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1 Introduction

A notion of a Gel'fand pair for compact quantum groups introduced by T.H. Koornwinder in [19] is a generalization of the classical one for a locally compact group G and its compact subgroup K such that for any irreducible unitary representation of G, the dimension of the space of K-bi-invariant matrix elements is not greater then 1; this is equivalent to the commutativity of the subalgebra of group algebra of G, formed by K-bi-invariant functions (see [11]). This classical notion of a Gel'fand pair can be formulated as the cocommutativity of the coproduct

$$\Delta(f)(g,h) := \int_K f(gkh)d\mu_K(k) \quad (\mu_K - \ Haar\ measure\ for\ K) \tag{1}$$

on the space of all K-bi-invariant functions on G. Considering such functions as functions on the set of double cosets $Q = K\backslash G/K$, one can rewrite (1) in the following form:

$$\Delta(f)(p,r) = \int_Q K(p,r,s)f(s)d\mu_Q(s) \qquad (p,r \in Q), \tag{2}$$

where $K(.,.,.)$ is some positive kernel, μ_Q is some positive Borel measure on Q (which can depend on p and r in general case). A function $\chi_\alpha(.)$ on Q (α is classifying parameter) is called a character of the coalgebra given by (2) if it satisfies a product formula:

$$\int_Q K(p,r,s)\chi_\alpha(s)d\mu_Q(s) = \chi_\alpha(p)\chi_\alpha(r) \qquad (p,r \in Q). \tag{3}$$

We will say that the coproduct (2) defines a hypergroup structure on the algebra of K-bi-invariant functions on G with the pointwise multiplication. One can find a discussion of hypergroups in [5],[6],[13],[22],[26] and in references given there. In many cases the χ_α are well known special functions. Very often we have a similar formula with respect to α - dual product formula. It shows that χ_α is also a character of a dual hypergroup by the variable α.

In this paper we consider double cosets of compact quantum group with respect to its subgroup and distinguish cases of a Gel'fand pair and a strict Gel'fand pair. We show that every strict Gel'fand pair of compact quantum groups generates a normal commutative hypercomplex system with compact basis [5],[6] and a commutative discrete hypergroup [13], which are in duality to one another, consider corresponding examples and describe characters of hypergroups in terms of q-orthogonal polynomials.

After this paper had gone to press, the essential development of the subject took place. On the one hand, Gel'fand pairs for non compact quantum groups were considered (see, for example, [29]). On the other hand, one can consider a notion of a quantum subgroup of a quantum group from more general point of view then in this paper, using a notion of a coideal (see, for example, [9],[12],[14],[15],[20], [21],[23],[28]). This permits to apply the Gel'fand pair approach to exceptionally interesting classes of q-special functions such as Macdonalds and Askey-Wilson polynomials and Jacksons q-Bessel functions. This development is described in the survey [28].

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2 Double cosets of quantum groups

2.1. Let $\mathbf{H}$:=$(H, d, 1, \Delta, \varepsilon, S), \widehat{\mathbf{H}} := (\widehat{H}, \widehat{d}, \widehat{1}, \widehat{\Delta}, \widehat{\varepsilon}, \widehat{S})$ be two Hopf algebras over C [1], with multiplications $d, \widehat{d}$, units $1, \widehat{1}$, comultiplications $\Delta, \widehat{\Delta}$, counits $\varepsilon, \widehat{\varepsilon}$, antipodes $S, \widehat{S}$

Definition 1 *We say that* $\mathbf{H}, \widehat{\mathbf{H}}$ *are in duality, if there exists a doubly non-degenerate pairing* $\langle \cdot, \cdot \rangle$*:* $H \times \widehat{H} \to C$ *such that:*

$$\langle 1, \zeta \rangle = \widehat{\varepsilon}(\zeta),\ \langle ab, \zeta \rangle = \langle a \otimes b, \widehat{\Delta}(\zeta) \rangle,\ \langle \Delta(a), \zeta \otimes \eta \rangle = \langle a, \zeta\eta \rangle,$$

$$\langle a, \widehat{1} \rangle = \varepsilon(a),\ \langle S(a), \zeta \rangle = \langle a, \widehat{S}(\zeta) \rangle \qquad (\forall a, b \in H,\ \zeta, \eta \in \widehat{H}).$$

We can define elements $\zeta * a := (id \otimes \zeta) \circ \Delta(a)$, $a * \zeta := (\zeta \otimes id) \circ \Delta(a)$, where the pairing is used in the first, respectively second part of the tensor product. It is possible to rewrite the last equalities as $\langle \zeta * a, \eta \rangle = \langle a, \eta\zeta \rangle$, $\langle a * \zeta, \eta \rangle = \langle a, \zeta\eta \rangle$. These operations yield left and right algebra actions of $\widehat{H}$ on H:

$$(\zeta\eta) * a = \zeta * (\eta * a),\ a * (\zeta\eta) = (a * \zeta) * \eta \qquad (\forall a \in H,\ \eta, \zeta \in \widehat{H}).$$

