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LOCAL TAME LIFTING FOR GL(n) II: WILDLY RAMIFIED SUPERCUSPIDALS

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Abstract. — Let F be a non-Archimedean local field with finite residue field of characteristic p. An irreducible representation σ of the Weil group $\mathcal{W}_F$ of F is called wildly ramified if $\dim \sigma$ is a power of p and $\sigma \not\cong \chi \otimes \sigma$ for any unramified quasicharacter $\chi \neq 1$ of $\mathcal{W}_F$. We write $\mathcal{G}_m^{\mathrm{wr}}(F)$ for the set of equivalence classes of such representations of dimension p^m. An irreducible supercuspidal representation π of $\mathrm{GL}_n(F)$ is wildly ramified if n is a power of p and $\pi \not\cong \pi \otimes (\chi \circ \det)$ for any unramified quasicharacter $\chi \neq 1$ of $F^\times$. We write $\mathcal{A}_m^{\mathrm{wr}}(F)$ for the set of equivalence classses of such representations of $\mathrm{GL}_{p^m}(F)$. In this paper, we do two things. First, we propose a definition of a base change map $\boldsymbol{l}_{K/F} : \mathcal{A}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(K)$ for any finite tame extension K/F. The construction is explicit and local, being based on the classification of supercuspidal representations of $\mathrm{GL}_n(F)$ (due to C. Bushnell and P.C. Kutzko) and a partial definition of (non-Galois) tame base change (due to the authors). The results apply to local fields F of positive characteristic. When F has characteristic zero and K/F is cyclic of degree prime to p we show that this map coincides with base change in the sense of Arthur and Clozel. Second, when F has characteristic zero, we construct a canonical bijection $\boldsymbol{\pi}_m^F : \mathcal{G}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(F)$, for each m. We show that this has many of the properties demanded of a Langlands correspondence.

Recently, M. Harris and R. Taylor have announced a proof of the local Langlands conjecture for $\mathrm{GL}_n(F)$, using a global geometric method. This implies the existence of a canonical bijection $\mathcal{L}_m : \mathcal{G}_m^{\mathrm{wr}}(F) \to \mathcal{A}_m^{\mathrm{wr}}(F)$. If $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, there is an unramified quasicharacter χ_σ of $\mathcal{W}_F$ of finite order dividing p^m such that $\boldsymbol{\pi}_m(\sigma) = \mathcal{L}_m(\sigma \otimes \chi_\sigma)$.

We expect that the methods of this paper will lead to another proof of the local Langlands conjecture for GL_n.

***Résumé* (Changement de base local modéré pour $\mathrm{GL}(n)$ II : représentations supercuspidales sauvages)**

Soit F un corps local non archimédien à corps résiduel fini de caractéristique p. Une représentation irréductible σ du groupe de Weil $\mathcal{W}_F$ de F est dite sauvagement ramifiée si $\dim\sigma$ est une puissance de p et $\sigma \not\cong \chi \otimes \sigma$ pour tout quasicaractère non ramifié $\chi \neq 1$ de $\mathcal{W}_F$. Notons $\mathcal{G}_m^{\mathrm{wr}}(F)$ l'ensemble des classes d'isomorphie de telles représentations de dimension p^m. Une représentation irréductible supercuspidale π de $\mathrm{GL}_n(F)$ est dite sauvagement ramifiée si n est une puissance de p et $\pi \not\cong \pi \otimes (\chi \circ \det)$ pour tout quasicaractère non ramifié $\chi \neq 1$ de $F^\times$. Notons $\boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ l'ensemble des classes d'isomorphie de telles représentations de $\mathrm{GL}_{p^m}(F)$. Dans cet article, nous faisons deux choses. En premier, nous proposons une définition d'une application de changement de base $\boldsymbol{l}_{K/F} : \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F) \to \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$, où K/F est une extension finie modérée. La méthode est locale et explicite, basée sur la classification des représentations supercuspidales due à C. Bushnell et Ph. Kutzko et une définition partielle du changement de base modéré (non galoisien), due aux auteurs. Les arguments s'étendent à des corps locaux de caractéristique non nulle. Si le corps F est de caractéristique nulle et que K/F est cyclique de degré premier à p, nous montrons que cette application coïncide avec le changement de base au sens de J. Arthur et L. Clozel. Deuxièmement, dans le cas où F est de caractéristique nulle, nous construisons une bijection canonique $\boldsymbol{\pi}_m^F : \mathcal{G}_m^{\mathrm{wr}}(F) \to \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ qui possède beaucoup des propriétés exigées d'une correspondance de Langlands.

Récemment, M. Harris et R. Taylor ont annoncé une preuve, par voie globale et géométrique, des conjectures de Langlands pour $\mathrm{GL}_n(F)$. Leurs résultats impliquent l'existence d'une bijection canonique $\mathcal{L}_m : \mathcal{G}_m^{\mathrm{wr}}(F) \to \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$. Pour $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, il existe un quasicaractère non ramifié χ_σ de $\mathcal{W}_F$, d'ordre fini divisant p^m, tel que $\boldsymbol{\pi}_m(\sigma) = \mathcal{L}_m(\sigma \otimes \chi_\sigma)$.

Nous espérons que les méthodes du présent article mèneront à une preuve alternative des conjectures locales de Langlands pour GL_n.

Contents

INTRODUCTION

1. Let F denote a non-Archimedean local field with finite residue field of characteristic p. For the time being, we assume that F has characteristic zero, and so is a finite extension of $\mathbb{Q}_p$. We fix an algebraic closure $\overline{F}/F$ of F and let $\mathcal{W}_F$ denote the Weil group of $\overline{F}/F$.

For an integer $n \geqslant 1$, let $\mathcal{G}_n(F)$ denote the set of equivalence classes of irreducible continuous (complex) representations of $\mathcal{W}_F$ of dimension n, and $\mathcal{A}_n(F)$ the set of equivalence classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. The local Langlands conjecture for GL_n **[30]** predicts, for each n, the existence of a canonical bijection

$$\boldsymbol{\lambda}_n : \mathcal{G}_n(F) \xrightarrow{\approx} \mathcal{A}_n(F)$$

satisfying an extensive list of properties (see also **[25]**). In particular, $\boldsymbol{\lambda}_1$ is the bijection implied by local class field theory.

2. We are here concerned with the problem of constructing $\boldsymbol{\lambda}_n$ in a special, but crucial and rather subtle, case.

For an integer $m \geqslant 0$, let $\mathcal{G}_m^{\mathrm{wr}}(F)$ denote the set of $\sigma \in \mathcal{G}_{p^m}(F)$ which remain irreducible on restriction to the wild inertia subgroup of $\mathcal{W}_F$. (Equivalently, $\boldsymbol{\sigma}$ has dimension p^m and $\sigma \not\cong \sigma \otimes \chi$ for any unramified quasicharacter $\chi \neq 1$ of $\mathcal{W}_F$.) From the point of view of Galois theory, the set $\bigcup_m \mathcal{G}_m^{\mathrm{wr}}(F)$ contains the "difficult" representations of $\mathcal{W}_F$, including the primitive ones.

On the other hand, let $\mathcal{A}_m^{\mathrm{wr}}(F)$ denote the set of $\pi \in \mathcal{A}_{p^m}(F)$ with the property that π is not equivalent to the representation $\chi\pi : g \mapsto \chi(\det g)\pi(g)$ for any unramified quasicharacter $\chi \neq 1$ of $F^{\times}$.

The aim of this paper is to produce, for each $m \geqslant 0$ and each finite field extension $F/\mathbb{Q}_p$, a canonical bijection

$$\boldsymbol{\pi}_m : \mathcal{G}_m^{\mathrm{wr}}(F) \xrightarrow{\approx} \mathcal{A}_m^{\mathrm{wr}}(F).$$

The bijection we construct exhibits many of the properties demanded of a Langlands correspondence $\boldsymbol{\lambda}_{p^m}$. In particular, when $m = 0$, the map $\boldsymbol{\pi}_0$ is that given by class field theory. For $m \geqslant 0$, $\boldsymbol{\pi}_m$ is natural with respect to topological isomorphisms of the base field. It respects contragredience and takes determinants to central quasicharacters. It is compatible with twisting by quasicharacters. Its deepest properties concern local constants: for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, the Deligne-Langlands local constant $\boldsymbol{\varepsilon}(\sigma, s, \psi_F)$ **[39]** equals the Godement-Jacquet local constant $\boldsymbol{\varepsilon}(\boldsymbol{\pi}_m(\sigma), s, \psi_F)$ **[15]**. (Here, ψ_F is a non-trivial continuous character of the additive group of F.)

3. Since the completion of the original version of this paper (November 1997), there has been considerable progress in this area. This stems from **[16]** in which, following ideas of Drinfeld and Carayol, Harris produces a canonical map $\boldsymbol{\sigma}_n$ from $\mathcal{A}_n(F)$ to the set of equivalence classes of *semisimple* n-dimensional representations of $\mathcal{W}_F$. An argument in **[6]** shows that, for each n, $\boldsymbol{\sigma}_n$ is in fact a bijection $\mathcal{A}_n(F) \to \mathcal{G}_n(F)$. Let us set $\boldsymbol{\lambda}_n = \boldsymbol{\sigma}_n^{-1}$. In **[18]**, it is shown that the family $\{\boldsymbol{\lambda}_n\}$ has all the properties required of a Langlands correspondence. In particular, it preserves local constants of pairs:

$$\boldsymbol{\varepsilon}(\sigma_1 \otimes \sigma_2, s, \psi_F) = \boldsymbol{\varepsilon}(\pi_1 \times \pi_2, s, \psi_F),$$

for $\sigma_i \in \mathcal{G}_{n_i}(F)$ and $\pi_i = \boldsymbol{\lambda}_{n_i}(\sigma_i)$. Here, the second $\boldsymbol{\varepsilon}$ is the local constant of **[24]**, **[35]**. (This property, in the case $p \nmid n_1 n_2$, was established earlier in **[17]**.) However, the construction of $\boldsymbol{\sigma}_n$ in **[16]** is geometric and makes extensive use of global constructions. It gives no information whatsoever about the nature of the correspondence, especially the way it interacts with the structure theory of supercuspidals in **[9]**. The rôle of the present paper has thus become to make the correspondences $\boldsymbol{\lambda}_n$ more explicit, at least in the important case to hand.

A critical question in this regard is therefore whether our family of maps $\{\boldsymbol{\pi}_m\}$ preserves local constants of pairs. We do not answer that question here, but we do show that it preserves *conductors* of pairs: if $\sigma_i \in \mathcal{G}_{m_i}^{\mathrm{wr}}(F)$ and $\pi_i = \boldsymbol{\pi}_{m_i}(\sigma_i)$, $i = 1, 2$, then the exponent $f(\sigma_1 \otimes \sigma_2)$ of the Artin conductor of the tensor product $\sigma_1 \otimes \sigma_2$ is equal to the conductor $f(\pi_1 \times \pi_2)$ of the pair (π_1, π_2).

From results here and properties of the $\{\boldsymbol{\lambda}_n\}$ given in **[6]**, it is straightforward to show that, for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, *there is an unramified quasicharacter* χ_σ *of* $F^\times$

such that $\boldsymbol{\lambda}_{p^m}(\sigma) \cong \chi_\sigma \boldsymbol{\pi}_m(\sigma)$. *Moreover,* χ_σ *has finite order, strictly dividing* p^m. In particular, $\boldsymbol{\lambda}_p = \boldsymbol{\pi}_1$ on $\mathcal{G}_1^{\mathrm{wr}}(F)$.

We will return elsewhere to the exact relation between $\boldsymbol{\pi}_m$ and $\boldsymbol{\lambda}_{p^m}$.

4. Our approach is based on the fact that the representations in $\mathcal{G}_m^{\mathrm{wr}}(F)$ exhibit a fairly uniform structure. We proceed by uncovering similar structures in $\mathcal{A}_m^{\mathrm{wr}}(F)$ and constructing the map $\boldsymbol{\pi}_m$ to preserve these. First, $\mathcal{G}_m^{\mathrm{wr}}(F)$ has a canonical subset $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ as follows: a representation $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$ lies in $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ if and only if there is a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_m$$

with each F_{i+1}/F_i cyclic and totally ramified of degree p, and a quasicharacter χ of $F_m^\times$, such that σ is induced from the representation of $\mathcal{W}_{F_m}$ afforded by χ.

There is an analogous subset $\mathcal{A}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ of $\mathcal{A}_m^{\mathrm{wr}}(F)$: a representation π lies in this set if and only if there is a tower of fields $F = L_0 \subset L_1 \subset \cdots \subset L_m$ and a quasicharacter ξ of $L_m^\times$, with each L_{i+1}/L_i cyclic and totally ramified of degree p, such that

$$\pi = \boldsymbol{i}_{L_1/F} \circ \boldsymbol{i}_{L_2/L_1} \circ \cdots \circ \boldsymbol{i}_{L_m/L_{m-1}}(\xi).$$

Here, $\boldsymbol{i}$ denotes the operation of *automorphic induction,* as in **[22]**. One knows (**[20]**, **[6]** 3.8) that, in the notation above,

$$\sigma = \mathrm{Ind}_{\mathcal{W}_{F_m}}^{\mathcal{W}_F}(\chi) \longmapsto \boldsymbol{i}_{F_1/F} \circ \boldsymbol{i}_{F_2/F_1} \circ \cdots \circ \boldsymbol{i}_{F_m/F_{m-1}}(\chi)$$

induces a bijection ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m$ between $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$ and $\mathcal{A}\boldsymbol{c}_m^{\mathrm{wr}}(F)$. The maps ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m$ exhibit a multitude of desirable properties; in particular, they *preserve local constants of pairs.* Our bijection $\boldsymbol{\pi}_m$ is to be an extension of ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m$:

$$\boldsymbol{\pi}_m(\sigma) = {}^{\boldsymbol{c}}\boldsymbol{\pi}_m(\sigma), \quad \sigma \in \mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F).$$

5. In general, given a representation $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, there is a finite, tamely ramified, field extension K/F such that the restriction $\sigma_{K/F}$ of σ to $\mathcal{W}_K$ lies in $\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$. There is a canonical choice of the extension K/F (up to isomorphism), such that the degree $[K{:}F]$ is prime to p. We specify $\boldsymbol{\pi}_m(\sigma)$ in terms of K/F and the representation ${}^{\boldsymbol{c}}\boldsymbol{\pi}_m(\sigma_{K/F})$.

If the tame extension K/F is *cyclic,* there is an operation on the other side analogous to the restriction process $\sigma \mapsto \sigma_{K/F}$. *Base change,* in the sense of **[1]**, gives a map

$$\boldsymbol{b}_{K/F} : \mathcal{A}_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}_m^{\mathrm{wr}}(K);$$

one can easily extend the definition of $\boldsymbol{b}_{K/F}$ to the case where the tame extension K/F is Galois (as in [**5**] 16.5). If the tame extension K/F attached to $\sigma \in \mathcal{G}^{\mathrm{wr}}_m(F)$ is Galois, then global considerations demand that the representation $\pi = \boldsymbol{\pi}_m(\sigma)$ satisfy ${}^{c}\boldsymbol{\pi}(\sigma_{K/F}) = \boldsymbol{b}_{K/F}(\pi)$. Also, the central quasicharacter of π must correspond to $\det\sigma$ via class field theory. These two conditions determine π uniquely. The real problem is that the extension K/F given by σ will *not,* in general, be Galois. We thus need to define a suitable operation

$$\boldsymbol{l}_{K/F} : \mathcal{A}^{\mathrm{wr}}_m(F) \longrightarrow \mathcal{A}^{\mathrm{wr}}_m(K),$$

for tame extensions K/F of degree prime to p, which generalizes base change. *The explicit construction of the map $\boldsymbol{l}_{K/F}$ is the main point of the paper.*

6. In fact, we shall define the algebraic tame lifting map $\boldsymbol{l}_{K/F}$ for an arbitrary finite tame extension K of any non-Archimedean local field F, characteristic zero or not.

The map $\boldsymbol{l}_{K/F}$ is transitive with respect to the field extension K/F and natural with respect to topological isomorphisms of K. It respects contragredience and twisting with quasicharacters. It "preserves" central quasicharacters, Godement-Jacquet local constants and conductors of pairs, in the sense that its effect on these objects is precisely that predicted by the Langlands conjectures. We give a complete account of the image and the fibres of $\boldsymbol{l}_{K/F}$. We further show that, for K/F cyclic of p-prime degree and F of characteristic 0, we have $\boldsymbol{l}_{K/F}(\pi) = \boldsymbol{b}_{K/F}(\pi)$ for every $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$ and every $m \geqslant 0$. This refines some of the more general results of [**5**], and gives a complete local algebraic description of base change in these circumstances.

A full list of those properties of $\boldsymbol{l}_{K/F}$ needed for this paper is given in §1. The proofs of these occupy §§3–10.

7. Once we have these properties of $\boldsymbol{l}_{K/F}$, the construction of $\boldsymbol{\pi}_m$ is fairly easy. We take $\sigma \in \mathcal{G}^{\mathrm{wr}}_m(F)$ with associated canonical tame extension K/F as above; thus $\sigma_{K/F}$ lies in $\mathcal{G}c^{\mathrm{wr}}_m(K)$ and $p \nmid [K{:}F]$. The representation $\pi_{K/F} = {}^{c}\boldsymbol{\pi}_m(\sigma_{K/F})$ is defined. There is a unique $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$ such that $\boldsymbol{l}_{K/F}(\pi) = \pi_{K/F}$ and whose central quasicharacter corresponds to $\det\sigma$ via class field theory. We put $\pi = \boldsymbol{\pi}_m(\sigma)$.

The details of the construction of $\boldsymbol{\pi}_m$, and the deduction of its properties from those of $\boldsymbol{l}_{K/F}$, are all contained in §2.

We note in passing that, using the maps $\boldsymbol{\pi}_m$ and automorphic induction, it is an easy matter to produce a canonical bijection $\mathcal{G}_{p^n}(F) \to \mathcal{A}_{p^n}(F)$, $n \geqslant 0$.

8. It is a consequence of our constructions that, for given $\pi \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$, there exists an extension K/F of degree prime to p such that $\boldsymbol{l}_{K/F}(\pi) \in \boldsymbol{\mathcal{A}c}_m^{\mathrm{wr}}(K)$. This is not easy to prove directly: see **[27]**, **[29]** for a very detailed analysis of the case $p^m = 2$.

One knows already from **[21]** (see also the discussion in **[6]** 3.2) that there is some tame *Galois* extension K'/F such that $\boldsymbol{b}_{K'/F}(\pi) \in \boldsymbol{\mathcal{A}c}_m^{\mathrm{wr}}(K')$. This weaker result underlies everything we do in §2, and also much of **[16]**. Indeed, the proof **[6]** that Harris's map $\boldsymbol{\sigma}_n$ (as in paragraph 3 above) gives a bijection $\boldsymbol{\mathcal{A}}_n(F) \to \mathcal{G}_n(F)$ depends crucially on it.

9. As noted above in paragraph 6, our construction of $\boldsymbol{l}_{K/F}$ is purely algebraic and works equally well in positive characteristic. Unfortunately one cannot use it to produce an explicit version of the characteristic p Langlands correspondences of **[31]** (the construction of which, we note, is again geometric in nature). The reason is simply that we start from the map ${}^{c}\boldsymbol{\pi}_m$. This relies for its definition on base change (or automorphic induction) and base change is not available in positive characteristic. (There is detailed discussion of such matters in **[5]**.)

Be that as it may, our construction of $\boldsymbol{\pi}$ makes no use of base change beyond the definition and basic properties of ${}^{c}\boldsymbol{\pi}$.

Also, while our construction of $\boldsymbol{\pi}$ from ${}^{c}\boldsymbol{\pi}$ can justly claim to be quite explicit, we say nothing concerning ${}^{c}\boldsymbol{\pi}$ itself. This seems to be quite a difficult problem: again see **[27]** for a detailed examination of the case $p^m = 2$, and **[33]** for the case $m = 1$, $p \neq 2$. The explicit conductor formulæ of **[7]** might yield further information in more general cases.

10. We now review the arrangement and more technical aspects of the paper. Our definition of $\boldsymbol{l}_{K/F}$ is necessarily quite novel. Global methods, of the sort used in **[1]**, **[22]**, yield no clues as to how to handle non-Galois extensions K/F. We therefore rely on the local methods of **[9]** and **[5]** to generalize the approach of **[26]**, **[28]**. These methods have the incidental advantage of being characteristic-independent.

Let us take $\pi_F \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$, for some $m \geqslant 1$. The main results of **[9]** give a canonical presentation of π_F as an induced representation, obtained as follows. We recall that a simple stratum $[\mathfrak{A}_F, n_F, 0, \beta]$ (**[9]** 1.5 or "Preliminaries" below) in $A_F = \mathbb{M}_{p^m}(F)$ defines a pair $H^1(\beta, \mathfrak{A}_F) \subset J^1(\beta, \mathfrak{A}_F)$ of compact open subgroups of $G_F = \mathrm{GL}_{p^m}(F)$ and a distinguished finite set $\mathcal{C}(\mathfrak{A}_F, \beta)$ of abelian characters of $H^1(\beta, \mathfrak{A}_F)$. (The elements of $\mathcal{C}(\mathfrak{A}_F, \beta)$ are called *simple* characters.)

We know from **[9]** that there exists $[\mathfrak{A}_F, n_F, 0, \beta]$ such that π_F contains some $\theta_F \in \mathcal{C}(\mathfrak{A}_F, \beta)$, and that θ_F is thereby uniquely determined up to G_F-conjugacy. (Our hypothesis $\pi_F \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ is then equivalent to the field $E = F[\beta]$ being of degree p^m and totally ramified over F.) There is a unique irreducible representation η_F of $J^1(\beta, \mathfrak{A}_F)$ which contains θ_F and so η_F must also occur in π_F. The representation η_F admits extension to a representation of the group $\boldsymbol{J}_F = E^\times J^1(\beta, \mathfrak{A}_F)$ and some such extension, call it Λ_F, must occur in π_F. We then have

$$\pi_F \cong c\text{-}\mathrm{Ind}_{\boldsymbol{J}_F}^{G_F}(\Lambda_F).$$

The representation $(\Lambda_F, \boldsymbol{J}_F)$ is an instance of what we call a "central type", and it is *maximal* in the sense that the underlying field E is a maximal subfield of the ambient matrix ring A_F. A representation $\pi_F \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ thus contains a maximal central type; this central type is determined up to conjugacy by π_F and the type in turn determines π_F.

One can formulate a more general notion of central type: this we do in §3 below. One has to start with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{End}_F(V)$, where V is some finite-dimensional F-vector space, and, for technical convenience, we always assume that the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ is *principal.* A central type (attached to the given stratum) is then an irreducible representation of the group $\boldsymbol{J}(\beta, \mathfrak{A}) = F[\beta]^\times J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains some $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$.

These central types have a significant "functorial" property. Suppose we have a maximal one Λ_F attached to $[\mathfrak{A}_F, n_F, 0, \beta]$ in A_F as above. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in some $A = \mathrm{End}_F(V)$; the central type Λ_F then determines a central type Λ attached to $[\mathfrak{A}, n, 0, \beta]$, whose conjugacy class in $G = \mathrm{Aut}_F(V)$ depends only on the G_F-conjugacy class of Λ_F (i.e., only on the equivalence class of the irreducible supercuspidal representation π_F induced by Λ_F). We write θ for the simple character occurring in Λ, $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$.

Now let K/F be a finite, tamely ramified field extension. In the preceding paragraph, we take for V the F-vector space $K^{p^m} = K \otimes_F F^{p^m}$; there is then a unique choice of $[\mathfrak{A}, n, 0, \beta]$ such that $\mathfrak{A}$ is stable under conjugation by $K^\times$. We next invoke a technique borrowed from the representation theory of finite groups, namely the *Glauberman correspondence* **[14]**. We set $A_K = \mathbb{M}_{p^m}(K)$, $G_K = \mathrm{GL}_{p^m}(K)$, $\mathfrak{A}_K = \mathfrak{A} \cap A_K$. We know from **[5]** that

(i) $[\mathfrak{A}_K, n, 0, \beta]$ is a simple stratum in A_K;

(ii) $H^1(\beta, \mathfrak{A}) \cap G_K = H^1(\beta, \mathfrak{A}_K)$ and $J^1(\beta, \mathfrak{A}) \cap G_K = J^1(\beta, \mathfrak{A}_K)$;
(iii) the character $\theta_K = \theta \mid H^1(\beta, \mathfrak{A}_K)$ lies in $\mathcal{C}(\mathfrak{A}_K, \beta)$.

The Glauberman correspondence then attaches to Λ an irreducible representation $\boldsymbol{g}_{K/F}(\Lambda)$ of the group $E^\times J^1(\beta, \mathfrak{A}_K)$, and this representation contains θ_K. Let ω_F denote the central quasicharacter of π_F; there is a unique irreducible representation Λ_K of $\boldsymbol{J}_K = \boldsymbol{J}(\beta, \mathfrak{A}_K)$ which extends $\boldsymbol{g}_{K/F}(\Lambda)$ and whose restriction to $K^\times$ is a multiple of $\omega_F \circ \mathrm{N}_{K/F}$. The representation $(\Lambda_K, \boldsymbol{J}_K)$ is then a maximal central type in G_K; the induced representation

$$\pi_K = c\text{-Ind}_{\boldsymbol{J}_K}^{G_K}(\Lambda_K)$$

is an irreducible supercuspidal representation of $G_K = \mathrm{GL}_{p^m}(K)$, and we put

$$\boldsymbol{l}_{K/F}(\pi_F) = \pi_K.$$

(A minor adjustment is necessary in the case $p = 2$, but we ignore that here.)

The process $\Lambda \mapsto \Lambda_K$ given by the Glauberman correspondence is quite explicit in some cases. Suppose that K/F is either unramified or totally tamely ramified. In the first case, let ζ denote a root of unity in K, of order prime to p, such that $K = F[\zeta]$; in the second, let ζ be a prime element of K such that $\zeta^{[K:F]} \in F$. There is a canonical way of extending Λ to a representation of the group generated by ζ and $E^\times J^1(\beta, \mathfrak{A})$; for $x \in E^\times J^1(\beta, \mathfrak{A}_K)$, we then have the character relation

$$\mathrm{tr}\, \Lambda_K(x) = \pm \mathrm{tr}\, \Lambda(\zeta x).$$

The sign here is a constant, independent of x.

11. The construction of $\boldsymbol{l}_{K/F}$ just outlined is carried through in detail in sections 3–5 below. This part of the argument is straightforward in principle (given the extensive machinery of [**5**] and [**9**]), but one has to check that the various constructions are intrinsic in nature, and independent of the many choices made on the way. Once this is done, most of the main properties of $\boldsymbol{l}_{K/F}$ are not hard to establish (or follow from results in [**5**]): for details see §5.

It is the effect of $\boldsymbol{l}_{K/F}$ on local constants which requires most effort. Using ideas from [**4**], one can attach a local constant $\boldsymbol{\varepsilon}(\Lambda)$ to a central type Λ in any $\mathrm{GL}_n(F)$. Indeed, this is a direct generalization of the local root number of Tate's thesis [**37**]. Apart from a straightforward exponential factor, $\boldsymbol{\varepsilon}(\Lambda)$ is given by an explicit formula reminiscent of a classical Gauss sum: see §6 for details. Further, if Λ occurs in an irreducible smooth representation π of $\mathrm{GL}_n(F)$, then $\boldsymbol{\varepsilon}(\Lambda)$ is the usual Godement-Jacquet local constant of π. In the notation of paragraph 10 above, the relation between $\boldsymbol{\varepsilon}(\Lambda_F)$ and $\boldsymbol{\varepsilon}(\Lambda)$ is transparent. Using the explicit character relation furnished by the Glauberman correspondence, we can extract the relation between $\boldsymbol{\varepsilon}(\Lambda)$ and $\boldsymbol{\varepsilon}(\Lambda_K)$ and hence that between $\boldsymbol{\varepsilon}(\pi_F, s, \psi_F)$ and $\boldsymbol{\varepsilon}(\pi_K, s, \psi_K)$. These calculations are to be found in sections 8 and 9.

Another application of this approach to local constants yields a congruence determining $\boldsymbol{\varepsilon}$ modulo p-power roots of unity, analogous to a property **[19]** of the Langlands-Deligne local constant. This is stated as Theorem 1.4, and proved in §7. It gives the vital step of the proof that our map $\boldsymbol{\pi}_m$ preserves local constants.

12. In §10, we show that if K/F is Galois of degree prime to p (and F has characteristic zero), then $\boldsymbol{l}_{K/F}$ is the same as base change:

$$\boldsymbol{l}_{K/F}(\pi) = \boldsymbol{b}_{K/F}(\pi), \quad \pi \in \mathcal{A}_m^{\mathrm{wr}}(F).$$

We already know from **[5]** that, given $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$, there is an unramified character χ_π of $K^\times$ such that $\boldsymbol{b}_{K/F}(\pi_F) = \chi_\pi \boldsymbol{l}_{K/F}(\pi_F)$; the local constant calculations show (irrespective of the degree of K/F) that χ_π has order strictly dividing p^m.

To establish that χ_π is trivial when $p \nmid [K{:}F]$, it is enough to treat the case where K/F is cyclic of prime degree $\neq p$. We have to interpret the Shintani character relation, which defines base change, in terms of our direct representation-theoretic approach. When translated in terms of central types (and using the notation of paragraph 9 above), the Shintani character relation shows that the process $\pi_F \mapsto \boldsymbol{b}_{K/F}(\pi_F)$ is given by composing a (metacyclic) Glauberman correspondence with an Adams operation. The Adams operation comes from the step $\Lambda_F \mapsto \Lambda$ in our construction.

The argument breaks down when K/F is unramified of degree p, for a selection of possibly complementary reasons. Nontheless, we can still establish the relation $\boldsymbol{b}_{K/F}(\pi_F) = \boldsymbol{l}_{K/F}(\pi_F)$ for an interesting class of representations, as a consequence of the local constant calculations.

In characteristic zero, some of the local constant relations proved in §§8 and 9 may be regarded as consequences of the relation $\boldsymbol{l}_{K/F} = \boldsymbol{b}_{K/F}$. However, even in characteristic zero, one still needs virtually all of the apparatus of those sections to deal with the cases not covered by the base change approach.

13. For the convenience of the reader, we have summarized the main properties of the Glauberman correspondence in a brief appendix. The first, and longer, appendix, contains an extremely detailed examination of the characters of cyclic extensions of extra-special p-groups of class 2. These calculations are quite general and, in some sense, quite elementary. However, they are not trivial and are used repeatedly throughout the paper; in particular, the properties of these characters intertwine in an intricate and fascinating way with the Gauss sum computations of §§6–9.

The arguments in the main parts of the paper involve a detailed knowledge of the internal structure of supercuspidal representations, as revealed in **[9]**. We have therefore summarized the main points at the beginning. Our earlier paper **[5]** also plays a crucial rôle; however, we do not often need to delve into its more technical aspects.

NOTATION AND PRELIMINARIES

The following notation will be standard throughout. It is chosen to be consistent with **[9]** and **[5]**, to which we shall often refer.

$F =$ a non-Archimedean local field;
$\mathfrak{o}_F =$ the discrete valuation ring in F;
$\mathfrak{p}_F =$ the maximal ideal of $\mathfrak{o}_F$;
$\mathrm{k}_F =$ the residue field $\mathfrak{o}_F/\mathfrak{p}_F$;
$\nu_F =$ the additive valuation $F^\times \twoheadrightarrow \mathbb{Z}$;
$p =$ the characteristic of k_F, $q_F = |\mathrm{k}_F| < \infty$;
$\psi_F =$ a continuous character of the additive group of F with conductor $\mathfrak{p}_F$. (This means that ψ_F is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$.)

If K/F is a finite field extension, we use similar notations $\mathfrak{o}_K$ etc. Also, $e(K|F)$ denotes the ramification index, $\mathrm{N}_{K/F}$ the relative norm, and $\mathrm{Tr}_{K/F}$ the relative trace.

Let V be a finite-dimensional F-vector space, $A = \mathrm{End}_F(V)$. We put $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$, where $\mathrm{tr}_{A/F}$ is the reduced trace. For an element $a \in A$, ψ_a denotes the function $x \mapsto \psi_A(a(x-1))$, $x \in A$, or (more usually) some restriction of it.

Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ with radical $\mathfrak{P}$. We write $\boldsymbol{U}(\mathfrak{A}) = \mathfrak{A}^\times$ for the unit group of $\mathfrak{A}$, and $\boldsymbol{U}^m(\mathfrak{A}) = 1+\mathfrak{P}^m$, $m \geqslant 1$. In the case $V = F$, $\mathfrak{A} = \mathfrak{o}_F$, we use the simpler notation $\boldsymbol{U}^1(\mathfrak{o}_F) = U_F^1$.

Let $[\mathfrak{A}, n, m, \beta]$ be a *simple stratum* in $\mathrm{End}_F(V)$. Thus $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$, $\beta \in \mathrm{Aut}_F(V)$, and the algebra $F[\beta]$ is a field. Moreover, $F[\beta]^\times$ normalizes $\mathfrak{A}$. Also, $n > m$ are integers, $\beta\mathfrak{A} = \mathfrak{P}^{-n}$, and there is a technical condition "$m < -k_0(\beta, \mathfrak{A})$". (These concepts are discussed fully in **[9]** Ch. 1.)

As in **[9]** 3.1, a simple stratum $[\mathfrak{A}, n, 0, \beta]$ defines a pair of compact open subgroups $H(\beta, \mathfrak{A}) \subset J(\beta, \mathfrak{A})$ of $G = \mathrm{Aut}_F(V)$, filtered by $H^m(\beta, \mathfrak{A}) = H(\beta, \mathfrak{A}) \cap \boldsymbol{U}^m(\mathfrak{A})$, $J^m(\beta, \mathfrak{A}) = J(\beta, \mathfrak{A}) \cap \boldsymbol{U}^m(\mathfrak{A})$, $m \geqslant 1$. The simple character set $\mathcal{C}(\mathfrak{A}, m, \beta)$ is as in **[9]** 3.2. The elements of $\mathcal{C}(\mathfrak{A}, m, \beta)$ are (very particular) abelian characters of the group $H^{m+1}(\beta, \mathfrak{A})$. For each $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$, there is a unique irreducible representation η of $J^{m+1}(\beta, \mathfrak{A})$ such that $\eta \mid H^{m+1}(\beta, \mathfrak{A})$ contains θ; indeed, this restriction is a multiple of θ.

In this paper, we abbreviate $\mathcal{C}(\mathfrak{A}, \beta) = \mathcal{C}(\mathfrak{A}, 0, \beta)$. Also, when speaking of a simple stratum $[\mathfrak{A}, n, m, \beta]$, say, *we will always assume that the hereditary order* $\mathfrak{A}$ *is* principal.

Warning. — A simple stratum $[\mathfrak{A}, n, 0, \beta]$ determines the group $H^1(\beta, \mathfrak{A})$ and the simple character set $\mathcal{C}(\mathfrak{A}, \beta)$, but the converse is false. These objects together determine $\mathfrak{A}$ and n, but not β. Certain elementary properties of the field $F[\beta]$, e.g., its degree $[F[\beta]{:}F]$ and ramification index $e(F[\beta]|F)$ over F, are invariant. Such matters are discussed in some detail in **[9]** Ch. 3 and **[10]**. Thus, when treating simple characters, one has to check that the properties under consideration are independent of the chosen defining element β.

Supercuspidal representations. — We recall briefly some features of the classification of supercuspidal representations **[9]**. Let π be an irreducible supercuspidal representation of G. Thus π contains a *maximal simple type* (J, λ), in the sense of **[9]** 5.5.10, 6.2. The pair (J, λ) is uniquely determined up to G-conjugacy, by **[9]** 8.4.1. There are two slightly different cases. In the first, $J \cong \mathrm{GL}_n(\mathfrak{o}_F)$ and λ is the inflation of a cuspidal irreducible representation of $\mathrm{GL}_n(\mathrm{k}_F)$; in this case, we say π *has level zero.* (Representations of level zero will not feature largely here.) Otherwise, J is a compact open subgroup of G and λ is an irreducible smooth representation of J as follows: there is a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A such that $J = J(\beta, \mathfrak{A})$; the order $\mathfrak{A}$ is principal, and is maximal for the property of being normalized by $F[\beta]^\times$; there is a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$ such that the restriction of the representation λ to $H^1(\beta, \mathfrak{A})$ is a multiple of θ. The representation π determines this simple character θ up to G-conjugacy. (There are further conditions on λ which do not concern us: for the full definition, see **[9]** 5.5.10, 6.2.) Moreover, if we write $N_G(\lambda)$ for the G-normalizer of (J, λ), there is a (unique) representation Λ of $N_G(\lambda)$ which extends λ and induces π.

CHAPTER 1

ALGEBRAIC TAME LIFTING

Let V be a finite-dimensional F-vector space, and put $G = \mathrm{Aut}_F(V)$, $A = \mathrm{End}_F(V)$. We write $\boldsymbol{A}(G)$ for the set of equivalence classes of irreducible smooth representations of G, and $\boldsymbol{A}^0(G)$ for the subset of $\boldsymbol{A}(G)$ consisting of classes of supercuspidal representations. If we identify V with F^n, say, and G with $\mathrm{GL}_n(F)$, we use the alternative notations $\boldsymbol{A}_n(F)$, $\boldsymbol{A}_n^0(F)$.

If $\pi \in \boldsymbol{A}_n(G)$ and χ is a quasicharacter of $F^\times$, we write $\chi\pi$ for the class of the representation

$$\chi\pi : g \longmapsto \chi(\det g)\, \pi(g), \qquad g \in G.$$

1.1. We isolate a subset of $\boldsymbol{A}_n^0(F)$. Let $\pi \in \boldsymbol{A}_n^0(F)$, and assume first that it does not have level zero. Thus π contains a maximal simple type with underlying simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathbb{M}_n(F)$. We put $E = F[\beta]$. When π has level zero, we unify the notation by setting $\beta = 0$, $E = F$.

