EDITH LIPKIN

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SUBSET SUMS OF SETS OF RESIDUES

by

Edith Lipkin

Dedicated to Grisha Freiman, with respect and affection

Abstract. — The number m is called the critical number of a finite abelian group G, if it is the minimal natural number with the property:
for every subset A of G with $|A| \geq m, 0 \notin A$, the set of subset sums A^* of A is equal to G. In this paper, we prove the conjecture of G. Diderrich about the value of the critical number of the group G, in the case $G = \mathbb{Z}_q$, for sufficiently large q.

Let G be a finite Abelian group, $A \subset G$ such that $0 \notin A$. Let $A = \{a_1, a_2, \ldots, a_{|A|}\}$, where $|A| = \text{card}A$.

Let

$$A^* := \{x \mid x = a_1\varepsilon_1 + a_2\varepsilon_2 + \cdots + \varepsilon_{|A|}a_{|A|},\ \ \varepsilon_j \in \{0,1\},\ \ 1 \leq j \leq |A|,\ \ \sum_{j=1}^{|A|} \varepsilon_j > 0\}$$

and

$$X := \{m \in \mathbf{N} \mid \forall A \subset G, |A| \geq m \Rightarrow A^* = G\}.$$

Since $|G| - 1 \in X$, then $X \neq \varnothing$ if $|G| > 2$. The number

$$c(G) = \min_{m \in X} m$$

was introduced by George T. Diderrich in **[1]** and called the critical number of the group G.

In this note we study the magnitude of $c(G)$ in the case $G = \mathbb{Z}_q$, where $\mathbb{Z}_q$ is a group of residue classes modulo q. We set $c(q) := c(\mathbb{Z}_q)$. A survey of the problem was given by G.T. Diderrich and H.B. Mann in **[2]**.

In the case when q is a prime number John Olson **[3]** proved that

$$c(q) \leq \sqrt{4q - 3} + 1.$$

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Recently J.A. Dias da Silva and Y.O. Hamidoune [4] have found the exact value of $c(q)$ for which an estimate

$$2q^{1/2} - 2 < c(q) < 2q^{1/2}$$

is valid.

If $q = p_1 p_2$, $p_1 \geq p_2$, p_1, p_2 – prime numbers, then

$$p_1 + p_2 - 2 \leq c(G) \leq p_1 + p_2 - 1$$

as was proved by Diderrich [1].

It was proved in [2] that for $q = 2\ell$, $\ell > 1$

$$c(G) = \ell \text{ if } \ell \geq 5 \text{ or } q = 8$$

$$c(G) = \ell + 1 \text{ in all other cases.}$$

Thus, to give thorough solution for $G = \mathbb{Z}_q$ we have to find $c(q)$ when q is a product of no less than three prime odd numbers.

G. Diderrich in [1] has formulated the following conjecture:

Let G be an Abelian group of odd order $|G| = ph$ where p is the least prime divisor of $|G|$ and h is a composite number. Then

$$c(G) = p + h - 2.$$

We prove here this conjecture for the case $G = \mathbb{Z}_q$ for sufficiently large q.

Theorem 1. — *There exists a positive integer q_0 that if $q > q_0$ and $q = ph$, $p > 2$, where p is the least prime divisor of q and h is a composite number, we have*

$$c(q) = p + h - 2.$$

To prove Theorem 1 we need the following results.

Lemma 1. — *Let $A = \{a_1, a_2, \ldots, a_{|A|}\} \subset N, N = \{1, 2, \ldots, \ell\}, S(A) = \sum_{i=1}^{|A|} a_i$, $A(g) = \{x \in A | x \equiv 0 (\text{mod } g)\}$, $B(A) = \frac{1}{2}\left(\sum_{i=1}^{|A|} a_i^2\right)^{1/2}$. Suppose that for some $\varepsilon > 0$ and $\ell > \ell_1(\varepsilon)$ we have $|A| \geq \ell^{2/3+\varepsilon}$ and*

$$|A(g)| \leq |A| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}}, \tag{1}$$

for every $g \geq 2$. Then for every M for which

$$|M - \frac{1}{2}S(A)| \leq B(A)$$

we have $M \subset A^*$.

Lemma 2. — *Let ε be a constant, $0 < \varepsilon \leq 1/3$. There exists $\ell_0 = \ell_0(\varepsilon)$ such that for every $\ell \geq \ell_0$ and every set of integers $A \subset [1, \ell]$, for which*

$$|A| \geq \ell^{\frac{2}{3}+\varepsilon}, \tag{2}$$

the set A^* contains an arithmetic progression of ℓ elements and difference d satisfying the condition

$$d < \frac{2\ell}{|A|}. \tag{3}$$

We cited as Lemma 1 the Proposition 1.3 on page 298 of [**5**].

Proof of Lemma 2. — Let us first assume that A fulfills the condition (1) in Lemma 1. Since we have

$$B(A) \geq \frac{1}{2}\sqrt{\sum_{i=1}^{|A|} i^2} > \frac{1}{2}\sqrt{\frac{|A|^3}{3}} > \frac{1}{2\sqrt{3}}\ell^{1+\frac{3}{2}\varepsilon}$$

and every M from the interval $(\frac{1}{2}S(A) - B(A), \frac{1}{2}S(A) + B(A))$ belong to A^*, there exists an arithmetic progression in A^* of the length $2B(A) > \ell$, if $\ell > \ell_0 = \ell_1(\varepsilon)$.

