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# ON THE STRUCTURE OF SETS OF LATTICE POINTS IN THE PLANE WITH A SMALL DOUBLING PROPERTY

*by*

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***Abstract.*** — We describe the structure of sets of lattice points in the plane, having a small doubling property. Let $\mathbb{K}$ be a finite subset of $\mathbb{Z}^2$ such that

$$|\mathbb{K}+\mathbb{K}| < 3.5|\mathbb{K}| - 7.$$

If $\mathbb{K}$ lies on three parallel lines, then the convex hull of $\mathbb{K}$ is contained in three compatible arithmetic progressions with the same common difference, having together no more than

$$|\mathbb{K}| + \frac{3}{4}\left(|\mathbb{K}+\mathbb{K}| - \frac{10}{3}|\mathbb{K}| + 5\right)$$

terms. This upper bound is best possible.

## Notation

We write $[m,n] = \{x \in \mathbb{Z} | \ m \le x \le n\}$. For any nonempty finite set $K \subseteq \mathbb{R}$, $K = \{u_1 < u_2 < \cdots < u_k\}$ we denote by $k = |K|$ the *cardinality* of $K$ and by $\ell(K)$ the *length* of $K$, that is the difference between its maximal and minimal elements. If $K \subseteq \mathbb{Z}$ and $k \ge 2$, by $d(K)$ we denote the *greatest common divisor* of $u_i - u_1$, $1 \le i \le k$. If $k = 1$, we put $d(K) = 0$. Let $h(K) = \ell(K) - |K| + 1$ denote the number of *holes* in $K$, that is $h(K) = | \ [u_1, u_k] \ \setminus \ K|$.

Let $A$ and $B$ be two subsets of an abelian group $(G, +)$. As usual, their *sum* is defined by $A + B = \{x \in G \mid x = a + b, \ a \in A, \ b \in B\}$ and we put $2A = A + A$. The *convex hull* of a set $\mathbb{S} \subseteq \mathbb{R}^2$ is denoted by $\operatorname{conv}(\mathbb{S})$. Vectors will be written in the form $u = (u_1, u_2)$, where $u_1$ and $u_2$ are the coordinates with respect to the canonical basis $e_1 = (1,0)$, $e_2 = (0,1)$.

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## 1. Introduction

In additive number theory we usually ask what may be said about $M + M$, for a given set $M$. As a counterbalance to this direct approach, consider now the inverse problem: we study the properties of $M$, when some characteristic of $M + M$ is given, for example, the cardinality of the sum set $M + M$. It was noticed by Freiman **[F1]** that the assumption that $|2M|$ is small compared to $|M|$, implies strong restrictions on the structure of the set $M$. If $|2M| = 2|M| - 1$ and $M \subseteq \mathbb{Z}$, then $M$ is an arithmetic progression. If we choose bigger values for $|2M|$, the problem ceases to be trivial. The fundamental theorem of G.A. Freiman **[F2]** gives the structure of finite sets of integers with small doubling property: $|2M| < c_0|M|$, where $c_0$ is any given positive number. This theorem was proved using geometric methods of number theory and a modification of the method of trigonometric sums. Y. Bilu recently studied in **[B]** a case when $c_0$ is a slowly growing function of $|M|$. The generalization to the case of different summands $M+N$, with a new proof, is to be found in the paper of I.Z. Ruzsa **[R]**.

However, in the case of small values of the constant $c_0$, elementary methods yield sharper results. Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a finite set of lattice points. Two cases have been studied by G. A. Freiman **[F1]**, pp.11, 28.

***Theorem A.*** — *If* $|\mathbb{K} + \mathbb{K}| < 3|\mathbb{K}| - 3$, *then*
*(1)* $\mathbb{K}$ *lies on a straight line.*
*(2)* $\mathbb{K}$ *is contained in an arithmetic progression of no more than* $v = |\mathbb{K}+\mathbb{K}| - |\mathbb{K}| + 1$ *terms.* □

***Theorem B.*** — *If* $|\mathbb{K} + \mathbb{K}| < \frac{10}{3}|\mathbb{K}| - 5$, $|\mathbb{K}| \geq 11$ *and* $\mathbb{K}$ *is not contained in a line, then*
*(1)* $\mathbb{K}$ *lies on two parallel straight lines.*
*(2)* $\mathbb{K}$ *is contained in two arithmetic progressions with the same common difference having together no more than* $v = |\mathbb{K} + \mathbb{K}| - 2|\mathbb{K}| + 3$ *terms.* □

The generalization of Theorems A(1) and B(1), to $s$ lines, $s \geq 3$, was obtained in **[S2]**:

***Theorem C.*** — *If* $|\mathbb{K} + \mathbb{K}| < \left(4 - \frac{2}{s+1}\right)|\mathbb{K}| - (2s + 1)$ *and* $|\mathbb{K}| \geq 16s(s + 1)(2s + 1)$, *then there exist $s$ parallel lines which cover the set* $\mathbb{K}$. □

A result which generalizes Theorems A(2) and B(2) was obtained in **[S3]**.

Theorems A(1), B(1) and C cannot be sharpened by increasing the upper bound for $|2\mathbb{K}|$. (see Example A in **[S2]**.) Assertion (2) of Theorems A and B gives the precise structure theorem for $s = 1$ and $s = 2$. In **[S2]** we obtained a sharpening of Theorem B(2) by giving the best possible value of the upper bound for $|2\mathbb{K}|$, under the additional assumption that $\mathbb{K}$ lies on $s = 2$ parallel lines. We proved that Theorem B(2) is true, even we replace $|2\mathbb{K}| < \frac{10}{3}|\mathbb{K}| - 5$ by $|2\mathbb{K}| < 4|\mathbb{K}| - 6$. More precisely:

***Theorem S.*** — *Let* $\mathbb{K} \subseteq \mathbb{Z}^2$ *be a finite set, which lies on the lines* $x_2 = 0$ *and* $x_2 = 1$. *Let the set of abscissae for* $x_2 = 0$ *and* $x_2 = 1$, *respectively be equal to $A$ and $B$.*

*(1) If* $\ell(A)+\ell(B)\le 2|\mathbb{K}|-5$, *then* $(d(A),d(B))=1$ *and*

$$|2\mathbb{K}|\ge(3|\mathbb{K}|-3)+h(A)+h(B)=(2|\mathbb{K}|-1)+\ell(A)+\ell(B).$$

*(2) If* $\ell(A)+\ell(B)\ge 2|\mathbb{K}|-4$ *and* $(d(A),d(B))=1$, *then* $|2\mathbb{K}|\ge 4|\mathbb{K}|-6$. □

It is not difficult to give examples to show that Theorems A(2), B(2) and Theorem S cannot be sharpened by reducing the quantity $v$ or by increasing the upper bound for $|2\mathbb{K}|$. (see Examples B1 and B2 of Section 3, **[S2]**)

The present paper is devoted to the generalization of Theorem A(2) and S to the case of $s=3$ parallel lines. Instead of condition $|2\mathbb{K}|<3k-3$, of Theorem A and condition $|2\mathbb{K}|<\frac{10}{3}k-5$ of Theorem B, we study now a set $\mathbb{K}$ of integer points on a plane, with the following small doubling property

$$|2\mathbb{K}|<3.5|\mathbb{K}|-7.$$

Take a lattice $\mathcal{L}$ generated by $\mathbb{K}$. We wish to obtain an estimate for the number of points of $\mathcal{L}$ that lie in $\operatorname{conv}(\mathbb{K})$; we are interested in an upper bound of $|\mathcal{L}\cap\operatorname{conv}(\mathbb{K})|$. Some estimate of this number was obtained in [S2, Theorem C]. In this paper we shall give the best possible estimate for $|\mathcal{L}\cap\operatorname{conv}(\mathbb{K})|$. The result implies an affirmative answer to a question of G.A. Freiman **[F3]** and generalizes previous results of **[F1]** and **[S2]**.

## 2. Main Result

An *arithmetic progression* in $\mathbb{Z}^2$ is a set of the form

$$P=P(a,\Delta)=\{a,a+\Delta,a+2\Delta,\dots,a+(p-1)\Delta\},$$

where $a,\ \Delta\in\mathbb{Z}^2$ and $p=|P|\ge 1$. The vector $\Delta$ is called the *common difference* of the progression and $a$ is the *initial term* . We say that $P_i=P_i(a_i,\ \Delta_i)$, $i=1,2,3$ are *compatible* arithmetic progressions, if $\Delta_1=\Delta_2=\Delta_3=\Delta$ and $a_1+a_3\equiv 2a_2(\operatorname{mod}\Delta)$.

Now we are ready to formulate our main result.

***Theorem 1.*** — *Let* $\mathbb{L}\subseteq\mathbb{Z}^2$ *be a finite set of lattice points with small doubling property:*

$$|\mathbb{L}+\mathbb{L}|<3.5|\mathbb{L}|-7. \tag{2.1}$$

*(1) If* $|\mathbb{L}|\ge 1344$, *then the set* $\mathbb{L}$ *lies on no more than three parallel lines.*
*(2) If* $\mathbb{L}$ *is not contained in any two parallel lines, then* $\operatorname{conv}(\mathbb{L})\cap\mathbb{Z}^2$ *is included in three compatible arithmetic progressions having together no more than*

$$v=|\mathbb{L}|+\frac{3}{4}\Big(|\mathbb{L}+\mathbb{L}|-\frac{10}{3}|\mathbb{L}|+5\Big)=\frac{3}{4}\Big(|\mathbb{L}+\mathbb{L}|-2|\mathbb{L}|+5\Big) \tag{2.2}$$

*terms.*

Assertion (1) of Theorem 1 is a partial case of Theorem C, for $s=3$. We shall reformulate our main result and prove that the new formulation implies assertion (2) of Theorem 1. We need some definitions. Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a finite set of lattice points that lies on three parallel lines:

$$\begin{aligned}&\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2 \cup \mathbb{K}_3,\\ &\mathbb{K}_1 \subseteq (x_2 = 0),\ \mathbb{K}_2 \subseteq (x_2 = 1),\ \mathbb{K}_3 \subseteq (x_2 = h),\ h \geq 2.\end{aligned} \tag{2.3}$$

Let the set of abscissae of $\mathbb{K}_i$ be respectively equal to $K_i$ and denote $d_i = d(K_i)$.

Put

$$\mathbb{K}^* = \operatorname{conv}(\mathbb{K}) \cap \mathbb{Z}^2,\ k = |K|,\ k^* = |\mathbb{K}^*| \tag{2.4}$$

and

$$d(\mathbb{K}) = \gcd(d_1, d_2, d_3). \tag{2.5}$$

Such a finite set of $\mathbb{Z}^2$ is called a *reduced set of lattice points,* if $h = 2$ and $d(\mathbb{K}) = 1$.

We would like to note at this point that this definition may be formulated in an obvious way, for sets that lie on $s \geq 2$ parallel lines. In this paper, however, a reduced set of lattice points will always be a set that lies on three parallel lines.

***Theorem 2.*** — *Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a reduced set of lattice points. If $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$, then*

$$k^* := |\operatorname{conv}(\mathbb{K}) \cap \mathbb{Z}^2| \leq |\mathbb{K}| + \frac{3}{4}\Big(|2\mathbb{K}| - \frac{10}{3}|\mathbb{K}| + 5\Big) = \frac{3}{4}\Big(|2\mathbb{K}| - 2|\mathbb{K}| + 5\Big).$$

*Proof of case (2) of Theorem 1, assuming Theorem 2.* — Since $\mathbb{L}$ lies on three parallel lines, there is an affine isomorphism of the plane which maps $\mathbb{L}$ onto a set $\mathbb{K}$ such that

(i) $\mathbb{K}$ lies on $(x_2 = 0),\ (x_2 = 1),\ (x_3 = h),\ h \geq 2$,

(ii) $m_1 = m_2 = 0$, where we put $m_i = \min(K_i)$, for $i = 1, 2, 3$.

