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0. Introduction

In **[HS]** we introduced a class of semi-classical Schrödinger operators of the form $-\frac{1}{2}h^2\Delta + V^{(m)}$ on $\mathbb{R}^m$ for $m = 1, 2, \dots$, where $V^{(m)}$ satisfy various assumptions, implying in particular convexity. If $\mu(m;h)$ denotes the first eigenvalue, we showed among other things that $\mu(m;h)/m$ tends to a limit $\mu(\infty;h)$ when $m \to \infty$ and that :

$$\mu(m;h)/m - \mu(\infty;h) = \mathcal{O}(h/m) \ . \tag{0.1}$$

We also proved (by adapting the methods of **[S1, 2]**) that $\mu(\infty;h)$ has an asymptotic expansion $\sim h(\mu_0 + \mu_1 h + \dots)$, when $h \to 0$. One element of the proof was the use of certain WKB-expansions, more precisely, we showed that if $h(\mu_0(m) + \mu_1(m)h + \dots)$ is the formal asymptotic expansion of $\mu(m;h)$, then $\mu_k(m)/m \to \mu_k$ when $m \to \infty$ with an exponential rate of convergence. A natural question is then wether (0.1) can be improved to :

$$\mu(m;h)/m - \mu(\infty;h) = \mathcal{O}(e^{-\kappa m}) \tag{0.2}$$

for some suitable $\kappa > 0$.

In this work, we establish estimates of the form (0.2) for certain sequences of $V^{(m)}$. A general result of this type is given in Theorem 3.1, and in Theorem 4.1 we obtain a better rate of exponential convergence for a somewhat more restricted class of potentials. In particular, we study in section 5 the same sequence of potentials related to statistical mechanics as in **[HS]**, and show that we get exponential convergence with a rate which seems to be optimal.

In **[HS]** we obtained exponential convergence at the level of WKB-eigenvalues by introducing exponential weights in the study of certain Hessians of the logarithm of certain WKB approximations to the first eigenfunction. These estimates were obtained by adapting the WKB-constructions in the complex domain of **[S1, 2]**, and by introducing certain exponential weights in these estimates. In the present work, we also establish exponentially weighted estimates of certain Hessians of the logarithm of the first eigenfunction, but this time we work with the exact first eigenfunctions, and inspired by the appendix b in **[SiWYY]**, we use systematically the maximum principle in order to obtain these estimates. In particular, we never use any small h expansions, and our results are uniform in h.

The plan of the paper is the following : In section 1, we make some estimates for the log. of the first eigenfunction near $|x| = \infty$, in the case when the potential is a compactly supported perturbation of $\frac{1}{2}x^2$. These estimates, which are not necessarily uniform with respect to the dimension, form a preparation for the more refined estimates that we obtain in section 2. In section 3 we get a first result about the validity of (0.2).

In section 4, we start by examining a sequence of simple quadratic potentials, and we see that Theorem 3.1 does not give the optimal κ in this case. Then after some further exponential estimates in the style of section 2, we obtain the sharper Theorem 4.1, which is valid under somewhat different assumptions. In section 5, we apply this result to the model problem from statistical mechanics already studied in **[HS]**, and establish (0.2) with a set of κ which seems to be optimal.

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1. Some estimates for the exterior Dirichlet problem for the harmonic oscillator

Let B be an open a ball in $\mathbb{R}^n$ centered at 0. Then the Dirichlet realization P of $-\Delta+x^2$ in $\mathbb{R}^n\setminus B$ has discrete spectrum. Choose $\mu\in\mathbb{R}$ such that $x^2-\mu>0$ in $\mathbb{R}^n\setminus B$. Then μ is also below the infimum of the spectrum of the operator P just defined, and we let $K : C^\infty(\partial B)\to C^\infty(\mathbb{R}^n\setminus B)$ be the operator such that $u=Kv$ belongs to the domain of P outside a compact set and solves the problem:

$$(1.1)\qquad (-\Delta+x^2-\mu)u=0\ ,\quad u|_{\partial B}=v\ .$$

Using weighted L^2 estimates we see that $\partial^\alpha Kv(x)\to 0$, $|x|\to\infty$, for every α. Using the maximum principle we then have that $v\geq 0\Rightarrow Kv\geq 0$. This implies that if $v_1\leq v_2$ then $Kv_1\leq Kv_2$, and also $Kv\leq \sup v$, if $\sup v\geq 0$, $Kv\geq \inf v$ if $\inf v\leq 0$. Of particular interest is $K(1)$ which is a radial function $u_0=u_0(|x|)$, with:

$$(1.2)\qquad (-\partial_r^2-((n-1)/r)\partial_r+r^2-\mu)u_0(r)=0\ ,\ u_0(1)=1\ .$$

Here and in the following we assume (without loss of generality) that B is the unit ball. Writing $u_0=r^{-(n-1)/2}f(r)$, we know that f is in $L^2([1,\infty[,dr)$ and satisfies the Schrödinger equation:

$$(1.3)\qquad (-\partial_r^2+r^2+(n-1)(n-3)/4r^2-\mu)f=0\ ,\quad f(1)=1\ .$$

We can construct $\varphi(r)$ with

$$(1.4)\qquad \varphi'(r)\sim r+a_{-1}r^{-1}+a_{-3}r^{-3}+\dots,\ r\to+\infty\ ,$$

such that

$$(1.5)\qquad (-\partial_r^2+r^2+(n-1)(n-3)/4r^2-\mu)(e^{-\varphi(r)})=e^{-\varphi(r)}\widetilde{R}(r)\ ,$$

where $\widetilde{R}$ is rapidly decreasing with all its derivatives when $r\to+\infty$. Actually we solve asymptotically the equation $(\varphi')^2-\varphi''=r^2+(n-1)(n-3)/4r^2-\mu$, and it is a routine procedure to verify that $f=e^{-(\varphi+R)}$, with $\partial^\alpha R=\mathcal{O}(r^{-\infty})$ for every $\alpha>0$. Replacing φ by $\varphi+R$, we still have (1.4). With $g(r)=\varphi(r)+((n-1)/2)\log r$, we get:

$$(1.6)\qquad u_0=e^{-g(|x|)}\ .$$

Here we note that $\partial_{x_\nu}g(|x|)=g'(|x|)x_\nu/|x|=x_\nu+\mathcal{O}(1/|x|)$,

$$(1.7)\qquad \partial_{x_\mu}\partial_{x_\nu}g=\delta_{\nu,\mu}+\mathcal{O}(|x|^{-2})\ .$$

