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Hopf structure on the Van Est spectral sequence in K-theory

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In this paper, we study the van Est spectral sequence and its close relationship to K-theory and cyclic homology. The bicommutative Hopf algebra structure on the Van Est spectral sequence induces a long exact sequence of indecomposables. This leads us to Diagram C below and a proof of Karoubi's conjecture on the duality relationship of multiplicative K-theory and smooth group cohomology in some restricted cases. In the last section we reinterpret the Van Est spectral sequence as the Serre spectral sequence of a fibration of simplicial spaces. This paper is a sequel to [Ti] to which the reader is refered for further motivation.

I would like to thank the organizers for the opportunity to present these results at the conference.

1. Results.

Here we describe briefly the main results. The reader familiar with the earlier paper will see easily how Diagram C below is an improvement on Diagram B in [Ti] as now all rows are exact. The unfamiliar reader might find it helpful to read Sections 3 and 4 first which we hope are of interest in their own right and where we review many definitions in more detail.

Let A be a Banach or Fréchet algebra over $\mathbb{R}$, $GLA = \lim_{n\to\infty} GL_nA$ be the general linear group over A, and $\mathfrak{gl}A = \lim_{n\to\infty} \mathfrak{gl}_nA$ be its Lie algebra. If M_1 and M_2 are two infinite matrices over A, denote by $M_1 \oplus M_2$ the infinite matrix that acts on even coordinates like M_1 and on odd coordinates like M_2. This sum operation defines a map of groups $\oplus : GLA \times GLA \longrightarrow GLA$ and also of Lie algebras $\oplus : \mathfrak{gl}A \times \mathfrak{gl}A \longrightarrow \mathfrak{gl}A$. It is well known that over a field (here to be taken $\mathbb{R}$ or $\mathbb{C}$) this product and the diagonal map give the structure of a bicommutative Hopf algebra to the homology H_*GLA of the underlying topological space, to the Lie algebra homology $H_*^{Lie}\mathfrak{gl}A$, and to the group homology H_*BGLA_δ of the discrete group GLA_δ. The product is commutative as $M_1 \oplus M_2$ and $M_2 \oplus M_1$ only differ by conjugation via a permutation matrix. (See [L] for a proof in the case of $H_*^{Lie}\mathfrak{gl}A$).

In order to study the Van Est spectral sequence, we are more interested in cohomology, i.e. de Rham cohomology H^*GLA, continuous Lie algebra cohomology $H^*_{Lie}\mathfrak{gl}A$, and smooth group cohomology $H^*_{sm}GLA$. We will assume that *these are of finite type.* This ensures that $\oplus$ induces a well-defined comultiplication on these three algebras. For the Van Est spectral sequence we will also have to assume that *the de Rham complex of differential forms is split.* (See Section 4.) Let GLA_0 denote the identity component of GLA. The invariant differential forms on GLA_0 can be identified with the exterior algebra on $\mathfrak{gl}A$, and all three cohomology groups associated with GLA_0 are connected, that is H^0 is one dimensional.

PROPOSITION 1. *The van Est spectral sequence for GLA_0 is a spectral sequence of connected, bicommutative Hopf algebras with*

$$E_2 = H^*_{sm}GLA_0 \otimes H^*GLA_0 \quad \Longrightarrow \quad H^*_{Lie}\mathfrak{gl}A.$$

PROOF: The direct sum operation $\oplus : GLA_0 \times GLA_0 \to GLA_0$ is a group homomorphism and its induced map on the Lie algebra $\mathfrak{gl}A$ is again $\oplus$ as defined above, for $\exp(M_1 \oplus M_2) = \exp M_1 \oplus \exp M_2$. The proposition then follows by naturallity of the van Est spectral sequence. (See Section 4 and [Be] for more details.) ◇

COROLLARY 2. *There is an exact sequence of indecomposables*

$$\ldots \longrightarrow Q(H^*_{sm}GLA_0) \longrightarrow Q(H^*_{Lie}\mathfrak{gl}A) \longrightarrow Q(H^*GLA_0) \longrightarrow \ldots$$

PROOF: This follow from an application of Theorem 3.1 to the Van Est spectral sequence of Propostion 1. ◇

