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ON THE ENVELOPES OF HOLOMORPHY OF STRICTLY LEVI-CONVEX HYPERSURFACES

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INTRODUCTION

We shall be concerned with the subject of holomorphic continuation of CR-functions from a relatively open part of the boundary of a strongly pseudoconvex domain.

Let M be a Stein manifold of dimension $n \geq 2$, $D \subset\subset M$ a $\mathcal{C}^2$-bounded strongly pseudoconvex domain and K a proper closed subset of the boundary bD of D.

It is well-known that, due to the strict Levi-convexity of $bD \setminus K$, there exists an open set $U \subset D$, having $bD \setminus K$ as a part of its boundary, such that every continuous CR-function on $bD \setminus K$ has a unique continuous extension to $(bD \setminus K) \cup U$ which is holomorphic on U. The existence of U is referred to as the H. Lewy's extension phenomenon.

More recents results yield sharper information on U; in particular it has been shown that the open set $D \setminus \widehat{K}_{\overline{D}}$ ($\widehat{K}_{\overline{D}} = \mathcal{O}(\overline{D})$-hull of K) is such a U with the mentioned features (see [11, 6] and the references therein).

For $n = 2$ it is also known that $D \setminus \widehat{K}_{\overline{D}}$ has another independent property: it is pseudoconvex (see [8, 9, 10]). This, combined with the above, implies at once the following noteworthy result:

(*I*) *For* $n = 2$ *the envelope of holomorphy of* $bD \setminus K$ *is* $\overline{D} \setminus \widehat{K}_{\overline{D}}$.

Remark. Here above and throughout the continuation we speak of envelopes of holomorphy of non-open subsets of M. We recall that in general the envelope of holomorphy $E(S)$ of an arbitrary subset S of a Stein manifold can be given a precise definition as the union of the components of $\widetilde{S} = spec(\mathcal{O}(S))$ which meet S (see [5]). However, in the case of our concern where $S = bD \setminus K$, for the purposes of this paper the envelope of holomorphy may be simply understood as the disjoint union of $bD \setminus K$ and the envelope of holomorphy $E(U)$ of an open set U as specified above, regarded as a holomorphic extension of U.

An immediate consequence of (I) is:

$(I)'$ For $n = 2$, in order that K be removable, in the sense that each continuous CR-function f on $bD \setminus K$ may have a continuous extension $F \in \mathcal{C}^0(\overline{D} \setminus K) \cap \mathcal{O}(D)$, it is necessary and sufficient that $\widehat{K}_{\overline{D}} = K$, i.e. that K be $\mathcal{O}(\overline{D})$-convex.

On the other hand, for $n \geq 3$ it is not true in general that $D \setminus \widehat{K}_{\overline{D}}$ is pseudoconvex, as simple examples show, and hence the extension of (I) to general $n \geq 2$ fails to be valid. Indeed Corollary 2 below specifies the necessary and sufficient conditions for $D \setminus \widehat{K}_{\overline{D}}$ to be pseudoconvex when $n \geq 3$. Also the extension of $(I)'$ to general $n \geq 2$ does not hold, since for $n \geq 3$ $\mathcal{O}(\overline{D})$-convexity is no longer necessary for removability: for example every Stein compactum on bD is removable for $n \geq 3$ (see [11]).

In fact, when $n \geq 3$ no theorem of the kind of (I), to the effect of describing the envelope of holomorphy of $bD \setminus K$ for an arbitrary compact set $K \subset bD$, is known, and it is even unknown, as far as we can say, whether it is always true that $bD \setminus K$ should have a single-sheeted envelope of holomorphy.[1]

As regards $(I)'$, on the contrary, an extension to $n \geq 2$ has been recently established (see [7]). It can be stated as follows:

(II) For $n \geq 2$, in order that K be removable it is necessary and sufficient that $H^{n-1}(K; \mathcal{O}) = 0$ and the restriction map $H^{n-2}(\overline{D}; \mathcal{O}) \to H^{n-2}(K; \mathcal{O})$ have dense image.

