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GILLES PISIER

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EXACT OPERATOR SPACES

Gilles Pisier

Plan

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Introduction

In this paper, we study *operator spaces* in the sense of the theory developed recently by Blecher-Paulsen [BP] and Effros-Ruan [ER1]. By an operator space, we mean a closed subspace $E \subset B(H)$, with H Hilbert. In the category of operator spaces, the morphisms are the *completely bounded maps* for which we refer the reader to [Pa1]. Let $E \subset B(H)$, $F \subset B(K)$ be operator spaces (H, K Hilbert). A map $u: E \rightarrow F$ is called completely bounded (*c.b.* in short) if

$$\sup_{n \geq 1} \|I_{M_n} \otimes u\|_{M_n(E) \rightarrow M_n(F)} < \infty$$

where $M_n(E)$ and $M_n(F)$ are equipped with the norms induced by $B(\ell_2^n(H))$ and $B(\ell_2^n(K))$ respectively. We denote

$$\|u\|_{cb} = \sup_{n \geq 1} \|I_{M_n} \otimes u\|_{M_n(E) \rightarrow M_n(F)}.$$

The map u is called a complete isomorphism if it is an isomorphism and if u and u^{-1} are *c.b.*. We say that $u: E \rightarrow F$ is a complete isometry if for each $n \geq 1$ the map $I_{M_n} \otimes u: M_n(E) \rightarrow M_n(F)$ is an isometry. We refer to [Ru, ER2-7, B1, B2] for more information on the rapidly developing Theory of Operator Spaces.

We will be mainly concerned here with the “geometry” of *finite dimensional* operator spaces. In the Banach space category, it is well known that every separable space embeds isometrically into ℓ_∞ . Moreover, if E is a finite dimensional normed space then for each $\varepsilon > 0$, there is an integer n and a subspace $F \subset \ell_\infty^n$ which is $(1+\varepsilon)$ -isomorphic to E , i.e. there is an isomorphism $u: E \rightarrow F$ such that $\|u\| \|u^{-1}\| \leq 1 + \varepsilon$.

Here of course, n depends on ε , say $n = n(\varepsilon)$ and usually (for instance if $E = \ell_2^k$) we have $n(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$.

Quite interestingly, it turns out that this fact is not valid in the category of operator spaces: although every operator space embeds completely isometrically into $B(H)$ (the non-commutative analogue of ℓ_∞) it is not true that a finite dimensional operator space must be close to a subspace of M_n (the non-commutative analogue of ℓ_∞^n) for some n . The main object of this paper is to study this phenomenon.

We will see that this phenomenon is very closely related to the remarkable work of E. Kirchberg on exact C^* -algebras. We will show that some of Kirchberg's ideas can be developed in a purely "operator space" setting. Our main result in the first section is Theorem 1, which can be stated as follows.

Let $B = B(\ell_2)$ and let $K \subset B$ be the ideal of all the compact operators on ℓ_2 .

If X, Y are operator spaces, we denote by $X \otimes_{\min} Y$ their minimal (or spatial) tensor product. If $X \subset B(H)$ and $Y \subset B(K)$, this is just the completion of the linear tensor product $X \otimes Y$ for the norm induced by $B(H \otimes_2 K)$.

Let $\lambda \geq 1$ be a fixed constant.

The following properties of an operator space X are equivalent:

(i) The sequence

$$\{0\} \rightarrow K \otimes_{\min} X \rightarrow B \otimes_{\min} X \rightarrow (B/K) \otimes_{\min} X \rightarrow \{0\}$$

is exact and the map

$$T_X: (B \otimes_{\min} X)/(K \otimes_{\min} X) \rightarrow (B/K) \otimes_{\min} X$$

has an inverse T_X^{-1} with norm $\|T_X^{-1}\| \leq \lambda$.

(ii) for each $\epsilon > 0$ and each finite dimensional subspace $E \subset X$, there is an integer n and a subspace $F \subset M_n$ such that $d_{cb}(E, F) < \lambda + \epsilon$.

Here $d_{cb}(E, F)$ denotes the *c.b.* analogue of the Banach-Mazur distance (see (0) below for a precise definition.) We will denote by $d_{SK}(E)$ the infimum of $d_{cb}(E, F)$ when F runs over all operator spaces F which are subspaces of M_k for some integer k .

One of the main results in section 2 can be stated as follows (see Theorem 7 below).

Consider $F \subset M_k$ with $\dim F = n$ and $k \geq n$ arbitrary, then for any linear isomorphism $u: \ell_\infty^n \rightarrow F^*$ we have

$$\|u\|_{cb} \|u^{-1}\|_{cb} \geq n[2(n-1)^{1/2}]^{-1}.$$

In particular this is > 1 for any $n \geq 3$. Here the space F^* is the dual of F with its "dual operator space structure" as explained in [BP, ER1, B1, B2].

Equivalently, if we denote by E_1^n the operator space dual of ℓ_∞^n , (this is denoted by $\max(\ell_1^n)$ in [BP]) then we have

$$d_{SK}(E_1^n) \geq \frac{n}{2\sqrt{n-1}}.$$

We also show a similar estimate for the space which is denoted by $R_n + C_n$ in [P1]. Moreover, we show that the n -dimensional operator Hilbert space OH_n (see [P1]) satisfies

$$d_{SK}(OH_n) \geq \left(\frac{n}{2\sqrt{n-1}}\right)^{1/2}.$$

These estimates are asymptotically sharp in the sense that $d_{SK}(E_1^n)$ and $d_{SK}(R_n + C_n)$ are $O(n^{1/2})$ and $d_{SK}(OH_n)$ is $O(n^{1/4})$ when n goes to infinity.

Later on in the paper, we show that the operator space analogue of the “Banach Mazur compactum” is not compact and we prove various estimates related to that phenomenon. (The noncompactness itself was known, at least to Kirchberg.) We will include several simple facts on ultraproducts of finite dimensional operator spaces which are closely connected to the discussion of “exact” operator spaces presented in section 1. Let us denote by OS_n the set of all n -dimensional operator spaces. We consider that two spaces E, F in OS_n are the same if they are completely isometric. Then the space OS_n is a metric space when equipped with the distance

$$\delta(E, F) = \log d_{cb}(E, F).$$

We include a proof that OS_n is complete but not compact (at least if $n \geq 3$) and we give various related estimates. As pointed out to me by Kirchberg, it seems to be an open problem whether OS_n is a *separable* metric space.¹

In passing, we recall that in [P1] we proved that $d_{cb}(E, OH_n) \leq n^{1/2}$ for any E in OS_n and therefore that

$$\sup\{d_{cb}(E, F) \mid E, F \in OS_n\} = n.$$

Actually, that supremum is attained on the subset $HOS_n \subset OS_n$ formed of all the *Hilbertian* operator spaces (i.e. those which, as normed spaces, are isometric to the Euclidean space ℓ_2^n). We also show that (at least for $n \geq 3$) HOS_n is a closed but non compact subset of OS_n . Perhaps the subset HOS_n is not even separable.¹ In section 5, we show the following result. Let E be any operator space and let $C \geq 1$ be a constant. Fix an integer $k \geq 1$. Then there is a compact set T and a subspace $F \subset C(T) \otimes_{\min} M_k$ such that $d_{cb}(E, F) \leq C$ iff for any operator space X and any $u: X \rightarrow E$ we have

$$\|u\|_{cb} \leq C\|u\|_k,$$

where $\|u\|_k = \|u\|_{M_k(X) \rightarrow M_k(E)}$.

Notation: Let (E_m) be a sequence of operator spaces. We denote by $\ell_\infty\{E_m\}$ the direct sum in the sense of ℓ_∞ of the family (E_m) . As a Banach space, this means that $\ell_\infty\{E_m\}$ is the set of all sequences $x = (x_m)$ with $x_m \in E_m$ for all m with $\sup_m \|x_m\|_{E_m} < \infty$ equipped with the norm $\|x\| = \sup_m \|x_m\|_{E_m}$. The operator space structure on $\ell_\infty\{E_m\}$ is defined by the identity

$$\forall n \quad M_n(\ell_\infty\{E_m\}) = \ell_\infty\{M_n(E_m)\}.$$

¹ See the Note added at the end of this paper.

Equivalently, if $E_m \subset B(H_m)$ (completely isometrically) then $\ell_\infty\{E_m\}$ embeds (completely isometrically) into $B(\oplus_m H_m)$ as block diagonal operators.

We will use several times the observation that if F is an other operator space then $\ell_\infty\{E_m\} \otimes_{\min} F$ embeds completely isometrically in the natural way into $\ell_\infty\{E_m \otimes_{\min} F\}$. In particular, if F is finite dimensional these spaces can be completely isometrically identified.

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§1. Exact operator spaces.

Let E, F be operator spaces. We denote

$$(0) \quad d_{cb}(E, F) = \inf\{\|u\|_{cb}\|u^{-1}\|_{cb}\}$$

where the infimum runs over all isomorphisms $u: E \rightarrow F$. If E, F are not completely isomorphic we set $d_{cb}(E, F) = \infty$. This is the operator space version of the Banach Mazur distance. We will study the smallest distance of an operator space E to a subspace of the space $K = K(\ell_2)$ of all compact operators on ℓ_2 . More precisely, this is defined as follows

$$(1) \quad d_{SK}(E) = \inf\{d_{cb}(E, F) \mid F \subset K\}.$$

Let F be a finite dimensional subspace of K . By an entirely classical perturbation argument one can check that for each $\varepsilon > 0$ there is an integer n and a subspace $\tilde{F} \subset M_n$ such that $d_{cb}(F, \tilde{F}) < 1 + \varepsilon$. It follows that for any finite dimensional operator space E we have

$$(1)' \quad d_{SK}(E) = \inf\{d_{cb}(E, F) \mid F \subset M_n, \quad n \geq 1\}.$$

In his remarkable work on exact C^* -algebras (cf. [Ki]) Kirchberg introduces a quantity which he denotes by $locfin(E)$ for any operator space E . His definition uses completely positive unit preserving maps. The number $d_{SK}(E)$ appears as the natural “c.b.” analogue of Kirchberg’s $locfin(E)$. Note that $d_{SK}(E)$ is clearly an invariant of the operator space E and we have obviously

$$(1)'' \quad d_{SK}(E) \leq d_{SK}(F)d_{cb}(E, F)$$

for all operator spaces E, F .