Let now $v \in C^\infty(\mathbb{S}^{n-1})$ be strictly positive everywhere and let $u = Kv$. If $0 < v_{\min} < v_{\max}$ denote the infimum and the supremum of v, then we have :

$$v_{\min} u_0 \leq u \leq v_{\max} u_0 \ , \tag{1.8}$$

and hence :

$$u = e^{-g(|x|)+k(x)} \ , \tag{1.9}$$

where k is a bounded function. If the vectorfield ν is an infinitesimal generator of a rotation of $\mathbb{S}^{n-1}$, and if we extend the definition of ν to $\mathbb{R}^n$ by means of polar coordinates, (r, θ), $x = r\theta$, then $\nu \circ K = K \circ \nu$. Since v is C^∞, it follows that $\partial_\theta^\alpha u = \mathcal{O}(1) e^{-\psi(r)}$ for every α. We conclude that

$$\partial_\theta^\alpha k = \mathcal{O}(1) \text{ for every } \alpha \ . \tag{1.10}$$

We also need to control some radial derivatives of k. Writing

$$\Big(- \partial_r^2 - ((n-1)/r)\partial_r + r^2 - \mu - r^{-2}\Delta_\theta \Big)(u_0(r) e^k) = 0 \ ,$$

and using (1.2), we get :

$$\Big(\partial_r^2 + (2(\partial_r u_0)/u_0 + (n-1)/r)\partial_r\Big)(e^k) = -r^{-2}\Delta_\theta e^k \ . \tag{1.11}$$

Here $\partial_\theta^\alpha(r^{-2}\Delta_\theta(e^k)) = \mathcal{O}(r^{-2})$, and we have $2(\partial_r u_0)/u_0 = -2\partial_r g$, so (1.11), (1.5) imply that

$$(\partial_r - f(r))\partial_r(e^k) = -r^{-2}\Delta_\theta(e^k) = \mathcal{O}(r^{-2}) \ , \tag{1.12}$$

where $f(r) = 2r + \mathcal{O}(1/r)$, $f'(r) = 2 + \mathcal{O}(1/r^2)$ etc. Let $F(r) = \int_1^r f(t)\,dt$. Then

$$\partial_r e^k = -\int_r^{+\infty} e^{F(r)-F(s)} \mathcal{O}(s^{-2})\,ds + C e^{F(r)} \ . \tag{1.13}$$

The first term is $\mathcal{O}(r^{-3})$ since $F(r) - F(s) \sim r^2 - s^2 \leq 2r(r-s)$, for $s \geq r$, and since we know that $\partial_r e^k$ cannot tend to $+\infty$ or $-\infty$, when $r \to \infty$, we conclude that $C = 0$ in (1.13), and hence :

$$\partial_r e^k = \mathcal{O}(r^{-3}) \ . \tag{1.14}$$

More generally,

$$\partial_r \partial_\theta^\alpha e^k = \mathcal{O}(r^{-3}) \ . \tag{1.15}$$

Differentiating (1.12) and using (1.15), we get :

$$(\partial_r - f(r))\partial_r^2(e^k) = \mathcal{O}(r^{-3}) \ , \tag{1.16}$$

and similarly for the $\theta-$ derivatives.

The same argument then shows that :

$$\partial_r^2 \partial_\theta^\alpha e^k = \mathcal{O}(r^{-4}) \ . \tag{1.17}$$

Continuing this way, we get by induction

$$\partial_r^\nu \partial_\theta^\alpha e^k = \mathcal{O}(r^{-2-\nu}) \ , \quad \nu = 1, 2, \ldots \tag{1.18}$$

and remembering that k is bounded, we deduce (by differentiating the identity $k = \log e^k$) that

$$\partial_r^\nu \partial_\theta^\alpha k = \mathcal{O}(r^{-2-\nu}) \ , \quad \nu = 1, 2, \ldots \tag{1.19}$$

Going back to the $x-$ coordinates, we get :

$$\partial_x^\alpha k = \mathcal{O}(|x|^{-|\alpha|}) \ , \ \text{for every } \alpha \neq 0 \ . \tag{1.20}$$

Using also the properties of g we get $-\log u = \frac{1}{2}x^2 + \psi(x)$, where ψ satisfies the estimates (1.20).

Let us finally remark that everything works equally well for the operator $-h^2\Delta + V$, when V satisfies the assumptions above. We then obtain $-h\log u = \frac{1}{2}x^2 + \psi(x)$, with ψ satisfying (1.20), not necessarily uniformly with respect to h.

2. Estimates on the logarithm of the first eigenfunction

Let $V : \mathbb{R}^m \to \mathbb{R}$ be a smooth potential which is equal to $x^2/2$ outside some bounded set. Let $u = e^{-\varphi(x)/h}$ be the first normalized eigenfunction of $-\frac{1}{2}h^2\Delta + V$. (Here φ also depends on h.) Let μ be the corresponding eigenvalue, and let $\mathbf{0}$ be an open ball centered at 0 with the property that $V = x^2/2 > \mu$ in the exterior of $\mathbf{0}$. If K is the exterior Poisson operator associated to $-\frac{1}{2}h^2\Delta + \frac{1}{2}x^2 - \mu$, then in the exterior of $\mathbf{0}$, we have $u = K(u|_{\partial\mathbf{0}})$, and after a scaling we are in the situation of section 1. We then know that $\varphi(x) = x^2/2 + \psi(x)$, where $\psi^{(\alpha)}(x) \to 0$, when $|x| \to \infty$, for $\alpha \neq 0$. Here, we have apriori no uniformity with respect to m or h, however, we shall use the maximum principle in a way inspired from the appendix B of **[SiWYY]**, to get some uniform estimates on the Hessian and on the third order derivatives of φ.

Proposition 2.1. *Let B be the space $\mathbb{R}^m$ equipped with some norm $\|\cdot\|_B$, and assume that for some fixed θ :*

$$\|V''(x) - I\|_{\mathcal{L}(B,B)} \leq \theta < 1 \ for \ every \ x \in \mathbb{R}^m \ . \tag{2.1}$$

Then for every $x \in \mathbb{R}^m$:

$$\|\varphi''(x) - I\|_{\mathcal{L}(B,B)} \leq \widetilde{\theta} \ , \tag{2.2}$$

where $\widetilde{\theta} = \theta/(1+(1-\theta)^{\frac{1}{2}})$.