Since for $n = 2$ the vanishing of $H^1(K; \mathcal{O})$ is equivalent to the condition that K be holomorphically convex (see [5]), it follows that (II) is indeed an extension of $(I)'$ to general $n \geq 2$. Note that, since $\overline{D}$ is a Stein compactum, and hence $H^q(\overline{D}; \mathcal{O}) = 0$ for $q \geq 1$, when $n \geq 3$ the condition on the restriction map amounts to having ${}^{\sigma}H^{n-2}(K; \mathcal{O}) = 0$, where the suffix σ means the associated separated space.

[1] *Added July 19, 1993.* Recently E.M. Chirka and E.L. Stout [Removable Singularities in the Boundary (to appear)] gave an example of a $\mathcal{C}^\infty$-bounded strongly pseudoconvex domain $D \subset\subset \mathbb{C}^{2m}$, $m \geq 2$, and a compact set $K \subset bD$, with $bD \setminus K$ being connected, such that the envelope of holomorphy of $bD \setminus K$ is not single-sheeted.

(II) gives a first answer to the question of finding, for general $n \geq 2$, the envelope of holomorphy of $bD \setminus K$. In fact it states a necessary and sufficient condition on K in order that the envelope may be the whole $\overline{D} \setminus K$. Here we shall establish a sharper result of this kind, which includes both (I) and (II) as particular cases, namely we shall prove the following theorem.

Theorem. *Let $n \geq 2$ and let E be a compact set such that $K \subset E \subset \widehat{K}_{\overline{D}}$. Then, in order that $\overline{D} \setminus E$ may be the envelope of holomorphy of $bD \setminus K$, it is necessary and sufficient that the following conditions should be satisfied:*

(1) *The restriction map $H^q(E; \mathcal{O}) \to H^q(K; \mathcal{O})$ is bijective for $q \leq n-3$ and is injective with closed image for $q = n-2$.*

(2) *$H^{n-1}(E; \mathcal{O}) = 0$ and the restriction map $H^{n-2}(\overline{D}; \mathcal{O}) \to H^{n-2}(E; \mathcal{O})$ has dense image.*

It is plain that this theorem implies (II): just take in it $E = K$. On the other hand, for $n = 2$ Condition (2) means that $E = \widehat{K}_{\overline{D}}$, and then Condition (1) amounts to saying that the restriction map $\mathcal{O}(\widehat{K}_{\overline{D}}) \to \mathcal{O}(K)$ should be injective with closed image, which indeed can be shown to be automatically true (see [8]); therefore for $n = 2$ the theorem does reduce to (I).

We wish to mention a couple of straightforward further consequences of the theorem. If we apply it to the case that $n \geq 3$ and E is holomorphically convex (*e.g.* a Stein compactum), on account of the vanishing of $H^q(E; \mathcal{O})$ for $q \geq 1$, we get at once:

Corollary 1. *Let $n \geq 3$ and let E be a holomorphically convex compact set such that $K \subset E \subset \widehat{K}_{\overline{D}}$. Then, in order that $\overline{D} \setminus E$ be the envelope of holomorphy of $bD \setminus K$, it is necessary and sufficient that $H^q(K; \mathcal{O}) = 0$ for $1 \leq q \leq n-3$, that $H^{n-2}(K; \mathcal{O})$ be separated and that E be the envelope of holomorphy of K.*

In particular we can state:

Corollary 2. *For $n \geq 3$, in order that $\overline{D} \setminus \widehat{K}_{\overline{D}}$ be the envelope of holomorphy of $bD \setminus K$, it is necessary and sufficient that $H^q(K; \mathcal{O}) = 0$ for $1 \leq q \leq n-3$, that $H^{n-2}(K; \mathcal{O})$ be separated and that $\widehat{K}_{\overline{D}}$ be the envelope of holomorphy of K.*

Remarks. (*i*) The cohomological conditions on K in the preceding corollaries can be shown to be equivalent to the following:

$$H^{n-2}(M \setminus K; \mathcal{O}) \text{ is separated, if } n = 3;$$
$$H^q(M \setminus K; \mathcal{O}) = 0 \text{ for } 2 \leq q \leq n-2, \text{ if } n \geq 4.$$

Moreover we recall that $H^2(M \setminus K; \mathcal{O})$ is separated if and only if $\bar{\partial}\mathcal{E}^{0,1}(M \setminus K)$ is a closed subspace of $\mathcal{E}^{0,2}(M \setminus K)$.