Proposition. — *Let $\pi \in \boldsymbol{A}_n^0(F)$ and let E/F be a field extension attached to π, as above. The following are equivalent:*

(i) *we have $\pi \ncong \chi\pi$, for any non-trivial unramified quasicharacter χ of $F^\times$, and $n = p^m$ for some $m \geqslant 0$;*
(ii) *the field extension E/F is totally wildly ramified of degree n.*

Proof. — By **[9]** 6.2.5, there are exactly $n/e(E|F)$ unramified quasicharacters χ of $F^\times$ such that $\chi\pi \cong \pi$. The result is now immediate. □

We write $\boldsymbol{A}_m^{\mathrm{wr}}(F)$ for the set of $\pi \in \boldsymbol{A}_{p^m}^0(F)$ satisfying the conditions of the proposition. Note that $\boldsymbol{A}_0^{\mathrm{wr}}(F) = \boldsymbol{A}_1^0(F) = \boldsymbol{A}_1(F)$.

1.2. In order to give a precise statement of the main result below (Theorem 1.3), we need to briefly recall a concept from **[5]**. We consider the class of all simple characters in $\mathcal{C}(\mathfrak{A}, \beta)$, allowing $[\mathfrak{A}, n, 0, \beta]$ to range over all simple strata in $\mathrm{End}_F(V)$ and V over all finite-dimensional F-vector spaces. We impose on this class the equivalence relation of *endo-equivalence,* **[5]** §8. It is not necessary to recall the details, except to mention that if we have $\theta_i \in \mathcal{C}(\mathfrak{A}, \beta_i)$, for simple strata $[\mathfrak{A}, n_i, 0, \beta_i]$ attached to the same order $\mathfrak{A}$ in some $\mathrm{End}_F(V)$, then the θ_i are endo-equivalent if and only if they are $\mathrm{Aut}_F(V)$-conjugate.

Suppose we are given an endo-equivalence class (or endo-class, for short) Θ of simple characters over F, and a finite, tamely ramified field extension K/F. In **[5]** §9, we showed how to attach to K/F and Θ a finite set $\boldsymbol{L}_{K/F}(\Theta)$ of endo-classes of simple characters over K. These are the *K/F-lifts of Θ*. If Θ contains $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$, the elements of $\boldsymbol{L}_{K/F}(\Theta)$ are in one-one correspondence with the simple components of the algebra $K \otimes_F F[\beta]$.

Now take $\pi \in \boldsymbol{\mathcal{A}}_n^0(F)$, and assume π does not have level zero. A maximal simple type occurring in π gives rise to a simple character $\theta_\pi \in \mathcal{C}(\mathfrak{A}, \beta)$ occurring in π, with $\mathfrak{A}$ principal and maximal for being normalized by $F[\beta]^\times$. The conjugacy class of θ_π is uniquely determined by π, so the endo-class $\Theta(\pi)$ of θ_π is uniquely determined.

If we have $\pi \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$, then $\Theta(\pi)$ has a *unique* K/F-lift, for any finite tame extension K/F: this is because the tensor product of a tame extension and a totally wildly ramified extension is a field. We shall see again in §4 how this lift is constructed.

1.3. We now fix an integer $m \geqslant 0$ and a finite, tamely ramified field extension K/F. We shall define an *algebraic tame lifting map*

$$\boldsymbol{l}_{K/F} : \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F) \longrightarrow \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(K). \tag{1.3.1}$$

The case $m = 0$ is easy; $\boldsymbol{\mathcal{A}}_0^{\mathrm{wr}}(F)$ is the set of quasicharacters of $F^\times$, and we define

$$\boldsymbol{l}_{K/F}(\chi) = \chi \circ \mathrm{N}_{K/F}, \qquad \chi \in \boldsymbol{\mathcal{A}}_0^{\mathrm{wr}}(F). \tag{1.3.2}$$

For the case $m \geqslant 1$, the construction of $\boldsymbol{l}_{K/F}$ is given in §§3–5 below. In this section, we list the main properties of $\boldsymbol{l}_{K/F}$, to be established later.

Theorem. — *Let K/F be a finite, tamely ramified field extension. The map $\boldsymbol{l}_{K/F}$ of* 1.3.1 *has the following properties.*

(i) *Let $\pi_F \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$, and put $\pi_K = \boldsymbol{l}_{K/F}(\pi_F)$. Then $\Theta(\pi_K)$ is the unique K/F-lift of $\Theta(\pi_F)$.*

(ii) *If $\iota : K \to \iota K$ is an isomorphism of local fields, the diagram*

$$\begin{array}{ccc} \mathcal{A}_m^{\mathrm{wr}}(F) & \xrightarrow{\boldsymbol{l}_{K/F}} & \mathcal{A}_m^{\mathrm{wr}}(K) \\ \downarrow & & \downarrow \\ \mathcal{A}_m^{\mathrm{wr}}(\iota F) & \xrightarrow[\boldsymbol{l}_{\iota K/\iota F}]{} & \mathcal{A}_m^{\mathrm{wr}}(\iota K) \end{array}$$

commutes, the vertical maps here being induced by the isomorphisms $\mathrm{GL}_{p^m}(F) \to \mathrm{GL}_{p^m}(\iota F)$, $\mathrm{GL}_{p^m}(K) \to \mathrm{GL}_{p^m}(\iota K)$ *given by* ι.

(iii) *If $K \supset K' \supset F$, then*

$$\boldsymbol{l}_{K/F} = \boldsymbol{l}_{K/K'} \circ \boldsymbol{l}_{K'/F}.$$

(iv) *If χ is a quasicharacter of $F^\times$, we have*

$$\boldsymbol{l}_{K/F}(\chi\pi) = (\chi \circ \mathrm{N}_{K/F})\, \boldsymbol{l}_{K/F}(\pi), \qquad \pi \in \mathcal{A}_m^{\mathrm{wr}}(F).$$

(v) *The map $\boldsymbol{l}_{K/F}$ respects contragredience, i.e.,*

$$\boldsymbol{l}_{K/F}(\pi)^\vee = \boldsymbol{l}_{K/F}(\check{\pi}),$$

for all $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$.

(vi) *If $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ has central quasicharacter ω_π, then $\boldsymbol{l}_{K/F}(\pi)$ has central quasicharacter $\omega_\pi \circ \mathrm{N}_{K/F}$.*

(vii) *For $\pi_1, \pi_2 \in \mathcal{A}_m^{\mathrm{wr}}(F)$, we have $\boldsymbol{l}_{K/F}(\pi_1) = \boldsymbol{l}_{K/F}(\pi_2)$ if and only if there exists a character α of $F^\times$ which is trivial on $\mathrm{N}_{K/F}(K^\times)$ and such that $\pi_2 = \alpha\pi_1$.*

(viii) *Let L/F be a normal closure of K/F. For $\pi_K \in \mathcal{A}_m^{\mathrm{wr}}(K)$, the following conditions are equivalent:*

  (a) *$\pi_K = \boldsymbol{l}_{K/F}(\pi_F)$, for some $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$;*

  (b) *the representation $\pi_L = \boldsymbol{l}_{L/K}(\pi_K)$ is invariant under the Galois group $\mathrm{Gal}(L/F)$ and the central quasicharacter ω_K of π_K factors through $\mathrm{N}_{K/F}$.*

  (c) *$\boldsymbol{l}_{L/K}(\pi_K) = \boldsymbol{l}_{L/F}(\pi_F)$, for some $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$.*

1.4. We now state a general result concerning the Godement-Jacquet local constant $\boldsymbol{\varepsilon}(\pi, s, \psi_F)$ **[15]** of a representation $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$. We first recall that the local constant takes the form

$$\boldsymbol{\varepsilon}(\pi, s, \psi_F) = q_F^{(\frac{1}{2}-s)f(\pi,\psi_F)}\, \boldsymbol{\varepsilon}(\pi, \tfrac{1}{2}, \psi_F),$$

for an integer $f(\pi, \psi_F)$.

We need a variant of the classical Gauss sum (*cf.* **[19]**). Since the character ψ_F has conductor $\mathfrak{p}_F$, it defines an additive character of the residue field $\mathrm{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ of F. We continue to denote this character ψ_F.

Suppose first that $p \neq 2$. Let $x \in F^\times$ have even valuation $-2b$, and fix a prime element ϖ of F. We put

$$\varphi(x) = \sum_{y \in \mathfrak{o}_F/\mathfrak{p}_F} \psi_F(x\varpi^{2b}y^2/2).$$

This only depends on x mod $U_F^1 F^{\times 2}$.

In general, we put

$$\mathfrak{g}(x) = \mathfrak{g}_F(x, \psi_F) = \begin{cases} q_F^{-1/2}\varphi(x) & \text{if } \nu_F(x) \text{ is even and } p \neq 2, \\ 1 & \text{if } p = 2 \text{ or } \nu_F(x) \text{ is odd,} \end{cases}$$

where $q_F = |\mathrm{k}_F|$.

We write $\boldsymbol{\mu}_{p^\infty}(\mathbb{C})$ for the group of all roots of unity in $\mathbb{C}$ of p-power order.

Theorem. — *Let $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$, $m \geqslant 1$, and write ω_π for the central quasicharacter of π.*

(i) *There exists $a_\pi \in F^\times$, uniquely determined mod U_F^1, such that*

$$\boldsymbol{\varepsilon}(\chi\pi, s, \psi_F) = \chi(a_\pi)^{-1}\,\boldsymbol{\varepsilon}(\pi, s, \psi_F),$$

for all tamely ramified quasicharacters χ of $F^\times$.

(ii) *We have*

$$\boldsymbol{\varepsilon}(\pi, \tfrac{1}{2}, \psi_F) \equiv \omega_\pi(a_\pi)^{-1/p^m}\,\mathfrak{g}(a_\pi)^{p^m} \pmod{\boldsymbol{\mu}_{p^\infty}(\mathbb{C})}.$$

(iii) *If K/F is a finite tame extension and $\pi_K = \boldsymbol{l}_{K/F}(\pi)$, then $a_{\pi_K} \equiv a_\pi \pmod{U_K^1}$.*

This theorem is analogous to a known result concerning the Langlands-Deligne local constant, which we recall in 2.5 below.

Remark. — It is easy to identify the element a_π occurring in the theorem. If π contains a maximal simple type with underlying simple stratum $[\mathfrak{A}, n, 0, \beta]$, we have $a_\pi \equiv \mathrm{N}_{F[\beta]/F}(\beta) \pmod{U_F^1}$: this follows easily from 6.1 Lemma 2 below.

1.5. We need a short digression. We fix a separable algebraic closure F^{sep}/F of F, and let $\mathcal{W}_F$ denote the Weil group of F^{sep}/F. We regard our finite tamely ramified extension K/F as a subfield of F^{sep} and $\mathcal{W}_K$ as a subgroup of $\mathcal{W}_F$.

Let 1_K denote the trivial representation of $\mathcal{W}_K$, and let $\boldsymbol{\rho}_{K/F}$ be the representation of $\mathcal{W}_F$ induced by 1_K. This gives us a one-dimensional representation $\boldsymbol{\delta}_{K/F}$ of $\mathcal{W}_F$ by

$$\boldsymbol{\delta}_{K/F} : x \longmapsto \det(\boldsymbol{\rho}_{K/F}(x)), \quad x \in \mathcal{W}_F.$$

We use the same notation for the corresponding character of $F^\times$ given by class-field theory. As such, $\boldsymbol{\delta}_{K/F}$ is tamely ramified and has order at most 2. It has the transitivity property

$$\boldsymbol{\delta}_{K/F} = \boldsymbol{\delta}_{K'/F}^{[K:K']} \cdot (\boldsymbol{\delta}_{K/K'} \mid F^\times), \tag{1.5.1}$$

for an intermediate extension $F \subset K' \subset K$.

1.6. If σ is a finite-dimensional semisimple representation of $\mathcal{W}_F$, we denote the Langlands-Deligne local constant **[12]**, **[39]** of σ by $\boldsymbol{\varepsilon}(\sigma, s, \psi_F)$. We form this local constant relative to Haar measure self-dual with respect to ψ_F. We use the analogous notation relative to the base field K. Observe that, when K/F is tame, the character $\psi_K = \psi_F \circ \mathrm{Tr}_{K/F}$ of K has conductor $\mathfrak{p}_K$.

Theorem. — *Let $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$, $m \geqslant 1$, let K/F be a finite tamely ramified field extension of degree d, and let $\pi_K = \boldsymbol{l}_{K/F}(\pi_F)$. Let $[\mathfrak{A}_F, n_F, 0, \beta]$ be a simple stratum underlying some maximal simple type in π_F, and write $E = F[\beta]$. We then have*

$$\frac{\boldsymbol{\varepsilon}(\pi_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\pi_F, s, \psi_F)^d} = \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\beta)) \left(\frac{\boldsymbol{\varepsilon}(1_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{K/F}, s, \psi_F)} \right)^{p^m}.$$

This relation is analogous to the behaviour of the Langlands-Deligne local constant under restriction of (suitable) representations from $\mathcal{W}_F$ to $\mathcal{W}_K$, as we shall recall in 2.6 below.

The proof of Theorem 1.3 is given in §5 and that of Theorem 1.4 in §7. The proof of 1.6 occupies §8 and §9.

Remark. — The theorem holds without change for $m = 0$ *except* in the following case: π_F is tamely ramified but not unramified, while $\pi_K = \pi_F \circ \mathrm{N}_{K/F}$ is unramified.

1.7. In this paragraph, we let m_1, m_2 be non-negative integers, and take $\pi_i \in \mathcal{A}_{m_i}^{\mathrm{wr}}(F)$, $i = 1, 2$. We consider the local constant $\boldsymbol{\varepsilon}(\pi_1 \times \pi_2, s, \psi_F)$ of the pair (π_1, π_2), in the sense of **[24]**. This takes the form

$$\boldsymbol{\varepsilon}(\pi_1 \times \pi_2, s, \psi_F) = q_F^{(\frac{1}{2}-s)f}\, \boldsymbol{\varepsilon}(\pi_1 \times \pi_2, \tfrac{1}{2}, \psi_F),$$

for an integer $f = f(\pi_1 \times \pi_2, \psi_F)$.

The aim of this section is to investigate the behaviour of $f(\pi_1 \times \pi_2, \psi_F)$ under tame lifting, in the case where $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$, $i = 1, 2$. One case is a little exceptional, and has to be dealt with via the following result.

Lemma. — *Let $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$, $m \geqslant 0$. Let χ be a tamely ramified quasicharacter of $F^\times$. Then:*

$$f(\check{\pi} \times \chi\pi, \psi_F) = \begin{cases} f(\check{\pi} \times \pi, \psi_F) & \text{if } \chi \text{ is unramified,} \\ f(\check{\pi} \times \pi, \psi_F) + 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

Proof. — See **[7]** 6.14. □

Taken together with the next result, this gives us a complete account of the behaviour of the conductor under tame lifting.

Theorem. — *Let K/F be a finite, tamely ramified field extension, and let $\pi_i \in \mathcal{A}^{\mathrm{wr}}_{m_i}(F)$, $i = 1, 2$.*

(i) *Suppose that $\pi_1 \not\cong \chi\check{\pi}_2$, for any tamely ramified quasicharacter χ of $F^\times$ such that $\chi \circ \mathrm{N}_{K/F}$ is unramified. Then:*

$$f(\boldsymbol{l}_{K/F}(\pi_1) \times \boldsymbol{l}_{K/F}(\pi_2), \psi_K) = f(\pi_1 \times \pi_2, \psi_F)\, e(K|F).$$

(ii) *Let $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$. We have*

$$f(\boldsymbol{l}_{K/F}(\check{\pi}) \times \boldsymbol{l}_{K/F}(\pi), \psi_K) + 1 = \Big(f(\check{\pi} \times \pi, \psi_F) + 1\Big)\; e(K|F).$$

We shall prove this result in 5.6 below.

1.8. We assume in this paragraph that F *has characteristic zero.* Let F'/F be a finite cyclic extension. Base change, in the sense of **[1]**, then gives us a map

$$\boldsymbol{b}_{F'/F} : \mathcal{A}_n(F) \longrightarrow \mathcal{A}_n(F'),$$

for each $n \geqslant 1$. In this section, we compare $\boldsymbol{b}_{K/F}(\pi)$ with $\boldsymbol{l}_{K/F}(\pi)$ for the case of a cyclic tame extension K/F and $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$. The main result is:

Theorem. — *Let K/F be a cyclic tame extension of degree d, and assume p does not divide d. Let $\pi \in \mathcal{A}^{\mathrm{wr}}_m(F)$. Then*

$$\boldsymbol{b}_{K/F}(\pi) \cong \boldsymbol{l}_{K/F}(\pi).$$

The proof starts with the following general result, which requires no restriction on the degree $[K{:}F]$.

Lemma. — *Let $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$, and let K/F be a finite, cyclic, tamely ramified field extension. Then $\boldsymbol{b}_{K/F}(\pi) \in \mathcal{A}_m^{\mathrm{wr}}(K)$. Moreover, there is an unramified character χ_π of $K^\times$, of order dividing p^m, such that*

$$\boldsymbol{b}_{K/F}(\pi) \cong \chi_\pi \, \boldsymbol{l}_{K/F}(\pi).$$

Proof. — This follows immediately from Theorem 1.3(i), (vi) and [**5**] 14.21. □

As we shall see later, Theorem 1.6 implies the following sharper version of this lemma.

Proposition. — *Using the notation of the lemma, the character χ_π has order* strictly *dividing p^m. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum underlying some maximal simple type occurring in π. If β is minimal over F, then*

$$\boldsymbol{b}_{K/F}(\pi) \cong \boldsymbol{l}_{K/F}(\pi).$$

In the constructions of the next section, we only directly use the tame lift $\boldsymbol{l}_{K/F}$ for extensions K/F of degree prime to p, although some of these extensions are not Galois.

The proofs of the proposition and the theorem will be given in §10.

CHAPTER 2

CORRESPONDENCE WITH GALOIS REPRESENTATIONS

We first describe a class $\mathcal{G}_m^{\mathrm{wr}}(F)$ of irreducible representations of the Weil group $\mathcal{W}_F$ of F analogous to the class $\mathcal{A}_m^{\mathrm{wr}}(F)$ introduced in 1.1. The aim of the section is to construct a canonical bijection between these sets, using the properties of the tame lifting map listed in 1.3–8.

Before starting, we recall that local class field theory gives a topological isomorphism

$$\boldsymbol{c}_F : \mathcal{W}_F^{\mathrm{ab}} \xrightarrow{\approx} F^\times$$

of the maximal abelian topological quotient $\mathcal{W}_F^{\mathrm{ab}}$ of $\mathcal{W}_F$ with $F^\times$. We normalize this map so that *geometric* Frobenius elements correspond to prime elements of F.

Given a quasicharacter χ of $F^\times$, we often denote the corresponding one-dimensional representation $\chi \circ \boldsymbol{c}_F$ of $\mathcal{W}_F$ simply by χ, as confusion is unlikely to arise.

We write $\boldsymbol{\varepsilon}(\sigma, s, \psi_F)$ for the Langlands-Deligne local constant of a finite-dimensional admissible representation σ of $\mathcal{W}_F$ **[12]**, **[39]**, **[38]**. (We always define this using the Haar measure on F which is self-dual relative to ψ_F.) It takes the form

$$\boldsymbol{\varepsilon}(\sigma, s, \psi_F) = q_F^{(\frac{1}{2}-s)f(\sigma,\psi_F)}\,\boldsymbol{\varepsilon}(\sigma, \tfrac{1}{2}, \psi_F),$$

for an integer $f(\sigma, \psi_F)$.

2.1. For each integer $n \geqslant 1$, we let $\mathcal{G}_n^0(F)$ denote the set of equivalence classes of irreducible admissible complex representations of $\mathcal{W}_F$ of dimension n. For $n = 1$, $\mathcal{A}_1^0(F)$ is just the set of quasicharacters of $F^\times$; we therefore have a bijection

$$\boldsymbol{\pi}_1 : \mathcal{G}_1^0(F) \xrightarrow{\approx} \boldsymbol{\mathcal{A}}_1^0(F),$$
$$\chi \longmapsto \chi \circ \mathbf{c}_F^{-1}.$$

We recall that in this case, the Langlands-Deligne local constant of χ is

$$\boldsymbol{\varepsilon}(\chi, s, \psi_F) = \boldsymbol{\varepsilon}(\boldsymbol{\pi}_1(\chi), s, \psi_F),$$

by definition.

2.2. We write $\mathcal{I}_F$ for the inertia subgroup of $\mathcal{W}_F$ and $\mathcal{P}_F$ for the first ("wild") ramification subgroup; thus $\mathcal{P}_F$ is the unique pro-p Sylow subgroup of $\mathcal{I}_F$. We isolate a distinguished class of irreducible representations of $\mathcal{W}_F$.

Proposition. — *Let (σ, V) be an irreducible representation of $\mathcal{W}_F$. The following are equivalent:*

(i) *$\sigma \mid \mathcal{P}_F$ is irreducible;*

(ii) *$\dim \sigma = p^m$, for some integer $m \geqslant 0$, and $\sigma \not\cong \sigma \otimes \chi$, for any unramified quasicharacter $\chi \neq 1$ of $F^\times$.*

Proof. — For any irreducible representation (σ, V) of $\mathcal{W}_F$, it is easy to see that the image of $\sigma(\mathcal{W}_F)$ in the projective group $\mathrm{Aut}_{\mathbb{C}}(V)/\mathbb{C}^\times$ is finite. This image is the Galois group of some finite Galois extension E/F. Let E_1/F be the maximal tamely ramified sub-extension of E/F. The Galois group G_1 of E/E_1 is the wild inertia subgroup of $\mathrm{Gal}(E/F)$; the restriction $\sigma \mid \mathcal{P}_F$ is irreducible if and only if $\sigma \mid \mathcal{W}_{E_1}$ is irreducible. Thus, if (i) holds, $\dim \sigma$ must be a power of p, since G_1 is a p-group. If, moreover, we have $\sigma \otimes \chi \cong \sigma$ for some non-trivial unramified character χ of $F^\times$, the restriction $\sigma \mid \mathcal{I}_F$ is reducible by Clifford theory. Thus (i) $\Rightarrow$ (ii).

Conversely, (ii) implies that $\sigma \mid \mathcal{I}_F$ is irreducible. Suppose, for a contradiction, that σ is reducible on $\mathcal{P}_F$. Let σ_0 be an irreducible component of $\sigma \mid \mathcal{P}_F$. Since $\mathcal{I}_F/\mathcal{P}_F$ is pro-cyclic, σ_0 admits extension to a representation σ_0' of the $\mathcal{I}_F$-stabilizer of σ_0. Call this stabilizer $\mathcal{H}$. Some choice of σ_0' induces $\sigma \mid \mathcal{I}_F$, giving us $\dim \sigma = (\mathcal{I}_F : \mathcal{H}) \dim \sigma_0$. Since p cannot divide the index $(\mathcal{I}_F : \mathcal{H})$, this forces $\mathcal{H} = \mathcal{I}_F$ and so $\sigma_0 = \sigma$, as required. □

We write $\mathcal{G}_m^{\mathrm{wr}}(F)$ for the set of equivalence classes of irreducible representations of $\mathcal{W}_F$ which satisfy the conditions of the Proposition and have dimension p^m.

Let $(\sigma, V) \in \mathcal{G}_m^{\mathrm{wr}}(F)$. We establish, for later use, a system of notation like that above:

$\mathcal{W}_E =$ the kernel of $\mathcal{W}_F \to \mathrm{Aut}_{\mathbb{C}}(V)/\mathbb{C}^\times$;

$G = \mathcal{W}_F/\mathcal{W}_E = \mathrm{Gal}(E/F)$;
$E_1/F =$ the maximal tame sub-extension of E/F;
$G_1 = \mathrm{Gal}(E/E_1)$;
$P =$ a p-Sylow subgroup of G and $F' = E^P$;
$F''/F =$ the normal closure of F'/F;
$H = \mathrm{Gal}(E/F'')$.

(All of these objects depend on σ.) We observe that F'/F is tame of degree prime to p. Also, $P \supset G_1$ and E_1/F is Galois, so $E_1 \supset F''$. The extension E_1/F' is unramified of p-power degree.

Lemma. — *Let $\sigma, \sigma' \in \mathcal{G}_m^{\mathrm{wr}}(F)$; suppose that $\sigma \mid \mathcal{W}_{F'} \cong \sigma' \mid \mathcal{W}_{F'}$ and $\det \sigma = \det \sigma'$. Then $\sigma \cong \sigma'$.*

Proof. — If σ, σ' agree on $\mathcal{W}_{F'}$, i.e., on P, we have $\sigma' \cong \sigma \otimes \chi$, for some character χ of $\mathcal{W}_F$ trivial on $\mathcal{W}_{F'}$. In particular, χ has order prime to p. However, $\det \sigma' = \chi^{p^m} . \det \sigma$, whence χ is trivial. □

2.3. We come to the second of our main results. Since it depends on base change [**1**], we must now assume that F *has characteristic zero.*

Theorem. — *For each $m \geqslant 0$, there is a bijection (to be described in the proof)*

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{m,F}^{\mathrm{wr}} : \mathcal{G}_m^{\mathrm{wr}}(F) \xrightarrow{\approx} \mathcal{A}_m^{\mathrm{wr}}(F).$$

It has the following properties:

(i) *$\boldsymbol{\pi}_{m,F}^{\mathrm{wr}}$ is natural with respect to topological isomorphisms of F;*
(ii) *if χ is a quasicharacter of $F^\times$, then $\boldsymbol{\pi}(\chi \otimes \sigma) = \chi\boldsymbol{\pi}(\sigma)$, $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$;*
(iii) *$\boldsymbol{\pi}(\check{\sigma}) = \boldsymbol{\pi}(\sigma)^\vee$, $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$;*
(iv) *the central quasicharacter of $\boldsymbol{\pi}(\sigma)$ is $(\det \sigma) \circ \boldsymbol{c}_F^{-1}$;*
(v) *$\boldsymbol{\varepsilon}(\boldsymbol{\pi}(\sigma), s, \psi_F) = \boldsymbol{\varepsilon}(\sigma, s, \psi_F)$, $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$.*
(vi) *let K/F be a finite, tamely ramified, Galois extension of degree prime to p, let $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, and write $\sigma_{K/F} = \sigma \mid \mathcal{W}_K$; we have*

$$\boldsymbol{\pi}_{m,K}(\sigma_{K/F}) = \boldsymbol{b}_{K/F}(\boldsymbol{\pi}_{m,F}(\sigma)).$$

(vii) *for $i = 1, 2$, let $\sigma_i \in \mathcal{G}_{m_i}^{\mathrm{wr}}(F)$; we have*

$$L(\boldsymbol{\pi}(\sigma_1) \times \boldsymbol{\pi}(\sigma_2), s) = L(\sigma_1 \otimes \sigma_2, s), \quad \text{and}$$
$$f(\boldsymbol{\pi}(\sigma_1) \times \boldsymbol{\pi}(\sigma_2), \psi_F) = f(\sigma_1 \otimes \sigma_2, \psi_F).$$

In the remainder of this section, we deduce Theorem 2.3 from properties of the tame lifting map stated in §1. We state and prove some further results.

The first step is to define $\boldsymbol{\pi}_m^{\mathrm{wr}}(\sigma)$, for $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$. Our approach is based on **[20]**.

Let $n \geqslant 1$ and write $\mathcal{G}\boldsymbol{c}_n^0(F)$ for the set of $\sigma \in \mathcal{G}_n^0(F)$ with the following property: there exists a sequence of fields,

$$F = L_0 \subset L_1 \subset \cdots \subset L_m = L, \tag{2.3.1}$$

with each L_i/L_{i-1} cyclic of prime degree, and a quasicharacter χ of $L^\times$ such that

$$\sigma \cong \mathrm{Ind}_{\mathcal{W}_L}^{\mathcal{W}_F}(\chi).$$

Théorème A of **[20]** defines a *canonical* map

$$\Psi_n^0(F) : \mathcal{G}\boldsymbol{c}_n^0(F) \longrightarrow \boldsymbol{\mathcal{A}}_n^0(F)$$

as follows. (We summarize here the arguments of **[20]** 7.2–7.6.) We put $d_i = [L{:}L_i]$ and define, by descending induction on i, a representation $\pi_i \in \boldsymbol{\mathcal{A}}_{d_i}^0(L_i)$. We start with $\pi_m = \chi$, and then π_{i-1} is obtained from π_i thus. The stabilizer of π_i in $\Gamma_i = \mathrm{Gal}(L_i/L_{i-1})$ is trivial, so the representation of $\mathrm{GL}_{d_{i-1}}(L_i)$ parabolically induced from the representation $\bigotimes_{\gamma\in\Gamma_i} \pi_i^\gamma$ of the relevant Levi subgroup is the base change from L_{i-1} to L_i of a (uniquely determined) supercuspidal representation of $\mathrm{GL}_{d_{i-1}}(L_{i-1})$ (**[1]** I Prop. 6.6 and Lemma 6.10), which is defined to be $\pi_{i-1} \in \boldsymbol{\mathcal{A}}_{d_{i-1}}^0(L_{i-1})$. (Equivalently, π_{i-1} is *automorphically induced* from π_i; see **[6]** 2.6 for a discussion of the relation between base change and automorphic induction.) The class of $\Psi_n^0(F)(\sigma)$ is then $\pi_0 \in \boldsymbol{\mathcal{A}}_n^0(F)$.

Now put

$$\mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F) = \mathcal{G}_m^{\mathrm{wr}}(F) \cap \mathcal{G}\boldsymbol{c}_{p^m}^0(F), \qquad m \geqslant 0.$$

This is the set of $\sigma \in \mathcal{G}_{p^m}^0(F)$ for which there exists a sequence of fields

$$F = L_0 \subset L_1 \subset \cdots \subset L_m = L$$

and a quasicharacter χ of $L^\times$, as in 2.3.1, with the additional property that each L_i/L_{i-1} is cyclic *of degree p and totally ramified.*

Lemma. — *Let $\sigma \in \mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F)$; then $\Psi^0(F)(\sigma)$ lies in $\boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$.*

Proof. — Since $\dim\sigma = p^m$, the construction of $\Psi^0(F)$ gives $\Psi^0(F)(\sigma) \in \boldsymbol{\mathcal{A}}_{p^m}^0(F)$. Moreover, the map $\Psi^0(F)$ is injective and compatible with twisting by quasicharacters **[20]** Théorème B. In particular, since $\sigma\chi \neq \sigma$ for any non-trivial unramified quasicharacter χ of $F^\times$, the representation $\Psi^0(F)(\sigma)$ has the same property and lies in $\boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ by 1.1. □

We write

$$ {}^{c}\boldsymbol{\pi} = {}^{c}\boldsymbol{\pi}_{m,F} : \mathcal{G}c_m^{\mathrm{wr}}(F) \longrightarrow \mathcal{A}_m^{\mathrm{wr}}(F) $$

for the map induced by $\Psi^0(F)$. By **[20]** 7.2 Théorème B it is compatible with twisting by quasicharacters and it preserves L- and ε- factors *for pairs,* in the sense of **[24]**. Because base change is compatible with contragredience, ${}^{c}\boldsymbol{\pi}$ is also compatible with contragredience. Since base change is natural with respect to topological isomorphisms of the base field (see its defining relation **[1]** Defn. 6.1), it follows that ${}^{c}\boldsymbol{\pi}$ is also natural: if $\iota : F \to \iota F$ is an isomorphism of extensions of $\mathbb{Q}_p$, then the following diagram, in which the vertical arrows are the natural identifications induced by ι, is commutative:

$$ \begin{array}{ccc} \mathcal{G}c_m^{\mathrm{wr}}(F) & \xrightarrow{{}^{c}\boldsymbol{\pi}} & \mathcal{A}_m^{\mathrm{wr}}(F) \\ \downarrow & & \downarrow \\ \mathcal{G}c_m^{\mathrm{wr}}(\iota F) & \xrightarrow{{}^{c}\boldsymbol{\pi}} & \mathcal{A}_m^{\mathrm{wr}}(\iota F). \end{array} $$

Finally, the map ${}^{c}\boldsymbol{\pi}$ is compatible with cyclic base change in the sense of **[20]** Th. 7.12.

We now take a general $(\sigma, V) \in \mathcal{G}_m^{\mathrm{wr}}(F)$ and use the notation of 2.2. We also put $\sigma_P = \sigma \mid \mathcal{W}_{F'}$. Thus $\sigma_P \in \mathcal{G}_m^{\mathrm{wr}}(F')$; indeed, since it is effectively a representation of a central extension of a finite p-group, it actually lies in $\mathcal{G}c_m^{\mathrm{wr}}(F')$. Applying the foregoing relative to the base field F', the representation

$$ \pi' = {}^{c}\boldsymbol{\pi}(\sigma_P) \in \mathcal{A}_m^{\mathrm{wr}}(F') $$

is defined.

Proposition. — *There exists a unique $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ such that $\boldsymbol{l}_{F'/F}(\pi) = \pi'$ and having central quasicharacter $\omega_\pi = \det\sigma \circ \boldsymbol{c}_F^{-1}$.*

Proof. — We use Theorem 1.3(viii). First, the central quasicharacter $\omega_{\pi'}$ of π' satisfies $\omega_{\pi'} = \det\sigma_P = \det\sigma \circ \mathrm{N}_{F'/F}$. Next, the representation $\pi'' = \boldsymbol{b}_{F''/F'}(\pi')$ is just ${}^{c}\boldsymbol{\pi}(\sigma|H)$; it is therefore $\mathrm{Gal}(F''/F)$-invariant. By Theorem 1.3(ii) and Lemma 1.8, the same applies to $\boldsymbol{l}_{F''/F'}(\pi')$. The existence of the desired representation π follows from Theorem 1.3(viii). The uniqueness follows from part (vii) of the same result. □

In the notation of this proposition, we now put

$$ \boldsymbol{\pi}(\sigma) = \pi \in \mathcal{A}_m^{\mathrm{wr}}(F). \tag{2.3.2} $$

We note that the naturality properties of ${}^{c}\boldsymbol{\pi}$ and $\boldsymbol{l}_{K/F}$ imply that the definition of $\boldsymbol{\pi}(\sigma)$ is independent of the choice of p-Sylow subgroup P used in the construction.

Thus we have a family of canonical maps

$$\boldsymbol{\pi} = \boldsymbol{\pi}^{\mathrm{wr}}_{m,F} : \mathcal{G}^{\mathrm{wr}}_m(F) \longrightarrow \boldsymbol{\mathcal{A}}^{\mathrm{wr}}_m(F), \qquad m \geqslant 0.$$

Since $\boldsymbol{\pi}$ was obtained by combining algebraic lifting with base change (or automorphic induction), it inherits their common properties. In particular, it is natural with respect to isomorphisms of the base field and respects both contragredience and twisting with quasicharacters. The map $\boldsymbol{\pi}$ therefore satisfies the requirements (i)–(iv) of Theorem 2.3.

2.4. We now prove the bijectivity assertion and part (v) of Theorem 2.3 (along with the first assertion of part (vii)).

Proposition. — *The maps $\boldsymbol{\pi}$ are injective and preserve L-functions of pairs:*

$$L(\sigma_1 \otimes \sigma_2, s) = L(\boldsymbol{\pi}(\sigma_1) \times \boldsymbol{\pi}(\sigma_2), s), \quad \sigma_i \in \mathcal{G}^{\mathrm{wr}}_{m_i}(F).$$

Proof. — Suppose first that we have $\sigma_1, \sigma_2 \in \mathcal{G}^{\mathrm{wr}}_m(F)$ with $\boldsymbol{\pi}(\sigma_1) \cong \boldsymbol{\pi}(\sigma_2)$. We use the same notation as above, appending subscripts 1 or 2 as necessary. We abbreviate $\pi_i = \boldsymbol{\pi}(\sigma_i)$. Let F''/F be the composite of the extensions F''_i/F; this is a tamely ramified Galois extension, so the base change operation $\boldsymbol{b}_{F''/F}$ is defined, *cf.* [5] 16.5. The representations $\boldsymbol{b}_{F''/F}(\pi_i)$, $\boldsymbol{l}_{F''/F}(\pi_i)$, $\boldsymbol{b}_{F''/F'_i}\boldsymbol{l}_{F'_i/F}(\pi_i)$ differ from each other by, at most, unramified twist (1.8 Lemma). The last of these three is ${}^{c}\boldsymbol{\pi}(\sigma_i \mid \mathcal{W}_{F''})$, by the compatibility of ${}^{c}\boldsymbol{\pi}$ with base change. The injectivity of ${}^{c}\boldsymbol{\pi}$ now implies that the representations $\sigma_i \mid \mathcal{W}_{F''}$ differ, at most, by an unramified twist,

$$\sigma_1 \mid \mathcal{W}_{F''} \cong (\sigma_2 \mid \mathcal{W}_{F''}) \otimes \chi''.$$

Now let F_0/F be the maximal unramified subextension of F''/F; we compare the representations $\sigma_i \mid \mathcal{W}_{F_0}$. These differ, at most, by a tamely ramified character, since they restrict to unramified twists of the same irreducible representation of $\mathcal{W}_{F''/F_0}$. They have the same determinants, so the p^m-th power of this character is unramified, i.e., it is unramified. It follows now that the σ_i differ, at most, by an *unramified* character, $\sigma_2 = \sigma_1 \otimes \chi$, say. This gives

$$\boldsymbol{\pi}(\sigma_1) = \boldsymbol{\pi}(\sigma_2) = \boldsymbol{\pi}(\sigma_1 \otimes \chi) = \chi\boldsymbol{\pi}(\sigma_1).$$

Since $\boldsymbol{\pi}(\sigma_1) \in \boldsymbol{\mathcal{A}}^{\mathrm{wr}}_m(F)$, this forces $\chi = 1$ and $\sigma_1 \cong \sigma_2$, as required.