Since the function $|2\mathbb{L}|$ is an affine invariant of the set $\mathbb{L}$, we see that

$$|2\mathbb{K}| = |2\mathbb{L}| < 3.5|\mathbb{L}| - 7 = 3.5|\mathbb{K}| - 7. \tag{2.6}$$

Denote $d = d(\mathbb{K})$. Remark that, thanks to the small doubling property (2.6) one has

$$h = 2 \text{ and } m_1 + m_3 \equiv 2m_2 (\operatorname{mod} d). \tag{2.7}$$

Indeed, if $h > 2$, then $(\mathbb{K}_1 + \mathbb{K}_3) \cap 2\mathbb{K}_2 = \varnothing$ and thus

$$\begin{aligned}|2\mathbb{K}| &\geq |2K_1| + |K_1 + K_2| + |K_1 + K_3| + |2K_2| + |K_2 + K_3| + |2K_3|\\ &\geq (2k_1 - 1) + (k_1 + k_2 - 1) + (k_1 + k_3 - 1)\\ &\quad + (2k_2 - 1) + (k_2 + k_3 - 1) + (2k_3 - 1)\\ &= 4k - 6 \geq 3.5k - 7.\end{aligned} \tag{2.8}$$

In the same way, if $m_1 + m_3 \not\equiv 2m_2 (\operatorname{mod} d)$, then for $x \in K_1; y', y'' \in K_2,\ z \in K_3$ we have $y' + y'' \equiv 2m_2 \not\equiv m_1 + m_3 = x + z (\operatorname{mod} d)$. Thus, $(\mathbb{K}_1 + \mathbb{K}_3) \cap 2\mathbb{K}_2 = \varnothing$ is valid and (2.8) follows again.

Consequently, $\mathbb{K}$ and $\mathbb{L}$ are contained each in three equidistant compatible arithmetic progressions.

Equation (2.7) and (ii) ensure that $m_3 \equiv 2m_2 - m_1 = 0 (\operatorname{mod} d)$. This yields $w \equiv 0 (\operatorname{mod} d)$ for every $w \in K_1 \cup K_2 \cup K_3$. We can now easily check that the linear isomorphism $(x,y) \to (x/d,y)$, maps $\mathbb{K}$ onto a reduced set $\mathbb{K}'$ of lattice points. Assertion (2) of Theorem 1 follows now easily, because of the inequality

$$v = |\mathbb{K}'^*| \le \frac{3}{4}(|2\mathbb{K}'^*| - 2k'^* + 5) = \frac{3}{4}(|2\mathbb{K}| - 2k + 5) = \frac{3}{4}(|2\mathbb{L}| - 2|\mathbb{L}| + 5),$$

due to Theorem 2 applied to the set $\mathbb{K}'^*$. □

As usual, the solution of an inverse problem allows us to obtain nontrivial lower bounds for $|\mathbb{K}+\mathbb{K}|$, thus solving at the same time a direct additive problem. By $L^* = L(\mathbb{K}^*) = \sum_{i=1}^{3} \ell_i^*$, we denote the *length* of $\mathbb{K}^*$, where $\ell_1^* = \ell_1 = \ell(K_1)$, $\ell_3^* = \ell_3 = \ell(K_3)$ and $\ell_2^* = \max(\mathrm{conv}(\mathbb{K}) \cap (x_2 = 1)) - \min(\mathrm{conv}(\mathbb{K}) \cap (x_2 = 1)) \ge \ell_2 = \ell(K_2)$. The assertion of Theorem 1 and 2 may be reworded as follows:

***Theorem 3.*** — *Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a finite set of lattice points which lies on three parallel lines $x_2 = 0$, $x_2 = 1$, $x_2 = 2$.*
*(1) If $L^* \le \frac{9}{8}(|\mathbb{K}| - 4)$, then $d(\mathbb{K}) = 1$ and $|2\mathbb{K}| \ge (2|\mathbb{K}| - 1) + \frac{4}{3}L^*$.*
*(2) If $L^* \ge \frac{9}{8}(|\mathbb{K}| - 4)$ and $d(\mathbb{K}) = 1$, then $|2\mathbb{K}| \ge 3.5|\mathbb{K}| - 7$.*

We conjecture that inequality $|2\mathbb{K}| < 3.5k - 7$ of Theorem 2 may be actually replaced by $|2\mathbb{K}| \le 4k - 7$.

***Conjecture.*** — *Let $\mathbb{K} \subseteq \mathbb{Z}^2$ be a reduced set of lattice points that lies on three parallel lines. If $|2\mathbb{K}| \le 4|\mathbb{K}| - 7$, then*

$$k^* := |\mathrm{conv}(\mathbb{K}) \cap \mathbb{Z}^2| \le |\mathbb{K}| + \frac{3}{4}\Big(|2\mathbb{K}| - \frac{10}{3}|\mathbb{K}| + 5\Big) = \frac{3}{4}\Big(|2\mathbb{K}| - 2|\mathbb{K}| + 5\Big). \qquad \square$$

We construct an example $\mathbb{K} \subseteq \mathbb{Z}^2$ such that
**(i)** $\mathbb{K}$ satisfies the small doubling property $|2\mathbb{K}| < 3.5k - 7$ or $|2\mathbb{K}| \le 4k - 7$.
**(ii)** The number of lattice points in $\mathrm{conv}(\mathbb{K})$ is exactly $k^* = \frac{3}{4}(|2\mathbb{K}| - 2k + 5)$.
This means that the upper bound (2.2) is best possible. Thus, Theorems 1 and 2 cannot be sharpened by reducing the quantity $v = k^*$.

***Example.*** — Choose $a \ge b$ two natural numbers and define $\mathbb{K} \subseteq \mathbb{Z}^2$ by :

$$K_1 = \{0,1,2,\dots,2a+b\} \cup \{2a+2b\},\ K_2 = \{0,1,2,\dots,a\} \cup \{a+b\},\ K_3 = \{0\}. \quad (2.9)$$

Then $k_1 = 2a + b + 2$, $k_2 = a + 2$, $k_3 = 1$, $k = 3a + b + 5$, $k^* = L^* + 3 = L + 3 = 3a + 3b + 3, 4k - 7 = 12a + 4b + 13$. Note that $2\mathbb{K}_2 = \mathbb{K}_1 + \mathbb{K}_3$ and therefore

$$\begin{aligned}|2\mathbb{K}| &= |2K_1| + |K_1 + K_2| + |K_1 + K_3| + |K_2 + K_3| + |2K_3| \\ &= (4a + 3b + 2) + (3a + 2b + 2) + (2a + b + 2) + (a + 2) + 1 \\ &= 10a + 6b + 9 = (2k - 1) + \frac{4}{3}L^* = (2k - 1) + \frac{4}{3}(k^* - 3).\end{aligned}$$

This proves (ii), that is $k^* = \frac{3}{4}(|2\mathbb{K}| - 2k + 5)$. Moreover, assertion (i) is also true because, if $a \ge b-2$, then $|2\mathbb{K}| \le 4|\mathbb{K}| - 7$ and if $a > 5b - 3$, then $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$. □

We shall need a generalization of Theorem A(2) to the case of distinct summands. The first result in this direction, due to G.A. Freiman ([**F4**]), was sharpened recently by Lev & Smeliansky [**L-S**] and Stanchescu [**S1**].

Let $A = \{0 = a_1 < a_2 < \cdots < a_k\}$, $B = \{0 = b_1 < b_2 < \cdots < b_\ell\}$ be two sets of integers. Define $\varepsilon = \varepsilon(A,B) \in \{0,1\}$ by $\varepsilon = 1$, if $\ell(A) = \ell(B)$, and $\varepsilon = 0$, if $\ell(A) \neq \ell(B)$.

***Theorem D***

*(1) If* $\max(\ell(A), \ell(B)) \leq |A| + |B| - 2 - \varepsilon$, *then* $|A + B| \geq (|A| + |B| - 1) + \max(h(A), h(B))$.

*(2) If* $\ell(A) \geq |A| + |B| - 1 - \varepsilon$, $\ell(A) \geq \ell(B)$ *and* $d(A) = 1$, *then* $|A + B| \geq |A| + 2|B| - 2 - \varepsilon$. *If* $\ell(A) \geq |A| + |B| - 2$, $\ell(A) \geq \ell(B)$ *and* $d(A \cup B) = 1$, *then* $|A + B| \geq |A| + |B| - 3 + \min(|A|, |B|)$.

*(3) If* $d = d(A) > 1$ *and* $B$ *intersects exactly* $s$ *residue classes modulo* $d$, *then* $|A+B| \geq |B| + s(|A| - 1)$. *If* $d(A \cup B) = 1$, *then* $|A + B| \geq |B| + 2(|A| - 1)$. □

The proof of D(1) is to be found in [**S1**] and of D(2) in [**L-S**]. We shall use this theorem for $A = K_i$ and $B = K_j$, $1 \leq i, j, \leq 3$. In this case we put $\varepsilon_{ij} = \varepsilon(K_i, K_j)$.

Denote by $k_i = |\mathbb{K}_i| = |K_i|$, $m_i = \min(K_i)$, $M_i = \max(K_i)$, $\ell_i = \ell(K_i)$, $d_i = d(K_i)$, $h_i = h(K_i)$, for every $1 \leq i \leq 3$. Denote by $H = h_1 + h_2 + h_3$, the number of *interior holes* of $\mathbb{K}$ and by $H^* = |K^*| - |K| = k^* - k$, the *total number of holes* of $\mathbb{K}$. By $L = L(\mathbb{K}) = \ell_1 + \ell_2 + \ell_3 = H + k - 3$, we denote the *length* of $\mathbb{K}$. For every pair $1 \leq i < j \leq 3$, we let $\mathbb{K}_{ij} = \mathbb{K}_i \cup \mathbb{K}_j$ and $d_{ij} = (d_i, d_j)$ the greatest common divisor of $d_i$ and $d_j$.

In the remaining sections the set $\mathbb{K} \subseteq \mathbb{Z}^2$ denotes a reduced set of lattice points (on three parallel lines). We note at this point two inequalities, which will be used in the paper:

$$\begin{aligned}
|2\mathbb{K}| &= |2K_1| + |K_1 + K_2| + \max(|2K_2|, |K_1 + K_3|) + |K_2 + K_3| + |2K_3| \\
&\geq (2k_1 - 1) + (k_1 + k_2 - 1) + \max(2k_2 - 1, k_1 + k_3 - 1) \\
&\qquad\qquad + (k_2 + k_3 - 1) + (2k_3 - 1),
\end{aligned}$$

which leads to

$$|2\mathbb{K}| \geq 3k_1 + 4k_2 + 3k_3 - 5, \tag{2.10}$$

$$|2\mathbb{K}| \geq 4k_1 + 2k_2 + 4k_3 - 5. \tag{2.11}$$

## 3. Some Lemmas

***Lemma 3.1.*** — *Suppose* $k_2 = 1$. *Then* $|2\mathbb{K}| \geq 4|\mathbb{K}| - 7$.

*Proof.* — Since $k_2 = 1$, inequality (2.11) yields $|2\mathbb{K}| \geq 4k_1 + 2k_2 + 4k_3 - 5 = 4k - 7$. □

***Lemma 3.2.*** — *Suppose that* $K_1$ *and* $K_3$ *lie each in one residue class modulo* $d$, $d > 1$. *Then* $|2\mathbb{K}| \geq 4|\mathbb{K}| - 7$.

