,x^2yx^{-1}yx^{-1}y\rangle$, $H/H'=\text{abel}(\infty), H'=\text{abel}(2,2)$;
162	$\langle (u_1,u_2)_4^-, (u_2,u_1)_4^-, (1,u_1u_2)_4^+\rangle$	$H=\text{abel}(2,2,4)$;
163	$\langle (u_1,u_2)_4^-, (u_2,u_1)_4^-, (u_1,u_2)_4^+\rangle$	$H=\langle x,y,z \mid x^2,y^2,zyzy,zxzx\rangle$, $H/H'=\text{abel}(2,2,2), H'=\text{abel}(\infty,\infty)$;
164	$\langle (u_1,u_2)_4^-, (u_2,u_1)_4^-, (u_2,u_1)_4^+\rangle$	$H=\langle x,y,z \mid x^2,y^2,zxzx,zyzy\rangle$, $H/H'=\text{abel}(2,2,2), H'=\text{abel}(\infty,\infty)$;
165	$\langle (u_1,u_2)_4^-, (u_2,u_1)_4^-, (u_1u_2,1)_4^-\rangle$	$H=\langle x,y,z \mid z^2x^2,y^2z^{-2},xz^{-1}xz,xyx^{-1}y, xy^{-1}xy,yzy^{-1}z,yz^{-1}yz,x^{-1}zxz\rangle$, $H/H'=\text{abel}(2,2,2), H'=\text{abel}(2)$;
166	$\langle (u_1,u_2)_4^-, (u_1u_2,1)_4^-, (1,u_1u_2)_4^+\rangle$	$H=\text{abel}(2,2,2)$;
167	$\langle (u_1,u_2)_4^-, (u_1u_2,1)_4^-, (u_1,u_2)_4^+\rangle$	$H=\text{abel}(2,2,2)$;
168	$\langle (u_1,u_2)_4^-, (u_1u_2,1)_4^-, (u_2,u_1)_4^+\rangle$	$H=\text{abel}(2,2,2)$;
169	$\langle (u_1,u_2)_4^-, (u_1u_2,1)_4^-, (u_1u_2,1)_4^+\rangle$	$H=\text{abel}(2,2,2)$;

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S. Brodsky, Eventus Logistics Ltd., 8 Dov Friedman st., P.O.B. 1930, Ramat-Gan 52118, Israel
E-mail : sergei@eventus.co.il