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§1. Introduction

Let X be a compact complex n-dimensional manifold with a smooth strictly-pseudoconvex boundary. Without loss of generality one can assume that X sits inside an open complex manifold, Z. A smooth function, $\phi : Z \longrightarrow \mathbb{R}$, is a *defining function* of X if it has the property:

$$\phi(p) \leq 1 \Longleftrightarrow p \in X$$

and if it has no critical points on the boundary. There are an infinity of different ways of choosing such a defining function, and it is a problem of considerable interest in the theory of pseudoconvex domains to find ways of making *canonical* choices. Jack Lee proved a result in his thesis which sheds some light on this problem: Suppose all the data above are real-analytic. Let S be the boundary of X and let $\Gamma \longrightarrow S$ be the bundle of outward-pointing conormal vectors to S. Given a real-analytic section, $\mu : S \longrightarrow \Gamma$, Lee proved that there exists a unique real-analytic defining function, ϕ, which satisfies the boundary condition, $d\phi = \mu$ on S and satisfies the homogeneous Monge-Ampere equation

$$(\bar{\partial}\partial\phi)^n = 0 \tag{1.1}$$

on a neighborhood of S.* One of the aims of this paper is to give a new proof of this result. This proof is similar to a proof that Matt Stenzel and I gave of an existence theorem for Monge-Ampere with a different set of boundary conditions in $[GS]_1$. I will give a brief description of this proof below; however, first I want to describe the other main result of this paper. Let X be a compact Riemannian manifold. Suppose that X is real-analytic, and suppose that $f : X \longrightarrow \mathbb{R}$ is a real-analytic function. Several years ago Boutet de Monvel proved the following surprising result:

Theorem. *[B] The following are equivalent*

1. *f can be extended holomorphically to a Grauert tube of radius r about X.*
2. *The wave equation*

$$\frac{\partial u}{\partial t} = \sqrt{\Delta} u \ , \qquad u(x,0) = f(x)$$

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*See [L]. Subsequently Jerison and Lee [JL] showed that there is a canonical way of choosing μ as well (by solving a CR variant of the Yamabe problem).

can be solved backwards in time over the interval $-r \leq t \leq 0$.

In other words Boutet's result says that the problem of extending f to a small neighborhood of X inside the complexification, $X_{\mathbb{C}}$, is equivalent to solving a diffusion problem in the wrong direction! Matt Stenzel and I showed in $[GS]_2$ that this result has some interesting connections with homogeneous Monge-Ampere. In this paper I will show that there is a form of Boutet's result which is true for an *arbitrary* real-analytic pseudoconvex domain; and this, too, will involve homogeneous Monge-Ampere in a fundamental way. The statement and proof of this result will be given in §5 and I will give my new proof of Lee's theorem in §4. As in $[GS]_1$ the main step in this proof will be the complexification of a solution of a certain *real* Monge-Ampere equation which I now want to describe: Let X and Y be real n-dimensional manifolds and consider the DeRham complex on $X \times Y$. By the Künneth theorem this complex is a double complex with an exterior derivative, d_x, that only involves the X-variables and an exterior derivative, d_y, that only involves the Y-variables. In particular, given a function, $\phi = \phi(x, y)$, on $X \times Y$ one gets a two-form, $d_x d_y \phi$, and, wedging this form with itself n times, a $2n$-form, $(d_x d_y \phi)^n$. Now let S be a hypersurface in $X \times Y$ and ϕ_0 a defining function for it. Suppose that ϕ_o satisfies:

$$(d_x d_y \phi_0)^{n-1} \wedge d_x \phi_0 \wedge d_y \phi_0 \neq 0 \tag{1.2}$$

on a neighborhood of S.* I will prove in §2 that, on every sufficiently small neighborhood of S, there exists a unique function, ϕ, such that $\phi - \phi_0$ vanishes to second order on S and

$$(d_x d_y \phi)^n = 0. \tag{1.3}$$

In other words given a surface, S, with the convexity property, (1.2), the Cauchy problem for (1.3), with initial data on S, can always be solved in a neighborhood of S. The proof will involve some ideas that have come up earlier in the work of Phong and Stein, [PS], and in my own work with Sternberg ([GS], Chapter 6) on Radon integral transforms; and I will explain what Monge-Ampere has to do with this subject in §2–3.

To conclude I would like to mention a number of recent articles on homogeneous Monge-Ampere dealing with issues that I've touched on here. These are, in addition to my two articles with Stenzel cited above, the article, [EM], of Epstein-Melrose and the articles, [LS] of Lempert-Szöke, [S] of Szöke and [Lem] of Lempert. In particular, in Lempert's article, it is shown that for the Monge-Ampere problem discussed in $[GS]_1$, $[GS]_2$ the analyticity assumptions are *necessary* as well as sufficient.

§2. Double fibrations.

*This condition depends only on S not on the choice of ϕ_0. It is the analogue in this "Künneth" theory of the Levi condition.

