

Astérisque

KARL OELJEKLAUS

**On the automorphism group of certain
hyperbolic domains in $\mathbb{C}^2$**

Astérisque, tome 217 (1993), p. 193-216

[<http://www.numdam.org/item?id=AST_1993__217__193_0>](http://www.numdam.org/item?id=AST_1993__217__193_0)

© Société mathématique de France, 1993, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (<http://smf4.emath.fr/Publications/Asterisque/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

On the automorphism group of certain hyperbolic domains in $\mathbf{C}^2$

Karl Oeljeklaus

1 Introduction and Results

Let $Q = Q(z, \bar{z})$ be a subharmonic and non-harmonic polynomial on the complex plane $\mathbf{C}$ with real values. Then the degree the non-harmonic part Q^N of Q is an even positive number $2k \in \mathbf{N}^*$. In their paper [1], F. Berteloot and G. Cœuré proved that the domain $\Omega_Q = \{(w, z) \in \mathbf{C}^2 \mid \operatorname{Re} w + Q(z, \bar{z}) < 0\}$ is **hyperbolic** for every Q like above. In this note, we consider the positive cone M of all such polynomials and the associated domains $\Omega_Q \subset \mathbf{C}^2$.

Let $Q_1, Q_2 \in M$ and $\Omega_{Q_1}, \Omega_{Q_2}$ be the associated domains. In what follows, we use also $\Omega, \Omega_1, \Omega_2$ instead of $\Omega_Q, \Omega_{Q_1}, \Omega_{Q_2}$ if there is no confusion possible. First, we introduce an equivalence relation on the cone M .

Definition 1.1 *Let $Q_1, Q_2 \in M$. We say that Q_1 and Q_2 are equivalent $Q_1 \sim Q_2$, if there is a real number $\rho > 0$, a holomorphic polynomial $p(z)$ and an automorphism $g(z)$ of $\mathbf{C}$ such that*

$$(1.1) \quad Q_1(z, \bar{z}) = \rho \operatorname{Re}(p(z)) + \rho Q_2(g(z), \overline{g(z)}).$$

On the other hand, there is another equivalence relation on M given by the biholomorphy of the domains Ω_{Q_1} and Ω_{Q_2} . The first results states that these two equivalence relations are the same.

Theorem 1.2 *Let $Q_1, Q_2 \in M$. Then Ω_1 and Ω_2 are biholomorphic, if and only if the two polynomials Q_1 and Q_2 are equivalent in the sense of definition 1.1. In particular the degrees of the non-harmonic parts Q_1^N and Q_2^N are equal, if the domains Ω_1 and Ω_2 are biholomorphic.*

The fact that Ω is hyperbolic implies that the holomorphic automorphism group $\operatorname{Aut}_{\mathcal{O}}(\Omega)$ is a real Lie group and that all isotropy groups of the action

of $\text{Aut}_{\mathcal{O}}(\Omega)$ on Ω are compact [3]. We denote by G, G_1, G_2 the connected identity components of $\text{Aut}_{\mathcal{O}}(\Omega)$, $\text{Aut}_{\mathcal{O}}(\Omega_1)$, $\text{Aut}_{\mathcal{O}}(\Omega_2)$. Clearly, if Ω_1 and Ω_2 are biholomorphic, then G_1 and G_2 are isomorphic.

Let $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ denote the Lie algebras of G, G_1, G_2 .

Let J, J_1, J_2 denote the subgroups of G, G_1, G_2 generated by the translation $\{(w, z) \mapsto (w+it, z) \mid t \in \mathbf{R}\}$ and j, j_1, j_2 their Lie algebras. Hence the dimension of G, G_1, G_2 is at least one.

The second result gives a “canonical” defining polynomial for the domain Ω if $\dim_{\mathbf{R}} \mathcal{G} \geq 2$.

Theorem 1.3 *Let $\Omega = \{\text{Re } w + Q(z) < 0\}$ as above. Assume that $\dim_{\mathbf{R}} G \geq 2$. Then there are the following cases :*

a) Ω is homogeneous. Then $\Omega \simeq \mathbf{B}_2 = \{|w|^2 + |z|^2 < 1\}$ and $Q \sim P_1 \sim P_2$, where $P_1(z, \bar{z}) = (\text{Re } z)^2$ and $P_2(z, \bar{z}) = |z|^2$.

b) Ω is not homogeneous.

1) $\dim_{\mathbf{R}} G = 2$. Then $\deg Q^N \geq 4$ and either i) $Q \sim P_1$ or ii) $Q \sim P_2$, or iii) $Q \sim P_3$, where

i) $P_1(z, \bar{z}) = P_1(\text{Re } z)$ is an element of M depending only on $\text{Re } z$ and $G \simeq (\mathbf{R}^2, +)$,

ii) $P_2(z, \bar{z}) = P_2(|z|^2)$ is an element of M depending only on $|z|^2$, and $G \simeq \mathbf{R} \times S^1$,

iii) $P_3(z, \bar{z})$ is a homogeneous polynomial of degree $2k$, $k \geq 2$, i.e. $P_3(\lambda z, \lambda \bar{z}) = \lambda^{2k} P_3(z, \bar{z})$ for all $\lambda \in \mathbf{R}$ and G is the non-abelian two dimensional real Lie group.

2) $\dim_{\mathbf{R}} G \geq 3$. Then $\deg Q^N \geq 4$ and either i) $Q \sim P_1$ or ii) $Q \sim P_2$ where

i) $P_1(z, \bar{z}) = (\text{Re } z)^{2k}$ and G is 3-dimensional and solvable,

ii) $P_2(z, \bar{z}) = |z|^{2k}$ and G is 4-dimensional and contains a finite covering of $SL_2(\mathbf{R})$.

We are going to prove the two theorems simultaneously by distinguishing the dimension of G . First we handle the one and two-dimensional cases, then the homogeneous case and we finish with the three and higher dimensional cases.

Before doing so, we prove the easy direction of theorem 1.1.

Lemma 1.4 *If $Q_1 \sim Q_2$, then Ω_1 and Ω_2 are biholomorphic.*

Proof : Assume (1.1). Let $\Psi = (\Psi_1, \Psi_2)$ be the biholomorphic map of $\mathbb{C}^2$ defined by

$$(*) \quad \begin{cases} \Psi_1(w, z) = \frac{1}{\rho}w + p(z) \\ \Psi_2(w, z) = g(z) \end{cases}$$

Then $\Psi(\Omega_1) = \Omega_2$. ■

Remark 1.5 *In what follows we will often make a global coordinate change in $\mathbb{C}^2$ like (*), which is coherent with the equivalence of the defining polynomials. In the following, we take the notation from above.*

2 The one-dimensional case

Let $\Psi : \Omega_1 \rightarrow \Omega_2$ be a biholomorphic map. For a subgroup $N \subset G_2$ let $\Psi^*(N)$ be the group $\Psi^{-1} \circ N \circ \Psi \subset G_1$.

Lemma 2.1 *Assume that $\Psi^*(J_2) = J_1$. Then $Q_1 \sim Q_2$.*

Proof : From our hypothesis it follows that there is a non-zero real number ρ such that

$$\Psi^{-1} \circ T_t \circ \Psi = T_{\rho t}, \quad (T_t(w, z) = (w + it, z)),$$

since Ψ^* is a continuous group isomorphism of two copies of $\mathbf{R}$.

So we get with $\Psi = (\Psi_1, \Psi_2)$

$$\begin{aligned} \Psi_1(w, z) + it &= \Psi_1(w + i\rho t, z) \\ \Psi_2(w, z) &= \Psi_2(w + i\rho t, z) \end{aligned}$$

which implies :

$$\begin{aligned} \Psi_1(w, z) &= \frac{1}{\rho}w + p(z) \\ \Psi_2(w, z) &= g(z) \end{aligned}$$

with $p \in \mathcal{O}(\mathbf{C})$ and $g \in \text{Aut}_{\mathcal{O}}(\mathbf{C})$, since the projection $\pi : \mathbb{C}^2 \rightarrow \mathbf{C}$, $(w, z) \mapsto z$ is surjective on Ω_1 and Ω_2 .

Therefore Ψ is a biholomorphic map of $\mathbb{C}^2$ which maps Ω_1 to Ω_2 and so we have

$$\begin{aligned} \Omega_1 &= \{\text{Re } w + Q_1(z, \bar{z}) < 0\} = \Psi^{-1}(\Omega_2) \\ &= \{\text{Re}(\frac{1}{\rho}w + p(z)) + Q_2(g(z), \overline{g(z)}) < 0\} \\ &= \{\text{Re } w + \rho \text{Re } p(z) + \rho Q_2(g(z), \overline{g(z)}) < 0\}. \end{aligned}$$

It follows that

$$Q_1(z, \bar{z}) = \rho \operatorname{Re} p(z) + \rho Q_2(g(z), \overline{g(z)}).$$

This equality implies the positivity of ρ and the fact that the holomorphic function $p(z)$ is already a polynomial. Hence $Q_1 \sim Q_2$. ■

We mention the following direct consequence, which is the statement of theorem 1.2 in the case $\dim_{\mathbf{R}} G_1 = 1$.

