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# CYCLIC HOMOLOGY AND MODULI SPACES OF RIEMANN SURFACES

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## 1 Introduction

Let  $\vec{\mathfrak{M}}(g)$  denote the moduli space of directed Riemann surfaces of genus  $g$ . It consists of conformal equivalence classes of triples  $[F, O, X]$ , where  $F$  is a closed Riemann surface and  $X$  is a tangent direction at some point  $O$  of  $F$ . Since the mapping class group  $\vec{\Gamma}(g) = \pi_0(\text{Diff}^+(F, O, X))$  acts freely on the contractible Teichmüller space  $\vec{\mathfrak{X}}(g)$  of marked directed surfaces, the quotient  $\vec{\mathfrak{M}}(g)$  is an orientable, open manifold of dimension  $6g - 3$  with the homotopy type of  $B\vec{\Gamma}(g)$ . The group  $\vec{\Gamma}(g)$  is better known as the mapping class group of genus  $g$  surfaces with one boundary curve.

This moduli space  $\vec{\mathfrak{M}}(g)$  can be described as a configuration space  $\mathfrak{P}(g)$  of slits in the complex plane; we recall this uniformization from [Bö 1]. A compactification  $P(g)$  was developed in [Bö 2]. It has a cell structure whose cellular chain complex resembles formally the Hochschild resolution of a non-commutative algebra without unit; and in addition, there is a cyclic operation and an involution on the set of cells.

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This analogy is strong enough to permit the definition of Hochschild homology groups  $HH_*(P(g))$  and cyclic homology groups  $HC_*(P(g))$  for these complexes. They are related to their (topological) homology groups  $H_*(P(g))$  by long exact sequences

$$\cdots \longrightarrow HHN_*(P(g)) \longrightarrow HH_*(P(g)) \longrightarrow H_{*-1}(P(g)) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow HH_*(P(g)) \longrightarrow HC_*(P(g)) \longrightarrow HC_{*-2}(P(g)) \longrightarrow \cdots$$

in which  $HHN_*(P(g))$  is the so-called naive Hochschild homology. Our intention is to use the apparatus of cyclic homology theory to study the spaces  $P(g)/W(g)$ , which are Poincaré dual to the moduli spaces  $\vec{\mathfrak{M}}(g)$ . Here we merely report on some basic ideas.

We point out that  $\vec{\mathfrak{M}}(g)$  carries a (non-free)  $\mathbb{S}^1$ -action given by rotation of the tangent vector or  $X$ . The quotient is the moduli space of genus  $g$  surfaces with one puncture. It seems difficult to describe this action on the homeomorphic space  $\mathfrak{P}(g)$ ; but we expect this action to be related to the cyclic action on cells. Complex conjugation of conformal structures is another symmetry on  $\vec{\mathfrak{M}}(g)$ ; it is easily seen to transform to the reflection operator mentioned above.

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## 2 Moduli and parallel slit domains

We recall a specific description of the moduli space  $\vec{\mathfrak{M}}(g)$ ; the reader is referred to [Bö 1] for more details.

Let  $[F, O, X] \in \vec{\mathfrak{M}}(g)$  be a directed Riemann surface of genus  $g$ . There is a function  $u : F \longrightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \infty$  with the following properties: (1)  $u$  is harmonic away from  $O$ , and (2)  $u(z) - \operatorname{Re}(1/z)$  is smooth and vanishes at  $O$  for any local parameter  $z$  defined around  $O$  such that  $z(O) = 0$  and  $dz(x)$  is positive-proportional to  $-dx$ , where  $x$  is any non-zero tangent vector

at  $O$  representing the direction  $X$ . Such a function exists by the standard existence theorems for differentials on Riemann surfaces; it is unique up to a positive real factor and up to a real additive constant.

