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Stable topological cyclic homology is topological Hochschild homology


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STABLE TOPOLOGICAL CYCLIC HOMOLOGY IS TOPOLOGICAL HOCHSCHILD HOMOLOGY

By Lars Hesselholt

1. Introduction

1.1. Topological cyclic homology is the codomain of the cyclotomic trace from algebraic $K$-theory
\[ \text{trc}: K(L) \to TC(L). \]

It was defined in [2] but for our purpose the exposition in [6] is more convenient. The cyclotomic trace is conjectured to induce a homotopy equivalence after $p$-completion for a certain class of rings including the rings of algebraic integers in local fields of positive residue characteristic $p$. We refer to [11] for a detailed discussion of conjectures and results in this direction.

Recently B. Dundas and R. McCarthy have proven that the stabilization of algebraic $K$-theory is naturally equivalent to topological Hochschild homology,
\[ KS(R; M) \simeq T(R; M) \]
for any simplicial ring $R$ and any simplicial $R$-module $M$, cf. [4]. We note that both functors are defined for pairs $(L; P)$ where $L$ is a functor with smash product and $P$ is an $L$-bimodule; cf. [12]. An outline of a proof in this setting and by quite different methods, has been given by R. Schwänzl, R. Staffelt and F. Waldhausen. Hence the following result is a necessary condition for the conjecture mentioned above to hold.

Theorem. Let $L$ be a functor with smash product and $P$ an $L$-bimodule. Then there is a natural weak equivalence, $TCS(L; P)^\wedge \simeq T(L; P)^\wedge$.

It is not surprising that we have to $p$-complete in the case of TC since the cyclotomic trace is really an invariant of the $p$-completion of algebraic $K$-theory, cf. 1.4 below. The rest of this paragraph recalls cyclotomic spectra, topological Hochschild homology, topological cyclic homology and stabilization. In paragraph 2 we decompose topological Hochschild homology of a split extension of FSP's and approximate TC in a stable range. Finally in paragraph 3 we study free cyclic objects and use them to prove the theorem.

Throughout $G$ denotes the circle group, equivalence means weak homotopy equivalence and a $G$-equivalence is a $G$-map which induces an equivalence of $H$-fixed sets for any closed subgroup $H \leq G$.

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1.2. Let $L$ be an FSP and let $P$ be an $L$-bimodule. Then $\text{THH}(L; P)_*$ is the simplicial space with $k$-simplices

$$\text{holim}_{I^{k+1}} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, P(S^{i_0}) \wedge L(S^{i_1}) \wedge \ldots \wedge L(S^{i_k}))$$

and Hochschild-type structure maps, cf. [12], and $\text{THH}(L; P)$ is its realization. When $P = L$, considered as an $L$-bimodule in the obvious way, $\text{THH}(L; L)$ is a cyclic space so $\text{THH}(L; L)$ has a $G$-action. In both cases we use a thick realization to ensure that we get the right homotopy type, cf. the appendix. More generally if $X$ is some space we let $\text{THH}(L; P; X)_*$ be the simplicial space

$$\text{holim}_{I^{k+1}} F(S^{i_0} \wedge \ldots \wedge S^{i_k}, P(S^{i_0}) \wedge L(S^{i_1}) \wedge \ldots \wedge L(S^{i_k}) \wedge X),$$

where $X$ acts as a dummy for the simplicial structure maps. If $X$ has a $G$-action then $\text{THH}(L; P; X)$ becomes a $G$-space and $\text{THH}(L; L; X)$ a $G \times G$-space. We shall view the latter as a $G$-space via the diagonal map $\Delta: G \to G \times G$ and then denote it $\text{THH}(L; X)$.

We define a $G$-prespectrum $t(L; P)$ in the sense of [9] whose 0'th space is $\text{THH}(L; P)$. Let $V$ be any orthogonal $G$-representation, or more precisely, any f.d. sub inner product space of a fixed ‘complete $G$-universe’ $U$. Then

$$t(L; P)(V) = \text{THH}(L; P; S^V),$$

with the obvious $G$-maps

$$\sigma: S^{W-V} \wedge t(L; P)(V) \to t(L; P)(W)$$

as prespectrum structure maps. Here $S^V$ is the one-point compactification of $V$ and $W - V$ is the orthogonal complement of $V$ in $W$. We also define a $G$-spectrum $T(L; P)$ associated with $t(L; P)$, i.e. a $G$-prespectrum where the adjoints $\tilde{\sigma}$ of the structure maps are homeomorphisms. We first replace $t(L; P)$ by a thickened version $t^\tau(L; P)$ where the structure maps $\sigma$ are closed inclusions. It has as $V$'th space the homotopy colimit over suspensions of the structure maps

$$t^\tau(L; P)(V) = \text{holim}_{Z \subset V} \Sigma^{V-Z} t(L; P)(Z)$$
and as structure maps the compositions (t=t(L;P))

$$\sum W^\rightarrow V \text{holim} \sum V^\rightarrow Z t(Z) \cong \text{holim} \sum W^\rightarrow Z t(Z) \rightarrow \text{holim} \sum W^\rightarrow Z t(Z).$$

Here the last map is induced by the inclusion of a subcategory and as such is a closed cofibration, in particular it is a closed inclusion. Furthermore since $V$ is terminal among $Z \subset V$ there is a natural map $\pi: t^r(L; P) \rightarrow t(L; P)$ which is spacewise a $G$-homotopy equivalence. Next we define $T(L; P)$ by

$$T(L; P)(V) = \lim_{W \subset U} \Omega W^\rightarrow V t^r(L; P)(W)$$

with the obvious structure maps.

We can replace $\text{THH}(L; P; S^V)$ by $\text{THH}(L; S^V)$ above and get a $G$-prespectrum $t(L)$ and a $G$-spectrum $T(L)$. These possess some extra structure which allows the definition of $\text{TC}(L)$ and we will now discuss this in some detail. For a complete account we refer to [6], see also [3].

1.3. Let $C$ be a finite subgroup of $G$ of order $r$ and let $J$ be the quotient. The $r$'th root $\rho_C: G \rightarrow J$ is an isomorphism of groups and allows us to view a $J$-space $X$ as a $G$-space $\rho_C^*X$. Recall that the free loop space $\mathcal{L}X$ has the special property that $\rho_C^*\mathcal{L}X \cong \mathcal{L}X$ for any finite subgroup of $G$. Cyclotomic spectra, as defined in [3] and [6], is a class of $G$-spectra which have the analogous property in the world of spectra. This section recalls the definition.