*Proof.* — Since $\mathbb{K}$ is a reduced set, it follows that in $K_2$ there are at least two elements in-congruent modulo $d$. We estimate $|K_1 + K_2|$ and $|K_2 + K_3|$ by Theorem D(3) and obtain

$$|2\mathbb{K}| \geq (2k_1 - 1) + (k_2 + 2k_1 - 2) + (2k_2 - 1) + (2k_3 + k_2 - 2) + (2k_3 - 1) \geq 4k - 7. \quad \square$$

**Lemma 3.3.** — *Suppose* $k_1 = \max(k_1, k_2, k_3)$, $d_1 = d(K_1) > 1$. *Then* $|2\mathbb{K}| \geq 4|\mathbb{K}| - 7$.

*Proof.* — If $k_2 = 1$, we use Lemma 3.1. If $k_3 = 1$, then $K_1$ and $K_3$ lie each in only one residue class modulo $d_1$ and we apply Lemma 3.2. Therefore, we assume

$$\min(k_1, k_2, k_3) \geq 2. \tag{3.1}$$

We distinguish three cases:

(a) Suppose that $(d_1, d_2) = (d_1, d_3) = 1$. Theorem D(3) gives

$$\begin{aligned}|2\mathbb{K}| &\geq |2K_1| + |K_1 + K_2| + |K_1 + K_3| + |K_2 + K_3| + |2K_3| \\ &\geq (2k_1 - 1) + (k_2 + 2k_1 - 2) + (2k_1 + k_3 - 2) + (k_2 + k_3 - 1) + (2k_3 - 1) \\ &= 6k_1 + 2k_2 + 4k_3 - 7 = 4k - 6 + 2(k_1 - k_2) - 1 \geq 4k - 7.\end{aligned}$$

(b) Suppose $d = (d_1, d_2) > 1$. It follows that $(d, d_3) = 1$, because $\mathbb{K}$ is reduced. Theorem D(3) yields

$$\begin{aligned}|2\mathbb{K}| &\geq (2k_1 - 1) + (k_1 + k_2 - 1) + (2k_1 + k_3 - 2) + (2k_2 + k_3 - 2) + (2k_3 - 1) \\ &= 4k - 7 + (k_1 - k_2) \geq 4k - 7.\end{aligned}$$

(c) Suppose $(d_1, d_3) > 1$. We apply Lemma 3.2. $\square$

**Lemma 3.4.** — *Suppose* $k_2 = \max(k_1, k_2, k_3)$ *and* $d_2 = d(K_2) > 1$. *Then* $|2\mathbb{K}| \geq 4|\mathbb{K}| - 7$.

*Proof*

(a) Suppose $1 = k_3 = k_1$. Then we apply Lemma 3.2.

(b) Suppose $1 = k_3 < k_1 \leq k_2$. It is clear that $(d_1, d_2) = 1$, because $\mathbb{K}$ is reduced. Theorem D(3) implies

$$|2\mathbb{K}| \geq (2k_1 - 1) + (2k_2 + k_1 - 2) + (2k_2 - 1) + k_2 + 1 = 3k_1 + 5k_2 - 3 \geq 4k - 7.$$

We may suppose now that $k_1 \geq k_3 \geq 2$.

(c) Suppose $(d_2, d_1) = (d_2, d_3) = 1$. Using Theorem D(3) we get

$$\begin{aligned}|2\mathbb{K}| &\geq (2k_1 - 1) + (2k_2 + k_1 - 2) + (2k_2 - 1) + (2k_2 + k_3 - 2) + (2k_3 - 1) \\ &\geq (4k - 6) + (k_2 - k_1) + (k_2 - k_3) - 1 \geq 4k - 7.\end{aligned} \tag{3.2}$$

(d) Suppose $d = (d_2, d_1) > 1$ (the case $(d_2, d_3) > 1$ is similar). It is clear that $(d, d_3) = 1$ and therefore $|K_2 + K_3| \geq 2k_2 + k_3 - 2$. Moreover $|(\mathbb{K}_1 + \mathbb{K}_3) \backslash 2\mathbb{K}_2| \geq k_1$. Indeed, $K_1$ and $2K_2$ each lie in only one residue class modulo $d$ and in $K_3$ there are at least two elements, say $x < y$, non-congruent modulo $d$. Thus, $(x + K_1) \cap 2K_2 = \varnothing$ or $(y + K_1) \cap 2K_2 = \varnothing$, which yields $|2\mathbb{K}_2 \cup (\mathbb{K}_1 + \mathbb{K}_3)| \geq 2k_2 - 1 + k_1$. In conclusion,

$$\begin{aligned}|2\mathbb{K}| &\geq (2k_1 - 1) + (k_1 + k_2 - 1) + (2k_2 - 1 + k_1) + (2k_2 + k_3 - 2) + (2k_3 - 1) \\ &= 4k_1 + 5k_2 + 3k_3 - 6 \geq 4k - 6.\end{aligned}$$

□

***Lemma 3.5.*** — *Suppose that $K_2$ and $K_3$ lie each in only one residue class modulo $d$, with $d > 1$. Then $|2\mathbb{K}| > 3.5|\mathbb{K}| - 7$.*

*Proof.* — If $k_2 = 1$, we use Lemma 3.1. Suppose $k_2 \geq 2$. $K_1$ intersects at least two residue classes modulo $d$, because $\mathbb{K}$ is a reduced set. Thanks to Theorem D(3), we get

$$\begin{aligned}|2\mathbb{K}| &\geq (2k_1 - 1) + (k_1 + 2k_2 - 2) + (k_1 + 2k_3 - 2) + (k_2 + k_3 - 1) + (2k_3 - 1) \\ &= (4k_1 + 3k_2 + 5k_3 - 7). \end{aligned} \tag{3.3}$$

Take the arithmetic mean between inequalities (3.3) and (2.10). We get

$$|2\mathbb{K}| \geq 3.5k_1 + 3.5k_2 + 4k_3 - 6 > 3.5k - 7.$$

□

Next we discuss what happens if $d_2 > 1$. By the previous Lemma it is enough to study the case $k_3 \geq 2, k_1 \geq 2, d_{23} = d_{21} = 1$.

***Lemma 3.6.*** — *Suppose that $d(K_2) > 1, (d_2, d_1) = (d_2, d_3) = 1$. Then $|2\mathbb{K}| \geq 4|\mathbb{K}| - 7$.*

*Proof.* — In view of Theorem D(3), we get

$$\begin{aligned}|2\mathbb{K}| \geq |2K_1| + |K_1 + K_2| + |K_1 + K_3| + |K_2 + K_3| + |2K_3| \geq (2k_1 - 1) + \\ + (k_1 + 2k_2 - 2) + (k_1 + k_3 - 1) + (k_3 + 2k_2 - 2) + (2k_3 - 1) = 4k - 7.\end{aligned}$$

□

***Lemma 3.7.*** — *If $d_2 = 1, d_1 > 1, d_3 > 1$, then $|2\mathbb{K}| \geq 4|\mathbb{K}| - 7$.*

*Proof.* — We apply Theorem D(3) and we get

$$\begin{aligned}|2\mathbb{K}| &\geq |2K_1| + |K_1 + K_2| + |2K_2| + |K_2 + K_3| + |2K_3| \\ &\geq (2k_1 - 1) + (k_2 + 2k_1 - 2) + (2k_2 - 1) + (k_2 + 2k_3 - 2) + (2k_3 - 1) \geq 4k - 7.\end{aligned}$$

□

*Conclusion.* — Lemmas 3.1-3.7 and inequality $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$ ensure that $k_2 \geq 2$, $d_2 = d(K_2) = 1$. Indeed, if $d_2 > 1$, then Lemma 3.6 yields $(d_2, d_3) > 1$ or $(d_2, d_1) > 1$ and this leads to a contradiction, in view of Lemma 3.5. We obtained that $d_2 = 1$. By Lemma 3.7, $d_1$ and $d_3$ cannot be simultaneously greater than one. Suppose that $d_2 = d_3 = 1$, $d_1 > 1$. Lemma 3.3 shows that $k_1 \neq \max(k_1, k_2, k_3)$. Similarly, one has $k_3 \neq \max(k_1, k_2, k_3)$, if $d_2 = d_1 = 1$, $d_3 > 1$. In consequence, one of the following situations holds

$$(\alpha) \quad d_1 = d_2 = d_3 = 1,\ k_2 \geq 2, \tag{3.4}$$

$$(\beta) \quad d_2 = d_3 = 1,\ d_1 > 1,\ k_1 \neq \max(k_1, k_2, k_3),\ k_2 \geq 2, \tag{3.5}$$

$$(\gamma) \quad d_2 = d_1 = 1,\ d_3 > 1,\ k_3 \neq \max(k_1, k_2, k_3),\ k_2 \geq 2. \tag{3.6}$$

We end Section 3, by proving a lemma which will be used several times in the sequel.

***Lemma 3.8.*** — *Suppose* $\max(h_1,h_2,h_3) \le \min(k_1,k_2,k_3) - 2$. *If*

$$|2\mathbb{K}| \le 4|\mathbb{K}| - 7, \tag{3.7}$$

*then*

*(a)* $|2\mathbb{K}| \ge (2|\mathbb{K}| - 1) + 2\ell_1 + 2\ell_3$ *and* $|2\mathbb{K}| \ge (2|\mathbb{K}| - 1) + \ell_1 + 2\ell_2 + \ell_3$,

*(b)* $|2\mathbb{K}| \ge (\frac{10}{3}|\mathbb{K}| - 5) + \frac{5}{3}H = \frac{5}{3}(|\mathbb{K}| + L)$,

*(c)* $|2\mathbb{K}| \ge (2|\mathbb{K}| - 1) + \frac{4}{3}L^*$.

*Proof.* — It is clear that $\ell_i \le 2k_i - 3, \max(\ell_i,\ell_j) \le k_i + k_j - 3$, for every $1 \le i,j \le 3$. Applying Theorem D(1) we obtain $|K_i + K_j| \ge k_i + k_j - 1 + \max(h_i,h_j)$. First, we estimate $|2\mathbb{K}|$ by using $2K_1,\ K_1+K_2,\ K_1+K_3,\ K_2+K_3,\ 2K_3$. We can write

$$\begin{aligned} |2\mathbb{K}| &\ge (2k_1 - 1 + h_1) + (k_1 + k_2 - 1 + \max(h_1,h_2)) + (k_1 + k_3 - 1 + \max(h_1,h_3)) \\ &\qquad + (k_2 + k_3 - 1 + \max(h_2,h_3)) + (2k_3 - 1 + h_3) \\ &= 4k_1 + 2k_2 + 4k_3 - 5 + h_1 + h_3 \qquad\qquad (3.8)\\ &\qquad + \max(h_1,h_2) + \max(h_2,h_3) + \max(h_3,h_1) \\ &\ge \begin{cases} 4k_1 + 2k_2 + 4k_3 - 5 + H + 2\max(h_1,h_2,h_3), & \text{if } h_2 \ne \max(h_1,h_2,h_3). \\ 4k_1 + 2k_2 + 4k_3 - 5 + 2H - \min(h_1,h_2,h_3), & \text{if } h_2 = \max(h_1,h_2,h_3). \end{cases} \\ &\ge 4k_1 + 2k_2 + 4k_3 - 5 + \frac{5}{3}H. \end{aligned}$$

Thus, $|2\mathbb{K}| \ge (2|\mathbb{K}|-1) + 2\ell_1 + 2\ell_3$. Moreover, inequality (b) is also true, if $k \ge 3k_2$. Second, using $2K_2$ instead of $K_1 + K_3$ we get

$$\begin{aligned} |2\mathbb{K}| &\ge (2k_1 - 1 + h_1) + (k_1 + k_2 - 1 + \max(h_1,h_2)) + (2k_2 - 1 + h_2) \\ &\qquad + (k_2 + k_3 - 1 + \max(h_2,h_3)) + (2k_3 - 1 + h_3) \\ &= 3k_1 + 4k_2 + 3k_3 - 5 + h_1 + h_2 + h_3 + \max(h_1,h_2) + \max(h_2,h_3) \\ &\ge \begin{cases} 3k_1 + 4k_2 + 3k_3 - 5 + H + 2\max(h_1,h_2,h_3), & \text{if } h_2 = \max(h_1,h_2,h_3). \\ 3k_1 + 4k_2 + 3k_3 - 5 + 2H - \min(h_1,h_2,h_3), & \text{if } h_2 \ne \max(h_1,h_2,h_3). \end{cases} \\ &\ge 3k_1 + 4k_2 + 3k_3 - 5 + \frac{5}{3}H. \end{aligned}$$

Thus, $|2\mathbb{K}| \ge (2|\mathbb{K}|-1) + \ell_1 + 2\ell_2 + \ell_3$. Moreover, inequality (b) is also true, if $k \le 3k_2$.