Let the critical graph  $K$  of the gradient flow of  $u$  consist of the dipole  $O$  and all zeroes of the flow as vertices, and all integral curves which leave zeroes are the edges. Since  $F_0 = F \setminus K$  is connected and simply-connected, the restriction of  $u$  to  $F_0$  is the real part of a holomorphic map  $w = u + i v : F_0 \rightarrow \mathbb{C}$ ;  $w$  is unique, up to another additive constant for the harmonic-conjugate  $v$  of  $u$ . The complement of  $w(F_0) \subseteq \mathbb{C}$ , — which can be described as a configuration of  $2g$  pairs of horizontal slits in the complex plane  $\mathbb{C}$ , — will comprise all moduli of the conformal class  $[F, O, X]$ .

A slit  $L_k$  is a horizontal half-line, which starts at some point  $z_k = (x_k, y_k) \in \mathbb{C}$ , and is unbounded to the left. There are always  $4g$  slits, paired by a fixed point free involution  $\lambda$  in the symmetric group  $\mathfrak{S}_{4g}$ , acting on the index set  $\mathbf{I} = \{1, \dots, 4g\}$ . A configuration is subject to two conditions:

- (1)  $y_k \leq y_{k+1}$
- (2)  $x_k = x_{\lambda(k)}$

So far no assumption is made about the slits being disjoint or different.

To associate a surface  $F(L)$  to  $L$  we glue, for each pair  $k$  and  $\lambda(k)$ , the upper (resp. lower) bank of  $L_k$  to the lower (resp. upper) bank of  $L_{\lambda(k)}$ . As basepoint we choose  $O = \infty$  and  $X$  is the direction of  $-dx$  under the local parameter  $\zeta \mapsto 1/\zeta$ . If  $F(L)$  is a (non-singular) surface, it inherits from  $\mathbb{C}$  a conformal structure, and thus  $[F, O, X] \in \mathfrak{M}(g)$ . In case  $F(L)$  has singularities, or if it is a surface of a genus smaller than  $g$ , we call  $L$  degenerate.

The following conditions (3) and (4) guarantee that neither  $O$  nor any finite point of  $F(L)$  is singular and that  $F(L)$  has maximal genus  $g$ . Define a new permutation  $\sigma = \lambda \circ t$ , where  $t$  denotes the cyclic rotation  $k \mapsto k+1 \pmod{4g}$ . Let  $\kappa(\lambda)+1$  denote the number of cycles of  $\sigma$ , which can be any even number between 0 and  $2g$ . We admit only pairings  $\lambda$  for which

- (3)  $\kappa(\lambda) = 0$

holds; such a  $\lambda$  is called connected.

The next condition excludes certain subconfigurations.

- (4) *There is no index  $k$  such that :*  
 $\lambda(k) = k + 2, \quad L_k = L_{k+2} \quad \text{and} \quad L_{k+1} \subseteq L_k.$

In [Bö 2] we examined in detail, what type of singular surfaces occur if (3) or (4) is violated.

It is obvious from the gluing process that two distinct configurations can lead to conformally equivalent surfaces. In this case they are connected by a chain of moves (called Rauzy-moves) of the following type: if  $L_{k-1} \subseteq L_k$  then  $L_{k-1}$  can jump to the upper bank of the slit  $L_{\lambda(k)}$ . In its new position it will be contained in  $L_{\lambda(k)}$ , and all slits overtaken by this move change their index by a cyclic rotation, and  $\lambda$  is conjugated accordingly. Such a move leaves  $F(L)$  certainly invariant. The equivalence classes generated by Rauzy moves are denoted by  $\mathfrak{L} = [L_1, \dots, L_{4g} | \lambda]$ . A class is called non-degenerate, if none of its representatives violates (3) or (4). In the older literatur such a class is called a parallel slit domains.

On the space of all parallel slit domains the contractible 3-dimensional group of similarities of  $\mathbb{C}$  acts freely as a group of conformal invariants. It is generated by translations in the  $x$ - and  $y$ -direction and by dilations; the parameters of such a transformation correspond precisely to the three undetermined constants in the complex potential  $w$ . We therefore introduce the following normalizations.