Let X and Y be n-dimensional manifolds and S a closed $(2n-1)$-dimensional submanifold of $X \times Y$. Let π and ρ be the restrictions to S of the projection maps of $X \times Y$ onto X and Y. The triple (S, π, ρ) is called a *double fibration* if both π and ρ are fiber mappings. I will assume that the conormal bundle of S is oriented and will denote by Γ the set of its positively-oriented vectors. Composing the inclusion, $\Gamma \longrightarrow T^*(X \times Y)$, with the projections of $T^*(X \times Y)$ and T^*X and T^*Y one gets maps

$$\pi_1 : \Gamma \longrightarrow T_0^*X \quad \text{and} \quad \rho_1 : \Gamma \longrightarrow T_0^*Y \tag{2.1}$$

of Γ onto the punctured cotangent bundles of X and Y.* The data, (S, π, ρ), are said to satisfy the *Bolker* condition if π_1 and ρ_1 are diffeomorphisms, in which case the composite mapping, $\rho_1 \circ \pi_1^{-1}$ is well-defined. Composing this mapping with the involution:

$$\sigma : T_0^*Y \longrightarrow T_0^*Y \quad , \quad \sigma(y, \eta) = (y, -\eta)$$

one gets a canonical transformation

$$\gamma : T_0^*X \longrightarrow T_0^*Y \tag{2.2}$$

which I will call the *canonical transformation associated with the double fibration* (S, π, ρ).

To check that the Bolker condition is satisfied, one has to check first that π_1 and ρ_1 are diffeomorphisms locally in the neighborhood of each point of Γ, and then check that they are one-one and onto. Often the second criterion is *implied* by the first. (This is so, for instance, if both X and Y are compact.) As for the first criterion, it is easy to see that if π_1 is locally a diffeomorphism at a point of Γ, ρ_1 is as well. This criterion can also be checked rather easily by the following means. Let $\phi = \phi(x, y)$ be a defining function of S i.e. let S be the subset of $X \times Y$ defined by the equation, $\phi(x, y) = 1$; and assume $d\phi_p \neq 0$ at all points, $p \in S$. Let $d_x d_y \phi$ be the two-form

$$\sum_{i,y=1}^{n} \frac{\partial^2 \phi}{\partial x_i \partial y_i} dx_i \wedge dy_j$$

Lemma. *For π_1 and ρ_1 to be local diffeomorphisms at all points of Γ it is necessary and sufficient that the $2n$-form*

$$(d_x d_y \phi)^{n-1} \wedge d_x \phi \wedge d_y \phi \tag{2.3}$$

be non-vanishing on a neighborhood of S.

I will leave the proof of this as an easy exercise. My goal in this section is to prove that if S satisfies the Bolker condition it has a defining function which satisfies, in addition to (2.3), the homogeneous Monge-Ampere equation described in the introduction:

*Given a manifold, M, we will denote by $T_0^*(M)$ the cotangent bundle of M with its zero section deleted.

Theorem 1. *Let $\mu : S \longrightarrow \Gamma$ be a section of Γ. Then there exists a unique defining function, ϕ, of S such that*

$$(d_x d_y \phi)^n \equiv 0 \tag{2.4}$$

on a neighborhood of S, and such that, in addition, $d\phi_p = \mu_p$ at all points, $p \in S$.*

Proof. Existence: There exists a unique homogeneous function of degree one on Γ which is identically equal to one on the image of μ. Lets denote this function by H_0. Under the diffeomorphism $T_0^* X \longrightarrow \Gamma$ this pulls back to a homogeneous function of degree one, H, on $T_0^* X$. Since (S, π, ρ) is a double fibration the fibers, $S_y = \rho^{-1}(y)$, above points of Y are $(n-1)$-dimensional submanifolds of X. Now, with y fixed, solve the Hamilton-Jacobi equation:

$$H(d\phi) = H\left(x, \frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n}\right) = 1 \tag{2.5}$$

with the initial condition $\phi = 1$ on S_y.* This solution depends parametrically on y so it is really a function, $\phi = \phi(x, y)$, of *both* the x and the y variables and is well-defined in a neighborhood, U, of S. Let's show that it satisfies the Monge-Ampere equation and the required initial conditions. That it satisfies the initial conditions is equivalent to the assertion that $H_0(d\phi) = 1$ on S and this is equivalent to the assertion that, for y fixed, the equation $H(d\phi) = 1$ holds on X. To check that ϕ satisfies Monge-Ampere, we note that because H doesn't depend on y we can differentiate the identify

$$H\left(\frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n}, x\right) = 1$$

with respect to y_i getting:

$$\sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j}(d_x \phi, x) \frac{\partial^2 \phi}{\partial x_j \partial y_i} = 0$$

Since $\frac{\partial H}{\partial \xi}(x, \xi) \neq 0$ when $\xi \neq 0$ this implies that

$$\det\left(\frac{\partial^2 \phi}{\partial x_i \partial y_j}\right) = 0.$$

Uniqueness: Let ϕ be a defining function of S satisfying the given initial conditions. By assumption the map

$$\pi_1 : S \times \mathbb{R}^+ \longrightarrow T_0^* X$$

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*In local coordinates this is just the Monge-Ampere equation $\det(\frac{\partial^2 \phi}{\partial x_i \partial y_j}) = 0$.

*Let's briefly review how this is done. The equation, $H = 1$, cuts out a hypersurface in the conormal bundle of S_y. This hypersurface is an isotropic submanifold of $T^* X$ of dimension $n-1$, so if we take its flow-out with respect to the Hamiltonian flow, $\exp t\Xi_H$, we get an n-dimensional Lagrangian submanifold, Λ, of $T^* X$. In the vicinity of S_y Λ is the graph of an exact one form, $d\phi$, and if we normalize ϕ to be one on S_y this determines it uniquely.

sending (x, y, s) to $(x, sd_x\phi)$ is a diffeomorphism; and, by our definition of H, it embeds S onto the hypersurface, $H = 1$. Suppose now that ϕ satisfies the Monge-Ampere equation on a neighborhood of S. This says that the map

$$p: X \times Y \longrightarrow T_0^*X, \quad p(x,y) = (x, d\phi_x), \tag{2.6}$$

is of rank $2n - 1$ in a neighborhood of S, and hence is a *fibering* of a neighborhood, U, of S onto the hypersurface, $H = 1$. In particular, for y fixed, $\phi(x, y)$ satisfies the Hamilton-Jacobi equation

$$H\left(x, \frac{\partial \phi}{\partial x}\right) = 1.$$

Moreover, since ϕ is a defining function of S, it takes the value, $\phi = 1$, on S. Therefore, if one fixes y and regards it as a function of x alone, it takes the initial value, $\phi = 1$ on S_y. Hence the uniqueness of ϕ follows from standard uniqueness results in the Hamilton-Jacobi theory. Q.E.D.