Corollary 2.2 *If $\dim_{\mathbf{R}} G_1 = 1$, then Q_1 and Q_2 are equivalent.*

Proof : Here we have $G_1 = J_1$ and $G_2 = J_2$, hence $\Psi^*(J_2) = J_1$. ■

3 The two-dimensional case

We are going to handle this case in a sequence of lemmas. We always assume that there is a two-dimensional subgroup $H \subset G$ such that $J \subset H$. Since $J \subset G$ is a closed subgroup isomorphic to $\mathbf{R}$ there are two possibilities for H :

- i) H is abelian and non-compact.
- ii) H is the solvable two dimensional non-abelian Lie group.

Lemma 3.1 *Suppose that H is abelian. Then $Q \sim P_1$ or $Q \sim P_2$, where $P_1(z, \bar{z}) = P_1(\operatorname{Re} z)$ is an element of M which depends only on $\operatorname{Re} z$, or $P_2(z, \bar{z}) = P_2(|z|^2)$ is an element of M which depends only on $|z|^2$.*

In the first case, the domain $\{\operatorname{Re} w + P_1(\operatorname{Re} z) < 0\}$ realizes the domain Ω as a tube domain.

Proof : Let $L = \{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbf{R}\}$ be a one parameter group of H such that L and J generate H . The group H being abelian implies that L and J commute and so we get for all $s, t \in \mathbf{R}$:

$$\begin{aligned} \sigma_1^t(w + is, z) &= \sigma_1^t(w, z) + is \\ \sigma_1^t(w + is, z) &= \sigma_1^t(w, z). \end{aligned}$$

The restriction of the projection $\pi : (w, z) \rightarrow z$ from $\mathbf{C}^2$ to Ω being surjective and the second equality imply that

$$\sigma_2^t(w, z) = \sigma_2^t(z)$$

is a non-trivial one-parameter subgroup of $\text{Aut}_{\mathcal{O}}(\mathbb{C}) \simeq \mathbb{C}^* \ltimes \mathbb{C}$. Furthermore $\sigma_1^t(w, z) = w + f(t, z)$, where $f(t, \cdot) \in \mathcal{O}(\mathbb{C})$. Since $\sigma^t \in \text{Aut}_{\mathcal{O}}(\mathbb{C}^2)$ and stabilises Ω , it follows that $f(t, \cdot)$ is a holomorphic polynomial for all $t \in \mathbb{R}$.

After a holomorphic change of coordinates in $\{z \in \mathbb{C}\}$, which is in fact polynomial and therefore coherent with the equivalence of defining polynomials, we have that

- a) $\sigma_2^t(z) = z + it$ or
- b) $\sigma_2^t(z) = e^{\alpha \cdot t} \cdot z$ for $\alpha \in \mathbb{C}^*$ fixed.

ad (a) : Here we have

$$\begin{aligned}\sigma_1^t(w, z) &= w + f(t, w) \\ \sigma_2^t(w, z) &= z + it \quad \text{for all } t \in \mathbb{R}.\end{aligned}$$

It follows that

$$(3.1) \quad f(t_1 + t_2, z) = f(t_1, z + it_2) + f(t_2, z) \quad \text{for all } t_1, t_2 \in \mathbb{R}.$$

and therefore there is a *holomorphic polynomial* $\tilde{f}$ such that

$$(3.2) \quad f(t, z) = \tilde{f}(z + it) - \tilde{f}(z).$$

After the change of coordinates in $\mathbb{C}^2$

$$\begin{pmatrix} \tilde{w} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} w - \tilde{f}(z) \\ z \end{pmatrix},$$

we have that Ω is given by $\{\text{Re } \tilde{w} + \tilde{Q}(\tilde{z}, \bar{\tilde{z}}) < 0\}$, with a polynomial $\tilde{Q}$ equivalent to Q . The action of L is then given by

$$\begin{aligned}\sigma_1^t(\tilde{w}, \tilde{z}) &= \tilde{w} \\ \sigma_2^t(\tilde{w}, \tilde{z}) &= \tilde{z} + it.\end{aligned}$$

This means that $\tilde{Q}(\tilde{z}, \bar{\tilde{z}})$ is invariant under translations of the form $\{\tilde{z} \mapsto \tilde{z} + it \mid t \in \mathbb{R}\}$, which implies that $\tilde{Q}(\tilde{z}, \bar{\tilde{z}}) = \tilde{Q}(\text{Re } \tilde{z})$ and that Ω is realized as a tube domain. The group H is isomorphic to $(\mathbb{R}^2, +)$.

ad (b) : In this case, we have

$$\begin{aligned}\sigma_1^t(w, z) &= w + f(t, z) \\ \sigma_2^t(w, z) &= e^{\alpha t} \cdot z\end{aligned}$$

for all $t \in \mathbf{R}$ with fixed $\alpha = a + ib \in \mathbf{C}^*$. By the same argument as in case (a), we see that $f(t, \cdot)$ is a holomorphic polynomial and that $\sigma^t \in \text{Aut}_{\mathcal{O}}(\mathbf{C}^2)$ for all $t \in \mathbf{R}$. So we have :

$$\begin{aligned}\Omega &= \{(w, z) \in \mathbf{C}^2 \mid \text{Re } w + Q(z, \bar{z}) < 0\} \\ &= \{(w, z) \in \mathbf{C}^2 \mid \text{Re } w + \text{Re } f(t, z) + Q(e^{\alpha t} \cdot z, e^{\bar{\alpha} t} \cdot \bar{z}) < 0\} \text{ for all } t \in \mathbf{R},\end{aligned}$$

i.e. $Q(z, \bar{z}) = \text{Re } f(t, z) + Q(e^{\alpha t} \cdot z, e^{\bar{\alpha} t} \cdot \bar{z})$. Without loss of generality, we may assume that the harmonic part of Q is trivial, which implies that $\text{Re } f(t, z) \equiv 0$ for all $t \in \mathbf{R}$, i.e. $f(t, z) = f(t) \in i\mathbf{R}$ for all $t \in \mathbf{R}$. Hence $f(t) = i\beta t$ with $\beta \in \mathbf{R}$. Then we have that $Q(z, \bar{z}) = Q(e^{\alpha t} \cdot z, e^{\bar{\alpha} t} \cdot \bar{z})$ for all $t \in \mathbf{R}$. This implies that $\alpha \in i\mathbf{R}^*$ and that $Q(z, \bar{z}) = Q(|z|^2)$, i.e. the polynomial Q depends only on $|z|^2$.

The action of L then is given by

$$\begin{aligned}\sigma_1^t(w, z) &= w + i\beta t \\ \sigma_2^t(w, z) &= e^{\alpha t} \cdot z, \quad \text{for all } t \in \mathbf{R}.\end{aligned}$$

The group H is isomorphic to $\mathbf{R} \times S^1$. ■

Lemma 3.2 *Suppose that H is the two dimensional solvable non-abelian Lie group. Then the polynomial Q is equivalent to a polynomial P_{2k} , which is homogeneous of degree $2k$, i.e. $P_{2k}(\lambda z, \lambda \bar{z}) = \lambda^{2k} P_{2k}(z, \bar{z})$ for all $\lambda \in \mathbf{R}$ and J is a normal subgroup of H .*

Proof : Let $L = \{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbf{R}\}$ be a one parameter subgroup of H such that L and J generate H . Then there are two cases :

- a) J is not the normal subgroup of dimension one in H .
- b) J is normal in H .

ad(a) : We may assume that L is normal in H .

Let $X = i\frac{\partial}{\partial w} - i\frac{\partial}{\partial \bar{w}}$ and $Y = f\frac{\partial}{\partial w} + g\frac{\partial}{\partial z} + \bar{f}\frac{\partial}{\partial \bar{w}} + \bar{g}\frac{\partial}{\partial \bar{z}}$ be the two holomorphic infinitesimal transformations induced by J and L on Ω . By our assumption there is a $\lambda \in \mathbf{R}^*$ such that $[X, Y] = \lambda \cdot Y$. This equation yields $f(w, z) = e^{-i\lambda w} h_1(z)$ and $g(w, z) = e^{-i\lambda w} h_2(z)$, $h_1, h_2 \in \mathcal{O}(\mathbf{C})$. It follows that Y is a global infinitesimal holomorphic transformation of $\mathbf{C}^2$, since $\pi : \Omega \rightarrow \mathbf{C}$, $(w, z) \mapsto z$ is surjective.

Furthermore h_2 vanishes nowhere, since $h_2(z_0) = 0$ implies that the set $\{(w, z_0) \mid \text{Re } w + Q(z_0, \bar{z}_0) < 0\}$ is stabilized by H with J as a non-normal subgroup which is impossible. Now we have $Y(\text{Re } w + Q(z, \bar{z}))|_{\{\text{Re } w + Q(z, \bar{z})=0\}} \equiv 0$.