- (5)  $y_1 = 0$   
 (6)  $y_{4g} = 1$   
 (7)  $\min\{x_k\} = 0$

These conditions are invariant under moves, and thus conditions on a class. For a non-degenerate class we always have  $y_1 < y_{4g}$ , enabling us to normalize as in (5) and (6). The main result of [Bö 1] is that the space of all non-degenerate, normalized configuration classes is homeomorphic to the moduli space  $\vec{\mathfrak{M}}(g)$ .

It will be convenient for the compactification to introduce the additional condition

- (8)  $\max\{x_k\} < 1.$

This selects a subspace, which is homeomorphic to the entire space by re-parametrizing the real parts of the slit end points. We denote this subspace of all classes satisfying (1) to (8) by  $\mathfrak{P}(g)$ , and use this configuration space as a model for the moduli space  $\vec{\mathfrak{M}}(g)$ .

### 3 The cyclic structure of the compactification

The space  $\mathfrak{P}(g)$  can be compactified by taking its closure in the space of all normalized classes of configurations: one simply forgets about the condition (4) and replaces (8) by the weak inequality

$$(9) \quad \max\{x_k\} \leq 1.$$

This compactification of the moduli space is denoted by  $P(g)$ . Let  $D(g)$  be the subspace of degenerate classes. The subspace  $N(g)$ , consisting of all classes with  $\max\{x_k\} = 1$ , is a partial boundary of the manifold  $\mathfrak{P}(g)$ . The subspace  $U(g)$  of all classes such that  $\max\{x_k\} = 0$  (which we call uni-level surfaces) is a homotopy retract of  $P(g)$ , see [Bö 2].  $U(g)$  and  $N(g)$  are disjoint, but  $N(g)$  and  $D(g)$  are not.  $W(g) = D(g) \cup N(g)$  is called the periphery of  $\mathfrak{P}(g)$ , because  $\mathfrak{P}(g) = P(g) \setminus W(g)$ . Since  $\mathfrak{P}(g)$  is an orientable manifold of dimension  $6g - 3$ , Poincaré duality implies  $H^*(P(g), W(g)) \cong H_{6g-3-*}(\mathfrak{P}(g))$  for all coefficients, see [Bö 2].

It was shown in [Bö 2] that  $P(g)$  is a finite cell complex, the cells of which were denoted by symbols

$$(10) \quad E = [a_0, a_1, \dots, a_{n+1} \mid \lambda \mid B_0, B_1, \dots, B_{m+1}] = [a \mid \lambda \mid B].$$

If the slits lie on  $n + 2$  distinct  $y$ -levels – the 0-th being  $y = 0$ , the  $(n + 1)$ -st being  $y = 1$  – then  $a_i$  is the number of slits on the  $i$ -th level. Thus  $0 < a_i < 4g$ ,  $\sum_{i=0}^{n+1} a_i = 4g$  and  $0 \leq n \leq 4g - 2$ . Similarly, if the slits start at  $m + 2$  distinct  $x$ -levels – the 0-th being  $x = 0$ , and the  $(m + 1)$ -st being  $x = 1$ , although there may be none on this last vertical – then  $B_j$  is the subset of indices whose slits start on the  $j$ -th vertical. The  $B_j$  are a  $\lambda$ -invariant decomposition of  $\mathbf{I}$ , non-empty for  $j = 0, \dots, m$ ; and  $0 \leq m \leq 2g - 1$ . Taking the distances between these horizontals resp.

verticals as barycentric coordinates, the cell  $E$  becomes a product of two open simplices,  $E \cong \Delta^n \times \Delta^m$ . We call  $n$  the vertical and  $m$  the horizontal dimension of  $E$ ; and  $\ell = n + m$  is its dimension.

Since a Rauzy move changes some of the numbers  $a_i$ , some of the sets  $B_j$  and conjugates  $\lambda$ , this notation (10) for a cell is not unique, which is indicated by the brackets referring to the equivalence relation generated by Rauzy moves.

