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Lattices in semi-simple Lie groups, and multipliers of group C^*- algebras

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1 Introduction, and some history.

Let G be a locally compact group, and H be a closed subgroup. Viewing $L^1(G)$ as a two-sided ideal in the measure algebra $M(G)$, and viewing elements of $L^1(H)$ as measures on G supported inside H, we obtain an action of $L^1(H)$ on $L^1(G)$ as double centralizers. It is easy to check (see e.g. proposition 4.1 in [Rie]) that this action extends to an action of the full group C^*-algebra $C^*(H)$ as double centralizers on C^*(G); this corresponds to a $*$-homomorphism $j_H : C^*(H) \to M(C^*(G))$, where $M(C^*(G))$ denotes the multiplier C^*-algebra of $C^*(G)$. We now quote from p. 209 of Rieffel's Advances paper [Rie]:

It does not seem to be known whether this homomorphism j_H is injective. It will be injective if and only if every unitary representation of H is weakly contained in the restriction to H of some unitary representation of G [Fe2]. J.M.G. Fell has pointed out to us that the example that he gave in which this appeared to fail (p. 445 of [Fe2]) depended on the completeness of the classification of the irreducible representations of $SL_3(\mathbb{C})$ given in [GeN], and there is now some doubt that this classification is complete [Ste].

Probably this quotation requires some word of explanation. In [Fe2], Fell studies extensions to the topological framework of Frobenius reciprocity for finite groups. Thus he introduces a list of weak Frobenius properties, the last and weakest one being (WF3):

The locally compact group G satisfies property (WF3) if, for any closed subgroup H of G, every representation σ in the dual $\hat{H}$ is weakly contained in the restriction $\pi|_H$ of some unitary representation π of G.

Property (WF3) is indeed equivalent to the injectivity of j_H for any closed subgroup H; for completeness, we shall give a proof in Proposition 2.1 below. In §6 of [Fe2], Fell wishes to show that even (WF3) may fail, by taking $G = SL_3(\mathbb{C})$ and $H = SL_2(\mathbb{C})$; to this end he appeals to the incomplete description of $\hat{G}$ given in [GeN]; Fell's proof was recently corrected in Remark 1.13(i) of [BLS].

In this paper, we take for G a semi-simple Lie group with finite centre and without compact factor, and as closed subgroup a lattice Γ. In section 3, we prove:

THEOREM 1.1 *Let G be a semi-simple Lie group without compact factors, with finite centre and with Kazhdan's property (T). Let Γ be an irreducible lattice in G, and let σ be a non-trivial irreducible unitary representation of Γ of finite dimension n. Then σ*

determines a direct summand of $C^*(\Gamma)$ which is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$; this direct summand is isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices.

If G is a non-compact *simple* Lie group with finite centre, then G has property (T) unless G is locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$ (see [HaV]). For these two families, we prove in section 4:

THEOREM 1.2 *Let G be locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$, for some $n \geq 2$. Let Γ be a lattice in G. Denote by $\hat{\Gamma}_f$ the set of (classes of) irreducible, finite-dimensional unitary representations of Γ. If the trivial representation 1_Γ is not isolated in $\hat{\Gamma}_f$ (for the induced Fell-Jacobson topology), then infinitely many elements of $\hat{\Gamma}_f$ are not weakly contained in the restriction to Γ of any unitary representation of G. In particular $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.*

In view of Theorems 1.1 and 1.2, it seems natural to formulate the following

Conjecture. If Γ is a lattice in a non-compact semi-simple Lie group G, then $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$ is not injective.

This conjecture means that, if ρ is a representation of G which is faithful on $M(C^*(G))$ (e.g. take for ρ either the universal representation of G, or the direct sum of all its irreducible representations), then $\rho|_\Gamma$ is never faithful on $C^*(\Gamma)$; this has bearing on a question of de la Harpe in his paper in these Proceedings (see immediately after Problem 13 in [Har]). In §5, we give examples of lattices Γ in $SO_o(n,1)$ or $SU(n,1)$ such that 1_Γ is not isolated in $\hat{\Gamma}_f$; this is the case for any lattice in $SL_2(\mathbb{R})$, any non-uniform lattice in $SL_2(\mathbb{C})$, and any arithmetic lattice in $SO_o(n,1)$ for $n \neq 3, 7$.