PROOF : Write $\mu = hE$ and recall that

$$\frac{1}{2}(\varphi')^2 = V + \frac{1}{2}h\Delta\varphi - hE \ . \tag{2.3}$$

Taking the Hessian of this relation, we get (as in **[SiWYY]**) :

$$\varphi' \cdot \partial_x(\varphi'') + \varphi''^2 = V'' + \frac{1}{2}h\Delta(\varphi'') \ . \tag{2.4}$$

Write $\varphi'' = 1 + \psi''$, $V'' = 1 + W''$:

$$\varphi' \cdot \partial_x(\psi'') + 2\psi'' + \psi''^2 = W'' + \frac{1}{2}h\Delta(\psi'') \ . \tag{2.5}$$

In section 1 we showed that $\|\psi''(x)\|_{\mathcal{L}(B,B)} \to 0$, $|x| \to \infty$, so there is a point x_0, where $\|\psi''(x)\|_{\mathcal{L}(B,B)}$ is maximal, and we let M denote the maximal value. Let $\nu \in B$ be a normalized vector such that $\|\psi''(x_0)\nu\|_B = M$, and let $\mu \in B^*$ be a normalized vector such that $\langle\psi''(x_0)\nu, \mu\rangle = M$. Then $x \mapsto \langle\psi''(x)\nu, \mu\rangle$ reaches its maximum value (M) at the point x_0. We apply the terms in (2.5) to ν and take the scalar product with μ. Then with $x = x_0$, we get :

$$2\langle\psi''(x_0)\nu, \mu\rangle + \langle\psi''(x_0)^2\nu, \mu\rangle \leq \langle W''(x_0)\nu, \mu\rangle \ , \tag{2.6}$$

and hence

$$2M - M^2 \leq \theta \ , \tag{2.7}$$

or equivalently :

$$\text{Either } M \leq \theta/\left(1+(1-\theta)^{\frac{1}{2}}\right) \text{ or } M \geq 1+(1-\theta)^{\frac{1}{2}} \ . \tag{2.8}$$

The last possibility can be excluded by a deformation argument : Putting $V_t = x^2/2 + tW$, we see that $M_t = \sup_x \|\psi''(x)\|_{\mathcal{L}(B,B)}$ depends continuously on t. $\square$

We also need to estimate the third derivatives of φ. In order to do so we assume that the assumptions of Proposition 2.1 are fulfilled also in the case $B = \ell^\infty$:

$$\begin{aligned} &\|V''(x) - I\|_{\mathcal{L}(B,B)} \text{ and } \|V''(x) - I\|_{\mathcal{L}(\ell^\infty,\ell^\infty)} \\ &\text{are } \leq \theta \text{ for all } x \in \mathbb{R}^m. \end{aligned} \tag{2.9}$$

Here it is assumed that $0 \leq \theta < 1$.

We can rewrite (2.4) as :

$$\left\langle \varphi^{(3)}, \varphi' \otimes t \otimes s \right\rangle + \langle \varphi'' t, \varphi'' s \rangle = \langle V'', t \otimes s \rangle + \frac{1}{2} h\Delta \langle \varphi'', t \otimes s \rangle \tag{2.10}$$

for all $t, s \in \mathbb{R}^N$, and if we take the derivative of this relation in the constant direction r, we get

$$\begin{aligned} &\left\langle \varphi' \cdot \partial_x(\psi^{(3)}) + 3\psi^{(3)}, r \otimes s \otimes t \right\rangle + \left\langle \psi^{(3)}, \psi''(r) \otimes s \otimes t \right\rangle + \\ &\qquad \left\langle \psi^{(3)}, r \otimes \psi''(s) \otimes t \right\rangle + \left\langle \psi^{(3)}, r \otimes s \otimes \psi''(t) \right\rangle = \\ &\qquad\qquad \left\langle V^{(3)}, r \otimes s \otimes t \right\rangle + \frac{1}{2} h\Delta \left\langle \psi^{(3)}, r \otimes s \otimes t \right\rangle . \end{aligned} \tag{2.11}$$

In section 1 we established that $\psi^{(3)}(x) \to 0$ when $x \to \infty$, and hence there is a point x_0 where $\|\psi^{(3)}(x)\|_{(B\otimes B^*\otimes\ell^\infty)^*}$ reaches its supremum that we shall denote by $M^{(3)}(\psi)$. Here we identify the dual space of a tensorproduct of normed spaces with the normed space of multilinear forms on the corresponding Cartesian product. Let $r \in B$, $s \in B^*$, $t \in \ell^\infty$ be normalized vectors such that $\langle \psi^{(3)}(x_0), r \otimes s \otimes t \rangle = M^{(3)}(\psi)$. The same argument as before gives :

$$3M^{(3)}(\psi) - 3M^{(3)}(\psi)\widetilde{\theta} \leq \left\langle V^{(3)}(x_0), r \otimes s \otimes t \right\rangle \leq M^{(3)}(V) , \tag{2.12}$$

where $M^{(3)}(V)$ is defined as $\sup_x \left\| V^{(3)}(x) \right\|_{(B\otimes B^*\otimes\ell^\infty)^*}$. We then get :

$$\sup_x \left\| \psi^{(3)}(x) \right\|_{(B\otimes B^*\otimes\ell^\infty)^*} \leq (3(1-\widetilde{\theta}))^{-1} \sup_x \left\| V^{(3)}(x) \right\|_{(B\otimes B^*\otimes\ell^\infty)^*} . \tag{2.13}$$

We shall now take two potentials V_0 and V_1, which satisfy the assumptions above and in particular the assumption (2.9). We shall estimate $\varphi_1' - \varphi_0'$

and $\varphi_1'' - \varphi_0''$, where φ_j denotes the phase associated to V_j, so that

$$\frac{1}{2}(\varphi_j')^2 - \frac{1}{2}h\Delta\varphi_j + hE_j = V_j \ , \ j = 0,1 \ . \tag{2.14}$$

Taking the difference of these two equations, we get :

$$\frac{1}{2}(\varphi_1' + \varphi_0') \cdot \partial_x(\varphi_1 - \varphi_0) + h(E_1 - E_0) = V_1 - V_0 + \frac{1}{2}h\Delta(\varphi_1 - \varphi_0), \tag{2.15}$$

and taking the gradient of this relation gives :

$$\frac{1}{2}(\varphi_1' + \varphi_0') \cdot \partial_x(\varphi_1' - \varphi_0') + \frac{1}{2}(\varphi_1'' + \varphi_0'')(\varphi_1' - \varphi_0') = V_1' - V_0' + \frac{1}{2}h\Delta(\varphi_1' - \varphi_0') \ . \tag{2.16}$$

From section 1 it follows that $\varphi_1'(x) - \varphi_0'(x) \to 0$ when $x \to \infty$, so $\sup\limits_x \|\varphi_1'(x) - \varphi_2'(x)\|_B \underset{\text{def}}{=} m$ is reached at some point x_0. Let $\nu \in B^*$ be a normalized vector such that $\langle \varphi_1'(x_0) - \varphi_2'(x_0), \nu\rangle = m$. Then applying (2.16) to ν and putting $x = x_0$, we get :

$$\left\langle \frac{1}{2}(\varphi_1''(x_0) + \varphi_0''(x_0))(\varphi_1'(x_0) - \varphi_0'(x_0)), \nu \right\rangle \leq \langle (V_1' - V_0')(x_0), \nu\rangle \ . \tag{2.17}$$

Here we use that $\varphi_j''(x) = 1 + \psi_j''(x)$ with $\left\|\psi_j''(x)\right\|_{\mathcal{L}(B,B)} \leq \widetilde{\theta}$, and obtain : $m - \widetilde{\theta}m \leq \|(V_1' - V_0')(x_0)\|_B$.