This yields the equation

$$h_1(z) + h_2(z) \frac{\partial Q}{\partial z}(z, \bar{z}) + e^{2i\lambda Q(z, \bar{z})} (\overline{h_1(z)} + \overline{h_2(z)} \frac{\partial Q}{\partial \bar{z}}(z, \bar{z})) \equiv 0.$$

The expression $h_1(z) + h_2(z) \frac{\partial Q}{\partial z}(z, \bar{z})$ being a polynomial in $\bar{z}$ implies that the expression $e^{2i\lambda Q(z, \bar{z})} (\overline{h_1(z)} + \overline{h_2(z)} \frac{\partial Q}{\partial \bar{z}}(z, \bar{z}))$ is also a polynomial in $\bar{z}$. By differentiating n times, $n \in \mathbb{N}$ with respect to $\bar{z}$ this yields that $\overline{h_2(z)} = 0$ for all $z \in \mathbb{C}$, a contradiction to the fact mentioned above.

ad (b) : Assume that J is normal in H . We get

$$\begin{aligned} \sigma_1^t(w + is, z) &= \sigma_1^t(w, z) + ie^{\alpha t} \cdot s \\ \sigma_2^t(w + is, z) &= \sigma_2^t(w, z), \alpha \in \mathbb{R}^* \text{ fixed.} \end{aligned}$$

So we have again $\sigma_2^t(w, z) = \sigma_2^t(z)$ and $\sigma_2^t \in \text{Aut}_{\mathcal{O}}(\mathbb{C})$ for all $t \in \mathbb{C}$.

Furthermore $\sigma_1^t(w, z) = e^{\alpha t} w + f(t, z)$ with $f(t, \cdot) \in \mathcal{O}(\mathbb{C})$ for all $t \in \mathbb{R}$. Hence $\sigma^t \in \text{Aut}_{\mathcal{O}}(\mathbb{C}^2)$ and $f(t, z)$ is a holomorphic polynomial for all $t \in \mathbb{R}$.

Since $\dim_{\mathbb{R}} H = 2$, the one parameter group $\{\sigma_2^t(z) \mid t \in \mathbb{R}\} \subset \text{Aut}_{\mathcal{O}}(\mathbb{C})$ cannot be trivial. So after a change of coordinates in the z -variable, we have

- (i) $\sigma_2^t(z) = z + it$ or
- (ii) $\sigma_2^t(z) = e^{\beta t} \cdot z$, $\beta \in \mathbb{C}^*$ fixed.

If (i) $\sigma_2^t = z + it$, we get

$$\begin{aligned} \sigma_1^t(w, z) &= e^{\alpha t} w + f(t, z) \\ \sigma_2^t(w, z) &= z + it \text{ and } \sigma^t \in \text{Aut}_{\mathcal{O}}(\mathbb{C}^2) \end{aligned}$$

This yields

$$Q(z, \bar{z}) = e^{-\alpha t} \text{Re } f(t, z) + e^{-\alpha t} Q(z + it, \bar{z} - it).$$

It is easy to see that this is not possible by considering the highest degree homogeneous summand of the non-harmonic part Q^N of Q .

So we may assume (ii), $\sigma_2^t(z) = e^{\beta t} \cdot z$, $\beta \in \mathbb{C}^*$ fixed.

Hence

$$\begin{aligned} \sigma_1^t(w, z) &= e^{\alpha t} \cdot w + f(t, z) \\ \sigma_2^t(w, z) &= e^{\beta t} \cdot z, \quad \alpha \in \mathbb{R}^*, \quad \beta = a + ib \in \mathbb{C}^* \text{ fixed} \end{aligned}$$

and it follows that

$$Q(z, \bar{z}) = e^{-\alpha t} \text{Re } f(t, z) + e^{-\alpha t} Q(e^{\beta t} z, e^{\bar{\beta} t} \bar{z}).$$

We may assume that Q has no harmonic summands and therefore

$$Q(z, \bar{z}) = e^{-\alpha t} Q(e^{\beta t} \cdot z, e^{\bar{\beta} t} \cdot \bar{z}), \text{ for all } t \in \mathbf{R}.$$

The highest degree of Q is an even number $2k$, $k \in \mathbf{N}^*$. Let

$Q_{2k}(z, \bar{z}) = \sum_{j=1}^{2k-1} a_j z^j \bar{z}^{2k-j}$, ($a_j = \bar{a}_{2k-j}$) be the highest degree homogeneous summand of Q . We get

$$\begin{aligned} Q_{2k}(z, \bar{z}) &= e^{-\alpha t} Q_{2k}(e^{\beta t} \cdot z, e^{\bar{\beta} t} \cdot \bar{z}), \text{ i.e.} \\ a_j &= a_j e^{-\alpha t} \cdot e^{j\beta t + (2k-j)\bar{\beta} t}, 1 \leq j \leq 2k-1. \end{aligned}$$

A necessary condition for this is

$$\alpha = 2k \cdot \operatorname{Re} \beta$$

and that there are no summands in Q of degree smaller than $2k$.

Hence $Q = Q_{2k} = P_{2k}$ and the lemma is proved. ■

Remark 3.3 *Lemma 3.1 and Lemma 3.2 give the proof of theorem 1.3 in the case $\dim_{\mathbf{R}} G = 2$.*

Lemma 3.4 *Let $\Omega_1 = \{\operatorname{Re} w + Q_1(z, \bar{z}) < 0\}$ and $\Omega_2 = \{\operatorname{Re} w + Q_2(z, \bar{z}) < 0\}$ like above. Assume that $\Psi : \Omega_1 \rightarrow \Omega_2$ is biholomorphic and that J_1 and $\Psi^*(J_2)$ are both contained in a two-dimensional subgroup $H \subset G_1$. Then $J_1 = \Psi^*(J_2)$ and $Q_1 \sim Q_2$.*

Proof : We have again to consider the following two cases :

- a) H is abelian,
- b) H is not abelian.

In both cases, we assume $J_1 \neq \Psi^*(J_2)$ and produce a contradiction. Let $\Psi^*(J_2) = \{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbf{R}\}$.

ad(a) : i) Assume that $H = (\mathbf{R}^2, +)$. Then by Lemma 3.1, we may suppose that $\Omega_1 = \{\operatorname{Re} w + Q_1(\operatorname{Re} w) < 0\}$ and $\Omega_2 = \{\operatorname{Re} w + Q_2(\operatorname{Re} z) < 0\}$ are already realized as tube domains and that the biholomorphism Ψ is equivariant with respect to the action of $H \simeq i\mathbf{R}^2$ as imaginary translations on both domains. Hence Ψ is an affine linear automorphism of $\mathbf{C}^2$, i.e. $\Psi_1(w, z) = aw + bz + e$, $\Psi_2(w, z) = cw + dz + f$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbf{R}) \quad \text{and} \quad \begin{pmatrix} e \\ f \end{pmatrix} \in \mathbf{R}^2$$

We get

$$\Omega_1 = \{\operatorname{Re} w + Q_1(\operatorname{Re} z) < 0\} = \{a \operatorname{Re} w + b \operatorname{Re} z + e + Q_2(c \operatorname{Re} w + d \operatorname{Re} z + f) < 0\},$$

which implies $c = 0$, i.e. $J_1 = \Psi^*(J_2)$ and that the two polynomials are equivalent.

ii) Assume that $H = \mathbf{R} \times S^1$. Then by Lemma 3.1, we may assume that $\Omega_1 = \{\operatorname{Re} w + Q_1(|z|^2) < 0\}$ and $\Omega_2 = \{\operatorname{Re} w + Q_2(|z|^2) < 0\}$ where Q_1 and Q_2 depend only on $|z|^2$. Furthermore, the action of S^1 on both domains is given by rotations in the z -variable. Hence there is an $\alpha \in \mathbf{R}^*$ such that

$$\begin{aligned}\Psi_1(w, e^{i\alpha t} \cdot z) &= \Psi_1(w, z) \\ \Psi_2(w, e^{i\alpha t} \cdot z) &= e^{it} \cdot \Psi_2(w, z), \quad \text{for all } t \in \mathbf{R}.\end{aligned}$$

We get $\alpha = 1$ and

$$(1) \quad \begin{cases} \Psi_1(w, z) = \Psi_1(w) \\ \Psi_2(w, z) = z \cdot g(w). \end{cases}$$

Furthermore there exist $b \in \mathbf{R}$, $\beta \in \mathbf{R}^*$ such that $\Psi^*(J_2) = \{\sigma^t \mid t \in \mathbf{R}\}$ looks like :

$$\begin{aligned}\sigma_1^t(w, z) &= w + i\beta t \\ \sigma_2^t(w, z) &= e^{ibt} z\end{aligned}$$

We get

$$\begin{aligned}\Psi_1(w + i\beta t, e^{ibt} \cdot z) &= \Psi_1(w, z) + it \\ \Psi_2(w + i\beta t, e^{ibt} \cdot z) &= \Psi_2(w, z).\end{aligned}$$

Now the above expression (1) yields

$$\begin{aligned}\Psi_1(w, z) &= \Psi_1(w) = \frac{1}{\beta} w \\ \Psi_2(w, z) &= z \cdot g(w) = e^{ibt} \cdot z \cdot g(w + i\beta t), \quad \text{for all } t \in \mathbf{R},\end{aligned}$$

i.e. $e^{-ibt} g(w) = g(w + i\beta t)$, for all $t \in \mathbf{R}$.