We prove now inequality (c).

*First Case.* — Suppose $[m_2, M_2] \supseteq \Big[\frac{1}{2}(m_1 + m_3), \frac{1}{2}(M_1 + M_3)\Big]$.

It is clear that $L^* = L = \ell_1 + \ell_2 + \ell_3$ and in this case inequality (c) follows from (a), in view of $2\ell_2 \ge \ell_1 + \ell_3$. We could have used (b). Indeed,

$$|2\mathbb{K}| \ge (\frac{10}{3}|\mathbb{K}| - 5) + \frac{5}{3}H \ge (\frac{10}{3}|\mathbb{K}| - 5) + \frac{4}{3}H = (2|\mathbb{K}| - 1) + \frac{4}{3}L = (2|\mathbb{K}| - 1) + \frac{4}{3}L^*.$$

*Second Case.* — Suppose $[m_2, M_2] \subseteq \left[\frac{1}{2}(m_1+m_3), \frac{1}{2}(M_1+M_3)\right]$.

It is clear that

$$L^* = \ell_1 + \frac{1}{2}(\ell_1+\ell_3) + \ell_3 = \frac{3}{2}(\ell_1+\ell_3),\ \frac{4}{3}L^* = 2\ell_1 + 2\ell_3.$$

Inequality (c) follows from (a). Actually, (3.8) shows that a sharper inequality is true:

$$\begin{aligned}|2\mathbb{K}| &\geq 4k_1 + 2k_2 + 4k_3 - 5 + 2h_1 + 2h_3 + \max(h_1,h_2,h_3)\\ &= (2k-1) + 2\ell_1 + 2\ell_3 + \max(h_1,h_2,h_3)\\ &\geq (2k-1) + 2\ell_1 + 2\ell_3 = (2k-1) + \frac{4}{3}L^*.\end{aligned}$$

*Third Case.* — Suppose $m_2 < \frac{1}{2}(m_1+m_3) \leq M_2 < \frac{1}{2}(M_1+M_3)$. Put

$$\delta = \frac{m_1+m_3}{2} - m_2.$$

Define

$$K_2^- = K_2 \cap \left[m_2, \frac{m_1+m_3}{2}\right), \quad k_2^- = |K_2^-|. \tag{3.9}$$

We improve inequality (3.8) by taking into account

$$|2\mathbb{K}_2 \setminus (\mathbb{K}_1 + \mathbb{K}_3)| \geq |2K_2^-| \geq (2k_2^- - 1).$$

One has $k_2^- \geq \delta - h_2$ and therefore (3.8) shows that

$$\begin{aligned}|2\mathbb{K}| &\geq \Big(|2\mathbb{K}_{13}| + |K_2+K_1| + |K_2+K_3|\Big) + |2\mathbb{K}_2 \setminus (\mathbb{K}_1+\mathbb{K}_3)|\\ &\geq \Big(4k_1 + 2k_2 + 4k_3 - 5 + h_1 + h_3 + \max(h_1,h_3) + 2h_2\Big) + (2k_2^- - 1)\\ &\geq 4k_1 + 2k_2 + 4k_3 - 6 + \frac{3}{2}(h_1+h_3) + 2\delta.\end{aligned} \tag{3.10}$$

If $\frac{2}{3}\delta \geq \frac{h_1+h_3}{2} + 1$, then inequality (c) is proved, because (3.10) ensures

$$\begin{aligned}|2\mathbb{K}| &\geq (4k_1 + 2k_2 + 4k_3 - 5) + 2h_1 + 2h_3 + \frac{4}{3}\delta\\ &= (2k-1) + 2\ell_1 + 2\ell_3 + \frac{4}{3}\delta\\ &= (2k-1) + \frac{4}{3}\Big(\ell_1 + \frac{1}{2}(\ell_1+\ell_3) + \delta + \ell_3\Big)\\ &= (2k-1) + \frac{4}{3}L^*.\end{aligned}$$

Now we may suppose that $\frac{2}{3}\delta < \frac{h_1+h_3}{2} + 1$. First of all, note that

$$2\delta - 1 \leq k_2 - 1 \leq \ell_2. \tag{3.11}$$

Indeed, in view of (3.10) one has

$$|2\mathbb{K}| > (4k_1 + 2k_2 + 4k_3 - 6) + 3(\frac{2}{3}\delta - 1) + 2\delta = (4k_1 + 2k_2 + 4k_3 - 9) + 4\delta \tag{3.12}$$

and if $2\delta \geq k_2 + 1$ we would obtain $|2\mathbb{K}| > 4k - 7$, in contradiction with (3.7).

We estimate now $|2\mathbb{K}_2\backslash(\mathbb{K}_1+\mathbb{K}_3)|$. It is clear that $m_2+(K_2\cap[m_2,m_1+m_3-m_2))$ is included in $2\mathbb{K}_2\backslash(\mathbb{K}_1+\mathbb{K}_3)$. The length of $[m_2,m_1+m_3-m_2)$ is exactly $2\delta$ and in view of inequality (3.11) we obtain $|K_2\cap[m_2,m_1+m_3-m_2)|\geq 2\delta-h_2$. Therefore

$$|2\mathbb{K}_2\backslash(\mathbb{K}_1+\mathbb{K}_3)|\geq 2\delta-h_2. \tag{3.13}$$

As in (3.10), we improve inequality (3.8) by taking into account $2\mathbb{K}_2\backslash(\mathbb{K}_1+\mathbb{K}_3)$. One has

$$\begin{aligned}|2\mathbb{K}| &\geq |2\mathbb{K}_{13}|+|K_2+K_1|+|K_2+K_3|+|2\mathbb{K}_2\backslash(\mathbb{K}_1+\mathbb{K}_3)|\\ &\geq (4k_1+2k_2+4k_3-5+2h_1+2h_3+h_2)+(2\delta-h_2)\\ &\geq (2k-1)+2\ell_1+2\ell_3+\frac{4}{3}\delta=(2k-1)+\frac{4}{3}L^*.\end{aligned}$$

□

In the remaining part of the proof, we shall distinguish *three main cases* according to $k_2=\min(k_1,k_2,k_3)$, $k_2=\max(k_1,k_2,k_3)$, $k_3<k_2<k_1$.

## 4. First Case : $k_2=\min(k_1,k_2,k_3)$

***Theorem 4.1***. — *Suppose $k_2\leq k_3\leq k_1$. If $|\mathbb{K}+\mathbb{K}|<3.5|\mathbb{K}|-7$, then $k_2\geq 2$ and*
*(i)* $d_1=d_2=d_3=1$ *and* $\max(h_1,h_2,h_3)\leq k_2-2$.
*(ii)* $|2\mathbb{K}|\geq\left(\frac{10}{3}|\mathbb{K}|-5\right)+\frac{5}{3}H=\frac{5}{3}(|\mathbb{K}|+L)$.
*(iii)* $|2\mathbb{K}|\geq(2|\mathbb{K}|-1)+\frac{4}{3}L^*$.

*Proof.* — In view of $k_2\leq k_3\leq k_1$, it is enough to prove only (i), because assertions (ii) and (iii) are direct consequences of Lemma 3.8 and inequality $\max(h_1,h_2,h_3)\leq k_2-2$. Using $k_1=\max(k_1,k_2,k_3)$, equations (3.4), (3.5), (3.6) and the small doubling hypothesis we deduce that $k_2\geq 2$, $d_1=1$, $d_2=1, d_3\geq 1$.

**1.** *We show that* $d_3=1$.
Suppose that $d_3>1$. By Theorem D(3) we have $|2\mathbb{K}|\geq(2k_1-1)+(k_1+k_2-1)+(k_1+2k_3-2)+(k_2+2k_3-2)+(2k_3-1)=4k-6+2(k_3-k_2)-1\geq 4k-7$.

**2.** *We show that* $\max(h_1,h_2,h_3)\leq k_2-2$.
Suppose that $\max(h_2,h_i)\geq k_2-1$, for $i=1$ or $i=3$. Using Theorem D, one has $|K_2+K_i|\geq\min\left(k_i+2k_2-3,k_i+k_2-1+\max(h_i,h_2)\right)\geq(k_i+k_2-1)+(k_2-2)$. In consequence, we improve (2.11) to $|2\mathbb{K}|\geq 4k_1+3k_2+4k_3-7$. Take the arithmetic mean between (2.10) and the previous inequality. We obtain $|2\mathbb{K}|\geq 3.5k-6>3.5k-7$, which contradicts (4.1). □

## 5. Second Case : $k_2=\max(k_1,k_2,k_3)$

***Theorem 5.1***. — *Suppose $k_3\leq k_1\leq k_2$ and $|2\mathbb{K}|<3.5|\mathbb{K}|-7$. Then, $k_3\geq 2$ and*
*(a)* $\max(h_1,h_2,h_3)\leq k_3-2$.
*(b)* $|2\mathbb{K}|\geq(\frac{10}{3}|\mathbb{K}|-5)+\frac{5}{3}H=\frac{5}{3}(|\mathbb{K}|+L)$.
*(c)* $|2\mathbb{K}|\geq(2|\mathbb{K}|-1)+\frac{4}{3}L^*$.

*Proof.* — Inequalities (b) and (c) follow from Lemma 3.8 and assertion (a). Therefore, we need to prove only $\max(h_1, h_2, h_3) \leq k_3 - 2$. Lemma 3.4 implies that $d_2 = 1$.

**1.** *We show that* $d_1 = 1$.

Suppose $d_1 \geq 2$ and prove that $|2\mathbb{K}| > 3.5k - 7$, which contradicts our small doubling hypothesis. If $k_3 = 1$, then $K_1$ and $K_3$ lie each in only one residue class modulo $d_1$ and we use Lemma 3.2. If $k_3 \geq 2$ and $d_3 \geq 2$, we apply Lemma 3.7. We may assume now that $k_3 \geq 2$, $d_2 = d_3 = 1$ and estimate $|K_1 + K_2|$ by Theorem D(3):

$$\begin{aligned}|2\mathbb{K}| &\geq |2K_1| + |K_1 + K_2| + |2K_2| + |K_2 + K_3| + |2K_3| \\ &\geq (2k_1 - 1) + (k_2 + 2k_1 - 2) + (2k_2 - 1) + (k_2 + k_3 - 1) + (2k_3 - 1) \\ &\geq 4k_1 + 4k_2 + 3k_3 - 6 = (4k - 6) - k_3 \geq \frac{11}{3}k - 6 > 3.5k - 7.\end{aligned}$$

**2.** *We show that* $\max(h_1, h_2, h_3) \leq k_3 - 2$.

Suppose that $\max(h_1, h_2, h_3) \geq k_3 - 1$. We use this inequality in order to improve (2.10) by $k_3 - 2$ and thus obtain

$$|2\mathbb{K}| \geq 3k_1 + 4k_2 + 4k_3 - 7 = (4k - 7) - k_1 \geq 3.5k - 7 + \frac{k_3}{2} > 3.5k - 7, \tag{5.1}$$

in contradiction with the small doubling hypothesis.