On the other hand, this notation makes it obvious, how similar this cell structure is to several well-known constructions like the bar-construction or the Hochschild resolution of an algebra, as we shall see by looking at the face operators.

There are face operators  $\partial'_i$  for the first factor  $\Delta^n$  and  $\partial''_j$  for the second factor  $\Delta^m$  of  $E$ , for  $i = 0, \dots, n$  resp. for  $j = 0, \dots, m$  :

$$(11) \quad \partial'_i(E) = [a_0, \dots, a_i + a_{i+1}, \dots, a_{n+1} \mid \lambda \mid B_0, \dots, B_{m+1}],$$

$$(12) \quad \partial''_j(E) = [a_0, \dots, a_{n+1} \mid \lambda \mid B_0, \dots, B_j \cup B_{j+1}, \dots, B_{m+1}].$$

The cyclic structure of this cell complex comes from the cyclic operator  $\tau$  defined by

$$(13) \quad \tau(E) = [a_{n+1}, a_0, \dots, a_n \mid t^{a_{n+1}} \circ \lambda \circ t^{-a_{n+1}} \mid t^{a_{n+1}} B_0, \dots, t^{a_{n+1}} B_{m+1}],$$

where  $t \in \mathfrak{S}_{4g}$  is the maximal cyclic permutation  $k \mapsto k + 1$  used earlier.  $\tau$  moves the last package of slits on the level  $y = 1$  to the bottom to become the first one; it follows that it is well-defined with respect to Rauzy moves. The cycle number of  $\sigma' = \tau^{a_{n+1}} \circ \lambda \circ \tau^{-a_{n+1}} \circ \tau$  is the same as that of  $\sigma = \lambda \circ \tau$ , thus  $\kappa(t^{a_{n+1}} \circ \lambda \circ t^{-a_{n+1}} \circ t) = \kappa(\lambda) = 0$ . The sets  $B_j$  are invariant under the new  $\lambda' = \tau^{a_{n+1}} \circ \lambda \circ \tau^{-a_{n+1}}$ .

$\tau$  acts essentially on the first factor of  $E$ , in accordance with the general philosophy that this factor seems to hold more information.

The order of  $\tau$  on a cell  $E$  is not its dimension, but determined by its vertical dimension,

$$(14) \quad \tau^{n+2} = id.$$

The subspaces  $D(g)$ ,  $N(g)$ ,  $U(g)$  and  $W(g)$  are subcomplexes of this cell decomposition and invariant under  $\tau$ . The presence of singular subconfigurations as described in (4) is independent of the values of barycentric coordinates and therefore a property of a cell  $E$ ; furthermore, such subconfigurations are then also present in each face of the cell and in the cell  $\tau(E)$ .  $N(g)$  resp.  $U(g)$  can be characterized by the properties  $B_{m+1} \neq \emptyset$  resp.  $m = 0$  of their cells; both properties are invariant under the face operators and the cyclic operator.

The relations between face operators and the cyclic operator are recorded in the following easily proved

**LEMMA 1.**

$$\begin{aligned}
 (15) \quad \partial'_i \circ \partial'_j &= \partial'_{j-1} \circ \partial'_i & \text{for } 0 \leq i < j \leq n, \\
 (16) \quad \partial'_i \circ \partial'_i &= \partial'_i \circ \partial'_{i+1} & \text{for } 0 \leq i \leq n, \\
 (17) \quad \partial''_i \circ \partial''_j &= \partial''_{j-1} \circ \partial''_i & \text{for } 0 \leq i < j \leq m, \\
 (18) \quad \partial''_j \circ \partial''_j &= \partial''_j \circ \partial''_{j+1} & \text{for } 0 \leq j \leq m, \\
 (19) \quad \partial'_i \circ \partial''_j &= \partial''_j \circ \partial'_i & \text{for } 0 \leq i \leq n, \quad 0 \leq j \leq m, \\
 (20) \quad \tau \circ \partial'_i &= \partial'_{i+1} \circ \tau & \text{for } 0 \leq i \leq n-1, \\
 (21) \quad \tau \circ \partial'_n &= \partial'_0 \circ \tau^2 \\
 (22) \quad \tau \circ \partial''_j &= \partial''_j \circ \tau & \text{for } 0 \leq j \leq m.
 \end{aligned}$$