In the final §6, we come back to property (WF3) and show that it always fails for almost connected, non-amenable groups:

THEOREM 1.3 *Let G be an almost connected, locally compact group. The following properties are equivalent:*

(i) G has Fell's property (WF3);
(ii) G is amenable.

Observe that Theorem 1.3 cannot hold for any locally compact group. Indeed, any discrete group G satisfies property (WF3) since, given a subgroup H of G, one checks easily that $C^*(H)$ is a C^*-subalgebra of $C^*(G) = M(C^*(G))$.

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A word about terminology: as usual, semi-simple Lie groups are assumed to be connected and non-trivial; group representations are assumed to be unitary, strongly continuous, and on non-zero Hilbert spaces.

2 On multipliers of C^*-algebras.

For a C^*-algebra B, we denote by $M(B)$ its multiplier algebra.

PROPOSITION 2.1 *Let A, B be C^*-algebras, and let $j : A \to M(B)$ be a $*$-homomorphism. The following properties are equivalent:*

(i) j is one-to-one;
(ii) for any $\sigma \in \hat{A}$, there exists a non-degenerate $*$-representation π of B such that σ is weakly contained in $\tilde{\pi} \circ j$, where $\tilde{\pi}$ denotes the extension of π to $M(B)$;
(iii) any $\sigma \in \hat{A}$ is weakly contained in $\{\tilde{\pi} \circ j \mid \pi \in \hat{B}\}$.

Fell's property (WF3), mentioned in §1, is deduced from property (ii) above by taking $B = C^*(G)$ and $A = C^*(H)$, for any closed subgroup H of the locally compact group G.

Proof of Proposition 2.1. (i) $\Rightarrow$ (ii) Let us assume that j is injective, so that we may identify A with a C^*-subalgebra of $M(B)$. Let π be a faithful representation of B. It is known that the extension $\tilde{\pi}$ of π to $M(B)$ is also faithful ([Ped], 3.12.5). Thus any representation of A is weakly contained in the restriction of $\tilde{\pi}$ to A.

(ii) $\Rightarrow$ (iii) This follows from decomposition theory.

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Fix a non-zero element x of A; choose $\sigma \in \hat{A}$ such that $\sigma(x) \neq 0$. Our assumption says that $Ker\,\sigma$ contains $\bigcap_{\pi\in\hat{B}} Ker\tilde{\pi} \circ j = Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$; in particular $x \notin Ker(\bigoplus_{\pi\in\hat{B}} \tilde{\pi} \circ j)$. It follows that $j(x) \neq 0$, i.e. that j is one-to-one.

3 Proof of theorem 1.1

We slightly generalize Theorem 1.1 in the following form:

THEOREM 3.1 *Let G be a non-compact semi-simple Lie group with finite centre and with Kazhdan's property (T). Let Γ be an irreducible lattice in G, and let σ be an irreducible representation of Γ of finite dimension n, which is not contained in the restriction to Γ of a unitary, finite-dimensional representation of G. Then σ determines a direct summand of $C^*(\Gamma)$ isomorphic to the algebra $M_n(\mathbb{C})$ of n-by-n matrices, which moreover is contained in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$.*

Observe that Theorem 1.1 is an immediate consequence of Theorem 3.1: indeed, if G has no compact factor, then any unitary, finite-dimensional representation of G is trivial.

Proof of Theorem 3.1. Since G has property (T), so has Γ (see [HaV], Théorème 4 in Chapter 3). Let σ be an irreducible representation of Γ, of finite dimension n. By Theorem 2.1 in [Wan], σ is isolated in the dual $\hat{\Gamma}$, hence determines a direct sum decomposition of $C^*(\Gamma)$:

$$C^*(\Gamma) = J \oplus M_n(\mathbb{C})$$

where J is the C^*-kernel of σ.

We assume from now on that σ is not contained in the restriction to Γ of a unitary, finite-dimensional representation of G, and wish to prove that the direct summand $M_n(\mathbb{C})$ lies in the kernel of $j_\Gamma : C^*(\Gamma) \to M(C^*(G))$. Suppose by contradiction that j_Γ is non-zero on $M_n(\mathbb{C})$. Choose $\pi \in \hat{G}$ such that $\tilde{\pi} \circ j_\Gamma$ is non-zero, hence faithful on $M_n(\mathbb{C})$ (here $\tilde{\pi}$ denotes the extension of π to $M(C^*(G))$, as in Proposition 2.1). Then the C^*-kernel of $\tilde{\pi} \circ j_\Gamma$ is contained in J, which means that σ is weakly contained in the restriction $\pi|_\Gamma$. As σ is isolated in $\hat{\Gamma}$, this implies that σ is actually a subrepresentation of $\pi|_\Gamma$ (see Corollary 1.9 in [Wan]). Our assumption shows that π is infinite-dimensional. Two cases may occur:

(a) π is a discrete series representation of G (if any); this would imply that σ is an irreducible subrepresentation of the left regular representation of Γ, which in turn implies that Γ is finite - and this is absurd.