(*ii*) It is not possible to omit, in the preceding corollaries, the requirement that $H^{n-2}(K; \mathcal{O})$ should be separated. As a matter of fact, consider the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, $n \geq 3$, and the compact sets $K = b\mathbb{B}_n \cap \{z \in \mathbb{C}^n : \mathcal{I}m(z_{n-1}) = 0, z_n = 0\}$, $E = \overline{\mathbb{B}}_n \cap \{z \in \mathbb{C}^n : \mathcal{I}m(z_{n-1}) = 0, z_n = 0\}$. It is readily seen that K is removable, and hence the envelope of holomorpy of $b\mathbb{B}_n \setminus K$ is not $\overline{\mathbb{B}}_n \setminus E$, but the whole $\overline{\mathbb{B}}_n \setminus K$. On the other hand E is both the envelope of holomorphy and the polynomial hull of K, moreover one has $H^q(K; \mathcal{O}) = 0$ for $1 \leq q \leq n-3$ and ${}^{\sigma}H^{n-2}(K; \mathcal{O}) = 0$. Indeed the point is that in this case $H^{n-2}(K; \mathcal{O})$ is not separated.

1. Preliminaries

Before going into the proof of the theorem we need some preliminary results. We shall use the notation that, given a compact set $E \subset M$, $\Phi(E)$, or simply Φ when no confusion can arise, denotes the paracompactifying family of supports in $M \setminus E$ of all the relatively closed subsets of $M \setminus E$ whose closure in M is compact, that is $\Phi = c \cap (M \setminus E)$, where c denotes the family of compact subsets of M.

Lemma 1. *For $n \geq 2$, if $M \setminus E$ is connected, the following facts are equivalent:*

(*a*) *$H^{n-1}(E; \mathcal{O}) = 0$ and the restriction map $H^{n-2}(M; \mathcal{O}) \to H^{n-2}(E; \mathcal{O})$ has dense image.*

(*b*) $H^1_{\Phi}(M \setminus E; \mathcal{O}) = 0$.

We have already established this result in [7], where it is needed for the proof of (*II*), so we refer to [7] for its proof.

Lemma 2. *For $n \geq 2$, if D, $E \subset M$ are a pseudoconvex domain and a compact set, respectively, the following facts are equivalent:*

(*α*) *The restriction map $H^q(\overline{D} \cap E; \mathcal{O}) \to H^q(bD \cap E; \mathcal{O})$ is bijective for $q \leq n-3$ and injective for $q = n-2$, moreover the space $H^{n-1}_c(D \cap E; \mathcal{O})$ is separated;*

(β) *$D \setminus E$ is pseudoconvex.*

Proof. It is known that $D \setminus E$ is pseudoconvex if and only if $H^q(D\setminus E; \mathcal{O}) = 0$ for $q \geq 1$. Moreover the vanishing of $H^q(D \setminus E; \mathcal{O})$ is equivalent to that of $H^q(D \setminus E; \Omega)$, where Ω is the sheaf of germs of holomorphic n-forms on M. This follows from the fact that, as M is Stein, a positive integer r and a locally free sheaf $\mathcal{R}$ of $\mathcal{O}$-modules on M of rank $r-1$ exist, such that the exact sequence $0 \to \mathcal{R} \to \mathcal{O}^r \to \Omega \to 0$ holds on M, and hence $\mathcal{O}^r \cong \mathcal{R} \oplus \Omega$ and $\Omega^r \cong Hom_{\mathcal{O}}(\mathcal{R}; \Omega) \oplus \mathcal{O}$. Furthermore the relative cohomology sequence

$$\cdots \to H^q_{D\cap E}(D; \Omega) \to H^q(D; \Omega) \to H^q(D \setminus E; \Omega) \to \cdots$$

implies that, for $q \geq 1$, $H^q(D \setminus E; \Omega) = 0$ if and only if $H^{q+1}_{D\cap E}(D; \Omega) = 0$.