We have then proved :

$$\sup_x \|\varphi_1'(x) - \varphi_0'(x)\|_B \leq (1 - \widetilde{\theta})^{-1} \sup_x \|V_1'(x) - V_0'(x)\|_B \ . \tag{2.18}$$

We shall also estimate $\varphi_1'' - \varphi_0''$ in $\mathcal{L}(\ell^\infty, B)$. We first apply (2.16) to a constant vector ν :

$$\left\langle \varphi_1'' - \varphi_0'', \frac{1}{2}(\varphi_1' + \varphi_0') \otimes \nu \right\rangle + \left\langle \frac{1}{2}(\varphi_1'' + \varphi_0''), (\varphi_1' - \varphi_0') \otimes \nu \right\rangle = \langle V_1' - V_0', \nu\rangle + \frac{1}{2}h\Delta(\langle \varphi_1' - \varphi_0', \nu\rangle) \ , \tag{2.19}$$

and differentiate in the constant direction μ :

$$\begin{aligned} &\left\langle \varphi_1''' - \varphi_0''', \frac{1}{2}(\varphi_1' + \varphi_0') \otimes \nu \otimes \mu \right\rangle + \\ &\left\langle \varphi_1'' - \varphi_0'', \frac{1}{2}(\varphi_1'' + \varphi_0'')(\mu) \otimes \nu \right\rangle + \left\langle \frac{1}{2}(\varphi_1''' + \varphi_0'''), (\varphi_1' - \varphi_0') \otimes \nu \otimes \mu \right\rangle + \\ &\left\langle \frac{1}{2}(\varphi_1'' + \varphi_0''), (\varphi_1'' - \varphi_0'')(\mu) \otimes \nu \right\rangle = \\ &\langle V_1'' - V_0'', \nu \otimes \mu\rangle + \frac{1}{2}h\Delta\left(\langle \varphi_1'' - \varphi_0'', \nu \otimes \mu\rangle\right) \ , \end{aligned} \tag{2.20}$$

which can be rewritten as :

$$\begin{aligned}(2.21)\quad &\frac{1}{2}(\varphi_1'+\varphi_0')\cdot\partial_x\langle(\varphi_1''-\varphi_0''),\nu\otimes\mu\rangle+2\langle\varphi_1''-\varphi_0'',\nu\otimes\mu\rangle+\\ &\left\langle\varphi_1''-\varphi_0'',\frac{1}{2}(\psi_1''+\psi_0'')(\mu)\otimes\nu\right\rangle+\left\langle\varphi_1''-\varphi_0'',\mu\otimes\frac{1}{2}(\psi_1''+\psi_0'')(\nu)\right\rangle+\\ &\left\langle\frac{1}{2}(\psi_1'''+\psi_0'''),(\varphi_1'-\varphi_0')\otimes\nu\otimes\mu\right\rangle=\langle V_1''-V_0'',\nu\otimes\mu\rangle+\\ &\frac{1}{2}h\Delta(\langle\varphi_1''-\varphi_0'',\nu\otimes\mu\rangle)\ .\end{aligned}$$

We know that $\sup_x\|\varphi_1''-\varphi_0''\|_{\mathcal{L}(\ell^\infty,B)}\underset{\text{def}}{=}M$ is attained at some point x_0. Let $\nu\in\ell^\infty$, $\mu\in B^*$ be normalized vectors with $\langle\varphi_1''(x_0)-\varphi_0''(x_0),\nu\otimes\mu\rangle=M$. Taking these vectors in (2.21) and $x=x_0$ gives :

$$\begin{aligned}(2.22)\quad 2M&+\left\langle\varphi_1''-\varphi_0'',\frac{1}{2}(\psi_1''+\psi_0'')(\mu)\otimes\nu\right\rangle+\\ &\left\langle\varphi_1''-\varphi_0'',\mu\otimes\frac{1}{2}(\psi_1''+\psi_0'')(\nu)\right\rangle+\\ &\left\langle\frac{1}{2}(\psi_1'''+\psi_0'''),(\varphi_1'-\varphi_0')\otimes\nu\otimes\mu\right\rangle\le\langle V_1''-V_0'',\nu\otimes\mu\rangle\ .\end{aligned}$$

Here we use that $\left\|\frac{1}{2}(\psi_1''+\psi_0'')(\mu)\right\|_{B^*}$, $\left\|\frac{1}{2}(\psi_1''+\psi_0'')(\nu)\right\|_{\ell^\infty}\le\widetilde{\theta}$ to bound the second and the third term of the LHS from below by $-M\widetilde{\theta}$. Using (2.13), (2.18), we can bound the fourth term from below by

$$-\frac{1}{3}(1-\widetilde{\theta})^{-2}\left(\max_{j=0,1}\sup_x\left\|V_j'''(x)\right\|_{(B\otimes B^*\otimes\ell^\infty)^*}\right)\cdot\sup_x\|V_1'(x)-V_0'(x)\|_B$$

and we end up with the estimate :

$$\begin{aligned}(2.23)\quad &\sup_x\|\varphi_1''-\varphi_0''\|_{\mathcal{L}(\ell^\infty,B)}\le\frac{1}{2}(1-\widetilde{\theta})^{-1}\sup_x\|V_1''-V_0''\|_{\mathcal{L}(\ell^\infty,B)}+\\ &(1/6)(1-\widetilde{\theta})^{-3}\left(\sup_{x,j}\left\|V_j'''(x)\right\|_{(B\otimes B^*\otimes\ell^\infty)^*}\right)\cdot\left(\sup_x\|V_1'(x)-V_0'(x)\|_B\right)\ .\end{aligned}$$

So far, all the estimates have been obtained under the assumption that $V-x^2/2$ and $V_j-x^2/2$ have compact support, and we shall now eliminate this assumption by means of an approximation procedure. We start by noticing that for every $\varepsilon\in\,]0,1]$, there exists $\chi=\chi_\varepsilon\in C_0^\infty(\mathbb{R})$ with values in $[0,1]$ such that $|\chi'(t)|\le\varepsilon/|t|$, $|\chi''(t)|\le\varepsilon/t^2$, $|\chi'''(t)|\le\varepsilon/|t|^3$, such that χ is equal to 1 on the interval $[-\varepsilon^{-1},\varepsilon^{-1}]$. (We can take $\chi_\varepsilon(t)=f(\varepsilon\log|t|)$

for a suitable f.) Let $V = \frac{1}{2}x^2 + W(x)$ with $W \in C^\infty(\mathbb{R}^m;\mathbb{R})$ and assume that W'' and W''' are uniformly bounded as functions of x. We also assume that (2.9) is satisfied. By symmetry and interpolation we then also have $\|W''(x) - I\|_{\mathcal{L}(\ell^2,\ell^2)} \leq \theta < 1$, and it follows that $V''(x) \geq 1 - \theta$ in the sense of symmetric matrices. We then know that V is a strictly convex function.