For a $G$-spectrum $T$ there are two $J$-spectra $T^C$ and $\Phi^C T$ each of which could be called the $C$-fixed spectrum of $T$. If $V \subset U^C$ is a $C$-trivial representation, then

$$T^C(V) = T(V)^C, \quad \Phi^C T(V) = \lim_{W \subset U} \Omega W^C \rightarrow V T(W)^C$$

and the structure maps are evident. There is a natural map $r_C: T^C \rightarrow \Phi^C T$ of $J$-spectra; $r_C(V)$ is the composition

$$T^C(V) \cong \lim_{W \subset U} F(S^{W^\rightarrow V}, T(W))^C \ni^* \lim_{W \subset U} F(S^{W^C \rightarrow V}, T(W)^C) = \Phi^C T(V)$$

where the map $\iota^*$ is induced by the inclusion of $C$-fixed points. The difference between $T^C$ and $\Phi^C T$ is well illustrated by the following example.

**Example.** Consider the case of a suspension $G$-spectrum $T = \Sigma^\infty_G X$,

$$T(V) = \lim_{W \subset U} \Omega W^\rightarrow V (S^W \wedge X).$$

We let $E_G H$ denote a universal $H$-free $G$-space, that is $E_G H^K \cong \ast$ when $H \cap K = 1$ and $E_G H^K = \emptyset$ when $H \cap K \neq 1$. Then on the one hand we have the tom Dieck splitting

$$(\Sigma^\infty_G X)^C \cong \bigvee_{H \leq C} \Sigma^\infty_J (E_G/H(C/H) \wedge C/H X^H),$$

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and on the other hand the lemma shows that $\Phi^C(\Sigma_G^\infty X) \simeq \Sigma_J^\infty X^C$. Moreover the natural map $r_C: (\Sigma_G^\infty X)^C \to \Phi^C(\Sigma_G^\infty X)$ is the projection onto the summand $H = C$. □

A $J$-spectrum $D$ defines a $G$-spectrum $\rho_G^*D$. However this $G$-spectrum is indexed on the $G$-universe $\rho_G^*U^C$ rather than on $U$. To get a $G$-spectrum indexed on $U$ we must choose an isometric isomorphism $f_C: U \to \rho_G^*U^C$, then $(\rho_G^*D)(f_C(V))$ is the $V$'th space of the required $G$-spectrum, which we denote it $\rho_G^G D$.

We want the $f_C$'s to be compatible for any pair of finite subgroups, that is the following diagram should commute

$$
\begin{array}{ccc}
U & \xrightarrow{f_{Cr}} & \rho_{Cr}^* U^{Cr} \\
\downarrow{f_C} & & \downarrow{} \\
\rho_{Cr}^* U^{Cr} & \xrightarrow{(\rho_{Cr}^* f_C)^{Cr}} & \rho_{Cr}^* (\rho_{Cs}^* U^{Cs})^{Cr}.
\end{array}
$$

Moreover the restriction of $f_C$ to the $G$-trivial universe $U^G$ induces an automorphism of $U^G$ which we request be the identity. We fix our universe,

$$U = \bigoplus_{n \in \mathbb{Z}, \alpha \in \mathbb{N}} C(n) \alpha,$$

where $C(n) = C$ but with $G$ acting through the $n$'th power map. The index $\alpha$ is a dummy. Since $\rho_G^* C(n) = C(nr)$, where $r$ is the order of $C$, we obtain the required maps $f_C$ by identifying $\mathbb{Z} = r \mathbb{Z}$.

**Definition.** ([6]) A *cyclotomic spectrum* is a $G$-spectrum indexed on $U$ together with a $G$-equivalence

$$\varphi_C: \rho_G^G \Phi^C T \to T$$

for every finite $C \subset G$, such that for any pair of finite subgroups the diagram

$$
\begin{array}{ccc}
\rho_{C_r}^G \Phi^C r T & \xrightarrow{\rho_{C_r}^G \Phi^C s T} & \rho_{C_r}^G \Phi^{Cr} T \\
\rho_{C_r}^G \Phi^C r \varphi_{C_s} & \downarrow & \varphi_{C_r} \\
\rho_{C_r}^G \Phi^C r T & \xrightarrow{\varphi_{C_r}} & T
\end{array}
$$

commutes.

We prove in [6] that the topological Hochschild spectrum $T(L)$ defined above is a cyclotomic spectrum. The rest of this section recalls the definition of the $\varphi$-maps for $T(L)$. The definition goes back to [2] and begins with the concept of edgewise subdivision.
The realization of a cyclic space becomes a $G$-space upon identifying $G$ with $\mathbb{R}/\mathbb{Z}$, and hence $C$ may be identified with $r^{-1}\mathbb{Z}/\mathbb{Z}$. Edgewise subdivision associates to a cyclic space $Z_*$ a simplicial $C$-space $\text{sd}_C Z_*$. It has $k$-simplices $\text{sd}_C Z_k = Z_{r(k+1)-1}$ and the generator $r^{-1} + \mathbb{Z}$ of $C$ acts as $\tau^{k+1}$. Moreover, there is a natural homeomorphism

$$D : |\text{sd}_C Z_*| \to |Z_*|,$$

an $\mathbb{R}/r\mathbb{Z}$-action on $|\text{sd}_C Z_*|$ which extends the simplicial $C$-action, and $D$ is $G$-equivariant when $\mathbb{R}/r\mathbb{Z}$ is identified with $\mathbb{R}/\mathbb{Z}$ through division by $r$.

We now consider the case of $\text{THH}(L; X)_*$. Let us write $G_k(i_0, \ldots, i_k)$ for the pointed mapping space $F(S^{i_0}, \ldots, S^{i_k}, L(S^{i_0}) \ldots \wedge L(S^{i_k}) \wedge X)$.