Remark. If ϕ satisfies Monge-Ampere, the function, $\psi = \phi^2$, satisfies

$$\det\left(\frac{\partial^2 \psi}{\partial x_i \partial y_j}\right) \neq 0 \tag{2.7}$$

everywhere on a neighborhood of S. To see this, note that since

$$d_x d_y \psi = 2(\phi d_x d_y \phi + d_x \phi \wedge d_y \phi),$$

Monge-Ampere implies that

$$(d_x d_y \psi)^n = n 2^n \phi^{n-1} (d_x d_y \phi)^{n-1} \wedge d_x \phi \wedge d_y \phi$$

and the expression on the right is non-zero on a neighborhood of S in view of (2.3). (Recall that $\phi = 1$ on S.) Therefore, by specifying a section, μ, of $\Gamma \longrightarrow S$ one gets, not only a solution of Monge-Ampere on a neighborhood of S, but also a *symplectic* form on that neighborhood, (i.e. $d_x d_y \psi$) and a *pseudo-Riemannian metric* of signature (n, n) :

$$\Sigma \frac{\partial^2 \psi}{\partial x_i \partial y_j} dx_i \circ dy_j. \tag{2.8}$$

§3. A dynamical interpretation at the Monge-Ampere equation (2.4).

For simplicity I will assume in this section that X, Y and S are compact. Let ϕ be a defining function of S and let S_t be the subset of $X \times Y$ defined by the equation, $\phi = 1 - t$.

(In particular, $S_0 = S$.) For t sufficiently small, S_t will also satisfy the Bolker condition and hence give rise to a canonical transformation

$$\gamma_t : T_0^*X \longrightarrow T_0^*Y.$$

I will prove below that if ϕ satisfies Monge-Ampere the canonical transformations, γ_t, are related to one another in a very simple way: As we saw in the preceding section, the initial data, $\mu : S \longrightarrow \Gamma$, determine a homogeneous function of degree one, H, on T_0^*X. Let Ξ_H be the Hamiltonian vector field corresponding to H. Because of the homogeneity of H the group of symplectomorphisms generated by Ξ_H is a one-parameter group of canonical transformations. Lets denote this one-parameter group by $\exp t\Xi_H, -\infty < t < \infty$.

Theorem 2. *The following are equivalent:*

1. *ϕ satisfies Monge-Ampere.*
2. *$\gamma_t = \gamma \circ \exp t\Xi_H$.*

Proof that 1 implies 2:. Suppose ϕ satisfies Monge-Ampere. For the moment lets fix $y \in Y$ and think of $\phi(x,y)$ as a function of x alone.

Lemma. *There exists a neighborhood, U, of S_y and, for every point, $x \in U$, a unique point, $x_0 \in S_y$ such that*

$$\exp t\Xi_H(x_0,\xi_0) = (x,\xi) \tag{3.1}$$

where $\xi_0 = d\phi_{x_0}$, $\xi = d\phi_x$ and $\phi(x) = 1 + t$.

Proof. Let Λ_0 be the set

$$\{(x_0,\xi_0);\ x_0 \in S_y,\ \xi_0 = d\phi_{x_0}\}.$$

For ϵ sufficiently small the map of $\Lambda_0 \times (-\epsilon,\epsilon)$ into T^*X which sends (x_0,ξ_0,t) onto $(\exp t\Xi_H)(x_0,\xi_0)$ is a diffeomorphism of $\Lambda_0 \times (-\epsilon,\epsilon)$ onto a Lagrangian submanifold, Λ, of T^*X. Moreover, Λ is also the image of a neighborhood, U of S_y in X with respect to the mapping

$$d\phi : U \longrightarrow T^*X, \quad x \longrightarrow (x, d\phi_x).*$$

Thus, if x is in U, there is a unique $x_0 \in S_y$ and a unique t on the interval $(-\epsilon,\epsilon)$ such that

$$\exp t\Xi_H(x_0,\xi_0) = (x,\xi)$$

with $\xi_0 = d\phi_{x_0}$ and $\xi = d\phi_x$. Thus all that is left to prove is that $t = \phi(x)$. This, however, follows easily from the homogeneity of H. Namely, by Euler's identity

$$H(x,\xi) = \Sigma \frac{\partial H}{\partial \xi_i}(x,\xi)\xi_i;$$

*See the footnote following the display (2.4) in§2.

so, if $\xi = d\phi_x$:

$$1 = H(x, d\phi_x) = \Sigma \frac{\partial H}{\partial \xi_i}(x,\xi)\frac{\partial \phi}{\partial x_i}. \tag{3.2}$$

Let $(x(t), \xi(t))$, $0 \leq t \leq \epsilon$, be the integral curve of Ξ_H starting at (x_0, ξ_0). Then by (3.2)

$$\frac{d}{dt}\phi(x(t)) = 1.$$

Hence since $\phi = 1$ at x_0, $\phi = 1 + t$ at $x(t)$. Q.E.D.