(b) π is not in the discrete series of G; then, by a result of Cowling and Steger (Proposition 2.4 in [CoS]), the restriction $\pi|_\Gamma$ is irreducible, which contradicts the fact that σ is a finite-dimensional subrepresentation.

With a contradiction reached in both cases, the proof of Theorem 3.1 is complete. We thank G. Skandalis for a helpful conversation that led to a more explicit version of Theorem 3.1.

Remark. Let us show that there are countably many finite-dimensional elements $\sigma \in \hat{\Gamma}$ satisfying the assumptions of Theorem 3.1.

Thus, let $G/Z(G)$ be the adjoint group of G; this is a linear group. Denote by Γ_1 the image of Γ in $G/Z(G)$; as a finitely generated linear group, Γ_1 is residually finite (see [Mal]); a non-trivial irreducible representation σ of Γ that factors through a finite quotient of Γ_1 cannot be contained in the restriction to Γ of a finite-dimensional unitary representation of G.

This argument shows that $Ker[j_\Gamma : C^*(\Gamma) \to M(C^*(G))]$ contains the C^*-direct sum of countably many matrix algebras.

4 The cases $SO_o(n,1)$ and $SU(n,1)$.

We begin with the following result, which is certainly known to many experts (see [Moo], Proposition 3.6; compare also with [Mar], Chap. III, (1.12), Remark 1).

PROPOSITION 4.1 *Let G be a simple Lie group with finite centre, and let Γ be a lattice in G. Denote by γ the quasi-regular representation of G on $L^2(G/\Gamma)$, and by γ_0 the restriction of γ to $L^2_o(G/\Gamma) = \{f \in L^2(G/\Gamma) | < f|1 >= 0\}$.*

(a) There exists $N \in \mathbb{N}$ such that the N-fold tensor product $\gamma_0^{\otimes N}$ is weakly contained in the left regular representation λ_G of G.

(b) The trivial representation 1_G is not weakly contained in γ_0.

Proof. (a) Suppose first that G has Kazhdan's property (T). Then, by Theorems 2.4.2 and 2.5.3 in [Cow], there exists $N \in \mathbb{N}$ such that $\pi^{\otimes N}$ is weakly contained in

λ_G for any unitary representation π of G which does not contain 1_G. This implies the result.

Suppose now that G is locally isomorphic either to $SO_o(n,1)$ or to $SU(n,1)$. Let K be a maximal compact subgroup of G. Let $\hat{G}_1 = \{\pi \in \hat{G} \mid \pi|_K \text{ contains } 1_K\}$ be the set of all spherical representations of G. Observe that $\hat{G}_1$ is open in $\hat{G}$ (because $\pi \in \hat{G}_1$ if and only if there exists $\xi \in H_\pi$ such that $\int_K < \pi(k)\xi|\xi > dk \neq 0$). For a unitary representation σ of G, set $Supp\,\sigma = \{\pi \in \hat{G} \mid \pi \text{ is weakly contained in } \sigma\}$. By Proposition 3.6 in [Moo], the existence of $N \in \mathbb{N}$ such that $\sigma^{\otimes N}$ is weakly contained in λ_G is equivalent to $1_G \notin Supp\,\sigma \cap \hat{G}_1$ (the proof of this uses the explicit description of the unitary duals of $SO_o(n,1)$ and $SU(n,1)$). So we must prove that 1_G is not in $Supp\,\gamma_0 \cap \hat{G}_1$ or, equivalently, that 1_G is isolated in $Supp\gamma \cap \hat{G}_1$.

Recall the standard parametrization of $\hat{G}_1$. Let ρ be half the sum of the positive roots associated with a maximal split torus of G. Then $\hat{G}_1$ identifies (topologically) with $i\mathbb{R}^+ \cup [0,\rho]$, the representations π_s with $s \in i\mathbb{R}^+$ being the spherical principal series representations, those π_s with $s \in]0,\rho[$ being the spherical complementary series representations, and π_ρ being the trivial representation 1_G.