Now we study the case when A does not satisfy (1). We can then find an integer $g_1 \geq 2$ such that $B_1 \subset A = A_0$ and B_1 contains those elements of A_0 which are divisible by g_1 and for the set $A_1 = \{x/g_1 | x \in B_1 \text{ and } x \equiv 0 (\text{mod } g_1)\}$ we have

$$|A_1| > |A_0| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}} .$$

Suppose that this process was repeated s times and numbers $g_1, g_2, \ldots, g_s$ were found and sets $A_1, A_2, \ldots, A_s$ defined inductively, B_j being a subset of A_{j-1} containing those elements of A_{j-1} which are divisible by g_j and

$$A_j = \{x/g_j | x \in B_j \text{ and } x \equiv 0 (\text{mod } g_j)\}$$

so that we have

$$|A_j| > |A_{j-1}| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}}, \quad j = 1, 2, \ldots, s .$$

From

$$|A_s| \geq |A_{s-1}| - \ell^{\frac{2}{3}+\frac{\varepsilon}{2}} > |A| - s\ell^{\frac{2}{3}+\frac{\varepsilon}{2}}$$

and

$$\ell_s = \left[\frac{\ell_{s-1}}{q_s}\right] \leq \frac{\ell}{2^s}$$

it follows that

$$|A_s| \geq \frac{1}{2}|A| \geq \frac{1}{2}\ell^{\frac{2}{3}+\frac{\varepsilon}{2}} > \ell_s^{\frac{2}{3}+\varepsilon}. \tag{4}$$

The condition (2) of Lemma 2 for A_s is verified, for some sufficiently large s the condition (3) is fulfilled and thus A_s^* contains an interval

$$\left(\frac{1}{2}S(A_s) - B(A_s), \frac{1}{2}S(A_s) + B(A_s)\right).$$

We have, in view of (4),

$$\begin{aligned} B(A_s) &\geq \frac{1}{2}\sqrt{\sum_{i=1}^{|A_s|} i^2} > \frac{1}{2}\sqrt{\frac{|A_s|^3}{3}} \\ &\geq \frac{1}{4\sqrt{6}}\ell^{1+\frac{3}{2}\varepsilon} > \ell. \end{aligned} \tag{5}$$

We have shown that A_s^* contains an arithmetic progression of length ℓ and difference $d = g_1 g_2 \cdots g_s$, and thus A^* has the same property.

We now prove (2). From

$$\ell_s = \left[\frac{\ell}{d}\right], \quad \ell_s \geq |A_s| \geq \frac{1}{2}|A|$$

we have

$$\left[\frac{\ell}{d}\right] \geq \frac{1}{2}|A|$$

or

$$d \leq \frac{2\ell}{|A|}.$$

Lemma 2 is proved.

***Lemma 3* (M. Chaimovich [6]).** — *Let $B = \{b_i\}$ be a multiset, $B \subset \mathbb{Z}_q$. Suppose that for every $s \geq 2$, s dividing q, we have*

(6) $$|B \backslash B(s)| \geq s - 1.$$

There exists $F \subset B$ for which

$$|F| \leq q - 1,$$
$$F^* = \mathbb{Z}_q.$$

Proof of Theorem 1. — Let $q = p_1 p_2 \cdots p_k$, $k \geq 4$, $p = p_1 \leq p_2 \leq \cdots \leq p_k$. We have

(7) $$p^k \leq q \Rightarrow p \leq q^{1/4} .$$

Let $A \subset \mathbb{Z}_q$ be such that $0 \notin A$ and

(8) $$|A| \geq \frac{q}{p} + p - 2;$$

we have to prove that $A^* = \mathbb{Z}_p$.

From (7) and (8) we get

(9) $$|A| > \frac{q}{p} \geq q^{3/4} .$$

Let us consider some divisor d of q, and denote by A_d a multiset A viewed as a multiset of residues mod d. Let us show that for every δ dividing d the number of residues in A_d which are not divisible by δ satisfies the condition of Lemma 3.

The number of residues in $\mathbb{Z}_q$ which are divisible by δ is equal to q/δ. Therefore the number of such residues in A (which are all different) is not larger than $q/\delta - 1$, because $0 \notin A$.

From this reasoning and from (7) we get the estimate

(10) $$|A_d \backslash A(\delta)| \geq |A| - \left(\frac{q}{\delta} - 1\right) \geq \frac{q}{p} + p - 2 - \frac{q}{\delta} + 1 = \frac{q}{p} + p - \left(\frac{q}{\delta} + \delta\right) + \delta - 1 .$$

The function $x + q/x$ is decreasing on the segment $[1, \sqrt{q}]$.