Using the notation Q for indecomposables and P for primitives as in Section 3, these vector spaces may now be identified as follows. By [LQ] or [T], the space of indecomposables $Q(H^*_{Lie}\mathfrak{gl}A)$ is isomorphic to the continuous cyclic cohomology groups $HC_c^{*-1}A$. For simplicity, we assume now that GLA *is connected.* Then, the cohomology H^*GLA is the dual Hopf algebra of H_*GLA and $Q(H^*GLA) = [P(H_*GLA)]^*$. Furthermore, as GLA is an associative H-space, rationally $P(H_*(GLA;\mathbb{Z}))$ is isomorphic to the homotopy groups $\pi_*GLA = K_{*+1}^{top}A$. Thus, $Q(H^*GLA)$ is essentially the dual of topological K-theory and we may use the suggestive notation $K_{top}^nA := [P(H_{n-1}GLA)]^*$. Similarly, we define $K_{alg}^nA := [P(H_nBGLA_\delta)]^*$ and $K_{rel}^n := [P(H_nGLA/GLA_\delta)]^*$. Hence, by passing to indecomposables, Diagram B in [Ti] may now be replaced by the commutative

DIAGRAM C.

$$\begin{array}{ccccccc}
HC_c^nA & \xrightarrow{I} & HH_c^nA & \xrightarrow{B} & HC_c^{n-1}A & \xrightarrow{S} & HC_c^{n+1}A \\
 & & \downarrow{\scriptstyle D_{sm}} & & \| & & \\
K_{top}^nA & \longrightarrow & Q(H_{sm}^nGLA) & \longrightarrow & HC_c^{n-1}A & \longrightarrow & K_{top}^{n+1}A \\
\| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle ch_{rel}} & & \| \\
K_{top}^nA & \longrightarrow & K_{alg}^nA & \longrightarrow & K_{rel}^nA & \longrightarrow & K_{top}^{n+1}A
\end{array}$$

Here the top row is Connes' exact sequence for continuous cyclic and Hochschild cohomology. The middle row is that of Corollary 2 reinterpreted. The bottom row is also exact and is by definition the dual of the exact sequence that relates relative, Quillen's algebraic, and periodic K-theory. The vertical maps are described as follows. D_{sm} composed with h is the dual of the Dennis trace map, and ch_{rel} is the dual of Karoubi's relative Chern character.[1]

PROOF: If indeed we had just passed to indecomposables, there would be nothing to prove as all maps would be well defined and the commutativity of Diagram C would follow from that of Diagram B in [Ti]. However, in order to stay closer to to the K-homology groups, K_{alg}^*A and K_{rel}^*A have not been defined in terms of indecomposables. We thus need to thus define h and ch_{rel}. This can be done as follows.

[1] The missing vertical lines may be filled in by the dual of Karoubi's topological Chern character. See [Ti, §5] for further comments.

In [Ti] we constructed a factorization

$$HH_c^n A \xrightarrow{D_{sm}} H_{sm}^n GLA \longrightarrow H^n BGLA_\delta$$

of the dual Dennis trace map. Composing D_{sm} with the projection onto indecomposables gives the map $HH_c^n A \to Q(H_{sm}^n GLA)$. Now, in a bicommutative Hopf algebra, the natural map from the primitives to the indecomposables is an isomorphism. Thus we may think of the indecomposables as the subspace $P(H_{sm}^n GLA)$ to get a well defined map to $H^n BGLA_\delta$. Its image is contained in $P(H^\circ)$ where H° denotes the continuous dual Hopf algebra of $H = H_* BGLA_\delta$, i.e. H° is the largest Hopf algebra contained in $H^* BGLA_\delta$. But for all Hopf algebras we have $P(H^\circ) = [Q(H)]^*$ by a theorem of Michaelis [Mi]. Hence,

$$Q(H_{sm}^n GLA) = P(H_{sm}^n GLA) \xrightarrow{h} P(H^\circ) = [Q(H)]^* = [P(H)]^* = K_{alg}^n A$$

is well defined. Similarly, the factorization of the dual relative Chern character $HC_c^{n-1} A \to H_{Lie}^n \mathfrak{gl} A \to H^n GLA/GLA_\delta$ gives rise to the map $ch_{rel} : HC_c^{n-1} A \to K_{rel}^n A$. ◊

In [K] Karoubi also defines multiplicative K-groups $MK_n A$ such that they fit into a long exact sequence

$$\ldots \longrightarrow HC_{n-1}^c A \longrightarrow MK_n A \longrightarrow K_n^{top} A \longrightarrow \ldots$$

The middle row of Diagram C is just the dual of this sequence. This gives us a partial solution to a conjecture by Karoubi that the continuous dual of the multiplicative K-theory is the smooth group cohomology. We illustrate this with an example in the next section for $A = \mathbb{C}$.