Then, by resorting to the relative version of the Serre duality theorem (see [1; p.287]), we can infer that the pseudoconvexity of $D \setminus E$ is also equivalent to the condition that for $q \geq 2$ ${}^{\sigma}H^{n-q}_c(D \cap E; \mathcal{O}) = 0$ and $H^{n-q+1}_c(D \cap E; \mathcal{O})$ be separated, *i.e.* $H^q_c(D \cap E; \mathcal{O}) = 0$ for $q \leq 2$ and $H^{n-1}_c(D \cap E; \mathcal{O})$ be separated.

Finally the cohomology sequence with compact supports

$$\cdots \to H^q_c(D \cap E; \mathcal{O}) \to H^q(\overline{D} \cap E; \mathcal{O}) \to H^q(bD \cap E; \mathcal{O}) \to \cdots$$

implies that having $H^q_c(D \cap E; \mathcal{O}) = 0$ for $q \leq n-2$ is equivalent to the condition that the restriction map $H^q(\overline{D} \cap E; \mathcal{O}) \to H^q(bD \cap E; \mathcal{O})$ be bijective for $q \leq n-3$ and injective for $q = n-2$.

The proof of the lemma is then completed.

Lemma 3. *Let X be a complex analytic manifold of dimension $N \geq 1$, $F \subset X$ a closed set, and consider the relative cohomology sequence*

$$\cdots \longrightarrow H^q(X, F; \mathcal{O}) \xrightarrow{i_*^{(q)}} H^q(X; \mathcal{O}) \xrightarrow{\rho_*^{(q)}} H^q(F; \mathcal{O}) \xrightarrow{\delta^{(q)}} H^{q+1}(X, F; \mathcal{O}) \longrightarrow \cdots,$$

where the cohomology spaces are equipped with the standard locally convex topologies. Then all the coboundary maps $\delta^{(q)}$ are continuous. Moreover, if the space $H^{q+1}(X; \mathcal{O})$ is separated, $\delta^{(q)}$ is a topological homomorphism ($0 \leq q \leq N-1$). In particular, if X is Stein, all the coboundary maps $\delta^{(q)}$ are topological isomorphisms.

Proof. We may argue in terms of Dolbeault's cohomology. The exact sequence under consideration can be regarded as the $\bar{\partial}$-cohomology sequence induced by the short exact sequences of spaces of $\mathcal{C}^\infty$ differential forms

$$0 \to \mathcal{E}^{0,q}(X, F) \xrightarrow{i^{(q)}} \mathcal{E}^{0,q}(X) \xrightarrow{\rho^{(q)}} \mathcal{E}^{0,q}(F) \to 0,$$

$0 \leq q \leq N$. Here $\mathcal{E}^{0,q}(F)$, the space of $\mathcal{C}^\infty$ $(0,q)$-forms around F, is the inductive limit of the Fréchet spaces $\mathcal{E}^{0,q}(U)$, as U ranges through a fundamental system of open neighbourhoods of F; whereas $\mathcal{E}^{0,q}(X,F)$, the space of $\mathcal{C}^\infty$ $(0,q)$-forms on X supported in $X \setminus F$, is the inductive limit of the subspaces $\mathcal{E}^{0,q}_G(X) \subset \mathcal{E}^{0,q}(X)$ of the $\mathcal{C}^\infty$ $(0,q)$-forms on X supported in the closed set G, as G ranges through a family of closed subsets of $X \setminus F$, whose complements in X form a fundamental system of open neighbourhoods of F.