Then the $k$-simplices of the edgewise subdivision is the homotopy colimit

$$\text{sd}_C \text{THH}(L; X)_k = \text{holim}_{f^{r(k+1)}} G_{r(k+1)-1}.$$

The $C$-action on $\text{sd}_C \text{THH}(L; X)_k$ is not induced by one on $G_{r(k+1)-1}$. We consider instead the composite functor $G_{r(k+1)-1} \circ \Delta_r$, where $\Delta_r : I^{k+1} \to (I^{k+1})^r$ is the diagonal functor. It has $C$-action and the canonical map of homotopy colimits

$$b_k : \text{holim}_{f^{r(k+1)}} G_{r(k+1)-1} \circ \Delta_r \to \text{holim}_{f^{r(k+1)}} G_{r(k+1)-1}$$

is a $C$-equivariant inclusion and induces a homeomorphism of $C$-fixed sets. Let $Y$ and $Z$ be two $C$-spaces and consider the mapping space $F(Y, Z)$. It is a $C$-space by conjugation and we have a natural map

$$t^* : F(Y, Z)^C \to F(Y^C, Z^C),$$

which takes a $C$-equivariant map $\psi : Y \to Z$ to the induced map of $C$-fixed sets. In the case at hand $t^*$ gives us a natural transformation

$$(G_{r(k+1)-1} \circ \Delta_r)^C \to G_k,$$

and the induced map on homotopy colimits defines a map of simplicial spaces

$$\tilde{\phi}_C : \text{sd}_C \text{THH}(L; X)_*^C \to \text{THH}(L; X^C)_*.$$

We define a $G$-equivariant map

$$\phi_C (V) : \rho_C^* t(L)(V)^C \to t(L)(f^{-1}_C(\rho_C^* V^C))$$
as the composite

$$
\rho_C^* \mid \text{THH}(L; S^V)^C \xrightarrow{D^{-1}} \text{sd}_C \text{THH}(L; S^V)^C \xrightarrow{\tilde{\varphi}_C} \mid \text{THH}(L; S^V)^C \mid
$$

Indeed it is $G$-equivariant by [2] lemma 1.11. Next we define a $G$-map

$$
\varphi_C(V) : \rho_C^* T(L)(V)^C \to T(f_C^{-1}(\rho_C^* V^C))
$$

as the map on colimits over $W \subset U$ induced by the composition

$$
\rho_C^*(\Omega^W - V \tau(L)(W))^C \xrightarrow{i^*} \rho_C^*(\Omega^{W^C - V^C} \tau(L)(W)^C)
\xrightarrow{\tilde{\varphi}_C(W)} \Omega^{\rho_C^*(W^C - V^C)} \tau(L)(f_C^{-1}(\rho_C^* W^C))
\xrightarrow{f_C^*} \Omega^{f_C^{-1}(\rho_C^*(W - V)^C)} \tau(L)(f_C^{-1}(\rho_C^* W^C)).
$$

Then the required maps $\varphi_C : \rho_C^# \Phi^C T \to T$ of $G$-spectra are evident in view of the definitions. Furthermore [2] 1.12 shows that the diagram which relates the $\varphi$-maps for a pair of finite subgroups of $G$ commutes. We refer to [6] for the proof that the $\varphi$-maps are $G$-equivalences.

1.4. Let $j : U^G \to U^C$ be the inclusion of the trivial $G$-universe and let $D$ be a $J$-spectrum. The underlying non-equivariant spectrum of $D$ is the spectrum $j^* D$ with its $J$-action forgotten. By abuse of notation we usually denote this $D$ again.

Let $T$ be a cyclotomic spectrum, then $r_{C^s}$ and $\varphi_{C^s}$ induce a map of $G$-spectra

$$
\rho_{C^s}^* T^{C^s} = \rho_{C^s}^* \left( \rho_{C^s}^* T^{C^s} \right)^C \to \rho_{C^s}^* \left( \rho_{C^s}^* \Phi_{C^s}^C T \right)^C \to \rho_{C^s}^# T^{C^s}.
$$

It gives a map $\Phi : T^{C^s} \to T^{C^s}$ of underlying non-equivariant spectra and the compatibility condition in definition 1.3 implies that $\Phi^s \Phi_{rs} = \Phi^s$. The inclusion of the fixed set of a bigger group in that of a smaller also defines a map of non-equivariant spectra $D_{rs} T^{C^s} \to T^{C^s}$, and these satisfies that $D_{rs} = D_{rs}$. Moreover $D_{rs} \Phi_{s} = \Phi_{s} D_{rs}.

Topological cyclic homology of an $FSP$ was defined in [2]; the presentation here is due to T. Goodwillie [5]. Let $\mathbb{I}$ be the category with $\text{ob} \mathbb{I} = \{1, 2, 3, \ldots \}$ and two morphisms $\Phi, D : n \to m$, whenever $n = rm$, subject to the relations

$$
\Phi_1 = D_1 = \text{id}_n,
\Phi_r \Phi_s = \Phi_{rs}, D_r D_s = D_{rs},
\Phi_r D_s = D_s \Phi_r.
$$

For a prime $p$ we let $\mathbb{I}_p$ denote the full subcategory with $\text{ob} \mathbb{I}_p = \{1, p, p^2, \ldots \}$. The discussion above shows that a cyclotomic spectrum $T$ defines a functor from $\mathbb{I}$ to the category of non-equivariant spectra, which takes $n$ to $T^{C^s}$.
Definition. ([2]) \( \text{TC}(T) = \text{holim} T^{C_n} \), \( \text{TC}(T; p) = \text{holim} T^{C_p^s} \).

If \( L \) is a functor with smash product then \( \text{TC}(L) \) and \( \text{TC}(L; p) \) are the connective covers of \( \text{TC}(T(L)) \) and \( \text{TC}(T(L); p) \) respectively. It is often useful to have the definition of \( \text{TC}(T; p) \) in the form it is given in [2],

\[
\text{TC}(T; p) \cong [\text{holim} T^{C_p^s}]^{h\langle D_p \rangle} \cong [\text{holim} T^{C_p^s}]^{h\langle D_p \rangle}.
\]

Here \( \langle D_p \rangle \) is the free monoid on \( D_p \) and \( X^{h\langle D_p \rangle} \) stands for the \( \langle D_p \rangle \)-homotopy fixed points of \( X \). It is naturally equivalent to the homotopy fiber of \( 1 - D_p \).

The functor \( \text{TC}(\cdot) \) is really not a stronger invariant than the \( \text{TC}(\cdot; p) \)'s. Indeed we have the following result, which will be proved elsewhere.

Proposition. The projections \( \text{TC}(T) \to \text{TC}(T; p) \) induce an equivalence of \( \text{TC}(T) \) with the fiber product of the \( \text{TC}(T; p) \)'s over \( T \). Moreover the \( p \)-complete theories agree, \( \text{TC}(T)^p \cong \text{TC}(T; p)^p \). \( \square \)

Remark. In [2] the authors define a space \( \text{TC}(L; p) \) and a \( \Gamma \)-space structure on it. Furthermore they show that the cyclotomic trace \( \text{trc}: K(L) \to \text{TC}(L; p) \) is a map of \( \Gamma \)-spaces. We show in [6] that the spectrum \( \text{TC}(L; p) \) defined above is equivalent to the one determined by the \( \Gamma \)-space structure. \( \square \)

1.5. Stable \( K \)-theory of simplicial rings was defined by Waldhausen in [15], see also [8]. We conclude this paragraph with the definition of stable \( \text{TC} \) of a \( FSP \) and leave it to reader to see that stable \( K \)-theory also may be defined in this generality.