Now let $\mathbf{H}, \widehat{\mathbf{H}}$ be two Hopf algebras in duality, $\zeta \in \widehat{H}$. In what follows we will suppose that another pair $(\mathbf{H}_1, \widehat{\mathbf{H}}_1)$ of Hopf algebras in duality exists together with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$ and embedding $i : \widehat{\mathbf{H}}_1 \to \widehat{\mathbf{H}}$ such that $\langle \pi(a), \zeta \rangle = \langle a, i(\zeta) \rangle$ $(\forall a \in H, \zeta \in \widehat{H}_1)$. Left and right coactions $\Delta^l := (\pi \otimes id) \circ \Delta$, $\Delta^r := (id \otimes \pi) \circ \Delta$ of $\mathbf{H}_1$ on H define the subsets of left-, right- and bi-ivariant elements: $H_1 \backslash H := \{h \in H | \Delta^l(h) = 1_1 \otimes h\}$, $H/H_1 := \{h \in H | \Delta^r(h) = h \otimes 1_1\}$, $H_1 \backslash H / H_1 := H_1 \backslash H \cap H/H_1$. All these sets are evidently unital algebras. Let an invariant integral ν_1 (such that $\nu_1(1) = 1$) on Hopf algebra $\mathbf{H}_1$ [1] exist (it always exists when $\mathbf{H}_1$ is a compact quantum group in the sense of [34]). Then we can introduce two projections $\pi^l := (\nu_1 \circ \pi \otimes id) \circ \Delta$, $\pi^r := (id \otimes \nu_1 \circ \pi) \circ \Delta$ from H to $H_1 \backslash H$ and H/H_1 correspondingly. They commute and $\pi^r \circ \pi^l$ is a projection from H to $H_1 \backslash H / H_1$ (see [7],[19]). A new coproduct may be introduced on $H_1 \backslash H / H_1$:

$$\tilde{\Delta} := (id \otimes \nu_1 \circ \pi \otimes id) \circ (id \otimes \Delta) \circ \Delta \tag{4}$$

This definition is a generalization of (1) for Hopf algebra case.

Theorem 1 *Let a mapping $\tilde{\Delta}$ be defined by (4). Then:*

(a) *$\tilde{\Delta}$ maps $H_1\backslash H/H_1$ into $H_1\backslash H/H_1 \otimes H_1\backslash H/H_1$;*

(b) *$\tilde{\Delta}$ is coassociative, i.e. $(id \otimes \tilde{\Delta}) \circ \tilde{\Delta} = (\tilde{\Delta} \otimes id) \circ \tilde{\Delta}$;*

(c) *ε is a counit with respect to $\tilde{\Delta}$: $(\varepsilon_1 \otimes id) \circ \tilde{\Delta} = (id \otimes \varepsilon_1) \circ \tilde{\Delta} = id$;*

(d) *if ν is an invariant integral on $\mathbf{H}$, then ν is invariant with respect to $\tilde{\Delta}$:*

$$(\nu \otimes id) \circ \tilde{\Delta}(h) = (id \otimes \nu) \circ \tilde{\Delta}(h) = \nu(h) \cdot 1;$$

(e) *the following relation holds: $\tilde{\Delta} \circ S = \Pi \circ (S \otimes S) \circ \tilde{\Delta}$.*

PROOF. a) Evidently, $\tilde{\Delta} = (id \otimes \pi^l)\Delta = (\pi^r \otimes id)\Delta$. On the other hand, $(id \otimes \pi^r)\Delta = \Delta \circ \pi^r, (\pi^l \otimes \pi^l)\Delta = \Delta \circ \pi^l$. So for every $h \in H_1\backslash H/H_1$ we have $\tilde{\Delta}(h) \in H_1\backslash H/H_1 \otimes H$. Similarly we see that $\tilde{\Delta}(h) \in H \otimes H_1\backslash H/H_1$. b) Both sides of needed equality coincide with $(\pi^r \otimes id \otimes \pi^l)(\Delta \otimes id)\Delta$. c) $(\varepsilon \otimes id)\tilde{\Delta} = (\varepsilon \otimes \pi^l)\Delta = \pi^l$, so that ε is right counit. Similarly one can see that it is also left counit. d) Replacing ε by ν, we can prove this statement exactly as previous. e) This is implied by the following chain of equalities: $\tilde{\Delta} \circ S = (id \otimes \pi^l)\Delta \circ S = (id \otimes \pi^l)\Pi(S \otimes S)\Delta = (id \otimes \nu_1 \circ \pi \otimes id)$ $(\Pi \otimes id)(id \otimes \Pi)(S \otimes S \otimes S)(\Delta \otimes id)\Delta = \Pi(S \otimes S)(id \otimes \nu_1 \circ S_1 \circ \pi \otimes id)(\Delta \otimes id)\Delta = \Pi(S \otimes S)\tilde{\Delta}$. □