We approximate W by the compactly supported functions $W_\varepsilon = \chi_\varepsilon(|x|)W$. Since $W_\varepsilon'' = \chi_\varepsilon W'' + 2\chi_\varepsilon' W' + \chi_\varepsilon'' W$, and since $\chi_\varepsilon' = \mathcal{O}(\varepsilon/|x|)$, $\chi_\varepsilon'' = \mathcal{O}(\varepsilon/|x|^2)$, $W' = \mathcal{O}(1+|x|)$, $W = \mathcal{O}((1+|x|)^2)$ (where for the moment the estimates are not necessarily uniform with respect to the dimension), we see that W_ε will satisfy (2.9) with θ replaced by $\theta_\varepsilon \to \theta$ when $\varepsilon \to 0$. Similarly we see that $\sup_x \left\|W_\varepsilon^{(3)}(x)\right\|_{(B\otimes B^*\otimes\ell^\infty)^*}$ tends to $\sup_x \left\|W^{(3)}(x)\right\|_{(B\otimes B^*\otimes\ell^\infty)^*}$ when $\varepsilon \to 0$. Let $u_\varepsilon = e^{-\varphi_\varepsilon/h}$ be the first normalized eigenfunction of $-\frac{1}{2}h^2\Delta + V_\varepsilon$, where $V_\varepsilon = \frac{1}{2}x^2 + W_\varepsilon$. Then all the estimates of this section that we obtained for a single potential of the form $\frac{1}{2}x^2 + W$ with W of compact support, apply to φ_ε when ε is small enough. Moreover it is easy to see (for instance by using exponential decay estimates) that $u_\varepsilon \to u$ in the C^∞ topology when $\varepsilon \to 0$, so $\varphi_\varepsilon \to \varphi$ in C^∞. From these remarks we see that the assumption that W have compact support can be eliminated in the estimates above, in the case of a single potential. Consider finally the case of two potentials of the form $V_j = \frac{1}{2}x^2 + W_j$ for $j = 1, 2$. We assume that V_j satisfy (2.9) and that W_j'' are uniformly bounded on $\mathbb{R}^m$, and that $\sup \|V_1' - V_0'\|_B$, and $\sup \|V_1'' - V_0''\|_{\mathcal{L}(\ell^\infty,B)}$ are finite. Then we can put $V_{j,\varepsilon} = \frac{1}{2}x^2 + \chi_\varepsilon(|x|)W_j$ and perform the same approximation argument and deduce the same estimates for the difference of the phases, as we had in the case when W_j had compact support. Let us sum up our results :

Theorem 2.1.

(A) *Let $V(x) = \frac{1}{2}x^2 + W(x)$ where W is real valued and smooth on $\mathbb{R}^m$. We assume that (2.9) holds and that the third derivatives of V are bounded on $\mathbb{R}^m$. Let $u = e^{-\varphi/h}$ be the normalized positive eigenfunction associated to the first eigenvalue of $-\frac{1}{2}h^2\Delta + V$. Then the conclusion of Proposition 2.1 holds as well as the estimate (2.13).*

(**B**) *Let* $V_j(x) = \frac{1}{2}x^2 + W_j(x)$, $j = 1,2$ *satisfy the assumptions of* (A) *(with the same* θ *in* (2.9)*), and assume in addition that* $\sup \|V_1' - V_0'\|_B$, *and* $\sup \|V_1'' - V_0''\|_{\mathcal{L}(\ell^\infty, B)}$ *are finite. Then if we let* $e^{-\varphi_j/h}$ *denote the first normalized eigenfunction of* $-\frac{1}{2}h^2\Delta + V_j$, *we have the estimates* (2.18) *and* (2.23).

3. Exponential convergence

We consider a sequence of potentials $V^{(m)}(x_1, \dots, x_m)$, $m = 1, 2, \dots$, and an associated sequence of functions $\rho = \rho^{(m)} : \mathbb{Z}/m\mathbb{Z} \to]0, \infty[$, with the following properties :

(3.1) $$V^{(m)}(0) = 0 \ , \quad \nabla V^{(m)}(0) = 0 \ ,$$

(3.2)
For $0 \le t \le 1$, $m, n \in \{1, 2, \dots\}$, we have :
$$\left\| I - \nabla^2((1-t)V^{(m)} \oplus V^{(n)} + tV^{(m+n)}) \right\|_{\mathcal{L}(\ell^\infty_\rho, \ell^\infty_\rho)} \le \theta,$$
for $\rho \equiv 1$ and for $\rho = \rho^{m,n}$ given by $\rho(j) = \rho^{(m)}(j)$ when $1 \le j \le m$, $\rho(j) = \rho^{(n)}(j-m)$, $m+1 \le j \le m+n$.

(3.3) $$\rho^{(m)}\left([\frac{1}{2}m]\right) \ge e^{m\kappa/2} \ , \quad \rho^{(m)}(1) = \rho^{(m)}(m) = 1 \ .$$

Here $0 \le \theta < 1$, $\kappa > 0$ are fixed in the following, and we let ℓ^p_ρ denote the space $\mathbb{C}^m$ equipped with the norm : $|x|_{p,\rho} = |\rho x|_p = (\Sigma\, |\rho(j) x_j|^p)^{1/p}$ (with the obvious modifications when $p = \infty$). The choice of m will be clear from the context. We write :

$$V^{(m)} \oplus V^{(n)}(x_1, \dots, x_{m+n}) = V^{(m)}(x_1, \dots, x_m) + V^{(n)}(x_{m+1}, \dots, x_{m+n}) \ .$$

We assume that there exists a constant C_0, such that :

(3.4) $$\sup_x \left\| \nabla^3 V^{(m)}(x) \right\|_{(\ell^\infty_\rho \otimes \ell^1_{1/\rho} \otimes \ell^\infty)^*} \le C_0,$$
$$\rho = \rho^{j,k}, \ j + k = m, \ \text{and } \rho = \rho^{(m)}.$$

We also assume that

(3.5) $V^{(m)}$ is invariant under cyclic perturbations of the coordinates :
$$V^{(m)}(x_m, x_1, \dots, x_{m-1}) = V^{(m)}(x_1, x_2, \dots, x_m),$$

and that $V^{(m)}$ is close to $V^{(m+n)}$ in the following sense: We have

$$\sup_x \left\| \nabla (V^{(m+n)} - V^{(m)} \oplus V^{(n)}) \right\|_{\ell^\infty_\rho} \leq C_0 \ , \tag{3.6}$$

$$\sup_x \left\| \nabla^2 (V^{(m+n)} - V^{(m)} \oplus V^{(n)}) \right\|_{\mathcal{L}(\ell^\infty, \ell^\infty_\rho)} \leq C_0 \ , \tag{3.7}$$

for $\rho = \rho^{m,n}$.