Let X be an operator space. We will say that X is exact if the sequence

$$(2) \quad \{0\} \rightarrow K \otimes_{\min} X \rightarrow B \otimes_{\min} X \rightarrow (B/K) \otimes_{\min} X \rightarrow \{0\}$$

is exact. In other words, X is an exact operator space if the natural completely contractive map

$$B \otimes_{\min} X \rightarrow (B/K) \otimes_{\min} X$$

in onto and its kernel coincides with $K \otimes_{\min} X$.

Note in particular that every finite dimensional space is trivially exact. Following Kirchberg, we will measure the “degree of exactness” of X via the number $ex(X)$ defined as follows: we consider the map

$$(3) \quad T_X: (B \otimes_{\min} X)/(K \otimes_{\min} X) \rightarrow (B/K) \otimes_{\min} X$$

associated to the exact sequence (2) and we define

$$(4) \quad ex(X) = \|T_X^{-1}\|.$$

Clearly $ex(X)$ is a (completely) isomorphic invariant of X in the following sense: if X and Y are completely isomorphic operator spaces we have

$$(4)' \quad ex(X) \leq ex(Y) d_{cb}(X, Y).$$

The main result of this section is the following which is proved by adapting in a rather natural manner the ideas of Kirchberg [Ki]. One simply needs to substitute everywhere in his argument “completely positive unital” by “completely bounded” and to keep track of the *c.b.* norms. The resulting proof is very simple.

Theorem 1. *For every finite dimensional operator space E , one has*

$$(5) \quad ex(E) = d_{SK}(E).$$

More generally, for any operator space X

$$(6) \quad ex(X) = \sup\{d_{SK}(E) \mid E \subset X, \dim E < \infty\},$$

and X is exact iff the right side of (6) is finite.

Remarks. (i) If X is a C^* -algebra then the maps appearing in (2) are (C^* -algebraic) representations. Recall that a representation necessarily has closed range and becomes isometric when we pass to the quotient modulo its kernel (cf. e.g. [Ta, p.22]). Hence if X is a C^* -algebra (2) is exact iff the kernel of the map

$$B \otimes_{\min} X \rightarrow (B/K) \otimes_{\min} X$$

coincides with $K \otimes_{\min} X$.

By a known argument, a sufficient condition for this to hold is a certain “slice map” property (cf. [W2]) which is a consequence of the *CBAP* (see [Kr]). (Recall that X has the *CBAP* if the identity on X is a pointwise limit of a net of finite rank maps $u_i: X \rightarrow X$ with $\sup_i \|u_i\|_{cb} < \infty$.)

Thus it is known that the reduced C^* -algebra of the free group $\mathbf{F}_N$ with N generators ($N \geq 2$) is exact, because by [DCH] it has the *CBAP*. On the other hand, it is known ([W1]) that the full C^* -algebras $C^*(\mathbf{F}_N)$ are not exact. This can also be

derived from Theorem 7 below after noticing that the space $E_1^n = (\ell_\infty^n)^*$ appearing in Theorem 7 is completely isometric to a subspace of $C^*(\mathbf{F}_n)$. By the same argument (using Corollary 10 below) if an operator space E is completely isometric to the space OH_n introduced in [P1], then the C^* -algebra generated by E is not exact if $n \geq 3$.

(ii) As explained to me by Kirchberg, if X is a C^* -algebra then we have $ex(X) < \infty$ iff $ex(X) = 1$. Indeed since (3) is a representation, it is isometric if it is injective. This shows if a C^* -algebra is completely isomorphic to an exact operator space then it is exact as a C^* -algebra.

We will use the following simple fact.

Lemma 2. *Let E be a separable operator space. There are operators $P_n: E \rightarrow M_n$ such that*

- (i) $\|P_n\|_{cb} \leq 1$ for all n .
- (ii) The embedding $J: E \rightarrow \ell_\infty\{M_n\}$ defined by $J(x) = (P_n(x))_{n \in \mathbb{N}}$ is a complete isometry.
- (iii) For all $k \leq n$, there is a map $a_{kn}: M_n \rightarrow M_k$ with $\|a_{kn}\|_{cb} \leq 1$ such that $P_k = a_{kn}P_n$.
- (iv) Assume E finite dimensional. Then for some $n_0 \geq 1$, the maps P_n are injective for all $n \geq n_0$.

Proof. We can assume $E \subset B(\ell_2)$. Then let $Q_n: B(\ell_2) \rightarrow M_n$ be the usual projection (defined by $Q_n(e_{ij}) = e_{ij}$ if $i, j \leq n$ and $Q_n(e_{ij}) = 0$ otherwise). Let $P_n = Q_n|_E$. Then (i), (ii), (iii) and (iv) are immediate. $\blacksquare$

The point of the preceding lemma is that we can write for all $N \geq 1$ and all (a_{ij}) in $M_N(E)$

$$(7) \quad \|(a_{ij})\|_{M_N(E)} = \lim_{n \rightarrow \infty} \uparrow \|(P_n(a_{ij}))\|_{M_N(M_n)}.$$

Indeed, by (ii) we have

$$\|(a_{ij})\|_{M_N(E)} = \sup_n \|(P_n(a_{ij}))\|_{M_N(M_n)}$$

and by (iii) this supremum is monotone nondecreasing, whence (7).

The following two lemmas are well known to specialists.

Lemma 3. *If X, Y are exact operator spaces and if $X \subset Y$, then $ex(X) \leq ex(Y)$.*

Proof. We will identify $B \otimes_{\min} X$ (resp. $(B/K) \otimes_{\min} X$) with a subspace of $B \otimes_{\min} Y$ (resp. $(B/K) \otimes_{\min} Y$). Consider u in the open unit ball of $(B/K) \otimes_{\min} X$. By definition of $ex(Y)$, there is an element v in $B \otimes_{\min} Y$ such that $\|v\| < ex(Y)$ and if $q: B \otimes_{\min} Y \rightarrow (B/K) \otimes_{\min} Y$ is the canonical mapping, we have $q(v) = u$. On the

other hand, since X is exact we know there is a $\tilde{u}$ in $B \otimes_{\min} X$ such that $q(\tilde{u}) = u$. Note that by exactness $\text{Ker } q = K \otimes_{\min} Y$, hence $v - \tilde{u} \in K \otimes_{\min} Y$. Let p_n be an increasing sequence of finite rank projections in B tending to the identity (in the strong operator topology). Consider the mapping $\sigma_n: B \rightarrow B$ defined by $\sigma_n(x) = (1 - p_n)x(1 - p_n)$. Clearly $\|\sigma_n\|_{cb} \leq 1$ and for all x in B we have $\sigma_n(x) - x \in K$. Moreover, for all x in K we have $\|\sigma_n(x)\| \rightarrow 0$. More generally, by equicontinuity, for any w in $K \otimes_{\min} Y$ we have $\|(\sigma_n \otimes I_Y)(w)\|_{K \otimes_{\min} Y} \rightarrow 0$ when $n \rightarrow \infty$.

Hence for any $\varepsilon > 0$, for some n large enough we have $\|(\sigma_n \otimes I_Y)(v - \tilde{u})\| < \varepsilon$. Therefore,

$$\|(\sigma_n \otimes I_Y)\tilde{u}\| \leq \|\sigma_n\|_{cb}\|v\| + \varepsilon < ex(Y) + \varepsilon.$$

But on the other hand, $\tilde{u} - (\sigma_n \otimes I_Y)\tilde{u} = ((1 - \sigma_n) \otimes I_Y)\tilde{u} \in K \otimes_{\min} X$ since $\tilde{u} \in B \otimes_{\min} X$. Hence $\text{dist}(\tilde{u}, K \otimes_{\min} X) < ex(Y) + \varepsilon$ and we conclude $ex(X) < ex(Y) + \varepsilon$. ■

Lemma 4. *Let X be an operator space. Then $\sup\{\|T_E^{-1}\| \mid E \subset X, \dim E < \infty\}$ is finite iff X is exact and we have*

$$(8) \quad ex(X) = \sup\{\|T_E^{-1}\| \mid E \subset X, \dim E < \infty\}.$$

Proof. Let λ be the right side of (8). By Lemma 3 we clearly have $\lambda \leq ex(X)$ hence it suffices to show that if λ is finite X is exact and (8) holds. Assume λ finite. Then clearly T_X is onto. Let $q: B \otimes_{\min} X \rightarrow (B/K) \otimes_{\min} X$ be the natural map. Consider u in $\text{Ker}(q)$. By density there is a sequence u_n in $B \otimes X$ such that $\|u - u_n\| < 2^{-n}$. Then $\|q(u_n)\| < 2^{-n}$. By definition of λ (since $u_n \in B \otimes E_n$ for some finite dimensional subspace $E_n \subset X$ and $\|T_{E_n}^{-1}\| \leq \lambda$) there is v_n in $B \otimes X$ which is a lifting of $q(u_n)$ so that $\|v_n\| < 2^{-n}\lambda$ and $u_n - v_n \in K \otimes X$. Therefore $\|u - (u_n - v_n)\| < 2^{-n} + 2^{-n}\lambda$ so that $u = \lim(u_n - v_n) \in K \otimes X$. This shows that $\text{Ker}(q) = K \otimes_{\min} X$. Thus we have showed that $\lambda < \infty$ implies X exact. By definition of T_X^{-1} it is then easy to check that $\|T_X^{-1}\| \leq \lambda$. ■

Lemma 5. *For any operator space X*

$$(9) \quad ex(X) \leq \sup\{d_{SK}(E) \mid E \subset X, \dim E < \infty\}.$$

Proof. By the preceding lemma it suffices to show that a finite dimensional operator space E satisfies $\|T_E^{-1}\| \leq d_{SK}(E)$.