If $k_3 = 1$ holds, then clearly (5.1) is true. Suppose $k_3 \geq 2$.

(i) If $h_2 \geq k_3 - 1$, then $|K_2 + K_3| \geq (k_2 + k_3 - 1) + (k_3 - 2)$, by using Theorem D(2).

(ii) If $h_1 \geq k_3 - 1$, then $|2K_1| \geq (2k_1 - 1) + \min(h_1, k_1 - 2) \geq (2k_1 - 1) + (k_3 - 2)$, thanks to Theorem D.

(iii) If $h_3 \geq k_3 - 1, d_3 > 1$, then $|K_2 + K_3| \geq (k_2 + k_3 - 1) + (k_3 - 1)$, by Theorem D(3). Finally, if $h_3 \geq k_3 - 1$, $d_3 = 1$, then $|2K_3| \geq (2k_3 - 1) + (k_3 - 2)$, due to Theorem D(2). The proof of Theorem 5.1 is now complete. □

## 6. Third Case : $k_3 \leq k_2 \leq k_1$

***Theorem 6.1.*** — *Suppose* $k_3 \leq k_2 \leq k_1$ *and* $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$. *Then,*

$$|2\mathbb{K}| \geq (2|\mathbb{K}| - 1) + \frac{4}{3}L^*.$$

This theorem is a consequence of lemmas 6.1, 6.2 and 6.3.

***Lemma 6.1.*** — *Suppose* $k_3 \leq k_2 \leq k_1$, $\max(h_1, h_2) \geq k_2 - 1$. *Then* $|2\mathbb{K}| \geq 3.5|\mathbb{K}| - 7$.

*Proof.* — Using Lemma 3.3, we deduce that $d_1 = 1$. We estimate $|K_1 + K_2|$ by Theorem D(1),(2). One has

$$|K_1 + K_2| \geq \min\Big(k_1 + k_2 - 1 + \max(h_1, h_2), k_1 + 2k_2 - 3\Big) \geq (k_1 + k_2 - 1) + (k_2 - 3).$$

Inequality (2.11) becomes $|2\mathbb{K}| \geq 4k_1 + 3k_2 + 4k_3 - 7$. Taking the arithmetic mean between this inequality and (2.10) we get $|2\mathbb{K}| \geq 3.5k - 6 > 3.5k - 7$. □

***Lemma 6.2.*** — *Suppose* $k_3 \leq k_2 \leq k_1$, $\max(h_1, h_2) \leq k_2 - 2$ *and* $\ell_2 \geq \ell_1$ *or* $\ell_3 \geq \ell_2$. *If* $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$, *then* $\max(h_1, h_2, h_3) \leq k_3 - 2$ *and* $|2\mathbb{K}| \geq (2k - 1) + \frac{4}{3}L^*$.

*Proof.* — We shall show that $|2\mathbb{K}| \geq 4k_1 + 3k_2 + 3k_3 - 5 + H$, which yields $|2\mathbb{K}| \geq 3.5k - 7 + H + 2 - \frac{1}{2}k_3$. This proves $\max(h_1, h_2, h_3) \leq H < \frac{1}{2}k_3 - 2$ and Lemma 6.2 follows thanks to Lemma 3.8.

(i) Assume first that $\ell_2 \geq \ell_1$. We get $|K_1 + K_2| \geq (2k_1 - 1)$ and thus

$$\begin{aligned} |2(\mathbb{K}_2 \cup \mathbb{K}_3)| &\leq |2\mathbb{K}| - (|2K_1| + |K_1 + K_2|) \\ &\leq |2\mathbb{K}| - (2k_1 - 1) - (2k_1 - 1) \\ &= |2\mathbb{K}| - (4k_1 - 2) \\ &\leq 4k_2 + 4k_3 - 7, \end{aligned}$$

in view of the small doubling property of $\mathbb{K}$. Since $d_2 = 1$, Theorem S yields

$$|2(\mathbb{K}_2 \cup \mathbb{K}_3)| \geq 3k_2 + 3k_3 - 3 + h_2 + h_3,$$

and thus

$$\begin{aligned} |2\mathbb{K}| &\geq (2k_1 - 1 + h_1) + (k_1 + k_2 - 1 + \max(h_1, h_2)) + (3k_2 + 3k_3 - 3 + h_2 + h_3) \\ &\geq (3k_1 + 4k_2 + 3k_3 - 5) + h_1 + 2h_2 + h_3 \\ &= (3k - 4) + H + \ell_2 \\ &\geq (4k_1 + 3k_2 + 3k_3 - 5) + H. \end{aligned}$$

(ii) Assume $\ell_3 \geq \ell_2$. We get $|K_2 + K_3| \geq 2k_2 - 1$ and thus

$$\begin{aligned} |2(\mathbb{K}_1 \cup \mathbb{K}_3)| &\leq |2\mathbb{K}| - (|K_1 + K_2| + |K_2 + K_3|) \\ &\leq |2\mathbb{K}| - (k_1 + k_2 - 1) - (2k_2 - 1) \\ &\leq |2\mathbb{K}| - (4k_2 - 2) \\ &\leq 4k_1 + 4k_3 - 7, \end{aligned}$$

by the small doubling property of $\mathbb{K}$. Since $d_1 = 1$, Theorem S gives $|2(\mathbb{K}_1 \cup \mathbb{K}_3)| \geq 3k_1 + 3k_3 - 3 + h_1 + h_3$ and thus

$$\begin{aligned} |2\mathbb{K}| &\geq |2(\mathbb{K}_1 \cup \mathbb{K}_3)| + |K_1 + K_2| + |K_2 + K_3| \geq \\ &\geq (3k_1 + 3k_3 - 3 + h_1 + h_3) + (k_1 + k_2 - 1 + \max(h_1, h_2)) + (2k_2 - 1) \\ &\geq (4k_1 + 3k_2 + 3k_3 - 5) + H. \end{aligned}$$

□

**Lemma 6.3.** — *Suppose $k_3 \leq k_2 \leq k_1$, $\max(h_1, h_2) \leq k_2 - 2$ and $\ell_3 \leq \ell_2 \leq \ell_1$. If $|2\mathbb{K}| < 3.5|\mathbb{K}| - 7$, then $|2\mathbb{K}| \geq (2k - 1) + \frac{4}{3}L^*$.*

*Proof.* — **(I)** We begin the proof by obtaining an upper bound for $\ell_3$, see (6.6), and by showing in (6.7), (6.8) that we may estimate $|2(\mathbb{K}_2 \cup \mathbb{K}_3)|$ and $|2(\mathbb{K}_1 \cup \mathbb{K}_3)|$ by using Theorem S(1). In the same time, we shall obtain (6.12) below, an inequality which will be used several times in the proof.

The hypothesis $\max(h_1, h_2) \leq k_2 - 2$ ensures that

$$\ell_1 \leq k_1 + k_2 - 3 \leq 2k_1 - 3,\ d_1 = 1, \tag{6.1}$$

$$\ell_2 \leq 2k_2 - 3 \leq k_1 + k_2 - 3,\ d_2 = 1. \tag{6.2}$$

Note that $\ell_3 \le \ell_i$, for $i = 1$ and $i = 2$ give

$$\begin{aligned}|K_i + K_3| &\ge |m_3 + K_i| + \left|M_3 + \Big(K_i \cap (M_i - \ell_3, M_i]\Big)\right| \\ &\ge k_i + \ell_3 - h_i = (k_i + k_3 - 1) + h_3 - h_i. \end{aligned} \tag{6.3}$$

Applying (6.3) and Theorem D(1) for $|2K_1|, |K_1 + K_2|$, inequalities (6.1), (6.2) yield

$$\begin{aligned}|2\mathbb{K}| &\ge |2K_1| + |K_1 + K_2| + |K_1 + K_3| + |K_2 + K_3| + |2K_3| \\ &\ge (2k_1 - 1 + h_1) + (k_1 + k_2 - 1 + \max(h_1, h_2)) + ((k_1 + k_3 - 1) \\ &\quad + \max(0, h_3 - h_1)) + ((k_2 + k_3 - 1) + \max(0, h_3 - h_2)) + (2k_3 - 1),\end{aligned}$$

and thus

$$|2\mathbb{K}| \ge (4k_1 + 2k_2 + 4k_3 - 5) + 2h_3, \tag{6.4}$$

$$|2\mathbb{K}| \ge (4k_1 + 2k_2 + 4k_3 - 5) + \max(h_1, h_2) + h_3. \tag{6.5}$$

We claim that

$$h_3 \le k_2 - 2, \ \ell_3 \le k_2 + k_3 - 3. \tag{6.6}$$

On the contrary, suppose that $h_3 \ge k_2 - 1$. Inequality (6.4) gives $|2\mathbb{K}| \ge 4k - 7 > 3.5k - 7$, a contradiction. By a similar argument, the small doubling property and inequality (6.5) lead to

$$h_2 + h_3 \le k_2 + k_3 - 3, \ h_1 + h_3 \le k_1 + k_3 - 3, \tag{6.7}$$

which shows that

$$|2(\mathbb{K}_2 \cup \mathbb{K}_3)| \ge 3k_2 + 3k_3 - 3 + h_2 + h_3, \ \ |2(\mathbb{K}_1 \cup \mathbb{K}_3)| \ge 3k_1 + 3k_3 - 3 + h_1 + h_3, \tag{6.8}$$

in view of $d_1 = d_2 = 1$ and Theorem S(1). We are now able to deduce

$$\begin{aligned}|2\mathbb{K}| &\ge |2K_1| + |K_1 + K_2| + |2(\mathbb{K}_2 \cup \mathbb{K}_3)| \\ &\ge (2k_1 - 1 + h_1) + (k_1 + k_2 - 1 + \max(h_1, h_2)) + (3k_2 + 3k_3 - 3 + h_2 + h_3) \\ &= (3k_1 + 4k_2 + 3k_3 - 5) + h_1 + \max(h_1, h_2) + h_2 + h_3.\end{aligned} \tag{6.9}$$

If $\max(h_1, h_2, h_3) \le k_3 - 2$, Lemma 6.3 follows from Lemma 3.8. Therefore, we have to examine only the case

$$\max(h_1, h_2, h_3) \ge k_3 - 1. \tag{6.10}$$

**(II)** *We prove (6.12), inequality which will be repeatedly used.*

In order to obtain (6.12), we need one more lower bound for $|2\mathbb{K}|$ (see (6.11) below). We use (6.10) and consider two cases :