Let's now compare the canonical transformations, $\gamma_t \circ \exp(-t\Xi_H)$ and γ. Because of the homogeneity properties of γ, γ_t and H it suffices to check that they are equal at all points, (x_0, ξ_0), on the hypersurface, $H = 1$. However, each point on this hypersurface is the image under the mapping (2.6) of a point, $(x_0, y) \in S$. In other words there is a unique $y \in Y$ such that $(d_x\phi)(x_0, y) = \xi_0$. Thus, by definition

$$\gamma(x_0, \xi_0) = (y, d_y\phi(x_0, y)). \tag{3.3}$$

On the other hand, by the lemma

$$\exp(-t\Xi_H)(x_0, \xi_0) = (x, \xi)$$

with $\xi = (d_x\phi)(x, y)$ and $\phi(x, y) = 1 - t$. Thus $(x, y) \in S_t$ and, hence

$$\gamma_t(x, \xi) = (y, d_y\phi(x, y)) = \gamma_t \circ \exp(-t\Xi_H)(x_0, \xi_0).$$

Therefore, if we denote by β the projection of T_0^* onto Y, we conclude from (3.3) and (3.4) that

$$\beta \circ \gamma_t \circ \exp(-t\Xi_H) = \beta \circ \gamma,$$

and hence that the canonical transforms, γ and $\gamma_t \circ \exp(-t\Xi_H)$ are themselves the same.*

Proof that 2 implies 1:. Not only does the hypersurface S_t determine the canonical transformation, γ_t, but it is clear from the definition of γ_t that the reverse is true: The canonical transformation, γ_t, determines the hypersurface, S_t. Thus if 2 holds, ϕ has the same level surfaces as does the corresponding solution of Monge-Ampere and hence has to be *equal* to this solution. Q.E.D.

I will conclude with a few words about the "quantum picture" that goes along with the result above. Let F_t be an elliptic Fourier integral operator whose underlying canonical transformation is γ_t. then by Theorem 2

$$F_t = F_0 U(t) Q + K \tag{3.4}$$

*Fact: If $\gamma_i : T_0^*X \longrightarrow T_0^*Y$, $i = 1, 2$, are canonical transforms, then $\beta \circ \gamma_1 = \beta \circ \gamma_2$ implies $\gamma_1 = \gamma_2$. (see, for instance [AM].)

K being a smoothing operator, Q an invertible elliptic pseudodifferential operator and $U(t)$ a one-parameter unitary group of the form

$$U(t) = \exp\sqrt{-1}tP \tag{3.5}$$

where P is a pseudodifferential operator with the function, H, as its leading symbol. We will see in section 5 that for he Monge-Ampere equation on a complex manifold there is no analogue of Theorem 2 per se, but there *is* an analogue of the statements (3.4) and (3.5). In the analogue of (3.5), however, the unitary group, $\exp\sqrt{-1}tP$, gets replaced by the corresponding heat semi-group, $\exp(-tP)$.

§4. The proof of Lee's theorem.

First of all note that if one replaces real C^∞ data by complex holomorphic data the existence theorem of §2 is still true and can be proved, with a few small changes, in exactly the same way. More explicitly, suppose Z and W are complex n-dimensional manifolds, $S_{\mathbb{C}}$ a complex hypersurface in $Z \times W$ and $\phi_{\mathbb{C}} = \phi_{\mathbb{C}}(z, w)$ a holomorphic defining function of $S_{\mathbb{C}}$ which satisfies the complex analogue of (2.3). Then one can modify $\phi_{\mathbb{C}}$, without changing $d\phi_{\mathbb{C}}$ at points of $S_{\mathbb{C}}$, so that it also satisfies

$$\det\left(\partial^2\phi_{\mathbb{C}} \middle| \partial z_i \partial w_j\right) = 0 \tag{4.1}$$

on a neighborhood of S. Moreover, this modified $\phi_{\mathbb{C}}$ is unique.

Now let X be a compact complex n-dimensional manifold with a real-analytic strictly pseudoconvex boundary and let Z be an open complex manifold containing it. Let S be the boundary of X and let ϕ be a real-analytic defining function of S. I will denote by W the manifold, Z, equipped with its conjugate complex structure, and by ι the diagonal imbedding of Z into $Z \times W$. It is clear that there exists an open neighborhood, U, of the image of S in $Z \times W$ and a (unique) holomorphic function, $\phi_{\mathbb{C}}$, on U such that $\iota^*\phi_{\mathbb{C}} = \phi$. Let $S_{\mathbb{C}}$ be defined by the equation, $\phi_{\mathbb{C}} = 1$. Then the Levi condition implies that $\phi_{\mathbb{C}}$ satisfies the holomorphic analogue of (2.3) at all points $\iota(p), p \in S$; and, therefore, if U is chosen small enough, this condition is satisfied at all points of $S_{\mathbb{C}}$. Therefore, by the holomorphic version of Theorem 1, one can modify $\phi_{\mathbb{C}}$, without changing the first derivatives of $\phi_{\mathbb{C}}$ along $S_{\mathbb{C}}$, so that it satisfies (4.1). This, however, implies that $d\iota^*\phi_{\mathbb{C}} = d\phi$ at points of S and, in addition,

$$\det\left(\partial^2\iota^*\phi_{\mathbb{C}}/\partial z_i \partial \bar{z}_j\right) = 0. \tag{4.2}$$

Moreover, $\iota^*\phi_{\mathbb{C}}$ is the unique real-analytic solution of (4.2) satisfying the given initial condition. However, since the initial data are real-valued, $\overline{\iota^*\phi_{\mathbb{C}}}$ is another solution of (4.2) with these initial data; so $\iota^*\phi_{\mathbb{C}} = \overline{\iota^*\phi_{\mathbb{C}}}$: i.e. $\iota^*\phi_{\mathbb{C}}$ is itself real-valued.