Now consider $F \subset M_n$. By Lemma 3 and by (4)' we have

$$\begin{aligned} \|T_E^{-1}\| &= ex(E) \leq ex(F)d_{cb}(E, F) \\ &\leq ex(M_n)d_{cb}(E, F) \end{aligned}$$

but trivially $ex(M_n) = 1$ hence we obtain $\|T_E^{-1}\| \leq d_{cb}(E, F)$ and taking the infimum over F , $\|T_E^{-1}\| \leq d_{SK}(E)$. ■

Proof of Theorem 1. Let $E \subset X$ be finite dimensional. We will prove

$$d_{SK}(E) \leq ex(E).$$

This is the main point. To prove this claim we consider the maps $P_n: E \rightarrow M_n$ appearing in Lemma 2. Let $E_n = P_n(E) \subset M_n$. For $n \geq n_0$ we consider the isomorphism $u_n: E \rightarrow E_n$ obtained by considering P_n with range E_n instead of M_n . Since E is finite dimensional u_n^{-1} is *c.b.* for each $n \geq n_0$. We claim that we have

$$(10) \quad \limsup_{n \rightarrow \infty} \|u_n^{-1}\|_{cb} \leq ex(E).$$

From (10) it is easy to complete the proof of Theorem 1. Indeed, if (10) holds, we have

$$d_{SK}(E) \leq \limsup_{n \rightarrow \infty} \|u_n\|_{cb} \|u_n^{-1}\|_{cb} \leq ex(E).$$

By (9) we have conversely $ex(E) \leq d_{SK}(E)$, whence (5). Then by (8) X is exact iff the right side of (6) is finite and (6) follows from (8). Thus to conclude it suffices to prove our claim (10).

Consider $\varepsilon_n > 0$ with $\varepsilon_n \rightarrow 0$. For each $n \geq n_0$, we choose h_n in $M_{k(n)}(E)$ such that

$$(11) \quad \|(I_{M_{k(n)}} \otimes u_n)h_n\|_{M_{k(n)}(E_n)} = 1 \text{ and } \|h_n\|_{M_{k(n)}(E)} > \|u_n^{-1}\|_{cb} - \varepsilon_n.$$

Then we form the direct sum $B_1 = \ell_\infty\{M_{k(n)}\}$ and consider the corresponding element

$$h = (h_n)_{n \geq n_0} \quad \text{in} \quad B_1 \otimes_{\min} E = \ell_\infty\{M_{k(n)}(E)\}.$$

Let $K_1 \subset B_1$ be the subspace formed of all the sequences (x_n) with $x_n \in M_{k(n)}$ which tend to zero when $n \rightarrow \infty$. By suitably embedding B_1 into $B(\ell_2)$ and K_1 into $K(\ell_2)$ we find that the natural map

$$T_1: (B_1 \otimes_{\min} E)/(K_1 \otimes_{\min} E) \rightarrow (B_1/K_1) \otimes_{\min} E$$

satisfies (note that it is an isomorphism since $\dim E < \infty$)

$$(12) \quad \|T_1^{-1}\| \leq \|T_E^{-1}\| = ex(E).$$

Let $q: B_1 \otimes_{\min} E \rightarrow (B_1 \otimes_{\min} E)/(K_1 \otimes_{\min} E)$ be the quotient mapping. Observe that we have clearly

$$(13) \quad \limsup_{n \rightarrow \infty} \|h_n\| \leq \|q(h)\|.$$

On the other hand we have $q(h) = T_1^{-1}T_1q(h)$ hence

$$(14) \quad \|q(h)\| \leq \|T_1^{-1}\| \|T_1q(h)\|$$

and since $J: E \rightarrow \ell_\infty\{E_m\}$ is a complete isometry (cf. Lemma 2) we have

$$(15) \quad \|T_1 q(h)\| = \|(I_{B_1/K_1} \otimes J)T_1 q(h)\|_{(B_1/K_1) \otimes_{\min} \ell_\infty\{E_m\}}.$$

Let $q_1: B_1 \otimes_{\min} E \rightarrow (B_1/K_1) \otimes_{\min} E$ be the natural map. Clearly $q_1 = T_1 q$, and the right side of (15) is the same as the norm of the corresponding element in the space $\ell_\infty\{(B_1/K_1) \otimes_{\min} E_m\}$, hence the right side of (15) is equal to

$$\sup_m \|(I_{B_1/K_1} \otimes u_m)q_1(h)\|_{(B_1/K_1) \otimes_{\min} E_m}$$

which is clearly

$$\leq \sup_m \limsup_{n \rightarrow \infty} \|(I_{M_{k(n)}} \otimes u_m)(h_n)\|_{M_{k(n)}(E_m)}.$$

For $m \leq n$, we have by Lemma 2 $u_m = a_{mn}u_n$ with $\|a_{mn}\|_{cb} \leq 1$. By (11) this implies

$$\|(I_{M_{k(n)}} \otimes u_m)(h_n)\|_{M_{k(n)}(E_m)} \leq \|(I_{M_{k(n)}} \otimes u_n)(h_n)\|_{M_{k(n)}(E_n)} = 1.$$

Hence we conclude that (15) is ≤ 1 . By (12), (13) and (14) we obtain that $\limsup_{n \rightarrow \infty} \|u_n^{-1}\|_{cb} = \limsup_{n \rightarrow \infty} \|h_n\| \leq ex(E)$. This proves (10) and concludes the proof of Theorem 1. ■

Remark. The reader may have noticed that our definition of exact operator spaces is not the most natural extension of “exactness” in the category of operator spaces. However the more natural notion is easy to describe. Let us say that an operator space X is *OS-exact* if for any exact sequence of operator spaces (note: here the morphisms are c.b. maps)

$$\{0\} \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow \{0\}$$

the sequence

$$\{0\} \rightarrow Y_1 \otimes_{\min} X \rightarrow Y_2 \otimes_{\min} X \rightarrow Y_3 \otimes_{\min} X \rightarrow \{0\}$$

is exact.

Then we claim that X is *OS-exact* iff there is a constant C such that for any finite dimensional subspace $E \subset X$, the inclusion $i_E: E \rightarrow X$ admits for some n a factorization of the form

$$i_E: E \xrightarrow{a} M_n \xrightarrow{b} X$$

with $\|a\|_{cb}\|b\|_{cb} \leq C$.

Equivalently, this means that there is a net (u_i) of finite rank maps on X of the form $u_i = b_i a_i$ with $a_i: X \rightarrow M_{n_i}$ and $b_i: M_{n_i} \rightarrow X$ such that

$$\sup \|a_i\|_{cb}\|b_i\|_{cb} < \infty \quad \text{and} \quad u_i(x) \rightarrow x$$

for all x in X .

This result was known to E. Kirchberg and G. Vaillant. It can be proved as follows. First if X is OS -exact, it is a fortiori exact in the above sense so that by Theorem 1 there is a constant C_1 such that $d_{SK}(E) \leq C_1$ for all finite dimensional subspaces $E \subset X$.

Secondly, if X is OS -exact, there is clearly a constant C_2 such that for any pair of finite dimensional operator spaces $E_1 \subset E_2$ (so that $E_1^\perp \subset E_2^*$) we have an isomorphism

$$T: (E_2^* \otimes_{\min} X)/(E_1^\perp \otimes_{\min} X) \longrightarrow (E_2^*/E_1^\perp) \otimes_{\min} X$$

such that $\|T^{-1}\| \leq C_2$.

In other words, since $E_2^* \otimes_{\min} X = cb(E_2, X)$ we have an extension property associated to the following diagram:

$$\begin{array}{ccc} E_2 & & \\ & \searrow \tilde{v} & \\ \cup & & \\ E_1 & \xrightarrow{v} & X \end{array}$$

More precisely, for any $v: E_1 \rightarrow X$ there is an extension $\tilde{v}: E_2 \rightarrow X$ such that $\|\tilde{v}\|_{cb} \leq C_2\|v\|_{cb}$. Consider now an arbitrary finite dimensional subspace $E \subset X$. Let $\varepsilon > 0$. Consider $E_1 \subset M_n$ such that there is an isomorphism $u: E_1 \rightarrow E$ with $\|u\|_{cb}\|u^{-1}\|_{cb} \leq d_{SK}(E) + \varepsilon \leq C_1 + \varepsilon$.

Using the preceding extension property (with $E_2 = M_n$) we find an operator $b: M_n \rightarrow X$ extending u and such that $\|b\|_{cb} \leq C_2\|u\|_{cb}$. Let $a: E \rightarrow M_n$ be the operator u^{-1} considered as acting into M_n . Then $i_E = ba$ and

$$\|a\|_{cb}\|b\|_{cb} \leq C_2\|u\|_{cb}\|u^{-1}\|_{cb} \leq C_2(C_1 + \varepsilon).$$

This proves our claim. Note in particular that if X is a C^* -algebra, it is OS -exact iff it is nuclear, by [P1, Remark before Theorem 2.10].

In the category of Banach spaces one can define a similar notion of exactness using the *injective* tensor product instead of the minimal one. Then a Banach space is “exact” iff it is a $\mathcal{L}_\infty$ -space in the sense of [LR]. We refer the reader to [LR, Theorem 4.1 and subsequent Remark].

§2. Ultraproducts.