2. Example $A = \mathbb{C}$.

We consider $\mathbb{C}$ as an algebra over $\mathbb{R}$. $GL\mathbb{C}$ is connected and it is well-known that its de Rham cohomology is an exterior algebra with one generator in each odd dimension:

$$H^*(GL\mathbb{C}, \mathbb{C}) = E_{\mathbb{C}}^*(x_1, x_3, \ldots, x_p, \ldots)$$

where x_p has degree $2p-1$ and transgresses to the universal Chern class c_p in the Serre spectral sequence associated to the universal $GL\mathbb{C}$-bundle.

Using the unitary trick, one can show that with coefficients in $\mathbb{C}$ the de Rham cohomology is isomorphic to the smooth group cohomology, that is:

$$H^*(GL\mathbb{C}, \mathbb{C}) = H^*_{sm}(GL\mathbb{C}, \mathbb{C})$$

where the isomorphism is given via multiplication by $(i)^n$ in dimension n. The composition of this isomorphism and the natural map $H^*_{sm}(GL\mathbb{C}, \mathbb{C}) \to H^*(BGL\mathbb{C}_\delta, \mathbb{C})$ maps x_p to the Borel regulator element in dimension $2p-1$. (More precisely, one considers its image in $H^*(BGL\mathbb{C}_\delta, \mathbb{C}/\mathbb{R}_p)$ where $\mathbb{R}_p = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathbb{Z}_p$ is the subgroup of $\mathbb{C}$ generated by $(2\pi i)^p$. See for example [DHZ].)

Finally, we compute the Lie algebra homology of $\mathfrak{gl}\mathbb{C}$ as a vector space over $\mathbb{R}$. Using the relation of Lie algebra cohomology to cyclic cohomology, one has

$$Q(H^*_{Lie}(\mathfrak{gl}\mathbb{C}; \mathbb{C})) = HC^*(\mathbb{C}; \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$$

in even dimensions and zero otherwise. Hence, for $n = 2p-1$, the middle row of Diagram C breaks up in short exact sequences

$$(*) \qquad 0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{C} \longrightarrow 0.$$

Recall [K], Karoubi's multiplicative K-groups MK_nA fit into a long exact sequence

$$\ldots \longrightarrow K^{top}_{n+1}A \longrightarrow HC^c_{n-1}A \longrightarrow MK_nA \longrightarrow \ldots$$

For $A = \mathbb{C}$ and $n = 2p-1$ this breaks into short exact sequences

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(2\pi i)^p} \mathbb{C} \longrightarrow \mathbb{C}/\mathbb{Z}_p \longrightarrow 0.$$

The above cohomology sequence $(*)$ is clearly the continuous dual of this sequence with

$$\mathrm{Hom}^{cont}_{\mathbb{Z}}(MK_{2p-1}\mathbb{C}; \mathbb{C}) = \mathrm{Hom}^{cont}_{\mathbb{Z}}(\mathbb{C}/\mathbb{Z}_p; \mathbb{C}) = \mathbb{C} = H^{2p-1}_{sm}(GL\mathbb{C}; \mathbb{C}).$$

3. Spectral sequence of Hopf algebras.

The goal of this section is to show how a spectral sequence of cocommutative Hopf algebras leads to a long exact sequence of primitives. For this we will assume that all Hopf algebras are connected and of finite type over a field of characteristic zero.