§5. Homogeneous Monge-Ampere and the extendibility problem.

Let X be a compact, complex n-dimensional manifold with a real-analytic strictly pseudoconvex boundary. As above I will assume that X is sitting inside an open complex manifold, Z, and that $\phi : Z \longrightarrow \mathbb{R}$ is a real analytic defining function of X. By choosing ϵ sufficiently small one can arrange that for all t on the interval, $(-\epsilon, \epsilon)$, $\phi - t$ is the defining function for a strictly pseudoconvex domain, X_{t+1} defined by the inequality, $\phi(z) \leq 1 + t$.

The problem I want to consider below is the extendibility problem: Given $s < t$ and given a holomorphic function, f, on X_s, can one extend f to a holomorphic function on X_t? I would like an answer to this question which is similar in spirit to the result of Boutet de Monvel that I quoted in the introduction: namely f can be extended providing one can solve some kind of diffusion process, with f as initial data, backwards in time over the interval $[s-t, 0]$. I will show that one can find a characterization of extendibility, in these terms, if ϕ satisfies homogeneous Monge-Ampere.* First of all, however, let me formulate the extendibility problem in a way that only involves the behavior of ϕ on the annulus

$$Z_\epsilon = \{z \in Z,\ 1 - \epsilon < \phi(z) < 1 + \epsilon\}$$

Let S_t be the boundary of the region, X_t, and ι_t the inclusion map of S_t into Z. Let $\mathcal{O}(X_t)$ be the space of functions which are holomorphic on $\mathrm{Int}(X_t)$ and smooth up to the boundary. Then the restriction map

$$\iota^* : \mathcal{O}(X_t) \longrightarrow C^\infty(S_t)$$

is injective, and its image is the space of *Cauchy-Riemann functions* on S_t. I will denote this space by $\mathbb{CR}(S_t)$. Thus for $s < t$ one gets a diagram:

$$(5.1) \qquad \begin{array}{ccc} \mathcal{O}(X_t) & \xrightarrow{\cong} & \mathbb{C}R(S_t) \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{O}(X_s) & \longrightarrow & \mathbb{CR}(S_s) \end{array}$$

the left hand arrow being the restriction map and $R_{s,t}$ being, by definition, the right hand arrow. Note that, for $-\epsilon < s < t < \mu < \epsilon$:

$$(5.2) \qquad R_{s,t}\, R_{t,u} = R_{s,u}.$$

It is clear that the extendibility problem is equivalent to the problem of characterizing the ranges of the mappings, $R_{s,t}$.

To formulate my main result I will have to discuss some geometric properties of the annulus, Z_ϵ, associated with the function, ϕ. I will think of the complex structure on Z as being given by a morphism, J, of the tangent bundle of Z, with $J^2 = -I$. Since $d\phi_p \neq 0$ at all points, $p \in Z_\epsilon$ the one-forms:

$$(5.3) \qquad d\phi \quad \text{and} \quad \alpha =: -d\phi \circ J$$

are non-vanishing and linearly independent everywhere. Corresponding to these one forms are a dual pair of vector fields, $\mathfrak{v}$ and $\mathfrak{w}$, which I will define by means of the following:

*And, in some sense, only if ϕ satisfies homogeneous Monge-Ampere.

Proposition. *There exists a unique vector field, $\mathfrak{w}$, on Z_ϵ with the following three properties*

$$(5.4)\qquad \begin{array}{ll} i. & \iota(\mathfrak{w})\alpha = 1 \\ ii. & \iota(\mathfrak{w})d\phi = 0 \\ iii. & \iota(\mathfrak{w})d\alpha \text{ is the product of } d\phi \text{ with a } C^\infty \text{ function.} \end{array}$$

Proof. To say that the hypersurfaces, $\phi = 1 + t$, are strictly pseudoconvex for $-\epsilon < t < \epsilon$ is equivalent to saying that

$$(5.5)\qquad (d\alpha)^{n-1} \wedge \alpha \wedge d\phi \neq 0$$

at all points of Z_ϵ, and the existence of a unique vector field, $\mathfrak{w}$, satisfying (5.4) can be deduced from (5.5) by elementary linear algebra. Q.E.D.

I will now define $\mathfrak{v}$ to be the vector field,

$$(5.6)\qquad \mathfrak{v} = J\mathfrak{w}.$$

From (5.3) and (5.4) one deduces

$$(5.7)\qquad \iota(\mathfrak{v})d\phi = 1 \quad \text{and} \quad \iota(\mathfrak{v})\alpha = 0.$$

Next I want to recall a standard criterion for the function, ϕ, to satisfy homogeneous Monge-Ampere:

Lemma. *ϕ satisfies homogeneous Monge-Ampere if and only if the condition, (5.4) iii., can be replaced by the stronger condition*

$$(5.8)\qquad \iota(\mathfrak{w})d\alpha = 0.$$

Proof. ϕ satisfies homogeneous Monge-Ampere if and only if the two-form, $d\alpha$, is of rank $n-1$ at all points of Z_ϵ, or in other words, if and only if there exists a nowhere vanishing vector field, $\mathfrak{w}$, satisfying (5.8). However, because of (5.5), if such a vector field exists, it can always be chosen to satisfy (5.4), i. and ii., as well. Q.E.D.