It is not clear, how this cyclic structure fits into the general theory of cyclic sets and cyclic spaces as developed in [C], [B], [DHK], [G], [J] and others. There are no degeneracy operators, since the complex  $P(g)$  is finite dimensional. The degeneracies seem to be important to put an  $\mathbb{S}^1$ -action on the geometric realization, see [J]. We point out that the cyclic structure restricted to the subcomplex  $U(g)$  is closer to the general theory; only the order of  $\tau$  is  $n+2$  instead of  $n+1$ , what can be regarded as an effect of our normalization, i.e. the cone of  $U(g)$  is a cyclic set in the sense of [C]. On the other hand, certain other cyclic constructions are used, where the order of  $\tau$  differs from the general theory, e.g. the edgewise subdivision in [BHM].

The  $\mathbb{S}^1$ -action on the moduli space  $\vec{\mathfrak{M}}(g)$  is given by rotating the tangent direction  $X$ , i.e.  $\alpha \cdot [F, O, X] = [F, O, \alpha X]$  for an angle  $\alpha \in \mathbb{S}^1$ . It is well-defined, since the tangent bundle of  $F$  is a complex vector bundle. This action is not free; whenever  $O$  is a fixed point under some (necessarilly)



cyclic automorphism of  $F$  order  $r$ , then  $\mathbb{Z}/r\mathbb{Z} \leq \mathbb{S}^1$  is the isotropy group of  $[F, O, X]$  for any direction  $X$ .

But at this point we do not know, how this action transforms to this specific parametrization  $\mathfrak{P}(g)$  of the moduli space.

**Remark.** There is also a free  $\mathbb{S}^1$ -action on the "homotopy-type" of  $\mathfrak{P}(g)$ . The mapping class group  $\vec{\Gamma}(g)$  is the central extension of the pointed mapping class group  $\Gamma^1(g) = \pi_0(\text{Diff}^+(F, O))$  by an infinite cyclic group generated by a Dehn-twist along a null-homotopic curve enclosing the point  $O$ . Thus  $B\vec{\Gamma}(g)$  is the total space of an  $\mathbb{S}^1$ -bundle. But the rotation does not lift to a free flow on the Teichmüller space  $\vec{\mathfrak{T}}(g)$ ; only the isotropy is disjoint from the integral part  $\mathbb{Z} \leq \mathbb{R}$ .

## 4 The chain complex of $P(g)$

Let  $\mathbb{K}$  be any commutative ring with unit, and let  $S(g)$  be the chain complex with  $S_\ell(g)$  the free  $\mathbb{K}$ -module generated by all cells  $E$  of  $P(g)$  of dimension  $\ell = n + m$ . The boundary operator  $\partial : S_\ell(g) \rightarrow S_{\ell-1}(g)$  is given by

$$(23) \quad \partial = \partial' + (-1)^n \partial''$$

with

$$(24) \quad \partial' = \sum_{i=0}^n (-1)^i \partial'_i \quad \text{and} \quad \partial'' = \sum_{j=0}^m (-1)^j \partial''_j$$

Since  $\partial'$  and  $\partial''$  commute by (19), we have  $\partial \circ \partial = 0$ . We call  $\partial$  the topological boundary operator, since  $H_*(S(g), \partial) = H_*(P(g))$ .