Then, to prove the first statement of the lemma, it suffices to show that, if U is any open neighbourhood of F, $\pi^{(q)}_U : Z^{0,q}_{\bar\partial}(U) \to H^{0,q}_{\bar\partial}(U)$ is the canonical projection and $\rho_{U*} : H^{0,q}_{\bar\partial}(U) \to H^{0,q}_{\bar\partial}(F)$ the map induced by restriction, then the composed map

$$\delta^{(q)}\rho_{U*}\pi^{(q)}_U : Z^{0,q}_{\bar\partial}(U) \to H^{0,q+1}_{\bar\partial}(X,F)$$

is continuous. As a matter of fact, if $\chi : X \to \mathbb{R}$ is any fixed $\mathcal{C}^\infty$ function with $\chi = 1$ on a neighbourhood of F and $supp(\chi) \subset U$, it is readily seen that, for every $\omega \in Z^{0,q}_{\bar\partial}(U)$, $\delta^{(q)}\rho_{U*}\pi^{(q)}_U(\omega)$ is the $\bar\partial$-cohomology class in $H^{0,q+1}_{\bar\partial}(X,F)$ represented by $\bar\partial(\chi\omega)$. Now, if $\{\omega_n\}$ is a sequence of elements of $Z^{0,q}_{\bar\partial}(U)$, convergent to an element $\omega \in Z^{0,q}_{\bar\partial}(U)$, it is plain that the sequence $\{\bar\partial(\chi\omega_n)\}$ converges to $\bar\partial(\chi\omega)$ in $Z^{0,q+1}_{\bar\partial}(X,F)$, and hence we infer that $\delta^{(q)}\rho_{U*}\pi^{(q)}_U$ is continuous.

Next, assume that the space $H^{q+1}(X,\mathcal{O}) \cong H^{0,q+1}_{\bar\partial}(X)$ is separated. This means that $\bar\partial\mathcal{E}^{0,q}(X)$ is a closed subspace of $\mathcal{E}^{0,q+1}(X)$ and hence, as the spaces $\mathcal{E}^{0,q}(X)$, $\mathcal{E}^{0,q+1}(X)$ are Fréchet, that $\bar\partial : \mathcal{E}^{0,q}(X) \to \mathcal{E}^{0,q+1}(X)$ is a topological homomorphism (see [4]). Therefore $\bar\partial$ transforms the open subsets of $\mathcal{E}^{0,q}(X)$ into open subsets of its image. Since the coboundary map $\delta^{(q)}$ can be explicited as $\delta^{(q)} = \pi^{(q+1)}_1(i^{(q+1)})^{-1}\bar\partial(\rho^{(q)})^{-1}(\pi^{(q)}_2)^{-1}$, where $\pi^{(q+1)}_1 : Z^{0,q+1}_{\bar\partial}(X,F) \to H^{0,q+1}_{\bar\partial}(X,F)$ and $\pi^{(q)}_2 : Z^{0,q}_{\bar\partial}(F) \to H^{0,q}_{\bar\partial}(F)$ are the canonical projections, it follows that $\delta^{(q)}$ transforms the open subsets of $H^{0,q}_{\bar\partial}(F)$ into open subsets of its image, and hence it is a topological homomorphism (see also [3]).

The lemma is proved.

2. Proof of the theorem

After shrinking M to a suitable Stein neighbourhood of $\overline{D}$, we may assume that $\overline{D}$ is $\mathcal{O}(M)$-convex and so Condition (2) is equivalent, also for $n = 2$, to Condition (a) of Lemma 1.

We first prove the sufficiency. Thus assume that (1) and (2) are valid.

To prove that $D \setminus E$ is pseudoconvex, it suffices, in view of Lemma 2, to show that the space $H^{n-1}_c(E \setminus K; \mathcal{O})$ is separated. Let us consider the exact sequence of relative cohomology

$$\cdots \to H^{n-1}(M,E;\mathcal{O}) \xrightarrow{i_*} H^{n-1}(M,K;\mathcal{O}) \xrightarrow{r_*} H^{n-1}_c(E\setminus K;\mathcal{O}) \xrightarrow{\delta} H^n(M,E;\mathcal{O}) \to \cdots$$

(see [2; p.60]). We claim that $i_*(H^{n-1}(M,E;\mathcal{O}))$ is closed in $H^{n-1}(M,K;\mathcal{O})$ and that $r_*: H^{n-1}(M,K;\mathcal{O}) \to H^{n-1}_c(E\setminus K;\mathcal{O})$ is a surjective open map.