For the second assertion, we recall ([24] Prop. 8.1) that

$$L(\boldsymbol{\pi}(\sigma_1) \times \boldsymbol{\pi}(\sigma_2), s) = \prod_{\chi} L(\chi, s),$$

where χ ranges over those unramified quasicharacters of $F^\times$ such that

$$\chi\boldsymbol{\pi}(\sigma_1) \cong \boldsymbol{\pi}(\sigma_2)^\vee.$$

There is an analogous description of $L(\sigma_1 \otimes \sigma_2, s)$. However, by 1.3(vi) and what we have just proved, we have $\chi\boldsymbol{\pi}(\sigma_1) \cong \boldsymbol{\pi}(\sigma_2)^\vee$ if and only if $\sigma_1 \otimes \chi \cong \check{\sigma}_2$, and the result follows. □

The next step of the proof is to show that the local constant relation

$$\boldsymbol{\varepsilon}(\boldsymbol{\pi}(\sigma), s, \psi_F) = \boldsymbol{\varepsilon}(\sigma, s, \psi_F), \qquad \sigma \in \mathcal{G}_m^{\mathrm{wr}}(F), \tag{2.4.1}$$

of Theorem 2.3(v) implies that $\boldsymbol{\pi}$ is bijective for all m.

We have

$$\boldsymbol{\varepsilon}(\sigma, s, \psi_F) = \boldsymbol{\varepsilon}(\sigma, 0, \psi_F)\, q_F^{-f(\sigma,\psi_F)s},$$

for an integer $f(\sigma, \psi_F)$. Likewise for representations of $\mathrm{GL}_n(F)$. Thus 2.4.1 gives:

$$f(\boldsymbol{\pi}(\sigma), \psi_F) = f(\sigma, \psi_F). \tag{2.4.2}$$

Théorème 1.2 of [20] now implies that $\boldsymbol{\pi} = \boldsymbol{\pi}_{m,F}^{\mathrm{wr}}$ is bijective.

2.5. We remarked in 2.3 above that:

$$\boldsymbol{\varepsilon}({}^{\mathbf{c}}\boldsymbol{\pi}(\sigma), s, \psi_F) = \boldsymbol{\varepsilon}(\sigma, s, \psi_F), \quad \sigma \in \mathcal{G}\boldsymbol{c}_m^{\mathrm{wr}}(F). \tag{2.5.1}$$

We need again the classical Gauss sum $\mathfrak{g}$ defined in 1.4.

Lemma. — *Let $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, $m \geqslant 1$. Then:*

(i) *There exists $a_\sigma \in F^\times$, uniquely determined mod U_F^1, such that*

$$\boldsymbol{\varepsilon}(\sigma \otimes (\chi \circ \mathbf{c}_F), s, \psi_F) = \chi(a_\sigma)^{-1}\, \boldsymbol{\varepsilon}(\sigma, s, \psi_F),$$

for all tamely ramified characters χ of $F^\times$.

(ii) *We have*

$$\boldsymbol{\varepsilon}(\sigma, \tfrac{1}{2}, \psi_F) \equiv \det\sigma(a_\sigma)^{-1/p^m}\, \mathfrak{g}(a_\sigma)^{p^m} \pmod{\boldsymbol{\mu}_{p^\infty}(\mathbb{C})},$$

where $\boldsymbol{\mu}_{p^\infty}(\mathbb{C})$ denotes the group of all p-power roots of unity in $\mathbb{C}$.

(iii) *If K/F is a finite tame extension, then $a_{\sigma|\mathcal{W}_K} \equiv a_\sigma \pmod{U_K^1}$.*

Proof. — Part (i) is [13], (ii) is [19], and (iii) follows easily from [13]. □

This lemma, we observe, is closely parallel to Theorem 1.4 above.

2.6. We deduce Theorem 2.3(v) from Lemma 2.5 and Theorems 1.4, 1.6. We take $\sigma \in \mathcal{G}_m^{\mathrm{wr}}(F)$, and keep the notation of 2.2 used in the construction of $\boldsymbol{\pi}(\sigma)$. We apply Lemma 2.5 and Theorem 1.4 to the tame extension F'/F. We set $r = [F':F]$; this is relatively prime to p. We write $\sigma' = \sigma \mid \mathcal{W}_{F'}$, $\pi = \boldsymbol{\pi}(\sigma)$, $\pi' = \boldsymbol{l}_{F'/F}(\pi)$. By definition, $\pi' = {}^{\boldsymbol{c}}\boldsymbol{\pi}(\sigma_{F'})$. If χ' is a quasicharacter of $F'^{\times}$, we thus have $\chi'\pi' = {}^{\boldsymbol{c}}\boldsymbol{\pi}(\sigma' \otimes (\chi' \circ \boldsymbol{c}_{F'}))$; by 2.5.1, therefore,

$$\boldsymbol{\varepsilon}(\chi'\pi', s, \psi_{F'}) = \boldsymbol{\varepsilon}(\sigma' \otimes (\chi' \circ \boldsymbol{c}_{F'}), s, \psi_{F'}).$$

By Lemma 2.5 and 1.4, we get $\chi'(a_{\pi'}) = \chi'(a_{\sigma_{F'}})$ for all tame χ', whence

$$a_{\pi'} \equiv a_{\sigma'} \pmod{U^1_{F'}}.$$

Parts (iii) of Lemma 2.5 and 1.4 now give

$$a_\pi \equiv a_\sigma \pmod{U^1_F}.$$

We recall the identity

$$\frac{\boldsymbol{\varepsilon}(\sigma', s, \psi_{F'})}{\boldsymbol{\varepsilon}(\sigma, s, \psi_F)^r} = \left(\frac{\boldsymbol{\varepsilon}(1_{F'}, s, \psi_{F'})}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{F'/F}, s, \psi_F)}\right)^{p^m} \boldsymbol{\delta}_{F'/F}(a_\sigma). \tag{2.6.1}$$

Here, $\boldsymbol{\rho}$, $\boldsymbol{\delta}$ are as in 1.5, $\psi_{F'} = \psi_F \circ \mathrm{Tr}_{F'/F}$, and $1_{F'}$ denotes the trivial character of $\mathcal{W}_{F'}$. ((2.6.1) follows from the defining property of $\boldsymbol{\varepsilon}$ and the Remarks in **[38]** §2.) The character $\boldsymbol{\delta}_{F'/F}$ is tame, so the above congruence between a_σ and a_π gives

$$\boldsymbol{\varepsilon}(\pi, s, \psi_F)^r = \boldsymbol{\varepsilon}(\sigma, s, \psi_F)^r,$$

by 2.6.1 and Theorem 1.6. In particular, these two local constants have the same exponential parts,

$$\frac{\boldsymbol{\varepsilon}(\pi, s, \psi_F)}{\boldsymbol{\varepsilon}(\pi, \frac{1}{2}, \psi_F)} = \frac{\boldsymbol{\varepsilon}(\sigma, s, \psi_F)}{\boldsymbol{\varepsilon}(\sigma, \frac{1}{2}, \psi_F)}.$$

By construction, the quasicharacters ω_π, $\det \sigma$ correspond by class field theory, so now we have

$$\boldsymbol{\varepsilon}(\pi, \tfrac{1}{2}, \psi_F) \equiv \boldsymbol{\varepsilon}(\sigma, \tfrac{1}{2}, \psi_F) \pmod{\boldsymbol{\mu}_{p^\infty}(\mathbb{C})}.$$

Since r is prime to p, we get $\boldsymbol{\varepsilon}(\pi, \frac{1}{2}, \psi_F) = \boldsymbol{\varepsilon}(\sigma, \frac{1}{2}, \psi_F)$, and hence

$$\boldsymbol{\varepsilon}(\pi, s, \psi_F) = \boldsymbol{\varepsilon}(\sigma, s, \psi_F),$$

as required to complete the proof of Theorem 2.3(v). □

2.7. We now prove Theorem 2.3(vi). It is enough, by transitivity, to treat the case where K/F is cyclic of prime degree $\ell \neq p$. We start with the representation σ, and denote its representation space by V. Let $\bar{\sigma}$ denote the associated homomorphism $\mathcal{W}_F \to \mathrm{Aut}_{\mathbb{C}}(V)/\mathbb{C}^\times$ and set $\mathcal{W}_E = \mathrm{Ker}\,\bar{\sigma}$, so that $\mathrm{Im}\,\bar{\sigma}$ may be identified with $\mathrm{Gal}(E/F)$, for some finite extension E/F. Choose a p-Sylow subgroup P of $\mathrm{Gal}(E/F)$ and let $F' = E^P$. Write $\sigma_{F'/F} = \sigma \mid \mathcal{W}_{F'}$, and similarly for other field extensions.

By definition, $\pi = \boldsymbol{\pi}(\sigma)$ is the unique element of $\mathcal{A}_m^{\mathrm{wr}}(F)$ with central quasicharacter corresponding to $\det\sigma$ and such that

$$\boldsymbol{l}_{F'/F}(\boldsymbol{\pi}(\sigma)) = {}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{F'/F}).$$

The assertion is thus obvious when $K \subset F'$ (and, since K/F is normal, this condition is independent of the choice of p-Sylow subgroup defining F').

We can therefore assume that the extensions K/F, E/F are linearly disjoint over F. Restriction of operators induces an isomorphism $\mathrm{Gal}(KE/K) \cong \mathrm{Gal}(E/F)$, so $F'K$ is the fixed field of a p-Sylow subgroup of the image of $\bar{\sigma}_{K/F}$. Thus $\boldsymbol{\pi}(\sigma_{K/F})$ is the unique element π_K, say, of $\mathcal{A}_m^{\mathrm{wr}}(K)$ whose central quasicharacter corresponds to $\det\sigma_{K/F} = (\det\sigma)\circ \mathrm{N}_{K/F}$ such that

$$\boldsymbol{l}_{KF'/K}(\pi_K) = {}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{KF'/F}).$$

The representation $\boldsymbol{l}_{K/F}(\pi)$ satisfies the first of these properties, by Theorem 1.3(vi). On the other hand, by transitivity of tame lifting,

$$\boldsymbol{l}_{KF'/K}\boldsymbol{l}_{K/F}(\pi) = \boldsymbol{l}_{KF'/F'}\boldsymbol{l}_{F'/F}(\pi).$$

Now, $\boldsymbol{l}_{F'/F}(\pi) = {}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{F'/F})$; by Theorem 1.8, we have $\boldsymbol{l}_{KF'/F'} = \boldsymbol{b}_{KF'/F'}$, and base change commutes with ${}^{\mathbf{c}}\boldsymbol{\pi}$. In all,

$$\boldsymbol{l}_{KF'/K}\boldsymbol{l}_{K/F}(\pi) = \boldsymbol{b}_{KF'/F'}({}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{F'/F})) = {}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{KF'/F}).$$

This implies $\pi_K = \boldsymbol{l}_{K/F}(\pi)$, as required. □

2.8. We now prove the second assertion of Theorem 2.3(vii), the first having been dealt with in 2.4 above. Thus, for $i = 1, 2$, we are given an integer $m_i \geqslant 0$ and a representation $\sigma_i \in \mathcal{G}_{m_i}^{\mathrm{wr}}(F)$. We set $\pi_i = \boldsymbol{\pi}(\sigma_i)$; we have to show that $f(\sigma_1 \otimes \sigma_2, \psi_F) = f(\pi_1 \times \pi_2, \psi_F)$.

Suppose first that $\sigma_i \in \mathcal{G}\mathbf{c}_{m_i}^{\mathrm{wr}}(F)$, $i = 1, 2$. Then, by definition, $\pi_i = {}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_i)$; we have already observed that the family $\{{}^{\mathbf{c}}\boldsymbol{\pi}\}$ preserves local constants of pairs, hence also conductors.

We reduce the general case to this one as follows. As in 2.7, we find a tame extension F_i'/F, of degree prime to p, such that

$$\boldsymbol{l}_{F_i'/F}(\pi_i) = {}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{i,F_i'/F}) \in \mathcal{A}\boldsymbol{c}_{m_i}^{\mathrm{wr}}(F).$$

Let K/F be the normal closure of $F_1'F_2'/F$. The extension K/F is tamely ramified and, by 1.8 Lemma, there are unramified quasicharacters χ_i and χ_i' of $K^\times$ such that

$$\boldsymbol{l}_{K/F}(\pi_i) = \boldsymbol{l}_{K/F_i'}\boldsymbol{l}_{F_i'/F}(\pi_i) = \chi_i'\boldsymbol{b}_{K/F_i'}\boldsymbol{l}_{F_i'/F}(\pi_i) = \chi_i\,\boldsymbol{b}_{K/F}(\pi_i), \quad i = 1, 2.$$

Thus

$$f(\boldsymbol{l}_{K/F}(\pi_1) \times \boldsymbol{l}_{K/F}(\pi_2), \psi_K) = f(\boldsymbol{b}_{K/F}(\pi_1) \times \boldsymbol{b}_{K/F}(\pi_2), \psi_K).$$

We have

$$\begin{aligned}\boldsymbol{b}_{K/F}(\pi_i) &= \chi_i^{-1}\chi_i'\,\boldsymbol{b}_{K/F_i'}\boldsymbol{l}_{F_i'/F}(\pi_i) \\ &= \chi_i^{-1}\chi_i'\,\boldsymbol{b}_{K/F_i'}{}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{i,F_i'/F}) = \chi_i^{-1}\chi_i'\,{}^{\mathbf{c}}\boldsymbol{\pi}(\sigma_{K/F}),\end{aligned}$$

since, by **[20]** Th. 7.12, the map ${}^{\mathbf{c}}\boldsymbol{\pi}$ is compatible with base change. So, by the first case, we have

$$f(\boldsymbol{l}_{K/F}(\pi_1) \times \boldsymbol{l}_{K/F}(\pi_2), \psi_K) = f(\sigma_{1,K/F} \otimes \sigma_{2,K/F}, \psi_K).$$

Theorem 1.7 gives the relation between

$$f(\boldsymbol{l}_{K/F}(\pi_1) \times \boldsymbol{l}_{K/F}(\pi_2), \psi_K) \quad \text{and} \quad f(\pi_1 \times \pi_2, \psi_F);$$

a simple exercise yields the same relation between

$$f(\sigma_{1,K/F} \otimes \sigma_{2,K/F}, \psi_K) \quad \text{and} \quad f(\sigma_1 \otimes \sigma_2, \psi_F).$$

The result follows. □

CHAPTER 3

CENTRAL TYPES

We now start the construction of the tame lifting map $\boldsymbol{l}_{K/F}$ required by Theorem 1.3. To this end, we introduce a family of representations of open, compact mod centre, subgroups of general linear groups. We call these "central types". They should be regarded as refinements of the simple characters which play a pivotal rôle in **[9]** and **[5]**. This chapter is mainly concerned with extending important properties of simple characters to the class of central types.

As with the simple characters which they refine, central types can only be described in terms of simple strata, but a simple stratum is not an invariant of the central types it defines. That is, some very different simple strata can give rise to the same families of central types. Thus, when treating central types, we have to take care that our arguments are invariant and, in particular, independent of the choice of simple stratum used to define the types.

To start with, we are given a finite-dimensional F-vector space V; we set $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A. *Here and throughout, we assume that the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ is principal.* We write $E = F[\beta]$, $B = \mathrm{End}_E(V)$, and $\mathfrak{B} = \mathfrak{A} \cap B$. In particular, $\mathfrak{B}$ is a principal $\mathfrak{o}_E$-order in B.

When there is no fear of ambiguity, we abbreviate

$$J = J(\beta, \mathfrak{A}), \quad J^1 = J^1(\beta, \mathfrak{A}) = J \cap \boldsymbol{U}^1(\mathfrak{A}), \quad H^1 = H^1(\beta, \mathfrak{A}) \cap \boldsymbol{U}^1(\mathfrak{A}).$$

3.1. We define an open compact mod centre subgroup $\boldsymbol{J}$ of G by

$$\boldsymbol{J} = \boldsymbol{J}(\beta, \mathfrak{A}) = E^\times J^1.$$

We can describe the group $\boldsymbol{J}$ in more intrinsic terms as follows. Choose $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$, and write $X(\theta)$ for the group of $g \in G$ which normalize H^1 and such that $\theta^g = \theta$. By **[9]** 3.3.17, we have $X(\theta) = \mathfrak{K}(\mathfrak{B})J^1$, where, we recall, $\mathfrak{K}(\mathfrak{B})$ is the $B^\times$-normalizer of the order $\mathfrak{B}$. In particular, this group depends only on $\mathcal{C}(\mathfrak{A}, \beta)$.

Lemma 1. — *In the notation above, the group J^1 is the unique maximal normal pro-p-subgroup of $X(\theta)$. Moreover, $\boldsymbol{J}/J^1$ is the centre of the group $X(\theta)/J^1$. If we have a simple stratum $[\mathfrak{A}, n, 0, \beta']$ with $\mathcal{C}(\mathfrak{A}, \beta') = \mathcal{C}(\mathfrak{A}, \beta)$, then $\boldsymbol{J}(\beta', \mathfrak{A}) = \boldsymbol{J}(\beta, \mathfrak{A})$.*

Proof. — Straightforward. □

This is a convenient moment to recall a remarkable property of simple characters.

Lemma 2. — *Let $\theta_1, \theta_2 \in \mathcal{C}(\mathfrak{A}, \beta)$, and suppose that θ_1 intertwines with θ_2 in G. There then exists $x \in \boldsymbol{U}(\mathfrak{A})$ which normalizes H^1 and such that $\theta_2 = \theta_1^x$. Moreover, this element x normalizes the group $\boldsymbol{J}$ and conjugation by x permutes the set $\mathcal{C}(\mathfrak{A}, \beta)$.*

Proof. — The first and last assertions are given by **[9]** 3.5.11. The element x must conjugate the $\mathfrak{K}(\mathfrak{A})$-normalizer $X(\theta_1)$ of θ_1 to that of θ_2. However, as above $X(\theta_1) = X(\theta_2) = \mathfrak{K}(\mathfrak{B})J^1$, so x normalizes this group. It likewise normalizes J^1 and hence the centre $\boldsymbol{J}/J^1$ of $X(\theta_i)/J^1$. It therefore normalizes $\boldsymbol{J}$. □

We recall (**[9]** 5.1.8) that, for $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$, there is a unique irreducible representation $\eta = \eta_\theta$ of J^1 containing θ; indeed, $\eta \mid H^1$ is a multiple of θ.

Lemma 3. — *There is an irreducible representation Λ of $\boldsymbol{J}$ such that $\Lambda \mid J^1 \cong \eta$.*

Proof. — The representation η admits extension to a representation κ of the group $J = J(\beta, \mathfrak{A})$ which is intertwined by every element of $B^\times$ **[9]** 5.2.2. In particular, κ is normalized by $X(\theta) = \mathfrak{K}(\mathfrak{B})J$. The quotient $X(\theta)/J$ is cyclic, so κ extends to $X(\theta)$. The restriction of this extension to $\boldsymbol{J}$ provides the desired extension of η. □

Since $E^\times$ is abelian, it follows that any irreducible representation of $\boldsymbol{J}$ containing η on J^1 is an extension of η. We refer to such a representation Λ as *a central type,* and to θ as the simple character *underlying* Λ. We write $\mathcal{CC}(\theta)$

for the set of equivalence classes of central types with underlying character θ and

$$\mathcal{CC}(\mathfrak{A},\beta) = \bigcup_{\theta} \mathcal{CC}(\theta),$$

with θ ranging over $\mathcal{C}(\mathfrak{A},\beta)$. Observe that the class of an irreducible representation ρ of $\boldsymbol{J}$ lies in $\mathcal{CC}(\theta)$ if and only if $\rho \mid H^1$ contains θ. Thus $\mathcal{CC}(\mathfrak{A},\beta)$ consists of equivalence classes of irreducible representations of $\boldsymbol{J}$ which contain some $\theta \in \mathcal{C}(\mathfrak{A},\beta)$ on restriction to H^1.

We shall need to know about the intertwining properties of central types. Let $\theta_i \in \mathcal{C}(\mathfrak{A},\beta)$ and $\Lambda_i \in \mathcal{CC}(\theta_i)$, $i = 1, 2$; if the Λ_i intertwine in G, so do the simple characters θ_i. By Lemma 2, there is $x \in \boldsymbol{U}(\mathfrak{A})$ such that $\theta_2 = \theta_1^x$, and, further, $\mathcal{CC}(\theta_2) = \mathcal{CC}(\theta_1)^x$. Thus we usually only need consider intertwining between elements of a given $\mathcal{CC}(\theta)$. At the moment, we can give only a partial result on this matter: it will be settled in 3.2 Corollary 1 below.

Lemma 4. — *Suppose that the $\mathfrak{o}_E$-order $\mathfrak{B}$ is maximal; let $\theta \in \mathcal{C}(\mathfrak{A},\beta)$ and let $\Lambda_1, \Lambda_2 \in \mathcal{CC}(\theta)$. The representations Λ_i then intertwine in G if and only if $\Lambda_1 \cong \Lambda_2$.*

Proof. — This follows from **[9]** 6.1.2. □

There is one other straightforward property which will be useful later. If $\Lambda \in \mathcal{CC}(\theta)$, the restriction of Λ to $F^\times$ is a multiple of a quasicharacter ω of $F^\times$; since $F^\times \cap H^1 = U_F^1$, this must satisfy

$$\omega \mid U_F^1 = \theta \mid U_F^1.$$

Given a quasicharacter ω satisfying this condition, we write $\mathcal{CC}(\theta,\omega)$ for the set of $\Lambda \in \mathcal{CC}(\theta)$ such that $\Lambda \mid F^\times$ is a multiple of ω.

Lemma 5. — *The set $\mathcal{CC}(\theta,\omega)$ has exactly $(E^\times : F^\times U_E^1) = (\boldsymbol{J} : F^\times J^1)$ elements.*

Proof. — Immediate from Clifford theory. □

3.2. We investigate relations between central types attached to different simple strata (in the same algebra A). This is weakly analogous to the theory of endo-equivalence of simple characters discussed in **[5]**.

Suppose first that we have simple strata $[\mathfrak{A}_i, n_i, 0, \beta]$, $i = 1, 2$, in A attached to the same element β. We recall from **[9]** §3.6 the existence of a canonical bijection

$$\tau = \tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta} : \mathcal{C}(\mathfrak{A}_1,\beta) \xrightarrow{\approx} \mathcal{C}(\mathfrak{A}_2,\beta).$$

As a consequence of [5] 8.7, this bijection respects intertwining. More precisely:

Lemma. — *Let $\theta, \theta' \in \mathcal{C}(\mathfrak{A}_1, \beta)$, and abbreviate $\tau = \tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}$. The following are equivalent:*

(i) *the characters θ, θ' intertwine in G;*
(ii) *the characters $\tau\theta, \tau\theta' \in \mathcal{C}(\mathfrak{A}_2, \beta)$ intertwine in G.*

The central types of 3.1 have a similar property:

Proposition. — *For $i = 1, 2$, let $[\mathfrak{A}_i, n_i, 0, \beta]$ be a simple stratum in A. Let $\theta_i \in \mathcal{C}(\mathfrak{A}_i, \beta)$, and suppose $\theta_2 = \tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}(\theta_1)$. There is a canonical bijection*

$$\mathcal{C}\tau : \mathcal{CC}(\theta_1) \xrightarrow{\approx} \mathcal{CC}(\theta_2)$$

and hence a canonical bijection

$$\mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta} : \mathcal{CC}(\mathfrak{A}_1, \beta) \xrightarrow{\approx} \mathcal{CC}(\mathfrak{A}_2, \beta).$$

Moreover, representations $\Lambda, \Lambda' \in \mathcal{CC}(\mathfrak{A}_1, \beta)$ intertwine in G if and only if the representations $\mathcal{C}\tau(\Lambda), \mathcal{C}\tau(\Lambda') \in \mathcal{CC}(\mathfrak{A}_2, \beta)$ intertwine in G.

Proof. — We proceed by examining a sequence of special cases. The first of these is:

(a) $\mathfrak{A}_1 \cong \mathfrak{A}_2$ as $\mathfrak{o}_F$-orders.

The principal $\mathfrak{o}_E$-orders $\mathfrak{B}_i = \mathfrak{A}_i \cap B$ are then isomorphic. It follows that there exists $b \in B^\times$ such that $\mathfrak{B}_2 = b^{-1}\mathfrak{B}_1 b$. Conjugation by b then induces a bijection between the sets $\mathcal{C}(\mathfrak{A}_i, \beta)$ and another between the sets $\mathcal{CC}(\mathfrak{A}_i, \beta)$. These bijections are in fact independent of the choice of b: if $b' \in B^\times$ also conjugates $\mathfrak{B}_1$ to $\mathfrak{B}_2$, then $b' = yb$, for some $y \in \mathfrak{K}(\mathfrak{B}_1)$. Each Λ is the restriction of a representation of $\mathfrak{K}(\mathfrak{B}_1)J^1(\beta, \mathfrak{A}_1)$ (as in the proof of 3.1 Lemma 3), and so conjugation by y fixes every $\Lambda \in \mathcal{CC}(\mathfrak{A}_1, \beta)$. The conjugation map $\mathcal{CC}(\mathfrak{A}_1, \beta) \to \mathcal{CC}(\mathfrak{A}_2, \beta)$ is then the desired map $\mathcal{C}\tau$; it certainly preserves intertwining.

We next treat the case:

(b) $\mathfrak{A}_1 \subset \mathfrak{A}_2$ and $\mathfrak{A}_2$ is maximal for the property of being normalized by $E^\times$.

This condition is equivalent to $\mathfrak{B}_2$ being a maximal $\mathfrak{o}_E$-order. It will be convenient to vary the notation: we set $\mathfrak{A}_2 = \mathfrak{A}_{\mathrm{M}}$, $\mathfrak{B}_2 = \mathfrak{B}_{\mathrm{M}}$, to emphasize their maximality. We then have a simple stratum $[\mathfrak{A}_{\mathrm{M}}, n_{\mathrm{M}}, 0, \beta]$, for some integer n_{M}. Take $\Lambda \in \mathcal{CC}(\mathfrak{A}_1, \beta)$ with underlying $\theta \in \mathcal{C}(\mathfrak{A}_1, \beta)$, and set

$$\theta_{\mathrm{M}} = \tau_{\mathfrak{A}_1,\mathfrak{A}_{\mathrm{M}},\beta}(\theta) \in \mathcal{C}(\mathfrak{A}_{\mathrm{M}}, \beta).$$

Let η (resp. η_{M}) be the unique irreducible representation of $J_1^1 = J^1(\beta, \mathfrak{A}_1)$ (resp. $J_{\mathrm{M}}^1 = J^1(\beta, \mathfrak{A}_{\mathrm{M}})$) containing θ (resp. θ_{M}). By **[9]** 5.1.16, there is a unique irreducible representation $\tilde{\eta}_{\mathrm{M}}$ of the group $J_{\mathrm{M}}^1 \boldsymbol{U}^1(\mathfrak{B}_1)$ such that $\tilde{\eta}_{\mathrm{M}} \mid J_{\mathrm{M}}^1 = \eta_{\mathrm{M}}$ and such that η, $\tilde{\eta}_{\mathrm{M}}$ induce the same (irreducible) representation ϱ of $\boldsymbol{U}^1(\mathfrak{A}_1)$: indeed, ϱ contains both η and η_{M} with multiplicity one. The uniqueness property of $\tilde{\eta}_{\mathrm{M}}$ shows that it must be stable under conjugation by $E^\times$. The same argument can then be taken a step further to establish the following: given $\Lambda \in \mathcal{CC}(\theta)$, there is a unique irreducible representation $\tilde{\Lambda}_{\mathrm{M}}$ of the group $E^\times J_{\mathrm{M}}^1 \boldsymbol{U}^1(\mathfrak{B}_1)$ which extends η_{M} and induces the same irreducible representation of $E^\times \boldsymbol{U}^1(\mathfrak{A}_1)$ as Λ. We put $\Lambda_{\mathrm{M}} = \tilde{\Lambda}_{\mathrm{M}} \mid E^\times J_{\mathrm{M}}^1$, and then $\Lambda \mapsto \Lambda_{\mathrm{M}}$ is the desired map $\mathcal{C}\tau_{\mathfrak{A}_1, \mathfrak{A}_{\mathrm{M}}, \beta}$. The construction shows it is bijective.

This construction of $\mathcal{C}\tau$ commutes with conjugation by $B^\times$. If $\Lambda, \Lambda' \in \mathcal{CC}(\mathfrak{A}_1, \beta)$ intertwine, then so do the corresponding representations Λ_{M}, Λ'_{M} in $\mathcal{CC}(\mathfrak{A}_{\mathrm{M}}, \beta)$.

At this point, we interrupt the proof to give a corollary.

Corollary 1

(i) *Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, let $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$, and let $\Lambda_1, \Lambda_2 \in \mathcal{CC}(\theta)$. Then Λ_1, Λ_2 intertwine in G if and only if $\Lambda_1 \cong \Lambda_2$.*

(ii) *For $i = 1, 2$, let $[\mathfrak{A}, n, 0, \beta_i]$ be a simple stratum in A, and let $\Lambda_i \in \mathcal{CC}(\mathfrak{A}, \beta_i)$. Suppose that the Λ_i intertwine in G. There then exists $x \in \boldsymbol{U}(\mathfrak{A})$ such that conjugation by x maps $\mathcal{CC}(\mathfrak{A}, \beta_1)$ bijectively to $\mathcal{CC}(\mathfrak{A}, \beta_2)$ and $\Lambda_2 \cong \Lambda_1^x$.*

Proof. — We first choose a hereditary order $\mathfrak{A}_{\mathrm{M}}$ containing $\mathfrak{A}$, normalized by $E^\times$ and maximal for this property. We use notation analogous to that above.

In part (i), if the Λ_i intertwine, the corresponding central types $\Lambda_{i,\mathrm{M}}$ intertwine. Then $\Lambda_{1,\mathrm{M}} \cong \Lambda_{2,\mathrm{M}}$ by 3.1 Lemma 4; since $\mathcal{C}\tau$ is a bijection, we have $\Lambda_1 \cong \Lambda_2$.

To prove part (ii), let $\theta_i \in \mathcal{C}(\mathfrak{A}, \beta_i)$ be the simple character underlying Λ_i, $i = 1, 2$. Since the Λ_i intertwine in G, so do the θ_i. So, by **[9]** 3.5.11, there exists $x \in \boldsymbol{U}(\mathfrak{A})$ such that $\theta_2 = \theta_1^x$ and, further, $\mathcal{C}(\mathfrak{A}, \beta_2) = \mathcal{C}(\mathfrak{A}, \beta_1^x) = \mathcal{C}(\mathfrak{A}, \beta_1)^x$. This implies (3.1 Lemma 1) that $\boldsymbol{J}(\beta_2, \mathfrak{A}) = \boldsymbol{J}(\beta_1^x, \mathfrak{A})$ and then that $\mathcal{CC}(\mathfrak{A}, \beta_2) = \mathcal{CC}(\mathfrak{A}, \beta_1^x) = \mathcal{CC}(\mathfrak{A}, \beta_1)^x$. This reduces us to the case where $\beta_1 = \beta_2$ and $\theta_1 = \theta_2$, and the result now follows from (i). □

We return to the proof of the proposition. To treat the general case, we can replace $\mathfrak{A}_2$ by a $B^\times$-conjugate and assume that there is a maximal $\mathfrak{o}_E$-order $\mathfrak{B}_{\mathrm{M}}$ containing both $\mathfrak{B}_i$. Defining $\mathfrak{A}_{\mathrm{M}}$ to be the $E^\times$-invariant principal $\mathfrak{o}_F$-order satisfying $\mathfrak{A}_{\mathrm{M}} \cap B = \mathfrak{B}_{\mathrm{M}}$, we can then put

$$\mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta} = \mathcal{C}\tau^{-1}_{\mathfrak{A}_2,\mathfrak{A}_{\mathrm{M}},\beta} \circ \mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_{\mathrm{M}},\beta}.$$

Any two choices of $\mathfrak{B}_{\mathrm{M}}$ are conjugate under both $\mathfrak{K}(\mathfrak{B}_i)$, and it follows that this definition is independent of $\mathfrak{B}_{\mathrm{M}}$.

We now prove the second assertion of the proposition. We fix a quasicharacter ω of $F^\times$ and put

$$\mathcal{CC}(\mathfrak{A},\beta,\omega) = \bigcup \mathcal{CC}(\theta,\omega).$$

This set is finite. The bijection $\mathcal{C}\tau$ takes $\mathcal{CC}(\mathfrak{A},\beta,\omega)$ to $\mathcal{CC}(\mathfrak{A}_{\mathrm{M}},\beta,\omega)$. By Corollary 1, intertwining is an *equivalence* relation on these sets, and it is respected by $\mathcal{C}\tau$. However, the number of intertwining classes in either of these sets is $(E^\times : F^\times U_E^1)$ times the number of conjugacy classes of $\theta \in \mathcal{C}(\mathfrak{A},\beta)$ which agree with ω on U_F^1, again by Corollary 1 and 3.1 Lemma 5. Thus $\mathcal{C}\tau$ is bijective on conjugacy classes, and the result follows. □

Comment. — The procedure used above to construct $\mathcal{C}\tau_{\mathfrak{A}_{\mathrm{M}},\mathfrak{A}_1,\beta}$ can also be used to give a direct construction of $\mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta}$ whenever $\mathfrak{A}_1 \supset \mathfrak{A}_2$. It is an easy exercise (*cf.* **[9]** 5.1.18) to check that this gives the same result as above:

$$\mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta} = \mathcal{C}\tau_{\mathfrak{A}_{\mathrm{M}},\mathfrak{A}_2,\beta} \circ \mathcal{C}\tau^{-1}_{\mathfrak{A}_{\mathrm{M}},\mathfrak{A}_1,\beta},$$

where $\mathfrak{A}_{\mathrm{M}}$ contains $\mathfrak{A}_1$ and $\mathfrak{A}_{\mathrm{M}} \cap B$ is maximal.

Corollary 2. — *Suppose we have simple strata $[\mathfrak{A}_i, n_{ij}, 0, \beta_j]$ in A, $i,j \in \{1,2\}$, with the property*

$$\mathcal{C}(\mathfrak{A}_i,\beta_1) = \mathcal{C}(\mathfrak{A}_i,\beta_2), \quad i = 1,2.$$

Abbreviate $\mathcal{C}\tau_j = \mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta_j}$. Then, for any $\Lambda \in \mathcal{CC}(\mathfrak{A}_1,\beta_1) = \mathcal{CC}(\mathfrak{A}_1,\beta_2)$, the representations $\mathcal{C}\tau_j(\Lambda) \in \mathcal{CC}(\mathfrak{A}_2,\beta_j)$ intertwine, and hence are conjugate, in G.

Proof. — We choose a principal $\mathfrak{o}_F$-order $\mathfrak{A}_{\mathrm{M}}$ containing $\mathfrak{A}_1$, normalized by $F[\beta_1]^\times$ and maximal for this property. This order $\mathfrak{A}_{\mathrm{M}}$ is obtained as follows: it is attached to a uniform lattice chain, contained in that defining $\mathfrak{A}_1$, of period $e(F[\beta_1]|F)$, e here denoting ramification index. However, the equality of the $\mathcal{C}(\mathfrak{A}_1,\beta_j)$ implies that the ramification indices $e(F[\beta_j]|F)$ are equal **[9]** 3.5.1. It follows that $\mathfrak{A}_{\mathrm{M}}$ is also normalized by $F[\beta_2]^\times$ and maximal for this property.

Let us assume first that $\mathfrak{A}_2 = \mathfrak{A}_{\mathrm{M}}$. By construction, the representations $\mathcal{C}\tau_j(\Lambda)$ both occur in the irreducible representation of the (j-independent) group $F[\beta_j]^\times \boldsymbol{U}^1(\mathfrak{A}_1)$ induced by Λ. They therefore intertwine, and indeed are $\boldsymbol{U}(\mathfrak{A}_{\mathrm{M}})$-conjugate, by Corollary 1.

We return to the general case and let $\mathfrak{A}_i^{\mathrm{M}} \supset \mathfrak{A}_i$ be maximal for the property of being normalized by $F[\beta_j]^\times$. In particular, the orders $\mathfrak{A}_i^{\mathrm{M}}$ are isomorphic. As we saw in the proof of the Proposition, the process of transferring central types between isomorphic orders is achieved by a conjugation, which certainly preserves intertwining. □

Remark. — In the context of the last corollary, it is quite possible for the two maps $\mathcal{C}\tau_{\mathfrak{A}_1,\mathfrak{A}_2,\beta_j}$ to be distinct; the corollary says that the maps they induce on conjugacy classes *are* the same.

3.3. We need a way of forming a direct sum of copies of a given central type. We modify our notation slightly; we start with a vector space V_0, and set $A_0 = \mathrm{End}_F(V_0)$, $G_0 = \mathrm{Aut}_F(V_0)$. We are given a simple stratum $[\mathfrak{A}_0, n_0, 0, \beta]$ in A_0 (and, as always, $\mathfrak{A}_0$ is assumed principal). We set $E = F[\beta]$. We consider the vector space

$$V = V_0 \oplus V_0 \oplus \cdots \oplus V_0 \quad (t \text{ copies})$$

and put $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. We let M be the Levi subgroup of G which stabilizes this decomposition of V, and choose a pair of mutually opposite parabolic subgroups $P_{\mathrm{u}} = MN_{\mathrm{u}}$, $P_\ell = MN_\ell$ of G with Levi component M.

Following the procedure of **[9]** 7.1, 7.2, we can find a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, in which $\mathfrak{A}$ is principal with period $e(\mathfrak{A}) = te(\mathfrak{A}_0)$, and $n = tn_0$, having the following properties. (We abbreviate $H_0^1 = H^1(\beta, \mathfrak{A}_0)$, $H^1 = H^1(\beta, \mathfrak{A})$, and likewise for J.) First,

$$H^1 = H^1 \cap N_\ell \cdot H^1 \cap M \cdot H^1 \cap N_{\mathrm{u}},$$

$$H^1 \cap M = H_0^1 \times H_0^1 \times \cdots \times H_0^1,$$

and likewise for J. Next, given $\theta_0 \in \mathcal{C}(\mathfrak{A}_0, \beta)$, there is a unique simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$ which is trivial on $H^1 \cap N_\ell$ and $H^1 \cap N_{\mathrm{u}}$, while

$$\theta \mid H^1 \cap M = \theta_0 \otimes \theta_0 \otimes \cdots \otimes \theta_0.$$

Indeed, $\theta_0 \mapsto \theta$ is the canonical bijection $\tau_{\mathfrak{A}_0,\mathfrak{A},\beta}$ of **[9]** 3.6.