Definition. Let \( P \) be an \( L \)-bimodule and \( K \) a space. The shift \( P[K] \) of \( P \) by \( K \) is the functor given by \( P[K](X) = K \wedge P(X) \) with structure maps

\[
l_{X,Y}^{P[K]} = \text{id}_K \wedge l_{X,Y}^P \circ \text{tw} \wedge \text{id}_{P(Y)}, \quad r_{X}^{P[K]}, Y = \text{id}_K \wedge r_{X,Y}^P.
\]

We shall write \( P[n] \) for \( P[S^n] \).

We define a new \( FSP \) denoted \( L \oplus P \) which is to be thought of as an extension of \( L \) by a square zero ideal \( P \).

Definition. Let \( L \) be an \( FSP \) and \( P \) an \( L \)-bimodule. We define the extension of \( L \) by \( P \) as \( L \oplus P(X) = L(X) \vee P(X) \) with multiplication

\[
L \oplus P(X) \wedge L \oplus P(Y) \to L(X) \wedge L(Y) \vee L(X) \wedge P(Y) \vee P(X) \wedge L(Y) \vee P(X) \wedge P(Y) \to L(X) \wedge L(Y) \vee P(X) \wedge P(Y) \to L \oplus P(X \wedge Y).
\]

The first map is the canonical homeomorphism, the second is \( \mu_{X,Y} \vee l_{X,Y} \vee r_{X,Y} \vee * \) and the last is convolution. Finally the unit in \( L \oplus P \) is the composite

\[
X \to L(X) \to L \oplus P(X).
\]

One verifies immediately that \( L \oplus P \) is in fact an \( FSP \) and that it contains \( L \) as a retract. We shall write \( \overline{\text{TC}}(L \oplus P) \) for the homotopy fiber of the induced retraction \( \text{TC}(L \oplus P) \to \text{TC}(L) \).
Lemma. If $K$ is contractible then so is $\tilde{\text{TC}}(L \oplus P[K])$. Furthermore a contraction of $K$ induces one of $\tilde{\text{TC}}(L \oplus P[K])$.

Proof. Let us write $F$ instead of $L \oplus P[K]$. If $h: I_+ \wedge K \to K$ is a contraction we can define $h(X): I_+ \wedge F(X) \to F(X)$ by the composition

$$I_+ \wedge (L(X) \vee K \wedge P(X)) \cong I_+ \wedge L(X) \vee I_+ \wedge K \wedge P(X) \xrightarrow{pr_2 \vee h \wedge \text{id}} L(X) \vee K \wedge P(X).$$

It is compatible with the multiplication and unit in $F$, that is the following diagrams commute

$$
\begin{array}{ccc}
I_+ \wedge (F(X) \wedge F(Y)) & \xrightarrow{id \wedge X \wedge Y} & I_+ \wedge F(X \wedge Y) \\
\Delta \wedge \text{id} \downarrow & & \downarrow h_{X \wedge Y} \\
(I \times I) \wedge F(X) \wedge F(Y) & & F(X \wedge Y) \\
\text{id} \wedge \text{tw id} \downarrow & & \mu_{X,Y} \uparrow \\
I_+ \wedge F(X) \wedge I_+ \wedge F(Y) & \xrightarrow{h_{X \wedge h_{Y}}} & F(X) \wedge F(Y).
\end{array}
$$

and

$$
\begin{array}{ccc}
I_+ \wedge X & \xrightarrow{id \wedge 1_X} & I_+ \wedge F(X) \\
pr_2 \downarrow & & \downarrow h(X) \\
X & \xrightarrow{1_X} & F(X).
\end{array}
$$

Therefore the composition

$$I_+ \wedge (F(S^0) \wedge \ldots \wedge F(S^k)) \xrightarrow{\text{tw} \circ (\Delta \wedge \text{id})} I_+ \wedge F(S^0) \wedge \ldots \wedge I_+ \wedge F(S^k) \xrightarrow{h(S^0) \wedge \ldots \wedge h(S^k)} F(S^0) \wedge \ldots \wedge F(S^k)$$

gives rise to a cyclic map $h_V*: I_+ \wedge \text{THH}(F; F; S^V) \to \text{THH}(F; F; S^V)$, whose realization is a $G$-equivariant homotopy

$$h_V: I_+ \wedge t(F)(V) \to t(F)(V).$$

Furthermore these are compatible with the structure maps in the prespectrum such that we get a $G$-equivariant homotopy

$$H: I_+ \wedge T(F) \to T(F).$$

This gives in turn a homotopy $I_+ \wedge \text{TC}(F) \to \text{TC}(F)$ from the identity to the retraction onto the image of $\text{TC}(L)$. 

\[\square\]
If we apply $\widetilde{TC}(L \oplus P[-])$ to the cocartesian square of spaces

$$
\begin{array}{ccc}
S^n & \longrightarrow & D^{n+1}_+ \\
\downarrow & & \downarrow \\
D^{n+1} & \longrightarrow & S^{n+1}.
\end{array}
$$

we get a map from $\widetilde{TC}(L \oplus P[n])$ to the homotopy limit of

$$
\widetilde{TC}(L \oplus P[D^{n+1}_+]) \rightarrow \widetilde{TC}(L \oplus P[S^{n+1}], p) \leftarrow \widetilde{TC}(L \oplus P[D^{n+1}_+]).
$$

By the lemma the radial contractions of the discs $D^{n+1}$ give a preferred contraction of $\widetilde{TC}(L \oplus P[n + 1])$. Hence we obtain a natural map from the homotopy limit above to $\Omega \widetilde{TC}(L \oplus P[n + 1])$. All in all we get a stabilization map

$$
\sigma: \widetilde{TC}(L \oplus P[n]) \rightarrow \Omega \widetilde{TC}(L \oplus P[n + 1])
$$

which is natural in $L$ and $P$.

**Definition.** Let $L$ be an FSP and $P$ an $L$-bimodule. Then

$$
TC^S(L; P) = \text{holim}_n \Omega_{n+1} \widetilde{TC}(L \oplus P[n]),
$$

with the colimit taken over the stabilization maps.

2. **Stable Approximation of $TC(L \oplus P)$**

2.1. In the rest of this paper the prime $p$ is fixed and we shall always consider the functor $TC(-; p)$ rather than $TC(-)$.