Let E be such a Hopf algebra with coproduct $\triangle$. Its set of primitives is the set $P(E) := \{x \in E| \quad \triangle(x) = 1 \otimes x + x \otimes 1\}$, and its set of indecomposables is the set $Q(E) := \bar{E}/\bar{E}^2$ where $\bar{E}$ is the augmentation ideal. As E is of finite type its dual E^* is also a Hopf algebra and

$$P(E)^* = Q(E^*).$$

We refer the reader to [MM] and [B] for the basic theory of differential graded Hopf algebras. Recall, though, that E is called primitive (coprimitive, biprimitive) if the natural map $P(E) \to Q(E)$ is surjective (injective, bijective), and that this is equivalent to E being cocommutative (commutative, bicommutative) [MM, 4.17].

THEOREM 3.1. *Let $\{E_r, d_r\}$ be a spectral sequence of Hopf algebras converging to A with $E_n = B \otimes C$ for some $n \geq 0$, where A, B, C are biprimitive Hopf algebras of finite type. Then there are long exact sequences*

$$(*) \qquad P(A)^{n-1} \longrightarrow P(C)^{n-1} \longrightarrow P(B)^n \longrightarrow P(A)^n \longrightarrow \dots$$

and equivalently

$$(**) \qquad Q(A)^{n-1} \longrightarrow Q(C)^{n-1} \longrightarrow Q(B)^n \longrightarrow Q(A)^n \longrightarrow \dots$$

PROOF: By the Hopf decomposition theorem [B, 3.9, 3.10], every biprimitive differential graded Hopf algebra E is isomorphic as a differential graded Hopf algebra to $\bigotimes_i K_i \otimes \bigotimes_j M_j \otimes Q$ where

$$K_i = \bigwedge(x_i) \otimes k[y_i] \quad \text{with} \quad dx_i = y_i \quad \text{and} \quad dy_i = 0$$
$$M_j = \bigwedge(x_j) \otimes k[y_j] \quad \text{with} \quad dy_j = x_j \quad \text{and} \quad dx_j = 0$$
$$Q = \bigwedge(x_s) \otimes k[y_t] \quad \text{with} \quad dx_s = 0 \quad \text{and} \quad dy_t = 0.$$

Then $H(E) = Q$ as Hopf algebras, and we note:

(1) *the differential maps primitives to primitives, and the preimage of a primitive is a primitive.*

By hypothesis, $P(E_n) = P(E_n^{*,0}) \oplus P(E_n^{0,*})$. Thus, using a bigraded version of the Hopf decomposition theorem, $P(H(E_n)) = P(E_{n+1}) = P(E_{n+1}^{*,0}) \oplus P(E_{n+1}^{0,*})$ and by induction

$$P(E_\infty) = P(E_\infty^{*,0}) \oplus P(E_\infty^{0,*}) \tag{2}$$

Furthermore, by (1) the familiar exact sequence

$$0 \longrightarrow E_\infty^{0,k-1} \longrightarrow E_k^{0,k-1} \xrightarrow{d_k} E_k^{k,0} \longrightarrow E_\infty^{k,0} \longrightarrow 0$$

induces an exact sequence on primitives for $k \geq n$:

(3)
$$0 \longrightarrow P(E_\infty)^{0,k-1} \longrightarrow P(E_k)^{0,k-1} \longrightarrow P(E_k)^{k,0} \longrightarrow P(E_\infty)^{k,0} \longrightarrow 0.$$

Again using (1),

$$\begin{aligned} P(E_k)^{0,k-1} &= P(E_n)^{0,k-1} = P(C)^{k-1} \\ P(E_k)^{k,0} &= P(E_n)^{k,0} = P(B)^k \end{aligned} \tag{4}$$

as d_r, leaves these unchanged for dimension reasons when $n \leq r \leq k$.

Finally, two graded, biprimitive Hopf algebras E and $\tilde{E}$ are isomorphic if and only if $\dim E^k = \dim \tilde{E}^k$ for all k. This is immediate from the Hopf decomposition theorem. Hence,

$$P(A)^k = P(E_\infty)^k = P(E_\infty^{0,*})^k \oplus P(E_\infty^{*,0})^k \tag{5}$$

Putting (3), (4), and (5) together, we yield the long exact sequence $(*)$. ◇

Corollary 3.2. *If in Theorem 3.1, the algèbra A is merely assumed to be primitive (resp. coprimitive) then sequence $(*)$ (resp. sequence $(**)$) is still exact.*