Two Hopf *-algebras $\mathbf{H}, \widehat{\mathbf{H}}$ are said to be in duality, if they are in duality as Hopf algebras and

$$\zeta^*(f) = \overline{\zeta(S(f^*))} \quad \forall \zeta \in \widehat{\mathbf{H}},\ f \in \mathbf{H},$$

where the same symbol denotes the involution in $\mathbf{H}$ and in $\widehat{\mathbf{H}}$. In what follows, we will be considering $\mathbf{H}, \mathbf{H}_1$ as Hopf *-algebras (see, for example, [25],[32],[34]) with the Hopf algebra structure and the involution *, π as an epimorphism of Hopf *-algebras, ν_1 as a state on the *-algebra H_1. Then $H_1\backslash H$, H/H_1 and $H_1\backslash H/H_1$ will be unital *-algebras, $\pi^l, \pi^r, \tilde{\Delta}$ map the cone of positive elements into the cones of positive elements of the corresponding *-algebras.

Definition 2 *A pair of Hopf algebras (resp. *-Hopf algebras) $(\mathbf{H}, \mathbf{H}_1)$ is called a Gel'fand pair if the coproduct $\tilde{\Delta}$ is cocommutative. A Gel'fand pair is called strict if the algebra $H_1\backslash H/H_1$ is commutative.*

2.2. Now let $\mathbf{H}$ be *-Hopf algebra associated with a compact quantum group and $\widehat{\mathbf{H}}$ is its algebraic dual. We know [34] that H can be represented as

$$H = \sum_{\alpha} \sum_{i,j=1}^{d_\alpha} \mathbb{C} u_{i,j}^{\alpha}, \tag{5}$$

where $u_{i,j}^{\alpha}$ are matrix elements of d_α- dimensional unitary corepresentation of $\mathbf{H}$ ($d_\alpha < \infty$ for all α running in some discrete set $\hat{Q}$) and there exists an invariant integral ν on H, which is a state and such that α-sum in (5) defines an orthogonal decomposition in the sense of the inner product given by $\langle f, g \rangle := \nu(f \cdot g^*)$ after a suitable choice of an orthonormal basis for each representation space. In this case, the comodules $H_1\backslash H$, H/H_1 and also $H_1\backslash H/H_1$ may be given by

$$H_1\backslash H = \sum_\alpha \sum_{i=1}^{d'_\alpha} \sum_{j=1}^{d_\alpha} \mathbf{C} u^\alpha_{i,j}, \quad H/H_1 = \sum_\alpha \sum_{i=1}^{d_\alpha} \sum_{j=1}^{d'_\alpha} \mathbf{C} u^\alpha_{i,j},$$

$$H_1\backslash H/H_1 = \sum_\alpha \sum_{i,j=1}^{d'_\alpha} \mathbf{C} u^\alpha_{i,j}$$

where $d'_\alpha \leq d_\alpha$ for all α. A notion of a Gel'fand pair for compact quantum groups was introduced in [19] as a pair $(\mathbf{H}, \mathbf{H}_1)$ with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$, such that for any irreducible unitary matrix corepresentation of $\mathbf{H}$, the dimension of the space of bi-invariant matrix elements is not greater then 1.

Lemma 1 *A pair of compact quantum groups $(\mathbf{H}, \mathbf{H}_1)$ with an epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$ is a Gel'fand pair in the sense of Definition 2, iff for any irreducible unitary matrix corepresentation of $\mathbf{H}$, the dimension of the space of bi-invariant matrix elements is not greater then 1.*

PROOF. Suppose that $2 \leq d_\beta$ for some fixed $\beta \in \widehat{Q}$. Set $\eta_1(u^\beta_{1,2}) := 1$, $\eta_1(u^\alpha_{i,j}) := 0$ otherwise and $\eta_2(u^\beta_{2,1}) := 1$, $\eta_2(u^\alpha_{i,j}) := 0$ otherwise. One can check that $\eta_1, \eta_2 \in (H_1\backslash H/H_1)^* \cap \widehat{H}_1$. Direct calculations show that $\langle \tilde{\Delta}(u^\beta_{1,1}), \eta_1 \otimes \eta_2 \rangle \neq \langle \tilde{\Delta}(u^\beta_{1,1}), \eta_2 \otimes \eta_1 \rangle$, i.e., $\tilde{\Delta}$ is not cocommutative. Conversely, if $d_\alpha = 1\ \forall \alpha \in \widehat{Q}$, then $\tilde{\Delta}$ is obviously cocommutative. □