Let X be the Riemannian symmetric space associated with G. The Laplace-Beltrami operator Δ on X is invariant for the left action of G, so it descends to a positive, unbounded operator on $L^2(\Gamma\backslash X)$. It is well-known that π_s is weakly contained in γ if and only if $\rho^2 - s^2$ belongs to the spectrum of Δ on $L^2(\Gamma\backslash X)$ (see §4 of Chap. I in [GGP] for $G = SL_2(\mathbb{R})$ and Γ uniform, or Theorem 1.7.10 in [GaV] for the general case; note that this Theorem is stated there for the quasi-regular representation of G on $L^2(X)$, but the proof extends word for word to our representation γ).

Denoting by $\lambda_1(\Gamma\backslash X)$ the bottom of the spectrum of the restriction of Δ to the orthogonal of constants in $L^2(\Gamma\backslash X)$, we see that our result follows from $\lambda_1(\Gamma\backslash X) > 0$. In turn, this is a consequence of the facts that the continuous spectrum of Δ on $L^2(\Gamma\backslash X)$ is the half-line $[\rho^2, \infty[$ (see [OsW]), and that its discrete spectrum is a sequence increasing to ∞ (see Theorem 3 in [BoG]). In our case, $\lambda_1(G\backslash X) > 0$ can also be deduced from the fact that $\lambda_1(M) > 0$ for any complete Riemannian manifold M with finite volume and pinched negative sectional curvature (see [Dod]).

(b) This follows from (a) and non-amenability of G.

Proof of Theorem 1.2

We shall use several times Fell's inner hull-kernel topology, which is defined on sets of unitary (not necessarily irreducible) representations of a locally compact group (cf. [Fel], section 2): a net $(\pi_i)_{i\in I}$ of representations converges to a representation π if and only if π is weakly contained in $\{\pi_j \mid j \in J\}$ for each subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$.

Assume that G and Γ satisfy the assumptions of Theorem 1.2. We are going to show that Fell's property (WF3) fails for the pair (G,Γ); i.e., we shall produce some $\sigma \in \hat{\Gamma}$ such that σ is not weakly contained in the set $\{\pi|_\Gamma \mid \pi \in \hat{G}\}$.

Since 1_Γ is not isolated in $\hat{\Gamma}_f$, there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ in $\hat{\Gamma}_f - \{1_\Gamma\}$ that converges to 1_Γ .

1st step: There exists a sequence of integers $n_1 < n_2 < ...$, and spherical complementary series representations π_{n_k} of G such that π_{n_k} is weakly contained in $Ind_\Gamma^G \sigma_{n_k}$ for any k, and $\lim_{k\to\infty} \pi_{n_k} = 1_G$.

Indeed, by continuity of induction ([Fel], Theorem 4.1),

$$\lim_{n\to\infty} Ind_\Gamma^G \sigma_n = Ind_\Gamma^G 1_\Gamma = \gamma.$$

Since 1_G is a subrepresentation of γ, we also have

$$\lim_{n\to\infty} Ind_\Gamma^G \sigma_n = 1_G.$$

This implies that there exists integers $n_1 < n_2 < \dots$, and irreducible representations π_{n_k} of G such that π_{n_k} is weakly contained in $Ind_\Gamma^G \sigma_{n_k}$ for any k, and such that $\lim_{k\to\infty} \pi_{n_k} = 1_G$ (cf. proof of Lemme 2, §1, in [Bur]). Since the spherical dual $\hat{G}_1$ is open in $\hat{G}$, and since G is not amenable, we can clearly assume that π_{n_k} is either 1_G, or a spherical complementary series representation. To exclude the case $\pi_{n_k} = 1_G$, we are going to show that 1_G is not weakly contained in $Ind_\Gamma^G \sigma_{n_k}$; this can be viewed as a form of Frobenius reciprocity.

Indeed, since σ_{n_k} is finite-dimensional, σ_{n_k} does not contain 1_G weakly. Moreover, we know by Proposition 4.1(b) that 1_G is isolated in $Supp\,\gamma$. Hence, by a result of Margulis ([Mar], Chap. III, (1.11)(b)), 1_G is not weakly contained in $Ind_\Gamma^G \sigma_{n_k}$. This proves the 1st step.