Proof: Recall (the dual of) Corollary 2.2 in [B]: If E and $\tilde{E}$ are two primitive Hopf algebras with $\dim E^k = \dim \tilde{E}^k$ then E and $\tilde{E}$ are isomorphic as coalgebras. The last part of the argument in the proof of Theorem 3.1 can then be changed to read as follows: As A and E_∞ are two primitive Hopf algebras of the same rank, they are isomorphic as coalgebras and hence $P(A) = P(E_\infty)$. ◇

THEOREM 3.3. *If in Theorem 3.1 all three algebras, A, B, C, are merely assumed to be primitive (resp. coprimitive), then sequence $(*)$ (resp. sequence $(**)$) is still exact.*

PROOF: Recall from [B, 2.2, 3.8] that every primitive differential graded Hopf algebra (E, d) is isomorphic to its biprimitive form $^\circ E$ as differential graded coalgebra. Furthermore, there is a spectral sequence $\{\mathcal{E}_r(E), d_r\}$ of biprimitive Hopf algebras with $\mathcal{E}_0(E) = {}^\circ E$ converging to the biprimitive form $^\circ H(E)$ of its homology.

The theorem now follows by induction on r where at each stage we compute E_r from E_{r-1} via the corresponding spectral sequence of biprimitive Hopf algebras to which we can apply Theorem 3.1. As E_r is isomorphic to $^\circ E_r$ as a differential graded coalgebra, $P(E_r) = P(^\circ E_r)$. This and the various exact sequences that result from the application of Theorem 3.1 prove the result after some diagram chasing. ◇

REMARK 3.4: The above theorems have obvious dual versions where the spectral sequence is replaced by a spectral sequence corresponding to a homology theory. In this case the arrows in sequences $(*)$ and $(**)$ are reversed.

EXAMPLE 3.5: The motivating example was that of the Serre spectral sequence of a multiplicative fibre map of H-spaces $F \to E \to B$ over the rationals with F and B connected.

Recall that the rational homology of an H-space X is a primitive Hopf algebra, the diagonal map inducing the commutative comultiplication and the H-space structure giving rise to the multiplication. Furthermore, its primitives $P(H_*(X, \mathbb{Q}))$ are isomorphic to the rational homotopy groups $\pi_* X \otimes \mathbb{Q}$ [MM, Appendix]. Hence, in the case of a multiplicative fibre map, Theorem 3.3 gives us the long exact sequence of rational homotopy groups

$$\ldots \longrightarrow \pi_n F \otimes \mathbb{Q} \longrightarrow \pi_n E \otimes \mathbb{Q} \longrightarrow \pi_n B \otimes Q \longrightarrow \ldots .$$

4. Van Est spectral sequence as Serre spectral sequence.

In this section it is shown that the Van Est spectral sequence may be interpreted as a Serre spectral sequence in continuous cohomology. More details on the simplicial spaces and complexes below may also be found in [Ti] Sections 2 and 4.

Recall, if $Y_\bullet$ is a simplicial space then its continuous cohomology with coefficients in a topological vector space V is defined by

$$H_c^*(Y_\bullet, V) = H_*(C^*(Y_\bullet, V), \delta)$$

where $C^n(Y_\bullet, V)$ is the set of continuous functions $f : Y_n \to V$, and δ is the usual boundary map induced by the face maps of $Y_\bullet$.

For example, the continuous cohomology of a topological group G is by definition the continuous cohomology (in the above sense) of the simplicial space $E_\bullet G/G = B_\bullet G$, where $E_\bullet G$ is the bar construction on G. That is,

$$H_c^*(G, V) = H_c^*(B_\bullet G, V).$$

Another important example is the simplicial space of singular smooth simplices of a manifold X with the C^∞ compact-open topology. We will simply denote this simplicial space by $X_\bullet$. Then the smooth singular cohomology of X is isomorphic to the continuous cohomology of $X_\bullet$:

$$H^*X = H_c^*(X_\bullet)$$

at least when X has the smooth homotopy type of an open subset of a Fréchet space [BS3, 1.6]. Here, and from now on, real coefficients are understood.

Recall from [Be], a complex of topological vector spaces (C^*, d) is called split if the complex is homotopy equivalent to its homology, or equivalently, if $C^n = B^n \oplus H^n(C^*) \oplus \tilde{B}^{n+1}$ with $B^n \subset ker(d)$ and $\tilde{B}^{n+1}$ isomorphic to B^{n+1}.