3 Connections with hypercomplex systems and hypergroups

3.1. We will use notions of a spatial tensor product for C^*-algebras, a unital Hopf C^*-algebra, a morphism, and a counit for unital Hopf C^*-algebras, as well as notions of a coaction of a unital Hopf C^*-algebra on a unital C^*- algebra and finite Haar measure on a unital Hopf C^*-algebra (see[4],[10],[34]). If $\mathbf{H}$ is a unital Hopf C^*-algebra, then the coproduct defines a structure of a Banach algebra in the conjugate space H^* for the C^*-algebra H:

$$\tilde{\omega} * \omega := (\tilde{\omega} \otimes \omega) \circ \Delta, \quad \forall \omega, \tilde{\omega} \in H^*.$$

$\mathbf{H}$ has a counit if and only if H^* is a unital algebra.

As in Section 2, we denote $\forall a \in H, \omega \in H^*$:

$$\omega * a := (id \otimes \omega) \circ \Delta(a), \ a * \omega := (\omega \otimes id) \circ \Delta(a).$$

Let ν be finite Haar measure on a unital Hopf C^*-algebra $\mathbf{H}$. One can introduce by means of GNS-construction a structure of the Hilbert space $L_2(H, \nu)$ and the corresponding representation of Λ_ν of H in this space. For every compact quantum group the completion of the initial Hopf *-algebra with respect to the C^{**}-norm $|\cdot| = \sup_\rho ||\rho(\cdot)||$, where ρ runs over the set of all irreducible representations of H, give a unital Hopf C^*-algebra $\mathbf{H}$ (see [34]).

Now we consider the initial situation of section 2, in which: 1) $\mathbf{H}$ is a unital Hopf C*-algebra, having a finite Haar measure ν and a counit ε; 2) $\mathbf{H_1}$ is a unital Hopf C*-algebra, having a finite Haar measure ν_1 and a counit ε_1 (such measures and counits always exist when $\mathbf{H}, \mathbf{H_1}$ are compact quantum groups (see [34]); 3) π is an epimorphism in the category of unital Hopf C^*-algebras. Then one can consider all the above mentioned algebras as unital C^*-algebras, all the above mentioned coactions and positive mappings as coactions of unital Hopf C^*-algebras and positive mappings of C^*-algebras. Particularly, $H_1\backslash H$, H/H_1, $H_1\backslash H/H_1$ are C^*-subalgebras of H, $\pi^l, \pi^r, \tilde{\Delta}$ are positive mappings of unital C^*-algebras. All the statements of section 2 are valid for unital Hopf C^*-algebras except for those which involve an antipode.

3.2. Let Q be a locally compact Hausdorff space, and let M be a Banach space of complex valued functions on Q. We denote by $\{L^s | s \in Q\}$ a family of left generalized shift operators (below Delsarte-Levitan hypergroup or simply hypergroup) [22] acting in M. One can find a discussion of hypergroups and their special classes in [5],[6],[13],[26] and references given there. The notion of a hypergroup can be formulated in terms of the coassociativity of the coproduct $(\tilde{\Delta} f)(t,s) := L^s f(t)$. In the case when $M = C_b(Q)$ is the C^*-algebra of all bounded continuous functions on Q and $\tilde{\Delta}$ is a continuous mapping from M to $M \otimes M$, there exists a structure of a Banach algebra on the dual space $M^* = M(Q)$ of all finite regular Borel measures on Q with the convolution

$$\langle f, \delta_s * \delta_t \rangle = \langle L^s f,\ \delta_t \rangle = (L^s f)(t) \qquad \forall t, s \in Q, f \in M, \tag{6}$$

(where $\langle \cdot, \cdot \rangle$ is the pairing for M and M^*, δ_t is the delta function concentrated at the point t), and with a unity δ_e. The hypergroup is called commutative if the algebra M^* is commutative. We will call a function $\chi(\cdot) \in M$ *a character* of hypergroup $\{L^p\}$ if it is a character of the algebra M^*, i.e., if $(L^p \chi)(r) = \chi(p)\chi(r)\ \forall p, r \in Q$.

In many applications, hypergroups satisfy some special conditions:

a) the action of a hypergroup preserves positivity of functions and the function which identically equals to unity;

b) there exists an involutive homeomorphism $x \mapsto x^\vee$ of Q (the analogue of taking the inverse in a group) such that

$$(L^s f)(t) = (L^t f^\vee)(s), \quad \forall f \in M, r, s, t \in Q, f^\vee(t) := f(t^\vee)$$

and $e^\vee = e$. The hypergroup having a property b) is called involutive. In this case M^* is a Banach *-algebra with the involution extending the mapping $\delta_x \to \delta_{x^\vee}$. If $x = x^\vee \quad \forall x \in Q$, the corresponding hypergroup is called Hermitian, it is automatically commutative. The definition of a character of an involutive hypergroup contains a condition $\chi(r^\vee) = \overline{\chi(r)} \quad \forall r \in Q$.