We can then apply Theorem 2.1 (B) with $V_0 = V^{(m)} \oplus V^{(n)}$, $V_1 = V^{(m+n)}$, $B = \ell^\infty_\rho$, $\rho = \rho^{m,n}$ and hence:

$$\sup_{x \in \mathbb{R}^{m+n}} \left\| \nabla (\varphi^{(m+n)} - \varphi^{(m)} \oplus \varphi^{(n)}) \right\|_{\infty, \rho} \leq C_0 / (1 - \widetilde{\theta}), \tag{3.8}$$

$$\sup_{x \in \mathbb{R}^{m+n}} \left\| \nabla^2 (\varphi^{(m+n)} - \varphi^{(m)} \oplus \varphi^{(n)}) \right\|_{\mathcal{L}(\ell^\infty, \ell^\infty_\rho)} \leq \tag{3.9}$$
$$C_0 / (2(1 - \widetilde{\theta})) + C_0^2 / (6(1 - \widetilde{\theta})^3) \ ,$$

where $\widetilde{\theta}$ is defined in Proposition 2.1. Choosing $\nu = \left[\frac{1}{2} m\right]$ we get:

$$\left| \partial_{x_\nu} \varphi^{(m+n)}(0) - \partial_{x_\nu} \varphi^{(m)}(0) \right| = \mathcal{O}(1) e^{-\kappa m / 2} \tag{$\widetilde{3.8}$}$$

$$\left| \partial^2_{x_\nu} \varphi^{(m+n)}(0) - \partial^2_{x_\nu} \varphi^{(m)}(0) \right| = \mathcal{O}(1) e^{-\kappa m / 2}. \tag{$\widetilde{3.9}$}$$

Let $\mu(m) = \mu(m; h)$ be the lowest eigenvalue of $-\frac{1}{2} h^2 \Delta + V^{(m)}$. From (2.3) and the fact that $V^{(m)}(0) = 0$, we get:

$$\mu(m) = \frac{1}{2} h \Sigma \partial^2_{x_\nu} \varphi^{(m)}(0) - \frac{1}{2} \Sigma (\partial_{x_\nu} \varphi^{(m)}(0))^2 \ ,$$

with $-\varphi^{(m)}/h$ being the logarithm of the first eigenfunction. From (3.5), we deduce that $\varphi^{(m)}$ is invariant under cyclic permutations of the coordinates, and hence the terms in each of the sums are independent of ν. For an arbitrary ν in $\{1, 2, \ldots, m\}$, we then get:

$$\mu(m)/m = \frac{1}{2} h \partial^2_{x_\nu} \varphi^{(m)}(0) - \frac{1}{2} (\partial_{x_\nu} \varphi^{(m)}(0))^2 \ . \tag{3.10}$$

Choosing ν so that $(\widetilde{3.8})$, $(\widetilde{3.9})$ hold, and noticing that $\partial_{x_\nu} \varphi^{(m)}(0) = \mathcal{O}(h^{\frac{1}{2}})$ by (2.3), we get:

$$|\mu(m+n)/(m+n) - \mu(m)/m| = \mathcal{O}(h^{\frac{1}{2}} + h) e^{-\kappa m / 2} \ . \tag{3.11}$$

This implies that $\lim_{m \to \infty} \mu(m)/m$ exists (as we already know from **[HS]**). If we denote the limit by $\mu(\infty)$, then (3.11) implies:

$$|\mu(m)/m - \mu(\infty)| = \mathcal{O}(h^{\frac{1}{2}} + h) e^{-\kappa m / 2} \ . \tag{3.12}$$

Summing up, we have proved :

Theorem 3.1. *Let* $V^{(m)}(x_1,\dots,x_m)$ *satisfy* (3.1)-(3.7) *and let* $\mu(m)$ *be the lowest eigenvalue of* $-\frac{1}{2}h^2\Delta+V^{(m)}$ *on* $\mathbb{R}^m$ *. Let* $\mu(\infty)=\lim_{m\to\infty}\mu(m)/m$ *(which exists according to* **[HS]***). Then uniformly with respect to* h *we have* (3.12).

REMARK. If $V^{(m)}$ are even, then $\varphi^{(m)}$ are even, and the second term of the RHS of (3.10) vanishes. Then we can replace $\mathcal{O}(h^{\frac{1}{2}}+h)$ in (3.12) by $\mathcal{O}(h)$.

4. Improved bounds on the speed of convergence

We first study the speed of convergence for the family of quadratic potentials, $V^{(m)}=\frac{1}{2}\Sigma_1^m x_j^2-\frac{1}{2}\alpha\Sigma_1^m x_j x_{j+1}$ (with the convention that subscripts are in $\mathbb{Z}/m\mathbb{Z}$). A similar discussion was given in **[HS]**. Here α is fixed in $[0,1[$. If we view $\nabla^2 V^{(m)}$ as a map from $\mathbb{C}^m$ to itself and identify $\mathbb{C}^m$ with $\ell^2(\mathbb{Z}/m\mathbb{Z})$, we have :

$$\nabla^2 V^{(m)}=1-\alpha\frac{1}{2}(\tau_1+\tau_{-1}) \ , \tag{4.1}$$

where $(\tau_k x)_j=x_{j-k}$. The eigenvectors $e_k=(x_0,x_1,\dots,x_{m-1})$ of $\nabla^2 V^{(m)}$ are given by $x_j=\exp(2\pi ikj/m)$, $0\le k<m$, and the corresponding eigenvalues are $1-\alpha\cos(2\pi k/m)$. The lowest eigenvalue $\mu(m)$ of $P^{(m)}=-\frac{1}{2}\Delta+V^{(m)}$ therefore satisfies :

$$\mu(m)/m=(2m)^{-1}\Sigma_0^{m-1}(1-\alpha\cos(2\pi k/m))^{\frac{1}{2}}\,, \tag{4.2}$$

and this is a Riemann sum corresponding to the integral :

$$(4\pi)^{-1}\int_0^{2\pi}(1-\alpha\cos x)^{\frac{1}{2}}\,dx \ . \tag{4.3}$$

Let $v(x)=(1-\alpha\cos x)^{\frac{1}{2}}$. Then the right hand side of (4.2) can be rewritten :

$$\frac{1}{2}\int_0^{2\pi} v(x)u_m(x)dx, \ \text{ with } u_m(x)=\sum_{k\in\mathbb{Z}} m^{-1}\delta(x-2\pi k/m). \tag{4.4}$$

The Fourier coefficients of u_m are given by :

$$\widehat{u}_m(j)=1/2\pi \ \text{ if } e^{-ij2\pi/m}=1 \tag{4.5}$$

(i.e. if j is a multiple of m) and $\widehat{u}_m(j)=0$ otherwise.