Let $\pi_{n_k} \in \hat{G}$ be a sequence as above. By Proposition 4.1(a), there exists $N \in \mathbb{N}$ such that $\gamma_0^{\otimes N}$ is weakly contained in λ_G. Since $\lim_{k\to\infty} \pi_{n_k} = 1_G$, we see that $\pi_{n_l}^{\otimes N}$ is not weakly contained in λ_G for $l \in \mathbb{N}$ big enough. Fix such an l, and set $\sigma = \sigma_{n_l}$ and $\pi = \pi_{n_l}$.

2nd step: σ is not weakly contained in $\{\rho|_\Gamma : \rho \in \hat{G}\}$. Indeed, assume by contradiction that there exists a sequence $\rho_n \in \hat{G}$ with $\lim_{n\to\infty} \rho_n|_\Gamma = \sigma$.

Then $\lim_{n\to\infty} Ind_\Gamma^G \rho_n|_\Gamma = Ind_\Gamma^G \sigma$. Hence, since π is weakly contained in $Ind_\Gamma^G \sigma$:

$$\lim_{n\to\infty} Ind_\Gamma^G (\rho_n|_\Gamma) = \pi\,.$$

But

$$Ind_\Gamma^G (\rho_n|_\Gamma) = \rho_n \otimes Ind_\Gamma^G 1_\Gamma = \rho_n \oplus (\rho_n \otimes \gamma_0).$$

Since π is irreducible, this implies (upon passing to a subsequence) that either $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$ or $\lim_{n\to\infty} \rho_n = \pi$.

We first exclude the case $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$. Indeed, $(\rho_n \otimes \gamma_0)^{\otimes N} = \rho_n^{\otimes N} \otimes \gamma_0^{\otimes N}$ is weakly contained in λ_G. Hence, $\lim_{n\to\infty} \rho_n \otimes \gamma_0 = \pi$ would imply that $\pi^{\otimes N} = \lim_{n\to\infty} (\rho_n \otimes \gamma_0)^{\otimes N}$ is weakly contained in λ_G; this would contradict our choice of π.

It remains to exclude the case $\lim_{n\to\infty} \rho_n = \pi$. Since the set $\hat{G}_1^{\,c} = \{\pi_s \mid s \in]0, r[\}$ of all spherical complementary series representations is open in $\hat{G}$ and since $\pi \in \hat{G}_1^{\,c}$, we can clearly assume that $\rho_n \in \hat{G}_1^{\,c}$ for all n. Then, there exists $s_o \in]0, r[$ such that, for all n:

$$\rho_n \in \{\pi_s : 0 < s < s_o\}.$$

Therefore, there exists $M \in \mathbb{N}$ such that $\rho_n^{\otimes M}$ is weakly contained in λ_G, for all $n \in \mathbb{N}$. Hence

$$\sigma^{\otimes M} = \lim_{n\to\infty} (\rho_n^{\otimes M})|_\Gamma$$

is weakly contained in λ_Γ. Since $\sigma^{\otimes M}$ is finite-dimensional, this contradicts non-amenability of G. This concludes the proof of Theorem 1.2.

Remark: In our previous paper [BeV], Theorem 1.2 was already proved for $G = PSL_2(\mathbb{R})$ and Γ the fundamental group of a closed Riemann surface of genus 2.

5 Some examples of lattices in $SO_o(n,1)$ and $SU(n,1)$.

Let Γ be a lattice in a simple Lie group locally isomorphic either to $SO_o(n,1)$ or $SU(n,1)$. Let us denote by $\hat{\Gamma}_{fq}$ the set of elements of $\hat{\Gamma}$ that factor through some finite quotient of Γ.

DEFINITION 5.1 *We say that Γ satisfies property (*) if the trivial representation 1_Γ is not isolated in $\hat{\Gamma}_{fq}$, for the induced Fell-Jacobson topology.*

Our property (*) is precisely the negation of property $(T;R(\Im))$ in the notation of Lubotzky-Zimmer [LuZ], where a lucid discussion of this property appears on pp. 291-292. Since $\hat{\Gamma}_{fq}$ is a subset of $\hat{\Gamma}_f$, it is clear that, if Γ satisfies (*), then 1_Γ is not isolated in $\hat{\Gamma}_f$; note the question at the bottom of p. 291 of [LuZ] whether or not the converse implication holds.

The purpose of this section is to give examples of lattices with property (*), i.e. for which Theorem 1.2 is true. We begin with a sufficient condition for property (*).