Usually the existence of some special positive regular Borel measure ν on Q with the property $\int_Q L^p f(q) d\nu(q) = \int_Q f(q) d\nu(q)$ - the analogue of a Haar measure on a group is assumed (or is proved under some additional conditions [5],[6],[13]). Such a measure is also called a Haar measure. Then one can consider the hypergroup as a family of bounded linear operators acting in the spaces $L_p(Q, \nu), (1 \le p \le \infty)$. The important special classes of hypergroups with the described properties were studied by Yu.M. Berezanskii, S.G. Krein, A.A. Kalyuzhnyi (hypercomplex systems with a locally compact basis) and also by Ch.Dunkl, R. Jewett, R. Spector and others (DJS-hypergroups). See a discussion in [5],[6],[13],[26].

Let us suppose that $\mathbf{H}$ is a unital Hopf C^*-algebra with a finite Haar measure ν and $H_1\backslash H/H_1$ is a unital commutative C^*-algebra. Then the restristion of ν to this algebra is generated by some finite measure on its spectrum Q, which is a compact topological space. We use $\tilde{\Delta}$ and ν to denote the restrictions of the corresponding mappings to $H_1\backslash H/H_1$ (as well as the measure on Q, generating ν). Since $H_1\backslash H/H_1$ is isomorphic to the unital C^*-algebra $C(Q)$ of all continuous functions on Q and any $p \in Q$ can be identified with a continuous homomorphism $p : H_1\backslash H/H_1 \to \mathrm{C}$, there exists a family of operators $L^p : H_1\backslash H/H_1 \to H_1\backslash H/H_1$ given by

$$L^p(f) := (id \otimes p) \circ \tilde{\Delta}(f) = p *' f \qquad \forall f \in H. \tag{7}$$

One can see that these operators generate hypergroup with $e = \varepsilon, M = H_1\backslash H/H_1$. We use $H_1\backslash H/H_1$ and the measure ν to construct the Hilbert space $L_2(Q,\nu)$ and consider L^p as operators acting in this space, defined first on $L_2(Q,\nu) \cap L_\infty(Q,\nu)$.

Lemma 2 *Let the hypergroup $L^p, p \in Q$, be given by (7). Then:*

1) L^p may be extended to a bounded operator for all $p \in Q$ and the mapping $p \mapsto L^p$ is strongly continuous;

2) $L^\varepsilon = id$;

3) for any positive $f \in L_2(Q,\nu)$, $L^p(f)$ is positive for all $p \in Q$;

4) $L^p(1)(q) = 1$ for all $p, q \in Q$.

PROOF. 1) It follows from the definition of L^p and the positivity property of $\tilde{\Delta}$ that L^p are bounded with $||L^p|| \le 1$. Moreover, for any $f \in H$ the mapping $p \mapsto L^p(f) = (p \otimes id)\tilde{\Delta}(f)$ is strongly continuous. 2) This is a direct consequence of part c) of the Theorem 1. 3) Since $\tilde{\Delta}$ is positive, this follows from the fact that p is a homomorphism and from the property 1). As a cosequence of this fact we have that L^p maps real functions into real. 4) Since $\tilde{\Delta}(1) = 1 \otimes 1$ and p, q are homomorphisms, we have $L^p(1)(q) = 1$. □

Remark 3.1. Let now S, S_1 be antipodes on $\mathbf{H}, \mathbf{H}_1$ respectively such that $\pi \circ S = S_1 \circ \pi$ and the restriction of S to $H_1\backslash H/H_1$ is continuous. Then one can define an involutive homeomorphism $\vee$ of Q: $p^\vee = p \circ S \quad \forall p \in Q$ such that the hypergroup has a property

5) $\varepsilon^\vee = \varepsilon$ and $(L^q S(f))(p) = (L^p f)(q)$ for all $f \in L_2(Q,\nu)$.

If additionally the Haar measure ν satisfies the relation

$$\nu((p *' a)^* b) = \nu(a^*(p *' b^\vee)), \qquad \forall a, b \in H_1\backslash H/H_1, p \in Q, \tag{8}$$

then the hypergroup have an additional property

6)$(L^p)^* = L^{p^\vee}$, where $(L^p)^*$ is the operator adjoint to L^p in $L_2(Q,\nu)$.

These considerations and Theorem 2.1 of [6] show that the described hypergroup satisfies all the properties of a commutative normal hypercomplex system with a compact basis and a basis unity. A dual hypergroup may be constructed using the considerations of [19] (in this paper a discrete, generally noncommutative, DJS-hypergroup was constructed for every Gel'fand pair of compact quantum groups, not obligatory strict).