Rewriting (4.4) with Plancherel's formula, we get :

$$(4.6)\quad \mu(m)/m = \frac{1}{2}\sum_{\nu\in\mathbb{Z}} \widehat{v}(\nu m) = (4\pi)^{-1}\int_0^{2\pi}(1-\alpha\cos x)^{\frac{1}{2}}\,dx + \frac{1}{2}\sum_{\nu\neq 0}\widehat{v}(\nu m).$$

Here,

$$(4.7)\qquad v(\nu m) = (2\pi)^{-1}\int_0^{2\pi}(1-\alpha\cos x)^{\frac{1}{2}}\,e^{-i\nu m x}\,dx\ ,$$

and depending on the sign of ν, we wish to deform the integration contour into the upper or the lower half plane. The amount of deformation is limited by the singularities of the function $x\mapsto(1-\alpha\cos x)^{\frac{1}{2}}$, i.e. by the points x such that $1-\alpha\cos x=0$. These are the points of the form $iy+2\pi k$, with $ch\,y = 1/\alpha$. The deformation argument then shows that

$$|\widehat{v}(\nu m)| \leq C_\varepsilon \exp[-(1-\varepsilon)\,|\nu|\,m\,ch^{-1}(1/\alpha)] \text{ for every } \varepsilon>0,$$

and (4.6) then gives :

$$(4.8)\qquad |\mu(m)/m - \mu(\infty)| \leq \widetilde{C}_\varepsilon \exp[-(1-\varepsilon)m\,ch^{-1}(1/\alpha)]\ ,$$

for every $\varepsilon>0$. Pushing the same method a little further would probably give an asympotic expansion of $(\mu(m)/m-\mu(\infty))\exp[m\,ch^{-1}(1/\alpha)]$ in decreasing powers of m.

Let us interpret the exponent in (4.8) in terms of exponential weights. If $\rho : \mathbb{Z}/m\mathbb{Z}\to\,]0,+\infty[$, then the norm of $\frac{1}{2}\alpha(\tau_1+\tau_{-1}) : \ell^p_\rho\to\ell^p_\rho$, or equivalently the norm of $\rho\circ\frac{1}{2}\alpha(\tau_1+\tau_{-1})\circ\rho^{-1} : \ell^p\to\ell^p$ can be bounded by

$$\alpha\max\Big[\sup_j\frac{1}{2}\Big(\rho(j)/\rho(j-1)+\rho(j)/\rho(j+1)\Big),\ \sup_k\frac{1}{2}\Big(\rho(k-1)/\rho(k) + \rho(k+1)/\rho(k)\Big)\Big]\ .$$

Put $\nu(j)=\rho(j+1)/\rho(j)$ and assume that $e^{-\delta}\leq\nu(j+1)/\nu(j)\leq e^{\delta}$ for some small δ. Then the quantity above can be estimated by $\alpha e^{\delta}\sup_k\frac{1}{2}(\nu(k)+1/\nu(k))$, and we are then naturally led to the assumption that $\alpha e^{\delta}\sup_k\frac{1}{2}(\nu(k)+\nu(k)^{-1})\leq\theta<1$, or equivalently : $|\log(\rho(k+1)/\rho(k))|\leq ch^{-1}(e^{-\delta}\theta/\alpha)$. Choosing ρ conveniently and approaching the limiting case $\theta=1$, $\delta=0$, we see that the estimate of section 3 gives :

$$(4.9)\qquad |\mu(m)/m-\mu(\infty)| \leq \widetilde{C}_\varepsilon\exp[-(1-\varepsilon)\frac{1}{2}m\,ch^{-1}(1/\alpha)],$$

which is not as good as (4.8).

In the remainder of this section we shall establish improved bounds of the form (4.8) for sequences of potentials which are not necessarily quadratic. As a preparation we need bounds on the fourth order derivatives of the phase. Let $V : \mathbb{R}^m \to \mathbb{R}$ satisfy the assumptions of Proposition 2.1, for some B and also for $B = \ell^\infty$, and let $u = e^{-\varphi/h}$ be the positive normalized eigenfunction associated to the first eigenvalue of $-\frac{1}{2}h^2\Delta + V$. We then have (2.2) and (2.13) (where $\varphi'' = 1 + \psi''$, so that $\varphi^{(3)} = \psi^{(3)}$).

Rewrite (2.4) as:

$$\left\langle \varphi^{(3)}, \varphi' \otimes t \otimes s \right\rangle + \langle \langle \varphi'', t\rangle, \langle \varphi'', s\rangle\rangle = \langle V'', t \otimes s\rangle + \frac{1}{2} h \langle \Delta\varphi'', t \otimes s\rangle , \tag{4.10}$$

where we use the following notation: if A is a symmetric $k-$tensor and B a $\ell-$tensor with $\ell \leq k$, then $\langle A, B\rangle$ is the symmetric $k - \ell$ tensor C with $\langle C, t\rangle = \langle A, B \otimes t\rangle$. We differentiate (4.10) in the constant direction r:

$$\begin{aligned} &\left\langle \varphi^{(4)}, \varphi' \otimes r \otimes s \otimes t \right\rangle + \left\langle \varphi^{(3)}, \langle \varphi'', r\rangle \otimes s \otimes t \right\rangle + \\ &\quad \left\langle \left\langle \varphi^{(3)}, r \otimes t \right\rangle, \langle \varphi'', s\rangle \right\rangle + \left\langle \langle \varphi'', t\rangle, \left\langle \varphi^{(3)}, r \otimes s \right\rangle \right\rangle = \\ &\qquad \left\langle V^{(3)}, r \otimes s \otimes t \right\rangle + \frac{1}{2} h\Delta \left\langle \varphi^{(3)}, r \otimes s \otimes t \right\rangle , \end{aligned}$$

which can be rewritten as:

$$\begin{aligned} &\left\langle \varphi^{(4)}, \varphi' \otimes r \otimes s \otimes t \right\rangle + \\ &\quad \left\langle \varphi^{(3)}, \langle \varphi'', r\rangle \otimes s \otimes t + r \otimes \langle \varphi'', s\rangle \otimes t + r \otimes s \otimes \langle \varphi'', t\rangle \right\rangle = \\ &\qquad \left\langle V^{(3)}, r \otimes s \otimes t \right\rangle + \frac{1}{2} h\Delta \left\langle \varphi^{(3)}, r \otimes s \otimes t \right\rangle . \end{aligned} \tag{4.11}$$