It follows :

$$\begin{aligned}-ibg(w) &= g'(w)i\beta, \text{ i.e.} \\ g'(w) &= -\frac{b}{\beta}g(w), \text{ hence} \\ g(w) &= c \cdot e^{-\frac{b}{\beta} \cdot w}\end{aligned}$$

and Ψ is a global automorphism of $\mathbf{C}^2$. This yields easily that $b = 0$ and $c \neq 0$, i.e. $\Psi_2(w, z) = c \cdot z$. But then $\Psi^*(J_2) = J_1$ and $Q_1 \sim Q_2$.

ad (b) : Assume that H is not abelian. By lemma 3.2, we have $J_1 \triangleleft H$. Suppose that $\Psi^*(J_2) \neq J_1$. Let $\Sigma = \Psi^{-1}$ the inverse of Ψ . Then we have that $J_2 = \Sigma^*(\Psi^*(J_2))$ is not normal in H . But lemma 3.2 applied to the domain Ω_2 gives a contradiction. Hence $\Psi^*(J_2) = J_1$ and Lemma 3.4 is proved. ■

Remark 3.5 : Lemma 3.4 gives the proof of Theorem 1.2 in the case $\dim_{\mathbb{R}} G_1 = \dim_{\mathbb{R}} G_2 = 2$.

4 The homogeneous case

Now we are going to handle the case when the domains in question are homogeneous, i.e. the group G acts transitively on them.

Assume that $\Omega = \{\operatorname{Re} w + Q(z, \bar{z}) < 0\}$ is a homogeneous complex manifold.

Then by a theorem of Rosay [5] the domain Ω is biholomorphic to the unit ball $\mathbf{B}_2 = \{|w|^2 + |z|^2 < 1\}$. As other “canonical” models for $\mathbf{B}^2$ we mention the two realisations $\{\operatorname{Re} w + (\operatorname{Re} z)^2 < 0\}$ and $\{\operatorname{Re} w + |z|^2 < 0\}$, which we use in the sequel. Here the polynomials $(\operatorname{Re} z)^2$ and $|z|^2$ are obviously equivalent.

So we assume that $\Omega_1 = \{\operatorname{Re} w + (\operatorname{Re} z)^2 < 0\}$ and $\Omega_2 = \{\operatorname{Re} w + Q_2(z, \bar{z}) < 0\}$.

Lemma 4.1 *Suppose that Ω_1 and Ω_2 are biholomorphic. Then $Q_2(z, \bar{z}) \sim (\operatorname{Re} z)^2$.*

Proof : Let $\Psi : \Omega_1 \rightarrow \Omega_2$ denote a biholomorphism. The group G_1 is isomorphic to $SU(2, 1)$ and J_1 and $\Psi^*(J_2)$ are two closed one-dimensional non-compact subgroups of $SU(2, 1)$. By investigating the structure of $SU(2, 1)$ one can show that the normaliser $N_{G_1}(J_1)$ of J_1 in G_1 is five-dimensional and closed and that there is an element $g \in G_1$ such that $g\Psi^*(J_2)g^{-1} \subset N_{G_1}(J_1)$. So one can replace the map Ψ by another biholomorphism $\tilde{\Psi}$, such that J_1 and $\tilde{\Psi}^*(J_2)$ are contained in a two dimensional subgroup H of G_1 . But then by lemma 3.4 $\Psi^*(J_2) = J_1$ and $Q_2(z, \bar{z}) = (\operatorname{Re} z)^2$. ■

Remark 4.2 : The above mentioned theorem of Rosay and lemma 4.1 prove theorem 1.2 and theorem 1.3 in the homogeneous case.

5 The three-dimensional case

We start with the following two useful lemmas.

Lemma 5.1 *Let $H \subset G$ be an at least three-dimensional subgroup of $G = \operatorname{Aut}_{\mathcal{O}}^0(\Omega)$. Then H is not abelian.*

Proof : By assumption G and therefore H act effectively on Ω . The lemma follows from the fact that Ω is a two-dimensional hyperbolic complex manifold. ■

Lemma 5.2 *Assume that $G = \text{Aut}_\Omega^0(\Omega)$ is not solvable and that Ω is not homogeneous. Let $\mathcal{G} = s \ltimes r$ be a Levi-Malcev decomposition of $\mathcal{G} = \text{Lie}(G)$. Then the semisimple part s is isomorphic to $sl_2(\mathbf{R})$, the Lie algebra of $SL_2(\mathbf{R})$.*

Proof : Let $\tilde{s}$ be a complex simple Lie algebra admitting a one or two codimensional complex subalgebra. Then $\tilde{s} \simeq sl_2(\mathbf{C})$ or $\tilde{s} = sl_3(\mathbf{C})$.

Hence our real semi-simple algebra s is isomorphic to $sl_2(\mathbf{R})$, $su(2)$, $sl_3(\mathbf{R})$, $su(2, 1)$ or $su(3)$.

In the last four cases, s admits a subalgebra, which is isomorphic to $su(2)$. This means that we have an almost effective action of $SU(2, \mathbf{C})$ on Ω . Then the generic orbit of this action is a compact 3-dimensional CR -hypersurface isomorphic to a finite quotient of S^3 . But we have also the non-compact closed subgroup $J \subset G$, which shows that G has an open orbit in Ω . This orbit is isomorphic to the unit ball $\mathbf{B}_2$ and for a point p in this orbit the isotropy group $I_G(p)$ is a maximal compact subgroup K . Assume that there is a point $q \in \Omega$ such that $\dim_{\mathbf{R}} G(q) < 4$. The Ω being hyperbolic implies that $I_G(q)$ is compact and of greater dimension than K . This is impossible. Hence Ω is already homogeneous. But this contradicts our assumption. Hence $s \simeq sl_2(\mathbf{R})$ and the lemma is proved. ■

Now we assume that $\dim_{\mathbf{R}} G \geq 3$ and that there is a three-dimensional subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$ such that $J \subset H$. In view of Lemmas 5.1 and 5.2, we have the following cases :

- I. $\mathfrak{h}$ is solvable and not abelian.
- II. $\mathfrak{h} \simeq sL_2(\mathbf{R})$.

5.1 Case I :

The Lie algebra $\mathfrak{h}$ is solvable and $\dim_{\mathbf{R}} \mathfrak{h} = 3$.

In view of lemma 5.1 $\mathfrak{h}$ cannot be abelian.

We use the classification of three-dimensional solvable Lie algebras given in [2]. Let $\mathfrak{h} = \langle a, b, c \rangle_{\mathbf{R}}$. Then there are the following cases :

- (1) $[a, b] = b$, $[a, c] = [b, c] = 0$;
- (2) $[a, c] = b$, $[a, b] = [c, b] = 0$, i.e. $\mathfrak{h}$ is nilpotent.
- (3) $[c, b] = 0$, $[a, b] = \alpha b + \beta c$, $[a, c] = \gamma b + \delta c$, where

$$D := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbf{R})$$

Lemma 5.3 *Assume that the structure of $\mathfrak{h}$ is given by (1) above. Then $j = \mathfrak{h}'$, the commutator of $\mathfrak{h}$ and $Q \sim P$, where $P(z, \bar{z}) = |z|^{2k}$, $k \geq 2$.*

Proof : In view of lemma 3.2, we have that $j \subset \langle b, c \rangle_{\mathbf{R}} \subset \mathfrak{h}$. Our first step of the proof will be to prove that the group H cannot be simply connected. So we assume this and produce a contradiction.

Then the group L associated to the Lie algebra $\mathfrak{l} = \langle b, c \rangle_{\mathbf{R}}$ is isomorphic to $(\mathbf{R}^2, +)$ and contains J .

Hence $\Omega = \{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\}$ by lemma 3.1. Since L is normal H we have by [4] that the group H acts as a subgroup of $GL_2(\mathbf{R}) \ltimes \mathbf{R}^2$ on $\mathbf{C}^2$ and hence on Ω . So we have a one parameter subgroup $\{(A(t), v(t)) \in H \mid t \in \mathbf{R}\}$ with $\{A(t) \mid t \in \mathbf{R}\} \subset GL_2(\mathbf{R})$ being a non-trivial one parameter subgroup of $GL_2(\mathbf{R})$. By considering the Lie algebra structure of $\mathfrak{h}$ and the shape of Ω , it is an easy calculation to see that this is impossible.

Hence H is not simply connected and isomorphic to $N \times S^1$ where N is the non-abelian two-dimensional Lie group. The group J is contained in $N' \times S^1 \simeq \mathbf{R} \times S^1$ and therefore we have that $\Omega = \{\operatorname{Re} w + Q(|z|^2) < 0\}$, the action of S^1 being given as the rotations in the z -variable.