Recall that by definition $L \oplus P(S^i) = L(S^i) \vee P(S^i)$. Thus we can decompose the smash product

$$
L \oplus P(S^{i_0}) \wedge \ldots \wedge L \oplus P(S^{i_k})
$$

into a wedge of summands of the form

$$
F_0(S^{i_0}) \wedge \ldots \wedge F_k(S^{i_k}),
$$

where $F_i = L, P$. A summand where $\# \{i | F_i = P\} = a$ will be called an $a$-configuration and the one-point space $\ast$ will be considered an $a$-configuration for any $a \geq 0$.

Recall from 1.3 the functor $G_k = G_k(L \oplus P; X)$ whose homotopy colimit is $\text{THH}(L \oplus P; X)_k$. The $a$-configurations define subspaces

$$
G_{a,k}(i_0, \ldots, i_k) \subset G_k(i_0, \ldots, i_k)
$$

preserved under $G_k(f_0, \ldots, f_k)$, i.e. we get a functor $G_{a,k} = G_{a,k}(L \oplus P; X)$.

The spaces

$$
\text{THH}_a(L \oplus P; X)_k = \text{holim}_{I_{k+1}} G_{a,k}(L \oplus P; X)
$$

form a cyclic subspace $\text{THH}_a(L \oplus P; X)_* \subset \text{THH}(L \oplus P; X)_*$ with realization $\text{THH}_a(L \oplus P; X)$. Like in 1.2 we can define a $G$-prespectrum $t_a(L \oplus P)$ and a $G$-spectrum $T_a(L \oplus P)$. Then $T_a(L \oplus P)$ is a retract of $T(L \oplus P)$. We show below that as a $G$-spectrum $T(L \oplus P)$ is the wedge sum of the $T_a(L \oplus P)$'s.
Lemma. Let \( j \) be a \( G \)-prespectrum and let \( J \) be the \( G \)-spectrum associated with \( j^\tau \). If \( J^\Gamma \simeq * \) for any finite subgroup \( \Gamma \subset G \) and \( j(V)^G \simeq * \) for any \( V \subset U \) then \( J \simeq_G * \).

Proof. Let \( \mathcal{F} \) be the family of finite subgroups of the circle, then \( J \) is \( \mathcal{F} \)-contractible. Since \( J \wedge E\mathcal{F}_+ \to J \) is an \( \mathcal{F} \)-equivalence, \( J \wedge E\mathcal{F}_+ \) is also \( \mathcal{F} \)-contractible. However \( J \wedge E\mathcal{F}_+ \) is \( G \)-equivalent to an \( \mathcal{F} \)-CW-spectrum and therefore it is in fact \( G \)-contractible by the \( \mathcal{F} \)-Whitehead theorem, [9] p.63. Now

\[
(J \wedge E\mathcal{F}_+)(V) \cong \lim_{W} \Omega^W j^\tau(V + W) \wedge E\mathcal{F}_+,
\]

and \( j^\tau(V) \wedge E\mathcal{F}_+ \to j^\tau(V) \) is an \( G \)-equivalence since \( j(V)^G \simeq * \). Therefore \( J \simeq_G J \wedge E\mathcal{F}_+ \) and we have already seen that the latter is \( G \)-contractible. \( \square \)

Lemma. Let \( H \) be a compact Lie group, let \( X \) a finite \( H \)-CW-complex and let \( Y_a \) a family of \( H \)-spaces. For \( K \leq H \) a closed subgroup we let

\[
n(K) = \min_a \{\text{conn}(Y^K_a)\}.
\]

Then the inclusion

\[
\bigvee_a F(X, Y_a)^H \to F(X, \bigvee_a Y_a)^H
\]

is \( 2 \min\{n(K) - \dim(X^K)|K \leq H\} + 1 \)-connected.

Proof. The inclusion above fits into a commutative square

\[
\begin{array}{ccc}
\bigvee_a F(X, Y_a)^C & \longrightarrow & F(X, \bigvee_a Y_a)^C \\
\downarrow & & \downarrow \\
\prod'_a F(X, Y_a)^C & \cong & F(X, \prod'_a Y_a)^C,
\end{array}
\]

where \( \prod' \) is the weak product, \textit{i.e.} the subspace of the product where all but a finite number of coordinates are at the basepoint. The lower horizontal map is a homeomorphism because \( X \) is finite, and the connectivity of the vertical maps may be estimated by elementary equivariant obstruction theory. For example the connectivity of an equivariant mapping space satisfies

\[
\text{conn}(F(X, Y)^H) \geq \min\{\text{conn}(Y^K) - \dim(X^K)|K \leq H\}.
\]

Therefore the left vertical map is \( 2 \min\{n(K) - \dim(X^K)|K \leq H\} + 1 \)-connected. \( \square \)

Proposition. \( T(L \oplus P) \simeq_G \bigvee_a T_a(L \oplus P) \).

Proof. We apply the first lemma with \( j \) the \( G \)-prespectrum whose \( V \)’th space is the homotopy fiber of the inclusion

\[
\bigvee_{a=0}^\infty t_a(L \oplus P)(V) \to t(L \oplus P)(V).
\]
We first consider a finite subgroup $\Gamma \subset G$ and show that $J^\Gamma \simeq \ast$. It suffices to show that $j(V)^C$ is $\dim(V^C) + k(V, C)$-connected, where $k(V, C) \to \infty$ as $V$ runs through the f.d. sub inner product spaces of $U$, for any subgroup $C \subset \Gamma$. We use edgewise subdivision to get a simplicial $C$-action, that is we can identify $j(V)^C$ with the homotopy fiber of

$$|\bigvee_a \sd C \THH_a(L \oplus P; S^V)^C| \to |\sd C \THH(L \oplus P; S^V)^C|.$$ 

As in the 1.3 we consider the diagonal functor $\Delta_r: I^{k+1} \to (I^{k+1})^r$. Then the second lemma shows that the inclusion

$$\bigvee_a (G_{a,r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k))^C \to (G_{r(k+1)-1} \circ \Delta_r(i_0, \ldots, i_k))^C$$

is $2 \dim(V^C) - 1$-connected. By [1] theorem 1.5 the same is true for the homotopy colimits over $I^{k+1}$. Hence the inclusion map

$$\bigvee_a \sd C \THH_a(L \oplus P; S^V)^C_k \to \sd C \THH(L \oplus P; S^V)^C_k$$

is $2 \dim(V^C) - 1$-connected. Finally the spectral sequence of [13] shows that the induced map on realizations is $2 \dim(V^C) - 1$-connected. It follows that $J^\Gamma \simeq \ast$.