(a) On the one hand, if $\max(h_2, h_3) \ge k_3 - 1$, then $|K_2 + K_3| \ge k_2 + 2k_3 - 2$. Indeed, if $\ell_2 \ge k_2 + k_3 - 2$, then $\varepsilon_{23} = 0$ thanks to (6.6); using Theorem D(1),(2) we get $|K_2 + K_3| \ge \min(k_2 + 2k_3 - 2, k_2 + k_3 - 1 + \max(h_2, h_3)) \ge k_2 + 2k_3 - 2$. If $\ell_2 \le k_2 + k_3 - 3$, then $|K_2 + K_3| \ge k_2 + k_3 - 1 + \max(h_2, h_3) \ge k_2 + 2k_3 - 2$, by Theorem D(1). Therefore, in each of these two cases, $k_2 + 2k_3 - 2$ is a lower bound for $|K_2+K_3|$. We may estimate $|K_2+K_1|$ by Theorem D(1), because of (6.1) and (6.2). Finally, in view of (6.8) one has

$$\begin{aligned}|2\mathbb{K}| &\geq |2(\mathbb{K}_1\cup\mathbb{K}_3)|+|K_2+K_1|+|K_2+K_3|\\ &\geq (3k_1+3k_3-3+h_1+h_3)+(k_1+k_2-1+\max(h_1,h_2))+(k_2+2k_3-2)\\ &\geq (4k_1+2k_2+4k_3-6)+h_1+\max(h_1,h_2)+h_3+k_3.\end{aligned}$$

(b) On the other hand, if $\max(h_2,h_3)\leq k_3-2$ and $h_1\geq k_3-1$, then we estimate $|2K_1|,|K_1+K_2|,|K_2+K_3|,|2K_3|$ by Theorem D(1) and $|K_1+K_3|$ by Theorem D(2), for $\varepsilon_{13}=0$. We get

$$\begin{aligned}|2\mathbb{K}| &\geq |2K_1|+|K_1+K_2|+|K_1+K_3|+|K_2+K_3|+|2K_3|\\ &\geq (2k_1-1+h_1)+(k_1+k_2-1+\max(h_1,h_2))+(k_1+2k_3-2)+\\ &\qquad +(k_2+k_3-1+\max(h_2,h_3))+(2k_3-1+h_3)\geq\\ &\geq (4k_1+2k_2+4k_3-6)+h_1+\max(h_1,h_2)+\max(h_2,h_3)+h_3+k_3.\end{aligned}$$

Thus, in both cases (a) and (b), we obtain

$$|2\mathbb{K}|\geq (4k_1+2k_2+4k_3-6)+h_1+\max(h_1,h_2)+h_3+k_3. \tag{6.11}$$

Taking the arithmetic mean between (6.9) and (6.11) we obtain

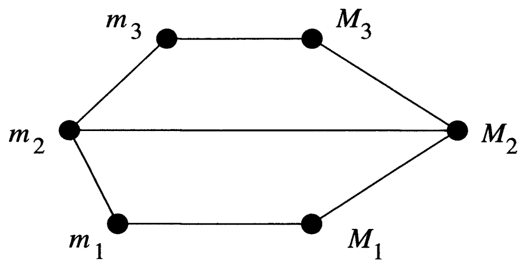
$$|2\mathbb{K}|\geq (3.5k-7)+\frac{1}{2}\Big(2h_1+2\max(h_1,h_2)+h_2+2h_3+k_3+3-k_2\Big).$$

Applying the small doubling property, we deduce immediately that

$$2h_1+2\max(h_1,h_2)+h_2+2h_3+k_3+4\leq k_2. \tag{6.12}$$

As in Lemma 3.8, we shall distinguish at this point three situations, depending on the relative position of $[m_2,M_2]$ and $[\frac{m_1+m_3}{2},\frac{M_1+M_3}{2}]$.

*First Case.* — $m_2\leq\frac{m_1+m_3}{2}\leq\frac{M_1+M_3}{2}\leq M_2$.

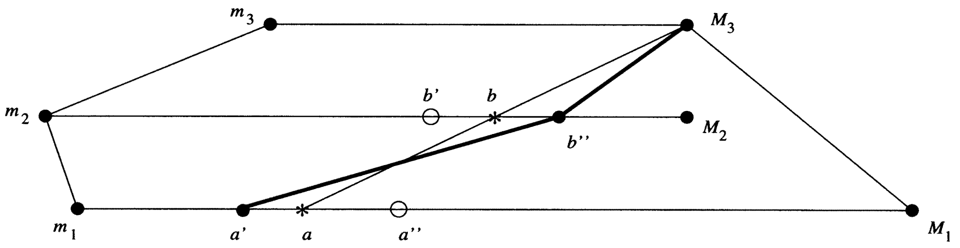



We already proved in (6.9) that $|2\mathbb{K}|\geq(2k-1)+\ell_1+2\ell_2+\ell_3$, and this implies $|2\mathbb{K}|\geq(2k-1)+\frac{4}{3}L^*$, in view of

$$\ell_1+2\ell_2+\ell_3-\frac{4}{3}L^*=(\ell_1+2\ell_2+\ell_3)-\frac{4}{3}(\ell_1+\ell_2+\ell_3)=\frac{2}{3}\Big(\ell_2-\frac{\ell_1+\ell_3}{2}\Big)\geq 0.$$

□

*Second Case.* — $m_2 \le \frac{m_1+m_3}{2} \le M_2 \le \frac{M_1+M_3}{2}$.




As usual, put $\delta = \frac{m_1+m_3}{2} - m_2 \ge 0$.

**(i)** *We split* $\mathbb{K}$ *into two subsets,* $\mathbb{K}'$ *and* $\mathbb{K}''$ *and get a lower estimate of* $|2\mathbb{K}|$ *by adding* $|2\mathbb{K}'|$ *and* $|2\mathbb{K}''|$.

Let us take a line $l$ which intersects $(x_2 = 0)$ at $(0, a)$, $(x_2 = 1)$ at $(1, b)$, $(x_2 = 2)$ at $(2, M_3)$. In the sequel we shall prove that we may choose $l$ such that

$$m_1 \le a \le M_1 \text{ and } \frac{1}{2}(m_1 + m_3) + \frac{\ell_3}{2} \le b \le M_2. \tag{6.13}$$

Take $b' \le b''$ two consecutive elements of $K_2$ such that $b' \le b \le b''$. Take $a' \le a''$ two consecutive elements of $K_1$ such that $a' < a \le a''$. Define

$$K_1' = K_1 \cap [m_1, a'], \quad K_2' = K_2 \cap [m_2, b''], \quad K_3' = K_3, \tag{6.14}$$
$$K_1'' = K_1 \cap [a', M_1], \quad K_2'' = K_2 \cap [b'', M_2], \quad K_3'' = \{M_3\}. \tag{6.15}$$

It will be shown in step (v) bellow, that we may choose $a$ and $b$ such that

$$\ell_1' \le 2k_1' - 3, \ \max(\ell_1', \ell_2') \le k_1' + k_2' - 3, \ \ell_2' + \ell_3' \le 2k_2' + 2k_3' - 5, \tag{6.16}$$
$$\ell_1'' \le 2k_1'' - 3, \ \max(\ell_1'', \ell_2'') \le k_1'' + k_2'' - 3. \tag{6.17}$$

Now Lemma 6.3 follows easily. Indeed, we estimate $|2\mathbb{K}|$ by adding $|2\mathbb{K}'|$ and $|2\mathbb{K}''|$ and paying attention to the points counted twice:

$$2\mathbb{K}_2' \cap (\mathbb{K}_1'' + \mathbb{K}_3'') \quad \text{and} \quad 2a',\ a' + b'',\ b'' + M_3,\ 2M_3.$$

Thus, if we denote by $x = |2\mathbb{K}_2' \cap (\mathbb{K}_1'' + \mathbb{K}_3'')|$, then

$$\begin{aligned}
|2\mathbb{K}| &\geq |2\mathbb{K}'| + |2\mathbb{K}''| - \Big(4 + |2\mathbb{K}_2' \cap (\mathbb{K}_1'' + \mathbb{K}_3'')\Big)| \\
&\geq |2K_1'| + |K_1' + K_2'| + |2(\mathbb{K}_2' \cup \mathbb{K}_3')| \\
&\quad + |2K_1''| + |K_1'' + K_2''| + |K_1'' + K_3''| + |K_2'' + K_3''| + |2K_3''| - (4 + x) \\
&\geq \Big[(\ell_1' + k_1') + (\ell_2' + k_1') + (2k_2' + 2k_3' - 1 + \ell_2' + \ell_3')\Big] \\
&\quad + \Big[(\ell_1'' + k_1'') + (\ell_1'' + k_2'') + k_1'' + k_2'' + 1\Big] - 4 - x \\
&= \Big[2(k_1' + k_2' + k_3') - 1 + \ell_1' + 2\ell_2' + \ell_3'\Big] \\
&\quad + \Big[2(k_1'' + k_2'' + k_3'') - 1 + 2\ell_1''\Big] - 4 - x \\
&= 2(k_1' + k_1'' + k_2' + k_2'' + k_3' + k_3'') - 2 + \ell_1' + 2\ell_2' + \ell_3' + 2\ell_1'' - 4 - x \\
&= 2k + \ell_1' + 2\ell_2' + \ell_3' + 2\ell_1'' - x.
\end{aligned} \tag{6.18}$$

In the last equality we used $k_i' + k_i'' = k_i + 1$, for $1 \leq i \leq 3$. It is clear that

$$x = |2\mathbb{K}_2' \cap (\mathbb{K}_1'' + \mathbb{K}_3'')| \leq 1 + |[2b, 2b'']| = 2 + 2(b'' - b). \tag{6.19}$$

In view of the collinearity of $a$ and $b$ we have $(a - m_1) + \ell_3 = 2b - (m_1 + m_3)$ and thus

$$\begin{aligned}
\ell_1' + 2\ell_2' + \ell_3' + 2\ell_1'' &= \ell_1' + 2\Big((b'' - b) + (b - \frac{m_1 + m_3}{2}) + (\frac{m_1 + m_3}{2} - m_2)\Big) \\
&\quad + \ell_3' + 2\ell_1'' \\
&= \ell_1' + 2(b'' - b) + 2\Big(b - \frac{m_1 + m_3}{2}\Big) + 2\Big(\frac{m_1 + m_3}{2} - m_2\Big) \\
&\quad + \ell_3' + 2\ell_1'' \\
&= \ell_1' + 2(b'' - b) + \Big((a - m_1) + \ell_3\Big) + 2\delta + \ell_3' + 2\ell_1'' \\
&= \ell_1' + 2(b'' - b) + \Big(\ell_1' + (a - a') + \ell_3\Big) + 2\delta + \ell_3' + 2\ell_1'' \\
&= 2(\ell_1' + \ell_1'') + (\ell_3 + \ell_3') + 2\delta + 2(b'' - b) + (a - a') \\
&= 2\ell_1 + 2\ell_3 + 2\delta + 2(b'' - b) + (a - a').
\end{aligned} \tag{6.20}$$

Thus, using (6.19), (6.20) in (6.18), we conclude that

$$|2\mathbb{K}| \geq 2k - 2 + 2\ell_1 + 2\ell_3 + 2\delta + (a - a') \geq (2k - 1) + 2\ell_1 + 2\ell_3 + \frac{4}{3}\delta.$$

**(ii)** *We put inequalities (6.16), (6.17) in a slightly different form:*

$$h_1' \leq k_1' - 2,\ h_1' \leq k_2' - 2,\ h_2' \leq k_1' - 2,\ h_2' + h_3' \leq k_2' + k_3' - 3 = k_2' + k_3 - 3,$$
$$h_1'' \leq k_1'' - 2,\ h_1'' \leq k_2'' - 2,\ h_2'' \leq k_1'' - 2.$$

Consequently, it is enough to choose $a$ and $b$ such that (6.13) and the following four inequalities are true:

$$k_1'' \geq \max(h_1, h_2) + 2,\ k_2'' \geq h_1 + 2, \tag{6.21}$$

$$k_1' \geq \max(h_1, h_2) + 2,\ k_2' \geq \max(h_1 + 2, h_2 + h_3 - k_3 + 3). \tag{6.22}$$