Proposition 4.1. *Let G be a Lie group such that*

(1) Ω^*G and $C^*(G_\bullet)$ are split, and
(2) $H^*G = H^*_c G_\bullet$.
Then integration $deR : \Omega^*G \to C^*(G_\bullet)$ restricted to the G-invariant subcomplexes induces an isomorphism in homology. Thus, if $\mathfrak{g}$ denotes the Lie algebra of G then

$$H^*_{Lie}\mathfrak{g} \simeq H^*_c(G_\bullet/G).$$

Condition (2) seems to be the weaker condition and is automatically satisfied by all groups known to satisfy condition (1). These include the diffeomorphism group Diff(M) of a compact manifold M, the loop group LG of a compact Lie group G, and the general linear group GL_nA when A is a seperable Banach space or $A = \mathcal{C}^\infty(M)$ for a compact manifold [Be, 7.5, 8.9] [BS3, 1.5].

PROOF: Consider the commutative diagram of (double) complexes

$$\begin{array}{ccc}
(\Omega^*G)^G & \longrightarrow & C^*_\infty(E_\bullet G, \Omega^*G)^G \\
{\scriptstyle deR}\downarrow & & {\scriptstyle deR}\downarrow \\
C^*(G_\bullet)^G & \longrightarrow & C^*_\infty(E_\bullet G, C^*(G_\bullet))^G
\end{array}$$

The suffix ∞ indicates here that we take smooth instead of continuous maps. Note that $H^*_{Lie}\mathfrak{g} = H((\Omega^*G)^G)$ and $H^*_c(G_\bullet/G) = H(C^*(G_\bullet)^G)$ as G acts freely on $G_\bullet$. The horizontal maps are edge maps induced by the inclusion of a G-module into the constant functions, $V \hookrightarrow C^0_\infty(E_\bullet G, V)$. As both Ω^*G and $C^*(G_\bullet)$ are smoothly injective G-modules, these are homology equivalences (see also [BS3, §8]).

We are left to show that the right hand vertical map is a homotopy equivalence. We replace the homogenous cochains by the non-homogenous cochains:

$$C^*_\infty(B_\bullet G, \Omega^*G) \xrightarrow{deR} C^*_\infty(B_\bullet G, C^*(G_\bullet)).$$

Both Ω^*G and $C^*(G_\bullet)$ are split so that the E_1-terms of the spectral sequences, which arise from the filtration of the double comlexes by columns, are $C^*_\infty(B_\bullet G, H^*_{deR}G)$ and $C^*_\infty(B_\bullet G, H^*_c(G_\bullet))$ respectively. But by the usual de Rham Theorem and condition (2), $H^*_{deR}G = H^*_c(G_\bullet)$. Hence, deR induces an isomorphism of E_1-terms, and hence on the abutments. ◇

CAVEAT: In general, we may not replace the continuous cohomolgy with the ordinary cohomology in Proposition 4.1. For example, if $G = SL_n\mathbb{R}$ then $H^2_c(G_\bullet/G) = H^2_{Lie}M_n\mathbb{R}$ is finite dimensional but $H^2(G_\bullet/G)$ is infinite dimensional [M, Lemma 7]. This also means that $ch_{rel} : H^*_{Lie}\mathfrak{gl}A \longrightarrow H^*(GLA_\bullet/GLA)$ is in general not an isomorphism. (Recall, GLA/GLA_δ was defined to be the realization of the simplicial set $GLA_\bullet/GLA$.)

PROPOSITION 4.2. *There is natural map of spectral sequences from the Van Est spectral sequence for a Lie group G to the Serre spectral sequence in continuous cohomology for the fibration of simplicial spaces*

$$G_\bullet \longrightarrow B_\bullet G \times_\tau G_\bullet \longrightarrow B_\bullet G.$$

Furthermore, if G is connected and satisfies the conditions of Proposition 4.1 then the transformation induces an isomorphism on E_2-terms.