Let η_0 be the irreducible representation of J_0^1 which contains θ_0. We obtain the irreducible representation η of J^1 which contains θ as follows. We form the group

$$\widetilde{J}^1 = H^1 \cap N_\ell \cdot J^1 \cap M \cdot J^1 \cap N_{\mathrm{u}}.$$

This carries a representation $\tilde{\eta}$ which is trivial on the unipotent factors $\widetilde{J}^1 \cap N_\ell$, $\widetilde{J}^1 \cap N_{\mathrm{u}}$ while

$$\tilde{\eta} \mid J^1 \cap M = \eta_0 \otimes \eta_0 \otimes \cdots \otimes \eta_0.$$

The representation of J^1 induced by $\tilde{\eta}$ is then η *loc. cit.*

Now suppose we are given $\Lambda_0 \in \mathcal{CC}(\theta_0)$. We define a representation $\tilde{\Lambda}$ of the group $E^\times \widetilde{J}^1$ by the following conditions: $\tilde{\Lambda}$ is trivial on $\widetilde{J}^1 \cap N_\ell$, $\widetilde{J}^1 \cap N_{\mathrm{u}}$, while

$$\tilde{\Lambda} \mid E^\times J^1 \cap M = (\Lambda_0 \otimes \Lambda_0 \otimes \cdots \otimes \Lambda_0) \mid E^\times J^1 \cap M.$$

The representation Λ of $E^\times J^1 = \boldsymbol{J}(\beta, \mathfrak{A})$ induced by $\tilde{\Lambda}$ then lies in $\mathcal{CC}(\theta)$. We denote it

$$\Lambda = \Lambda_0^{(t)}.$$

Proposition

(i) *If* $\Lambda_i \in \mathcal{CC}(\mathfrak{A}_0, \beta_i)$, $i = 1, 2$, *intertwine in* G_0, *then the representations* $\Lambda_i^{(t)}$ *intertwine in* G.

(ii) *For integers* $r, s \geqslant 1$ *with* $rs = t$, *the representations* $\Lambda_0^{(t)}$, $(\Lambda_0^{(r)})^{(s)}$ *are conjugate in* G.

Proof. — As in the construction above, we obtain $\Lambda_i^{(t)}$ by induction from a representation $\tilde{\Lambda}_i$ of $F[\beta_i]^\times \widetilde{J}^1(\beta_i, \mathfrak{A})$. Suppose the Λ_i intertwine, so there exists $x \in \boldsymbol{U}(\mathfrak{A}_0)$ such that $\Lambda_2 = \Lambda_1^x$. We view G_0 as embedded in M on the diagonal; the representations $\tilde{\Lambda}_2$, $\tilde{\Lambda}_1$ are then visibly intertwined by x. The induced representations $\Lambda_i^{(t)}$ are then intertwined by x.

In part (ii), the orders underlying the central types $\Lambda_0^{(t)}$, $(\Lambda_0^{(r)})^{(s)}$ are $B^\times$-conjugate, and any such conjugation takes $\Lambda_0^{(t)}$ to $(\Lambda_0^{(r)})^{(s)}$. □

Remark. — The definition of $\Lambda_0^{(t)}$ is motivated by the following consideration. Let π_0 be an irreducible smooth representation of G_0 which contains Λ_0. Assume for simplicity that E is a maximal subfield of A_0, so that π_0 is supercuspidal. We can form the representation $\pi_0 \otimes \pi_0 \otimes \cdots \otimes \pi_0$ of M and parabolically induce to get an irreducible representation π of G. The point is that π will contain the central type $\Lambda_0^{(t)}$ (*cf.* **[9]** 7.3.14).

CHAPTER 4

BASE FIELD EXTENSION FOR CENTRAL TYPES

As in §3, we start with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in some $\mathrm{End}_F(V)$, with $n > 0$ and $\mathfrak{A}$ principal. We set $E = F[\beta]$. Throughout this section, *we assume that the field extension E/F is totally wildly ramified.* That is, the maximal tame sub-extension of E/F is F.

We fix a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$, and choose a quasicharacter ω of $F^\times$ agreeing with θ on $H^1(\beta, \mathfrak{A}) \cap F^\times = U_F^1$. As before, we write $\mathcal{CC}(\theta, \omega)$ for the set of $\Lambda \in \mathcal{CC}(\theta)$ which restrict to a multiple of ω on $F^\times$.

We use our standard abbreviations $\boldsymbol{J} = \boldsymbol{J}(\beta, \mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$, etc.

4.1. We are now given a finite, *tamely ramified* field extension K/F. The algebra $K \otimes_F E$ is then a field, which we denote more briefly KE. We assume that the underlying vector space V is a KE-space, this structure extending the given E-structure on V. We also assume that $\mathfrak{A}$ is normalized by $(KE)^\times$.

We set $A_K = \mathrm{End}_K(V)$, $G_K = \mathrm{Aut}_K(V)$. Following the procedures of **[5]** (especially sections 2 and 7), we form the principal $\mathfrak{o}_K$-order $\mathfrak{A}_K = \mathfrak{A} \cap A_K$; this gives us a simple stratum $[\mathfrak{A}_K, n, 0, \beta]$ in A_K **[5]** (2.4). By **[5]** (7.1), the groups

$$J_K^1 = J^1(\beta, \mathfrak{A}_K), \quad H_K^1 = H^1(\beta, \mathfrak{A}_K)$$

are given by

$$H_K^1 = H^1 \cap G_K, \quad J_K^1 = J^1 \cap G_K.$$

The character $\theta_K = \theta \mid H_K^1$ then lies in $\mathcal{C}(\mathfrak{A}_K, \beta)$ *ibid.* (7.7). We write η_K for the unique irreducible representation of J_K^1 which contains θ_K.

Remark. — The character θ_K is the tame lift of the simple character θ, in the sense of **[5]**. See especially *ibid.* §11. To use the precise language of that paper, under our hypothesis that $K \otimes_F E$ is a field, the endo-equivalence class of θ_K is the unique K/F-lift of the endo-equivalence class of θ.

We use the obvious analogues of the notations above when working relative to the base field K. In particular, $\boldsymbol{J}_K$ denotes the group $(KE)^\times J^1_K$ (which, under our present hypotheses, equals $K^\times E^\times J^1_K$).

The groups $\boldsymbol{J}$, J^1, H^1 are stable under conjugation by $K^\times$, the groups of fixed points being respectively $E^\times J^1_K$, J^1_K, H^1_K. This action stabilizes θ and $\operatorname{Ker}\theta$; we have $\operatorname{Ker}\theta \cap \boldsymbol{J}_K = \operatorname{Ker}\theta_K$.

Lemma 1. — *The group of fixed points for the natural action (by conjugation) of $K^\times$ on $\boldsymbol{J}/\operatorname{Ker}\theta$ is $E^\times J^1_K/\operatorname{Ker}\theta_K$.*

Proof. — We choose a prime element ϖ_K of K such that $\varpi_K^{e(K|F)}$ is of the form $\varpi_F\mu$, where ϖ_F is some prime element of F and $\mu \in K$ is a root of unity of order prime to p. Let C denote the group generated by ϖ_K and the group $\mu'_p(K)$ of roots of unity in K of order prime to p. Then C acts on $\boldsymbol{J}$ via the finite quotient $C/C \cap F^\times$, which has order prime to p. The set of fixed points is $E^\times J^1_K$. We take the cohomology of the sequence

$$1 \to \operatorname{Ker}\theta \to \boldsymbol{J} \to \boldsymbol{J}/\operatorname{Ker}\theta \to 1.$$

The group $\operatorname{Ker}\theta$ is a pro-p group; it has an obvious $K^\times$-stable filtration, in which the factors are finite abelian p-groups. A routine filtration argument shows that the cohomology set $\mathrm{H}^1(C/C \cap F^\times, \operatorname{Ker}\theta)$ reduces to a singleton, and the result follows. □

The action of $K^\times$ on $\boldsymbol{J}/\operatorname{Ker}\theta$ factors through the finite quotient $K^\times/F^\times U^1_K$ which, since K/F is tame, has order prime to p.

We apply the Glauberman correspondence of Appendix A2 (see especially A2.2 and A2.4), first to the action of $K^\times$ on $J^1/\operatorname{Ker}\theta$. We thus get a canonical bijection $\boldsymbol{g}^1_{K/F}$ between (equivalence classes of) $K^\times$-stable irreducible representations of J^1 trivial on $\operatorname{Ker}\theta$ and (equivalence classes of) irreducible representations of J^1_K trivial on $\operatorname{Ker}\theta_K$.

Lemma 2. — *The correspondence $\boldsymbol{g}^1_{K/F}$ takes η to η_K.*

Proof. — We first assume that there is an element $\zeta \in K^\times$ such that the group $\langle\zeta\rangle$ maps onto $K^\times/F^\times U^1_K$. This will hold if, for example, K/F is either unramified or totally tamely ramified: in the first case we take for ζ a root of unity of order prime to p such that $K = F[\zeta]$ and, in the second, ζ is a prime element of K such that $\zeta^{[K:F]} \in F$.

With this hypothesis, if η' corresponds to η, we have A2.4:

$$\mathrm{tr}\eta'(j) = \epsilon\, \mathrm{tr}\tilde{\eta}(\zeta j), \quad j \in J_K^1,$$

where $\epsilon = \pm 1$ and $\tilde{\eta}$ is the canonical extension of η to $\langle\zeta\rangle \ltimes J^1$, in the sense of A2.3. Taking $j \in H_K^1$, we get the relation $\mathrm{tr}\eta'(j) = \epsilon\, \mathrm{tr}\tilde{\eta}(\zeta)\theta(j)$, while $\theta(j) = \theta_K(j)$. We deduce that η' is an irreducible representation of J_K^1 whose restriction to H_K^1 is a multiple of θ_K. This implies $\eta' \cong \eta_K$.

The general case of the lemma now follows by transitivity of the Glauberman correspondence. □

Proposition. — *Let ω_K be a quasicharacter of $K^\times$ such that $\omega_K \mid F^\times = \omega$ and $\omega_K \mid U_K^1 = \theta_K \mid U_K^1$.*

(i) *There is a canonical bijection (to be described in the proof)*

$$\boldsymbol{g}_{K/F} : \mathcal{CC}(\theta,\omega) \xrightarrow{\approx} \mathcal{CC}(\theta_K,\omega_K). \tag{4.1.1}$$

(ii) *Let $F \subset K' \subset K$, and put*

$$\omega_{K'} = \omega_K \mid K'^\times, \qquad \theta_{K'} = \theta \mid H^1 \cap \mathrm{Aut}_{K'}(V).$$

Then $\boldsymbol{g}_{K/F}$ is the composite map

$$\boldsymbol{g}_{K/F} : \mathcal{CC}(\theta,\omega) \xrightarrow{\boldsymbol{g}_{K'/F}} \mathcal{CC}(\theta_{K'},\omega_{K'}) \xrightarrow{\boldsymbol{g}_{K/K'}} \mathcal{CC}(\theta_K,\omega_K).$$

Proof. — We first observe that $\Lambda_K \mapsto \Lambda_K \mid E^\times J_K^1$ induces a bijection between $\mathcal{CC}(\theta_K,\omega_K)$ and the set of irreducible representations of $E^\times J_K^1$ which extend η_K and restrict to a multiple of ω on $F^\times$.

Choose a prime element ϖ_F of F and write $\mu_p'(F)$ for the group of roots of unity in F of order prime to p. Assume to start with that ω is trivial on the group generated by ϖ_F and $\mu_p'(F)$. If ϖ_E is a prime element of E, then $\varpi_E^{p^m} \in \varpi_F \mu_p'(F) U_E^1$. Any $\Lambda \in \mathcal{CC}(\theta,\omega)$ is then effectively a representation of the finite p-group $\boldsymbol{J}/\langle \mathrm{Ker}\,\theta, \varpi_F, \mu_p'(F)\rangle$. The group of $K^\times$-fixed points here is $E^\times J_K^1/\langle \mathrm{Ker}\,\theta, \varpi_F, \mu_p'(F)\rangle$. We apply the Glauberman correspondence to get a bijection between $\mathcal{CC}(\theta,\omega)$ and the set of equivalence classes of irreducible representations of $E^\times J_K^1$ which extend η_K and restrict to a multiple of ω on $F^\times$. This set, as we have seen, is in bijection with $\mathcal{CC}(\theta_K,\omega_K)$, so we get the desired bijection $\boldsymbol{g}_{K/F}$ in this case.

In the general case, there is a tamely ramified quasicharacter χ_0 of $F^\times$ such that $\Lambda \otimes \chi_0 \circ \det$ satisfies the conditions of the special case above, for any $\Lambda \in \mathcal{CC}(\theta,\omega)$. We set

$$\boldsymbol{g}_{K/F}(\Lambda) \mid E^\times J_K^1 = \boldsymbol{g}_{K/F}(\Lambda \otimes \chi_0 \circ \det) \otimes (\chi_0^{-1} \circ \det) \mid G_K.$$

This has the desired properties (and is independent of the choices of ϖ_F and χ_0).

Part (ii) follows from the corresponding property of the Glauberman correspondence A2.2. □

One can set this up slightly differently, using "complementary subgroups". A complementary subgroup of $F^\times$ is a subgroup C_F such that $F^\times = C_F \times U_F^1$. Obviously, to define such a thing, we need only choose a prime element ϖ_F and set $C_F = \langle \varpi_F, \mu_p'(F)\rangle$. Given a finite tame extension K/F, one can find a complementary subgroup C_K of $K^\times$ such that $C_K \cap F^\times$ is a complementary subgroup of $F^\times$. Indeed, one can arrange for $C_K \cap K'^\times$ to be a complementary subgroup of $K'^\times$, for every intermediate field $F \subset K' \subset K$. One can then define a Glauberman correspondence using the action of C_K on $\boldsymbol{J}$. This factors through $C_K/C_K \cap F^\times$. One obtains the same correspondence.

4.2. Let us exhibit the Glauberman correspondence 4.1.1 in a particularly important special case, using the general character relation A2.4.

Special case. — *Suppose there is an element $\zeta \in K^\times$ whose centralizer in $J^1/\operatorname{Ker}\theta$ is $J_K^1/\operatorname{Ker}\theta_K$. Let $\Lambda \in \mathcal{CC}(\theta,\omega)$, and let $\Lambda_K = \mathfrak{g}_{K/F}(\Lambda)$. Write $\widetilde{\Lambda}$ for the canonical extension (see* (A2.3)*) of Λ to $\langle\zeta\rangle \ltimes \boldsymbol{J}$. We then have*

$$\operatorname{tr}\Lambda_K(j) = \epsilon\,\operatorname{tr}\widetilde{\Lambda}(\zeta j), \tag{4.2.1}$$

for all $j \in E^\times J_K^1$. The factor $\epsilon \in \{\pm 1\}$ depends only on the subgroup of $K^\times/F^\times U_K^1$ generated by ζ; it is, in particular, independent of the choice of representation $\Lambda \in \mathcal{CC}(\mathfrak{A},\beta)$.

As we observed in the proof of 4.1 Lemma 2, such an element ζ will exist when K/F is either unramified or totally tamely ramified.

Of the assertions above, only the last requires any comment. We can retrieve the 'Glauberman sign' ϵ from the representations η, η_K as follows. Let $\widetilde{\eta}$ denote the canonical (in the sense of (A2.3)) extension of η to $\langle\zeta\rangle \ltimes J^1$. Then

$$\epsilon = \operatorname{tr}\widetilde{\eta}(\zeta)/\dim\eta_K. \tag{4.2.2}$$

Comment. — There is a procedure **[14]** for computing the sign ϵ. Since we never need to know it, we do not pursue the matter.

4.3. We return to the situation of 4.1. We take a quasicharacter χ of $F^\times$ and set $N = \dim_K(V)$, $d = [K{:}F]$. We put $\chi_K = \chi \circ \mathrm{N}_{K/F}$. There is an element $a \in F$ such that $\theta \otimes \chi \circ \det_G$ lies in $\mathcal{C}(\mathfrak{A}, \beta{+}a)$ (*cf.* [**8**] Appendix). Note *loc. cit.* that $H^1(\beta{+}a, \mathfrak{A}) = H^1$ and likewise for J^1. The procedure of 4.1 gives a bijection

$$\boldsymbol{g}_{K/F} : \mathcal{CC}(\theta \otimes \chi \circ \det_G, \omega\chi^{Nd}) \xrightarrow{\approx} \mathcal{CC}(\theta_K \otimes \chi_K \circ \det_{G_K}, \omega_K \chi_K^N).$$

Proposition. — *Use the notation of* 4.1. *For any quasicharacter χ of $F^\times$, we have*

$$\boldsymbol{g}_{K/F}(\Lambda \otimes \chi \circ \det_G) = \boldsymbol{g}_{K/F}(\Lambda) \otimes \chi_K \circ \det_{G_K}, \quad \Lambda \in \mathcal{CC}(\theta, \omega),$$

where $\chi_K = \chi \circ \mathrm{N}_{K/F}$.

One verifies this using transitivity and the trace relation 4.2.1.

4.4. It will be useful to have a slightly different description of the map $\boldsymbol{g}_{K/F}$. Here, it is preferable to use a complementary subgroup C_K of $K^\times$ (as at the end of 4.1) to define $\boldsymbol{g}_{K/F}$.

Let us take $\Lambda \in \mathcal{CC}(\theta) \subset \mathcal{CC}(\mathfrak{A}, \beta)$ as above. The representation η induces an irreducible representation ρ of the group $\boldsymbol{U}^1(\mathfrak{A})$, so Λ induces an irreducible representation R of $E^\times \boldsymbol{U}^1(\mathfrak{A})$ such that $R \mid \boldsymbol{U}^1(\mathfrak{A}) = \rho$.

Likewise, $\Lambda_K \mid E^\times J_K^1$ induces an irreducible representation R_K of the group $E^\times \boldsymbol{U}^1(\mathfrak{A}_K)$. The restriction of R_K to $\boldsymbol{U}^1(\mathfrak{A}_K)$ is the irreducible representation ρ_K induced by η_K.

On the other hand, we can apply the Glauberman correspondence to the action of C_K on $E^\times \boldsymbol{U}^1(\mathfrak{A})$; we get a bijection $\boldsymbol{G}_{K/F}$ between C_K-invariant irreducible smooth representations of $E^\times \boldsymbol{U}^1(\mathfrak{A})$ and irreducible smooth representations of $E^\times \boldsymbol{U}^1(\mathfrak{A}_K)$. The Glauberman correspondence commutes with irreducible induction A2.6(i), so we have

$$R_K = \boldsymbol{G}_{K/F}(R). \tag{4.4.1}$$

4.5. We return to the situation of (4.1) and record two more useful technical properties of the Glauberman correspondence $\boldsymbol{g}_{K/F}$. The first concerns the "power operation" $\Lambda \mapsto \Lambda^{(t)}$ defined in 3.3.

Proposition

(i) *Let $\Lambda \in \mathcal{CC}(\theta, \omega)$, and $t \geqslant 1$. Then*

$$\boldsymbol{g}_{K/F}(\Lambda^{(t)}) = \boldsymbol{g}_{K/F}(\Lambda)^{(t)}.$$

(ii) *Let* $[\mathfrak{A}', n', 0, \beta]$ *be a simple stratum in* V *with* $\mathfrak{A}'$ *principal and stable under conjugation by* $(KE)^\times$. *Let* $\theta' = \tau\theta \in \mathcal{C}(\mathfrak{A}', \beta)$, *let* $\Lambda \in \mathcal{CC}(\theta, \omega)$ *and set* $\Lambda' = \mathcal{C}\tau(\Lambda) \in \mathcal{CC}(\theta', \omega)$. *Then* $\boldsymbol{g}_{K/F}(\Lambda')$ *intertwines with* $\mathcal{C}\tau(\boldsymbol{g}_{K/F}(\Lambda))$, *and so* $\boldsymbol{g}_{K/F}(\Lambda')$, $\mathcal{C}\tau(\boldsymbol{g}_{K/F}(\Lambda))$ *are conjugate in* $\mathrm{Aut}_K(V)$.

Proof. — (i) One has only to check that the Glauberman correspondence commutes with the induction step (from $E^\times \widetilde{J}^1$ to $E^\times J^1$) in the definition (3.3) of $\Lambda^{(t)}$. This, however, follows from A2.6(i).

(ii) The construction 3.2 of the bijections $\mathcal{C}\tau$ reduces us to two cases. In the first, the $\mathfrak{o}_F$-orders $\mathfrak{A}$, $\mathfrak{A}'$ are isomorphic. They are then conjugate by an element of the G-centralizer of the field KE, and such a conjugation respects our subsequent constructions. We can thus assume $\mathfrak{A} \supset \mathfrak{A}'$ and that $\mathfrak{A}$ is maximal for the property of being normalized by $(KE)^\times$. We recall that Λ, Λ' are related as follows. Let η be the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ containing θ, and define η' analogously. There is a unique extension $\tilde{\eta}$ to $\boldsymbol{U}^1(\mathfrak{B}')$, $\mathfrak{B}' = \mathfrak{A}' \cap B$, such that $\tilde{\eta}$ and η' induce the same representation of $\boldsymbol{U}^1(\mathfrak{A}')$. We extend Λ to a representation $\tilde{\Lambda}$ of $\boldsymbol{J}\boldsymbol{U}^1(\mathfrak{B}')$ via $\tilde{\eta}$, and then $\tilde{\Lambda}$ and Λ' induce the same representation R of $E^\times \boldsymbol{U}^1(\mathfrak{A}')$.

The variant construction 4.4 shows that the Glauberman correspondent R_K of R is induced by $\boldsymbol{g}_{K/F}(\Lambda')$. However, it is also induced by some extension of $\boldsymbol{g}_{K/F}(\Lambda)$, so $\boldsymbol{g}_{K/F}(\Lambda)$ intertwines with $\boldsymbol{g}_{K/F}(\Lambda')$, as required. The final assertion follows from 3.2 Corollary 1(ii). □

CHAPTER 5

CONSTRUCTION OF THE TAME LIFT

5.1. We arrive at the construction of the tame lifting map $\boldsymbol{l}_{K/F}$ of 1.3.1. We need to vary our notation a little. We start with a finite-dimensional F-vector space V, and set $A_F = \mathrm{End}_F(V)$, $G_F = \mathrm{Aut}_F(V)$. To avoid the trivial case 1.3.2, we assume $\dim V > 1$.

Take $\pi_F \in \boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_F)$. Let $[\mathfrak{A}_F, n_F, 0, \beta]$ be a simple stratum in A_F underlying some maximal simple type occurring in π_F. Thus $\mathfrak{A}_F$ is principal and maximal for being normalized by $F[\beta]^\times$. Set $J_F = J(\beta, \mathfrak{A}_F)$. Let $\theta_F \in \mathcal{C}(\mathfrak{A}_F, \beta)$ be the simple character appearing in the maximal simple type. By definition (see 1.1), the field $E = F[\beta]$ is maximal in A_F and totally wildly ramified over F. Moreover, there is a central type $\Lambda_F \in \mathcal{CC}(\theta_F)$ occurring in π_F. The representation π_F is induced by Λ_F, and π_F determines $(\boldsymbol{J}_F, \Lambda_F)$ uniquely, up to G_F-conjugacy ([**9**] 6.2.4 or 3.2 Corollary 1 above). Indeed, the maximal simple type in π_F is just the restriction of Λ_F to $J_F = J(\beta, \mathfrak{A}_F)$.

We now abbreviate

$$H^1_F = H^1(\beta, \mathfrak{A}_F), \quad J^1_F = J^1(\beta, \mathfrak{A}_F), \quad \boldsymbol{J}_F = E^\times J^1_F = E^\times J(\beta, \mathfrak{A}_F).$$

We write η_F for the unique irreducible representation of J^1_F such that $\eta_F \mid H^1_F$ contains θ_F.

5.2. We use the notation of 5.1, and set $\dim_F V = [E{:}F] = p^m$.

Let K/F be a finite, tamely ramified, field extension of degree d. We consider the F-vector space $V \otimes_F K$, and set

$$A = \mathrm{End}_F(V \otimes K), \quad G = \mathrm{Aut}_F(V \otimes K).$$

Then $KE = K \otimes_F E$ is a field; indeed it is a maximal subfield of A and is totally wildly ramified over K. There is consequently a unique principal $\mathfrak{o}_F$-order $\mathfrak{A}$ in A which is normalized by $KE^\times$. This gives rise to a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in A, for which $n = n_F e(K|F)$.

We take $\theta_F \in \mathcal{C}(\mathfrak{A}_F, \beta)$ and a central type $\Lambda_F \in \mathcal{CC}(\theta_F, \omega_F)$, for some quasicharacter ω_F of $F^\times$ agreeing with θ_F on U_F^1. We put

$$\theta = \tau_{\mathfrak{A}_F, \mathfrak{A}, \beta}(\theta_F) \in \mathcal{C}(\mathfrak{A}, \beta).$$

Using the procedure of 3.3, we can form the central type $\Lambda_F^{(d)}$ in G: there is a simple stratum $[\mathfrak{A}_{\mathrm{m}}, n_{\mathrm{m}}, 0, \beta]$ in A and $\theta_{\mathrm{m}} = \tau(\theta_F) \in \mathcal{C}(\mathfrak{A}_{\mathrm{m}}, \beta)$ such that $\Lambda_F^{(d)} \in \mathcal{CC}(\theta_{\mathrm{m}}, \omega_F^d)$. We then put

$$\Lambda^0 = \boldsymbol{\lambda}^0_{K/F}(\Lambda_F) = \mathcal{C}\tau_{\mathfrak{A}_{\mathrm{m}}, \mathfrak{A}, \beta}(\Lambda_F^{(d)}) \in \mathcal{CC}(\theta, \omega_F^d). \tag{5.2.1}$$

Up to G-conjugacy, the central type Λ^0 is independent of the choices implicit in its definition 3.3 Proposition and 3.2 Corollary 2. However, there is an extra element of structure, namely the conjugation action of $K^\times$ on the group $\boldsymbol{J}(\beta, \mathfrak{A})$ which stabilizes Λ^0. This is also independent of choices, as is implied by the following lemma.

Lemma. — *Let $\Lambda_1, \Lambda_2 \in \mathcal{CC}(\beta, \mathfrak{A})$ intertwine in G. There then exists $x \in \boldsymbol{U}(\mathfrak{A})$, commuting with K, such that $\Lambda_2 = \Lambda_1^x$.*

Proof. — Let θ_i be the simple character underlying Λ_i. By **[5]** 7.19, 9.3 we have $\theta_2 = \theta_1^x$, for x of the required form. This same element x must then conjugate Λ_1 to Λ_2, by 3.2 Corollary 1. The lemma now follows. □

5.3. We put

$$A_K = \mathrm{End}_K(V \otimes K), \quad G_K = \mathrm{Aut}_K(V \otimes K), \quad \mathfrak{A}_K = \mathfrak{A} \cap A_K,$$

much as in §4. Thus we have a simple stratum $[\mathfrak{A}_K, n, 0, \beta]$ in A_K, and we can form the group $H_K^1 = H^1(\beta, \mathfrak{A}_K) = H^1 \cap G_K$, and likewise for J_K^1 (*cf.* 4.1). We also need $\boldsymbol{J}_K = (KE)^\times J_K^1$.

The situation is now parallel to that of §4. We put $\theta_K = \theta \mid H_K^1 \in \mathcal{C}(\mathfrak{A}_K, \beta)$, and we let η_K be the unique irreducible representation of J_K^1 containing θ_K. We note that, in the language of **[5]** §9, the endo-class of θ_K is the unique K/F-lift of the endo-class of θ_F.

Let ω_F be the central quasicharacter of Λ_F. We set $\omega_K = \omega_F \circ \mathrm{N}_{K/F}$. The simple character θ_K agrees with ω_K on $K^\times \cap H_K^1$ **[5]** 11.9, so we can form the set $\mathcal{CC}(\theta_K, \omega_K)$. We now define a representation $\Lambda_K^0 \in \mathcal{CC}(\theta_K, \omega_K)$ by

$$\Lambda_K^0 = \boldsymbol{g}_{K/F}(\Lambda^0), \tag{5.3.1}$$

where Λ^0 is defined by 5.2.1 and $\boldsymbol{g}_{K/F}$ is given by Proposition 4.1.

We recall at this point that we were given $\pi_F \in \boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_F)$, and the central type Λ_F was chosen to satisfy

$$\pi_F = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_F}^{G_F}(\Lambda_F).$$

Proposition. — *Use the notation above, and define*

$$\pi_K^0 = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_K}^{G_K}(\Lambda_K^0).$$

The representation π_K^0 then lies in $\boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_K)$, and its equivalence class depends only on that of π_F.

Proof. — The first assertion follows from the observation that KE is a maximal subfield of A_K and is totally wildly ramified over K. We have already observed that the equivalence class of π_F determines the G_F-conjugacy class of Λ_F. That class determines the G_K-conjugacy class of Λ^0, taken together with its action by $K^\times$ (Lemma 5.2), and this conjugacy class determines the G_K-conjugacy class of Λ_K^0 by Proposition 4.5(ii). □

We put

$$\boldsymbol{l}_{K/F}^0(\pi_F) = \pi_K^0. \tag{5.3.2}$$

We thus obtain a well-defined map

$$\boldsymbol{l}_{K/F}^0 : \boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_F) \longrightarrow \boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_K).$$

This map $\boldsymbol{l}_{K/F}^0$ is *not* invariably the lifting map we seek; it can be thought of as a sort of "un-normalized lift".

Remark. — Above, we excluded the case $\dim V = 1$. In this situation, π_F is just a quasicharacter of $F^\times$. If π_F is not tamely ramified, all of the above constructions apply without change, and yield $\pi_K^0 = \pi_F \circ \mathrm{N}_{K/F}$. When π_F is tame, we use this relation to define π_K^0, as in 1.3.2.

We can identify $\boldsymbol{J}/J^1$ with $E^\times/U_E^1$; thus, if we have $\Lambda \in \mathcal{CC}(\mathfrak{A}, \beta)$ and a tamely ramified character χ of $E^\times$, the representation $\Lambda \otimes \chi$ is defined. We now set

$$\Lambda = \boldsymbol{\lambda}_{K/F}(\Lambda_F) = \Lambda^0 \otimes (\boldsymbol{\delta}_{K/F} \circ \mathrm{N}_{E/F})^{p^m-1},$$

where $\boldsymbol{\delta}_{K/F}$ is defined in 1.5. Thus $\boldsymbol{\lambda}_{K/F}$ differs from $\boldsymbol{\lambda}_{K/F}^0$ only in the case where $p = 2$ and $m \geqslant 1$.

We define

$$\Lambda_K = \boldsymbol{g}_{K/F}(\Lambda), \qquad \text{(5.3.3)}$$
$$\boldsymbol{l}_{K/F}(\pi_F) = \pi_K = c\text{-Ind}_{\boldsymbol{J}_K}^{G_K}(\Lambda_K).$$

This map

$$\boldsymbol{l}_{K/F} : \boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_F) \longrightarrow \boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_K) \qquad \text{(5.3.4)}$$

is the *algebraic tame lift*, relative to K/F.

Remark. — When K/F is of prime degree $\ell \neq p$, one can describe the process $\Lambda_F \mapsto \Lambda_K$ somewhat differently and more directly: see Corollary 10.3 and Proposition 10.4.

5.4. We now make a preliminary investigation of the fibres of the map $\boldsymbol{l}_{K/F}$ of 5.3.4. Let us write $\boldsymbol{\mathcal{A}}(\theta_F, \omega_F)$ for the set of equivalence classes of irreducible smooth representations π_F of G_F which contain θ_F and have central quasicharacter ω_F. This is a subset of $\boldsymbol{\mathcal{A}}^{\mathrm{wr}}(G_F)$; it is in canonical bijection with $\mathcal{CC}(\theta_F, \omega_F)$ and so has $p^m = [E{:}F]$ elements by 3.1 Lemma 5.

Let $\mathrm{X}(F, p^a)$ denote the group of unramified characters of $F^\times$ of order dividing p^a, $a \geqslant 0$. For chosen $\pi_F \in \boldsymbol{\mathcal{A}}(\theta_F, \omega_F)$, the map $\chi \mapsto \pi_F \cdot \chi$ gives a bijection (*cf.* 1.1)

$$\mathrm{X}(F, p^m) \xrightarrow{\approx} \boldsymbol{\mathcal{A}}(\theta_F, \omega_F),$$

i.e., $\boldsymbol{\mathcal{A}}(\theta_F, \omega_F)$ is a principal homogeneous space over $\mathrm{X}(F, p^m)$.

Likewise define $\boldsymbol{\mathcal{A}}(\theta_K, \pi_K)$. The tame lift $\boldsymbol{l}_{K/F}$ restricts to a map

$$\boldsymbol{l}_{K/F} : \boldsymbol{\mathcal{A}}(\theta_F, \omega_F) \longrightarrow \boldsymbol{\mathcal{A}}(\theta_K, \omega_K). \qquad \text{(5.4.1)}$$

Theorem

(i) *Let p^a be the largest power of p dividing $d = [K{:}F]$. The fibres in $\boldsymbol{\mathcal{A}}(\theta_F, \omega_F)$ of the map $\boldsymbol{l}_{K/F}$ then have order p^a. Each is a principal homogeneous space over the group $\mathrm{X}(F, p^a)$.*

(ii) *If we have an intermediate extension $F \subset K' \subset K$, then*

$$\boldsymbol{l}_{K/F} = \boldsymbol{l}_{K/K'} \circ \boldsymbol{l}_{K'/F}.$$

Proof. — If we take $\chi \in \mathrm{X}(F, p^m)$ and $\pi_F \in \boldsymbol{\mathcal{A}}(\theta_F, \omega_F)$, we have $\boldsymbol{l}_{K/F}(\pi_F \cdot \chi) = \boldsymbol{l}_{K/F}(\pi_F) \cdot (\chi \circ \mathrm{N}_{K/F})$, by 4.3. This can only be equivalent to $\boldsymbol{l}_{K/F}(\pi_F)$ if the character $\chi \circ \mathrm{N}_{K/F}$ is trivial, by 1.1. This proves assertion (i).

The second assertion follows from (4.2), (4.5) and the transitivity property 1.5.1 of $\boldsymbol{\delta}$. □

It is sometimes more convenient to consider simultaneously all quasicharacters ω' of $F^\times$ such that $\omega' \circ \mathrm{N}_{K/F} = \omega_K$. Since K/F is tame, all such ω' agree on U_F^1. Write $\mathfrak{n}_{K/F} = \mathrm{N}_{K/F}(K^\times)$ (which is also the group of norms from the maximal abelian sub-extension of K/F ([**34**] §2 Proposition 4)), and put

$$d_0 = (F^\times : \mathfrak{n}_{K/F}), \quad \omega_{K/F} = \omega_F \mid \mathfrak{n}_{K/F}.$$

Observe that p^a divides d_0. Let $\omega_F^{(i)}$, $1 \leqslant i \leqslant d_0$, be the quasicharacters of $F^\times$ which extend $\omega_{K/F}$. The quasicharacter ω_K only determines $\omega_{K/F}$, so we have a map

$$\boldsymbol{l}^{\mathfrak{n}}_{K/F} : \bigcup_{1\leqslant i\leqslant d_0} \boldsymbol{A}(\theta_F, \omega_F^{(i)}) \longrightarrow \boldsymbol{A}(\theta_K, \omega_K). \tag{5.4.2}$$

Corollary 1. — *The map $\boldsymbol{l}^{\mathfrak{n}}_{K/F}$ of* (5.4.2) *is surjective. Its fibres have order d_0, and are principal homogeneous spaces over the dual group of $F^\times/\mathfrak{n}_{K/F}$.*

Proof. — We first prove the assertion concerning fibres. Let

$$\pi_1, \pi_2 \in \bigcup_i \boldsymbol{A}(\theta_F, \omega_F^{(i)}).$$

We show there is a tamely ramified character χ of $F^\times$ such that $\pi_2 \cong \pi_1 \cdot \chi$. First, the central quasicharacters of the π_i agree on U_F^1. Since the π_i are representations of $\mathrm{GL}_{p^m}(F)$, we can choose a tame character χ_1 so that the central quasicharacters of $\pi_1 \cdot \chi_1$, π_2 agree on $\mathfrak{o}_F^\times$. We can then twist by an unramified quasicharacter to get the desired relation.

If we have $\boldsymbol{l}_{K/F}(\pi_1) \cong \boldsymbol{l}_{K/F}(\pi_2)$, we get

$$\boldsymbol{l}_{K/F}(\pi_1) \cong \boldsymbol{l}_{K/F}(\pi_1 \cdot \chi) \cong \boldsymbol{l}_{K/F}(\pi_1) \cdot \chi_K,$$

where $\chi_K = \chi \circ \mathrm{N}_{K/F}$. Comparing central quasicharacters, the character χ_K must be unramified, and then trivial, as required.

A similar argument shows that, for $\pi_F \in \boldsymbol{A}(\theta_F, \omega_F^{(i)})$, the representations $\pi_F \cdot \chi$, $\chi \in (F^\times/\mathrm{N}_{K/F}(K^\times))\widehat{\ }$, are distinct. This completes the description of the fibres.

Each set $\boldsymbol{A}(\theta_F, \omega_F^{(i)})$ has p^m elements. Likewise, $\boldsymbol{A}(\theta_K, \omega_K)$ has p^m elements; it follows that $\boldsymbol{l}^{\mathfrak{n}}_{K/F}$ is surjective, as desired. □

The set $\boldsymbol{A}^{\mathrm{wr}}(G_F)$ is the disjoint union of the subsets $\boldsymbol{A}(\theta_F, \omega_F)$, with θ_F ranging over conjugacy classes of simple characters attached to maximal totally ramified subfields of A_F and ω_F over quasicharacters of $F^\times$ compatible with θ_F. Similarly over K. Therefore:

Corollary 2. — *The fibres of the map*

$$\boldsymbol{l}_{K/F} : \mathcal{A}^{\mathrm{wr}}(G_F) \to \mathcal{A}^{\mathrm{wr}}(G_K)$$

are principal homogeneous spaces over the dual of the norm residue group $F^\times/\mathrm{N}_{K/F}(K^\times)$.