PROPOSITION 5.2 *If Γ has a finite index subgroup Γ_o that maps homomorphically onto $\mathbb{Z}$, then Γ has property (*).*

Proof. Let $(\chi_m)_{m\in\mathbb{N}}$ be a sequence of non-trivial characters of finite order of $\mathbb{Z}$, viewed as characters of Γ_o, that converges to the trivial character. Set:

$$\pi_m = Ind_{\Gamma_o}^{\Gamma}\chi_m .$$

Claim: π_m factors through some finite quotient of Γ. Indeed, since χ_m has finite order, the subgroup $Ker\,\chi_m$ of Γ_o has finite index in Γ, so there exists a normal subgroup N_m of Γ, of finite index and contained in $Ker\,\chi_m$. Then π_m factors through the finite group Γ/N_m , which establishes the claim.

The rest of the proof is similar in spirit to the first step of the proof of Theorem 1.2, but considerably easier: by continuity of induction, the sequence $(\pi_m)_{m\in\mathbb{N}}$ converges to the quasi-regular representation λ_o of Γ on $l^2(\Gamma/\Gamma_o)$. Since λ_o contains the trivial representation 1_Γ, we may select for any $m \in \mathbb{N}$ an irreducible component σ_m of π_m in such a way that the sequence $(\sigma_m)_{m\in\mathbb{N}}$ converges to 1_Γ in $\hat{\Gamma}$. By the claim, each σ_m lies in $\hat{\Gamma}_{fq}$; finally, no σ_m may be trivial, by Frobenius reciprocity. This shows that 1_Γ is not isolated in $\hat{\Gamma}_{fq}$.

Because Γ is finitely generated, the condition that Γ_o maps homomorphically onto $\mathbb{Z}$ is equivalent to the non-vanishing of the first cohomology $H^1(\Gamma_o, \mathbb{C})$. This is known to have deep representation-theoretic consequences, as it gives information on the decomposition of $L^2(G/\Gamma_o)$ into irreducibles (see the whole of Chapter VII in [BoW], and especially Propositions 4.9 and 4.11). There is a conjecture, sometimes attributed to Thurston (see e.g. [Bor], 2.8), according to which any uniform lattice Γ in $SO_o(n,1)$ $(n \geq 2)$ admits a finite index subgroup Γ_o such that $H^1(\Gamma_o, \mathbb{C}) \neq 0$. Next proposition summarizes what we know about this problem, both in the uniform and non-uniform cases.

PROPOSITION 5.3 *The following lattices Γ in $SO_o(n,1)$ admit a finite index subgroup Γ_o such that $H^1(\Gamma_o, \mathbb{C}) \neq 0$, and hence satisfy property (*):*

(i) any lattice in $PSL_2(\mathbb{R}) \simeq SO_o(2,1)$;

(ii) any non-uniform lattice in $PSL_2(\mathbb{C}) \simeq SO_o(3,1)$;

(iii) any uniform lattice Γ in $PSL_2(\mathbb{C})$ such that, for some x in the 3-dimensional real hyperbolic space $H_3(\mathbb{R})$, the orbit $\Gamma.x$ is invariant under some orientation-reversing involutive isometry of $H_3(\mathbb{R})$;

(iv) any arithmetic lattice, provided $n \neq 3, 7$; any non-uniform arithmetic lattice, without restriction on n.

Proof. The proof is compilation; however, it makes constant use of Selberg's lemma asserting that any lattice has a torsion-free subgroup of finite index.

(i) A torsion-free lattice in $PSL_2(\mathbb{R})$ is either a surface group (in the uniform case) or a non-abelian free group (in the non-uniform case); in any case, it surjects onto $\mathbb{Z}$.

(ii) Any torsion-free non-uniform lattice in $PSL_2(\mathbb{C})$ surjects onto $\mathbb{Z}$, by Propositions 5.1 and 3.1 in [Lub].

(iii) See Theorem 3.3 and Corollary 3.4 in [Hem]. Explicit examples of such lattices are given in §4 of [Hem].

(iv) The first statement is the main result of [LiM]. For the second one, combine the main result in [Mil] with the remarks on p. 365 of [LiM].