Now let $\{\sigma^t \mid t \in \mathbf{R}\}$ be the one parameter subgroup of H with Lie algebra $\langle a \rangle_{\mathbf{R}}$. Since S^1 is central in H , it follows

$$\begin{aligned} \sigma_1^t(w, e^{is} \cdot z) &= \sigma_1^t(w, z) \\ \sigma_2^t(w, e^{is} \cdot z) &= e^{is} \sigma_2^t(w, z) \quad \text{for all } s, t \in \mathbf{R}, \end{aligned}$$

i.e.

$$\begin{aligned} \sigma_1^t(w, z) &= \sigma_1^t(w) \\ \sigma_2^t(w, z) &= g(t, w) \cdot z \quad \text{with } g(t, \cdot) \text{ holomorphic in } w. \end{aligned}$$

Furthermore there is a non-compact one parameter group of the form

$$\left\{ \begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} w + is \\ e^{i\alpha s} \cdot z \end{pmatrix} \mid \alpha \in \mathbf{R} \text{ fixed}, t \in \mathbf{R} \right\} \triangleleft N$$

which generates together with $\{\sigma^t\}$ the group N i.e. there is a $\rho \in \mathbf{R}^*$ such that

$$\begin{aligned} \sigma_1^t(w + is, e^{i\alpha s} \cdot z) &= \sigma_1^t(w, z) + ie^{\rho t} \cdot s \\ \sigma_2^t(w + is, e^{i\alpha s} \cdot z) &= \sigma_1^t(w, z) \cdot e^{i\alpha e^{\rho t} \cdot s} \end{aligned}$$

and so

$$\begin{aligned} \sigma_1^t(w, z) &= e^{\rho t} \cdot w \\ \sigma_2^t(w, z) &= g(t, w) \cdot z \end{aligned}$$

with $g(t, w) \cdot e^{i\alpha e^{\rho t} \cdot s} = g(t, w + is) \cdot e^{i\alpha s}$ for all $s, t \in \mathbf{R}$ i.e. $g(t, w + is) = e^{i\alpha s(e^{\rho t} - 1)} \cdot g(t, w)$ and so

$$\frac{\partial g}{\partial w}(t, w) = \alpha(e^{\rho t} - 1) \cdot g(t, w)$$

$$g(t, w) = c(t)e^{(\alpha(e^{\rho t} - 1)) \cdot w}.$$

Hence is a global automorphism of $\mathbf{C}^2$ stabilizing Ω . But this is only possible if $g(t, w)$ does not depend on w , i.e. $g(t, w) = g(t)$ and then

$$\sigma_1^t(w, z) = e^{\rho t} \cdot w, \quad \text{and} \quad \sigma_2^t(w, z) = g(t) \cdot z, \quad \text{with} \quad g(t + \tilde{t}) = g(t) \cdot g(\tilde{t}).$$

This implies $g(t) = c \cdot e^{\nu \cdot t}$, $\nu \in \mathbf{R}$. Then it is easy to conclude that $Q(z, \bar{z}) \sim |z|^{2k}$ and it is obvious that $J = N' \triangleleft H$. The lemma is proved. $\blacksquare$

Remark 5.4 : In the setting of lemma 5.3, i.e. $\Omega = \{\operatorname{Re} w + |z|^{2k} < 0\}$, the automorphism group G of Ω is $S \cdot T$, where S is a finite covering of $SL_2(\mathbf{R})$ and T is a central subgroup isomorphic to S^1 , i.e. $\dim G = 4$. This case will also appear below.

Lemma 5.5 Assume that the structure of $\mathfrak{h}$ is given by (2) above. Then Ω is biholomorphic to the unit ball $\mathbf{B}_2$.

Proof : Here $\mathfrak{h}$ is isomorphic to the Lie algebra of the three-dimensional Heisenberg group H_3 . First we consider the case that H is not simply-connected. Then $H = H_3/\Gamma$, where Γ is a discrete subgroup of H_3 isomorphic to $\mathbf{Z}$ lying in the center C of H_3 . Hence H contains a central subgroup $L = C/\Gamma \simeq S^1$. Then J and L generate a two-dimensional subgroup isomorphic to $\mathbf{R} \times S^1$ and by lemma 3.1 we may assume that $\Omega = \{\operatorname{Re} w + Q(|z|^2) < 0\}$ with the natural $\mathbf{R} \times S^1$ action. The polynomial Q depends only on $|z|^2$, is subharmonic and can be assumed to satisfy $Q(0) = 0$ and $Q \geq 0$. Then $\tau(\Omega) = \{\operatorname{Re} w < 0\}$, where $\tau : (w, z) \rightarrow z$ from $\mathbf{C}^2$ to $\mathbf{C}$ denotes the projection on the first component.

This map is an equivariant H -map since $L \simeq S^1$ is central in H and the L -action is given by rotations in the Z -variable. Therefore the two-dimensional group H/L acts on $\{\operatorname{Re} w < 0\} = \tau(\Omega)$. But this action cannot be effective, since there is no two-dimensional abelian subgroup in the automorphism group of the half-plane. Hence a two-dimensional subgroup of H containing L stabilizes all fibers of τ and acts effectively on the fibers. But the τ -fibers in Ω are also half-planes and every two-dimensional subgroup of H is abelian. This is again not possible. So we have proven that H is isomorphic to the simply-connected Heisenberg group H_3 . Then there is a two-dimensional subgroup A containing

J which is isomorphic to $(\mathbf{R}^2, +)$. By lemma 3.1, the domain Ω is given by $\{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\}$ a tube domain.

Let $\{\sigma^t = (\sigma_1^t, \sigma_2^t) \mid t \in \mathbf{R}\}$ be a one-parameter group in H which together with A generates H . Since $A \subset H$ is normal, we have by [4] that $\{\sigma^t \mid t \in \mathbf{R}\}$ is a subgroup of the affine linear group $GL_2(\mathbf{R}) \ltimes \mathbf{R}^2$.

So let $\left\{ \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \begin{pmatrix} e(t) \\ f(t) \end{pmatrix} = \sigma^t \right\} \subset GL_2(\mathbf{R}) \ltimes \mathbf{R}^2$ denote this group.

The group $\{A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \mid t \in \mathbf{R}\}$ is not trivial in $GL_2(\mathbf{R})$. We have

$$\sigma^t(w, z) = \begin{pmatrix} a(t)w + b(t)z + e(t) \\ c(t)w + d(t)z + f(t) \end{pmatrix}$$

and σ^t stabilizing Ω implies :

$$\begin{aligned} \Omega &= \{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\} \\ &= \{a(t)\operatorname{Re} w + b(t)\operatorname{Re} z + e(t) + Q(c(t)\operatorname{Re} w + d(t)\operatorname{Re} z + f(t)) < 0\} \end{aligned}$$

It follows immediately that $c(t) = 0$ for all $t \in \mathbf{R}$ and that

$$Q(\operatorname{Re} z) = \frac{b(t)}{a(t)} \operatorname{Re} z + \frac{e(t)}{a(t)} + \frac{1}{a(t)} Q(d(t)\operatorname{Re} z + f(t)).$$

The group H being nilpotent implies that $a(t) = d(t) = 1$ for all $t \in \mathbf{R}$, i.e.

$$Q(\operatorname{Re} z) = b(t)\operatorname{Re} z + e(t) + Q(\operatorname{Re} z + f(t)).$$

Since $b(t)$ is not identically zero, this equation implies that $\deg Q = 2$ and that Ω is biholomorphic to $\mathbf{B}_2$. ■

Lemma 5.6 *Assume that the structure of $\mathfrak{h}$ is given by (3) above and that Ω is not homogeneous. Then $\Omega = \{\operatorname{Re} w + (\operatorname{Re} w)^{2k} < 0\}$, $k \geq 2$ and $G = H$.*

Proof : The structure of h implies that $\dim_{\mathbf{R}} h' = 2$ and that the associated group $H' \subset H$ is isomorphic to $(\mathbf{R}^2, +)$. So Ω as a simply-connected hyperbolic Stein manifold of dimension two with an action of $(\mathbf{R}^2, +)$, therefore it is biholomorphic to a tube domain $\Omega' = F + i\mathbf{R}^2$, where F is a convex domain in $\mathbf{R}^2$ containing no complex lines (see [7]). The group $H' \simeq (\mathbf{R}^2, +)$ being normal in H implies that H acts on Ω' as a subgroup of $GL_2(\mathbf{R}) \ltimes \mathbf{C}^2$ (see [4]).

Let $\{\sigma^t = (\sigma_1^t, \sigma_2^t)\}$ be a one-parameter subgroup of H generating together with H' the group H . Then

$$\sigma^t = \left(\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \begin{pmatrix} e(t) \\ f(t) \end{pmatrix} \right) \in GL_2(\mathbf{R}) \ltimes \mathbf{R}^2$$

$$= (A(t), \vec{v}(t)),$$

with $A(t) = e^{t \cdot D}$, where

$$D = \begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}.$$

Since $D \in GL_2(\mathbf{R})$, after a conjugation with an element of

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ltimes \mathbf{R}^2 \right\},$$

we have that $\vec{v}(t) = 0$ for all $t \in \mathbf{R}$, i.e.

$$\sigma^t = \left(\begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \text{ for all } t \in \mathbf{R}$$

Now assume that D is not triangulisable over $\mathbf{R}$. Then $\{\sigma^t \mid t \in \mathbf{R}\}$ is isomorphic to S^1 , since any one dimensional subgroup of $GL_2(\mathbf{R})$, which is not compact, is triangulisable over $\mathbf{R}$. So the domain $F \subset \mathbf{R}^2$ is invariant by a linear S^1 -action and must therefore be bounded.

On the other hand we have that J has to lie in H' because otherwise it would be a compact group. Then $\Omega = \{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\}$ and H acts affinely on Ω . Since the set $\{(y, x) \in \mathbf{R}^2 \mid y + Q(x) < 0\}$ is not bounded we get a contradiction.

So we can assume that the matrix D is triangulisable over $\mathbf{R}$. Hence H' contains a one-dimensional normal subgroup of H . If $J \notin H'$, then this group and J generate a two-dimensional non-abelian group, which is impossible by lemma 3.2.