As a matter of fact, there is a commutative diagram

$$\begin{array}{ccc} H^{n-2}(E;\mathcal{O}) & \longrightarrow & H^{n-1}(M,E;\mathcal{O}) \\ {\scriptstyle\rho_*}\downarrow & & {\scriptstyle i_*}\downarrow \\ H^{n-2}(K;\mathcal{O}) & \longrightarrow & H^{n-1}(M,K;\mathcal{O}) \end{array},$$

where the horizontal arrows are given by coboundary maps and hence, by Lemma 3, are topological isomorphisms. This implies at once that the image of i_* is closed, since, by assumption, so is that of ρ_*. Moreover, as $H^{n-1}(E;\mathcal{O})=0$, Lemma 3 also implies that $H^n(M,E;\mathcal{O})=0$ too, and so r_* is surjective. There remains to prove that r_* is an open map, *i.e.*, being continuous, that it is a topological homomorphism. It is a matter of proving that the inverse of the bijective linear map $\widetilde{r}_*: \frac{H^{n-1}(M,K;\mathcal{O})}{Ker(r_*)} \to H^{n-1}_c(E\setminus K;\mathcal{O})$ induced by r_* is continuous. We may argue in terms of Dolbeault's cohomology, identifying $H^{n-1}(M,K;\mathcal{O})$ with $H^{0,n-1}_{\bar\partial}(M,K)$ and $H^{n-1}_c(E\setminus K;\mathcal{O})$ with the inductive limit of the spaces $H^{0,n-1}_{\bar\partial}(U,K)$ as U ranges through the open neighbourhoods of E. For every such U, let us choose a $\mathcal{C}^\infty$ function $\chi: M\to\mathbb{R}$ with $\chi=1$ on a neighbourhood of E and $supp(\chi)\subset U$. If $\alpha\in Z^{0,n-1}_{\bar\partial}(U,K)$, then $\bar\partial(\chi\alpha)\in Z^{0,n}_{\bar\partial}(M,E)=\mathcal{E}^{0,n}(M,E)=\bar\partial\mathcal{E}^{0,n-1}(M,E)$, and hence one can find a $\beta\in\mathcal{E}^{0,n-1}(M,E)$ with $\bar\partial\beta=\bar\partial(\chi\alpha)$. Moreover, if β' is another choice of a $\bar\partial$-primitive of $\bar\partial(\chi\alpha)$ in $\mathcal{E}^{0,n-1}(M,E)$, one sees that the class of $\beta'-\beta$ in $H^{0,n-1}_{\bar\partial}(M,K)$ belongs to $Ker(r_*)$. Therefore one obtains a well-defined linear map $s_U: Z^{0,n-1}_{\bar\partial}(U,K)\to\frac{H^{0,n-1}_{\bar\partial}(M,K)}{Ker(r_*)}$ by mapping every $\alpha\in Z^{0,n-1}_{\bar\partial}(U,K)$ into the class, in $\frac{H^{0,n-1}_{\bar\partial}(M,K)}{Ker(r_*)}$, represented by $\chi\alpha-\beta$, with β being any $\bar\partial$-primitive of $\bar\partial(\chi\alpha)$ in $\mathcal{E}^{0,n-1}(M,E)$. One can readily check that, if $\alpha\in\bar\partial\mathcal{E}^{0,n-2}(U,K)$, then $\chi\alpha-\beta\in\bar\partial\mathcal{E}^{0,n-2}(M,K)+Z^{0,n-1}_{\bar\partial}(M,E)$ and the latter sum space projects into $H^{0,n-1}_{\bar\partial}(M,K)$ as a subspace of $Ker(r_*)$; hence s_U induces a linear map $\widetilde{s}_U: H^{0,n-1}_{\bar\partial}(U,K)\to\frac{H^{0,n-1}_{\bar\partial}(M,K)}{Ker(r_*)}$. It turns out that $(\widetilde{r}_*)^{-1}$ is the inductive limit of the maps $\widetilde{s}_U$ as U ranges through the open neighbourhoods of E, and consequently one is reduced to prove that each map