Thus, every strict Gel'fand pair of quantum groups generates two commutative hypergroup structures dual to one another: a discrete DJS-hypergroup and a normal hypercomplex system with a compact basis.

3.3. The described construction of double cosets for compact quantum groups may be generalized, if we replace a Hopf algebra $\widehat{\mathbf{H}}_1$ by a coideal (see[23]) in a Hopf algebra $\widehat{\mathbf{H}}$. Such a generalization permits to establish general point of view based on the notion of strict Gel'fand pair, to the interesting examples of Askey-Wilson [3],[21] and Macdonald [23] polynomials (see also [12],[9]). It will be described in a separate paper. In this more general situation we do not know if the mapping $\tilde{\Delta}$ is positive, so that we can not refer neither to hypercoplex systems in the sense of [5],[6] nor to DJS-hypergroups [13]. We can only use a duality principle for real hypercomplex systems with compact and discrete basis described in [27].

4 Examples

4.1. It is known [25] that quantum group $SL_q(2,C)$ is generated by elements $\alpha,\beta,\gamma,\delta$ such that:

$$\alpha\beta = q\beta\alpha, \alpha\gamma = q\gamma\alpha, \beta\gamma = \gamma\beta, \beta\delta = q\delta\beta,$$
$$\gamma\delta = q\delta\gamma, \alpha\delta - q\beta\gamma = \delta\alpha - q^{-1}\beta\gamma = 1,$$
$$\Delta(\alpha) := \alpha\otimes\alpha + \beta\otimes\gamma, \Delta(\beta) := \alpha\otimes\beta + \beta\otimes\delta,$$
$$\Delta(\gamma) := \gamma\otimes\alpha + \delta\otimes\gamma, \Delta(\delta) := \gamma\otimes\beta + \delta\otimes\delta,$$
$$\varepsilon(\alpha) = \varepsilon(\delta) := 1, \varepsilon(\beta) = \varepsilon(\gamma) := 0, \tag{9}$$
$$S(\alpha) := \delta,\ S_1(\beta) := -q^{-1}\beta,\ S(\gamma) := -q\gamma,\ S(\delta) := \alpha, \quad q \in \{C\backslash 0\}.$$

The dual Hopf algebra (quantized enveloping algebra) $U_q(sl(2,C))$ is generated by elements A,B,C,D and relations:

$$AB = qBA,\ AC = q^{-1}CA,\ AD = DA = \hat{1}_1,\ BC - CB = (q-q^{-1})^{-1}$$
$$(A^2 - D^2), \hat{\Delta}(A) := A\otimes A,\ \hat{\Delta}(B) := A\otimes B + B\otimes D, hat\Delta(C) := A\otimes C + C\otimes D,$$
$$\hat{\Delta}(D) := D\otimes D, \hat{\varepsilon}(A) = \hat{\varepsilon}(D) := 1,\ \hat{\varepsilon}(B) = \hat{\varepsilon}(C) := 0, \tag{10}$$
$$\hat{S}(A) := D,\ \hat{S}(B) := -q^{-1}B,\ \hat{S}(C) =:= -qC,\ \hat{S}(D) := A\ (q^2 \neq 1).$$

. In [31] it was shown that these Hopf algebras are in duality with respect to the pairing

$$\langle\alpha, A\rangle = \langle\delta, D\rangle := q^{\frac{1}{2}}, \langle\delta, A\rangle = \langle\alpha, D\rangle = q^{-\frac{1}{2}}, \langle\beta, B\rangle = \langle\gamma, C\rangle = 1, \tag{11}$$

which equals to 0 for other pairs of generators. A real form $\mathbf{H} = \mathbf{SU_q(2)}$ of $SL_q(2,C)$ distinguished by an involution $\alpha^* = \delta$, $\delta^* = \alpha$, $\beta^* = -q\gamma$, $\gamma^* = -q^{-1}\beta$ $(0 < q < 1)$ may be equipped with a structure of a compact quantum group in the sense of [34] (see also [31],[35]). Let $\mathbf{H}_1 = U(1)$ be a Hopf *-algebra generated by commuting variables $t, t^{-1} = t^*$ and mappings: $\Delta_1(t) := t\otimes t$, $S_1(t) := t^{-1}$, $\varepsilon_1(t) := 1$. An epimorphism $\pi : \mathbf{H} \to \mathbf{H}_1$ is defined as $\pi(\alpha) := t$, $\pi(\delta) := t^{-1}$, $\pi(\beta) = \pi(\gamma) := 0$. One can consider π as an epimorphism of unital Hopf C^*-algebras and Δ as a positive continuous mapping of commutative unital C^*-algebra $H_1\backslash H/H_1$ to its tensor square. The spectrum of the C^*-algebra $H_1\backslash H/H_1$ is a compact Hausdorff space $Q = \{q^{2k}|k \in Z_+\} \cup \{0\}$ and restriction of the invariant integral ν of $SU_q(2)$ to this C^*-algebra is given by the Jackson integral:

$$\nu(f) = (1-q^2)\sum_{k\in Z_+} f(q^{2k})q^{2k} \qquad \forall f \in C(Q). \tag{12}$$