5.5. We can now prove Theorem 1.3. Parts (i) and (vi) are built into the construction. Part (ii) is immediate, (iii) follows from the transitivity of the Glauberman correspondence as in 4.1. Part (iv) has been proved in 4.3. The Glauberman correspondence certainly preserves contragredience (by 4.2.1 and transitivity), whence (v). Part (vii) is given by 5.4 Corollary 2.

It remains only to prove (viii). We first treat the implication (b) $\Rightarrow$ (a). We choose a maximal simple type in π_K given by a simple stratum $[\mathfrak{B}, n_K, 0, \beta_K]$ in $\mathbb{M}_{p^m}(K)$ and a simple character $\theta_K \in \mathcal{C}(\mathfrak{B}, \beta_K)$. There is then a central type $\Lambda_K \in \mathcal{CC}(\theta_K, \omega_K)$ which induces π_K. Likewise, we take a simple stratum $[\mathfrak{C}, n_L, 0, \beta_L]$ in $\mathbb{M}_{p^m}(L)$ and a simple character $\theta_L \in \mathcal{C}(\mathfrak{C}, \beta_L)$ appearing in π_L. In fact, we can take β_L to be β_K and θ_L the unique L/K-lift of θ_K, up to endo-equivalence. (In our present language, θ_L is obtained from θ_K by the procedures of 5.2, 5.3.) The endo-class of θ_L is Galois-invariant by hypothesis; using **[5]** 9.13 *et seq.*, there is a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $\mathbb{M}_{p^m}(F)$ and $\theta_F \in \mathcal{C}(\mathfrak{A}, \beta)$ such that, up to endo-equivalence, θ_L is the unique L/F-lift of θ_F. The field extension $L[\beta_K]/L$ is totally wildly ramified, so the uniqueness of the lift implies that $F[\beta]/F$ is totally wildly ramified and then that θ_K is the unique K/F-lift of θ_F, i.e., θ_K comes from θ_F as in 5.2, 5.3. Statement (a) now follows from 5.4 Corollary 1.

Next, (a) $\Rightarrow$ (c) is trivial. Finally, if (c) holds, the representation $\pi_L = \boldsymbol{l}_{L/K}(\pi_K)$ is certainly $\mathrm{Gal}(L/F)$-invariant and the simple character θ_K occurring in π_K is lifted from a unique simple character θ_F over F, and θ_F occurs in π_F. We write ω_L, ω_K, ω_F for central quasicharacters in the obvious way. We have $\omega_L = \omega_K \circ \mathrm{N}_{L/K}$. The extension L/K is unramified, so the restriction of ω_K to $\boldsymbol{U}(\mathfrak{o}_K)$ is uniquely determined by ω_L. By hypothesis, ω_L is of the form $\omega_F \circ \mathrm{N}_{L/F}$, so ω_K, $\omega_F \circ \mathrm{N}_{K/F}$ agree on units. Put another way, they differ by an unramified quasicharacter. We can therefore adjust π_F by an unramified quasicharacter to ensure $\omega_K = \omega_F \circ \mathrm{N}_{K/F}$, and we have shown (c) $\Rightarrow$ (b). This completes the proof of Theorem 1.3.

5.6. We now prove Theorem 1.7, using the explicit formula for the conductor given in **[7]** Theorem 6.5.

We first take some irreducible smooth representation π of $\mathrm{GL}_n(F)$ and write its Godement-Jacquet local constant in the form

$$\varepsilon(\pi, s, \psi_F) = q_F^{-sf(\pi,\psi_F)}\varepsilon(\pi, 0, \psi_F).$$

By Theorem 1.6, we have

$$f\left(\boldsymbol{l}_{K/F}(\pi), \psi_K\right) = f(\pi, \psi_F)\, e(K|F), \quad \pi \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F). \tag{5.6.1}$$

Also, for $\pi \in \boldsymbol{\mathcal{A}}_n^0(F)$, $\chi \in \boldsymbol{\mathcal{A}}_1^0(F)$, we have $f(\pi \times \chi, \psi_F) = f(\pi \cdot \chi, \psi_F)$, so we can ignore those cases of Theorem 1.7 in which some $m_i = 1$.

We now prove (i). Assume first that the representations π_1, $\check{\pi}_2$ are completely distinct, in the sense of **[7]** 6.2. By Theorem 1.3 (i) and **[5]** Theorem 3.5, the representations $\boldsymbol{l}_{K/F}(\pi_1)$ and $\boldsymbol{l}_{K/F}(\check{\pi}_2) = \boldsymbol{l}_{K/F}(\pi_2)^\vee$ are also completely distinct. The result now follows from 5.6.1 and **[7]** 6.1.2, Theorem 6.5(ii).

We therefore assume that the representations π_1, $\check{\pi}_2$ are not completely distinct. Using the language of **[7]** 6.3, let $([\Lambda, m, 0, \gamma], l, \vartheta)$ be a best common approximation to this pair of representations. It is only the quantities l and γ which are significant; the method of finding them is given in *ibid.* 6.15. We first have to check that the corresponding quantities for $\boldsymbol{l}_{K/F}(\pi_1)$, $\boldsymbol{l}_{K/F}(\check{\pi}_2)$ are respectively $le(K|F)$ and γ. However, this follows from Theorem 1.3(i) and **[5]** Theorem 9.8.

Next, we need to know the behaviour under tame base field extension of the quantity $C(\gamma)$ **[7]** 6.4; here, we have to indicate the dependence on the base field, so we denote it $C_F(\gamma)$. Our result will follow from **[7]** Theorem 6.5(iii) once we show

$$C_K(\gamma) = C_F(\gamma)^{[K:F]}. \tag{5.6.2}$$

Since the field extension $F[\gamma]/F$ is totally wildly ramified and K/F is tame, the identity 5.6.2 is an immediate consequence of **[7]** 6.13.

Similarly, part (ii) of the theorem follows from parts (i) and (v) of Theorem 1.3, 5.6.2, and **[7]** Theorem 6.5(i).

CHAPTER 6

AUTOMORPHIC LOCAL CONSTANTS

We turn to the proofs of the two major results 1.4, 1.6 concerning the Godement-Jacquet local constant of $\pi \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$. In this section, we develop general procedures for computing local constants in terms of central types and classical Gauss sums. The basic ideas are outgrowths of **[4]**, **[2]** (but we use slightly different conventions here).

6.1. Initially, we work in some generality. Let V be a finite-dimensional F-vector space, and set $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. We set $\psi_A = \psi_F \circ \mathrm{tr}_{A/F}$.

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in A, with $\mathfrak{A}$ principal and $n > 0$. *For the time being, we make no assumption concerning the field extension* $F[\beta]/F$. Let $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$ and $\Lambda \in \mathcal{CC}(\mathfrak{A}, \beta)$. We abbreviate $\boldsymbol{U}^m = \boldsymbol{U}^m(\mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$ etc., as in the earlier sections. We write $\mathfrak{P}$ for the Jacobson radical of $\mathfrak{A}$.

Lemma 1. — *Let W be the representation space of Λ. The quantity*

$$\mathrm{T}(\Lambda, \beta, \psi_F) = \sum_{x \in J^1/\boldsymbol{U}^{n+1}} \Lambda^\vee(\beta x)\psi_A(\beta x)$$

is a scalar operator on $W^\vee$,

$$\mathrm{T}(\Lambda, \beta, \psi_F) = \tau(\Lambda, \beta, \psi_F)\, 1_{W^\vee},$$

for some $\tau(\Lambda, \beta, \psi_F) \in \mathbb{C}$. In particular

$$\tau(\Lambda, \beta, \psi_F) = \frac{1}{\dim \Lambda} \sum_x \mathrm{tr}\, \Lambda^\vee(\beta x)\psi_A(\beta x),$$

with the same range of summation as before.

Proof. — For $y \in \boldsymbol{U}^{n+1}$, $x \in J^1$, we have $\psi_A(\beta xy) = \psi_A(\beta x)$, while $\boldsymbol{U}^{n+1} \subset \operatorname{Ker} \Lambda$. It follows that the definition of T is independent of the choice of representatives x for $J^1/\boldsymbol{U}^{n+1}$. For $j \in J^1$ and x ranging over $J^1/\boldsymbol{U}^{n+1}$, we have

$$\mathrm{T}(\Lambda, \beta, \psi_F)\, \Lambda^\vee(j) = \sum_x \Lambda^\vee(\beta x j)\psi_A(\beta x) = \sum_x \Lambda^\vee(\beta x)\psi_A(\beta x j^{-1}).$$

However, $\psi_A(\beta x j^{-1}) = \psi_A(j^{-1}\beta x) = \psi_A(\beta k x)$, with $k = \beta^{-1}j^{-1}\beta \in J^1$. Thus

$$\mathrm{T}(\Lambda, \beta, \psi_F)\, \Lambda^\vee(j) = \sum_x \Lambda^\vee(\beta k^{-1} x)\psi_A(\beta x) = \Lambda^\vee(j)\, \mathrm{T}(\Lambda, \beta, \psi_F).$$

That is, $\mathrm{T}(\Lambda, \beta, \psi_F)$ commutes with $\Lambda^\vee(J^1)$; since $\Lambda^\vee \mid J^1$ is irreducible, it follows from Schur's Lemma that T is a scalar, as required. □

Lemma 2. — *Let π be an irreducible smooth representation of G containing the central type Λ. We then have*

$$\varepsilon(\pi, s, \psi_F) = (\mathfrak{P}^{-n} : \mathfrak{A})^{(\frac{1}{2}-s)/\dim V}\, \frac{\tau(\Lambda, \beta, \psi_F)}{(\mathfrak{A} : \mathfrak{P}^{n+1})^{\frac{1}{2}}}.$$

Proof. — Let (ϱ, X) be an irreducible representation of $\mathfrak{K}(\mathfrak{A})$ which occurs in π and contains Λ on restriction to the group $\boldsymbol{J}(\beta, \mathfrak{A}) = F[\beta]^\times J^1$. The representation ϱ is then nondegenerate, in the sense of [4], and we have *ibid.* 3.3.8

$$\varepsilon(\pi, s, \psi_F) = (\mathfrak{P}^{-n} : \mathfrak{A})^{(\frac{1}{2}-s)/\dim V}\, \frac{\tau(\varrho, \psi_F)}{(\mathfrak{A} : \mathfrak{P}^{n+1})^{\frac{1}{2}}}.$$

Here, $\tau(\varrho, \psi_F)$ is the unique eigenvalue of the scalar operator

$$\mathrm{T}(\varrho, \psi_F) = \sum_{x \in \boldsymbol{U}/\boldsymbol{U}^{n+1}} \check{\varrho}(cx)\psi_A(cx).$$

In this sum, c is any element of $\mathfrak{K}(\mathfrak{A})$ such that $c\mathfrak{A} = \mathfrak{P}^{-n}$. In particular, we can (and shall) take $c = \beta$. We evaluate the Gauss sum by realizing $\mathrm{T}(\varrho, \psi_F)$ as a matrix. We choose a basis of $X^\vee$ starting with a basis of the representation space of $\Lambda^\vee$. Setting $n'' = [n/2] + 1$, $n' = [(n+1)/2]$ (where $x \mapsto [x]$ is the usual greatest integer function), and arguing as in [4] 2.7, we get that $\tau(\varrho, \psi_F)$ is the $(1,1)$-entry of the matrix

$$(\boldsymbol{U}^{n''} : \boldsymbol{U}^{n+1}) \sum_{x \in \boldsymbol{U}^{n'}/\boldsymbol{U}^{n''}} \check{\varrho}(\beta x)\psi_A(\beta x).$$

We can apply exactly the same procedure to the sum $\mathrm{T}(\Lambda, \beta, \psi_F)$ to get a similar result (noting that, by definition [9] 3.1, $\boldsymbol{U}^{n'} = J^{n'}$). Thus $\tau(\Lambda, \beta, \psi_F) = \tau(\varrho, \psi_F)$ and the lemma follows. □

Remark. — Combining these last two lemmas, we see that, if χ is an unramified quasicharacter of $F^\times$, then

$$\varepsilon(\chi\pi, s, \psi_F) = \chi(\mathrm{N}_{F[\beta]/F}(\beta))^{-1}\varepsilon(\pi, s, \psi_F),$$

thereby justifying Remark 1.4.

As a consequence of the last argument in the proof of Lemma 2, we see that $\tau(\Lambda, \beta, \psi_F)$ is independent of the choice of the element β underlying Λ: we can replace β by any element $\beta' \in E^\times J^1$ satisfying $\beta'\mathfrak{A} = \mathfrak{P}^{-n}$ without changing anything. We therefore drop β from the notation.

We will have further use for one of the identities just uncovered. We therefore exhibit it:

$$\tau(\Lambda, \psi_F) = c \sum_{x \in \boldsymbol{U}^{n'}/\boldsymbol{U}^{n''}} \operatorname{tr} \Lambda^\vee(\beta x)\psi_A(\beta x), \tag{6.1.1}$$
$$c > 0,\ n' = [(n+1)/2],\ n'' = [n/2]+1.$$

The positive factor c is given by

$$c = \frac{(\boldsymbol{U}^{n''} : \boldsymbol{U}^{n+1})}{\dim \Lambda} = \frac{(\mathfrak{P}^{n''} : \mathfrak{P}^{n+1})}{\dim \Lambda}.$$

We can now read off a useful consequence of **[4]** 2.5.11.

Lemma 3. — *The complex number $\tau(\Lambda, \psi_F)$ has absolute value*

$$|\tau(\Lambda, \psi_F)| = (\mathfrak{A} : \mathfrak{P}^{n+1})^{1/2}\, |\omega_\Lambda(x)|, \tag{6.1.2}$$

for any $x \in \mathfrak{K}(\mathfrak{A})$ such that $x\mathfrak{A} = \mathfrak{P}^{-n}$. In particular, if Λ is unitary, we have

$$|\tau(\Lambda, \psi_F)| = (\mathfrak{A} : \mathfrak{P}^{n+1})^{1/2}.$$

Over the next few paragraphs, we give some more evolved versions of the formula 6.1.1. The first step is to choose a simple stratum $[\mathfrak{A}, n, n-1, \alpha]$ equivalent to $[\mathfrak{A}, n, n-1, \beta]$, as we may, by **[9]** Theorem 2.4.1. In particular, the element α is *minimal* over F.

The form of the result depends on the nature of α, so we have to treat two separate cases.

6.2. We first assume that $\alpha \in F$. Set $\gamma = \beta - \alpha$. It may happen that in fact $\beta \in F$, in which case we take $\alpha = \beta$, and we have $H^1(\beta, \mathfrak{A}) = \boldsymbol{U}^1(\mathfrak{A})$. Otherwise, there is an integer $0 < n_1 < n$ such that $[\mathfrak{A}, n_1, 0, \gamma]$ is a simple stratum; we then get $H^1(\beta, \mathfrak{A}) = H^1(\gamma, \mathfrak{A})$. Either way, we have $H^1(\beta, \mathfrak{A}) \supset \boldsymbol{U}^{n'}(\mathfrak{A})$. The definition of simple character **[9]** 3.2 shows that $\theta \mid \boldsymbol{U}^{n'}$ is of the form $\psi_\gamma.\chi \circ \det$, for a character χ of $F^\times$ such that $\chi \circ \det \mid \boldsymbol{U}^{n''} = \psi_\alpha$.

Proposition. — *With the notation above, suppose that $\alpha \in F$. We then have*

$$\tau(\Lambda, \psi_F) = c\,\mathrm{tr}(\Lambda^\vee(\beta))\,\chi(\alpha)^{\dim V}\,\psi_A(\gamma)\,\tau(\chi, \psi_F)^{\dim V},$$

for some $c > 0$.

Proof. — We start with the expression 6.1.1:

$$\tau(\Lambda, \psi_F) = c \sum_{x \in \boldsymbol{U}^{n'}/\boldsymbol{U}^{n''}} \mathrm{tr}\,\Lambda^\vee(\beta x)\psi_A(\beta x).$$

However, $\Lambda^\vee(\beta x) = \Lambda^\vee(\beta)\theta(x)^{-1}$. On the other hand,

$$\psi_A(\beta x) = \psi_A(\alpha x)\psi_A(\gamma x) = \psi_A(\alpha x)\psi_\gamma(x)\psi_A(\gamma).$$

We recall that $\theta(x) = \psi_\gamma(x)\chi(\det x)$, so the typical term in the sum is

$$\mathrm{tr}\,\Lambda^\vee(\beta)\,\psi_A(\gamma)\,\chi(\det x)^{-1}\,\psi_A(\alpha x).$$

The sum therefore reduces to

$$c\,\mathrm{tr}\,\Lambda^\vee(\beta)\,\psi_A(\gamma)\,\chi(\det \alpha) \sum_{x \in \boldsymbol{U}^{n'}/\boldsymbol{U}^{n''}} \chi(\det(\alpha x))^{-1}\,\psi_A(\alpha x).$$

The inner sum is

$$c\,\tau(\chi \circ \det, \psi_F) = c\,\tau(\chi, \psi_F)^{\dim V},$$

by [4] 2.8.13. The result now follows. □

6.3. We continue with the notation of the beginning of 6.2, and now treat the case $\alpha \notin F$ (although we will not use this hypothesis for a little while). We need some more notation for a subsidiary purpose. We write $\mathcal{V} = J^1/H^1$ and denote by $\mathcal{V}_0$ the canonical image of $J^{n'} = \boldsymbol{U}^{n'}$ in $\mathcal{V}$. We also write $\mathcal{Z}_\mathcal{V}(\beta)$ for the group of fixed points of β in $\mathcal{V}$ (with β acting by conjugation), and similarly for $\mathcal{V}_0$.

When the integer n is odd, the group $\boldsymbol{U}^{n'}/\boldsymbol{U}^{n''}$ is trivial, so we obtain (irrespective of the hypothesis on α):

Proposition 1. — *Suppose that the integer n is odd. We then have*

$$\tau(\Lambda, \psi_F) = \frac{(\mathfrak{P}^{(n+1)/2} : \mathfrak{P}^{n+1})}{\dim \Lambda}\,\mathrm{tr}(\Lambda^\vee(\beta))\,\psi_A(\beta).$$

We observe that, when Λ is unitary, this implies $|\mathrm{tr}\,\Lambda^\vee(\beta)| = \dim \Lambda$. Invoking A1.8, this gives us

$$|\mathcal{Z}_\mathcal{V}(\beta)|^{1/2} = \dim \Lambda = |\mathcal{V}|^{1/2}.$$

We deduce:

Corollary 1. — *If the integer n is odd, the element β acts trivially on $\mathcal{V} = J^1/H^1$.*

We therefore assume henceforward that the integer n is even, $n = 2l$, say. We further assume that the field extension $F[\alpha]/F$ is *not tamely ramified.* This simplifies the situation, and is the only case we need. We start with the expression 6.1.1:

$$\tau(\Lambda, \psi_F) = \frac{(\mathfrak{P}^{l+1} : \mathfrak{P}^{2l+1})}{\dim \Lambda} \sum_{x \in \boldsymbol{U}^l/\boldsymbol{U}^{l+1}} \operatorname{tr}(\Lambda^\vee(\beta x))\, \psi_A(\beta x).$$

Since the Gauss sum is not zero, we deduce there exists $h_0 \in \boldsymbol{U}^l$ such that $\operatorname{tr} \Lambda^\vee(\beta h_0) \neq 0$.

Let us abbreviate $E = F[\beta]$. The restriction of Λ to $E^\times \boldsymbol{U}^l$ is a direct sum of irreducible representations λ_i. Each $\lambda_i \mid \boldsymbol{U}^l$ is irreducible, and is the unique irreducible representation of $\boldsymbol{U}^l$ whose restriction to $H^l = H^l(\beta, \mathfrak{A})$ contains θ (*cf.* **[9]** 3.4). Appealing to A1.8, the term $\operatorname{tr} \Lambda^\vee(\beta x)$ vanishes unless βx is of the form $g\beta h_0 z g^{-1}$, for some $z \in H^l$ and some $g \in \boldsymbol{U}^l$. The volume of the $\boldsymbol{U}^l$-conjugacy class of $\beta h_0 z$ is independent of the element $z \in H^l$ so we can rearrange our sum as

$$\tau(\Lambda, \psi_F) = c \sum_{z \in H^l/\boldsymbol{U}^{l+1}} \sum_{g \in \boldsymbol{U}^l/H^l} \operatorname{tr}(\Lambda^\vee(g\beta h_0 z g^{-1}))\, \psi_A(g\beta h_0 z g^{-1}),$$

where

$$c = \frac{(\mathfrak{P}^{l+1} : \mathfrak{P}^{2l+1})}{\dim \Lambda} |\mathcal{Z}_{\mathcal{V}_0}(\beta)|^{-1},$$

and $\mathcal{Z}_{\mathcal{V}_0}(\beta)$ is the group of fixed points of β acting by conjugation on $\mathcal{V}_0$. This expression simplifies further to

$$\tau(\Lambda, \psi_F) = c \sum_{z \in H^l/\boldsymbol{U}^{l+1}} \operatorname{tr}(\Lambda^\vee(\beta h_0 z))\, \psi_A(\beta h_0 z),$$

with

$$c = \frac{(\mathfrak{P}^{l+1} : \mathfrak{P}^{2l+1})\, (\boldsymbol{U}^l : Z_{\mathcal{V}_0}(\beta))}{\dim \Lambda},$$

where $Z_{\mathcal{V}_0}(\beta)$ is the inverse image in $\boldsymbol{U}^l$ of $\mathcal{Z}_{\mathcal{V}_0}(\beta)$. In order to reduce the expression further, we need a lemma (which is where our hypothesis on the ramification of $F[\alpha]/F$ takes effect).

Lemma 4. — *Suppose that $\alpha \notin F$, that $n = 2l$ is even and that the field extension $F[\alpha]/F$ is not tamely ramified. The map*

$$z \longmapsto \psi_{\beta h_0}(z), \quad z \in H^l,$$

is then a character of H^l agreeing with θ on $\boldsymbol{U}^{l+1}$.

Proof. — Temporarily write $B = \mathrm{End}_{F[\alpha]}(V)$, $\mathfrak{B} = \mathfrak{A} \cap B$, $\mathfrak{Q} = \mathfrak{P} \cap B$. We have $H^l = 1 + \mathfrak{H}^l(\beta, \mathfrak{A})$, and $\mathfrak{H}^l(\beta, \mathfrak{A}) = \mathfrak{Q}^l + \mathfrak{P}^{l+1}$.

We have to show that $\psi_{\beta h_0}(z_1 z_2) = \psi_{\beta h_0}(z_1)\psi_{\beta h_0}(z_2)$, $z_i \in H^l$. Put $z_i = 1 + x_i + y_i$, with $x_i \in \mathfrak{Q}^l$, $y_i \in \mathfrak{P}^{l+1}$. Expanding,

$$z_1 z_2 \equiv 1 + x_1 + x_2 + y_1 + y_2 + x_1 x_2 \pmod{\mathfrak{P}^{n+1}}.$$

We thus have to show that $\psi_A(\beta h_0 x_1 x_2) = 1$, i.e., that $\psi_{\beta h_0}(t) = 1$ for $t \in \mathfrak{Q}^n$.

For such t, we have $\psi_A(\beta h_0 t) = \psi_A(\alpha t)$. Choose a character $\psi_{F[\alpha]}$ of the field $F[\alpha]$ with conductor $\mathfrak{p}_{F[\alpha]}$ and form $\psi_B = \psi_{F[\alpha]} \circ \mathrm{tr}_{B/F[\alpha]}$. There is then a tame corestriction s_α on A relative to $F[\alpha]/F$ (see **[9]** 1.3) such that

$$\psi_A(\alpha b) = \psi_B(s_\alpha(\alpha) b), \quad b \in B.$$

However, since $F[\alpha]/F$ is not tamely ramified, we have $s_\alpha(\alpha) \in \mathfrak{Q}^{1-n}$ (*ibid.* (1.3.8)(iii)), so $\psi_A(\alpha t) = 1$, as required. □

Returning to $\tau(\Lambda, \psi_F)$, we have $\Lambda^\vee(\beta h_0 z) = \Lambda^\vee(\beta h_0)\theta(z)^{-1}$, so the sum will vanish unless we have $\psi_{\beta h_0}(z) = \theta(z)$ for all $z \in H^l$. In all, we have shown:

Proposition 2. — *With the notation of* 6.2, *suppose $\alpha \notin F$, $n = 2l$ and that the field extension $F[\alpha]/F$ is not tamely ramified. There exists $h_0 \in \boldsymbol{U}^l$ such that*

(a) $\mathrm{tr}\, \Lambda^\vee(\beta h_0) \neq 0$, *and*
(b) $\psi_{\beta h_0}(z) = \theta(z)$, *for all $z \in H^l$.*

For any such h_0, we have

$$\tau(\Lambda, \psi_F) = \mathrm{tr}(\Lambda^\vee(\beta h_0))\, \psi_A(\beta h_0) \frac{(\mathfrak{P}^{l+1} : \mathfrak{P}^{2l+1})\,(\boldsymbol{U}^l : Z_{V_0}(\beta))\,(H^l : \boldsymbol{U}^{l+1})}{\dim \Lambda}.$$

Suppose for the moment that Λ is unitary. We then have

$$|\tau(\Lambda, \psi_F)| = (\mathfrak{A} : \mathfrak{P}^{2l+1})^{1/2}.$$

Thus

$$|\mathrm{tr}\, \Lambda^\vee(\beta h_0)| = \frac{\dim(\Lambda)\, |\mathcal{Z}_{V_0}(\beta)|\, (\mathfrak{A} : \mathfrak{P})^{1/2}}{(\boldsymbol{U}^l : \boldsymbol{U}^{l+1})} = \frac{(J^1 : H^1)^{1/2}\, |\mathcal{Z}_{V_0}(\beta)|}{(\mathfrak{A} : \mathfrak{P})^{1/2}}.$$

In this situation, we recall from A1.8 that

$$|\mathrm{tr}\, \Lambda^\vee(\beta h_0)| = |\mathcal{Z}_V(\beta)|^{1/2},$$

so we get the relation

$$(6.3.1) \qquad |\mathcal{Z}_{\mathcal{V}}(\beta)|^{1/2} (\mathfrak{A} : \mathfrak{P})^{1/2} = |\mathcal{V}|^{1/2} |\mathcal{Z}_{\mathcal{V}_0}(\beta)|.$$

We can apply this result to the element α; we have

$$J^1(\alpha, \mathfrak{A})/H^1(\alpha, \mathfrak{A}) = J^l(\alpha, \mathfrak{A})/H^l(\alpha, \mathfrak{A}) = J^l(\beta, \mathfrak{A})/H^l(\beta, \mathfrak{A}) = \mathcal{V}_0,$$

and we obtain

$$(6.3.2) \qquad |\mathcal{Z}_{\mathcal{V}_0}(\alpha)| = \frac{(\mathfrak{A} : \mathfrak{P})}{|\mathcal{V}_0|} = (\boldsymbol{U}^l : H^l).$$

We have $\alpha \equiv \beta \pmod{\boldsymbol{U}^1(\mathfrak{A})}$ and so $\mathcal{Z}_{\mathcal{V}_0}(\beta) = \mathcal{Z}_{\mathcal{V}_0}(\alpha)$. Substituting the relation 6.3.2 in 6.3.1, we get:

Corollary 2. — *If $\alpha \notin F$, $F[\alpha]/F$ is not tamely ramified, and $n = 2l$ is even, we have*

$$|\mathcal{Z}_{\mathcal{V}}(\beta)| = (\mathcal{V} : \mathcal{V}_0)\, |\mathcal{Z}_{\mathcal{V}_0}(\beta)|$$

where $\mathcal{V} = J^1/H^1$ and $\mathcal{V}_0$ is the image of J^l/H^l in $\mathcal{V}$.

More informally, this says that the non-fixed points of β on $\mathcal{V}$ all appear already in the subspace $\mathcal{V}_0$.

6.4. We record, for later use, a couple of general properties of the Gauss sums attached to central types. In this paragraph, we are given a vector space V_0, with $A_0 = \mathrm{End}_F(V_0)$, $G_0 = \mathrm{Aut}_F(V_0)$, a simple stratum $[\mathfrak{A}_0, n_0, 0, \beta]$ in A_0, a simple character $\theta_0 \in \mathcal{C}(\mathfrak{A}_0, \beta)$, and a central type $\Lambda_0 \in \mathcal{CC}(\theta_0)$. We also assume that $E = F[\beta]$ is a *maximal* subfield of A_0.

We take an integer $t \geqslant 1$ and, as in 3.3, form the central type $\Lambda_0^{(t)}$. Thus we have the vector space $V = V_0 \oplus V_0 \oplus \cdots \oplus V_0$ (t copies); we put $A = \mathrm{End}_F(V)$, $G = \mathrm{Aut}_F(V)$. We have a simple stratum $[\mathfrak{A}_{\mathrm{m}}, n_{\mathrm{m}}, 0, \beta]$, a simple character $\theta_{\mathrm{m}} \in (\mathfrak{A}_{\mathrm{m}}, \beta)$, and $\Lambda_0^{(t)} \in \mathcal{CC}(\theta_{\mathrm{m}})$. It will simplify the notation to set $\Lambda_0^{(t)} = \Lambda_{\mathrm{m}}$.

Proposition. — *We have $\tau(\Lambda_{\mathrm{m}}, \psi_F) = c\tau(\Lambda_0, \psi_F)^t$, for some $c > 0$.*

Proof. — We use the notation of 3.3, where Λ_{m} is defined (except that $\mathfrak{A}_{\mathrm{m}}$ is there denoted $\mathfrak{A}$). The representation Λ_{m} is induced by the representation $\tilde{\Lambda}$ of $E^\times \tilde{J}^1$. Abbreviating $J^1 = J^1(\beta, \mathfrak{A}_{\mathrm{m}})$ etc., and using the Mackey formula, we have:

$$\begin{aligned}\tau(\Lambda_{\mathrm{m}},\psi_F) &= c \sum_{x\in J^1/\boldsymbol{U}^{n+1}} \mathrm{tr}\Lambda_{\mathrm{m}}^\vee(\beta x)\,\psi_A(\beta x)\\ &= c \sum_{x\in \widetilde{J}^1/\boldsymbol{U}^{n+1}} \sum_{\substack{y\in E^\times \widetilde{J}^1\backslash E^\times J^1/E^\times\widetilde{J}^1,\\ y\beta x y^{-1}\in E^\times\widetilde{J}^1}} \mathrm{tr}\widetilde{\Lambda}^\vee(y\beta x y^{-1})\,\psi_A(\beta x).\end{aligned}$$

In the inner sum, we can take the coset representatives y from $J^1\cap N_\ell$. The element $y\beta x y^{-1}$ then lies in $E^\times \widetilde{J}^1$ if and only if y is a fixed point of β in $J^1\cap N_\ell/H^1\cap N_\ell$. In this case,

$$\mathrm{tr}\widetilde{\Lambda}^\vee(y\beta x y^{-1}) = \mathrm{tr}\widetilde{\Lambda}^\vee(\beta x\,[x^{-1},y^\beta]\,[\beta^{-1},y]),$$

where $y^\beta = \beta^{-1}y\beta$ and $[\,,\,]$ denotes the commutator, $[a,b]=aba^{-1}b^{-1}$. The term $[\beta^{-1},y]$ lies in $H^1\cap N_\ell\subset \operatorname{Ker}\theta$, so we are reduced to

$$\mathrm{tr}\widetilde{\Lambda}^\vee(y\beta x y^{-1}) = \mathrm{tr}\widetilde{\Lambda}^\vee(\beta x)\,\theta[x^{-1},y^\beta].$$

The map

$$\begin{aligned} J^1/H^1\times J^1/H^1 &\longrightarrow \mathbb{C}^\times,\\ (x,y)&\longmapsto \theta[x,y],\end{aligned}$$

is bi-multiplicative. For fixed x, the map $\chi_x: y\mapsto \theta[x^{-1},y^\beta]$ is therefore a character of the group of fixed points of β on $J^1\cap N_\ell/H^1\cap N_\ell$. We can now write $x=x_0x_1$, with $x_0\in J^1\cap M$, $x_1\in J^1\cap N_{\mathrm{u}}$. The character χ_{x_0} is trivial; if χ_{x_1} is non-trivial, the sum over y vanishes, and otherwise contributes a positive constant factor. In all,

$$\tau(\Lambda_{\mathrm{m}},\psi_F) = c\sum_{x\in J^1\cap M/H^1\cap M} \mathrm{tr}\widetilde{\Lambda}^\vee(\beta x)\,\psi_A(\beta x),$$

for some $c>0$. However, $J^1\cap M$ is the direct product of t copies of $J^1(\beta,\mathfrak{A}_0)$, and similarly for $H^1\cap M$. It follows that

$$\tau(\Lambda_{\mathrm{m}},\psi_F) = c\tau(\Lambda_0,\psi_F)^t,$$

for some $c>0$ as required. □

Suppose next that we have a simple stratum $[\mathfrak{A},n,0,\beta]$ in A. We can form

$$\begin{aligned}\theta &= \tau_{\mathfrak{A}_{\mathrm{m}},\mathfrak{A},\beta}(\theta_{\mathrm{m}})\in \mathcal{C}(\mathfrak{A},\beta),\\ \Lambda &= \mathcal{C}\tau(\Lambda_{\mathrm{m}})\in \mathcal{CC}(\theta).\end{aligned}$$

Corollary. — *There are positive constants c, c' such that*

$$\tau(\Lambda,\psi_F) = c\tau(\Lambda_{\mathrm{m}},\psi_F) = c'\tau(\Lambda_0,\psi_F)^t.$$

Proof. — As we observed, $\mathfrak{A}_{\mathrm{m}}$ is minimal for the property of being normalized by $E^\times$. Thus, without loss of generality, we can assume $\mathfrak{A} \supset \mathfrak{A}_{\mathrm{m}}$. We use the variant construction (Comment, 1.2) of Λ from Λ_{m}. This makes it clear that any irreducible smooth representation π of G which contains Λ_{m} must also contain Λ. The result now follows from 6.1 Lemma 2. □

CHAPTER 7

GAUSS SUMS MOD ROOTS OF UNITY

The object of this section is to prove Theorem 1.4. We therefore revert to the notation of paragraph 1.4. We are given $\pi \in \mathcal{A}_m^{\mathrm{wr}}(F)$ and we assume, to exclude trivial cases, that $m \geqslant 1$. There is then a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathbb{M}_{p^m}(F)$, a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$, and a central type $\Lambda \in \mathcal{CC}(\theta)$, such that Λ occurs in π. Moreover, the field $E = F[\beta]$ is a maximal subfield of A, totally wildly ramified over F. We choose a simple stratum $[\mathfrak{A}, n, n-1, \alpha]$ equivalent to $[\mathfrak{A}, n, n-1, \beta]$. For brevity, we put $\boldsymbol{\varepsilon}'(\pi) = \boldsymbol{\varepsilon}(\pi, \frac{1}{2}, \psi_F)$, and $\boldsymbol{\mu} = \boldsymbol{\mu}_{p^\infty}(\mathbb{C})$.

7.1. Remark 1.4 (proved in Remark 6.1) tells us we may take $a_\pi = \mathrm{N}_{E/F}(\beta)$. For the same reason, the construction of π_K allows us to take

$$a_{\pi_K} = \mathrm{N}_{K[\beta]/K}(\beta) = a_\pi.$$

This proves parts (i) and (iii) of the theorem.

7.2. We now prove part (ii) of the theorem. We observe that, since the two sides of the desired congruence have the same absolute values 6.1.2, it is enough to verify it modulo $\mathbb{R}_+^\times \boldsymbol{\mu}$. We therefore use the symbol c to denote a positive real number whose value varies from line to line.

We have to divide into cases, following the scheme introduced at the beginning of 6.2. In this paragraph, *we assume that* $\alpha \in F$. We set $\gamma = \beta - \alpha$; we choose a character χ of $F^\times$ such that $\theta \mid \boldsymbol{U}^{n'} = \psi_\gamma\, \chi \circ \det$, with $n' = [(n+1)/2]$, as at the beginning of 6.2. Proposition 6.2 gives us

$$\tau(\Lambda, \psi_F) = c\, \mathrm{tr}(\Lambda^\vee(\beta))\, \chi(\alpha)^{p^m}\, \psi_A(\gamma)\, \tau(\chi, \psi_F)^{p^m}.$$

We can write $\beta = \alpha(1+\alpha^{-1}\gamma)$, and the factor $u = 1+\alpha^{-1}\gamma$ lies in $H^1 = H^1(\beta,\mathfrak{A})$. Thus $\Lambda^\vee(\beta) = \omega_\pi(\alpha)^{-1}\theta(u)^{-1}1$. Substituting,

$$\begin{aligned}\varepsilon'(\pi) = c\,\tau(\Lambda,\psi_F) &= c\,\omega_\pi(\alpha)^{-1}\theta(u)^{-1}\,\psi_A(\gamma)\,(\chi(\alpha)\tau(\chi,\psi_F))^{p^m}\\ &\equiv \omega_\pi(\alpha)^{-1}\,(\chi(\alpha)\tau(\chi,\psi_F))^{p^m} \pmod{\mathbb{R}_+^\times\boldsymbol{\mu}}.\end{aligned}$$

Now we simply have to observe that $\chi(\alpha)\tau(\chi,\psi_F) \equiv \mathfrak{g}(\alpha)$ (which can be regarded as a special case of Lemma 2.5(ii)) and that $a_\pi = \alpha^{p^m}$.

This completes the proof in the case $\alpha \in F^\times$.