The notion of exactness for operator spaces is closely connected to a commutation property involving ultraproducts. To explain this let us recall a few facts about ultraproducts. Let $(F_i)_{i \in I}$ be a family of operator spaces and let $\mathcal{U}$ be a nontrivial ultrafilter on I . We denote by $\hat{F} = \Pi F_i / \mathcal{U}$ the associated ultraproduct in the category of Banach spaces (cf. e.g. [Hei]). Recall that if $\dim(F_i) = n$ for all i in I , then the ultraproduct $\hat{F}$ clearly also is n -dimensional.

Clearly $\hat{F}$ can be equipped with an operator space structure by defining

$$(16) \quad M_n(\hat{F}) = \Pi M_n(F_i) / \mathcal{U}.$$

It is easy to check that Ruan's axioms [Ru] are satisfied so that $\widehat{F}$ with the matricial structure (16) is an operator space. Alternatively, one may view F_i as embedded into $B(H_i)$ (H_i Hilbert) and observe that $\widehat{F} \subset \Pi B(H_i)/\mathcal{U}$. Since C^* -algebras are stable by ultraproduct we obtain $\widehat{F}$ embedded in a C^* -algebra. It is easy to see that the resulting operator space structure is the same as the one defined by (16). Note that (16) can be written as a commutation property between ultraproducts and the minimal tensor product, as follows

$$(17) \quad M_n \otimes_{\min} [\Pi F_i/\mathcal{U}] = \Pi[M_n \otimes_{\min} F_i]/\mathcal{U}.$$

It is natural to wonder which operator spaces E can be substituted to M_n in this identity (17). It turns out that this property is closely related to the invariant $d_{SK}(E)$, as we will now show.

We first observe that there is for any finite dimensional operator space E a canonical map

$$(18) \quad v_E: \Pi(E \otimes_{\min} E_i)/\mathcal{U} \rightarrow E \otimes_{\min} \widehat{E}$$

with $\|v_E\| \leq 1$. Indeed, we clearly have a norm one mapping

$$(18)' \quad E \otimes_{\min} \ell_{\infty}\{E_i\} \rightarrow E \otimes_{\min} \widehat{E}$$

but if E is finite dimensional $E \otimes_{\min} \ell_{\infty}\{E_i\} = \ell_{\infty}\{E \otimes_{\min} E_i\}$ and the map (18)' vanishes on the subspace of elements with $\mathcal{U}$ limit zero. Hence, after passing to the quotient by the kernel of (18)', we find the map (18) with norm ≤ 1 . More generally (recall the isometric identity $F^* \otimes_{\min} E = cb(F, E)$, cf. [BP, ER1]) if $(E_i)_{i \in I}$ (resp. $(F_i)_{i \in I}$) is a family of n -dimensional (resp. m -dimensional) operator spaces, we clearly have a norm one canonical map

$$(18)'' \quad \Pi cb(E_i, F_i)/\mathcal{U} \rightarrow cb(\widehat{E}, \widehat{F}),$$

where $\widehat{E} = \Pi E_i/\mathcal{U}$ and $\widehat{F} = \Pi F_i/\mathcal{U}$.

Proposition 6. *Let E be a finite dimensional operator space and let $C \geq 1$ be a constant. The following are equivalent.*

- (i) $d_{SK}(E) \leq C$.
- (ii) For all ultraproducts $\widehat{F} = \Pi F_i/\mathcal{U}$ the canonical isomorphism (which has norm ≤ 1)

$$v_E: \Pi(E \otimes_{\min} F_i)/\mathcal{U} \rightarrow E \otimes_{\min} (\Pi F_i/\mathcal{U})$$

satisfies $\|v_E^{-1}\| \leq C$.

- (iii) Same as (ii) but with all ultraproducts $(F_i)_{i \in I}$ on a countable set and such that $\sup_{i \in I} \dim F_i \leq \dim E$.

Proof. First observe that if $G \subset F$ are operator spaces then we have isometric embeddings

$$G \otimes_{\min} \widehat{F} \rightarrow F \otimes_{\min} \widehat{F}$$

and

$$\Pi(G \otimes_{\min} F_i)/\mathcal{U} \rightarrow \Pi(F \otimes_{\min} F_i)/\mathcal{U}.$$

Therefore in the finite dimensional case we have clearly $\|v_G^{-1}\| \leq \|v_F^{-1}\|$.

(i) $\Rightarrow$ (ii): Assume (i). Then consider $G \subset M_n$ isomorphic to E . We have clearly $\|v_{M_n}^{-1}\| = 1$ by (16), hence we can write

$$\begin{aligned} \|v_E^{-1}\| &\leq d_{cb}(E, G) \|v_G^{-1}\| \leq d_{cb}(E, G) \|v_{M_n}^{-1}\| \\ &\leq d_{cb}(E, G) \end{aligned}$$

hence $\|v_E^{-1}\| \leq d_{SK}(E)$, whence (ii).

(ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i): This is proved by an argument similar to the proof of (10) in Theorem 1. We merely outline the argument. Let $u_n: E \rightarrow E_n = P_n(E)$ be given by Lemma 2, as in the above proof of (10). For $n \geq n_0$ we consider $u_n^{-1}: E_n \rightarrow E$ and we identify u_n^{-1} with an element of $E_n^* \otimes E$. Recall $\|u_n^{-1}\|_{E_n^* \otimes_{\min} E} = \|u_n^{-1}\|_{cb(E_n, E)}$. Then

$$\|(u_n^{-1})_n\|_{\Pi(E_n^* \otimes_{\min} E)/\mathcal{U}} = \lim_{\mathcal{U}} \|u_n^{-1}\|_{cb}$$

and on the other hand since J is a complete isometry and since we have the monotonicity property (7) we have

$$\begin{aligned} \|(u_n^{-1})_n\|_{(\Pi E_n^*/\mathcal{U}) \otimes_{\min} E} &= \sup_m \|(P_m u_n^{-1})_n\|_{(\Pi E_n^*/\mathcal{U}) \otimes_{\min} M_m} \\ &= \sup_m \lim_{n, \mathcal{U}} \|P_m u_n^{-1}\|_{E_n^* \otimes_{\min} M_m} \\ &\leq \sup_m \lim_{n, \mathcal{U}} \|a_{mn}\|_{cb} \leq 1. \end{aligned}$$

Hence (iii) implies $\lim_{\mathcal{U}} \|u_n^{-1}\|_{cb} \leq C$ if $\mathcal{U}$ is any nontrivial ultrafilter on $\mathbb{N}$, and we conclude $d_{SK}(E) \leq \lim_{\mathcal{U}} \|u_n\|_{cb} \|u_n^{-1}\|_{cb} \leq C$. ■

§3. How large can $d_{SK}(E)$ be?

We now wish to produce finite dimensional operator spaces E with $d_{SK}(E)$ as large as possible. It follows from Theorem 9.6 in [P1] that for any n -dimensional operator space E we have

$$d_{SK}(E) \leq \sqrt{n}.$$

We will show that this upper bound in general cannot be improved, at least asymptotically, when n goes to infinity. We will consider the space ℓ_∞^n with its natural operator

space structure. We will denote by E_1^n the dual in the category of Banach spaces, so that as a Banach space E_1^n is the usual space ℓ_1^n , however it is embedded into $B(H)$ in such a way that the canonical basis $e_1, \dots, e_n$ of E_1^n satisfies for all $a_1, \dots, a_n$ in $B = B(\ell_2)$.

$$(19) \quad \left\| \sum_1^n e_i \otimes a_i \right\|_{E_1^n \otimes_{\min} B} = \sup_{\substack{u_i \\ \text{unitary}}} \left\| \sum_1^n u_i \otimes a_i \right\|_{B \otimes_{\min} B}$$

where the supremum runs over all unitary operators u_i in B .

Another remarkable representation of E_1^n appears if we consider the full C^* -algebra $C^*(\mathbf{F}_n)$ of the free group with n generators. If we denote by $\delta_1, \dots, \delta_n$ the generators of $\mathbf{F}_n$ viewed as unitary operators in $C^*(\mathbf{F}_n)$ in the usual way then $\|\sum \delta_i \otimes a_i\|_{C^*(\mathbf{F}_n) \otimes_{\min} B}$ is equal to (19), which shows that the map $u: E_1^n \rightarrow \text{span}(\delta_i)$ which takes e_i to δ_i is a complete isometry. Our main result is the following.

Theorem 7. *For all $n \geq 2$*

$$d_{SK}(E_1^n) \geq \frac{n}{2\sqrt{n-1}}.$$

Hence in particular $d_{SK}(E_1^n) > 1$ for all $n \geq 3$.

Remark. It is easy to check (this was pointed out to me by Paulsen) that $d_{SK}(E_1^n) = 1$ for $n = 2$. Indeed, in that case (19) becomes (after multiplication by u_1^{-1})

$$\|e_1 \otimes a_1 + e_2 \otimes a_2\| = \sup_{u \text{ unitary}} \|I \otimes a_1 + u \otimes a_2\|$$

and since (I, u) generate a commutative C^* -algebra, this is the same as

$$\sup_{\substack{|z|=1 \\ z \in \mathbb{C}}} \|a_1 + za_2\|$$

which shows that E_1^2 is completely isometric to the span of $\{1, e^{it}\}$ in $C(\mathbb{T})$. Therefore (since $C(\mathbb{T})$ is nuclear) $d_{SK}(E_1^2) = 1$. We will use an idea similar to Wassermann's argument in [W1]: we consider the direct sum $M = M_1 \oplus M_2 \oplus \dots$, or equivalently $M = \ell_\infty\{M_n\}$ in our previous notation, and we denote by $I_{\mathcal{U}}$ the set

$$I_{\mathcal{U}} = \{(x_\alpha) \in M \mid \lim_{\mathcal{U}} \tau_\alpha(x_\alpha^* x_\alpha) = 0\}$$

where τ_α is the normalized trace on M_α and where $\mathcal{U}$ is a nontrivial ultrafilter on $\mathbb{N}$. Then the group von Neumann algebra $VN(\mathbf{F}_n)$ is isomorphic to a von Neumann subalgebra of the quotient $N = M/I_{\mathcal{U}}$. It is well known that N is a finite von Neumann algebra with normalized trace τ given by $\tau(x) = \lim_{\mathcal{U}} \tau_\alpha(x_\alpha)$ where x denotes the equivalence class in N of (x_α) . Let us denote by

$$\Phi: M \rightarrow M/I_{\mathcal{U}}$$

the quotient mapping.