The least divisor of q is equal to p, and the maximal one to q/p. Therefore

$$p \le \delta \le \frac{q}{p} .$$

If $p \le \delta \le \sqrt{q}$, we have

$$\frac{q}{p} + p \ge \frac{q}{\delta} + \delta . \tag{11}$$

In the case $\sqrt{q} \le \delta \le \frac{q}{p}$, let $\rho = \frac{q}{\delta}$. Then $\delta = \frac{q}{\rho}$, $\sqrt{q} \le \frac{q}{\rho} \le \frac{q}{p}$ and $p \le \rho \le \sqrt{q}$ and we have

$$\frac{q}{p} + p \ge \frac{q}{\rho} + \rho = \delta + \frac{q}{\delta} . \tag{12}$$

From (11) and (12) it follows from (10) that we have

$$|A_d \backslash A(\delta)| \ge \delta - 1 . \tag{13}$$

Let us apply the Lemma 3 to A_d. Condition (13) is condition (6) of Lemma 3. Therefore there exists $F_d \subset A_d$ such that $|F_d| \le d-1$ and $F_d^* = \mathbb{Z}_d$.

Viewing F_d as a set of residues mod q, let

$$A' = \bigcup_{\substack{d/q \\ p \le d < q^{1/3}}} F_d .$$

It is well known that the number of divisors $d(q) = O(q^\varepsilon)$ for every $\varepsilon > 0$ so that

$$|A'| < q^{\frac{1}{3}+\varepsilon}$$

for sufficiently large q.

Take now $A'' = A \backslash A'$. Take the least positive integer from each class of residues of the set A'' and denote this set by $\widehat{A}''$. We have $\widehat{A}'' \subset [1, q-1]$. We set $\ell = q$ and see that all conditions of Lemma 1 are valid for $\widehat{A}''$. Thus, $(\widehat{A}'')^*$ contains an arithmetic progression $\mathcal{L}$ with a length q and a difference Δ such that

$$\Delta < \frac{2q}{q^{\frac{3}{4}}} = 2q^{1/4} . \tag{14}$$

If $(\Delta, q) = 1$ then $(A'')^* = \mathbb{Z}_q$. Suppose that $D = (\Delta, q) > 1$. Then $\mathcal{L}$ (and therefore $(\widehat{A}'')^*$ which contains $\mathcal{L}$) contains the residues of $\mathbb{Z}_q$ which are divisible by D. If $\mathbb{Z}_D$ is a system of residues mod q representing a system of all residues mod D/q, then $(\widehat{A}'')^* + \mathbb{Z}_D = \mathbb{Z}_q$. But $F_D \subset A'$ and $F_D^* = \mathbb{Z}_D$. Thus

$$A^* \supset (\widehat{A}'')^* + (A')^* = \mathbb{Z}_q .$$

Theorem 1 is proved in the case $k \ge 4$.

Now we have to study the case when q is a product of three primes. Let $q = p_1 p_2 p_3$, $p = p_1 \le p_2 \le p_3$. Suppose that for some positive ε we have $p < p^{\frac{1}{3+\varepsilon}}$. The proof may be completed in a similar way to what was done.

In the general case we can use a stronger result than Lemma 2. Namely, the formulation of Lemma 2 is valid if in (2) we replace the number $2/3$ in the exponent by $1/2$ (see G. Freiman [**7**] and A. Sárkőzy [**8**]). So, in the case of q being a product of three primes, we can use this stronger version and prove Theorem 1.

As we have seen, the version of Lemma 1 with the exponent 2/3 was sufficient in the majority of cases. It is preferable to use this version, for its proof is much simpler than the case 1/2. Secondly, in the case 2/3 estimates of error terms have been obtained explicitly by M. Chaimovich. It provides us with the possibility to get an explicit range of validity for Theorem 1.

Lemma 4. — *Define a function of ℓ in the following manner:*

(15) $$m_0(\ell) = \Big(\frac{12}{\pi^2}\Big)^{1/3} \ell^{2/3} (\log \ell + 1/6)^{1/3} \Big(2 - \frac{4\gamma}{3}\Big)^{1/3}$$

where $\gamma = \Big(\frac{12}{\pi^2}\ \frac{\log \ell + 1/6}{\ell}\Big)^{1/3}$.

Then for $\ell > 155$ a subset sum of each subset $A \subset \{1, 2, \dots, \ell\}$ with $|A| = m > m_0(\ell)$ contains an arithmetic progression of cardinality ℓ.

Simplifying (15) we can take

$$m_0(\ell) = 1.3\, \ell^{2/3} (\log \ell + 1/6)^{1/3}.$$

In the case of four or more primes in a representation of q we have to verify an inequality

(16) $$\ell^{3/4} > 1.3\, \ell^{2/3} (\log \ell + 1/6)^{1/3}$$

which is fulfilled for

$$\ell \geq 3000.$$

In some special cases we can give better estimates. For example, if $p = 3$ we have $m > q/3$ and instead of (16) we have

$$\ell/3 > 1.3\, \ell^{2/3} (\log \ell + 1/6)^{1/3},$$

$$\ell > 64 (\log \ell + 1/6)$$

which is valid for

$$\ell \geq 500.$$

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E. Lipkin, School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel-Aviv, Israel