We now exploit the fact that  $S(g)$  looks formally similiar to the Hochschild resolution of some algebra  $A$ , if we interpret the entry  $a = (a_0, \dots, a_{n+1})$  as a tensor in  $A^{\otimes(n+2)}$ . The Hochschild boundary operator  $b : S_\ell(g) \rightarrow S_{\ell-1}(g)$  is defined as

$$(25) \quad b = \partial' + (-1)^{n+1} \partial'_0 \tau + (-1)^n \partial''$$

Using the commutation relations of Lemma 1 it is straightforward that  $b \circ b = 0$ . We denote the complex  $S(g)$  with the boundary operator  $b$  by  $S^b(g)$ , and call its homology  $HHN_*(P(g)) = H_*(S(g), b)$  the naive Hochschild homology of  $P(g)$ .

Denote by  $T = (-1)^{n+2}\tau$  be the (signed) cyclic operator, by  $D = id - T$  the invariance operator, and by  $N = id + T + T^2 + \dots + T^{n+1}$  the norm operator. Then we obtain

**LEMMA 2.**

$$(26) \quad b \circ D = D \circ \partial$$

$$(27) \quad \partial \circ N = N \circ b$$

Proof: The arguments in [LQ] carry over verbatim.

Thus we can form the double complex  $C(g)$  with  $C_{p,q}(g) = S_q(g)$  for all  $p$ , and its boundary operator  $d : C_{p,q}(g) \longrightarrow C_{p-1,q} \oplus C_{p,q-1}(g)$  is given by  $d = D - \partial$  for odd  $p$ , and  $d = N + b$  for even  $p$ . Let the cyclic homology of  $P(g)$  be the homology of the associated total complex,  $HC_*(P(g)) = H_*(Tot(C(g)), d)$ .

Different from the classical situation is that the complex  $S(g)$  is not acyclic; in fact, its homology is precisely what interests us. Note that the total complex is periodic in dimensions above  $6g - 3$ , since  $S(g)$  vanishes there.

There is an earlier definition of cyclic homology, which uses the complex of  $\tau$ -coinvariants instead of the double complex  $C(g)$ . Denote by  $\tilde{S}(g)$  the quotient complex with  $\tilde{S}_\ell(g) = S_\ell(g)/im(D)$ . Because of (26)  $\tilde{S}(g)$  inherits from  $S(g)$  a boundary operator  $\tilde{b}$ .

**PROPOSITION 1.**

**If  $\mathbb{Q} \subseteq \mathbb{K}$ , then  $\tilde{S}(g)$  and  $Tot(C(g))$  are chain equivalent.**

Proof: One applies the usual argument, that the rows of  $C(g)$  form a free resolution of the cyclic groups  $\mathbb{Z}/\ell\mathbb{Z}$ . In our case, however, there are several groups involved per row. Let the terms  $S_\ell(g) = \bigoplus_{n+m=\ell} S_{n,m}(g)$  in the  $\ell$ -th row be decomposed according the vertical and horizontal bigrading of their cells. The summands are no subcomplexes (neither for  $\partial$  nor for  $b$ ), but they are invariant under both  $D$  and  $N$ . Thus for each  $\ell$  and  $n$  the summands  $S_{n,\ell-n}(g)$  form in the  $\ell$ -th row of  $C(g)$  a free, periodic resolution of the group  $\mathbb{Z}/\ell\mathbb{Z}$  with alternating differential  $D$  and  $N$ . The "row spectral sequence" of

$C(g)$ , which converges to  $H_*(\mathbb{Z}/\ell\mathbb{Z}; \mathbb{K})$ , is trivial as soon as  $\mathbb{Q} \subseteq \mathbb{K}$ . It follows that the homology of  $Tot(C(g))$  is isomorphic to the homology of  $\tilde{S}(g)$ .