**(iii)** *We define now the line $l$.*
To define $l$, we need only to choose $b$ as

$$b = \min\left\{M_2 - (h_1 + h_2 + 2), \frac{M_1 + M_3}{2} - \frac{\max(h_1, h_2) + h_1 + 1}{2}\right\}. \tag{6.23}$$

Using the collinearity condition $a$ is defined by

$$(a - m_1) + \ell_3 = 2b - (m_1 + m_3). \tag{6.24}$$

We shall show in step **(v)** that this choice ensures (6.13), (6.21) and (6.22). But first, we need some more estimates.
**(iv)** *We estimate $\delta$ and compare $h_1, h_2, h_3$ to $k_1$.*
(a) We prove that

$$2\delta + 2h_1 + 2\max(h_1, h_2) + 2h_3 + k_3 + 3 \leq k_2 + h_2. \tag{6.25}$$

Improve (6.11) by taking into account $|2\mathbb{K}_2 \setminus (\mathbb{K}_1 + \mathbb{K}_3)| \geq 2(\delta - h_2) - 1$. We get

$$|2\mathbb{K}| \geq (4k_1 + 2k_2 + 4k_3 - 7) + h_1 + \max(h_1, h_2) + h_3 + k_3 + 2\delta - 2h_2. \tag{6.26}$$

We take the arithmetic mean between (6.26) and (6.9) and obtain

$$|2\mathbb{K}| \geq (3.5k - 7) - \frac{1}{2}(k_2 + h_2) + \frac{1}{2}\Big(2h_1 + 2\max(h_1, h_2) + 2h_3 + k_3 + 2\delta + 2\Big).$$

In view of the small doubling property, we deduce that (6.25) holds.
(b) We prove that

$$3h_1 + 4\max(h_1, h_2) + 2h_2 + 3h_3 + k_3 + 8 \leq k_1. \tag{6.27}$$

Remark that in the Second Case, one has $\frac{\ell_1 + \ell_3}{2} > \ell_2 - \delta$. This gives $2\delta > 2\ell_2 - (\ell_1 + \ell_3)$. Thanks to (6.25), the last inequality implies

$$\Big(2\ell_2 - (\ell_1 + \ell_3)\Big) + 2h_1 + 2\max(h_1, h_2) + 2h_3 + k_3 + 3 < k_2 + h_2,$$

$$(2k_2 - k_1 - k_3) + (2h_2 - h_1 - h_3) + 2h_1 + 2\max(h_1, h_2) + 2h_3 + k_3 + 3 < k_2 + h_2,$$

$$k_2 + h_2 + h_1 + 2\max(h_1, h_2) + h_3 + 4 \leq k_1. \tag{6.28}$$

Combining inequalities (6.12) and (6.28), we obtain the desired result (6.27).

**(v)** *We prove (6.13), (6.21) and (6.22).*
We begin with (6.13). In view of the collinearity condition (6.24), we shall prove now $\frac{m_1 + m_3}{2} + \frac{\ell_3}{2} \leq b \leq M_2$, which ensures that $m_1 \leq a \leq M_1$. Since $b$ is defined by

(6.23), we have actually to check the two inequalities stated below.

$$
\begin{aligned}
(1)\quad & \Big(M_2 - (h_1 + h_2 + 2)\Big) - \Big(\frac{m_1 + m_3}{2} + \frac{\ell_3}{2}\Big) \\
&= (M_2 - \frac{m_1 + m_3}{2}) - (h_1 + h_2 + 2 + \frac{\ell_3}{2}) \\
&= (\ell_2 - \delta) - (h_1 + h_2 + \frac{k_3}{2} + \frac{h_3}{2} + 1.5) \\
&= (k_2 + h_2) - (\delta + h_1 + h_2 + \frac{k_3}{2} + \frac{h_3}{2} + 2.5) \geq 0,
\end{aligned}
$$

in view of (6.25);

$$
\begin{aligned}
(2)\quad & \left(\frac{M_1 + M_3}{2} - \frac{\max(h_1, h_2) + h_1 + 1}{2}\right) - \Big(\frac{m_1 + m_3}{2} + \frac{\ell_3}{2}\Big) \\
&= \frac{1}{2}\Big((M_1 + M_3) - (m_1 + m_3) - (\max(h_1, h_2) + h_1 + 1) - \ell_3\Big) \\
&= \frac{1}{2}\Big(\ell_1 + \ell_3 - (\max(h_1, h_2) + h_1 + 1) - \ell_3\Big) \\
&= \frac{1}{2}(k_1 - \max(h_1, h_2) - 2) \geq 0,
\end{aligned}
$$

by hypothesis $k_1 \geq k_2$ and inequality (6.12).

We estimate $k_1', k_2', k_1'', k_2''$. First of all, we verify (6.21) :

$$
\begin{aligned}
k_1'' &= |K_1 \cap [a', M_1]| \geq M_1 - a' + 1 - h_1 \geq M_1 - a + 1 - h_1 = \\
&= 2\Big(\frac{M_1 + M_3}{2} - b\Big) + 1 - h_1 \geq 2\frac{\max(h_1, h_2) + h_1 + 1}{2} + 1 - h_1 = \\
&= \max(h_1, h_2) + 2. \\
k_2'' &= |K_2 \cap [b'', M_2]| = |K_2 \cap (b, M_2]| \geq M_2 - [b] - h_2 \geq M_2 - b - h_2 \geq \\
&\geq M_2 - (M_2 - h_1 - h_2 - 2) - h_2 = h_1 + 2.
\end{aligned}
$$

Further, using (6.24) one has

$$
\begin{aligned}
k_1' &= |K_1 \cap [m_1, a']| = |K_1 \cap [m_1, a)| \geq [a] - m_1 + 1 - h_1 \geq a - m_1 - h_1 = \\
&= 2\Big(b - \frac{m_1 + m_3}{2}\Big) - \ell_3 - h_1 \geq \max(h_1, h_2) + 2,
\end{aligned}
$$

in view of the following two inequalities

$$
\begin{aligned}
(1)\quad & \left(2\Big(M_2 - h_1 - h_2 - 2 - \frac{m_1+m_3}{2}\Big) - \ell_3 - h_1\right) - \Big(\max(h_1,h_2)+2\Big) \\
&= 2\Big(M_2 - \frac{m_1+m_3}{2}\Big) - (3h_1 + 2h_2 + \ell_3 + \max(h_1,h_2) + 6) \\
&= 2(\ell_2 - \delta) - (3h_1 + 2h_2 + \ell_3 + \max(h_1,h_2) + 6) \\
&= 2k_2 - (2\delta + 3h_1 + \ell_3 + \max(h_1,h_2) + 8) \\
&= 2k_2 - (2\delta + 3h_1 + h_3 + k_3 + \max(h_1,h_2) + 7) \\
&\geq 2k_2 - (2\delta + 2h_1 + 2\max(h_1,h_2) + h_3 + k_3 + 7) \\
&= \Big((k_2+h_2) - (2\delta + 2h_1 + 2\max(h_1,h_2) + 2h_3 + k_3 + 3)\Big) + \Big(k_2 + h_3 - h_2 - 4\Big) \\
&\qquad \geq k_2 + h_3 - h_2 - 4 \geq 0, \quad \text{because of (6.25) and (6.12),}
\end{aligned}
$$

$$
\begin{aligned}
(2)\quad & \left(2\Big(\frac{M_1+M_3}{2} - \frac{\max(h_1,h_2)+h_1+1}{2} - \frac{m_1+m_3}{2}\Big) - \ell_3 - h_1\right) \\
&\qquad\qquad - \Big(\max(h_1,h_2)+2\Big) \\
&= (M_1 - m_1 + M_3 - m_3) - (2\max(h_1,h_2) + 2h_1 + \ell_3 + 3) \\
&= (\ell_1 + \ell_3) - (2\max(h_1,h_2) + 2h_1 + \ell_3 + 3) \\
&= k_1 - (h_1 + 2\max(h_1,h_2) + 4) \geq 0,
\end{aligned}
$$

due to (6.12) and $k_1 \geq k_2$.

It only remains to estimate $k_2'$. Note that

$$
\begin{aligned}
k_2' &= |K_2 \cap [m_2, b'']| \geq b'' - m_2 + 1 - h_2 \\
&\geq b - m_2 + 1 - h_2 \geq \max(h_1+2, h_2+h_3-k_3+3),
\end{aligned}
$$

because we may write the following four inequalities

$$
\begin{aligned}
(1)\quad & \Big((M_2 - h_1 - h_2 - 2) - m_2 + 1 - h_2\Big) - (h_1+2) \\
&= (M_2 - m_2) - (2h_1 + 2h_2 + 3) \\
&= \ell_2 - (2h_1 + 2h_2 + 3) \\
&= k_2 - (2h_1 + h_2 + 4) \\
&\geq 0, \quad \text{because of (6.12),}
\end{aligned}
$$

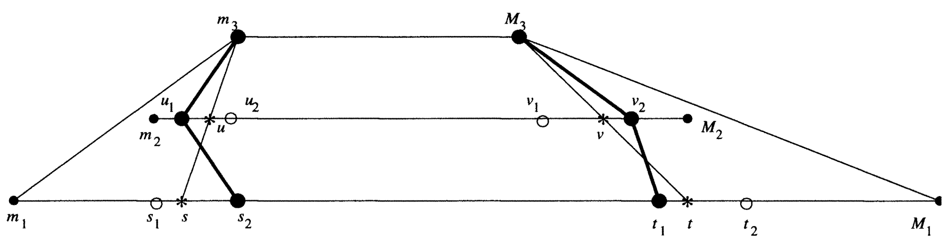
$$
\begin{aligned}
(2)\quad & \Big((M_2-h_1-h_2-2)-m_2+1-h_2\Big)-(h_2+h_3-k_3+3)\\
&= (M_2-m_2)+k_3-(h_1+3h_2+h_3+4)\\
&= \ell_2+k_3-(h_1+3h_2+h_3+4)\\
&= (k_2+k_3)-(h_1+2h_2+h_3+5)\\
&\geq 0, \quad \text{due to (6.12)},
\end{aligned}
$$

$$
\begin{aligned}
(3)\quad & \Big(\frac{M_1+M_3}{2}-\frac{\max(h_1,h_2)+h_1+1}{2}-m_2+1-h_2\Big)-(h_1+2)\\
&= \Big(\frac{M_1+M_3}{2}-m_2\Big)-\frac{1}{2}(\max(h_1,h_2)+3h_1+2h_2+3)\\
&= (\delta+\frac{\ell_1+\ell_3}{2})-\frac{1}{2}(\max(h_1,h_2)+3h_1+2h_2+3)\\
&= \frac{1}{2}\Big((k_1+k_3+h_3)-(\max(h_1,h_2)+2h_1+2h_2+5)\Big)+\delta\\
&\geq 0, \quad \text{thanks to (6.12)},
\end{aligned}
$$

$$
\begin{aligned}
(4)\quad & \Big(\frac{M_1+M_3}{2}-\frac{\max(h_1,h_2)+h_1+1}{2}-m_2+1-h_2\Big)-(h_2+h_3-k_3+3)\\
&= k_3+\Big(\frac{M_1+M_3}{2}-m_2\Big)-\frac{\max(h_1,h_2)+h_1+4h_2+2h_3+5}{2}\\
&= k_3+(\delta+\frac{\ell_1+\ell_3}{2})-\frac{\max(h_1,h_2)+h_1+4h_2+2h_3+5}{2}\\
&= k_3+\delta+\frac{1}{2}\Big((k_1+k_3)-(\max(h_1,h_2)+4h_2+h_3+7)\Big)\\
&\geq 0, \quad \text{because of (6.27)}.
\end{aligned}
$$

The proof of case 2 is now complete. □

*Third Case.* — $\frac{m_1+m_3}{2} \leq m_2 \leq M_2 \leq \frac{M_1+M_3}{2}$.