PROOF: Let $E_\bullet G \times_G G_\bullet$ be the orbit space of the cartesian product $E_\bullet G \times G_\bullet$ under the diagonal action of G. We can identify $E_\bullet G \times_G G_\bullet$ with the twisted cartesian product $B_\bullet G \times_\tau G_\bullet$ via

$$(g_0, \ldots, g_n, \sigma) \xrightarrow{\phi} (g_0^{-1} g_1, \ldots, g_{n-1}^{-1} g_n, g_0^{-1}\sigma)$$

where $g_i \in G$ and σ is a smooth n-simplex of G. It is easy to check that ϕ is a homeomorphism of simplicial spaces when $E_\bullet G$, $B_\bullet G$, and $G_\bullet$ are given the usual simplicial structure and the twisting function $\tau : B_n G \longrightarrow G$ is defined by

$$(g_1, \ldots, g_n) \xrightarrow{\tau} g_1^{-1}.$$

Then clearly $C^*(E_\bullet G \times_G G_\bullet) = C^*(B_\bullet G \times_\tau G_\bullet)$, and the natural map

$$C^*(B_\bullet G, C^*(G_\bullet)) = C^*(E_\bullet G, C^*(G_\bullet))^G \xrightarrow{\Delta} C^*(E_\bullet G \times G_\bullet)^G = C^*(B_\bullet G \times_\tau G_\bullet)$$

that takes the filtration by columns of the double complex on the left hand side to the usual filtration associated to a fibration on the right hand side. (We refer the reader to [BS1, §8] for details on the Serre spectral sequence in continuous cohomology. Note that in our case τ is the same as $\bar{\tau}$ as the twisting group is the constant simplicial group G.) Recall that the van Est Spectral sequence is associated to the filtration by columns of the double complex

$$C^*_\infty(B_\bullet G, \Omega^* G).$$

Thus, composition of the de Rham map deR with Δ gives us the desired natural map of spectral sequences.

If G satisfies the conditions of Proposition 4.1 then $C^*G_\bullet$ is split and Δ induces an isomorphism on E_2-terms [BS1, 8.3] with

$$E_2 = H^*_c(G, H^*G),$$

where we identified $H^*_c(G_\bullet)$ with H^*G using condition (2). Furthermore, filtering the double complex $C^*(B_\bullet G, C^*(G_\bullet))$ by rows, we get another spectral sequence which collapses at the E_2-term to $H^*_c(G_\bullet/G)$ as $C^*(G_\bullet)$ is G-injective. When G is connected, H^*G is an invariant G-module. Thus, in this case the Serre spectral sequence takes the form

$$E_2 = H^*_c G \otimes H^*G \quad \Longrightarrow \quad H^*_c(G_\bullet/G) = H^*_c(B_\bullet G \times_\tau G_\bullet).$$

while the Van Est spectral sequence has the form

$$E_2 = H^*_{sm} G \otimes H^*G \quad \Longrightarrow \quad H^*_{Lie}\mathfrak{g}.$$

By Proposition 4.1, the abutments are the same, which forces $H^*_{sm}G = H^*_c G$ by the Zeeman Comparison Theorem. This proves the second part of the proposition. $\diamondsuit$

REMARK 4.3: It follows from the proof of Proposition 4.2 and Theorem 4.3 in [Ti] that the natural projection $G_\bullet \times_G E_\bullet G \to E_\bullet G/G$ and $eval$: $G_\bullet/G \to E_\bullet G/G$ which takes an n-simplex σ to its evaluation at the vertices $(\sigma(0), \ldots, \sigma(n))$ are identical in continuous cohomology.

REMARK 4.4: It is well-known (e.g. [BW, IX §5]) that for locally compact groups G which are countable at infinity the continuous and the smooth cohomology are the same, that is

$$H^*_{sm}G = H^*_c G.$$

We proved indirectly above that this also holds for every group G satisfying the conditions (1) and (2).

Returning to the case when $G = GLA$, the direct sum product induces a map of simplicial spaces

$$B_\bullet(GLA \times GLA) \times_\tau (GLA \times GLA)_\bullet \xrightarrow{\oplus} B_\bullet GLA \times_\tau GLA_\bullet.$$

Thus we recover the Hopf structure of the Van Est spectral sequence in Section 1 as the Hopf structure on the Serre spectral sequence in continuous cohomology of a multiplicative fibre map of 'simplicial H-spaces'. Theorem 3.3 applies.

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