**Remark.** In search of an algebra behind all this we may perhaps first concentrate on the subspace  $U(g)$ . The cells are then all of type  $(n, 0)$ , and we write  $E = [a|\lambda] = [a_0, \dots, a_{n+1}|\lambda]$ . Let  $\mathcal{A} \subseteq \mathbb{K}[X]$  be the ideal generated by  $X$ . We write  $X^a$  for the tensor  $X^{a_0} \otimes \dots \otimes X^{a_{n+1}}$  in  $\mathcal{A}_n = \mathcal{A}^{\otimes(n+2)}$ . To involve  $\lambda$ , we consider the  $\mathbb{K}$ -module  $\Lambda$  in the group ring  $\mathbb{K}[\mathfrak{S}_{4g}]$  generated by all connected pairings, i.e. free involutions  $\lambda \in \mathfrak{S}_{4g}$  with  $\kappa(\lambda) = 0$ . Consider now the ideal  $\mathcal{I}_n \subseteq \mathcal{A}_n \otimes \Lambda$  generated by all differences  $X^a \otimes \lambda - X^{a'} \otimes \lambda'$  such that  $[a|\lambda]$  and  $[a'|\lambda']$  are related by a Rauzy-move. If  $Q_n$  is the quotient by this ideal, then the complex  $Q = (Q_n)$  inherits from the Hochschild resolution of  $\mathcal{A}$  a "topological" boundary  $\partial$  and a Hochschild boundary  $b$ . This is the complex we study.

## 5 The Connes - Gysin diagram

To relate the three homologies  $H$ ,  $HHN$  and  $HC$  we consider the following complexes and chain maps. Let the complex  $S^\sharp(g)$  consist of the first two columns of  $C(g)$ , with  $d$  as boundary operator; its homology is denoted by  $HH_*(P(g))$  and called the Hochschild homology of  $P(g)$ . Let  $Tot'(C(g))$  be the total complex associated to the double complex  $C(g)$  minus the first column, graded such that  $C_{1,0}(g) = S_0(g)$  is the degree zero part; its homology will be denoted by  $HC'(P(g))$ . In the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & S^b(g) & \longrightarrow & S^\sharp(g) & \longrightarrow & \Sigma S(g) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S^b(g) & \longrightarrow & Tot(C(g)) & \longrightarrow & \Sigma Tot'(C(g)) \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Sigma^2 Tot(C(g)) & = & \Sigma^2 Tot(C(g)) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

all maps are inclusions, except for two shift maps.

The first,  $sh : Tot(C(g)) \longrightarrow \Sigma Tot'(C(g))$ , is the chain map induced by shift-

ting the columns of the double complex  $C(g)$  one column to the left. And  $sh^2 : Tot(C(g)) \rightarrow \Sigma^2 Tot(C(g))$  is the periodicity self-map of  $Tot(C(g))$ . Here  $\Sigma$  denotes the suspension (or shift) of a complex. The diagram is commutative, and we obtain

**PROPOSITION 2.**

**There is the a diagram of long exact sequences**

$$\begin{array}{ccccccc}
 \longrightarrow & HHN_*(P(g)) & \longrightarrow & HH_*(P(g)) & \longrightarrow & H_{*-1}(P(g)) & \longrightarrow \\
 & \parallel & & \downarrow & & \downarrow & \\
 \longrightarrow & HHN_*(P(g)) & \longrightarrow & HC_*(P(g)) & \longrightarrow & HC_{*-1}^*(P(g)) & \longrightarrow \\
 & & & \downarrow & & \downarrow & \\
 & & & HC_{*-2}(P(g)) & = & HC_{*-2}^*(P(g)) & \\
 & & & \downarrow & & \downarrow & 
 \end{array}$$

In the classical situation of an algebra the term  $H_*(P(g))$  would vanish and the diagram would reduce to the long exact Connes-Gysin sequence.  $sh_*^2$  is then Connes' periodicity operator. Since the total complex  $Tot(C(g))$  is periodic in high dimensions, the shift induces an isomorphism.

**PROPOSITION 3.**

**The double shift**

$$sh_*^2 : HC_*(P(g)) \longrightarrow HC_{*-2}(P(g))$$

**is an isomorphism for  $* > 6g - 5$ .**

As a consequence, the cyclic homology  $HC_*(P(g))$  differs from the periodic cyclic homology  $HC_*^{per}(P(g)) = \lim_{\ell} HC_{*+2\ell}(P(g))$  only in dimensions  $* < 6g - 3$ .