We split $\mathbb{K}$ into three sets $\mathbb{K}', \mathbb{K}'', \mathbb{K}'''$. Choose $s \leq t$ between $m_1$ and $M_1$ and $u \leq v$ between $m_2$ and $M_2$ such that the points $(0,s),(1,u),(2,m_3)$ and $(0,t),(1,v),(2,M_3)$ are collinear.

Take $s_1, s_2, t_1, t_2$ in $K_1$ such that $s_1 \leq s < s_2$, $t_1 < t \leq t_2$ and there is no point of $K_1$ in the intervals $(s_1, s), (s, s_2)$, $(t_1, t), (t, t_2)$.

Take $u_1, u_2, v_1, v_2$ in $K_2$ such that $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$ and there is no point of $K_2$ in the intervals $(u_1, u), (u, u_2)$, $(v_1, v), (v, v_2)$. Define

$$K_1' = K_1 \cap [m_1, s_2],\ K_1'' = K_1 \cap [s_2, t_1],\ K_1''' = K_1 \cap [t_1, M_1], \tag{6.29}$$

$$K_2' = K_2 \cap [m_2, u_1],\ K_2'' = K_2 \cap [u_1, v_2],\ K_2''' = K_2 \cap [v_2, M_2], \tag{6.30}$$

$$K_3' = \{m_3\},\ K_3'' = K_3,\ K_3''' = \{M_3\}. \tag{6.31}$$

It will be shown that we may choose the points $s, t, u, v$ such that

$$\ell_1' \leq 2k_1' - 3, \quad \max(\ell_1', \ell_2') \leq k_1' + k_2' - 3, \tag{6.32}$$

$$\ell_1'' \leq 2k_1'' - 3, \quad \max(\ell_1'', \ell_2'') \leq k_1'' + k_2'' - 3,\ \ell_2'' + \ell_3'' \leq 2k_2'' + 2k_3'' - 5, \tag{6.33}$$

$$\ell_1''' \leq 2k_1''' - 3, \quad \max(\ell_1''', \ell_2''') \leq k_1''' + k_2''' - 3. \tag{6.34}$$

We can easily deduce Lemma 6.3 from (6.32-34). Denote $x = |2\mathbb{K}_2' \cap (\mathbb{K}_1' + \mathbb{K}_3')| + |2\mathbb{K}_2' \cap (\mathbb{K}_1''' + \mathbb{K}_3''')|$. It is clear that

$$\begin{aligned}|2\mathbb{K}| \geq{}& |2\mathbb{K}_1'| + |\mathbb{K}_1' + \mathbb{K}_2'| + |\mathbb{K}_1' + \mathbb{K}_3'| + |\mathbb{K}_2' + \mathbb{K}_3'| + |2\mathbb{K}_3'| \\ &+ |2\mathbb{K}_1''| + |\mathbb{K}_1'' + \mathbb{K}_2''| + |2(\mathbb{K}_2'' \cup \mathbb{K}_3'')| \\ &+ |2\mathbb{K}_1'''| + |\mathbb{K}_1''' + \mathbb{K}_2'''| + |\mathbb{K}_1''' + \mathbb{K}_3'''| + |\mathbb{K}_2''' + \mathbb{K}_3'''| + |2\mathbb{K}_3'''| - 8 - x.\end{aligned} \tag{6.35}$$

Indeed, the above inequality is true, because the points $2s_2, 2t_1$, $u_1 + s_2, v_2 + t_1, u_1 + m_3, M_3 + v_2, 2m_3, 2M_3, 2\mathbb{K}_2' \cap (\mathbb{K}_1' + \mathbb{K}_3'), 2\mathbb{K}_2' \cap (\mathbb{K}_1''' \cup \mathbb{K}_3''')$ are counted twice. By Theorem D we get

$$\begin{aligned}|2\mathbb{K}| &\geq (\ell_1' + k_1') + (\ell_1' + k_2') + k_1' + k_2' + 1 \\ &\qquad + (\ell_1'' + k_1'') + (\ell_2'' + k_1'') + (2k_2'' + 2k_3'' - 1 + \ell_2'' + \ell_3'') \\ &\qquad + (\ell_1''' + k_1''') + (\ell_1''' + k_2''') + k_1''' + k_2''' + 1 - (8 + x) \\ &= (2k_1' + 2k_2' + 1 + 2\ell_1') + (2k_1'' + 2k_2'' + 2k_3'' - 1 + \ell_1'' + 2\ell_2'' + \ell_3'') \\ &\qquad + (2k_1''' + 2k_2''' + 1 + 2\ell_1''') - (8 + x) \\ &= 2(k_1' + k_1'' + k_1''') + 2(k_2' + k_2'' + k_2''') + 2k_3'' - 7 \\ &\qquad + (2\ell_1 - \ell_1'') + 2\ell_2'' + \ell_3'' - x \\ &= 2k + 1 + 2\ell_1 + 2\ell_3 + [2\ell_2'' - (\ell_1'' + \ell_3)] - x.\end{aligned} \tag{6.36}$$

We have used here $k_3'' = k_3$, $\ell_3'' = \ell_3$ and $k_i' + k_i'' + k_i''' = k_i + 2$, for $i = 1, 2$. Note that the collinearity condition gives $2(v - u) = (t - s) + \ell_3$ and thus we have

$$\begin{aligned}2\ell_2'' - (\ell_1'' + \ell_3) &= 2[(v - u) + (u - u_1) + (v_2 - v)] \\ &\qquad - [(t - s) - (t - t_1) - (s_2 - s) + \ell_3] \\ &= 2(u - u_1) + 2(v_2 - v) + (t - t_1) + (s_2 - s).\end{aligned} \tag{6.37}$$

It is clear that

$$|(\mathbb{K}_1' + \mathbb{K}_3') \cap 2\mathbb{K}_2'| \leq 1 + |[2u_1, 2u]| = 2 + 2(u - u_1),$$
$$|2\mathbb{K}_2' \cap (\mathbb{K}_1''' + K_3''')| \leq 1 + |[2v, 2v_2]| = 2 + 2(v_2 - v).$$

Therefore, $x \leq 4+2(u-u_1)+2(v_2-v)$; applying this inequality and (6.37) in (6.36), we obtain the desired lower bound:

$$|2\mathbb{K}| \geq 2k-3+2\ell_1+2\ell_3+(t-t_1)+(s_2-s) \geq (2k-1)+2\ell_1+2\ell_3. \qquad (6.38)$$

The last step in the proof is to choose $u, s, v, t$ such that (6.32-6.34) are valid.

First of all, we rewrite these inequalities in the form

$$\begin{gathered} h_1' \leq k_1'-2,\ h_1' \leq k_2'-2,\ h_2' \leq k_1'-2, \\ h_1'' \leq k_1''-2,\ h_1'' \leq k_2''-2,\ h_2'' \leq k_1''-2, \\ h_2''+h_3'' \leq k_2''+k_3''-3 = k_2''+k_3-3, \\ h_1''' \leq k_1'''-2,\ h_1''' \leq k_2'''-2,\ h_2''' \leq k_1'''-2. \end{gathered}$$

Consequently, it will be enough to find $u, s, t, v$ such that

$$k_2' \geq h_1+2,\ k_2''' \geq h_1+2, k_2'' \geq \max(h_1+2, h_2+h_3-k_3+3), \qquad (6.39)$$

$$k_1' \geq \max(h_1, h_2)+2,\ k_1'' \geq \max(h_1, h_2)+2,\ k_1''' \geq \max(h_1, h_2)+2. \qquad (6.40)$$

Define $u, v$ between $m_2$ and $M_2$ by

$$u = m_2+(h_1+h_2+2), \qquad v = M_2-(h_1+h_2+2). \qquad (6.41)$$

Take $s, t$ between $m_1$ and $M_1$ so that $(0,s),(1,u),(2,m_3)$ and $(0,t),(1,v),(2,M_3)$ are collinear. We obtain

$$v-u = \ell_2-2(h_1+h_2+2) \text{ and } t-s = 2(v-u)-\ell_3. \qquad (6.42)$$

In order to prove (6.39-40), it will suffice to estimate $k_2'$, $k_2''$, $k_2'''$, $k_1'$, $k_1''$, $k_1'''$. We begin by establishing (6.39).

$$\begin{aligned} k_2' &= |K_2 \cap [m_2, u_1]| = |K_2 \cap [m_2, u)| \geq (u-m_2)-h_2 = h_1+2. \\ k_2''' &= |K_2 \cap [v_2, M_2]| = |K_2 \cap (v_1, M_2]| \geq (M_2-v)-h_2 = h_1+2. \\ k_2'' &= |K_2 \cap [u_1, v_2]| = 2+(v-u)-h_2 = 2+\Big(\ell_2-2(h_1+h_2+2)\Big)-h_2 \\ &= k_2-2h_1-2h_2-3 \geq \max(h_1+2, h_2+h_3-k_3+3). \end{aligned}$$

Indeed, $3h_1+2h_2+5 \leq k_2$ and $2h_1+3h_2+h_3+6 \leq k_2+k_3$ follow from (6.12). Remark that $v-u = \ell_2-2(h_1+h_2+2) \geq \ell_3$, because it is equivalent to $2h_1+h_2+k_3+h_3+4 \leq k_2$, which follows again from (6.12). Therefore, we may choose $s \leq t$ between $m_1$ and $M_1$ such that the points $(2, m_3)$, $(1, u)$, $(0, s)$ and $(2, M_3), (1, v), (0, t)$ are collinear. Define $s_1 \leq s < s_2$ and $t_1 < t \leq t_2$ such that $s_1, s_2$ and $t_1, t_2$ are consecutive points in $K_1$.

We check inequalities (6.40). Note that $s-m_1 \geq 2(u-m_2)$ and $M_1-t \geq 2(M_2-v)$. Using (6.42) we have

$$\begin{aligned} k_1' &= |K_1 \cap [m_1, s_2]| \geq s_2-m_1+1-h_1 \geq (s-m_1)+1-h_1 \\ &\geq 2(u-m_2)+1-h_1 = 2(h_1+h_2+2)+1-h_1 = h_1+2h_2+5 \\ &\geq \max(h_1, h_2)+2, \end{aligned}$$

$$\begin{aligned}k_1''' &= |K_1 \cap [t_1, M_1]| \geq (M_1 - t_1 + 1) - h_1 \geq (M_1 - t) + 1 - h_1 \\ &\geq 2(M_2 - v) + 1 - h_1 = 2(h_1 + h_2 + 2) + 1 - h_1 = h_1 + 2h_2 + 5 \\ &\geq \max(h_1, h_2) + 2,\end{aligned}$$

$$\begin{aligned}k_1'' &= |K_1 \cap [s_2, t_1]| = |K_1 \cap (s,t)| \geq [t] - [s] - h_1 \geq (t - s) - 1 - h_1 \\ &= \Big(2(v-u) - \ell_3\Big) - 1 - h_1 = 2\Big((\ell_2 - 2h_1 - 2h_2 - 4) - \ell_3\Big) - 1 - h_1 \\ &= 2(k_2 - 2h_1 - h_2 - 5) - k_3 - h_3 - h_1 = 2k_2 - (5h_1 + 2h_2 + h_3 + k_3 + 10) \\ &\geq \max(h_1, h_2) + 2, \quad \text{because of } (6.12).\end{aligned}$$

□

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