s_U is continuous. To this end, it suffices to prove that, if $\{\alpha_\nu\}$ is a sequence of elements of $Z^{0,n-1}_{\bar\partial}(U,K)$, convergent to an element $\alpha \in Z^{0,n-1}_{\bar\partial}(U,K)$, then the sequence $\{s_U(\alpha_\nu)\}$ converges to $s_U(\alpha)$ in $\frac{H^{0,n-1}_{\bar\partial}(M,K)}{Ker(r_*)}$. As a matter of fact, the map $\bar\partial : \mathcal{E}^{0,n-1}(M,E) \to \mathcal{E}^{0,n}(M,E)$, being continuous and surjective, is a topological homomorphism, hence is open, since the source space and the target space are both of type $(\mathcal{LF})$ (see [4; p.148]), and consequently one can check the possibility of finding a sequence $\{\beta_\nu\}$ of elements of $\mathcal{E}^{0,n-1}(M,E)$, convergent to an element $\beta \in \mathcal{E}^{0,n-1}(M,E)$, in such a way that $\bar\partial\beta_\nu = \bar\partial(\chi\alpha_\nu)$, for every ν, and $\bar\partial\beta = \bar\partial(\chi\alpha)$. Hence the sequence $\{\chi\alpha_\nu - \beta_\nu\}$ converges to $\chi\alpha - \beta$ in $Z^{0,n-1}_{\bar\partial}(M,K)$, which implies the desired conclusion.

Now, since $Im(i_*) = Ker(r_*)$ is closed in $H^{n-1}(M,K;\mathcal{O})$ and r_* is a surjective open map, it follows that $r_*(H^{n-1}(M,K;\mathcal{O}) \setminus Ker(r_*)) = H^{n-1}_c(E \setminus K;\mathcal{O}) \setminus \{0\}$ is open in $H^{n-1}_c(E \setminus K;\mathcal{O})$, which proves that the latter space is separated.

Next, we have to prove that every continuous CR-function f on $bD \setminus K$ has a unique extension $F \in \mathcal{C}^0(\overline{D} \setminus E) \cap \mathcal{O}(D \setminus E)$. Consider a function $\widetilde{f} \in \mathcal{C}^0(\overline{D} \setminus E) \cap \mathcal{C}^\infty(D \setminus E)$ which is equal to f on $bD \setminus K$ and is holomorphic on the interior of a neighbourhood, in $\overline{D} \setminus E$, of $bD \setminus K$, and consider the $(0,1)$-form η on $M \setminus E$ defined by

$$\eta = \bar\partial\widetilde{f} \text{ on } D \setminus E, \ \eta = 0 \text{ on } (M \setminus E) \setminus D,$$

which is $\mathcal{C}^\infty$, $\bar\partial$-closed and supported in the family $\Phi(E)$. Now, the vanishing of $H^{n-1}(E;\mathcal{O})$ implies the connectedness of $M \setminus E$ (see [6]) and therefore, on account of Lemma 1, one has that $H^1_\Phi(M \setminus E;\mathcal{O}) = 0$, hence there exists a function $u \in \mathcal{C}^\infty_\Phi(M \setminus E)$ with $\bar\partial u = \eta$ on $M \setminus E$. This function u is holomorphic on a neighbourhood of $(M \setminus E) \setminus D = (M \setminus \overline{D}) \cup (bD \setminus K)$ and hence, as $supp(u) \in \Phi$ and $M \setminus \overline{D}$ is connected, it follows that $u = 0$ on $(M \setminus \overline{D}) \cup (bD \setminus K)$. Then set $F = \widetilde{f} - u|_{\overline{D}\setminus E}$. It is plain that $F \in \mathcal{C}^0(\overline{D} \setminus E) \cap \mathcal{O}(D \setminus E)$ and $F|_{bD\setminus K} = f$. Finally, the extension F of f is unique, since the connectedness of $M \setminus E$ implies that $H^0_\Phi(M \setminus E;\mathcal{O}) = 0$.