So we have that $J \subset H' \simeq (\mathbf{R}^2, +)$, $\Omega = \{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\}$ a tube domain and that H acts affinely on $\mathbf{C}^2$ and on Ω with $H' \subset H$ the group of imaginary translations as a normal subgroup.

We have that

$$\begin{aligned} \sigma^t = A(t) &= \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \in GL_2(\mathbf{R}) \\ &= e^{t \cdot D}, D = \begin{pmatrix} \alpha & \beta \\ \mu & \delta \end{pmatrix}, t \in \mathbf{R}. \end{aligned}$$

Then

$$\begin{aligned} \Omega &= \{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\} \\ &= \{a(t) \operatorname{Re} w + b(t) \operatorname{Re} z + Q(c(t) \operatorname{Re} w + d(t) \operatorname{Re} z) < 0\} \end{aligned}$$

which shows that $c(t) \equiv 0$, i.e.

$$Q(\operatorname{Re} z) = \frac{b(t)}{a(t)} \operatorname{Re} z + \frac{1}{a(t)} Q(d(t) \operatorname{Re} z) \quad \text{for all } t \in \mathbf{R}.$$

Since we may assume that Q has no harmonic summands we get

$$Q(\operatorname{Re} z) = \frac{1}{a(t)} Q(d(t) \operatorname{Re} z).$$

This implies that $Q(\operatorname{Re} z) = (\operatorname{Re} z)^{2k}$, $k \geq 2$ and that the action of σ^t is given by

$$\sigma^t(w, z) = (e^{2kt} \cdot w, e^t \cdot z), t \in \mathbf{R}.$$

Now we prove that $G = H$. First we show that G is solvable. Assume to the contrary that G is not solvable. Then, since Ω is not homogeneous, the semisimple part of G is isomorphic to a covering of $SL_2(\mathbf{R})$. Then by checking the possibilities for G as an automorphism group of a 2-dimensional hyperbolic manifold (see Case II) it is easy to see that G' does not contain a two-dimensional abelian subgroup. So G is solvable and $\dim_{\mathbf{R}} G' \geq 2$. Furthermore G' is nilpotent and contains $H' \simeq (\mathbf{R}^2, +)$. Then it is easy to see (by checking the possibilities for G') that $H' \triangleleft G'$, which implies that $H' = G'$ (lemma 5.1 and lemma 5.5). Then $H' \triangleleft G$ and by applying again [4] one concludes that $G = H$. ■

5.2 Case II : $\mathfrak{h} \sim sl_2(\mathbf{R})$

Here we are going to handle completely the situation where Ω is not homogeneous and G is not solvable.

By lemma 5.2, there is a three-dimensional subgroup H of G such that the Lie algebra $\mathfrak{h}$ is isomorphic to $sl_2(\mathbf{R})$.

Since Ω is not homogeneous we have that $3 \leq \dim_{\mathbf{R}} \mathcal{G} \leq 5$, in view of the possibilities of a maximal compact subgroup K : $K = (e)$, $K = S^1$, $K = (S^1)^2$.

Let $\mathcal{G} = \mathfrak{h} \ltimes r$ be a Levi-Malcev decomposition of $\mathcal{G}$. Here r denotes the radical of $\mathcal{G}$. Hence $\dim_{\mathbf{R}} r = 1$ or 2 . If $\dim_{\mathbf{R}} r = 2$, then r is abelian, because otherwise the center of $SL_2(\mathbf{R}) \ltimes R$ is too small to admit a discrete central quotient with maximal compact subgroup $(S^1)^2$. But then $\mathcal{G} = \mathfrak{h} \times \tau$ is a direct product again because otherwise there is no central subgroup with quotient $(S^1)^2$. The existence of a three-dimensional abelian subgroup excludes this case (Lemma 5.1). If $\dim_{\mathbf{R}} r = 1$, then $\mathcal{G} = h \times r$ a direct product.

Hence we have only two possibilities for $\mathcal{G}$:

$$\mathcal{G} = \mathfrak{h} = sl_2(\mathbf{R}) \text{ or } \mathcal{G} = \mathfrak{h} \times \mathbf{R} = sl_2(\mathbf{R}) \times \mathbf{R}.$$

We consider these cases in the following lemmas.

Lemma 5.7 *Assume that $j \subset \mathfrak{h} \subset \mathcal{G}$. Then J is contained in a two-dimensional subgroup of H .*

Proof : If H is modulo a finite covering isomorphic to $SL_2(\mathbf{R})$, then J as a non-compact subgroup of H is contained in a two-dimensional subgroup of H . So assume that $H \simeq \widetilde{SL_2(\mathbf{R})}$, the universal covering of $SL_2(\mathbf{R})$, and let C denote the center of H which is isomorphic to $\mathbf{Z}$. If $J \cap C = (e)$, then J is also contained in a two-dimensional subgroup of H . So assume that $J \cap C \neq (e)$, i.e. $J \cap C \simeq \mathbf{Z}$.

First this implies that H is a closed subgroup of G . (If $H \simeq \widetilde{SL_2(\mathbf{R})}$ is not closed in G , then the maximal compact subgroup K of G is $(S^1)^2$ and contains C . But $J \subset G$ is a closed, non-compact subgroup of G and therefore $J \cap C = (e)$, which is a contradiction.)

Hence H acts freely on Ω and all orbits are closed and isomorphic to $\mathbf{R}^3$. We may assume that $J \cap C = \{(w, z) \mapsto (w + 2\pi ik, z) \mid k \in \mathbf{Z}\}$. This group acts freely and properly discontinuous on Ω and we can consider the quotient

$$\Omega = \{\operatorname{Re} w + Q(z, \bar{z}) < 0\} \xrightarrow{(e^w, z)} \{0 < |w|^2 e^{2Q(z, \bar{z})} < 1\} = \Omega'.$$

Then there is an action of a group $S = SL_2(\mathbf{R})/J \cap C$ on Ω' and the group $J/J \cap C$ acts as rotations in the w -variable. Furthermore the S -action is free and all orbits are closed.

Now let (X_1, X_2, X_3) be a basis of the three-dimensional vector space of holomorphic vector fields induced by the S -action on Ω' . We take the exterior products $\sigma_1 = X_1 \wedge X_2$, $\sigma_2 = X_1 \wedge X_3$, $\sigma_3 = X_2 \wedge X_3$. The σ_i are sections in the anticanonical bundle $\det(T_{\mathcal{O}}^{1,0}\Omega') = \kappa^{-1}$ and generate an S -invariant subspace of $\Gamma_{\mathcal{O}}(\Omega', \kappa^{-1})$. For every point $p \in \Omega'$, there is σ_i such that $\sigma_i(p) \neq 0$. Hence we get an S -equivariant holomorphic mapping $\alpha : \Omega' \rightarrow \mathbf{P}_2(\mathbf{C})$ defined by

$$\alpha(p) = (\sigma_1(p) : \sigma_2(p) : \sigma_3(p)),$$

where the S -action on $\mathbf{P}_2(\mathbf{C})$ is given by the natural $S/C(S) \simeq PSL_2(\mathbf{R})$ -action which is of course projective-linear.

Since there is no $PSL_2(\mathbf{R})$ -fix-point in $\mathbf{P}_2(\mathbf{C})$ the map α cannot be trivial.

Hence the map α is either locally biholomorphic or the dimension of the fibers is one.

In the latter case, the restriction of α to every S -orbit is an S^1 -principal Cauchy-Riemann bundle (see [5]) and this fact yields that there is an additional holomorphic S^1 -action on Ω' which commutes with the S -action. Hence $\dim_{\mathbf{R}} G = 4$ and we get a

2-dimensional abelian subgroup of G containing J , i.e. by Lemma 3.1, $\Omega = \{\operatorname{Re} w + Q(|z|^2) < 0\}$ or $\Omega = \{\operatorname{Re} w + Q(\operatorname{Re} z) < 0\}$. In both cases, one can assume that $Q(z, \bar{z}) \geq 0$ for all $z \in \mathbf{C}$.

But then an automorphism of Ω' extends to an automorphism of $\Omega' \cup \{w = 0\}$ and we get an S -action on $\mathbf{C} \simeq \{w = 0\}$. This is impossible.

So we have to consider the case where the map α is locally biholomorphic. By considering the $PSL_2(\mathbf{R})$ -invariant domains in $\mathbf{P}_2$, with the property that all $PSL_2(\mathbf{R})$ -orbits are 3-dimensional, one sees that the image of Ω' by α is contained in a domain biholomorphic to $\Delta \times \Delta \setminus \text{Diag}(\Delta \times \Delta)$ with the diagonal $PSL_2(\mathbf{R})$ -action. (Here $\Delta = \{y \in \mathbf{C} \mid |y| < 1\}$).

Furthermore the associated map of S resp. $PSL_2(\mathbf{R})$ -orbits is injective, since they are 3-dimensional in a 2-dimensional complex manifold and α is locally biholomorphic.