In the sequel, we will make use of the operator space version of the projective tensor product introduced in [ER5]. However, to facilitate the task of the reader, we include in the next few lines the simple facts that we use with indication of proof. Consider a finite dimensional algebra M_k equipped with the normalized trace which we denote by τ . We denote by $L_1(\tau)$ the space M_k equipped with the norm $\|x\|_1 = \tau(|x|)$. Let E be an operator space. Since $M_k = L_1(\tau)^*$ we have $(L_1(\tau) \otimes E)^* = M_k(E^*)$. We then denote by $L_1(\tau) \otimes_\wedge E$ the space $L_1(\tau) \otimes E$ equipped with the norm induced on $L_1(\tau) \otimes E$ by $M_k(E^*)^*$. We will use the following two facts which are easy to check:

- (a) If $F \subset E$ (completely isometric embedding) then $L_1(\tau) \otimes_\wedge F \subset L_1(\tau) \otimes_\wedge E$ (completely isometric embedding).
- (b) If $E = E_1^n$ and $e_1, e_2, \dots, e_n$ is the canonical basis of $E_1^n = \ell_\infty^n$, then for any $x_1, \dots, x_n$ in $L_1(\tau)$ we have

$$\left\| \sum_{i=1}^n x_i \otimes e_i \right\|_{L_1(\tau) \otimes_\wedge E_1^n} = \sum_{i=1}^n \|x_i\|_{L_1(\tau)}.$$

- (c) We have a norm one inclusion

$$M_k(E) \rightarrow L_1(\tau) \otimes_\wedge E.$$

These facts can be checked as follows:

- (a) follows by duality from the isometric identity

$$M_k(F^*) = M_k(E^*/F^\perp) = M_k(E^*)/M_k(F^\perp).$$

- (b) follows again by duality from the identity

$$M_k(\ell_\infty^n) = \ell_\infty^n(M_k).$$

- (c) follows from the inequality $\forall \xi_{ij} \in E^* \forall x_{ij} \in E$

$$(d) \quad k^{-1} \left| \sum_{ij \leq k} \xi_{ij}(x_{ij}) \right| \leq \|(x_{ij})\|_{M_k(E)} \|(\xi_{ij})\|_{M_k(E^*)}.$$

The latter inequality can be checked using the factorization of *c.b.* maps (cf. [Pa1, p. 100]) since $\|(\xi_{ij})\|_{M_n(E^*)} = \|(\xi_{ij})\|_{cb(E, M_n)}$: if $\|(\xi_{ij})\|_{M_n(E^*)} \leq 1$ then we can write $\xi_{ij}(x) = \langle \pi(x)x_j, y_i \rangle$ with $\pi: E \rightarrow B(H)$ restriction of a representation and with $x_j, y_i \in H$ such that $\|\sum \alpha_j x_j\| \leq 1$ and $\|\sum \alpha_i y_i\| \leq 1$ whenever $\sum |\alpha_j|^2 \leq 1$. This implies $(\sum \|x_j\|^2 \sum \|y_i\|^2)^{1/2} \leq k$ whence

$$\left| \sum \xi_{ij}(x_{ij}) \right| = \left| \sum \langle \pi(x_{ij})x_j, y_i \rangle \right| \leq k \|(x_{ij})\|_{M_k(E)}$$

which proves the inequality (d).

We denote below by $C_\lambda^*(\mathbf{F}_n)$ the reduced C^* -algebra associated to the left regular representation for the free group $\mathbf{F}_n$ with n generators. Then the key result for our subsequent estimates can be stated as follows.

Theorem 8. Fix $n \geq 2$. There is a family of unitary matrices (u_i^α) with $u_i^\alpha \in M_\alpha$, ($i = 1, \dots, n, \alpha \in \mathbf{N}$) and a nontrivial ultrafilter $\mathcal{U}$ on $\mathbf{N}$ such that for all $m \geq 1$ and all $x_1, \dots, x_n$ in M_m we have

$$(20) \quad \lim_{\mathcal{U}} \left\| \sum_{i=1}^n u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha) \otimes_\wedge M_m} \leq \left\| \sum_{i=1}^n \lambda(g_i) \otimes x_i \right\|_{C_\lambda^*(\mathbf{F}_n) \otimes_{\min} M_m}$$

Remark. By results included in [HP], the right side of (20) is

$$\leq 2 \max \left\{ \left\| \sum_{i=1}^n x_i^* x_i \right\|^{1/2}, \left\| \sum_{i=1}^n x_i x_i^* \right\|^{1/2} \right\}.$$

Proof of Theorem 8. As explained in [W1], for each i there is a unitary $u_i = (u_i^\alpha)_{\alpha \in \mathbf{N}}$ in M such that $\Phi(u_i) = \lambda(g_i)$. Let us denote

$$b = \sum \lambda(g_i) \otimes x_i \in C_\lambda^*(\mathbf{F}_n) \otimes_{\min} M_m.$$

Then $\sum u_i \otimes x_i$ is a lifting of b in $M \otimes M_m$ and $M_m(VN(\mathbf{F}_n))$ embeds isometrically (see [W1]) into $M_m(M/I_{\mathcal{U}})$ or equivalently into $M_m(M)/M_m(I_{\mathcal{U}})$. It follows that we have

$$\|b\| = \inf \left\{ \left\| \sum_{i=1}^m x_i \otimes u_i + \gamma \right\|_{M_m(M)} \mid \gamma \in M_m(I_{\mathcal{U}}) \right\}.$$

Hence there is a sequence $(\gamma^\alpha)_{\alpha \in \mathbf{N}}$ with $\gamma^\alpha \in M_m(M_\alpha)$ satisfying

$$\forall i, j \leq m \quad \lim_{\alpha \xrightarrow[\mathcal{U}]{\infty}} \tau_\alpha(\gamma_{ij}^{\alpha*} \gamma_{ij}^\alpha) = 0$$

and such that

$$(21) \quad \lim_{\mathcal{U}} \left\| \sum_{i=1}^n x_i \otimes u_i^\alpha + \gamma^\alpha \right\|_{M_m(M_\alpha)} \leq \|b\|.$$

Now observe that $M_m(M_\alpha) = M_\alpha(M_m)$. We will use the norm one inclusion (see fact (c) above) $M_\alpha(M_m) \hookrightarrow L_1(\tau_\alpha) \otimes_\wedge M_m$. Note that the inclusion $L_2(\tau_\alpha) \rightarrow L_1(\tau_\alpha)$ has norm ≤ 1 so that

$$\lim_{\alpha \xrightarrow[\mathcal{U}]{\infty} \|\gamma^\alpha\|_{L_1(\tau_\alpha) \otimes_\wedge M_m} = 0.$$

Therefore (21) yields

$$(22) \quad \lim_{\mathcal{U}} \left\| \sum_{i=1}^n u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha) \otimes_\wedge M_m} \leq \|b\|,$$

which is the announced inequality. ■

To prove Theorem 7, we will use the following lemma.

Lemma 9. *Consider the operator*

$$T_n: \ell_\infty^n \rightarrow C_\lambda^*(\mathbf{F}_n)$$

defined by $T_n(\alpha_1, \dots, \alpha_n) = \sum_{i=1}^n \alpha_i \lambda(g_i)$. Then for any m , any n -dimensional subspace $F \subset M_m$ and any factorization

$$\begin{array}{ccc} & F^* & \\ a \nearrow & & \searrow b \\ \ell_\infty^n & \xrightarrow{T_n} & C_\lambda^*(\mathbf{F}_n) \end{array}$$

with $T_n = ba$ we have

$$(23) \quad n \leq \|a\|_{cb} \|b\|_{cb}.$$

Proof. Consider a, b as above. We identify $b: F^* \rightarrow C_\lambda^*(\mathbf{F}_n)$ with an element of $F \otimes_{\min} C_\lambda^*(\mathbf{F}_n)$. Then we can write $b = \sum_{i=1}^n x_i \otimes \lambda(g_i)$ with $x_i \in F$ such that $a^*(x_i) = e_i$. (Recall that e_i is the canonical basis of $E_1^n = (\ell_\infty^n)^*$.)

Now by Theorem 8 we have

$$\lim_{\mathcal{U}} \left\| \sum_{i=1}^n u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha) \otimes_\wedge M_m} \leq \|b\|_{cb}.$$

By fact (a) recalled above, this implies

$$\lim_{\mathcal{U}} \left\| \sum_{i=1}^n u_i^\alpha \otimes x_i \right\|_{L_1(\tau_\alpha) \otimes_\wedge F} \leq \|b\|_{cb},$$

hence since $\|a^*\|_{cb} = \|a\|_{cb}$

$$\lim_{\mathcal{U}} \left\| \sum_{i=1}^n u_i^\alpha \otimes a^*(x_i) \right\|_{L_1(\tau_\alpha) \otimes_\wedge E_1^n} \leq \|a\|_{cb} \|b\|_{cb}.$$

This gives the conclusion since $a^*(x_i) = e_i$ and by fact (b)

$$\left\| \sum_{i=1}^n u_i^\alpha \otimes e_i \right\|_{L_1(\tau_\alpha) \otimes_\wedge E_1^n} = \sum_{i=1}^n \|u_i^\alpha\|_{L_1(\tau_\alpha)} = n.$$

■

Remark. The same operator T_n as in Lemma 9 was already considered in [H]. By [H, Lemma 2.5] we have $\|T_n\|_{dec} = n$, but this does not seem related to (23).