Theorem 2 *[19],[30] The strict Gel'fand pair $(SU_q(2), U(1))$ generates a Hermitian normal hypercomplex system with a compact basis Q, $e = 0$ and Haar measure of the form (12). The corresponding operation has a form*

$$(L^p f)(r) = (\Delta f)(p,r) = (1-q^2)\sum_{k\in Z_+} K(q^{2p'}, q^{2r'}; q^{2k}|q^2)f(q^{2k})q^{2k},$$

where $p', r' \in Z_+, p = q^{2p'}, r = q^{2r'} \in Q, K(q^{2p'}, q^{2r'}; q^{2k}|q^2)$ may be expressed by means of ${}_3\varphi_2$-q-hypergeometric series. The series in the right-hand side of the latter equality converges absolutely. The corresponding complete orthogonal in $L_2(Q,\nu)$ system of characters is formed by the little q-Legendre polynomials $p_n(z;1,1|q^2)$, $n \in Z_+, z \in Q$.

Corollary. We have the following product formula for the little q-Legendre polynomials:

$$(1-q^2)\sum_{k\in Z_+} K(q^{2p'}, q^{2r'}; q^{2k}|q^2)p_l(q^{2k};1,1|q^2) = p_l(q^{2p'};1,1|q^2)p_l(q^{2r'};1,1|q^2),$$

where $p', r', l \in Z_+$. One can find this formula and the expression for $K(q^{2p'}, q^{2r'}; q^{2k}|q^2)$ in [18].

Now we can find a dual hypercomplex system applying the general construction from [5],[6]. It has a discrete basis Z_+ and, hence, it is a DJS-hypergroup. Here we have

$$p_l(q^{2n};1,1|q^2)p_m(q^{2n};1,1|q^2) = \sum_{s=|l-m|}^{l+m} \widehat{K}(l,m,s)p_s(q^{2n};1,1|q^2),$$

where the numbers $\widehat{K}(l,m,s)$ are Clebsch-Gordan coefficients for the irreducible corepresentations of $SU_q(2)$. One can look at this formula as at the dual product formula for the little q-Legendre polynomials. The kernel $\widehat{K}(l,m,s)$ was expressed in [16] by means of q-Hahn polynomials.

4.2. Now consider a strict Gel'fand pair $(SU_q(n), U_q(n-1)), n \geq 2$. This is a generalization of the example considered in 4.1. It is known that [25] $U_q(n) := \mathrm{C}\langle t_{ij}, t, 1\rangle / I_R$, where $\mathrm{C}\langle t_{ij}, t, 1\rangle$ is a free algebra generated by the elements of the matrix $T = (t_{ij})$, $i, j = 1, \ldots, n$, the elements t, 1 and I_R is a two-sided ideal generated by the relations

$$RT_1T_2 = T_2T_1R, \; t\cdot t_{ij} = t_{ij}\cdot t, \; t\cdot \det_q(T) = \det_q(T)\cdot t = 1.$$

Here $T_1 = T\otimes I$, $T_2 = I \otimes T$, I is the identity matrix in R^n, the matrix R is given by

$$R := \sum_{1\leq i,j\leq n} q^{\Delta_{ij}} e_{ii}\otimes e_{jj} + (q-q^{-1})\sum_{1\leq i<j\leq n} e_{ij}\otimes e_{ji},$$

$e_{ij} \in Mat(n \times n)$ are matrix units, $\det_q(T) = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1\sigma_1} \cdot \ldots \cdot t_{n\sigma_n}$, S_n is the permutation group, $l(\sigma)$ - the length of the permutation σ, and $q \in \mathrm{C}$.

For $q \in \mathrm{R}$ and $|q| < 1$, the structure of a $*$-Hopf is given by

$$\Delta(t_{i,j}) := \sum_k t_{i,k} \otimes t_{k,j}, \ \Delta(t) := t \otimes t, \ \varepsilon(t_{i,j}) := \delta_{i,j}, \ \varepsilon(t) := 1,$$

$$S(t_{i,j}) := (-q)^{i-j} \sum_{\sigma \in S_{n-1}} (-q)^{l(\sigma)} t_{0\sigma_0} \cdot \ldots \cdot t_{j-1\sigma_{j-1}} t_{j+1\sigma_{j+1}} \cdot \ldots \cdot t_{n-1}\sigma_{n-1},$$

where $\sigma_k \in [0, i-1] \cup [i+1, n-1]$, and the antipode $S(t) := \det_q(T)$, and the involution $t^*_{i,j} := S(t_{j,i})$, $t^* := t$.