We have only left to show that $j(V)^G \simeq \ast$. If $X_\ast$ is a cyclic space, then $|X_\ast|^G$ is homeomorphic to the subspace $\{x \in X_0|s_0 x = \tau_1 s_1 x\}$ of the 0-simplices. For the domain and the codomain of $j(V)$ this is $S^V G$ and $j(V)$ is the identity. □

2.2. Let us write $a = p^s k$ with $(k, p) = 1$ and denote $T_a(L \oplus P)$ by $T^k_s(L \oplus P)$. Then the cyclotomic structure map $\varphi = \varphi_{C_p}$ induces a $G$-equivalence

$$\varphi_s: \rho_{C_p}^\# \Phi^{C_p} T^k_s(L \oplus P) \to T^k_{s-1}(L \oplus P), \quad s \geq 0,$$

where for convenience $T^k_{-1}(L \oplus P)$ denotes the trivial $G$-spectrum $\ast$.

Lemma. i) The cyclotomic structure map induces a map of underlying non-equivariant spectra

$$T^k_s(L \oplus P[n])^{C_p} \to T^k_0(L \oplus P[n])^{C_p}$$

which is $kpn$-connected.

ii) $T^k_0(L \oplus P[n])^{C_p}$ is $kn$-connected.

Proof. Let $\tilde{E}G$ be the mapping cone of the map $\pi: EG_+ \to S^0$ which collapses $EG$ to the non-basepoint of $S^0$. It comes with a $G$-map $\nu: S^0 \to \tilde{E}G$ and a $G$-null homotopy of the composition

$$EG_+ \xrightarrow{\pi} S^0 \xrightarrow{\nu} \tilde{E}G.$$
We can also describe $\tilde{E}G$ as the unreduced suspension of $EG$ and $\iota$ as the inclusion of $S^0$ as the two cone vertices. Finally we note that $\tilde{E}G$ is non-equivariantly contractible while $E^G_C = S^0$ for any non-trivial subgroup $C \leq G$.

Let us write $T_s$ for $T_s^k(L \oplus P[n])$. We can smash the sequence above with $T_s$ and take $C_p^r$-fixed points. Then we get maps of underlying non-equivariant spectra

$$[EG_+ \wedge T_s]^{C_p^r} \xrightarrow{\pi_*} T_s^{C_p^r} \xrightarrow{\iota_*} [\tilde{E}G \wedge T_s]^{C_p^r}$$

and a preferred null homotopy of their composition. These data specifies a map from $[EG_+ \wedge T_s]^{C_p^r}$ to the homotopy fiber of $\iota_*$ and this an equivalence.

We identify the right hand term. Recall the natural map $r_{C_p}: T_p^{C_p} \to \Phi_{C_p} T_s$ from 1.3. It factors as a composition

$$T_s^{C_p} \xrightarrow{\pi_*} [\tilde{E}G \wedge T_s]^{C_p^r} \xrightarrow{\tilde{r}_C} \Phi_{C_p} T_s,$$

where $\tilde{r}_C(V)$ is induced from the restriction map

$$F(S^{W-V}, \tilde{E}G \wedge T_s(W))^{C_p} \to F(S^{W_{C_p}-V}, T(W)^{C_p}).$$

Moreover $\tilde{r}_{C_p}(V)$ is a fibration with fiber the equivariant (pointed) mapping space

$$F(S^{W-V}/S^{W_{C_p}-V}, \tilde{E}G \wedge T(W))^{C_p}.$$ 

If we regard $W$ as a $C_{p^r}$-space, then $W^{C_p}$ is the singular set, so $S^{W-V}/S^{W_{C_p}-V}$ is a free $C_{p^r}$-CW-complex in the pointed sense. Since $\tilde{E}G$ is non-equivariantly contractible it follows that $\tilde{r}_{C_p}$ is a $C_{p^r}/C_p$-equivalence. The map $\Phi_{C_p}$ of underlying non-equivariant spectra defined in 1.4 restricts to a map

$$T_s^{C_p} \xrightarrow{r_{C_p}^{C_p^r/C_p}} (\Phi_{C_p} T_s)^{C_p^r/C_p} \xrightarrow{\varphi_{C_p}^{r_{C_p}^{C_p^r-1}}} T_s^{C_p^r-1}.$$ 

Our calculation above shows that its homotopy fiber is equivalent to the underlying non-equivariant spectrum of $[EG_+ \wedge T_s]^{C_p^r}$. We contend that this is as highly connected as is $T_s$. Indeed the skeleton filtration of $EG$ gives rise to a first quadrant spectral sequence

$$E^2_{s,t} = H_s(C_{p^r}; \pi_t(T_s)) \Rightarrow \pi_{s+t}([EG_+ \wedge T_s]^{C_p^r}),$$

where $\pi_t(T_s)$ is a trivial $C_{p^r}$-module. The identification of the $E^2$-term uses the transfer equivalence of [9] p. 89.

\[\Box\]
Proposition. In the stable range $\leq 2n$ we have
\[ \tilde{\text{TC}}(L \oplus P[n]) \simeq_{2n} \text{holim}_r T_1(L \oplus P[n]; p)_{C_r^r}, \]
with the limit taken over the inclusion maps $D$.

Proof. We get from the connectivity statements in the lemma that
\[ \tilde{T}(L \oplus P[n])_{C_r^r} \simeq_{2n} T^1(L \oplus P[n])_{C_r^r} = \bigvee_{s=0}^{\infty} T^1_s(L \oplus P[n])_{C_r^r} \]
\[ \simeq_{2n} \bigvee_{s=0}^{r} T^1_0(L \oplus P[n])_{C_{r-s}^r} = \bigvee_{t=0}^{r} T^1_0(L \oplus P[n])_{C_{r-t}^r}. \]
Under these equivalences $\Phi: \tilde{T}(L \oplus P[n])_{C_r^r} \to \tilde{T}(L \oplus P[n])_{C_{r-1}^r}$ becomes projection onto the first $r$ summands. Therefore
\[ \tilde{\text{TC}}(L \oplus P[n]; p) = [\text{holim}_r \tilde{T}(L \oplus P[n])_{C_r^r}]^{h(D)} \simeq_{2n} \prod_{t=0}^{\infty} T^1_0(L \oplus P[n])_{C_{r-t}^r}^{h(D)}. \]
The latter spectrum is naturally equivalent to the homotopy limit stated above. 

Remark. When $P = L$ there is an unstable formula for $\tilde{\text{TC}}(L \oplus L[n])$. It was found in [6] and used to evaluate TC of rings of dual numbers over finite fields.