So we have a locally biholomorphic, S -equivariant map

$$\tilde{\alpha} : \Omega' \rightarrow \Delta \times \Delta \setminus \text{Diag}(\Delta \times \Delta).$$

Using the S -equivariance and the concrete description of $PSL_2(\mathbf{R})$ -orbits in $\Delta \times \Delta \setminus \text{Diag}(\Delta \times \Delta)$, one can see that this is impossible. The lemma is proved. $\blacksquare$

Lemma 5.8 *Assume that $j \subset \mathfrak{h} \subset \mathcal{G}$ and that J is contained in a two-dimensional subgroup of H . Then H is a finite covering of $SL_2(\mathbf{R})$ and $Q \sim P$, with $P(z, \bar{z}) = |z|^{2k}$, $k \geq 2$.*

Proof : We assume that J is contained in a two dimensional subgroup of H . We are going to prove $Q \sim P$, with $P(z, \bar{z}) = |z|^{2k}$ directly. Then it follows that H is modulo a finite covering isomorphic to $SL_2(\mathbf{R})$, by an investigation of the automorphism group of $\{\text{Re } w + |z|^{2k} < 0\}$.

By lemma 3.2, we have the two holomorphic vector fields $X = i \frac{\partial}{\partial w}$ and $Z = -2w \frac{\partial}{\partial w} - \frac{z}{k} \frac{\partial}{\partial z}$ induced by J and the group $\{(w, z) \mapsto (e^{2kt} \cdot w, e^t \cdot z) \mid t \in \mathbf{R}\}$. In view of structure of H there is a third holomorphic vector field Y induced by a one parameter subgroup of H such that

$$\begin{aligned} [Z, X] &= 2X \\ [X, Y] &= Z \\ [Z, Y] &= -2Y. \end{aligned}$$

Furthermore $\langle \text{Re } X, \text{Re } Y, \text{Re } Z \rangle_{\mathbf{R}}$ is the Lie algebra of real infinitesimal holomorphic transformations induced by H on Ω .

Now let $Y(w, z) = f(w, z) \frac{\partial}{\partial w} + g(w, z) \frac{\partial}{\partial z}$. Using the commutator relations we calculate f and g :

$$\begin{aligned} [X, Y] &= \left[i \frac{\partial}{\partial w}, f \frac{\partial}{\partial w} + g \frac{\partial}{\partial z} \right] \\ &= i \frac{\partial f}{\partial w} \frac{\partial}{\partial w} + i \frac{\partial g}{\partial w} \frac{\partial}{\partial z} \\ &= -2w \frac{\partial}{\partial w} - \frac{z}{k} \frac{\partial}{\partial z} = Z \end{aligned}$$

Hence $\frac{\partial f}{\partial w} = -2iw$, $\frac{\partial g}{\partial w} = -\frac{iz}{k}$ and so

$$f(w, z) = -iw^2 + f_1(z) \quad \text{and} \quad g(w, z) = -\frac{izw}{k} + g_1(z).$$

Furthermore :

$$\begin{aligned} [Z, Y] &= \left[-2w \frac{\partial}{\partial w} - \frac{z}{k} \frac{\partial}{\partial z}, f \frac{\partial}{\partial w} + g \frac{\partial}{\partial z} \right] \\ &= -2w \frac{\partial f}{\partial w} \frac{\partial}{\partial w} - 2w \frac{\partial g}{\partial w} \frac{\partial}{\partial z} - \frac{z}{k} \frac{\partial f}{\partial z} \frac{\partial}{\partial w} \\ &\quad - \frac{z}{k} \frac{\partial g}{\partial z} \frac{\partial}{\partial z} + 2f \frac{\partial}{\partial w} + \frac{g}{k} \frac{\partial}{\partial z} \\ &= -2f \frac{\partial}{\partial w} - 2g \frac{\partial}{\partial z} = -2Y. \end{aligned}$$

and therefore

$$\begin{aligned} -2f &= -2w \frac{\partial f}{\partial w} + 2f - \frac{z}{k} \frac{\partial f}{\partial z} \\ -2g &= -2w \frac{\partial g}{\partial w} - \frac{z}{k} \frac{\partial g}{\partial z} + \frac{g}{k} \end{aligned}$$

and finally

$$4f = 2w \frac{\partial f}{\partial w} + \frac{z}{k} \frac{\partial f}{\partial z}, \quad (2k+1)g = 2kw \frac{\partial g}{\partial w} + z \frac{\partial g}{\partial z}.$$

It follows that :

$$\begin{aligned} 4(-iw^2 + f_1(z)) &= -4iw^2 + \frac{z}{k} f_1'(z) \\ (2k+1)\left(-\frac{izw}{k} + g_1(z)\right) &= -2izw - \frac{izw}{k} + zg_1'(z), \quad \text{i.e.} \end{aligned}$$

$4f_1(z) = \frac{z}{k} f_1'(z)$ and $(2k+1)g_1(z) = zg_1'(z)$, which implies

$$\begin{aligned} f_1(z) &= c \cdot z^{4k} \\ g_1(z) &= d \cdot z^{2k+1}, \quad c, d \in \mathbb{C}. \end{aligned}$$

The vector field Y is therefore given by

$$Y = (-iw^2 + cz^{4k}) \frac{\partial}{\partial w} + \left(-\frac{izw}{k} + d \cdot z^{2k+1}\right) \frac{\partial}{\partial z}.$$

In particular Y is a global holomorphic vector field on $\mathbb{C}^2$ and $\text{Re } Y$ stabilizes the CR-hypersurface $M = \{\text{Re } w + P_{2k}(z, \bar{z}) = 0\}$, which means that

$$(Y + \bar{Y})(\text{Re } w + P_{2k}(z, \bar{z}))|_M \equiv 0.$$

We will compute this expression now :

$$\begin{aligned}
 (Y + \bar{Y})(\operatorname{Re} w + P_{2k}(z, \bar{z})) &= \frac{1}{2}(-iw^2 + cz^{4k}) + \frac{1}{2}(i\bar{w}^2 + \bar{c}\bar{z}^{4k}) \\
 &+ \left(-\frac{izw}{k} + dz^{2k+1}\right)\frac{\partial P_{2k}}{\partial z} + \left(\frac{i\bar{z}\bar{w}}{k} + \bar{d}\bar{z}^{2k+1}\right)\frac{\partial P_{2k}}{\partial \bar{z}} \\
 &= \frac{1}{2}(cz^{4k} + \bar{c}\bar{z}^{4k}) + \frac{1}{2}i(-(\operatorname{Re} w + i\operatorname{Im} w)^2 + (\operatorname{Re} w - i\operatorname{Im} w)^2) \\
 &+ \left(dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}}\right) - \frac{iz}{k}(\operatorname{Re} w + i\operatorname{Im} w)\frac{\partial P_{2k}}{\partial z} \\
 &+ \frac{i\bar{z}}{k}(\operatorname{Re} w - i\operatorname{Im} w)\frac{\partial P_{2k}}{\partial \bar{z}} \\
 &= \frac{1}{2}(cz^{4k} + \bar{c}\bar{z}^{4k}) + \left(dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}}\right) \\
 &+ \left(2\operatorname{Re} w \operatorname{Im} w + \frac{z}{k}\operatorname{Im} w\frac{\partial P_{2k}}{\partial z} + \frac{\bar{z}}{k}\operatorname{Im} w\frac{\partial P_{2k}}{\partial \bar{z}}\right) \\
 &+ \left(-\frac{iz}{k}\operatorname{Re} w\frac{\partial P_{2k}}{\partial z} + \frac{i\bar{z}}{k}\operatorname{Re} w\frac{\partial P_{2k}}{\partial \bar{z}}\right).
 \end{aligned}$$

We put $\operatorname{Re} w = -P_{2k}$ and observe that P_{2k} being homogeneous implies that $P_{2k} = \frac{1}{2k}(z\frac{\partial P_{2k}}{\partial z} + \bar{z}\frac{\partial P_{2k}}{\partial \bar{z}})$ to get that the expression

$$\begin{aligned}
 &\frac{1}{2}(cz^{4k} + \bar{c}\bar{z}^{4k}) + \left(dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}}\right) \\
 &+ \left(\frac{iz}{k}P_{2k}\frac{\partial P_{2k}}{\partial z} - \frac{i\bar{z}}{k}P_{2k}\frac{\partial P_{2k}}{\partial \bar{z}}\right) = 0 \quad \text{for all } z \in \mathbb{C}.
 \end{aligned}$$

We may assume that P_{2k} has no harmonic summands and reduce to

$$dz^{2k+1}\frac{\partial P_{2k}}{\partial z} + \bar{d}\bar{z}^{2k+1}\frac{\partial P_{2k}}{\partial \bar{z}} + \frac{iz}{k}\frac{\partial P_{2k}}{\partial z} \cdot P_{2k} - \frac{i\bar{z}}{k}\frac{\partial P_{2k}}{\partial \bar{z}} P_{2k} = 0,$$

for all $z \in \mathbb{C}$, with $P_{2k}(z, \bar{z}) = \sum_{j=1}^{2k-1} a_j z^j \bar{z}^{2k-j}$, $a_j = \overline{a_{2k-j}}$ and $k \geq 2$.