Corollary. *ϕ satisfies homogeneous Monge-Ampere if and only if $[\mathfrak{v}, \mathfrak{w}] = 0$.*

Proof. Since $\iota(\mathfrak{w})d\alpha$ is a multiple of $d\phi$ the condition (5.8) holds if and only if $d\alpha(\mathfrak{v}, \mathfrak{w}) = 0$. Note, however, that

$$(5.9)\qquad d\alpha(\mathfrak{v}, \mathfrak{w}) = D_{\mathfrak{v}}(\alpha(\mathfrak{w})) = D_{\mathfrak{w}}(\alpha(\mathfrak{v})) - \alpha([\mathfrak{v}, \mathfrak{w}]) = -\alpha([\mathfrak{v}, \mathfrak{w}]).$$

Therefore, if $[\mathfrak{v}, \mathfrak{w}] = 0$, ϕ satisfies Monge-Ampere. Conversely, suppose ϕ satisfies Monge-Ampere. Then $\iota(\mathfrak{w})d\alpha = 0$ by the Lemma. Moreover, since $d\alpha = (1/2i)\bar{\partial}\partial\phi$, $d\alpha$ is J-invariant;

so $\iota(\mathfrak{w})d\alpha = 0$. As we have just remarked, ϕ satisfies Monge-Ampere if and only if $d\alpha$ is of rank $n-1$ everywhere, in which case the annihilator of $d\alpha$ is an integrable two-dimensional subbundle of the tangent bundle of Z_ϵ. Since $\mathfrak{v}$ and $\mathfrak{w}$ are sections of the subbundle it follows from the Frobenius condition that

$$[\mathfrak{v}, \mathfrak{w}] = f_1 \mathfrak{v} + f_2 \mathfrak{w}$$

for appropriately chosen C^∞ functions, f_1 and f_2. However, by (5.9), $f_2 = 0$, and f_1 is zero in view of the identity:

$$d\phi([\mathfrak{v}, \mathfrak{w}]) = D_{\mathfrak{v}}(D_{\mathfrak{w}}\phi) - D_{\mathfrak{w}}D_{\mathfrak{v}}\phi = 0.$$

Q.E.D.

Since $\mathfrak{w} = J\mathfrak{v}$ this result can be interpreted as saying that there is a (local) action of the complex, group, $\mathbb{C}$, on Z_ϵ generated by the complex vector field, $\mathfrak{v} + \sqrt{-1}\mathfrak{w}$. This action of $\mathbb{C}$ does *not* preserve the complex structure of Z_ϵ; however, the fact that $\mathfrak{w} = J\mathfrak{v}$ implies that the orbits of $\mathbb{C}$ are one-dimensional complex submanifolds of Z_ϵ. The existence of a $\mathbb{C}$-action with this property is probably the single most important consequence of the fact that ϕ satisfies Monge-Ampere.*

From now on I will assume that ϕ satisfies homogeneous Monge-Ampere and will describe some of the implications this has for the extendibility problem. Using the results above I will derive a formula for the restriction operator

$$R_{s,u} : \mathbb{CR}(S_u) \longrightarrow \mathbb{CR}(S_s)$$

in terms of an infinite series which will, in general, *not* converge; however, I will extract from this formula a meaningful expression by the insertion of Szegö projectors, and the main theorem of this paper will say that what I get is still a good approximation to $R_{s,u}$.

Let f be in $\mathcal{O}(X_u)$ and let $t = u - s$. Since $\mathfrak{v} - \sqrt{-1}\mathfrak{w}$ is an anti-holomorphic vector field,

$$\left(D_{\mathfrak{v}} - \sqrt{-1}D_{\mathfrak{w}}\right) f = 0$$

and hence, formally,

$$f = \exp t \left(D_{\mathfrak{v}} - \sqrt{-1}D_{\mathfrak{w}}\right) f.$$

Since $D_{\mathfrak{v}}$ and $D_{\mathfrak{w}}$ commute one can formally rewrite this equation in the form,

$$f = \exp(-\sqrt{-1}tD_{\mathfrak{w}})(\exp t\mathfrak{v})^* f,$$

hence if g is the restriction of f to the surface, S_u, we get for $R_{s,u}g$ the following formal expression

$$R_{s,u}g = \exp\left(-\sqrt{-1}tD_{\mathfrak{w}}\right)(\exp t\mathfrak{v})^* g. \tag{5.10}$$

*For some of its implications see Dan Burn's article [Bu]. This $\mathbb{C}$-action also plays an important role in the articles of Lempert and Szöke mentioned in the introduction.

Let's see to what extent this formula makes sense as a representation of $R_{s,u}$. Since $\exp t\mathfrak{v}$ is a diffeomorphism of S_s onto S_u, the operator

$$(5.11) \qquad (\exp t\mathfrak{v})^* : \mathbb{CR}(S_u) \longrightarrow C^\infty(S_s)$$

makes perfectly good sense. As for the operator, $\exp(-\sqrt{-1}tD_{\mathfrak{w}})$, if we substitute for it the infinite series

$$(5.12) \qquad I + (-\sqrt{-1}tD_{\mathfrak{w}}) + \frac{1}{2!}\left(\sqrt{-1}tD_{\mathfrak{w}}\right)^2 \ldots$$

each of the terms in this series is well defined as an operator on $C^\infty(S_s)$ since the vector field, $\mathfrak{w}$, is tangent to S_s. Indeed, if h is a real-analytic function on S_s, then

$$(\exp tD_{\mathfrak{w}})h = (\exp t\mathfrak{w})^* h$$

and the term on the left is real analytic in t as well as in the manifold variables, so it can be analytically continued to a small neighborhood of the origin on the imaginary t-axis. Thus, $\exp(-\sqrt{-1}tD_{\mathfrak{w}})$ is well-defined as an operator, but its domain of definition is a rather small subspace of $C^\infty(S_s)$.