3. Free cyclic objects

3.1. In this paragraph we examine the cyclic spaces $t_1(L \oplus P)(V)$, we introduced in 2.2. They turn out to be the free cyclic spaces generated by the simplicial spaces $t(L; P)(V)$, from 1.2. First we study free cyclic objects.

Suppose $K: I \to J$ is a functor between small categories and $\mathbb{C}$ a category which have all colimits. Then the functor $K^*: \mathbb{C}^J \to \mathbb{C}^I$ has a left adjoint $F$. If $X: I \to \mathbb{C}$ is a functor then
\[ FX(j) = \lim_i((K \downarrow j) \xrightarrow{pr_1} I \xrightarrow{X} \mathbb{C}), \]
where $(K \downarrow j)$ is the category of objects $K$-over $j$. It is called the left Kan extension of $X$ along $K$, cf. [10]. As an instance of this construction suppose $I$ and $J$ are monoids, i.e. categories with one object, and $\mathbb{C}$ the category of (unbased) spaces. Then a functor $X: I \to \mathbb{C}$ is just an $I$-space and $FX$ is the $J$-space $J \times_I X$.

Definition. Let $X_*$ be a simplicial object in $\mathbb{C}$. The free cyclic object generated by $X_*$ is the left Kan extension of $X_*$ along the forgetful functor $K: \Delta^{op} \to \Lambda^{op}$. It is denoted $FX_*$. If $X$ is an object in $\mathbb{C}$ and $S$ is a set, then we let $S \times X$ denote the coproduct of copies of $X$ indexed by $S$. We give a concrete description of $FX_*$. 

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Lemma. Let \( C_{n+1} = \{1, \tau_n, \tau_n^2, \ldots, \tau_n^n\} \). Then \( FX \) has \( n \)-simplices

\[
FX_n \cong C_{n+1} \ltimes X_n,
\]

and the cyclic structure maps are

\[
\begin{align*}
d_i(\tau_n^s \times x) &= \tau_n^{s-1} \times d_i + s x, & \text{if } i + s \leq n \\
      &= \tau_n^{s-1} \times d_i + s x, & \text{if } i + s > n \\
s_i(\tau_n^s \times x) &= \tau_n^{s+1} \times s_i + s x, & \text{if } i + s \leq n \\
      &= \tau_n^{s+1} \times s_i + s x, & \text{if } i + s > n \\
t_n(\tau_n^s \times x) &= \tau_n^{s-1} \times x.
\end{align*}
\]

All indices are to be understood as the principal representatives modulo \( n + 1 \).

Proof. Both \( \Delta \) and \( \Lambda \) has objects the finite ordered sets \( n = \{0, \ldots, n\} \) but \( \Lambda \) has more morphism than \( \Delta \). Specifically \( \Lambda(n, m) = \Delta(n, m) \times \text{Aut}_\Lambda(n) \) and \( \text{Aut}_\Lambda(n) \) is a cyclic group of order \( n + 1 \). As a generator for \( \text{Aut}_\Lambda(n) \) we choose the cyclic permutation \( \tau_n : n \to n; \tau_n(i) = i - 1 \) (mod \( n + 1 \)).

Consider the full subcategory \( C(n) \subset (K \downarrow n) \) whose objects are the automorphisms \( Kn \to n \), i.e. \( \text{ob} C(n) = C_{n+1} \). The restriction of colimits comes with a map

\[
\lim(C(n) \xrightarrow{\text{pr}_1} \Delta^\text{op} \xrightarrow{X_*} \mathcal{C}) \to \lim((K \downarrow n) \xrightarrow{\text{pr}} \Delta^\text{op} \xrightarrow{X_*} \mathcal{C}) = FX_n,
\]

and from the definitions one may readily show that this is an isomorphism. Since in \( \Delta^\text{op} \) there are no automorphisms of \( n \) apart from the identity, the category \( C(n) \) is a discrete category, i.e. any morphism is an identity. We conclude that

\[
FX_n \cong \bigsqcup_{\text{ob} C(n)} X_n = C_{n+1} \ltimes X_n.
\]

It is straightforward to check that the cyclic structure maps are as claimed. \( \square \)

Example. Suppose \( C \) is the category of commutative rings, where the coproduct is tensor product of rings, and \( R_* = R \) is a constant simplicial ring. Then the complex associated with \( FR \) is the standard Hochschild complex \( Z(R) \) whose homology is \( \text{HH}_*(R) \).

3.2. We now take \( C \) to be the category of pointed topological spaces and study the relation between \( F \) and realization.

Lemma. There is a natural \( G \)-homeomorphism \( |FX_*| \cong G_+ \wedge |X_*| \).

Proof. Consider the standard cyclic sets \( \Lambda[n] = \Lambda(-, n) \) and their realizations \( \Lambda^n \). From [7], 3.4 we know that as cocyclic spaces \( \Lambda^* \cong G \times \Delta^* \), so we may view
Λ∗ as a cocyclic G-space. Now suppose Y is a (based) G-space. We can define a cyclic space C∗(Y) as the equivariant mapping space

\[ C∗(Y) = F_G(Λ∗, Y), \]

with the compact open topology. Then one immediately verifies that C∗ is right adjoint to the realization functor \( |−| \). The realization functor for simplicial spaces also has a right adjoint. It is given as \( S∗(X) = F(Δ∗, X) \) with the compact open topology. Finally the forgetful functor U from G-spaces to spaces is right adjoint to the functor \( Gϕ− \).

By a very general principle in category theory called conjunction, to prove the lemma we may as well show that \( Sm(UY) = K∗C∗(Y) \) for any G-space Y. But this is evident since \( F_G(Gϕ−X, Y) = F(X, UY) \).

**Proposition.** There is a natural equivalence of G-spectra

\[ Gϕ− T(L; P) \simeq_G T1(L ⊕ P). \]

The V′th space in the smash product G-spectrum on the left is naturally homeomorphic to \( \lim_{W} \Omega^{W−V}(Gϕ−\tau(L; P)(W)) \), where G acts diagonally on \( Gϕ−\tau(L; P)(W) \).