7.3. We now assume that $\alpha \notin F$; in this paragraph, we also assume that *the integer n is odd.* We then have (6.3 Proposition 1):

$$\varepsilon'(\pi) = c\,\mathrm{tr}(\Lambda^\vee(\beta))\psi_A(\beta).$$

Without loss of generality, we can assume that Λ is unitary. The restriction of Λ to the group D generated by β and H^1 is then a direct sum of unitary characters ϕ_i, $1 \leqslant i \leqslant \dim\Lambda$. As in the proof of 6.3 Corollary 1, we have $|\Lambda^\vee(\beta)| = \dim\Lambda$. It follows that the ϕ_i are all the same, equal to some ϕ , say. Thus

$$\varepsilon'(\pi) = c\,\phi(\beta)^{-1}\psi_A(\beta).$$

However, $\beta^{p^m} \in F^\times H^1$, so $\phi(\beta) \equiv \omega_\pi(\beta^{p^m})^{1/p^m} \pmod{\boldsymbol{\mu}}$. Since E/F is totally wildly ramified, we have $\beta^{p^m} \equiv \mathrm{N}_{E/F}(\beta) \pmod{U_E^1}$, giving us

$$\varepsilon'(\pi) \equiv \omega_\pi(a_\pi)^{-1/p^m} \pmod{\boldsymbol{\mu}}.$$

Since n is odd, $\mathfrak{g}(a_\pi) = 1$ by definition, and the result follows in this case.

7.4. We now assume that n is even, $n = 2l$ say, and $\alpha \notin F$. We first treat the special case in which β is *minimal* over F. We can therefore take $\alpha = \beta$. This combination of hypotheses, we observe, forces $p \neq 2$.

We start with the formula $\tau(\Lambda,\psi_F) = c\,\mathrm{tr}(\Lambda^\vee(\beta h_0))\psi_A(\beta h_0)$ given by 6.3 Proposition 2. We use **[36]** Theorem 3.3.2. In our notation, this says

$$\varepsilon'(\pi) \equiv \Theta(\beta)^{-1}\mathfrak{G} \pmod{\mathbb{R}_+^\times\boldsymbol{\mu}},$$

where Θ is some character of $E^\times H^1$ agreeing with θ on H^1 and ω_π on $F^\times$, and $\mathfrak{G}$ is a Gauss sum as follows. (Note that the precise choice of Θ is irrelevant: any two choices are congruent mod $\boldsymbol{\mu}$.) We identify the residue class fields $\mathrm{k}_E = \mathrm{k}_F$ and choose a prime element ϖ_E of E. We write ψ for the character of k_F induced by ψ_F. Then

$$\mathfrak{G} = \sum_{x \in \mathrm{k}_F} \psi(\delta x^2/2),$$

$$\delta \equiv (-1)^{(p^m-1)/2} \beta \varpi_E^n \pmod{\mathfrak{p}_E}.$$

This simplifies to $\mathfrak{G} = \mathfrak{g}_F\left((-1)^{(p^m-1)/2} a_\pi\right)$, to use the notation of 1.4, and we have

$$\mathfrak{g}_F\left((-1)^{(p^m-1)/2} a_\pi\right) = \left(\frac{-1}{q_F}\right)^{(p^m-1)/2} \mathfrak{g}_F(a_\pi),$$

where $q_F = |\mathrm{k}_F|$. However, $\mathfrak{g}_F(a_\pi)^2 = \left(\frac{-1}{q_F}\right)$, so we obtain

$$\boldsymbol{\varepsilon}'(\pi)^{p^m} \equiv \omega_\pi(a_\pi)^{-1}\, \mathfrak{g}_F(a_\pi)^{p^{2m}} \pmod{\boldsymbol{\mu}},$$

as required.

7.5. We now abandon the hypothesis that β is minimal over F. (This, we note, permits the case $p = 2$.) We continue to assume $n = 2l$ is even.

We first make some adjustments to the representation π, and hence to the central type Λ. We choose a prime element ϖ_F of F; thus there is a root of unity ζ_0 in F, of order prime to p, such that

$$\beta^{p^m} \equiv \zeta_0 \varpi_F^{-n} \pmod{U_E^1}.$$

The element α has the analogous property relative to the same prime ϖ_F. We next twist π by a tamely ramified quasicharacter of $F^\times$ to ensure that ω_π is trivial on ϖ_F and on the group of p-prime roots of unity in F. This has no effect on the relation to be proved.

Let $\mathcal{G}$ be the group generated by the kernel of the simple character θ and the kernel of ω_π. Set $\mathfrak{J} = E^\times J^1/\mathcal{G}$, $\mathfrak{J}^1 = J^1/\mathcal{G}$. Then Λ is effectively a representation of the finite p-group $\mathfrak{J}$, which is itself a cyclic p-power extension of the extra-special p-group $\mathfrak{J}^1$ of class 2. The value of the character of $\Lambda^\vee$, taken mod p-power roots of unity and positive reals, at the element βh_0 depends only on the action of β on the alternating space $\mathcal{V} = J^1/H^1$ by A1.10.

The subspace $\mathcal{V}_0 = J^l/H^l$ of $\mathcal{V}$ is nondegenerate **[9]** 3.4. The alternating space $\mathcal{V}$ therefore decomposes as an orthogonal sum $\mathcal{V} = \mathcal{V}_0 \perp \mathcal{V}_1$, and, by 6.3 Corollary 2, β acts trivially on $\mathcal{V}_1$. Let $\mathcal{K}_j$ denote the inverse image in $\mathfrak{J}^1$ of $\mathcal{V}_j$. There is a unique irreducible representation η_j of $\mathcal{K}_j$ whose restriction to (the image of) H^1 is a multiple of θ. We have a canonical surjection $s : \mathcal{K}_0 \times \mathcal{K}_1 \to \mathfrak{J}^1$ and $\eta \circ s \cong \eta_0 \otimes \eta_1$. We therefore abuse notation and write $\eta = \eta_0 \otimes \eta_1$. We can similarly realize the representation Λ as a tensor product $\Lambda_0 \otimes \Lambda_1$, where

Λ_j extends η_j to the group generated by β and $\mathcal{K}_j$. By A1.10, the values of the character of Λ_1 lie in $\mathbb{R}^\times \boldsymbol{\mu}$. We need only concern ourselves with the character of Λ_0.

We now consider the group $\boldsymbol{J}(\alpha, \mathfrak{A})$. We take some simple character $\theta_\alpha \in \mathcal{C}(\mathfrak{A}, \alpha)$ and a central type $\Lambda_\alpha \in \mathcal{CC}(\theta_\alpha)$. We arrange for ϖ_F to act trivially as before. The definition [9] 3.1 of the groups J^1 gives $J^l(\alpha, \mathfrak{A}) = J^l(\beta, \mathfrak{A})$ and $J^1(\alpha, \mathfrak{A}) = H^1(\alpha, \mathfrak{A}) J^l(\beta, \mathfrak{A})$. Thus the alternating space $J^1(\alpha, \mathfrak{A})/H^1(\alpha, \mathfrak{A})$ is canonically identified with $\mathcal{V}_0$; moreover, the elements α and β induce the same automorphism of $\mathcal{V}_0$. Choose an element $k_0 \in \boldsymbol{U}^l$ such that $\mathrm{tr}(\Lambda_\alpha^\vee(\alpha k_0)) \neq 0$. By A1.10 therefore, we have

$$\mathrm{tr}(\Lambda^\vee(\beta h_0)) \equiv \mathrm{tr}(\Lambda_0^\vee(\beta h_0)) \equiv \mathrm{tr}(\Lambda_\alpha^\vee(\alpha k_0)) \pmod{\mathbb{R}_+^\times \boldsymbol{\mu}}.$$

The result now follows from Corollary 6.4 and the minimal case 7.4.

This completes the proof of Theorem 1.4.

CHAPTER 8

GAUSS SUM RELATIONS

We prove Theorem 1.6. The considerations of §6 reduce this to a comparison of Gauss sums of the kind introduced in 6.1. First, however, we have to introduce a new species of Gauss sum, formed relative to certain commutator relations. This is done in 8.1, where the main properties of this "commutator Gauss sum" are stated. It is explicitly related to the Gauss sums occurring in Theorem 1.6: see 8.2. The next step, in 8.3, is to prove an additivity property of the commutator Gauss sum. We can then complete the proof of Theorem 1.6, modulo the calculation of the commutator Gauss sum in certain fundamental cases. This is done in §9.

Throughout this section, we shall mainly be concerned with computing the arguments of various complex numbers: for $x, y \in \mathbb{C}^\times$, we write $x = cy$ to mean $x \equiv y \pmod{\mathbb{R}_+^\times}$.

8.1. We first need a *non-trivial* totally wildly ramified field extension $E = F[\alpha]/F$, generated by an element α which is minimal over F and of negative valuation. (In this situation, the minimality of α amounts to saying that the valuation $\nu_E(\alpha)$ is not divisible by p.) We also require a tamely ramified extension K/F *of prime degree d*. We abbreviate $KE = K \otimes_F E$.

We fix an element ζ such that $K = F[\zeta]$; if K/F is unramified, we take ζ to be a root of unity of order prime to p, and, if K/F is totally ramified, we let ζ be a prime element of K such that $\zeta^d \in F$. (Compare with the choice of ζ in 4.2.)

We are given the following data: a finite-dimensional KE-vector space V and a simple stratum $[\mathfrak{A}, n, n-1, \alpha]$ in $A = \mathrm{End}_F(V)$ such that $\mathfrak{A}$ is principal and normalized by $(KE)^\times$. We write $x^\zeta = \zeta^{-1}x\zeta$ and $\mathrm{A}_\alpha(x) = \alpha x \alpha^{-1} - x$, $x \in A$. If we write $\mathfrak{P}$ for the radical of $\mathfrak{A}$, we have $(\mathfrak{P}^a)^\zeta = \mathfrak{P}^a$ and $\mathrm{A}_\alpha(\mathfrak{P}^a) \subset \mathfrak{P}^a$, for all $a \in \mathbb{Z}$. We define

$$\mathfrak{G}(\mathfrak{A}, \alpha, \zeta) = \sum_{\substack{x \in \mathfrak{P}^l/\mathfrak{P}^{l+1}, \\ x - x^\zeta \in \mathrm{A}_\alpha(\mathfrak{P}^l) + \mathfrak{P}^{l+1}}} \psi_A(\alpha x(x - x^\zeta))$$

when $n = 2l$ is even and

$$\mathfrak{G}(\mathfrak{A}, \alpha, \zeta) = 1$$

if n is odd.

We consider the special case where $V = KE$; let $\mathfrak{A}_0$ be the unique hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(KE)$ normalized by $(KE)^\times$. This gives a simple stratum of the form $[\mathfrak{A}_0, n_0, n_0-1, \alpha]$, allowing us to form $\mathfrak{G}(\mathfrak{A}_0, \alpha, \zeta)$ according to the formula above.

The properties we need of these objects are:

Lemma. — *With the notation above, we have*

$$\mathfrak{G}(\mathfrak{A}, \alpha, \zeta) = c\,\mathfrak{G}(\mathfrak{A}_0, \alpha, \zeta)^{\dim_{KE}(V)},$$

$$\mathfrak{G}(\mathfrak{A}_0, \alpha, \zeta) = c\left(\frac{\boldsymbol{\varepsilon}(1_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{K/F}, s, \psi_F)}\right)^{-[E:F]} \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\alpha))^p,$$

for (possibly different) positive constants c.

The proof of the first statement will be given in 8.3, that of the second in §9.

8.2. In this section, we use the general set-up of §4. We are given an F-vector space V and a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{End}_F(V)$, with $\mathfrak{A}$ principal and $n \geqslant 0$. We assume that the field $E = F[\beta]$ is totally wildly ramified over F of degree > 1. We take K/F as above, and assume that V is, in fact, a vector space over the field KE, this structure extending the given E-structure on V. Moreover, we assume that $\mathfrak{A}$ is normalized by $KE^\times$.

We choose $\theta \in \mathcal{C}(\mathfrak{A}, \beta)$ and a central type $\Lambda \in \mathcal{CC}(\theta)$.

Now set $A_K = \mathrm{End}_K(V)$, $\mathfrak{A}_K = \mathfrak{A} \cap A_K$. This gives us a simple stratum $[\mathfrak{A}_K, n, 0, \beta]$ in A_K, along with a simple character $\theta_K \in \mathcal{C}(\mathfrak{A}_K, \beta)$ obtained by restricting θ, as in §4. We let $\Lambda_K = \boldsymbol{g}_{K/F}(\Lambda) \in \mathcal{CC}(\theta_K)$ (the choice of central quasicharacter for Λ_K is irrelevant here.)

The next step is to choose a simple stratum $[\mathfrak{A}_K, n, n-1, \alpha]$ equivalent to $[\mathfrak{A}_K, n, n-1, \beta]$. By **[5]** 3.8, we can choose α so that $[\mathfrak{A}, n, n-1, \alpha]$ is also simple; it is certainly equivalent to $[\mathfrak{A}, n, n-1, \beta]$. *We further assume that α does not lie in F,* (or, equivalently, in K).

Lemma. — *With the notation above, we have*

$$\tau(\Lambda,\psi_F) = c\,\tau(\Lambda_K,\psi_K)\,\mathfrak{G}(\mathfrak{A},\alpha,\zeta).$$

Proof. — We extend $\Lambda^\vee$ to a representation of $\langle\zeta\rangle \ltimes \boldsymbol{J}(\beta,\mathfrak{A})$ as in A2, and consider

$$\mathrm{T}_\zeta = \mathrm{T}_\zeta(\Lambda,\psi_F) = \sum_{x\in J^1/\boldsymbol{U}^{n+1}} \Lambda^\vee(\zeta\beta x)\,\psi_A(\beta x),$$

where $J^1 = J^1(\beta,\mathfrak{A})$ and $\boldsymbol{U}^t = \boldsymbol{U}^t(\mathfrak{A})$, $t \geqslant 1$. This is an operator on the space of $\Lambda^\vee$. Indeed

$$\mathrm{T}_\zeta = \Lambda^\vee(\zeta)\,\mathrm{T}(\Lambda,\psi_F),$$

and hence

$$\mathrm{tr}\mathrm{T}_\zeta = \tau(\Lambda,\psi_F)\,\mathrm{tr}(\Lambda^\vee(\zeta)).$$

We note that $\mathrm{tr}(\Lambda^\vee(\zeta)) \neq 0$, as follows from, e.g., 4.2.1.

Just as in 6.1, we can reduce the range of summation in the definition of T_ζ; this gives the convenient form

$$\tau(\Lambda,\psi_F) = \frac{c}{\mathrm{tr}\Lambda^\vee(\zeta)} \sum_{x\in \boldsymbol{U}^{n'}/\boldsymbol{U}^{n''}} \mathrm{tr}(\Lambda^\vee(\zeta\beta x))\,\psi_A(\beta x). \tag{8.2.1}$$

Here, we have abbreviated $n' = [(n+1)/2]$, $n'' = [n/2]+1$.

We now treat the case where *the integer n is odd.* Thus $n' = n''$. We are assuming that $\alpha \notin F$; it follows that $\alpha \notin K$ and we can use 6.1.1 to get

$$\tau(\Lambda_K,\psi_K) = c\,\mathrm{tr}(\Lambda_K^\vee(\beta))\,\psi_{A_K}(\beta).$$

Similarly, using 4.2.1,

$$\begin{aligned}\tau(\Lambda,\psi_F) &= c\,\mathrm{tr}(\Lambda^\vee(\zeta))^{-1}\,\mathrm{tr}(\mathrm{T}_\zeta)\\ &= c\,\mathrm{tr}(\Lambda^\vee(\zeta))^{-1}\,\mathrm{tr}(\Lambda^\vee(\zeta\beta))\,\psi_A(\beta)\\ &= c\,\mathrm{tr}(\Lambda^\vee(\zeta))^{-1}\,\mathrm{tr}(\Lambda_K^\vee(\beta))\,\psi_A(\beta)\\ &= c\,\tau(\Lambda_K,\psi_K),\end{aligned}$$

which proves the lemma in this case.

We now assume that n is even, $n = 2l$ say. By 6.3 Proposition 2, we have

$$\tau(\Lambda_K,\psi_K) = c\,\mathrm{tr}(\Lambda_K^\vee(\beta h_0))\,\psi_{A_K}(\beta h_0), \tag{8.2.2}$$

for a certain element $h_0 \in \boldsymbol{U}_K^l = \boldsymbol{U}^l(\mathfrak{A}_K)$.

We next evaluate

$$\mathrm{tr}(\mathrm{T}_\zeta(\Lambda,\psi_F)) = \mathrm{tr}(\Lambda^\vee(\zeta))\,\tau(\Lambda,\psi_F).$$

We have

$$\mathrm{tr}\mathrm{T}_\zeta = c \sum_{x\in \boldsymbol{U}^l/\boldsymbol{U}^{l+1}} \mathrm{tr}\Lambda^\vee(\zeta\beta x)\,\psi_A(\beta x).$$

Here, however, it will be more convenient to use $\boldsymbol{U}^l/\boldsymbol{U}^{n+1}$ as the range of summation: this only changes the value of c. There certainly exists $h\in \boldsymbol{U}^l$ such that

$$\sum_{z\in H^l/\boldsymbol{U}^{n+1}} \mathrm{tr}\Lambda^\vee(\zeta\beta hz)\,\psi_A(\beta hz)\neq 0,$$

where $H^l = H^l(\beta,\mathfrak{A})$. Exactly as in 6.3 Lemma 2, the map

$$z\longmapsto \psi_{\beta h}(z) = \frac{\psi_A(\beta hz)}{\psi_A(\beta h)}$$

is a character of H^l and, since the last sum does not vanish,

$$\psi_{\beta h}(z) = \theta(z), \qquad z\in H^l. \tag{8.2.3}$$

Let us write $H^l = 1+\mathfrak{H}$; the last condition determines the coset $\beta h + \mathfrak{H}^*$, where $\mathfrak{H}^*$ denotes the lattice $\{x\in A : \psi_A(xy)=1,\ y\in\mathfrak{H}\}$. As before, write A_α for the map $x\mapsto \alpha x\alpha^{-1}-x$, $x\in A$. Let $\mathfrak{P}$ denote the radical of $\mathfrak{A}$. The definition of $\mathfrak{H}$ (*cf.* **[9]** 3.1) gives

$$\begin{aligned}\mathfrak{H}^* &= \mathfrak{P}^{-l}\cap\Big(\mathrm{A}_\alpha(A)+\mathfrak{P}^{1-l}\Big)\\ &= \mathfrak{P}^{1-l}+\Big(\mathrm{A}_\alpha(A)\cap\mathfrak{P}^{-l}\Big)\\ &= \mathfrak{P}^{1-l}+\mathrm{A}_\alpha(\mathfrak{P}^{-l}),\end{aligned}$$

after an easy calculation along the lines of **[9]** 1.4. Put another way,

$$\mathfrak{H}^* = \Big(\mathfrak{P}^{l+1}+\mathrm{A}_\alpha(\mathfrak{P}^l)\Big)\,\alpha = \Big(\mathfrak{P}^{l+1}+\mathrm{A}_\alpha(\mathfrak{P}^l)\Big)\,\beta.$$

We have already observed that 8.2.3 determines the coset $\beta h+\mathfrak{H}^*$; since $\mathfrak{H}^*$ is invariant under conjugation by β, the relation 8.2.3 therefore determines the coset $h(1+\mathrm{A}_\alpha(\mathfrak{P}^l)+\mathfrak{P}^{l+1}) = h(1+\mathrm{A}_\alpha(\mathfrak{P}^l))\boldsymbol{U}^{l+1}$. However, if $h=1+x$ and we take $y\in\mathfrak{P}^l$, $h' = 1+x-\mathrm{A}_\alpha(\alpha^{-1}y\alpha)$, we have

$$\beta h'\boldsymbol{U}^{l+1} = (1-y)\beta h\boldsymbol{U}^{l+1}(1-y)^{-1}.$$

On the other hand, we assert

$$\mathrm{A}_\alpha(\mathfrak{P}^l)+\mathfrak{P}^{l+1}\subset \mathrm{A}_{\zeta\alpha}(\mathfrak{P}^l)+\mathfrak{P}^{l+1}. \tag{8.2.4}$$

To prove this, we first observe that α acts on $\mathfrak{P}^l/\mathfrak{P}^{l+1}$ (by conjugation) as an automorphism of p-power order, while ζ (which commutes with α) acts with p-prime order, r, say. Thus the fixed points of $\zeta\alpha$ are the common fixed points of α and ζ. A tame corestriction s_α on A relative to $F[\alpha]/F$ (see **[9]** 1.3) provides us with a surjection to the fixed points of α, whose kernel is the image of A_α. However, if we compose s_α with the map $x \mapsto \sum_{0\leqslant i\leqslant r-1} \zeta^i x \zeta^{-i}$, we get a surjective map to the fixed points of $\zeta\alpha$ whose kernel is the image of $\mathrm{A}_{\zeta\alpha}$. Hence, as endomorphisms of $\mathfrak{P}^l/\mathfrak{P}^{l+1}$, we get $\operatorname{Im}\mathrm{A}_\alpha \subset \operatorname{Im}\mathrm{A}_{\zeta\alpha}$, as asserted.

The identity 8.2.4 shows that we can write $h' = 1 + x + \mathrm{A}_{\zeta\alpha}(\alpha^{-1}\zeta^{-1}v\alpha\zeta)$, $v \in \mathfrak{P}^l$, so that

$$\zeta\beta h' \boldsymbol{U}^l = (1-v)\zeta\beta h \boldsymbol{U}^{l+1}(1-v)^{-1}.$$

Thus $\mathrm{tr}\Lambda(\zeta\beta h') \neq 0$. In other words, the property 8.2.3 of h is shared by all elements of the coset $h(1+\mathrm{A}_\alpha(\mathfrak{P}^l))\boldsymbol{U}^{l+1}$.

However, property 8.2.3 is also valid for the element h^ζ, since $\theta^\zeta = \theta$. We deduce that $x = h - 1$ is a fixed point of ζ on $\mathfrak{P}^l/(\mathrm{A}_\alpha(\mathfrak{P}^l) + \mathfrak{P}^{l+1})$. We can therefore choose x to commute with ζ. We have shown:

There exists $h_0 \in \boldsymbol{U}_K^l$ such that

$$\sum_{z\in H^l/\boldsymbol{U}^{n+1}} \mathrm{tr}\Lambda^\vee(\zeta\beta h_0 z)\,\psi_A(\beta h_0 z) \neq 0.$$

We might as well, therefore, take this h_0 to be the same as the one in 8.2.2. Now we can use our support criterion A1.8. Arguing as in 6.3, we have

$$\mathrm{trT}_\zeta = c \sum_{z\in H^l/\boldsymbol{U}^{n+1}} \sum_{g\in \boldsymbol{U}^l/\boldsymbol{U}^{n+1}} \mathrm{tr}\Lambda^\vee(g\zeta\beta h_0 z g^{-1})\,\psi_A(\zeta^{-1}g\zeta\beta h_0 z g^{-1}).$$

It will be more convenient to write the second factor in the summand as $\psi_A([g^{-1},\zeta^{-1}]\beta h_0 z)$ (using square brackets to denote the commutator). We sum first over z. The contribution from g vanishes unless

$$\psi_A([g^{-1},\zeta^{-1}]\beta h_0 z) = \psi_A([g^{-1},\zeta^{-1}]\beta h_0)\,\theta(z),$$

for all $z \in H^l$. By our arguments above, this forces

$$[g^{-1},\zeta^{-1}] \in (1+\mathrm{A}_\alpha(\mathfrak{P}^l))\boldsymbol{U}^{l+1}.$$

Our sum is thus reduced to

$$\mathrm{trT}_\zeta = c\,\mathrm{tr}\Lambda^\vee(\zeta\beta h_0)\,\psi_A(\beta h_0)\cdot \boldsymbol{\Sigma},$$

where

$$\boldsymbol{\Sigma} = \sum_g \frac{\psi_A([g^{-1},\zeta^{-1}]\beta h_0)}{\psi_A(\beta h_0)}$$

with g ranging over $\boldsymbol{U}^l/\boldsymbol{U}^{n+1}$ subject to the condition

$$[g^{-1}, \zeta^{-1}] \in (1 + \mathrm{A}_\alpha(\mathfrak{P}^l))\boldsymbol{U}^{l+1}.$$

If we write $g = 1 - x$, we get

$$\boldsymbol{\Sigma} = \sum_x \psi_A(\beta h_0(x - x^\zeta + x(x - x^\zeta)))$$

with x now ranging over $\mathfrak{P}^l/\mathfrak{P}^{l+1}$ subject to the condition

$$x - x^\zeta \in \mathrm{A}_\alpha(\mathfrak{P}^l) + \mathfrak{P}^{l+1}.$$

Since ζ commutes with both β and h_0, we get $\psi_A(\beta h_0(x - x^\zeta)) = 1$. Therefore only the quadratic term matters, and we find

$$\begin{aligned}\boldsymbol{\Sigma} &= c \sum_x \psi_A(\beta h_0 x(x - x^\zeta)) \\ &= c \sum_x \psi_A(\alpha x(x - x^\zeta)) \\ &= c\, \mathfrak{G}(\mathfrak{A}, \alpha, \zeta),\end{aligned}$$

as required for the lemma. □

8.3. We now prove the first equality in Lemma 8.1. In the notation of that section, we choose $\theta^0 \in \mathfrak{CC}(\mathfrak{A}_0, \alpha)$, $\Lambda^0 \in \mathfrak{CC}(\theta_0)$, $\Lambda^0_K = \boldsymbol{g}_{K/F}(\Lambda^0)$. By Lemma 8.2, we have

$$\tau(\Lambda^0, \psi_F) = c\,\tau(\Lambda^0_K, \psi_K)\, \mathfrak{G}(\mathfrak{A}_0, \alpha, \zeta). \tag{8.3.1}$$

Now let us use the notation of 8.1, and set $\dim_{KE}(V) = t$. We can form the central type $\Lambda_\mathrm{m} = (\Lambda^0)^{(t)}$ as in 3.3; this is attached to a principal order $\mathfrak{A}_\mathrm{m}$ minimal for the property of being normalized by $(KE)^\times$. We set

$$\Lambda = \mathfrak{C}\tau_{\mathfrak{A}_\mathrm{m}, \mathfrak{A}, \beta}(\Lambda_\mathrm{m}),$$

and form $\Lambda_K = \boldsymbol{g}_{K/F}(\Lambda)$. This is related to Λ^0_K in exactly the same way: it is the transfer (via a map $\mathfrak{C}\tau$) of the t-fold multiple of Λ^0_K, since the Glauberman correspondence $\boldsymbol{g}_{K/F}$ commutes with these constructions 4.5. By 6.4 Corollary 2, we have $\tau(\Lambda, \psi_F) = c\,\tau(\Lambda^0, \psi_F)^t$ and $\tau(\Lambda_K, \psi_K) = c\,\tau(\Lambda^0_K, \psi_K)^t$, whence the assertion follows.

As a consequence of this, we show:

Lemma. — *Suppose that $p = 2$. Then $\mathfrak{G}(\mathfrak{A}, \alpha, \zeta)$ is real and positive.*

Proof. — Since α is minimal over F and $F[\alpha]/F$ is totally ramified of degree $2^m > 1$, the valuation of α in $F[\alpha]$ is odd. Thus, in our standard notation, $\mathfrak{G}(\mathfrak{A}_0, \alpha, \zeta) = 1$ by definition, and the result follows from what we have just proved. □

8.4. We now prove Theorem 1.6, modulo the second equality in Lemma 8.1. The notation of 1.6 is identical to that of 8.2 except that KE is now a *maximal* subfield of A and 1.6 allows $\alpha \in F$. We continue to exclude this possibility and treat it separately below. Moreover, in 1.6, the tame extension K/F is arbitrary.

The identity we have to prove is:

$$\frac{\boldsymbol{\varepsilon}(\pi_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\pi_F, s, \psi_F)^d} = \left(\frac{\boldsymbol{\varepsilon}(1_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{K/F}, s, \psi_F)} \right)^{p^m} \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\beta)). \tag{8.4.1}$$

The left hand side here is independent of s, as follows from a simple computation based on 6.1 Lemma 2. Since K/F is tame, the same applies to the right hand side. Next we observe that both sides of 8.4.1 are transitive in K/F, so can reinstate our hypothesis that $[K{:}F] = d$ is prime.

The left hand side of 8.4.1 is $c\,\tau(\Lambda_K, \psi_K)/\tau(\Lambda_F, \psi_F)^d$. On the other hand, Lemma 8.2 gives us

$$\tau(\Lambda_K, \psi_K) = c\,\tau(\Lambda, \psi_F)\, \mathfrak{G}(\mathfrak{A}, \alpha, \zeta)^{-1}.$$

We recall that $\Lambda = \Lambda^0 \otimes \boldsymbol{\delta}_{K/F}^{p^m-1}$, where $p^m = [E{:}F]$, by its definition in 5.3, while $\tau(\Lambda^0, \psi_F) = c\,\tau(\Lambda_F, \psi_F)^d$ by Corollary 6.4. Thus $\tau(\Lambda, \psi_F) = \tau(\Lambda_F, \psi_F)^d \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\beta))^{p^m-1}$. So far, we have

$$\frac{\tau(\Lambda_K, \psi_K)}{\tau(\Lambda_F, \psi_F)^d} = c\, \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\beta))^{p-1}\, \mathfrak{G}(\mathfrak{A}, \alpha, \zeta)^{-1},$$

recalling that $m \geqslant 1$ and that $\boldsymbol{\delta}_{K/F}$ has order at most 2. Next we note that

$$\boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\beta)) = \boldsymbol{\delta}_{K/F}(\mathrm{N}_{F[\alpha]/F}(\alpha))^{[E:F]/[F[\alpha]:F]},$$

since $\alpha \equiv \beta \pmod{\boldsymbol{U}^1(\mathfrak{A})}$. Theorem 1.6 now follows from Lemma 8.2.

8.5. We still have to deal with the case of Theorem 1.6 in which $\alpha \in F$. We write $\beta = \alpha + \gamma$ and assume that the stratum $[\mathfrak{A}_F, n_1, 0, \gamma]$ is simple, for some $n_1 > 0$. (The contrary only arises in the uninteresting case $m = 0$.) We then have $\theta \mid \boldsymbol{U}^{n'} = \psi_\gamma\, (\chi \circ \det)$, for some quasicharacter χ of $F^\times$. Replacing F

by K, we have exactly the same situation, with $\chi_K = \chi \circ \mathrm{N}_{K/F}$ replacing χ. Proposition 6.2 gives us

$$\tau(\Lambda_K, \psi_K) = c\,\mathrm{tr}(\Lambda_K^\vee(\beta))\,\chi_K(\alpha)^{p^m}\,\psi_{A_K}(\gamma)\,\tau(\chi_K, \psi_K)^{p^m},$$
$$\tau(\Lambda, \psi_F) = c\,\mathrm{tr}(\Lambda^\vee(\beta))\,\chi(\alpha)^{dp^m}\,\psi_A(\gamma)\,\tau(\chi, \psi_F)^{dp^m},$$

with possibly different values of $c > 0$. Comparing these expressions, we get

$$\frac{\tau(\Lambda, \psi_F)}{\tau(\Lambda_K, \psi_K)} = c\left(\frac{\tau(\chi, \psi_F)^d}{\tau(\chi_K, \psi_K)}\right)^{p^m}.$$

Theorem 1.6 certainly holds for one-dimensional representations of GL_1, so we have

$$\frac{\tau(\chi_K, \psi_K)}{\tau(\chi, \psi_F)^d} = c\,\frac{\boldsymbol{\varepsilon}(1_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{K/F}, s, \psi_F)}\,\boldsymbol{\delta}_{K/F}(\alpha),$$

whence

$$\frac{\tau(\Lambda_K, \psi_K)}{\tau(\Lambda, \psi_F)} = c\left(\frac{\boldsymbol{\varepsilon}(1_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{K/F}, s, \psi_F)}\right)^{p^m}\boldsymbol{\delta}_{K/F}(\alpha)^{p^m}.$$

We have $\boldsymbol{\delta}_{K/F}(\alpha)^{p^m} = \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\beta))$ since $\alpha \equiv \beta \pmod{\boldsymbol{U}^1}$ and $\boldsymbol{\delta}_{K/F}$ is tame quadratic. The theorem follows in this final case.

CHAPTER 9

CALCULATION OF THE COMMUTATOR GAUSS SUM

We have to calculate the Gauss sum $\mathfrak{G}(\mathfrak{A}_0, \alpha, \zeta)$ of 8.1. Since $\mathfrak{A}_0$ will now be fixed, we drop all the attached 0's. We recall $E = F[\alpha]$ and set $[E{:}F] = p^m$. It will also be simpler to put

$$\mathcal{E}(K/F) = \frac{\boldsymbol{\varepsilon}(1_K, s, \psi_K)}{\boldsymbol{\varepsilon}(\boldsymbol{\rho}_{K/F}, s, \psi_F)}.$$

9.1. Suppose first that n is *odd*. Thus, by definition, $\mathfrak{G} = 1$, and we have to check that

$$\mathcal{E}(K/F)^{-p^m} \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\alpha))^p = c > 0. \tag{9.1.1}$$

Suppose first that K/F is unramified. The character $\boldsymbol{\delta}_{K/F}$ is then unramified and $\boldsymbol{\delta}_{K/F}(\mathfrak{p}_F) = (-1)^{d-1}$. Since $n = -\nu_{KE}(\alpha)$ is odd, we have

$$\boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\alpha)) = (-1)^{d-1}.$$

On the other hand, since ψ_F has conductor $\mathfrak{p}_F$, we get $\mathcal{E}(K/F) = c\,(-1)^{d-1}$ and the result follows. Taking K/F totally ramified, since d divides n, we see that d is odd. By **[3]** 10.1.6, the character $\boldsymbol{\delta}_{K/F}$ is unramified and

$$\boldsymbol{\delta}_{K/F}(\mathfrak{p}_F) = \left(\frac{q}{d}\right),$$

where $\left(\frac{a}{b}\right)$ is the Legendre symbol. Also, *loc cit.*,

$$\mathcal{E}(K/F) = c\left(\frac{q}{d}\right).$$

This proves 9.1.1.

9.2. We are thus left with the case in which $n = -\nu_{KE}(\alpha) = 2l$ is even. Since α is minimal over K, this case cannot arise when $p = 2$. Thus, *we assume from now on that p is odd.* We need some preliminaries on classical Gauss sums.

We are given a finite field k of odd characteristic and a non-trivial additive character ψ_{k} of k. We put

$$\mathfrak{g}_2(\psi_{\mathrm{k}}) = \sum_{x\in\mathrm{k}^\times} \chi_2(x)\psi_{\mathrm{k}}(x),$$

where χ_2 is the non-trivial quadratic character of $\mathrm{k}^\times$.

Now let V be a finite-dimensional k-vector space and $Q : V \to \mathrm{k}$ a quadratic form. We define

$$\mathfrak{g}(V, Q, \psi_{\mathrm{k}}) = \sum_{x\in V} \psi_{\mathrm{k}}(Q(X)).$$

By a simple exercise, we have:

Lemma 1. — *Suppose that V has dimension n and Q has rank r. Let Q' denote the nonsingular part of Q. Then*

$$\mathfrak{g}(V, Q, \psi_{\mathrm{k}}) = \chi_2(\det Q')\, |\mathrm{k}|^{n-r}\, \mathfrak{g}_2(\psi_{\mathrm{k}})^r.$$

Another elementary result will be useful:

Lemma 2. — *Let V be a k-vector space of finite dimension n and let Q be a nondegenerate quadratic form on V. Let W be a subspace of V of codimension r and suppose that the radical of $Q \mid W$ has dimension at least r. Then:*

(i) *the rank of $Q \mid W$ is exactly $n - 2r$;*

(ii) *the determinant of the nonsingular part of $Q \mid W$ is $(-1)^r \det Q$.*

9.3. We return to the commutator Gauss sum

$$\mathfrak{G} = \mathfrak{G}(\mathfrak{A}, \alpha, \zeta) = \sum_{\substack{x\in\mathfrak{P}^l/\mathfrak{P}^{l+1},\\ x-x^\zeta\in\mathrm{A}_\alpha(\mathfrak{P}^l)+\mathfrak{P}^{l+1}}} \psi_A(\alpha x(x - x^\zeta)).$$

Lemma 3. — *There exists $\gamma \in KE$ such that $\gamma^2\alpha$ is a root of unity $\mu \in F$ of order prime to p. If $\nu_E(\alpha)$ is even, we may choose $\gamma \in E$.*

Proof. — Since p is odd, the group U_F^1 is 2-divisible. If $\nu_E(\alpha)$ is even, we can therefore find $\gamma \in E$ such that $\gamma^2\alpha$ is a p-prime root of unity $\mu \in E$. Since E/F is totally ramified, we have $\mu \in F$.

If, on the other hand, $\nu_E(\alpha)$ is odd, the extension K/F is ramified quadratic. As before, we can find $\gamma \in KE$ with $\mu = \gamma^2\alpha$ a p-prime root of unity. Since KE/F is totally ramified, we have $\mu \in F$. □

This element γ has KE-valuation l, so $\gamma\mathfrak{A} = \mathfrak{P}^l$ and $x \mapsto \gamma x$ gives an isomorphism $\mathfrak{A}/\mathfrak{P} \cong \mathfrak{P}^l/\mathfrak{P}^{l+1}$ which commutes with the obvious conjugation actions of α and ζ. We thus have

$$\mathfrak{G} = \sum_{\substack{y\in\mathfrak{A}/\mathfrak{P},\\ y-y^\zeta\in\mathrm{A}_\alpha(\mathfrak{A})+\mathfrak{P}}} \psi_A(\mu y^\gamma(y-y^\zeta)).$$

Since α and ζ act on $\mathfrak{A}/\mathfrak{P}$ with relatively prime orders and commute with each other, the condition $y-y^\zeta \in \mathrm{A}_\alpha(\mathfrak{A})+\mathfrak{P}$ is equivalent to $y \in \mathrm{A}_\alpha(\mathfrak{A})+\mathfrak{A}_K+\mathfrak{P}$, where $\mathfrak{A}_K = \mathfrak{A}\cap \mathrm{End}_K(KE)$. For $z \in \mathfrak{A}_K$, we have $\psi_A(\mu z(y-y^\zeta)) = 1$, so

$$\mathfrak{G} = c \sum_{x\in\mathrm{A}_\alpha(\mathfrak{A}/\mathfrak{P})} \psi_A(\mu x^\gamma(x-x^\zeta)),$$

where $\mathrm{A}_\alpha(\mathfrak{A}/\mathfrak{P})$ is short-hand for $(\mathrm{A}_\alpha(\mathfrak{A})+\mathfrak{P})/\mathfrak{P}$.