It is clear that everything said so far is also true, if we replace the space  $P(g)$  by any of the spaces  $P(g)/W(g)$ ,  $P(g)/D(g)$ , or  $U(g)$  note that the dimension of the last two spaces is  $4g - 2$ . Recall that  $P(g)/W(g)$  is a Poincaré dual to  $\vec{\mathfrak{M}}(g)$ ; thus for  $\mathbb{K}$  a field we obtain results about the homology of the moduli space.

## 6 Dihedral and quaternionic homology

We mention another operator, the reflection  $W = (-1)^{\frac{1}{2}(n+1)(n+2)}\omega$ , where  $\omega$  acts on a cell  $E$  by

$$(28) \quad \omega(E) = [a_{n+1}, a_n, \dots, a_1, a_0 \mid \omega \circ \lambda \circ \omega^{-1} \mid \omega(B_0), \dots, \omega(B_{m+1})]$$

Here  $\omega \in \mathfrak{S}_{4g}$  is the involution  $k \mapsto 4g + 1 - k$ .

**LEMMA 3.**

$$(29) \quad W \circ \partial'_i = \partial'_{n-i} \circ W \quad \text{for} \quad 0 \leq i \leq n,$$

$$(30) \quad W \circ \partial''_j = \partial''_j \circ W \quad \text{for} \quad 0 \leq j \leq m.$$

It follows that

$$(31) \quad W \circ \partial' = (-1)^n \partial' \circ W,$$

$$(32) \quad W \circ \partial'' = \partial'' \circ W.$$

One can define dihedral and quaternionic homology groups for the complexes  $P(g)$  and the various subcomplexes and quotient complexes, following [L 1]. The easiest case is  $U(g)$ . It is obvious, that  $W$  corresponds to the complex conjugation in  $\mathfrak{P}(g)$ ; the conjugation of the conformal structure of a parallel slit domain is just the complex conjugation of its slits in  $\mathbb{C}$ , which amounts to reading the slits in reversed order from top to bottom.

## REFERENCES

- [Bö 1 ] C.-F. Bödigheimer: *On the topology of moduli spaces of Riemann surfaces. Part I : Hilbert Uniformization.* Math. Gott. SFB 170 Göttingen, Hefte 7+8 (1990).
- [Bö 2 ] C.-F. Bödigheimer: *The harmonic compactification of the moduli space of Riemann surfaces.* preprint (1992).
- [BHM ] M. Bökstedt, W. C. Hsiang, I. Madsen: *The cyclotomic trace and algebraic K-theory of spaces.* preprint (1989/90).

- [B ] D. Burghlea: *Cyclic homology of the group rings*. Comment. Math. Helv. 60 (1985), 354-365.
- [C ] A. Connes: *Cohomologie cyclique et foncteurs  $\text{Ext}^n$* . C. R. Acad. Sc. Paris 296 (1983), 953-958.
- [DHK ] W. G. Dwyer, M. J. Hopkins, D. M. Kan: *The homotopy theory of cyclic sets*. Trans. AMS 291 (1985), 281-289.
- [G ] T. Goodwillie: *Cyclic homology, derivations and the free loop space*. Topology 24 (1985), 187-215.
- [J ] J. D. S. Jones: *Cyclic homology and equivariant homology*. Invent. Math. 87 (1987), 403-423.
- [LQ ] J.-L. Loday, D. Quillen: *Cyclic homology and the Lie algebra homology of matrices*. Comment. Math. Helv. 59 (1984), 565-591.
- [L 1 ] J.-L. Loday: *Homologies diédrale et quaternionique*. Adv. Math. 66 (1987), 119-148.
- [L 2 ] J.-L. Loday: *Cyclic Homology*. Springer Grundlehren math. Wiss. Bd. 301 (1992).

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