Let me next recall the definition of the *Szegö projector* on $C^\infty(S_s)$. Restricting to S_s the $(2n-1)$-form, $\alpha \wedge (d\alpha)^{n-1}$, it becomes a volume form and provides $L^2(S_s)$ with a Hilbert space structure. Let $H^2(S_s)$ be the L^2 completion of $\mathbb{CR}(S_s)$ in $L^2(S_s)$ and let π_s be the orthogonal projection of $L^2(S_s)$ onto $H^2(S_s)$. This projection maps $C^\infty(S_s)$ onto $C^\infty(S_s) \cap H^2(S_s)$; and the latter space is $\mathbb{C}R(S_s)$; so, by restricting π_s to $C^\infty(S_s)$ one gets an operator:

$$\pi_s : C^\infty(S_s) \longrightarrow \mathbb{C}R(S_s);$$

and this is, by definition, the Szegö projector. I will now modify the right hand side of (5.12) by replacing the operator, $D_{\mathfrak{v}}$, wherever it occurs, by its Szegö cut-off: $\pi_s D_{\mathfrak{w}} \pi_s$. With this modification the right hand side of (5.12) becomes $\exp\left(-\sqrt{-1}t\pi_s D_{\mathfrak{w}}\pi_s\right)$ and since $t = u - s$, the right hand side of the formula, (5.10), for $R_{s,u}$ becomes

$$(5.13) \qquad \exp\left(-(u-s)T_s\right) F_{s,u}$$

where

$$(5.14) \qquad T_s = \pi_s \sqrt{-1} D_{\mathfrak{w}} \pi_s$$

and

$$(5.15) \qquad F_{s,u} = \pi_s(\exp(u-s)\mathfrak{v})^*.$$

For the following I will refer to [BG]:

Proposition A. *T_s is a positive first order self-adjoint elliptic Toeplitz operator. In particular it has real discrete spectrum:*

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots,$$

with the λ_i's tending to $+\infty$ and satisfying the Weyl asymptotics:

$$N(\lambda) = \text{ volume } (S_s)\lambda^n + O(\lambda^{n-1})$$

where $N(\lambda) = \#\{\lambda_i < \lambda\}$.

For the proof of this see §1 of [BG]. The point of the proposition is that T_s has exactly the same kind of spectral behavior as a positive definite elliptic pseudodifferential operator of order one on a compact n-dimensional manifold.

As for $F_{s,u}$ one has:

Proposition B. *The operator, $F_{s,u} : \mathbb{CR}(S_u) \longrightarrow \mathbb{CR}(S_s)$ is an elliptic Fourier-Toeplitz operator of order zero with $\exp(u-s)\mathfrak{v} : S_s \longrightarrow S_u$ as its underlying canonical transformation. Moreover, for $u - s$ sufficiently small, it is invertible.*

This also follows easily from the theory of Fourier-Toeplitz operators developed in [BG] or from the more general theory of Fourier integral operators with positive phase function developed in [MS]. I won't bother to give a proof of it here.

Thanks to these two propositions, the operator, (5.13), has very nice analytic properties, and this brings up the question: To what extent is it still a good approximation to $R_{s,u}$. The main result of this paper is that it is still a good approximation in the following sense.

Theorem 4. *For u and s close to one and $u - s$ small there exists an invertible zeroth order elliptic Toeplitz operator*

$$\mathcal{Q}_{s,u} : \mathbb{CR}(S_s) \longrightarrow \mathbb{CR}(S_s)$$

which depends real-analytically on u and s and satisfies:

$$R_{s,u} = \exp(-(u-s)T_s)\mathcal{Q}_{s,u}F_{s,u}. \tag{5.16}$$

This theorem says, in particular, that for $u - s$ small the range of $R_{s,u}$ agrees with the range of $\exp(-(u-s)T_s)$ so, in particular, one obtains from Theorem 4 the following result on extendibility.

Theorem 5. *Let f be a holomorphic function on X_s, which is smooth up to the boundary. Then it extends to a holomorphic function on X_u, which is smooth up to the boundary, iff the restriction of f to the boundary of X_s is in the range of $\exp(-(u-s)T_s)$.*

§6. The proof of the extendibility theorem.

Let $u > s > 1$ and let $S_1 = S$, $\pi_1 = \pi$ and $T_1 = T$. For $u < 1 + \epsilon$ consider the operator

$$W_{s,u} = F_{1,s} R_{s,u} F_{1,u}^{-1}. \tag{6.1}$$

This operator maps $\mathbb{CR}(S)$ onto $\mathbb{CR}(S)$ and, for $s < t < u$, satisfies the semigroup property

$$W_{s,t} W_{t,u} = W_{s,u} \tag{6.2}$$

In particular

$$\frac{d}{ds} W_{s,u} = P_s W_{s,u} \tag{6.3}$$

where

$$P_s = -\left(\frac{d}{d\epsilon}\right)_+ W_{s-\epsilon,s} \quad \text{at } \epsilon = 0. \tag{6.4}$$

we will prove:

Lemma. *P_s is a first order Toeplitz operator with the same leading symbol as T.*

Proof. Given a $\mathbb{CR}$-function, $h \in \mathbb{CR}(S)$, let $g = F_{1,s}^{-1} h$ and let f be the unique element of $\mathcal{O}(X_s)$ whose restriction to S_s is g. Finally let $\iota : S \longrightarrow Z$ be the inclusion map. Then

$$\begin{aligned} W_{s-\epsilon,s} h &= F_{1,s-\epsilon} R_{s-\epsilon,s} g \\ &= \pi \iota^* (\exp(s-\epsilon)\mathfrak{v})^* f \\ &= \pi \iota^* (\exp s\mathfrak{v})^* (\exp -\epsilon \mathfrak{v})^* f \end{aligned}$$