Proof of Theorem 7. By [AO] (see [H] for more details) we have $\|T_n\|_{cb} \leq 2\sqrt{n-1}$, hence for any $F \subset M_m$ we can write by (23)

$$(24) \quad n \leq \|T_n\|_{cb} d_{cb}(F^*, \ell_\infty^n)$$

$$(25) \quad n \leq \|T_n\|_{cb} d_{cb}(F^*, E_n^\lambda)$$

where $E_n^\lambda = \text{span}\{\lambda(g_i) \mid i = 1, \dots, n\}$ in $C_\lambda^*(\mathbb{F}_n)$. Since $d_{cb}(F^*, \ell_\infty^n) = d_{cb}(F, \ell_\infty^{n*})$ and $\ell_\infty^{n*} = E_1^n$ we obtain

$$n(2\sqrt{n-1})^{-1} \leq d_{cb}(F, E_1^n)$$

so that Theorem 7 follows.

Remark. By the same argument we have

$$d_{SK}(E_n^{\lambda*}) \geq n(2\sqrt{n-1})^{-1}.$$

Here again this is > 1 if $n \geq 3$ but $d_{SK}(E_n^{\lambda*}) = 1$ if $n = 2$ for the same reason as above for E_1^n . We can also derive an estimate for the n dimensional operator Hilbert space which is denoted by OH_n . This space was introduced in [P1] to which we refer for more details. It is isometric to ℓ_2^n and has an orthonormal basis $(\theta_i)_{i \leq n}$ such that for all $a_1, \dots, a_n$ in B we have

$$(26) \quad \left\| \sum_1^n \theta_i \otimes a_i \right\|_{OH_n \otimes_{\min} B} = \left\| \sum_1^n a_i \otimes \bar{a}_i \right\|_{B \otimes_{\min} \bar{B}}^{1/2}.$$

Corollary 10. For each $n \geq 2$, we have

$$(27) \quad d_{SK}(OH_n) \geq [n(2\sqrt{n-1})^{-1}]^{1/2}.$$

Proof. By (26) we have (we denote simply by $\| \cdot \|$ the minimal tensor norm everywhere)

$$\left\| \sum \theta_i \otimes \lambda(g_i) \right\| = \left\| \sum \lambda(g_i) \otimes \overline{\lambda(g_i)} \right\|^{1/2}$$

and by [AO] we have

$$\left\| \sum_1^n \lambda(g_i) \otimes \overline{\lambda(g_i)} \right\| = 2\sqrt{n-1}.$$

Hence we have a factorization of T_n of the form $\ell_\infty^n \xrightarrow{a} OH_n \xrightarrow{b} E_n$ with $\|b\|_{cb}^2 \leq 2\sqrt{n-1}$. On the other hand

$$\|a\|_{cb} = \|a^*\|_{cb} = \left\| \sum_1^n e_i \otimes \bar{e}_i \right\|_{E_1^n \otimes_{\min} \bar{E}_1^n}^{1/2} \leq n^{1/2}$$

hence we have by (23)

$$n \leq d_{SK}(OH_n) \|a\|_{cb} \|b\|_{cb}$$

which implies (27). ■

Remark. It is easy to verify that these estimates are asymptotically best possible. More precisely, we have with the notation of [P1], $d_{SK}(E_1^n) \leq d_{cb}(E_1^n, R_n) \leq \sqrt{n}$.

Similarly, $d_{SK}(E_n^{\lambda*}) \leq d_{cb}(E_n^{\lambda*}, R_n) \leq \sqrt{n}$, and finally $d_{SK}(OH_n) \leq d_{cb}(OH_n, R_n \cap C_n) \leq n^{1/4}$.

Remark. It is natural to raise the following question: Is there a function $n \rightarrow f(n)$ and a constant C such that for any E in OS_n satisfying $d_{SK}(E) = 1$, there is a subspace $F \subset M_m$ with $m \leq f(n)$ and $d_{cb}(F, E) \leq C$? It is easy to derive from the preceding construction that the answer is negative (contrary to the commutative case with ℓ_∞^n in the place of M_n). A negative answer can also be derived easily from the fact (due to Szankowski [Sz]) that the space of compact operators on ℓ_2 fails the *uniform* approximation property.

Remark. It is rather natural to introduce the following quantity for E in OS_n .

$$d_{QSK}(E) = \inf\{d_{cb}(E, F)\}$$

where the infimum runs over all the spaces F which are quotient of a subspace of K , i.e. there are $S_2 \subset S_1 \subset K$ such that $F = S_1/S_2$. Clearly $d_{QSK}(E) \leq d_{SK}(E)$. We do not know much about this new parameter. Note however that in view of the lifting property of E_1^n we have by Theorem 7

$$d_{QSK}(E_1^n) \geq \frac{n}{2\sqrt{n-1}}.$$

Moreover, it can be shown that OH_n embeds completely isometrically into the direct sum $L_\infty(R_n) \oplus L_\infty(C_n)$, and a fortiori into $L_\infty(M_{2n})$, so that $d_{QSK}(OH_n) = 1$, indeed this follows from the identity $OH_n = (R_n, C_n)_{1/2}$ proved in [P1, Corollary 2.6]. Note that for the quotient of a subspace case, the identity between (1) and (1)' might no longer hold, so that it might be necessary to distinguish between the quotients of a subspace of K and those of M_n for some $n \geq 1$.

§4. On the metric space of all n -dimensional operator spaces.

It will be convenient in the sequel to record in the next statement several elementary facts on ultraproducts.

Proposition 11. *Let E and F be two n -dimensional operator spaces. Let $(E_i)_{i \in I}$ and $(F_i)_{i \in I}$ be two families of n dimensional operator spaces. Let $\mathcal{U}$ be an ultrafilter on I and let $\widehat{E}$, $\widehat{F}$ be the corresponding ultraproducts.*

(i) We have

$$(28) \quad d_{cb}(\widehat{E}, \widehat{F}) \leq \lim_{\mathcal{U}} d_{cb}(E_i, F_i).$$

Moreover, if $d_{cb}(E_i, F_i) \rightarrow 1$ then $\widehat{E}$ is completely isometric to $\widehat{F}$.

(ii) If $F_i = F$ for all $i \in I$ then $\widehat{F}$ is completely isometric to F .

(iii) If $d_{cb}(E_i, F) \rightarrow 1$ then $\widehat{E}$ is completely isometric to F .

(iv) If $d_{cb}(E, F) = 1$, then E and F are completely isometric.

Proof. We have isomorphisms $u_i: E_i \rightarrow F_i$ such that $v_i = u_i^{-1}$ satisfies $\|u_i\|_{cb} \leq 1$ and $\lim_{\mathcal{U}} \|v_i\|_{cb} = \lim_{\mathcal{U}} d_{cb}(E_i, F_i)$. Then the maps $\hat{u}: \hat{E} \rightarrow \hat{F}$ and $\hat{v}: \hat{F} \rightarrow \hat{E}$ (associated respectively to (u_i) and (v_i)) are inverse of each other and satisfy $\|\hat{u}\|_{cb}\|\hat{v}\|_{cb} \leq \lim_{\mathcal{U}} d_{cb}(E_i, F_i)$. Hence we have (28). The preceding also shows that $\hat{u}$ is a complete isometry between $\hat{E}$ and $\hat{F}$ when $\lim_{\mathcal{U}} d_{cb}(E_i, F_i) = 1$, whence (i). But on the other hand it is easy to check that, if $F_i = F$ for all i , then $\hat{F}$ and F are completely isometric via the map $T: \hat{F} \rightarrow F$ defined by $T(x) = \lim_{\mathcal{U}} x_i$ if x is the equivalence class of (x_i) modulo $\mathcal{U}$. This justifies (ii). Then (iii) is clear. Finally, taking $E_i = E$ and $F_i = F$ for all i in what precedes, we obtain (iv). (Actually, (iv) is clear, by a direct argument based on the compactness of the unit ball of $cb(E, F)$.) ■

In this section, we will include some remarks on the set OS_n formed of all n dimensional operator spaces. More precisely OS_n is the set of all equivalence classes when we identify two spaces if they are completely isometric. We equip OS_n with the metric

$$\delta(E, F) = \text{Log } d_{cb}(E, F).$$

This is the analogue for operator spaces of the classical “Banach-Mazur compactum” formed of all n dimensional normed spaces equipped with the Banach-Mazur distance.

However, contrary to the Banach space situation the metric space OS_n is not compact. The next result was known, it was mentioned to me by Kirchberg.

Proposition 12. *The set OS_n equipped with the metric $\text{Log } d_{cb}$ is a complete metric space, but it is not compact at least if $n \geq 3$.*

Proof. We sketch a proof using ultrafilters. Let (E_i) be a Cauchy sequence in OS_n . Let $\hat{E}$ be an ultraproduct associated to a nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then by the Cauchy condition for each $\varepsilon > 0$ there is an i_0 such that for all $i, j > i_0$ we have

$$\text{Log } d_{cb}(E_i, E_j) \leq \varepsilon.$$

By Proposition 11 this implies $\forall i > i_0$

$$\text{Log } d_{cb}(\hat{E}, E_i) \leq \varepsilon$$

hence $E_i \rightarrow \hat{E}$ when $i \rightarrow \infty$ and OS_n is complete.

We now show that OS_n is not compact by exhibiting a sequence without converging subsequences if $n \geq 3$. Consider any space E_0 in OS_n such that

$$(29) \quad d_{SK}(E_0) > 1.$$

We know that such spaces exist if $n \geq 3$ by Theorem 7 and Corollary 10.