Concerning uniform lattices in $PSL_2(\mathbb{C})$, it seems appropriate to mention here the connection with a somewhat (in)famous question which is for sure due to Thurston (question 18 in [Thu]): does any complete, finite-volume, hyperbolic 3-manifold have a finite-sheeted cover that fibers over the circle S^1? An affirmative answer would imply that any lattice Γ in $PSL_2(\mathbb{C})$ satisfies property (*) (indeed, let Γ_1 be a torsion-free subgroup of finite index in Γ; then Γ_1 is the fundamental group, $\pi_1(M)$, of a complete finite-volume hyperbolic 3-manifold M; if N is a finite-sheeted cover of M which fibers over S^1, then $\Gamma_o = \pi_1(N)$ is a finite-index subgroup of Γ_1 that maps onto $\pi_1(S^1) = \mathbb{Z}$). For an example of a compact hyperbolic 3-manifold that does not fiber over S^1 but with a finite-sheeted cover that does, see example 2.1 in [Gab][1].

[1] Clearly, for a 3-dimensional closed hyperbolic manifold M, fibering over S^1 is a much stronger condition than having non-zero first Betti number. Algebraically, this can be seen by Stallings'fibration theorem [Sta]: if N is a normal subgroup of $\pi_1(M)$ such that $\pi_1(M)/N = \mathbb{Z}$, then N comes from a fibration of M over S^1 if and only if N is a finitely generated subgroup. Also, surface groups in $\pi_1(M)$ that come from some finite-sheeted cover of M fibering over S^1 (so-called virtual fibre groups) have been characterized algebraically in Corollary 1 of [Som].

In contrast with Proposition 5.3, we are not aware of any "large" class of lattices in $SU(n,1)$ that satisfies property (*). Essentially the only result we know is that, for any $n \geq 2$, there exists a uniform arithmetic lattice Γ in $SU(n,1)$ such that $H^1(\Gamma, \mathbb{C}) \neq 0$ (see Theorem 1 in [Kaz], or Theorem 1.4(b) in [Li]).

To conclude, let us indicate why, for a given lattice Γ in $SO_o(n,1)$, it is usually difficult to check that Γ satisfies property (*). Assume that Γ is arithmetic. It is then easy to construct elements of $\hat{\Gamma}_{fq}$: take a congruence subgroup $\Gamma(\wp)$ and consider irreducible representations of Γ that factor through the finite group $\Gamma/\Gamma(\wp)$ (for $\Gamma = SL_2(\mathbb{Z})$, these are representations that factor through some $SL_2(\mathbb{Z}/n\mathbb{Z})$). Denote by $\hat{\Gamma}_{arith}$ the subset of elements in $\hat{\Gamma}_{fq}$ that factor through some $\Gamma/\Gamma(\wp)$; then it follows from Selberg's inequality (see [Sel] for $n = 2$, and corollary 1.3 in [BuS] for $n > 2$) that the trivial representation 1_Γ is isolated in $\hat{\Gamma}_{arith}$. Thus, if Γ verifies property (*), any non-stationary net in $\hat{\Gamma}_{fq}$ that converges to 1_Γ will have to leave $\hat{\Gamma}_{arith}$ eventually.

We have been informed by M. Burger that, in unpublished work with P. Sarnak, similar phenomena have been obtained for a large class of arithmetic lattices in $SU(n,1)$.

6 Proof of Theorem 1.3

We begin with hereditary properties of the class of groups satisfying Fell's property (WF3).

LEMMA 6.1 *Let G be a locally compact group with property (WF3).*

(a) Any closed subgroup of G has property (WF3).

(b) Let K be a compact normal subgroup of G; then G/K has property (WF3).

Proof. (a) is obvious. To see (b), denote by $p : G \to G/K$ the quotient map. Let L be a closed subgroup of G/K; fix $\tau \in \hat{L}$. Set $H = p^{-1}(L)$ and $\sigma = \tau \circ (p|_H)$. Let π be a representation of G on a Hilbert space $\mathcal{H}$ such that $\pi|_H$ weakly contains σ. Let $\mathcal{H}^K$ be the space of K-fixed vectors in $\mathcal{H}$. Since K is a normal subgroup, $\mathcal{H}^K$ is an invariant subspace of π, and we denote by π_o the restriction of π to $\mathcal{H}^K$. Since K is compact and σ is irreducible, it is easy to see that σ is weakly contained in $\pi_o|_H$. But $\pi_o|_H$ can be viewed as a representation of $L = H/K$, that weakly contains τ.

Next lemma is probably well-known.