Thus, if we take the right hand derivative with respect to ϵ we get, at $\epsilon = 0$:

$$\left(\frac{d}{d\epsilon}\right)_+ W_{s-\epsilon,s} h = -\pi \iota^* (\exp s\mathfrak{v})^* D_{\mathfrak{v}} f.$$

Since f is holomorphic on the interior of X_s and smooth up to the boundary, and $\mathfrak{v} - \sqrt{-1}\mathfrak{w}$ is an anti-holomorphic vector field, $D_{\mathfrak{v}} f = \sqrt{-1} D_{\mathfrak{w}} f$. Moreover, since $\mathfrak{w}$ is tangent to S_s, $\sqrt{-1} D_{\mathfrak{w}} f$ is equal to $\sqrt{-1} D_{\mathfrak{w}} g$ on S_s; so the right hand side of the equation above is equal to

$$\pi (\exp s\mathfrak{v})^* (-\sqrt{-1} D_{\mathfrak{w}}) g$$

or

$$F_{1,s} (-\sqrt{-1} D_{\mathfrak{w}}) F_{1,s}^{-1} h.$$

Thus we obtain for P_s the formula

$$P_s = F_{1,s} (\sqrt{-1} D_{\mathfrak{w}}) F_{1,s}^{-1}. \tag{6.5}$$

By proposition B of §5 $F_{1,s}$ is a Fourier-Toeplitz operator whose underlying canonical transformation is $\exp(s-1)\mathfrak{v}$. Since $[\mathfrak{v}, \mathfrak{w}] = 0$

$$(\exp t\mathfrak{v})^* D_{\mathfrak{w}} = D_{\mathfrak{w}} (\exp t\mathfrak{v})^* \tag{6.6}$$

for all t. Thus, by the composition formula for Fourier-Toeplitz operators described in [BG] §7, the operator (6.5) is a Toeplitz operator, and has the same leading symbol as the Toeplitz operator, $\pi(\sqrt{-1}D_{\mathfrak{v}})\pi$. Q.E.D.

Let $A(s) = P_s - T$. This operator is a zeroth order Toeplitz operator depending analytically on the parameter, s, and, by (6.3), it satisfies the operator equation

$$\frac{d}{ds} W_{s,u} = T W_{s,u} + A(s) W_{s,u}.$$

With u fixed, let $s = u - t$, and let $W(t) = W_{u-t,u}$ and $B(t) = -A(u-t)$. Then the equation above can be rewritten in the form

$$\frac{d}{dt} W(t) = -T W(t) + B(t) W(t), \tag{6.7}$$

on the interval $0 \le t \le u-1$, with $W(0) = I$. Formally one can solve this equation by "variation of constants": i.e. setting

$$B^{\#}(t) = (\exp tT) B(t) \exp(-tT), \tag{6.8}$$

one can express the solution of (6.7) in the form:

$$W(t) = \exp(-tT) Q(t), \tag{6.9}$$

where $Q(t)$ is the solution of the operator equation,

$$\frac{dQ(t)}{dt} = B^{\#}(t) Q(t) \quad \text{with} \quad Q(0) = I. \tag{6.10}$$

To make sense of this formal solution we must first make sense of (6.8), and this we will do as follows: Since $B(t)$ depends real-analytically on t, it extends to a holomorphic function of t on a small neighborhood of the origin in the complex t-plane. Thus in particular $B(\sqrt{-1}t)$ is well defined, by analytic continuation, for real values of t close to zero. Now notice that when we replace t by $\sqrt{-1}t$ in (6.8), (6.8) becomes:

$$B^{\#}(\sqrt{-1}t) = \exp\sqrt{-1}tT\ B(\sqrt{-1}t) \exp(-\sqrt{-1}tT). \tag{6.11}$$

Since $\exp\sqrt{-1}tT$ is an elliptic zeroth order Fourier-Toeplitz operator, it follows from Egorov's theorem that (6.11) is a zeroth order Toeplitz operator, also depending in a real analytic fashion on t. Thus we can again, for $|t|$ small, replace t by $-\sqrt{-1}t$ in (6.11), and we end up with a well-defined zeroth order Toeplitz operator which is, formally, the operator (6.8). This we will now *define* to be the operator, $B^{\#}(t)$. With this definition of $B^{\#}(t)$ the equation (6.8) holds in the sense that for all $a > t$

$$\exp(-aT)B^{\#}(t) = \exp(t-a)TB(t)\exp(-tT). \tag{6.12}$$

Plugging this Toeplitz operator, $B^{\#}(t)$, that we have just defined, into (6.10) and solving for $\mathcal{Q}(t)$ we end up with a putative solution, $W(t) = \exp(-tT)\mathcal{Q}(t)$, to the equation, (6.7). To show, by means of (6.12), that this is an actual solution is not hard. We leave details to the reader.*

Inserting (6.10) into (6.1) and remembering that $\mathcal{Q}(t)$ depends analytically on the parameter, μ, as well as on t we get

$$\exp -(u-s)Q_u(u-s) = F_{1,s}R_{s,u}F_{1,s}^{-1}$$

or, in particular, setting $s = 1$,

$$R_{1,u} = \exp(-(u-1)T)Q_u(u-1)F_{1,u}$$

for $1 < u < \epsilon$. This proves Theorem 4 for $s = 1$; and the theorem, for arbitrary s, can be deduced from this special case by replacing ϕ, in the discussion above, by $\phi - (s-1)$.

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