We can now apply the elementary considerations of 9.2 to the field k_F, the character $\psi_\mu : x \mapsto \psi_F(\mu x)$ of k_F, the vector space $V = \mathrm{A}_\alpha(\mathfrak{A}/\mathfrak{P})$ and the quadratic form

$$Q(v) = \mathrm{tr}(v^\gamma(v-v^\zeta)), \quad v \in V. \tag{9.3.1}$$

Thus, in the language of 9.2, we have $\mathfrak{G} = \mathfrak{g}(V, Q, \psi_\mu)$. By 9.2 Lemma 2, we need to compute the rank and the determinant of the nonsingular part of Q.

For the time being, it will be convenient to regard Q as being the restriction to V of a quadratic form Q_0 on $\mathfrak{A}/\mathfrak{P}$ defined by the same formula 9.3.1:

$$Q_0(v) = \mathrm{tr}(v^\gamma(v-v^\zeta)), \quad v \in \mathfrak{A}/\mathfrak{P}.$$

We first suppose that K/F is totally ramified. The algebra $\mathfrak{A}/\mathfrak{P}$ is then the direct sum of $p^m d$ copies of k_F. The group $(KE)^\times$ acts by permutation of the coordinates, or, if preferred, by permuting the set of indecomposable idempotents of $\mathfrak{A}/\mathfrak{P}$. In particular, the action of $x \in (KE)^\times$ depends only on the valuation of x. Our element α has d orbits of length p^m and ζ has p^m orbits of length d in this set of idempotents. The element γ has the same orbits as α *except* in the case where $\nu_E(\alpha)$ is odd (and so K/F is ramified quadratic). In this exceptional case, the element γ has a single orbit of length $2p^m$.

Let us deal first with this exceptional case:

$$K/F \text{ ramified}, \quad d=2, \quad \nu_E(\alpha) \text{ odd}. \tag{9.3.2}$$

The element γ acts transitively on the indecomposable idempotents (call them e_i) of the algebra $\mathfrak{A}/\mathfrak{P}$; we may as well number these so that $e_i^\gamma = e_{i+1}$ (taking indices mod $2p^m$). Thus $e_i^\alpha = e_{i-2}$, and the image of A_α consists of elements $\sum_i x_i e_i$ such that

$$\sum_{i=1}^{p^m} x_{2i} = 0 = \sum_{i=1}^{p^m} x_{2i-1}.$$

On the other hand, ζ interchanges e_i with e_{p^m+i}, $1 \leqslant i \leqslant p^m$. The ζ-fixed subspace of $\mathfrak{A}/\mathfrak{P}$ is contained in the radical of Q_0, so we need only look at the restriction of Q_0 to the space of x such that $x^\zeta = -x$. Making an obvious choice of basis in this space, Q_0 is there given by

$$Q_0(x_1, x_2, \dots, x_{p^m}) = 4(x_2x_1 + x_3x_2 + \cdots + x_{p^m}x_{p^m-1} - x_1x_{p^m}).$$

A pleasant computation shows that this form is nondegenerate with determinant -1 (mod squares). The image of A_α in this space consists of vectors (x_i) such that

$$\sum_{i=1}^{p^m} (-1)^i x_i = 0.$$

In particular, $\operatorname{Im}\mathrm{A}_\alpha$ contains the vector $\boldsymbol{x}_1 = (1, -1, 1, -1, \dots)$, which is orthogonal to all of $\operatorname{Im}\mathrm{A}_\alpha$. On the other hand, the orthogonal complement of $\boldsymbol{x}_2 = (1, 0, 0, \dots, 0)$ is spanned by $\boldsymbol{x}_2$ and a subspace of $\operatorname{Im}\mathrm{A}_\alpha$ of codimension one. The space spanned by $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is nondegenerate with determinant -1. We deduce that the nonsingular part of Q on $\operatorname{Im}\mathrm{A}_\alpha$ has determinant $+1$.

In the case (9.3.2), we therefore get

$$\mathfrak{G} = c\,\chi_2(\mu)\mathfrak{g}_2(\psi_{\mathrm{k}})^{p^m-2},$$

where we have written ψ_{k} for the character of k_F induced by ψ_F. We can re-write this as

$$\mathfrak{G} = c\,\chi_2(-\mu)\mathfrak{g}_2(\psi_{\mathrm{k}})^{p^m}.$$

Now, we defined $\gamma \in KE$ by the relation $\gamma^2\alpha = \mu$, and $\mu \in F$. Thus $\gamma^2 \in E$ while $\gamma \notin E$ (since γ^2 has odd valuation in E). Thus $\mathrm{N}_{KE/E}(\gamma) = -\gamma^2 = -\mu\alpha^{-1}$. The character $\boldsymbol{\delta}_{K/F}$ is the norm-residue character for the extension K/F **[3]** (10.1.6) so

$$\begin{aligned} \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\alpha)) &= \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(-\mu\mathrm{N}_{KE/E}(\gamma^{-1}))) \\ &= \boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(-\mu)) \\ &= \chi_2(-\mu). \end{aligned}$$

On the other hand, by direct computation from Tate's thesis **[37]** (3.2.6), we find that

$$\mathcal{E}(K/F) = c\,\mathfrak{g}_2(\psi_{\mathrm{k}}).$$

This gives the desired result in the present case (9.3.2).

Now consider the case

$$\text{(9.3.3)} \qquad K/F \text{ ramified}, \quad d = 2, \quad \nu_E(\alpha) \text{ even.}$$

This is very similar to (9.3.2), except that the form $Q_0 : v \mapsto \mathrm{tr}(v^\gamma(v - v^\zeta))$ on $\mathfrak{A}/\mathfrak{P}$ has determinant $+1$ in the ζ-skew part of $\mathfrak{A}/\mathfrak{P}$, while $\boldsymbol{\delta}_{K/F}(\mathrm{N}_{E/F}(\alpha)) = \chi_2(\mu)$: we leave the details as an exercise.

Now we turn to the case

$$\text{(9.3.4)} \qquad K/F \text{ ramified}, \quad d \text{ odd.}$$

To deal with the case 9.3.4, we need another digression.

9.4. In this paragraph, we are given a finite field k with characteristic $p \neq 2$ and a cyclic group C of odd prime order $\ell \neq p$. We consider pairs (V, h), where V is a finite $\mathrm{k}C$-module and $h : V \times V \to \mathrm{k}$ is a nondegenerate symmetric bilinear form which is C-invariant:

$$h(cv_1, cv_2) = h(v_1, v_2), \quad v_i \in V, \ c \in C.$$

Such objects are semisimple. The simple objects of this kind are as follows, modulo the obvious concept of isomorphism:

9.4.1

(i) $\dim V = 1$, *and the action of* C *on* V *is trivial;*

(ii) *there is a simple* $\mathrm{k}C$*-module* W, *not isomorphic to its contragredient* $W^\vee$, *such that* $V = W \oplus W^\vee$, *and the form* h *is given by*

$$h((w_1, \check{w}_1), (w_2, \check{w}_2)) = \langle w_1, \check{w}_2 \rangle + \langle w_2, \check{w}_1 \rangle.$$

Here, $\langle\,,\,\rangle$ *is the canonical* C*-invariant pairing* $W \times W^\vee \to \mathrm{k}$.

(iii) V *is simple as* $\mathrm{k}C$*-module and the action of* C *is non-trivial.*

In cases (ii) *and* (iii), *the isomorphism class of the pair* (V, h) *is determined by that of the underlying module* V, *and the dimension of* V *is even. In case* (ii), *we have* $\det h = (-1)^{\dim V/2}$. *In case* (iii), $\det h$ *is of the form* $(-1)^{\dim V/2}\phi$, *where* $\phi \in \mathrm{k}^\times$ *is not a square.*

Any irreducible, self-contragredient $\mathrm{k}C$*-module appears in a pair* (V, h) *of type* (iii).

The description 9.4.1 is compatible with the following description of the irreducible $\mathrm{k}C$-modules. We first fix a generator c_0 of C. We next fix an algebraic closure $\bar{\mathrm{k}}/\mathrm{k}$ and write $\Gamma = \mathrm{Gal}(\bar{\mathrm{k}}/\mathrm{k})$. Write $\boldsymbol{\mu}'_\ell$ for the set of primitive

ℓ-th roots of unity in $\bar{\mathrm{k}}$. The isomorphism classes of non-trivial simple $\mathrm{k}C$-modules are then in bijection with the set of Γ-orbits in $\boldsymbol{\mu}'_\ell$. The underlying module V is simply the field extension k'/k generated by a primitive ℓ-th root of unity, and c_0 acts as multiplication by a root of unity ζ in the given orbit. In this description, the self-contragredient modules are those corresponding to orbits containing both some ζ and its inverse.

All of these assertions are quite easy to establish; the analogous case of *alternating* $\mathrm{k}C$-modules is treated fully in **[3]** §8.2. The ideas involved are so similar that we can omit the details here.

9.5. We return to the case (9.3.4). The first step, as before, is to choose $\gamma \in E$ and a p-prime root of unity $\mu \in F$ such that $\gamma^2\alpha = \mu$. Since α is minimal over F, its valuation is relatively prime to $e(E|F)$, whence $\mathrm{Ad}\,\alpha$ generates the image of $\mathrm{Ad}(F[\alpha]^\times)$ in $\mathrm{Aut}(\mathfrak{A}/\mathfrak{P})$. In particular, $\mathrm{Ad}\,\gamma$ is a power of $\mathrm{Ad}\,\alpha$. We have to consider the restriction, Q, of the quadratic form

$$Q_0(x) = \mathrm{tr}(x^\gamma(x - x^\zeta)), \quad x \in \mathfrak{A}/\mathfrak{P},$$

to $\mathrm{Im}\,\mathrm{A}_\alpha$.

In this case, we can make a useful simplification. Let $\mathfrak{A}_\alpha$ denote the unique $\mathrm{Ad}\,F[\alpha]^\times$-stable hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(F[\alpha])$. Write $\mathfrak{P}_\alpha$ for the radical of $\mathfrak{A}_\alpha$. Likewise, let $\mathfrak{A}_\zeta$ denote the unique $\mathrm{Ad}\,K^\times$-stable hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(K)$, and $\mathfrak{P}_\zeta$ its radical.

As k_F-algebra, we have

$$\mathfrak{A}/\mathfrak{P} \cong \mathfrak{A}_\alpha/\mathfrak{P}_\alpha \otimes_{\mathrm{k}_F} \mathfrak{A}_\zeta/\mathfrak{P}_\zeta.$$

The first (resp. second) factor here carries a natural conjugation action by α (resp. ζ); the isomorphism of algebras is then also an isomorphism of $\langle \mathrm{Ad}\,\alpha, \mathrm{Ad}\,\zeta\rangle$-modules, where α (resp. ζ) acts trivially on the second (resp. first) factor. Our quadratic form decomposes as a tensor product:

$$\begin{aligned} Q_0 &= Q_\alpha \otimes Q_\zeta, \\ Q_\alpha(x) &= \mathrm{tr}_{\mathfrak{A}_\alpha/\mathfrak{P}_\alpha}(x^\gamma x), \\ Q_\zeta(x) &= \mathrm{tr}_{\mathfrak{A}_\zeta/\mathfrak{P}_\zeta}(x(x - x^\zeta)). \end{aligned}$$

Here, for example, $\mathrm{tr}_{\mathfrak{A}_\alpha/\mathfrak{P}_\alpha}$ denotes the algebra trace $\mathfrak{A}_\alpha/\mathfrak{P}_\alpha \to \mathrm{k}_F$. Further,

$$\mathrm{A}_\alpha(\mathfrak{A} + \mathfrak{P}/\mathfrak{P}) = (\mathrm{A}_\alpha(\mathfrak{A}_\alpha + \mathfrak{P}_\alpha/\mathfrak{P}_\alpha)) \otimes \mathfrak{A}_\zeta/\mathfrak{P}_\zeta.$$

We can identify $\mathfrak{A}_\alpha/\mathfrak{P}_\alpha$ with $\mathrm{k}_F^{p^m}$ in such a way that $(x_1, x_2, \ldots, x_{p^m})^\gamma = (x_2, x_3, \ldots, x_1)$; we then have

$$Q_\alpha(x_1, \ldots, x_{p^m}) = x_1x_2 + x_2x_3 + \cdots + x_{p^m}x_1,$$

and this form has determinant 1. Restricting to the image of A_α, it has rank p^m-2 and the determinant of its nonsingular part is -1, by (9.2) Lemma 2.

On the other hand, Q_ζ is null on the $\mathrm{Ad}\,\zeta$-fixed points in $\mathfrak{A}_\zeta/\mathfrak{P}_\zeta$, but is nondegenerate on the image of A_ζ. It is, moreover, $\mathrm{Ad}\,\zeta$-invariant. The space $\mathfrak{A}_\zeta/\mathfrak{P}_\zeta$ is free of rank one over the k_F-group ring of the cyclic group (of order d) generated by $\mathrm{Ad}\,\zeta$. Thus Q_ζ has rank $d-1$.

In all, Q has rank $(p^m-2)(d-1)$, and, taken mod squares, the determinant of its nonsingular part is the determinant (call it Δ) of Q_ζ on the image of A_ζ. This gives us

$$\mathfrak{G} = c\,\chi_2(\Delta)\mathfrak{g}_2(\psi_\mu)^{(p^m-2)(d-1)} = c\,\chi_2(\Delta)\chi_2(-1)^{(d-1)/2}.$$

On the other hand, appealing to **[3]** (10.1.6), we have

$$\mathcal{E}(K/F) = c\left(\frac{q_F}{d}\right),$$

and the Legendre symbol $\left(\frac{q_F}{d}\right)$ equals $(-1)^s$, where s is the number of non-trivial self-contragredient Galois orbits of d-th roots of unity over k_F. Let k'/k_F be the field extension generated by a primitive d-th root of unity, and set $d'=[\mathrm{k}':\mathrm{k}_F]$. If d' is odd, no Galois orbit here is self-contragredient. In this case, $s=0$, and $\Delta=(-1)^{(d-1)/2d'}$ whence $\mathfrak{G}=1$, as required.

On the other hand, suppose that d' is even. All Galois orbits are then self-contragredient, and $s=(d-1)/d'$. Thus, choosing a non-square $\phi\in\mathrm{k}_F^\times$, we get

$$\Delta = \phi^s(-1)^{sd'/2} = \phi^s(-1)^{(d-1)/2},$$

and $\mathfrak{G}=\left(\frac{q_F}{d}\right)^s$, as required.

9.6. We are left only with the case

$$\text{(9.6.1)} \qquad K/F \text{ unramified}, \quad \nu_E(\alpha) \text{ even}.$$

We define $\mathfrak{A}_\alpha$ as in 9.5, and let $\mathfrak{A}_\zeta$ be the unique hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(K)$ stable under conjugation by $K^\times$. Again,

$$\mathrm{A}_\alpha(\mathfrak{A}+\mathfrak{P}/\mathfrak{P}) = (\mathrm{A}_\alpha(\mathfrak{A}_\alpha+\mathfrak{P}_\alpha/\mathfrak{P}_\alpha))\otimes\mathfrak{A}_\zeta/\mathfrak{P}_\zeta.$$

The algebra $\mathfrak{A}_\alpha/\mathfrak{P}_\alpha$ is $\mathrm{k}_F^{p^m}$, the same as before. But, this time, $\mathfrak{A}_\zeta/\mathfrak{P}_\zeta\cong\mathbb{M}(d,\mathrm{k}_F)=\mathrm{End}_{\mathrm{k}_F}(\mathrm{k}_K)$ with the natural conjugation action by ζ.

The quadratic form Q_0 again decomposes as a tensor product $Q_\alpha\otimes Q_\zeta$. The first factor is as before. The second is $Q_\zeta(x)=\mathrm{tr}(x(x-x^\zeta))$; this is null on the centralizer of ζ and nondegenerate on the image of A_ζ. Thus its rank is d^2-d, and that of Q is $(p^m-2)(d^2-d)$. The determinant of the nonsingular part of Q_α is -1. Writing Δ for the discriminant of the nonsingular part of Q_ζ, we therefore have

$$\mathfrak{G} = c\,\chi_2(\Delta) g_2(\psi_\mu)^{(p^m-2)(d^2-d)} = c\,\chi_2((-1)^{(d^2-d)/2}\Delta).$$

On the other hand,

$$\varepsilon(\boldsymbol{\rho}_{K/F}, s, \psi_F) = c\,(-1)^{d-1}.$$

Take first the case $d = 2$. The space $\operatorname{Im} \mathrm{A}_\zeta$ is then k_K, with action

$$x^\zeta = \zeta\sigma(\zeta)^{-1}x, \quad \zeta \in \mathrm{k}_K^\times,$$

where σ is the non-trivial element of $\mathrm{Gal}(\mathrm{k}_K/\mathrm{k}_F)$. This space admits a unique nondegenerate symmetric bilinear form invariant under this action of $\mathrm{k}_K^\times$ 9.4; the determinant of this form is $-\phi$, ϕ a non-square in k_F. In other words, $\mathfrak{G} = -c$, as required.

Now we take d odd. In this case, we identify $\mathfrak{A}_\zeta/\mathfrak{P}_\zeta = \mathrm{End}_{\mathrm{k}_F}(\mathrm{k}_K)$ with the "twisted group ring" $\widetilde{\mathrm{k}_K\Gamma}$, where $\Gamma = \mathrm{Gal}(\mathrm{k}_K/\mathrm{k}_F)$. As left k_K-vector space, this has a basis $[\sigma]$, $\sigma \in \Gamma$, and multiplication $[\sigma]x = x^{\sigma^{-1}}[\sigma]$, $\sigma \in \Gamma$. We therefore have

$$\mathrm{A}_\zeta(\mathfrak{A}_\zeta/\mathfrak{P}_\zeta) = \sum_{\substack{\sigma\in\mathrm{Gal}(\mathrm{k}_K/\mathrm{k}_F),\\ \sigma\neq 1}} \mathrm{k}_K[\sigma].$$

The action of Ad_ζ on the factor $\mathrm{k}_K[\sigma]$ is left multiplication by $\zeta^{-1}\zeta^{\sigma^{-1}}$. The k_F-contragredient of $\mathrm{k}_K[\sigma]$ is $\mathrm{k}_K[\sigma^{-1}]$; the form Q_ζ is therefore hyperbolic, with determinant $(-1)^{(d^2-d)/2}$. This implies $\mathfrak{G} = c$, and we have completed the proof of Theorem 1.6.

CHAPTER 10

COMPARISON WITH BASE CHANGE

We now prove the proposition and theorem of 1.8.

10.1. We start with the proof of Proposition 1.8. We are given a *cyclic* tame extension K/F and $\pi_F \in \boldsymbol{\mathcal{A}}^{\mathrm{wr}}_m(F)$. The case $m = 0$ is trivial, so we exclude it. Let $[\mathfrak{A}_F, n_F, 0, \beta]$ be a simple stratum underlying some maximal simple type occurring in π_F, and choose a simple stratum $[\mathfrak{A}_F, n_F, n_F{-}1, \alpha]$ equivalent to $[\mathfrak{A}_F, n_F, n_F{-}1, \beta]$. Since both tame lift and base change commute with twisting by quasicharacters of $F^\times$, we can exclude the case $\alpha \in F$.

Next, put $\pi_K = \boldsymbol{l}_{K/F}(\pi_F)$; this contains a maximal simple type with underlying stratum $[\mathfrak{A}_K, n_K, 0, \beta]$. Also, $[\mathfrak{A}_K, n_K, n_K{-}1, \alpha]$ is simple and equivalent to $[\mathfrak{A}_K, n_K, n_K{-}1, \beta]$. Again we have $\alpha \notin K$ and $[K[\alpha]{:}K] = [F[\alpha]{:}F] = p^t > 1$.

Lemma 1.8 gives us an unramified character χ of $K^\times$, of order dividing p^m, such that $\pi_K \cdot \chi \cong \boldsymbol{b}_{K/F}(\pi_F)$. By Theorem 1.6 and its analogue for base change, we have

$$\boldsymbol{\varepsilon}(\boldsymbol{b}_{K/F}(\pi_F), s, \psi_K) = \boldsymbol{\varepsilon}(\pi_K, s, \psi_K).$$

Thus $\boldsymbol{\varepsilon}(\pi_K, s, \psi_K) = \boldsymbol{\varepsilon}(\pi_K \cdot \chi, s, \psi_K)$ while, by Theorem 1.4,

$$\boldsymbol{\varepsilon}(\pi_K \cdot \chi, s, \psi_K) = \boldsymbol{\varepsilon}(\pi_K, s, \psi_K)\, \chi(\det \beta)^{-1}.$$

Thus

$$\chi(\det \beta) = \chi(\mathrm{N}_{K[\alpha]/K}(\alpha))^{p^{m-t}} = 1.$$

However, since α is minimal over K, its valuation in $K[\alpha]$ is prime to p, so $\nu_K(\mathrm{N}_{K[\alpha]/K}(\alpha))$ is likewise prime to p. It follows that χ has order dividing p^{m-t}. In particular, if β is minimal over F, i.e., $p^m = p^t$, the character χ must be trivial.

This completes the proof of Proposition 1.8.

10.2. For the moment, let the characteristic of F and the finite tame extension K/F be arbitrary. The sort of argument used in 10.1 yields some strong uniqueness results when the underlying element β is minimal. For example:

Proposition. — *Let $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$. Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum underlying some maximal simple type occurring in π_F, and assume that β is minimal over F. Let $\theta_F \in \mathcal{C}(\mathfrak{A}, \beta)$ occur in π_F, and let $\Theta(\pi_F)$ denote the endo-class of θ_F.*

Let $\pi_K \in \mathcal{A}_m^{\mathrm{wr}}(K)$, and define $\Theta(\pi_K)$ similarly. Suppose:

(a) *$\Theta(\pi_K)$ is the K/F-lift of $\Theta(\pi_F)$;*
(b) *the central quasicharacter of π_K is $\omega_F \circ \mathrm{N}_{K/F}$, where ω_F is the central quasicharacter of π_F;*
(c) $\boldsymbol{\varepsilon}(\pi_K, s, \psi_K) = \boldsymbol{\varepsilon}(\boldsymbol{l}_{K/F}(\pi_F), s, \psi_K)$.

Then $\pi_K \cong \boldsymbol{l}_{K/F}(\pi_F)$.

10.3. We reinstate our hypothesis that F has characteristic zero, and prove Theorem 1.8. (We remark that properties of local constants play no rôle in this proof.)

We are given $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$ and a cyclic tame extension K/F of degree d, with $p \nmid d$. We have to show that $\boldsymbol{b}_{K/F}(\pi_F) \cong \boldsymbol{l}_{K/F}(\pi_F)$.

Both base change and tame lift are transitive in K/F, so we can assume that d is *prime.* We write $\pi_K = \boldsymbol{l}_{K/F}(\pi_F)$, and retain the notation of §5 used in the original definition of π_K. In particular, π_F contains the central type Λ_F and π_K contains the central type $\Lambda_K = \boldsymbol{g}_{K/F}(\Lambda)$ constructed as there. Note that, under our present hypotheses, the character $\boldsymbol{\delta}_{K/F}^{p^m-1}$ is trivial and so, in the notation of 5.3, we have $\Lambda = \Lambda^0$.

We choose prime elements ϖ_F of F, ϖ_E of E, so that $\varpi_E^{p^m} \equiv \varpi_F \bmod U_E^1$. After twisting by a tamely ramified character, we can assume that the central quasicharacter ω_F of π_F is trivial on ϖ_F and the group $\mu_p'(F)$ of roots of unity in F of order prime to p. This means that the central type Λ_F is effectively a representation of the finite p-group $\boldsymbol{J}_F/\langle \operatorname{Ker}\theta_F, \varpi_F, \mu_p'(F)\rangle$. Similar comments apply to the representations Λ and $\Lambda_K \mid E^\times J_K^1$.

We now write $\Gamma = \mathrm{Gal}(K/F)$, and fix a generator σ of Γ. As in §5, we view $\boldsymbol{J}_F$ as embedded in $E^\times J^1$ "on the diagonal". This image lies in $E^\times J_K^1$, and is the set of σ-fixed points in $E^\times J_K^1$. The restriction of θ_K to $H_F^1 \subset H_K^1$ is then θ_F^d. Since d is prime to p, one shows easily (from the definitions in [**9**]) that the stratum $[\mathfrak{A}_F, n_F, 0, d\beta]$ is simple, and $\theta_F^d \in \mathcal{C}(\mathfrak{A}_F, d\beta)$.

These considerations lead us naturally to the *Adams operation* $\boldsymbol{\psi}_d$. If X is a (say) finite group and χ is a virtual character of X, then $\boldsymbol{\psi}_d(\chi)$ is the virtual character $x \mapsto \chi(x^d)$. If X happens to be of order p^n (and, as above, p does not divide d), any irreducible character χ of X takes values in the field $\mathbb{Q}(\alpha)$, where α is a primitive p^n-th root of unity in $\mathbb{C}$, and $\boldsymbol{\psi}_d\chi(x) = \chi(x)^\gamma$, where γ is the automorphism $\alpha \mapsto \alpha^d$ of $\mathbb{Q}(\alpha)$. In this case therefore, $\boldsymbol{\psi}_d$ takes irreducible characters to irreducible characters (bijectively), and commutes with induction. (For a full discussion of the operator $\boldsymbol{\psi}_d$, see, for example, **[11]**.)

Returning to the main argument, the Glauberman correspondence relative to the automorphism σ then gives us a bijection $\boldsymbol{g}_\sigma$ between the irreducible representations of $E^\times J_K^1$ containing θ_K and the irreducible representations of $\boldsymbol{J}_F = E^\times J_F^1$ containing the character $\theta_F^d : x \mapsto \theta_F(x^d)$.

Lemma 1. — *There is a unique* $\Lambda_1 \in \mathcal{CC}(\theta_F, \omega_F)$ *such that*

$$\boldsymbol{g}_\sigma(\Lambda_K \mid E^\times J_K^1) = \boldsymbol{\psi}_d(\Lambda_1),$$

that is,

$$\operatorname{tr}(\boldsymbol{g}_\sigma\, \Lambda_K(g)) = \operatorname{tr} \Lambda_1(g^d)$$

for all $g \in \boldsymbol{J}_F$.

Proof. — Since d is prime to p, the operation $\boldsymbol{\psi}_d$ gives a bijection between the sets $\mathcal{CC}(\theta_F, \omega_F)$, $\mathcal{CC}(\theta_F^d, \omega_F^d)$. The lemma then follows from the character relation A2.4. □

Now let R_K (resp. R_1) denote the (irreducible) representation of the group $E^\times \boldsymbol{U}^1(\mathfrak{A}_K)$ (resp. $E^\times \boldsymbol{U}^1(\mathfrak{A}_F)$) induced by Λ_K (resp. Λ_1). We can use σ to define a Glauberman correspondence between Γ-stable irreducible representations of $E^\times \boldsymbol{U}^1(\mathfrak{A}_K)$ with central quasicharacter ω_K (restricted to $F^\times U_K^1$) and irreducible representations of $E^\times \boldsymbol{U}^1(\mathfrak{A}_F)$ with central quasicharacter ω_F. The Adams operation and the Glauberman correspondence both commute with induction (*cf.* 4.4, A2.2). Under this bijection, R_K therefore corresponds to $\boldsymbol{\psi}_d(R_1)$.

Lemma 2. — *There exists* $u \in \boldsymbol{U}^1(\mathfrak{A}_F)$ *such that*

$$\operatorname{tr} \boldsymbol{\psi}_d(R_1)(\varpi_E u) \neq 0.$$

Proof. — Let χ range over the set of unramified characters of $F^\times$ of order dividing p^m; the representations $R_1 \otimes \chi \circ \det$ are then distinct by 3.2 Corollary 1. The same applies to their images under the Adams operation $\boldsymbol{\psi}_d$, and so the lemma follows. □

Now let $\pi_1 \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F)$ denote the representation of G_F induced by Λ_1.

Lemma 3. — *We have* $\pi_K \cong \boldsymbol{b}_{K/F}(\pi_1)$.

Proof. — We extend Λ_K to a representation of $\Gamma \ltimes KE^\times J_K^1$ so that $\mathrm{tr}\Lambda_K(\sigma)$ is positive (*cf.* [**5**], remark following 14.21). This induces a representation π_K of $\Gamma \ltimes G_K$, extending the original π_K, whose character is given by

$$\mathrm{tr}\pi_K(g\sigma) = \sum_{x\in J_K^1\backslash G_K/\boldsymbol{J}_K} \sum_{y\in J_K^1 x\boldsymbol{J}_K/\boldsymbol{J}_K} \mathrm{tr}\Lambda_K(y^{-1}g\sigma y), \tag{10.3.1}$$

for appropriate elements $g \in G_K$ (see, for example, [**5**] A.14). We evaluate this expression for $g = \varpi_E u$, where $u \in \boldsymbol{U}^1(\mathfrak{A}_F)$ satisfies Lemma 2.

For $y \in G_K$, the term $\mathrm{tr}\Lambda_K(y^{-1}g\sigma y)$ is, by definition, zero unless $y^{-1}g\sigma y$ lies in $\boldsymbol{J}_K\sigma$. Thus, any y which intervenes in the sum 10.3.1 must conjugate $g^d = (g\sigma)^d$ into $\boldsymbol{J}_K$. However, g^d generates a maximal subfield of A_K and is minimal over K. The condition $y^{-1}g^d y \in \mathfrak{K}(\mathfrak{A}_K)$ then forces $y \in \mathfrak{K}(\mathfrak{A}_K)$. We are allowed to adjust y by an element of $\boldsymbol{J}_K$, so we can assume $y \in \boldsymbol{U}(\mathfrak{A}_K)$. We have $y^{-1}g^d y \in g^d\boldsymbol{U}(\mathfrak{A}_K) \cap \boldsymbol{J}_K$, whence $y^{-1}g^d y = g^d\mu u$, for some $u \in J_K^1$ and a root of unity $\mu \in K$ of order prime to p. Comparing determinants, we get $\mu = 1$. In particular, $y^{-1}g^d y \in g^d\boldsymbol{U}^1(\mathfrak{A}_K)$ and $y \in \boldsymbol{U}(\mathfrak{A}_K)$. Since g^d is minimal over K, this implies that y lies in the $\boldsymbol{U}(\mathfrak{A}_K)$-centralizer of g^d times $\boldsymbol{U}^1(\mathfrak{A}_K)$. The centralizer of g^d in $\boldsymbol{U}(\mathfrak{A}_K)/\boldsymbol{U}^1(\mathfrak{A}_K)$ is exactly $\mathfrak{o}_E^\times/U_E^1$, and the cosets here are represented by the group of roots of unity in K of order prime to p. So, after adjusting y by an element of J_K, we can assume $y \in \boldsymbol{U}^1(\mathfrak{A}_K)$. Thus the expression 10.3.1 reduces to

$$\mathrm{tr}\pi_K(g\sigma) = \mathrm{tr}R_K(g\sigma),$$

for our element g.

A similar argument yields

$$\mathrm{tr}\pi_1(g^d) = \mathrm{tr}R_1(g^d).$$

Invoking Lemma 2, we now have

$$\mathrm{tr}\pi_K(g\sigma) = \mathrm{tr}\pi_1(g^d) \neq 0. \tag{10.3.2}$$

As in the proof of Lemma 1.8, the representations $\boldsymbol{b}_{K/F}(\pi_1)$, π_K can only differ by an unramified character of order dividing p^m. The relation 10.3.2 then implies $\pi_K \cong \boldsymbol{b}_{K/F}(\pi_1)$, as required for Lemma 3. □

To complete the proof of Theorem 1.8, it remains therefore to check that the representation Λ_1 of Lemma 1 is equivalent to Λ_F.

The central type Λ_K is obtained from the central type $\Lambda = \boldsymbol{\lambda}_{K/F}(\Lambda_F) \in \mathcal{CC}(\theta)$ by a Glauberman correspondence relative to an element $\zeta \in K$, just as in §5. (Recall that, in all cases to hand, we have $\Lambda = \Lambda^0$.) However, the action of the automorphism group generated by ζ and σ gives us a Glauberman correspondence $\boldsymbol{g} = \boldsymbol{g}_{\sigma,\zeta}$ connecting $\mathcal{CC}(\theta)$ and $\mathcal{CC}(\theta_F^d)$. By the transitivity of the Glauberman correspondence, we need to check that

$$\boldsymbol{g}_{\sigma,\zeta}(\Lambda) \cong \boldsymbol{\psi}_d(\Lambda_F). \tag{10.3.3}$$

We observe that there is a character χ of $E^\times J_F^1/F^\times J_F^1$ such that $\boldsymbol{g}_{\sigma,\zeta}(\Lambda) = \boldsymbol{\psi}_d(\Lambda_F \otimes \chi)$; thus 10.3.3 holds at least up to a p-power root of unity.

Let p^a denote the order of the group

$$\mathcal{G} = E^\times J^1/\langle \varpi_F, \mu_p'(F), \operatorname{Ker}\theta\rangle.$$

The representations Λ, Λ_F and $\Lambda_K \mid E^\times J_K^1$ can then all be realized over the field

$$C = \mathbb{Q}(e^{2\pi i/p^a}) \subset \mathbb{C}. \tag{10.3.4}$$

Suppose first that K/F is ramified. The group of automorphisms of $\mathcal{G}$ generated by σ and ζ then has order d^2. By **[14]** Corollary 6 (applied twice), there is a representation Λ_2 of $\boldsymbol{J}_F$ and a sign ϵ such that

$$\operatorname{tr}\Lambda \mid \boldsymbol{J}_F = \epsilon \operatorname{tr}\boldsymbol{g}\Lambda + d \operatorname{tr}\Lambda_2.$$

We evaluate $\operatorname{tr}\Lambda(g)$, for an element $g = \varpi_E^j h_j$, where $h_j \in J_F^1$ is chosen so that $\operatorname{tr}\Lambda_F(g) \neq 0$. We apply the Mackey formula for the character of an induced representation to the construction of $\Lambda^0 = \Lambda$ from Λ_F. This gives

$$\operatorname{tr}\Lambda(g) = \sum_{x \in \tilde{J}^1\backslash J^1} \operatorname{tr}\tilde{\Lambda}(xgx^{-1}),$$

in the notation of 3.3. We can take the coset representatives x from $J^1 \cap N_\ell$ (modulo $H^1 \cap N_\ell$). The element xgx^{-1} then lies in the inducing subgroup $E^\times \widetilde{J}^1$ if and only if x represents a fixed point of g, i.e., of ϖ_E^j, on $J^1 \cap N_\ell/H^1 \cap N_\ell$. For such x, we have $[x,g] \in H^1 \cap N_\ell$, and so

$$\operatorname{tr}\Lambda(xgx^{-1}) = \theta[x,g]\operatorname{tr}\tilde{\Lambda}(g).$$

However, θ is null on $H^1 \cap N_\ell$ (as in 3.3), so we have

$$\operatorname{tr}\Lambda(g) = q_F^{\tilde{n}(g)} \operatorname{tr}\Lambda_F(g)^d,$$

where $\tilde{n}(g)$ is the k_F-dimension of the space of fixed points of ϖ_E^j on the k_F-vector space $J^1/\widetilde{J}^1$. The field F contains a primitive d-th root of unity, so we have $q_F \equiv 1 \pmod{d}$ and hence

$$\mathrm{tr}\Lambda(g) \equiv \mathrm{tr}\Lambda_F(g)^d \pmod{d}.$$

The quantity $\mathrm{tr}\Lambda_F(g)$ is a sum of p^a-th roots of unity; thus, reducing modulo a prime divisor $\mathfrak{d}$ of d in the field C, we have

$$\mathrm{tr}\Lambda_F(g)^d \equiv \mathrm{tr}\Lambda_F(g^d) \pmod{\mathfrak{d}}.$$

Thus

$$\mathrm{tr}\Lambda(g) \equiv \mathrm{tr}\boldsymbol{\psi}_d(\Lambda_F)(g) \pmod{\mathfrak{d}};$$

reduction mod $\mathfrak{d}$ is injective on the irreducible characters of the finite p-group $\mathcal{G}$, so this is enough to give $\boldsymbol{\psi}_d(\Lambda_F) \cong \boldsymbol{g}\Lambda$, as required.