LEMMA 6.2 *Let G be a Lie group, and let S be a semisimple analytic subgroup. The closure $\bar{S}$ of S is reductive.*

Proof. We begin with a

Claim: Let h be a finite-dimensional Lie algebra, and let s be a semisimple ideal; then there exists an ideal j of h such that $h = s \oplus j$. Indeed, let $Der(s)$ be the Lie algebra of derivations of s. Since s is an ideal in h, we have a Lie algebra homomorphism:

$$\alpha : h \to Der(s) \; : \; X \to ad(X)_{|s}$$

the kernel of which is precisely the centralizer of s in h; set $j = Ker\alpha$. Since s is semisimple, $Der(s)$ is canonically isomorphic to s, so that α is onto and $h = s \oplus j$; this establishes the claim.

To prove Lemma 6.2, denote by s and h the Lie algebras of S and $\bar{S}$ respectively. Clearly $Ad(x)(s) = s$ for any x in S, so by density the same is true for any x in $\bar{S}$. This shows that s is an ideal in h. By the claim, there exists an ideal j of h such that $h = s \oplus j$. To see that h is reductive, it is enough to prove that j is central in h. But, for $X \in j$, we have $Ad(x)(X) = X$ for any x in S; again by density, this remains true for any x in $\bar{S}$; so X is central in h.

Proof of Theorem 1.3 It is easy to see that any amenable group G satisfies property (WF3); indeed, for a closed subgroup H of G, by amenability of H any representation of H is weakly contained in the left regular representation of H, which is itself contained in the restriction to H of the left regular representation of G; see also Corollary 1.5 in [BLS] for another proof.

Let us now prove the converse, namely that any almost connected locally compact group G with property (WF3) is amenable. In this proof, the stability of amenability under short exact sequences will be used constantly.

1st step: reduction to the connected case. Let G_0 be the connected component of the identity of G. By Lemma 6.1(a), G_0 has property (WF3). If G_0 is amenable, then so is G, since G/G_0 is compact.

2nd step: reduction to the Lie group case. Let G be a connected group with property (WF3). By the structure theory for connected groups, G admits a compact normal subgroup K such that G/K is a Lie group. By Lemma 6.1(b), G/K has property (WF3). If G/K is amenable, then so is G, since K is compact.

3rd step: reduction to the reductive case. Let G be a connected Lie group with property (WF3). Let $G = RS$ be a Levi decomposition, with R the solvable radical and S a semisimple analytic subgroup. Then the closure $\bar{S}$ is reductive with property (WF3), by Lemmas 6.1(a) and 6.2. If $\bar{S}$ is amenable, then so is $\bar{S}/(\bar{S} \cap R) = G/R$, hence so is G.

Coda. Let G be a connected, reductive Lie group with property (WF3). The adjoint group $G/Z(G)$ is a semisimple Lie group without centre, so it decomposes as a direct product

$$G/Z(G) = G_1 \times \cdots \times G_n$$

of simple Lie groups without centre. To prove that G is amenable, we have to show that G_j is compact for $j = 1, \cdots, n$. So suppose by contradiction that some G_j, say G_1, is not compact. By root theory, G_1 then contains a 3-dimensional analytic subgroup L which is locally isomorphic to $SL_2(\mathbb{R})$. Because G_1 is centreless, hence linear, L is closed in G_1 (any semisimple analytic subgroup in a linear group is closed, see Theorem 2 in [Got]). By the proof of Theorem 3 in [BeV], there exists a lattice Γ in L and a representation $\tau \in \hat{\Gamma}_f$ such that $\tau \otimes \bar{\tau}$ is not weakly contained in the restriction to Γ of any unitary representation of L. Denote by $p : G \to G_1$ the homomorphism obtained by composing the quotient map $G \to G/Z(G)$ with the projection of $G/Z(G)$ onto G_1. Set $H = p^{-1}(\Gamma)$ and $\sigma = \tau \circ (p|_H)$. Because H is closed in G, we find by property

(WF3) a net $(\rho_i)_{i\in I}$ in $\hat{G}$ such that

$$lim_i\, \rho_i|_H = \sigma\,.$$

Then:

$$lim_i(\rho_i \otimes \bar{\rho}_i)|_H = \sigma \otimes \bar{\sigma}\,.$$

But, since ρ_i is irreducible, the representation $\rho_i \otimes \bar{\rho}_i$ of G is trivial on $Z(G)$, so it factors through a representation π_i of $G/Z(G)$. Last formula then reads:

$$lim_i\, \pi_i|_\Gamma = \tau \otimes \bar{\tau}$$

and this contradicts our choice of G and τ.

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