By Lemma 2, we can find a sequence of spaces $E_i \subset M_i$ with $\dim E_i = n$ such that for any nontrivial ultrafilter on $\mathbb{N}$ the ultraproduct $\widehat{E} = \prod E_i / \mathcal{U}$ is completely isometric to E_0 . (Indeed, this is clear by (7).) Assume that some subsequence of E_i converges in OS_n . Then, by Proposition 11 (ii), its limit must be $\widehat{E}$ which is the same as E_0 . In other words the subsequence can only converge to E_0 but on the other hand by (29) we have (since $E_i \subset M_i$)

$$\forall i \in I \quad 1 < d_{SK}(E_0) \leq d_{cb}(E_0, E_i),$$

which is the desired contradiction. ■

Remark. It is well known that every separable Banach space E embeds isometrically into the space of all continuous functions on the unit ball of X^* equipped with the weak*-topology. Let us denote simply by $\mathcal{C}$ the latter space, and let $k: E \rightarrow \mathcal{C}$ be the isometric embedding. Given P_n as in Lemma 2, we can introduce $\tilde{P}_n: E \rightarrow \mathcal{C} \oplus_\infty M_n$ by setting $\tilde{P}_n(x) = (k(x), P_n(x))$. Then each $\tilde{P}_n$ is an *isometric* isomorphism of E into $\tilde{E}_n$
 $\oplus_\infty M_n$. Moreover the embedding $\tilde{J}: E \rightarrow \ell_\infty\{\tilde{E}_n\}$ is a completely isometric embedding with the same properties as in Lemma 2. Finally the C^* -algebras $\mathcal{C} \oplus_\infty M_n$ are nuclear. Using this it is easy to modify the preceding reasoning, replacing E_n by $\tilde{E}_n$ (note $d_{SK}(\tilde{E}_n) = 1$ since $\mathcal{C} \oplus_\infty M_n$ is nuclear) and to demonstrate the following

Corollary 13. *Let E_0 be any n dimensional operator space such that $d_{SK}(E_0) > 1$. Then the (closed) subset of OS_n formed of all the spaces isometric to E_0 is not compact.*

Remark. Actually we obtain a sequence E_i of spaces each isometric to E_0 and such that E_0 is completely isometric to $\widehat{E} = \prod E_i / \mathcal{U}$ but $d_{cb}(E_i, E_0) \not\rightarrow 0$. More precisely (in answer to a question of S. Szarek), the preceding argument shows that the “metric entropy” of OS_n is quite large in the following sense: Let $\delta = d_{SK}(E_0)$ with E_0 as in Corollary 13. Then for any $\varepsilon > 0$ there is a sequence E_i in OS_n such that

$$d_{cb}(E_i, E_j) > \delta - \varepsilon \quad \text{for any } i \neq j.$$

(Moreover, E_i is isometric to E_0 and the ultraproduct $\widehat{E} = \prod E_i / \mathcal{U}$ is completely isometric to E_0). This suggests the following question: does there exist such a sequence if δ is equal to the diameter of the set OS_n (or of the subset of OS_n formed of all the spaces which are isometric to E_0)? If not, what is the critical value of δ ?

Corollary 13 is of course particularly striking in the case $E_0 = OH_n$, it shows that the set of all possible operator space structures on the Euclidean space ℓ_2^n is very large. We refer to [Pa2] for more information of the latter set.

These results lead to the following question (due to Kirchberg).

Problem. Is the metric space OS_n separable¹?

¹ See the Note added at the end of this paper.

Equivalently, is there a separable operator space X such that for any $\epsilon > 0$ and any n -dimensional operator space E , there is a subspace $F \subset X$ such that $d_{cb}(E, F) < 1 + \epsilon$?

More generally, let E be an n -dimensional normed space. Let $OS_n(E)$ be the subset of OS_n formed of all the spaces which are isometric to E . For which spaces E is the space $OS_n(E)$ compact or separable? Note that $OS_n(E)$ can be a singleton, this happens in the 2-dimensional case if $E = \ell_1^2$ or $E = \ell_\infty^2$, however it never happens if $\dim(E) \geq 5$ (see [Pa2, Theorem 2.13]).

We will now characterize the spaces E_0 for which the conclusion of Corollary 13 holds. We will use the following simple observation.

Lemma 14. *Fix $n \geq 1$. Let $\widehat{E} = \Pi E_i / \mathcal{U}$ be an ultraproduct of n -dimensional spaces. Then $(\widehat{E})^* = \Pi E_i^* / \mathcal{U}$ completely isometrically.*

Proof. Let F_i be a family of m -dimensional spaces with $m \geq 1$ fixed and let $\widehat{F}$ be their ultraproduct. It is well known that

$$(30) \quad \Pi B(E_i, F_i) / \mathcal{U} = B(\widehat{E}, \widehat{F})$$

isometrically. Now observe that for any integer k we have (by [Sm]) for any $u: \widehat{E} \rightarrow M_k$, associated to a family $(u_i)_{i \in I}$ with $u_i \in B(E_i, M_k)$

$$\|u\|_{cb} = \|I_{M_k} \otimes u\|_{M_k(\widehat{E}) \rightarrow M_k(M_k)}.$$

By (30) it follows that

$$\|u\|_{cb} = \lim_{\mathcal{U}} \|I_{M_k} \otimes u_i\|_{M_k(E_i) \rightarrow M_k(M_k)}.$$

Hence

$$\|u\|_{cb(\widehat{E}, M_k)} = \lim_{\mathcal{U}} \|u_i\|_{cb(E_i, M_k)}.$$

Equivalently

$$\begin{aligned} M_k((\widehat{E})^*) &= \Pi M_k(E_i^*) / \mathcal{U} \\ &= M_k(\Pi E_i^* / \mathcal{U}) \end{aligned}$$

and we conclude that $(\widehat{E})^*$ and $\Pi E_i^* / \mathcal{U}$ are completely isometric. ■

Corollary 15. *Consider E in OS_n . The following are equivalent:*

- (i) $d_{SK}(E) = d_{SK}(E^*) = 1$.
- (ii) For any sequence E_i in OS_n such that $\widehat{E} = \Pi E_i / \mathcal{U}$ is completely isometric to E we have

$$\lim_{\mathcal{U}} d_{cb}(E, E_i) = 1.$$

(iii) Same as (ii) with each E_i isometric to E .

Proof. Assume (ii) (resp. (iii)). Then the proof of Proposition 12 (resp. Corollary 13) shows that necessarily $d_{SK}(E) = 1$. By Lemma 14 it is clear that (ii) and (iii) are self dual properties, hence we must also have $d_{SK}(E^*) = 1$. Conversely assume (i). We will use Proposition 6 together with the identity $E \otimes_{\min} F = cb(F^*, E)$ valid when F is finite dimensional. By Proposition 6 and Lemma 14, if $d_{SK}(E) = 1$ we have

$$(31) \quad \Pi cb(E_i, E)/\mathcal{U} = cb(\widehat{E}, E)$$

whenever $\widehat{E}$ is an ultraproduct of n dimensional spaces. Now if $d_{SK}(E^*) = 1$ this implies

$$\Pi cb(E_i^*, E^*)/\mathcal{U} = cb(\widehat{E}^*, E^*)$$

hence after transposition

$$(32) \quad \Pi cb(E, E_i)/\mathcal{U} = cb(E, \widehat{E}).$$

It is then easy to conclude: let $u: \widehat{E} \rightarrow E$ be a complete isometry. Let $u_i: E_i \rightarrow E$ be associated to u via (31) and let $v_i: E \rightarrow E_i$ be associated to u^{-1} via (32) in such a way that $\|u\|_{cb} = \lim_{\mathcal{U}} \|u_i\|_{cb} = 1$ and $\|u^{-1}\|_{cb} = \lim_{\mathcal{U}} \|v_i\|_{cb} = 1$.

Then $I_E = \lim_{\mathcal{U}} u_i v_i$ hence we have $\lim_{\mathcal{U}} \|I_E - u_i v_i\| = 0$, hence $(u_i v_i)^{-1}$ exists for i large and its norm tends to 1, so that u_i^{-1} exists and (since $\lim_{\mathcal{U}} \|v_i\| < \infty$) $\lim_{\mathcal{U}} \|u_i^{-1}\| < \infty$, whence $\lim_{\mathcal{U}} \|u_i^{-1} - v_i\| = 0$. Since all norms are equivalent on a finite dimensional space, we also have $\lim_{\mathcal{U}} \|I_E - u_i v_i\|_{cb} = 0$, finally (repeating the argument with the cb -norms) we obtain $\lim_{\mathcal{U}} \|u_i^{-1} - v_i\|_{cb} = 0$ and we conclude that

$$\lim_{\mathcal{U}} d_{cb}(E, E_i) \leq \lim_{\mathcal{U}} \|u_i\|_{cb} \|u_i^{-1}\|_{cb} \leq 1.$$

This shows that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) is trivial. ■

Remark. The row and column Hilbert spaces R_n and C_n obviously satisfy the properties in Corollary 15. Also, the two dimensional spaces ℓ_1^2 and ℓ_∞^2 , which (see [Pa2]) admit only one operator space structure (so that any space isometric to either one is automatically completely isometric to it), must clearly satisfy these properties. At the time of this writing, these are the only examples I know of spaces satisfying the properties in Corollary 15.

It is natural to describe the spaces appearing in Corollary 15 as points of continuity with respect to a weaker topology (on the metric space OS_n) which can be defined as follows. For any $k \geq 1$ and any linear map $u: E \rightarrow F$ between operator spaces, we denote as usual

$$\|u\|_k = \|I_{M_k} \otimes E\|_{M_k(E) \rightarrow M_k(F)}.$$

Then for any E, F in OS_n we define

$$d_k(E, F) = \inf \{ \|u\|_k \|u^{-1}\|_k \}$$

where the infimum runs over all isomorphisms $u: E \rightarrow F$.