Now we prove the necessity of the two conditions of the theorem. Thus assume that $D \setminus E$ is pseudoconvex and that every continuous CR-function f on $bD \setminus K$ has a unique extension $F \in \mathcal{C}^0(\overline{D} \setminus E) \cap \mathcal{O}(D \setminus E)$. In the first place it follows that $M \setminus E$ is connected, for, if A were a relatively compact connected component of $M \setminus E$, it would be contained in $D \setminus E$ and its boundary would not meet $bD \setminus K$. Consequently $F|_A$ could be any function in $\mathcal{O}(A)$, in contradiction with the uniqueness assumption.

Hence, in view of Lemma 2 and Lemma 1, what we have to show is that the image of the restriction map $\rho_* : H^{n-2}(E;\mathcal{O}) \to H^{n-2}(K;\mathcal{O})$ is closed and that $H^1_\Phi(M \setminus E;\mathcal{O}) = 0$.

The former fact is again a straightforward consequence of Lemma 2, which gives the separation of $H_c^{n-1}(D \cap E; \mathcal{O})$: then the cohomology sequence

$$\cdots \to H^{n-2}(E;\mathcal{O}) \xrightarrow{\rho_*} H^{n-2}(K;\mathcal{O}) \xrightarrow{\delta} H_c^{n-1}(E \setminus K;\mathcal{O}) \to \cdots$$

implies, since δ is continuous, that $\rho_*(H^{n-2}(E;\mathcal{O}))$ is closed. Note that the continuity of δ follows from Lemma 3, since the preceding sequence can be obtained, by taking inductive limits, from the exact sequences

$$\cdots \to H^{n-2}(U_j;\mathcal{O}) \xrightarrow{\rho_{j*}} H^{n-2}(K;\mathcal{O}) \xrightarrow{\delta_j} H^{n-1}(U_j, K;\mathcal{O}) \to \cdots,$$

where $\{U_j\}_{j\in\mathbb{N}}$ is a fundamental system of open neighbourhoods of E.

In order to prove the latter fact, we have to show that, if α is any $\mathcal{C}^\infty$ $\bar\partial$-closed $(0,1)$-form on $M \setminus E$, supported in Φ, there exists a function $g \in \mathcal{C}^\infty_\Phi(M \setminus E)$ with $\bar\partial g = \alpha$. Let $\Delta \subset\subset M$ be a pseudoconvex domain such that $D \subset \Delta$ and $bD \cap b\Delta = K$, as can be obtained by pushing bD away from D with a small $\mathcal{C}^2$- perturbation leaving K fixed pointwise. Since $D \setminus E$ and Δ are pseudoconvex, so is $\Delta \setminus E$, hence there exists $f_1 \in \mathcal{C}^\infty(\Delta \setminus E)$ with $\bar\partial f_1 = \alpha$ on $\Delta \setminus E$. On the other hand, since $\overline{D}$ is an $\mathcal{O}(M)$-convex Stein compactum, by Lemma 1, applied to $\overline{D}$ in place of E, one has that $0 = H^1_{\Phi(\overline{D})}(M \setminus \overline{D};\mathcal{O}) = H^1_{\Phi(E)\cap(M\setminus\overline{D})}(M \setminus \overline{D};\mathcal{O})$; hence there exists also $f_2 \in \mathcal{C}^\infty_{\Phi(\overline{D})}(M \setminus \overline{D})$ with $\bar\partial f_2 = \alpha$ on $M \setminus \overline{D}$. Then $f_2 - f_1$ is a holomorphic function on $\Delta \setminus \overline{D}$ and, since $bD \setminus K$ is strictly Levi-convex, it extends to an $f \in \mathcal{O}(\Delta \setminus D)$. The latter function in turn extends, by hypothesis, to an $F \in \mathcal{O}(\Delta \setminus E)$, and hence a function $g \in \mathcal{C}^\infty_\Phi(M \setminus E)$, such that $\bar\partial g = \alpha$, as is required, is that defined by

$$g = f_1 + F \text{ on } \Delta \setminus E, \; g = f_2 \text{ on } M \setminus \overline{D}.$$

The theorem is proved.

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