**Proof.** The smash product \( P(S^{i0}) ∩ L(S^{i1}) ∩ ... ∩ L(S^{ik}) \) is a 1-configuration, cf. 2.1. Thus we have an inclusion map \( THH(L; P; X)_k \hookrightarrow THH1(L ⊕ P; X)_k \) and these commutes with the simplicial structure maps. By definition we get a map of cyclic spaces

\[ j(X)_∗: F THH(L; P; X) → THH1(L ⊕ P; X). \]

and lemma 3.2 shows that on realizations this gives rise to a G-equivariant map

\[ j(X): Gϕ− THH(L; P; X) → THH1(L ⊕ P; X). \]

When X runs through the spheres \( S^v \) these maps form a map j of G-prespectra. Let us write \( Gϕ−\tau(L; P) \) for the G-spectrum whose V′th space is the colimit

\[ \lim_{W \in U} \Omega^{W−V}(Gϕ−\tau(L; P)(W)). \]

Then j induces a map \( J: Gϕ−\tau(L; P) → T1(L ⊕ P) \) and an argument completely analogous to the proof of proposition 2.1 shows that this is a G-equivalence. Finally the canonical inclusion

\[ Gϕ−\tau(L; P)(V) → Gϕ− T(L; P)(V) \]

gives a map \( Gϕ−\tau(L; P) → Gϕ− T(L; P) \) and this is a homeomorphism, cf. the appendix.
3.3. Before we prove our main theorem we need the following key lemma, also used extensively in [6].

**Lemma.** Let $T$ be a $G$-spectrum. Then there is a natural equivalence of non-equivariant spectra

$$[T \wedge G_+]^{C_p} \simeq T \vee \Sigma T,$$

and the inclusion $D: [T \wedge G_+]^{C_p} \hookrightarrow [T \wedge G_+]^{C_{p-1}}$ becomes $p \vee \text{id}$. Here $p$ denotes multiplication by $p$.

**Proof.** The Thom collaps $t: S^C \to S^{cP} \wedge G_+$ of $S(C) \subset C$ gives rise to a $G$-equivariant transfer map

$$\tau: F(G_+, \Sigma T) \to G_+ \wedge T$$

which is a $G$-homotopy equivalence, cf. [9], p.89. There is a cofibration sequence of $C_{p_+}$-spaces

$$C_{p_+} \to G_+ \to C_{p_+} \wedge S^1$$

where $S^1$ is $C_{p_+}$-trivial. We may apply $F_{C_{p_+}}(-, \Sigma T)$ and get a cofibration sequence of spectra

$$F(S^1, \Sigma T) \to F_{C_{p_+}}(G_+, \Sigma T) \xrightarrow{\text{ev}_\zeta} \Sigma T.$$  

Finally ev$\zeta$ is naturally split by the adjoint of the $G$-action $G_+ \wedge \Sigma T \to \Sigma T$. \qed

**Proof of theorem.** If we compare proposition 3.2 and lemma 3.3 we find that

$$T_1(L \oplus P)^{C_p} \simeq T(L; P) \vee \Sigma T(L; P).$$

Now holim of a string of maps

$$\ldots \xrightarrow{f_i} X_n \xrightarrow{f_{i-1}} \ldots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0$$

where every $f_i = pg_i$ for some $g_i$ vanishes after $p$-completion, so by proposition 2.2 and lemma 3.3 we get

$$\widetilde{T}(L \oplus P[n]) \simeq T(L; P[n]).$$

The functor $T(L; P)$ is linear in the second variable, cf. [12] 2.13, so therefore

$$\Omega^{n+1}\widetilde{T}(L \oplus P[n]) \simeq \Omega^{n+1}\Sigma T(L; P[n]) \simeq T(L; P).$$

It remains only to check that the stabilization maps defined in 1.5 induce an equivalence of $T(L; P)$. They do. \qed
A.1. Let $C$ be either of the categories $\Delta$ or $\Lambda$ and let $X: C \to \text{Top}_*$ be a functor to pointed spaces. We define a new functor $\tilde{X}: C \to \text{Top}_*$ by the homotopy colimit

$$\text{holim}(( - \downarrow C)^{\text{op}} \xrightarrow{\text{pr}^2_{\text{op}}} C^{\text{op}} \xrightarrow{X} \text{Top}_*),$$

where $(n \downarrow C)$ is the category under $n$, cf. [10]. If $\theta: n \to m$ is a morphism in $\Delta$ (not $C$), which is surjective, then $\theta^*: (m \downarrow C) \to (n \downarrow C)$ is an inclusion functor. In general inclusions of index categories induces closed cofibrations on homotopy colimits. In particular $\theta^*: \tilde{X}_m \to \tilde{X}_n$ is a closed cofibration, so $\tilde{X}$ is good in the sense of [14]. Moreover we have a homotopy equivalence $\tilde{X}_n \to X_n$ because $\text{id}: n \to n$ is initial in $(n \downarrow C)$.

A.2. This section explains a technical point in the passage from $G$-prespectra to $G$-spectra. Let $GPU$ denote the category of $G$-prespectra indexed on the universe $U$ and let $GSU$ be the full subcategory of $G$-spectra. In [9] the authors prove that the forgetful functor $l: GSU \to GPU$ has a left adjoint $L: GPU \to GSU$. We call this functor specification and if $t \in GPU$ then we call $Lt$ the associated $G$-spectrum. Such a functor is needed since many constructions such as $X \wedge -$ and any (homotopy) colimits do not preserve $G$-spectra. However $L$ has the serious drawback that in general it looses (weak) homotopy type, i.e. the homotopy type of $(Lt)(V)$ cannot be described in terms of that of the spaces $t(W)$. To control the homotopy type the $G$-prespectrum $t$ has to be an inclusion $G$-prespectrum, that is the structure maps $\tilde{\sigma}: t(V) \to \Omega^{W-V} t(W)$ must be inclusions, then

$$(Lt)(V) = \lim_{W \in U} \Omega^{W-V} t(W).$$

This is the case for example if the adjoints $\sigma: \Sigma^{W-V} t(V) \to t(W)$ are closed inclusions. The thickening functor $(-)^T$ defined in 1.2 produces $G$-prespectra of this kind. Therefore $L(t^T)$ has the right homotopy type.

If $a: GPU \to GPU$ is a functor we define $A: GSU \to GSU$ as the composite functor $La$ and if $a$ has a right adjoint $b$, then $B$ is the right adjoint of $A$. Suppose $b$ preserves $G$-spectra, then $b(lT) \cong lB(T)$ for any $T \in GSU$. By conjugation we get

$$A(Lt) \cong La(t)$$

for any $t \in GPU$. The functors $a$ we consider take a $G$-prespectrum, whose structure maps $\sigma$ are closed inclusions, to a $G$-prespectrum of the same kind. Hence the homotopy type of $La(t^T)$ and therefore $A(L(t^T))$ may be calculated. This shows that all $G$-spectra considered in this paper have the right homotopy type.

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