We will say that a sequence $\{E_i\}$ in OS_n tends weakly to E if, for each $k \geq 1$, $\text{Log } d_k(E_i, E) \rightarrow 0$ when $i \rightarrow \infty$. This notion of limit clearly corresponds to a topology (namely to the topology associated to the metric $\tilde{\delta} = \sum_{k \geq 1} 2^{-k} \text{Log } d_k$) which we will

call the weak topology. Let us say that E, F are k -isometric if there is an isomorphism $u: E \rightarrow F$ such that $I_{M_k} \otimes u$ is an isometry. Clearly this holds (by a compactness argument) iff $d_k(E, F) = 1$. Moreover (again by a compactness argument) E and F are completely isometric iff they are k -isometric for all $k \geq 1$. This shows that the weak topology on OS_n is Hausdorff. We observe

Proposition 16. *Let E and E_i ($i = 1, 2, \dots$) be operator spaces in OS_n . Then E_i tends weakly to E iff for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the ultraproduct $\hat{E} = \prod E_i / \mathcal{U}$ is completely isometric to E .*

Proof. Clearly if E_i tends weakly to E then $\hat{E}$ is k -isometric to E for each $k \geq 1$, hence $\hat{E}$ is completely isometric to E . Conversely, if E_i does not tend weakly to E , then for some $k \geq 1$ and some $\varepsilon > 0$ there is a subsequence E_{n_i} such that

$$(33) \quad d_k(E_{n_i}, E) > 1 + \varepsilon \quad \text{for all } i \neq j.$$

Let $\mathcal{U}$ be an ultrafilter refining this subsequence, let $\hat{E}$ be the corresponding ultraproduct, and let $u: \hat{E} \rightarrow E$ be any isomorphism. Clearly there are isomorphisms $u_i: E_i \rightarrow E$ which correspond to u and we have for each $k \geq 1$ (by compactness) $\|u\|_k = \lim_{\mathcal{U}} \|u_i\|_k$ and $\|u^{-1}\|_k = \lim_{\mathcal{U}} \|u_i^{-1}\|_k$. Therefore we obtain by (33) $d_k(\hat{E}, E) \geq 1 + \varepsilon$ and we conclude that E and $\hat{E}$ are not completely isometric. ■

We can now reformulate Corollary 15 as follows

Corollary 17. *Let $i: OS_n \rightarrow OS_n$ be the identity considered as a map from OS_n equipped with the weak topology to OS_n equipped with the metric d_{cb} . Then an element E in OS_n is a point of continuity of i iff $d_{SK}(E) = d_{SK}(E^*) = 1$.*

§5. On the dimension of the containing matrix space.

It is natural to try to connect our study of $d_{SK}(E)$ with a result of Roger Smith [Sm]. Smith's result implies that for any subspace $F \subset M_k$ we have for any operator space X and for any linear map $u: X \rightarrow F$

$$(34) \quad \|u\|_{cb} \leq \|u\|_k.$$

More generally, if T is any compact set and $F \subset C(T) \otimes_{\min} M_k$ then we also have (34).

Now let E be any finite dimensional operator space. For each $k \geq 1$ we introduce

$$\delta_k(E) = \inf\{d_{cb}(E, F) \mid F \subset C(T) \otimes_{\min} M_k\}$$

where the infimum runs over all possible compact sets T .

Note that

$$d_{SK}(E) = \inf_{k \geq 1} \delta_k(E).$$

Clearly if $C = \delta_k(E)$, then by (34) we have for all $u: X \rightarrow E$

$$(35) \quad \|u\|_{cb} \leq C\|u\|_k.$$

It turns out that the converse is true: if (35) holds for all X and all $u: X \rightarrow E$ then necessarily $\delta_k(E) \leq C$. This is contained in the next statement which can be proved following the framework of [P2], but using an idea of Marius Junge [J].

Theorem 18. *Let E be any operator space and let $C \geq 1$ be a constant. Fix an integer $k \geq 1$. Then the following are equivalent:*

- (i) *There is a compact set T and a subspace $F \subset C(T) \otimes_{\min} M_k$ such that $d_{cb}(E, F) \leq C$.*
- (ii) *For any operator space X and any $u: X \rightarrow E$ we have*

$$\|u\|_{cb} \leq C\|u\|_k.$$

- (iii) *For all finite dimensional operator spaces X , the same as (ii) holds.*

Proof. (Sketch) (i) $\Rightarrow$ (ii) is Smith's result [Sm] as explained above. (ii) $\Rightarrow$ (iii) is trivial. Let us prove (iii) $\Rightarrow$ (i). Assume (iii). Note that (for k fixed) the class of spaces of the form $C(T) \otimes_{\min} M_k$ is stable by ultraproduct, since the class of commutative unital C^* -algebras is stable by ultraproducts. In particular we may and will assume (for simplicity) that E is finite dimensional. Let G be an other operator space and consider a linear map $v: E \rightarrow G$. We introduce the number $\alpha_k(v)$ as follows. We consider all factorizations of v of the form

$$E \xrightarrow{a} \ell_{\infty}^N \otimes_{\min} M_k \xrightarrow{b} G$$

where $N \geq 1$ is an arbitrary integer, and we set

$$\alpha_k(v) = \inf\{\|a\|_{cb}\|b\|_{cb}\}$$

where the infimum runs over all possible N and all possible such factorizations.

Now, using Lemma 2 and an ultraproduct argument it suffices to prove that for any n , any $\varepsilon > 0$ and any map $v: E \rightarrow M_n$, there is an integer $N \geq 1$ and a factorization of v of the form

$$E \xrightarrow{a} \ell_\infty^N \otimes_{\min} M_k \xrightarrow{b} M_n$$

with $\|a\|_{cb}\|b\|_{cb} \leq C(1 + \varepsilon)\|v\|_{cb}$.

In other words, to conclude the proof, it suffices to show that if (iii) holds we have for all n and all $v: E \rightarrow M_n$

$$(36) \quad \alpha_k(v) \leq C\|v\|_{cb}.$$

Now we observe that (for any G) α_k is a norm on $cb(E, G)$ (left to the reader, this is where the presence of ℓ_∞^N is used). Hence (36) is equivalent to a statement on the dual norms. More precisely, (36) is equivalent to the fact that for any $T: M_n \rightarrow E$ we have

$$(37) \quad |tr(vT)| \leq C\|v\|_{cb}\alpha_k^*(T),$$

where α_k^* is the dual norm to α_k , i.e.

$$\alpha_k^*(T) = \sup\{|tr(vT)| \mid v: E \rightarrow M_n, \alpha_k(v) \leq 1\}.$$

We will now use the operator space version of the absolutely summing norm which was first introduced in [ER6]. In the broader framework of [P2], these operators are called completely 1-summing and the corresponding norm is denoted by π_1^o . We will use a version of the “Pietsch factorization” for these maps which is presented in [P2]. As observed by M. Junge, in the present situation the proof of Theorem 2.1 and Remark 2.7 in [P2] yield a factorization of T of the following form

$$M_n \xrightarrow{w} X \xrightarrow{u} E$$

where X is an n^2 -dimensional operator space and where the maps w and u satisfy

$$\pi_1^o(w) \leq 1 \quad \text{and} \quad \|u\|_k \leq \alpha_k^*(T).$$

Hence if (iii) holds we find

$$\pi_1^o(T) = \pi_1^o(uw) \leq \|u\|_{cb}\pi_1^o(w) \leq \|u\|_{cb} \leq C\|u\|_k \leq C\alpha_k^*(T).$$

But since T is defined on M_n , we clearly have by definition of π_1^o

$$|tr(vT)| \leq \pi_1^o(vT) \leq \pi_1^o(T)\|v\|_{cb},$$

hence we obtain (37). ■

Junge’s idea can also be used to obtain many more variants. For instance let $k \geq 1$ be fixed. Consider the following property of an operator space E : There is a constant C such that for any n and any bounded operator $v: M_n \rightarrow M_n$ we have

$$\|v \otimes I_E\|_{M_n(E) \rightarrow M_n(E)} \leq C\|v\|_k = C\|v\|_{M_n(M_k) \rightarrow M_n(M_k)}.$$

Then this property holds iff there is an operator space F completely isomorphic to E with $d_{cb}(E, F) \leq C$ such that for some compact set T , F is a quotient of a subspace of $C(T) \otimes_{\min} M_k$. In particular if $k = 1$ this result answers a question that I had raised in the problem book of the Durham meeting in July 92. Here is a sketch of a proof: By ultraproduct arguments, an operator space E satisfies $d_{cb}(E, F) \leq C$ for some F subspace of a quotient of $C(T) \otimes_{\min} M_k$ for some T iff for any integer n and any maps $v_1: M_n^* \rightarrow E$ and $v_2: E \rightarrow M_n$ we have

$$\alpha_k(v_2 v_1) \leq C \|v_1\|_{cb} \|v_2\|_{cb}.$$

Equivalently, this holds iff

$$|tr(v_2 v_1 T)| \leq C \|v_1\|_{cb} \|v_2\|_{cb} \alpha_k^*(T)$$

for any $T: M_n \rightarrow M_n^*$. The proof can then be completed using the factorization property of α_k^* as above. (See [Her] for similar results in the category of Banach spaces.)

We refer the reader to M. Junge's forthcoming paper for more results of this kind.

Note. (added December 11 1993) In a very recent joint paper with Marius Junge, we prove that OS_n (and even HOS_n) is non-separable when $n > 2$. See [JP].

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Gilles PISIER
Texas A. and M. University
College Station
TX 77843, U. S. A.

Université Paris 6
Equipe d'Analyse, Boîte 186
75252 Paris Cedex 05