The dual Hopf *-algebra, $U_q(u(n))$, is defined as [25] $U_q(u(n)) := \mathrm{C}\langle l^+_{ij}, l^-_{ij}, 1\rangle / I_{R^+}$, where the free algebra $\mathrm{C}\langle l^+_{ij}, l^-_{ij}, 1\rangle$ is generated by the elements l^+_{ij}, l^-_{ij}, 1, $i, j = 1, \ldots n$, and the two-sided ideal I_{R^+} is generated by the relations

$$R^+ L^\pm_1 L^\pm_2 = L^\pm_2 L^\pm_1 R^+, \ R^+ L^+_1 L^-_2 = L^-_2 L^+_1 R^+,$$

where $L^\pm_1 = L^\pm \otimes I$, $L^\pm_2 = I \otimes L^\pm$, $R^+ = PRP$ $(P(l_1 \otimes l_2) = l_2 \otimes l_1)$. The coalgebra structure is given by

$$\hat{\Delta}(l^\pm_{ij}) := \sum_{k=1}^n l^\pm_{ik} \otimes l^\pm_{kj},$$

and a nondegenerate pairing is defined to be $\langle L^\pm, T_1 \ldots T_k\rangle := R^\pm_1 \cdot \ldots \cdot R^\pm_k$, where

$$T_i = I \otimes \ldots \otimes \underbrace{T}_{i} \otimes \ldots \otimes I,$$

$R^- = R^{-1}$, and $R^\pm_i$ acts as $R^\pm$ on the 0^{th} and i^{th} component of the tensor product $(\mathrm{R}^n)^{\otimes(k+1)}$. $SU_q(n)$ is a Hopf *-algebra distinguished by the condition $\det_q(T) = 1$, with the same $\Delta_1, \varepsilon_1, S_1, *$.

Let the generators of $U_q(n-1)$ be s_{ij}, s and Hopf *-algebra epimorphism π : $SU_q(n) \to U_q(n-1)$ be$\pi(t_{ij}) := s_{ij}$, $\pi(t_{0j}) := \pi(t_{i0}) = 0$, $i, j = 1, \ldots, n-1$, $\pi(t_{00}) := s$. It was shown in [8] that $U_q(n-1) \backslash SU_q(n)$ is generated by the elements $x_{ij} = t_{0i} t^*_{0j}$, $i, j = 0, \ldots, n-1$, $SU_q(n)/U_q(n-1)$ - by the elements $y_{ij} = t_{i0} t^*_{j0}$, $i, j = 0, \ldots, n-1$,$U_q(n-1) \backslash SU_q(n)/U_q(n-1)$ is generated by an element $z = t_{00} t^*_{00}$. Since the latter unital algebra is commutative and S_1 is trivial on it, $(SU_q(n)$,$U_q(n-1))$ is strict Gel'fand pair.

Let H be the completion of the algebra $U_q(n-1) \backslash SU_q(n)/U_q(n-1)$ with respect to C^*-norm $|\cdot| = \sup_\rho ||\rho(\cdot)||$, where ρ runs over the set of all the irreducible representations of $SU_q(n)$. The spectrum of this C^*-algebra is $Q = \{q^{2k}\}_{k \in Z_+} \cup \{0\}$ and the restriction to it of the Haar measure ν of $SU_q(n)$ is given by the Jackson integral:

$$\nu(F(z_1)) = (1 - q^{2n-2}) \sum_{k=0}^\infty q^{2k(n-1)} F(q^{2k}) \tag{13}$$

(see [32]). It was shown in [8], that we have here a structure of a hypercomplex system with compact basis Q, its characters φ_m can be expressed by the little q-Jacobi polynomials P_m $(m = 1, 2, \ldots)$:

$$\varphi_m(z_1) = A_m P_m(z_1/q^2; q^{2n-4}, 1|q^2), \ A_m = \frac{(1 - q^{2n-2})(1 - q^2)}{(1 - q^{-2m})(1 - q^{2m+2n-2})} \tag{14}$$

After that we can again consider a dual hypercomplex system with the discrete basis Z_+ (DJS-hypergroup), whose structural constants are the Clebsch-Gordan coefficients for the irreducible corepresentations of $SU_q(n)$. Thus we have:

Theorem 3 *There are two dual to each other structures associated with the strict Gelfand pair $(SU_q(n), U_q(n-1))$: a commutative hypercomplex system with the compact basis $\{q^{2k}\}_{k \in Z_+} \cup \{0\}$ and the Haar measure given by (13), and the discrete commutative DJS-hypergroup with the basis Z_+. Their characters are expressed by (14).*

For $n > 2$ the expressions for the kernels in the product formula for the little q-Jacobi polynomials and in dual product formula are not known.

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