In case K/F is unramified, we choose the root of unity $\zeta \in K$ to have *prime* order d' mod $\mathrm{k}_F^\times$. In particular, $q_F^d \equiv 1 \pmod{d'}$. We can construct the representation $\Lambda = \Lambda^0$ from Λ_F in the same way as in the ramified case, using an Iwahori decomposition for J^1. This comes about as follows. First, the identification $F^{p^m} \otimes_F K = K^{p^m}$ induces the algebra decomposition

$$\mathrm{End}_F(K^{p^m}) = A_F \otimes_F \mathrm{End}_F(K),$$

where $A_F = \mathrm{End}_F(F^{p^m})$; this in turn gives

$$\mathfrak{A} = \mathfrak{A}_F \otimes_{\mathfrak{o}_F} \mathrm{End}_{\mathfrak{o}_F}(\mathfrak{o}_K).$$

Now write $J^1 = 1 + \mathfrak{J}^1$, $J_F^1 = 1 + \mathfrak{J}_F^1$; this last decomposition gives us

$$\mathfrak{J}^1 = \mathfrak{J}_F^1 \otimes_{\mathfrak{o}_F} \mathrm{End}_{\mathfrak{o}_F}(\mathfrak{o}_K).$$

We choose an ordered $\mathfrak{o}_F$-basis of $\mathfrak{o}_K$ to identify $\mathrm{End}_{\mathfrak{o}_F}(\mathfrak{o}_K)$ with $\mathfrak{M} = \mathbb{M}_{p^m}(\mathfrak{o}_F)$. We write $\mathfrak{M} = \mathfrak{M}_- \oplus \mathfrak{M}_0 \oplus \mathfrak{M}_+$, where the factors are respectively strictly lower triangular, diagonal, strictly upper triangular matrices. We thus obtain a Levi subgroup $L = (A_F \otimes \mathfrak{M}_0) \cap G$ of G and unipotent radicals $N_\ell = 1 + A_F \otimes \mathfrak{M}_-$, $N_\mathrm{u} = 1 + A_F \otimes \mathfrak{M}_+$. This gives an Iwahori decomposition

$$J^1 = J^1 \cap N_\ell \cdot J^1 \cap L \cdot J^1 \cap N_\mathrm{u}.$$

Also, $J^1 \cap L$ is identified with $J_F^1 \times \cdots \times J_F^1$ in the obvious way. We have a similar decomposition for H^1. Thus we can form the group $\widetilde{J}^1 = H^1 \cap N_\ell \cdot J^1 \cap L \cdot J^1 \cap N_\mathrm{u}$ and construct Λ from Λ_F as before. (This method is equivalent to the one used in §4.) We get

$$\mathrm{tr}\Lambda(g) = q_F^{\tilde{n}(g)}\, \mathrm{tr}\Lambda_F(g)^d,$$

where $\tilde{n}(g)$ is the dimension of the space of fixed points of g (or ϖ_E^j) on $J^1/\widetilde{J}^1$. Writing $H_F^1 = 1 + \mathfrak{H}_F^1$, we have

$$J^1/\widetilde{J}^1 \cong \mathfrak{J}_F^1/\mathfrak{H}_F^1 \otimes \mathfrak{M}_- \cong (\mathfrak{J}_F^1/\mathfrak{H}_F^1)^{d(d-1)/2},$$

and g commutes with the factor $\mathfrak{M}_-$. Thus $\tilde{n}(g) = n_F(g)d(d-1)/2$, where $n_F(g)$ is the dimension of the space of fixed points of ϖ_E^j on J_F^1/H_F^1.

We first apply [14] Corollary 6 to the Glauberman correspondence $\boldsymbol{g}_\zeta$. Since $\Lambda_K = \boldsymbol{g}_\zeta(\Lambda)$, this gives us a representation Λ_3 of $E^\times J_K^1$ and a sign ϵ' such that

$$\operatorname{tr} \Lambda \mid E^\times J_K^1 = \epsilon' \operatorname{tr} \Lambda_K + d' \operatorname{tr} \Lambda_3.$$

We note that $\Lambda_3 \mid H_K^1$ is a multiple of θ_K; hence it is a sum of irreducible representations of the form $\Lambda_K \otimes \chi$, where χ is an abelian character of $E^\times J_K^1/J_K^1$.

We abbreviate $Q(g) = q_F^{n_F(g)}$. For $g \in \boldsymbol{J}_F$, we therefore have

$$Q(g)^{d(d-1)/2} \operatorname{tr} \Lambda_F(g)^d = \epsilon' \operatorname{tr} \Lambda_K(g) + d' \operatorname{tr} \Lambda_3(g). \tag{10.3.5}$$

We can rewrite this as

$$Q(g)^{d(d-1)/2} \operatorname{tr} \Lambda_F(g)^d = c(g) \operatorname{tr} \Lambda_K(g);$$

the quantity $c(g)$ is a $\mathbb{Z}$-linear combination of roots of unity in the cyclotomic field C of 10.3.4. By A1.4 below, we have

$$|\operatorname{tr} \Lambda_F(g)| = Q(g)^{1/2}, \quad |\operatorname{tr} \Lambda_K(g)| = Q(g)^{d/2},$$

where $|\ |$ is the ordinary complex absolute value, whence

$$|c(g)| = Q(g)^{d(d-1)/2}.$$

Let $\gamma \in \operatorname{Gal}(C/\mathbb{Q})$; we have (by, for example, the defining character relation for the Glauberman correspondence) the relation $\Lambda_K^\gamma = \boldsymbol{g}_\zeta(\Lambda^\gamma)$. Consequently,

$$\left| c(g)^\gamma \Big/ Q(g)^{d(d-1)/2} \right| = 1, \quad \gamma \in \operatorname{Gal}(C/\mathbb{Q}).$$

Write $\mathfrak{p}_C$ for the unique place of C above p. The quantity $c(g)/Q(g)^{d(d-1)/2}$ is integral at every finite place of C except possibly $\mathfrak{p}_C$. We have just seen that it has absolute value 1 at every infinite place; by the product formula for valuations, it is an algebraic integer, and indeed a root of unity, in C. We have proved:

Lemma 4. — *For each $g \in \boldsymbol{J}_F$ as above, there is a root of unity $c_0(g)$, of order dividing $2p^a$, such that*

$$\operatorname{tr} \Lambda_F(g)^d = c_0(g) \operatorname{tr} \Lambda_K(g).$$

Now we compare with the relation 10.3.5. Let $\mathfrak{d}'$ be some prime divisor of d' in C. Since d' divides $q_F^d - 1$, we get

$$\operatorname{tr} \Lambda_F(g)^d \equiv \pm \operatorname{tr} \Lambda_K(g) \pmod{\mathfrak{d}'}.$$

The quantities $\operatorname{tr} \Lambda_F(g)$, $\operatorname{tr} \Lambda_K(g)$ are algebraic integers; by A1.4, their absolute values at infinite places of C are all powers of $\sqrt{p}$, whence they are divisible only by $\mathfrak{p}_C$. In particular, neither is divisible by $\mathfrak{d}'$, so

$$c_0(g) = \pm 1.$$

We now apply a similar argument to the Glauberman correspondence $\boldsymbol{g}_\sigma$. By [14] Corollary 6, there is a sign ϵ and a representation Λ_4 of $\boldsymbol{J}_F$ such that

$$\operatorname{tr} \Lambda_K \mid \boldsymbol{J}_F = \epsilon \operatorname{tr} \boldsymbol{g}_\sigma(\Lambda_K) + d \operatorname{tr} \Lambda_4.$$

Reducing modulo a prime divisor $\mathfrak{d}$ of d in C, we now have

$$\operatorname{tr} \Lambda_F(g)^d \equiv c_1(g) \operatorname{tr} \boldsymbol{g}_\sigma \Lambda_K(g) \pmod{\mathfrak{d}},$$

for a function c_1 taking values ± 1. We conclude that $\boldsymbol{g}_\sigma(\Lambda_K) = c_1 \boldsymbol{\psi}_d(\Lambda_F)$. However, as we observed at the start, $\boldsymbol{g}_\sigma(\Lambda_K)$, $\boldsymbol{\psi}_d(\Lambda_F)$ can only differ by a function taking values in p-power roots of unity. Thus, when $p \neq 2$, we have $c_1 = 1$ and the result.

Suppose now that $p = 2$. The integers d, d' are therefore both odd and $\geqslant 3$. Comparing dimensions above, we find $\epsilon' = \epsilon = +1$. Combining 10.3.5 with Lemma 5, we now have

$$c_0(g) \equiv Q(g)^{d(d-1)/2} \equiv \pm 1 \pmod{d'}.$$

We have $q_F^d \equiv 1 \pmod{d'}$, and $(d-1)/2$ is an integer. Thus $c_0(g) = +1$, and the result follows.

This completes the proof of Theorem 1.8.

We record as a corollary the conclusion of the last step of the proof above. Those arguments, we observe, are independent of the characteristic of F.

Corollary. — *Let K/F be cyclic of prime degree $\ell \neq p$, and write $\Gamma = \operatorname{Gal}(K/F)$. Let $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$ contain a central type $\Lambda_F \in \mathcal{CC}(\theta_F)$. Let $\Lambda_K \in \mathcal{CC}(\theta_K)$ be the K/F-lift of Λ_F. The Glauberman correspondence (relative to some generator of Γ) induces a map $\boldsymbol{g} : \mathcal{CC}(\theta_K)^\Gamma \to \mathcal{CC}(\theta_F^\ell)$. We have*

$$\boldsymbol{g}(\Lambda_K) = \boldsymbol{\psi}_\ell(\Lambda_F).$$

10.4. We give a result complementary to Corollary 10.3, in which we can allow the extension K/F to be non-Galois. We assume that K/F is totally ramified of prime degree $\ell \neq p$, and that ℓ is odd. We are given $\pi_F \in \mathcal{A}_m^{\mathrm{wr}}(F)$ containing a central type $(\boldsymbol{J}_F, \Lambda_F) \in \mathcal{CC}(\theta_F)$. Let $(\boldsymbol{J}_K, \Lambda_K)$ be the K/F-lift of this central type, as constructed in sections 4 and 5. (We use the notation of those sections, especially 5.1–5.3.) As above, we can view $\boldsymbol{J}_F$ as embedded in $\boldsymbol{J}$ "on the diagonal", and hence in $\boldsymbol{J}_K$.

Proposition. — *In the situation above, the representation $\Lambda_K \mid \boldsymbol{J}_F$ is irreducible, and lies in $\mathcal{CC}(\theta_F^\ell)$. Indeed,*

$$\Lambda_K \mid \boldsymbol{J}_F = \boldsymbol{\psi}_\ell(\Lambda_F) \otimes v^{p^m-1},$$

where $v : \boldsymbol{J}_F \to \pm 1$ is given by

$$v(xj) = \left(\frac{q_F}{\ell}\right)^{\nu_E(x)}, \quad x \in E^\times,\ j \in J_F^1.$$

Proof. — We can first tensor π_F with a tamely ramified quasicharacter to ensure that the central quasicharacter ω_F of π_F is trivial on a given prime element ϖ_F and on roots of unity in F of order prime to p, just as in 10.3.

We use the notation of 5.3 for the construction of Λ_K from Λ_F. In the case to hand, the character $\boldsymbol{\delta}_{K/F}$ is given by

$$\boldsymbol{\delta}_{K/F}(x) = \left(\frac{q_F}{\ell}\right)^{\nu_E(x)}, \quad x \in K^\times,$$

as in 9.1. Thus

$$\begin{aligned} \Lambda_K &= \boldsymbol{g}_{K/F}(\Lambda), \\ \Lambda &= \Lambda^0 \otimes \tilde{v}^{p^m-1}, \quad \text{where} \\ \tilde{v}(xj) &= \left(\tfrac{q_F}{\ell}\right)^{\nu_E(x)}, \end{aligned}$$

for $x \in E^\times$ and $j \in J^1$.

One can compute the dimensions of the representations Λ_F, Λ, Λ_K directly, using relations like

$$\Lambda_F \mid J_F^1 = \eta_F, \quad \dim \eta_F = (J_F^1 : H_F^1)^{1/2}.$$

Since ℓ is odd and K/F, E/F are totally ramified, we find

$$\begin{aligned} \dim \Lambda_K &= \dim \Lambda_F, \\ \dim \Lambda &= (\dim \Lambda_F)^\ell. \end{aligned} \tag{10.4.1}$$

Since $\Lambda_K \mid \boldsymbol{J}_F$ contains the simple character θ_F^ℓ, the first assertion follows. Thus, in particular, $\Lambda_K \mid \boldsymbol{J}_F$ is of the form $\boldsymbol{\psi}_\ell(\Lambda_F) \otimes \chi$, where $\chi : \boldsymbol{J}_F \to \mathbb{C}^\times$ is trivial on $J_F = U_E J_F^1$ and on $F^\times$.

Note that this relation also shows that $\widetilde{J}^1 = J^1$, and there is no induction step in the construction of Λ^0 (see 5.2).

Now we use the relation [**14**]

$$\mathrm{tr}\Lambda \mid \boldsymbol{J}_K = \epsilon\, \mathrm{tr}\Lambda_K + \ell\, \mathrm{tr}\Lambda_2,$$

for some $\epsilon = \pm 1$ and some representation Λ_2 of $\boldsymbol{J}_K$. Evaluating at the identity, we get $\epsilon = 1$. Now we choose a prime element ϖ_E of E and $u \in J_F^1$ so that

$$\mathrm{tr}\Lambda(\varpi_E u) = (\mathrm{tr}\Lambda_F(\varpi_E u))^{\ell}\, \tilde{v}(\varpi_E)^{p^m - 1} \neq 0.$$

Evaluating at this element and arguing as in 10.3, we get the result. □

Remark. — One can deal similarly with the case $\ell = 2$ provided the relation 10.4.1 holds. This is equivalent to the jumps of β over F, as in [**9**], being all even. This will hold, for example, if F is a ramified quadratic extension of a field F_0 and $\pi_F = \boldsymbol{l}_{F/F_0}(\pi_0)$, for some $\pi_0 \in \boldsymbol{\mathcal{A}}_m^{\mathrm{wr}}(F_0)$.

APPENDIX: REPRESENTATIONS OF FINITE GROUPS

Much of the foregoing relies on adaptations of rather specialized results from the representation theory of finite p-groups. For convenience, we gather them here.

Throughout, we use the following notational conventions. For a subgroup H of a group G, we set

$$\begin{aligned}
\mathrm{Cl}_H(g) &= \{h^{-1}gh : h \in H\}, \quad g \in G;\\
\mathrm{N}_G(H) &= \{g \in G : g^{-1}Hg = H\};\\
\mathcal{Z}_S(H) &= \{h \in H : s^{-1}hs = h,\ \forall s \in S\}, \quad S \subset \mathrm{N}_G(H),
\end{aligned}$$

and we use the commutator convention

$$[x, y] = xyx^{-1}y^{-1}, \quad x, y \in G.$$

A1. Characters of certain group extensions

In this section, we generalize the results of [5] §13. We are given the following data. First, p is an arbitrary prime number (in particular, we allow the possibility $p = 2$), and G is an extra special finite p-group of class two. We set $Z = \mathcal{Z}_G(G)$ (which is the centre of G); this group is cyclic, and we fix a faithful character $\chi : Z \to \mathbb{C}^\times$. We write $V = G/Z$, and $\bar{g}$ for the image in V of $g \in G$. Thus V is an elementary abelian p-group; the commutator $[g, h] \in Z$, $g, h \in G$, actually only depends on $\bar{g}$, $\bar{h}$ and has order dividing p. The pairing

$$\begin{aligned}
\langle\, ,\, \rangle : V \times V &\longrightarrow \boldsymbol{\mu}_p,\\
(\bar{g}, \bar{h}) &\mapsto \chi[g, h],
\end{aligned}$$

(where $\boldsymbol{\mu}_p$ is the group of p-th roots of unity in $\mathbb{C}$) is a nondegenerate alternating $\mathbb{F}_p$-bilinear form on V.

Let ξ denote the unique irreducible representation of G containing χ.

To start with, we are given a cyclic group Γ of automorphisms of G which fix Z elementwise. *We impose no restriction on the order of* Γ. The representation ξ then extends to a representation of the semi-direct product $\Gamma \ltimes G = \Gamma G$. There are exactly $|\Gamma|$ such extensions, and no two of these are equivalent, by Clifford theory. We fix some such extension, and continue to denote it by ξ. We write

$$y \mapsto \mathrm{tr}\xi(y), \quad y \in \Gamma G,$$

for the character of ξ. We are mainly concerned in this section with identifying the *support* of the character $\mathrm{tr}\xi$; this set clearly does not depend on the choice of extension ξ.

If $\gamma \in \Gamma$, we write $\mathfrak{I}(\gamma)$ for the space $\{v\gamma(v)^{-1} : v \in V\}$; one sees that the orthogonal complement in V of $\mathfrak{I}(\gamma)$ is the group $\mathfrak{Z}_\gamma(V)$ of γ-fixed points in V.

Proposition A1.1. — *Let* $\gamma \in \Gamma$. *Let* $h_\gamma \in G$ *and suppose that* $\mathrm{tr}\xi(h_\gamma\gamma) \neq 0$. *The element* $h = h_\gamma$ *then has the following property: if* $v \in \mathfrak{Z}_\gamma(V)$, *there exists* $g \in \mathfrak{Z}_{h\gamma}(G)$ *such that* $\bar{g} = v$.

Proof. — We have to prove that $\mathfrak{Z}_{h\gamma}(G)$ maps onto $\mathfrak{Z}_\gamma(V)$ under the natural map $G \to V$. For any $g \in G$, we have $\mathrm{tr}\xi(g\gamma h_\gamma g^{-1}) = \mathrm{tr}\xi(\gamma h_\gamma) \neq 0$. Choose g so that $\bar{g} \in \mathfrak{Z}_\gamma(V)$. We expand

$$\begin{aligned} g\gamma h g^{-1} &= g\gamma g^{-1}\gamma^{-1} \cdot \gamma h \cdot h^{-1} g h g^{-1} \\ &= [g,\gamma] \cdot \gamma h \cdot [h^{-1}, g]. \end{aligned}$$

The outer commutator factors here lie in Z, so

$$\mathrm{tr}\xi(g\gamma h g^{-1}) = \chi[g,\gamma]\,\chi[h^{-1},g]\,\mathrm{tr}\xi(\gamma h).$$

Since χ is, by assumption, faithful, we have

$$1 = [g,\gamma][h^{-1},g] = g\gamma g^{-1}\gamma^{-1}h^{-1}ghg^{-1}.$$

Conjugating this expression by $g\gamma g^{-1}$, we get

$$1 = \gamma^{-1}h^{-1}ghg^{-1}g\gamma g^{-1} = [(h\gamma)^{-1}, g].$$

Thus g commutes with $h\gamma$, as required. □

We do not yet assert the existence of an element $h_\gamma \in G$ satisfying the conditions of A1.1: this will be established below.

Proposition A1.2. — *For $\gamma \in \Gamma$, let $h_\gamma \in G$ satisfy* A1.1. *For an element $g \in G$, the following conditions are equivalent:*

(i) $\mathrm{tr}\xi(\gamma g) \neq 0$;

(ii) *γg is conjugate in ΓG to an element of $\gamma h_\gamma Z$;*

(iii) *γg is G-conjugate to an element of $\gamma h_\gamma Z$.*

Proof. — The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. We therefore assume that (i) holds and deduce (iii). We fix an element $h = h_\gamma \in G$ as in A1.1. We take $g \in G$ and assume that γhg is not G-conjugate to an element of γhZ. The first point to observe here is that this implies $\bar{g} \notin \mathcal{I}(\gamma)$. For, if $\bar{g} \in \mathcal{I}(\gamma)$, there exist $k \in G$ and $z \in Z$ such that $gz = (\gamma h)^{-1}k\gamma hk^{-1}$, and this relation translates to

$$\gamma hg = k\gamma hz^{-1}k^{-1}.$$

Thus $\bar{g} \notin \mathcal{I}(\gamma)$. In particular, $\bar{g}$ is not orthogonal to $\mathcal{Z}_\gamma(V)$, and by A1.1 we can choose $k \in G$, commuting with γh, such that $\langle \bar{g}, \bar{k}\rangle \neq 1$. This gives us

$$k\gamma hgk^{-1} = \gamma hg[g^{-1}, k],$$

whence

$$\mathrm{tr}\xi(\gamma hg) = \mathrm{tr}\xi(k\gamma hgk^{-1}) = \mathrm{tr}\xi(\gamma hg)\langle \bar{g}^{-1}, \bar{k}\rangle.$$

However, $\langle \bar{g}^{-1}, \bar{k}\rangle = \langle \bar{g}, \bar{k}\rangle^{-1} \neq 1$. The only possible conclusion is that $\mathrm{tr}\xi(\gamma hg) = 0$. Thus (i) $\Rightarrow$ (iii), and we have finished the proof. □

Lemma A1.3. — *Let $\gamma \in \Gamma$, and suppose there exists $h = h_\gamma \in G$ as in* A1.1. *Then, for $z_1, z_2 \in Z$, the elements γhz_1, γhz_2 are conjugate in ΓG if and only if $z_1 = z_2$.*

Proof. — For, if the γhz_i are conjugate, we have

$$\chi(z_1)\mathrm{tr}\xi(\gamma h) = \mathrm{tr}\xi(\gamma hz_1) = \mathrm{tr}\xi(\gamma hz_2) = \chi(z_2)\mathrm{tr}\xi(\gamma h).$$

This implies $\chi(z_1) = \chi(z_2)$, whence $z_1 = z_2$ since χ is faithful. □

Let $z \in \mathbb{C}$; we write $\|z\|$ for the usual absolute value of z (since we are using $|S|$ for the cardinality of a set S).

Theorem A1.4. — *For each $\gamma \in \Gamma$, there exists $h_\gamma \in G$ such that $\mathrm{tr}\xi(\gamma h_\gamma)$ is non-zero. For any such h_γ, we have*

$$\|\mathrm{tr}\xi(\gamma h_\gamma)\| = |\mathcal{Z}_\gamma(V)|^{1/2}.$$

Proof. — Fix a generator σ of Γ, and let C_{ij}, $0 \leqslant i \leqslant |\Gamma| - 1$, $1 \leqslant j \leqslant n_i$, be the conjugacy classes in ΓG whose projection into Γ is σ^i. The map $\alpha \mapsto \alpha \mathrm{tr}\xi$, $\alpha \in \widehat{\Gamma}$, induces an isometric injection of the character ring of Γ into that of ΓG. The argument given in **[5]** §13 applies in this situation to give us

$$(A1.5) \qquad \sum_{j=1}^{n_i} \|\mathrm{tr}\xi(C_{ij})\|^2 \, |C_{ij}| = |G|, \quad 0 \leqslant i \leqslant |\Gamma| - 1.$$

Thus, for each i, there exists $h_i \in G$ such that $\mathrm{tr}\xi(\sigma^i h_i) \neq 0$. This proves the first assertion.

For each i, fix a choice of h_i, and number the C_{ij} so that $C_{i1} = \mathrm{Cl}_{\Gamma G}(\sigma^i h_i)$. By A1.2, only those C_{ij} which meet $\sigma^i h_i Z$ contribute to the sum A1.5, i.e., those C_{ij} of the form zC_{i1} with $z \in Z$. Of course, $\|\mathrm{tr}\xi(zC_{i1})\| = \|\mathrm{tr}\xi(C_{i1})\|$, and the number of distinct conjugacy classes of this form is $|Z|$, by A1.3. Abbreviating $\mathcal{Z} = \mathcal{Z}_{\sigma^i h_i}$, we get

$$|Z| \cdot \|\mathrm{tr}\xi(C_{i1})\|^2 \cdot \frac{|G|}{|\mathcal{Z}|} = |G|,$$

for each i. However, by A1.1, we have $\mathcal{Z}/Z = \mathcal{Z}_{\sigma^i}(V)$, and the result follows. □

Remarks A1.6

(i) It is clear that Theorem A1.4 holds without change when Γ is replaced by any group of automorphisms of G and ξ by an irreducible representation of $\Gamma \ltimes G$ extending the given representation ξ of G.

(ii) Suppose that Γ is a p-group, and let p^a be the exponent of ΓG. Number-theoretic techniques, of the sort used in 10.3 above, allow one to deduce from A1.4 that

$$\mathrm{tr}\xi(\gamma h_\gamma) = \zeta(\gamma h_\gamma)\, |\mathcal{Z}_\gamma(V)|^{1/2},$$

where ζ is a root of unity of order dividing $2p^a$ (or even p^a in the case $p = 2$).

We now derive a more general version of A1.4. We need the following hypotheses:

Notation A1.7

(i) $\mathbb{G}$ is a finite group, G is a normal subgroup of $\mathbb{G}$, Γ is an abelian subgroup of $\mathbb{G}$, such that $\Gamma \cap G \subset \mathcal{Z}_G(G)$ and $\mathbb{G} = \Gamma G$;

(ii) there is a character χ of $Z = \mathcal{Z}_G(G)$, stable under conjugation by Γ, such that $G/\mathrm{Ker}\,\chi$ is an extra-special p-group of class 2, for some prime $p \geqslant 2$;

(iii) let ξ be an irreducible representation of $\mathbb{G}$ such that $\xi \mid G$ is irreducible and $\xi \mid Z$ is a multiple of χ.

In this situation, we have:

Corollary A1.8. — *Use the notation* A1.7. *Then:*

(i) *For $\gamma \in \Gamma$, there exists $h_\gamma \in G$ such that* $\mathrm{tr}\xi(\gamma h_\gamma) \neq 0$. *Moreover,*

$$\|\mathrm{tr}\xi(\gamma h_\gamma)\| = |\mathfrak{Z}_\gamma(G/Z)|^{1/2}.$$

(ii) *Let h_γ be as in* (i), *and let $g \in G$. The following are equivalent:*

- (a) $\mathrm{tr}\xi(\gamma g) \neq 0$;
- (b) $\mathrm{Cl}_{\mathbb{G}}(\gamma g) \cap \gamma h_\gamma Z \neq \varnothing$;
- (c) $\mathrm{Cl}_G(\gamma g) \cap \gamma h_\gamma Z \neq \varnothing$.

Proof. — We first remark that the restriction $\xi \mid G$ is the *unique* irreducible representation of G which contains χ. Next we note that the presence of a non-trivial $\mathrm{Ker}\,\chi$ is irrelevant, so we may as well assume that χ is faithful.

We can form the groups

$$\widetilde{\mathbb{G}} = \Gamma \ltimes G,$$
$$\overline{\mathbb{G}} = \frac{\Gamma}{\Gamma \cap Z} \ltimes G.$$

Note that we can apply A1.4 directly to the group $\overline{\mathbb{G}}$. We have canonical surjections

$$(A1.9) \qquad \begin{array}{ccc} \widetilde{\mathbb{G}} & \longrightarrow & \mathbb{G} \\ \big\downarrow & & \\ \overline{\mathbb{G}} & & \end{array}$$

We can inflate ξ to a representation $\tilde{\xi}$ of $\widetilde{\mathbb{G}}$; if we choose an abelian character θ of Γ which agrees with χ on $\Gamma \cap Z$, the representation $\theta^{-1}\tilde{\xi}$ (where we think of θ as a character of $\widetilde{\mathbb{G}}$ via the obvious map $\widetilde{\mathbb{G}} \to \Gamma$) is then the inflation of some representation $\bar{\xi}$ of $\overline{\mathbb{G}}$. We have the character relation

$$\mathrm{tr}\xi(\gamma g) = \theta(\gamma)\mathrm{tr}\bar{\xi}(\gamma g), \quad \gamma \in \Gamma,\ g \in G.$$

The result now follows. □

We continue with the situation of A1.7, *but we now assume that* $\mathbb{G}$ *is a finite p-group.* We write $\boldsymbol{\mu} = \boldsymbol{\mu}_{p^\infty}(\mathbb{C})$ for the group of roots of unity in $\mathbb{C}$ of order a power of p. We define a function $\mathrm{T}_\xi : \Gamma \to \mathbb{C}^\times/\boldsymbol{\mu}$ by

$$\mathrm{T}_\xi(\gamma) = \mathrm{tr}\xi(\gamma g) \pmod{\boldsymbol{\mu}},$$

where $g \in G$ is chosen so that $\mathrm{tr}\xi(\gamma g) \neq 0$. Observe that this is independent of the choice of g within the stated conditions, and also of the choice of ξ extending the irreducible representation of G which contains χ.

The next result shows that the function T_ξ in fact only depends on the alternating space $V = G/Z$ and the image of γ in $\mathrm{Aut}\, V$. We assume given another set of data $(\mathbb{G}', G', Z', \Gamma', \xi')$ satisfying the same hypotheses A1.7, with $\mathbb{G}'$ a finite p-group. Write $V' = G'/Z'$.

Theorem A1.10. — *Let $\gamma \in \Gamma$, $\gamma' \in \Gamma'$, and suppose there is an isomorphism of alternating spaces $\phi : V \to V'$ such that $\phi(\gamma v \gamma^{-1}) = \gamma' \phi(v) {\gamma'}^{-1}$, for all $v \in V$. Then*

$$\mathrm{T}_\xi(\gamma) = \mathrm{T}_{\xi'}(\gamma').$$

Proof. — Let us work first with the data $\mathbb{D} = (\mathbb{G}, G, Z, \Gamma, \chi, \xi)$. We set

$$\begin{aligned} G_1 &= G/\mathrm{Ker}\, \chi, \\ Z_1 &= Z/\mathrm{Ker}\, \chi, \\ \Gamma_1 &= \Gamma/\Gamma \cap Z, \\ \mathbb{G}_1 &= \Gamma_1 \ltimes G_1. \end{aligned}$$

Let ξ_1 denote some irreducible representation of $\mathbb{G}_1$ whose restriction to Z_1 contains χ. The last identity in the proof of A1.8 implies $\mathrm{T}_{\xi_1} = \mathrm{T}_\xi$. We therefore simplify the situation by assuming $G = G_1$, $\mathbb{G} = \mathbb{G}_1$ etc.

We impose the analogous simplification on $\mathbb{D}' = (\mathbb{G}', G', Z', \Gamma', \xi')$.

We return to the (simplified) data $(\mathbb{G}, G, Z, \Gamma, \chi, \xi)$. Now let Z_1 be some finite cyclic p-group and χ_1 a faithful character of Z_1; assume there is an embedding $Z \to Z_1$ such that χ_1 extends χ. We form the group

$$\mathbb{G}(Z_1) = \frac{\mathbb{G} \times Z_1}{\{(z, z^{-1}) : z \in Z\}}.$$

We define $G(Z_1)$ analogously; the representation ξ of $\mathbb{G}$ extends uniquely to an irreducible representation ξ_1 of $\mathbb{G}(Z_1)$ containing χ_1. The system $\mathbb{D}(Z_1) = (\mathbb{G}(Z_1), G(Z_1), Z_1, \Gamma, \chi_1, \xi_1)$ satisfies A1.7 and we have $\mathrm{T}_{\xi_1} = \mathrm{T}_\xi$.

Returning to the original data $\mathbb{D}$, $\mathbb{D}'$ as simplified above, we note that we can choose ξ to be trivial on the kernel of $\Gamma \to \mathrm{Aut}\, V$; we assume this done and likewise for $\mathbb{D}'$. That is, we can assume that Γ, Γ' act *faithfully* on V, V'.

Now we use the given isometry ϕ to identify V with V' and γ with γ' (hence also Γ with Γ'). This process also identifies the groups of values of χ, χ' on commutators. We can therefore find a finite cyclic p-group Z_1 and a faithful character χ_1 of Z_1, together with injections $Z \to Z_1$, $Z' \to Z_1$ so that χ_1 extends both χ and χ'. We replace $\mathbb{D}$ by $\mathbb{D}(Z_1)$ and $\mathbb{D}'$ by $\mathbb{D}'(Z_1)$.

This reduces us to the case in which the groups G, G', are given by exact sequences

$$1 \to Z \to G \to V \to 1,$$
$$1 \to Z \to G' \to V \to 1,$$

with $\chi = \chi'$. From these exact sequences, we obtain 2-cocycles $\alpha, \alpha' \in \mathrm{H}^2(V, Z)$. These two cocycles induce the same alternating form on V:

$$\alpha(v_1, v_2)\alpha(v_2, v_1)^{-1} = \alpha'(v_1, v_2)\alpha'(v_2, v_1)^{-1},$$

for all $v_i \in V$. The difference $\beta = \alpha/\alpha'$ is thus symmetric, in the sense that $\beta(v_1, v_2) = \beta(v_2, v_1)$, $v_i \in V$. The cocycle β thus corresponds to an *abelian* extension H of V by Z:

$$1 \to Z \to H \to V \to 1.$$

It is elementary to find an embedding $Z \to Z_1$ of Z in a finite cyclic p-group Z_1 so that the sequence

$$1 \to \widetilde{Z} \to H(Z_1) \to V \to 1$$

splits. Here, $H(Z_1)$ is defined as above. The splitting of this sequence implies that β becomes trivial in $\mathrm{H}^2(V, Z_1)$. Passing to $\mathbb{D}(Z_1)$, $\mathbb{D}'(Z_1)$, we are reduced to the case where the extensions G, G' of V by Z are isomorphic. This isomorphism extends to an isomorphism $\Gamma \ltimes G = \mathbb{G} \cong \mathbb{G}' = \Gamma \ltimes G'$ which carries ξ to an irreducible representation ξ'' of $\mathbb{G}'$ extending the unique representation of G' containing χ. We thus have $\mathrm{T}_\xi = \mathrm{T}_{\xi''} = \mathrm{T}_{\xi'}$, as required. □

A2. Glauberman correspondence

For the convenience of the reader, we summarize some facts from **[14]**, of fundamental importance to the work above. We give them in their original context, but we use them in the slightly more general situation of A2.5 below.

To start with, therefore, we assume given the following:

Notation A2.1. — G is a finite group and Γ is a soluble subgroup of $\operatorname{Aut} G$ such that $|G|$, $|\Gamma|$ are relatively prime.

We can thus form the semi-direct product $\Gamma \ltimes G$ relative to the given action of Γ on G. We tend to abbreviate $\Gamma \ltimes G = \Gamma G$. We write $\mathrm{Irr}(G)$ for the set of equivalence classes of irreducible representations of G, and use a similar notation for other groups. The group Γ acts on $\mathrm{Irr}(G)$ in the obvious way; we denote the set of fixed points here by $\mathrm{Irr}(G)^\Gamma$.

In this situation, we have a fundamental result of Glauberman [**14**]. The following formulation of it is taken from [**32**] 0.15:

A2.2. — *There is a uniquely determined bijection*

$$\boldsymbol{g} = \boldsymbol{g}_{\Gamma,G} : \mathrm{Irr}(G)^\Gamma \xrightarrow{\approx} \mathrm{Irr}(\mathcal{Z}_\Gamma(G))$$

with the following properties:

(i) *If Γ is an ℓ-group, for some prime number ℓ and $\rho \in \mathrm{Irr}(G)^\Gamma$, then $\boldsymbol{g}(\rho)$ satisfies*

$$\langle \mathrm{tr}\, \boldsymbol{g}(\rho), \mathrm{tr}\, \rho \rangle_{\mathcal{Z}_\Gamma(G)} \not\equiv 0 \pmod{\ell},$$

where $\langle\ ,\ \rangle_{\mathcal{Z}_\Gamma(G)}$ denotes the inner product of characters of the group $\mathcal{Z}_\Gamma(G)$.

(ii) *If Δ is a normal subgroup of Γ, then*

$$\boldsymbol{g}_{\Gamma,G} = \boldsymbol{g}_{\Gamma/\Delta,\mathcal{Z}_\Delta(G)} \circ \boldsymbol{g}_{\Delta,G}.$$

In fact, in case (i), one has

$$\langle \mathrm{tr}\, \boldsymbol{g}(\rho), \mathrm{tr}\, \rho \rangle_{\mathcal{Z}_\Gamma(G)} \equiv \pm 1 \pmod{\ell},$$

as follows from [**14**] Corollary 6.

It will be useful for us to have a more explicit result in certain special cases. We first recall [**14**] Theorem 1:

A2.3. — *Let G be a finite group and Γ a subgroup of* $\mathrm{Aut}\, G$ *of order relatively prime to $|G|$. Let $\rho \in \mathrm{Irr}(G)^\Gamma$. There exists an irreducible representation $\tilde\rho = e_\Gamma(\rho)$ of ΓG such that*

(i) *$\tilde\rho \mid G \cong \rho$, and*

(ii) *$\det \tilde\rho(\gamma) = 1$, for all $\gamma \in \Gamma$.*

These conditions determine the representation $\tilde\rho$ uniquely, up to equivalence.

We often refer to $\tilde\rho$ as the "canonical" extension of ρ to ΓG. We next recall [**14**] Th. 3:

A2.4. — *Suppose that the group Γ above is cyclic. Let $\rho \in \mathrm{Irr}(G)^\Gamma$ and define $\tilde\rho$ as in* A2.3. *The representation $\zeta = \boldsymbol{g}_{\Gamma,G}(\rho)$ satisfies the following; there is a sign $\epsilon = \epsilon(\rho, \Gamma)$ such that*

$$\mathrm{tr}\zeta(x) = \epsilon\, \mathrm{tr}\tilde\rho(\gamma x),$$

for all generators γ of Γ and all $x \in \mathcal{Z}_\Gamma(G)$.

We emphasize that the sign ϵ is independent of the elements γ, x in this relation. Also, it is possible to have a subgroup Δ of Γ such that $\mathcal{Z}_\Delta(G) = \mathcal{Z}_\Gamma(G)$; the correspondences $\boldsymbol{g}_\Gamma$, $\boldsymbol{g}_\Delta$ are then the same, but the signs $\epsilon(\rho, \Gamma)$, $\epsilon(\rho, \Delta)$ need not be equal.

Remark A2.5. — In the foregoing, it is just as easy to work with a triple (G, Γ, φ), where $\varphi : \Gamma \to \operatorname{Aut} G$ is a homomorphism with soluble image of order prime to $|G|$. Everything goes through without change. (The canonical extension of ρ to $\Gamma \ltimes G$ in (A2.3) is now the inflation to ΓG of the canonical extension of ρ to $\varphi(\Gamma) \ltimes G$.)

Under certain circumstances, the Glauberman correspondence interacts well with induction and restriction. We use the notation A2.1, and let H be a Γ-stable subgroup of G. The following is taken from **[23]** Theorem A:

A2.6. — *Let* $\rho \in \operatorname{Irr}(G)^\Gamma$, $\sigma \in \operatorname{Irr}(H)^\Gamma$, *and abbreviate* $\mathcal{Z}(H) = \mathcal{Z}_\Gamma(H)$, $\mathcal{Z}(G) = \mathcal{Z}_\Gamma(G)$.

(i) *If* $\rho \cong \operatorname{Ind}_H^G(\sigma)$, *then*

$$\boldsymbol{g}_{\Gamma,G}(\rho) \cong \operatorname{Ind}_{\mathcal{Z}(H)}^{\mathcal{Z}(G)}(\boldsymbol{g}_{\Gamma,H}(\sigma)).$$

(ii) *If* $\rho \mid H \cong \sigma$, *then* $\boldsymbol{g}_{\Gamma,G}(\rho) \mid \mathcal{Z}(H) \cong \boldsymbol{g}_{\Gamma,H}(\sigma)$.

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