If the constant $d = 0$, then it follows that

$$\bar{z}\frac{\partial P_{2k}}{\partial \bar{z}} = z\frac{\partial P_{2k}}{\partial z}, \text{ which forces } P_{2k}(\bar{z}, \bar{z}) = a_k |z|^{2k}, a_k \in \mathbf{R}^{>0}.$$

So assume that $d \neq 0$. Then we have

$$d \cdot \sum_{j=1}^{2k-1} j a_j z^{2k+j} \bar{z}^{2k-j} + \bar{d} \sum_{j=1}^{2k-1} a_j (2k-j) z^j \bar{z}^{4k-j}$$

$$\begin{aligned}
 & + \frac{2i}{k} \left[\left(\sum_{j=1}^{2k-1} a_j z^j \bar{z}^{2k-j} \right) \left(\sum_{j=1}^{2k-1} a_j (j-k) z^j \bar{z}^{2k-j} \right) \right] \\
 & = \bar{d} \sum_{j=1}^{2k-1} a_j (2k-j) z^j \bar{z}^{4k-j} + d \sum_{j=2k+1}^{4k-1} a_{j-2k} (j-2k) z^j \bar{z}^{4k-j} \\
 & + \frac{2i}{k} \left[\sum_{j=2}^{4k-2} \left(\sum_{l+n=j} a_l a_n (n-k) \right) z^j \bar{z}^{4k-j} \right] = 0 \text{ for all } z \in \mathbb{C}.
 \end{aligned}$$

Let $\tau \in \{1, \dots, k\}$ be the smallest number such that $a_\tau \neq 0$. Then our expression becomes

$$\begin{aligned}
 & \bar{d} \sum_{j=\tau}^{2k-\tau} a_j (2k-j) z^j \bar{z}^{4k-j} + d \sum_{j=2k+\tau}^{4k-\tau} a_{j-2k} (j-2k) z^j \bar{z}^{4k-j} \\
 & + \frac{2i}{k} \left[\sum_{j=2\tau}^{4k-2\tau} \left(\sum_{l+n=j} a_l a_n (n-k) \right) z^j \bar{z}^{4k-j} \right] = 0.
 \end{aligned}$$

But then $a_\tau = 0$, which is a contradiction.

So we have that $\mathcal{P}(z, \bar{z}) = |z|^{2k}$, $k \geq 2$ and the lemma is proved. $\blacksquare$

Lemma 5.9 *Assume that $\mathcal{G} = \mathfrak{h} \times r$, $\dim r = 1$. Then $j \subset \mathfrak{h}$.*

Proof : Assume that $\mathcal{G} = \mathfrak{h} \times r$ and $j \not\subset \mathfrak{h}$. In view of lemma 5.3, we have $j \neq r$. Let $\pi : \mathcal{G} \rightarrow \mathfrak{h}$ be the projection of $\mathcal{G}$ onto $\mathfrak{h}$ with kernel r . Again in view of lemma 5.3, we have that $\pi(j)$ is the Lie algebra of a maximal compact subgroup of $SL_2(\mathbf{R})$. Let L be the two-dimensional subgroup of G whose Lie algebra $\mathfrak{l}$ is generated by r and $\pi(j)$. It is clear that L is a two-dimensional Lie group containing J and the center C of G . Therefore $L = S^1 \times \mathbf{R}$, since otherwise $G = SL_2(\tilde{\mathbf{R}}) \times \mathbf{R}$, which is impossible. Hence $\Omega = \{\operatorname{Re} w + Q(|z|^2) < 0\}$, where we may assume that $Q(|z|^2) \geq 0$ for all $z \in \mathbb{C}$. The action of the connected component of C^0 the center of G is given by

$$(w, z) \mapsto (w + it, e^{i\rho t} \cdot z), t \in \mathbf{R}, \rho \in \mathbf{R}^* \text{ fixed.}$$

We consider the function $(w, z) \xrightarrow{f} z \cdot e^{-\rho w} \in \mathbb{C}$, which is invariant under this action. We have

$$|z \cdot e^{-\rho w}|^2 = |z|^2 \cdot e^{-\rho^2 \operatorname{Re} w} \geq |z|^2 e^{\rho^2 Q(|z|^2)}.$$

The expression on the right side tends to $+\infty$ when $|z| \rightarrow +\infty$ and the image of f is S^1 -invariant. Hence $f : \Omega \rightarrow \mathbb{C}$ is surjective and has maximal rank

everywhere. Hence we get an G/C^0 action on $\mathbf{C}$ which is impossible. The lemma is proved. $\blacksquare$

Remark 5.10 a) *The automorphism group of a domain $\Omega = \{\operatorname{Re} w + |z|^{2k} < 0\}$, $k \geq 2$ is a product $S \cdot S^1$, where S is modulo a finite group isomorphic to $SL_2(\mathbf{R})$ and S^1 is a central one-dimensional group. Hence G is four-dimensional.*

b) *In the case $\dim_{\mathbf{R}} G = 3$ the lemmas 5.3 to 5.9 prove theorem 1 and theorem 2.*

c) *We mention that from now on we may assume that G is solvable since the non-solvable case is completely handled by the lemmas 5.2 to 5.9 .*

6 The case $\dim_{\mathbf{R}} G \geq 4$

Lemma 6.1 *Let $\Omega = \{\operatorname{Re} w + Q(z, \bar{z}) < 0\}$ and assume that $G = \operatorname{Aut}_{\mathcal{O}}^0(\Omega)$ is solvable. Then $\dim_{\mathbf{R}} G \leq 3$.*

Proof : We assume that $\dim_{\mathbf{R}} G \geq 4$ and that Ω is not homogeneous. So we have that $\dim G = 4$ or 5 , since the highest dimensional compact subgroup of G is $(S^1)^2$.

Let $N \subset G$ be the largest nilpotent normal connected subgroup of G . Clearly, N contains $(G')^0$, the connected component of the commutator G' of G .

We first show that $\dim_{\mathbf{R}} N \leq 3$. Assume the contrary, i.e. $\dim N \geq 4$. Then the maximal compact subgroup of N is not trivial, i.e. isomorphic to S^1 or $(S^1)^2$. But compact subgroups of nilpotent Lie groups are always central, in view of the bijectivity of the exponential map. Then N as a subgroup of G doesnot act effectively, a contradiction. So $\dim_{\mathbf{R}} N \leq 3$. So we have to consider three cases :

- i) $n = h_3$ the three-dimensional Heisenberg algebra ;
- ii) $\dim N = 2$ and N is abelian ;
- iii) $\dim N = 1$.

Cas i) : $n = h_3$. By similar arguments as above and using the fact that all maximal compact subgroups are conjugate one sees that N is simply connected. Hence all N and therefore all G -orbits in Ω are closed CR-hypersurfaces isomorphic to $\mathbf{R}^3$. Using the results of [4], [7], it is not hard to check that a simply connected hyperbolic Stein manifold acted on by H_3 is biholomorphic to the ball ; this contradicts our assumption.

Cas ii) : $\dim_{\mathbb{R}} N = 2$ and N is abelian.

If $J \not\subset N$ then J and N generate a three-dimensional solvable group. Using the lemmas of Section V, we see that G cannot be solvable and of dimension four or greater, if Ω is not homogeneous.

So we have $J \subset N$ and we can find a 3-dimensional solvable group containing J . Using again the lemmas of Section V we conclude like above.

Case iii) : $\dim_{\mathbb{R}} N = 1$. Then either $J = N$ or J and N generate a two dimensional abelian group. In both cases we can take the complex-analytic quotient of Ω by N , which is either the upper half plane or $\mathbb{C}$. But G/N is at least 3-dimensional and abelian. This is impossible. ■

Remark 6.2 *Using the same methods as above it can be shown that the number of connected components of $\text{Aut}_{\mathcal{O}}(\Omega)$ is always finite.*

Acknowledgments : This paper was motivated by a result of F. Berteloot and G. Cœuré[1]. I would like to thank both for their interest and useful discussions. My thanks are also due to R. Bérat for the L^AT_EX-typeset.

Bibliographie

- [1] Berteloot, F. et Cœuré G., *Domaines de $\mathbb{C}^2$, pseudoconvexes et de type fini ayant un groupe non compact d'automorphismes*, Ann. Inst. Fourier Grenoble 41 (1), (1991), pp. 77-86.
- [2] Jacobson, N., *Lie algebras*. Wiley & Sons, 1962.
- [3] Kobayashi, S., *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, Inc., New York (1970).
- [4] Oeljeklaus, K., *Une remarque sur le groupe des automorphismes holomorphes de domaines tubes dans $\mathbb{C}^n$* , C.R.A.S. 312 (I), (1991), pp. 967-968.
- [5] Richthofer, W., *Homogene CR-Mannigfaltigkeiten*, Dissertation Bochum (1985).

- [6] Rosay, J.P., *Sur une caractérisation de la boule parmi les domaines de $\mathbf{C}^n$ par son groupe d'automorphismes*, Ann. Inst. Fourier Grenoble, 29 (4), (1979), pp. 91-97.
- [7] Yang, P., *Geometry of Tube Domains*. Proc. Symp. Pure Math. 41 (1984), AMS Providence, Rhode Island, pp. 277-283.

Karl Oeljeklaus
Université des Sciences et Technologies de Lille
U.R.A. 751 "GAT" associée au CNRS
UFR de Mathématiques Pures et Appliquées
F-59655 - Villeneuve d'Ascq Cedex (France)
New address (after the 1.10.1993) :
U.F.R. de Mathématiques, Informatiques et Mécanique
Université de Provence (Aix-Marseille I)
3, place Victor Hugo
F-13331 - Marseille Cedex (France)