We differentiate this in the constant direction u and get:

$$\begin{aligned} &\varphi' \cdot \partial_x \Big(\left\langle \varphi^{(4)}, u \otimes r \otimes s \otimes r \right\rangle \Big) + 4 \left\langle \varphi^{(4)}, u \otimes r \otimes s \otimes t \right\rangle + \\ &\quad \Big\langle \varphi^{(4)}, \langle \psi'', u\rangle \otimes r \otimes s \otimes t + u \otimes \langle \psi'', r\rangle \otimes s \otimes t + \\ &\quad u \otimes r \otimes \langle \psi'', s\rangle \otimes t + u \otimes r \otimes s \otimes \langle \psi'', t\rangle \Big\rangle = \\ &\quad \left\langle V^{(4)}, u \otimes r \otimes s \otimes t \right\rangle + \frac{1}{2} h\Delta \left\langle \varphi^{(4)}, u \otimes r \otimes s \otimes t \right\rangle - \\ &\quad \Big[\left\langle \left\langle \varphi^{(3)}, s \otimes t \right\rangle, \left\langle \varphi^{(3)}, u \otimes r \right\rangle \right\rangle + \left\langle \left\langle \varphi^{(3)}, u \otimes s \right\rangle, \left\langle \varphi^{(3)}, r \otimes t \right\rangle \right\rangle + \\ &\qquad \left\langle \left\langle \varphi^{(3)}, r \otimes s \right\rangle, \left\langle \varphi^{(3)}, u \otimes t \right\rangle \right\rangle \Big] . \end{aligned} \tag{4.12}$$

Let $M_B^3(\varphi)=M_B^3(\psi)$ be defined as in (2.12) and recall (2.13) :

$$M_B^3(\varphi)\le(3(1-\widetilde{\theta}))^{-1}M_B^3(V)\ . \tag{4.13}$$

Since everything also works in the case when $B=\ell^\infty$, we have :

$$M_{\ell^\infty}^3(\varphi)\le(3(1-\widetilde{\theta}))^{-1}M_{\ell^\infty}^3(V)\ . \tag{4.14}$$

Put $M_B^4(\varphi)=\sup_x\left\|\varphi^{(4)}(x)\right\|_{(B\otimes B^*\otimes\ell^\infty\otimes\ell^\infty)^*}$, where the norm is the one for multilinear forms on $B\times B^*\times\ell^\infty\times\ell^\infty$. Let x_0 be a point where the supremum is attained and let $u\in B$, $r\in B^*$, $s,t\in\ell^\infty$ be corresponding normalized vectors. Then

$$\begin{aligned}
&\left|\left\langle\varphi^{(3)},s\otimes t\right\rangle\right|_\infty\le M_{\ell^\infty}^3(\varphi), &&\left|\left\langle\varphi^{(3)},u\otimes r\right\rangle\right|_1\le M_B^3(\varphi),\\
&\left|\left\langle\varphi^{(3)},u\otimes s\right\rangle\right|_B\le M_B^3(\varphi), &&\left|\left\langle\varphi^{(3)},r\otimes t\right\rangle\right|_{B^*}\le M_B^3(\varphi),\\
&\left|\left\langle\varphi^{(3)},r\otimes s\right\rangle\right|_{B^*}\le M_B^3(\varphi), &&\left|\left\langle\varphi^{(3)},u\otimes t\right\rangle\right|_B\le M_B^3(\varphi).
\end{aligned}$$

Hence the last term in (4.12) can be bounded by

$$M_{\ell^\infty}^3(\varphi)\cdot M_B^3(\varphi)+2(M_B^3(\varphi))^2\ .$$

The usual argument gives :

$$\begin{aligned}4(1-\widetilde{\theta})M_B^4(\varphi)\le M_B^4(V)+M_{\ell^\infty}^3(\varphi)M_B^3(\varphi)+2M_B^3(\varphi)^2\le\\
M_B^4(V)+(9(1-\widetilde{\theta})^2)^{-1}[M_{\ell^\infty}^3(V)M_B^3(V)+2M_B^3(V)^2].\end{aligned} \tag{4.15}$$

Here we used (4.13), (4.14) in order to get the last inequality. Hence

$$\begin{aligned}M_B^4(\varphi)\le(4(1-\widetilde{\theta}))^{-1}M_B^4(V)+\\
(36(1-\widetilde{\theta})^3)^{-1}[M_{\ell^\infty}^3(V)M_B^3(V)+2M_B^3(V)^2].\end{aligned} \tag{4.16}$$

Everything works the same way with $B=\ell^\infty$ and we get :

$$M_{\ell^\infty}^4(\varphi)\le(4(1-\widetilde{\theta}))^{-1}M_{\ell^\infty}^4(V)+(12(1-\widetilde{\theta})^3)^{-1}M_{\ell^\infty}^3(V)^2. \tag{4.17}$$

As before, these estimates extend to the case of potentials of the form $\frac{1}{2}x^2+W(x)$, where W need not have compact support, but with (2.9) fulfilled and with $\nabla^3V(x)$, $\nabla^4V(x)$ bounded as functions of x.

Let $V^{(m)}$, $m=1,2,\dots$ be a sequence of strictly convex smooth potentials on $\mathbb{R}^m$ with $V^{(m)}(x)\to+\infty$ when $|x|\to\infty$. More assumptions will be made later, for the moment, we only assume that for m sufficiently large and for some fixed $k \geq 2$:

$$V^{(km)}(x, x, \ldots, x) = kV^m(x), \ x \in \mathbb{R}^m \ , \tag{4.18}$$

$$V^{(m)}(x_m, x_1, \ldots, x_{m-1}) = V^{(m)}(x_1, \ldots, x_m) \ . \tag{4.19}$$

Let $u^{(m)} = e^{-\varphi^{(m)}/h}$ be the positive normalized eigenfunction associated to the first eigenvalue, $hE_m(h)$ of $-\frac{1}{2}h^2\Delta + V^{(m)}$. Our goal is to estimate $(E^{(km)}/km) - E^{(m)}/m$, when m tends to infinity, and in order to do so, we shall show that $k^{-1}\varphi^{(km)}(x, x, \ldots, x)$ is close to $\varphi^{(m)}(x)$ when m is large.

If $f(x) = \varphi^{(km)}(x, x, \ldots, x)$, $x \in \mathbb{R}^m$, then :

$$\begin{aligned}
\Delta f = &\sum_{1 \leq \nu \leq m} \Big((\partial_{x_\nu} + \partial_{x_{\nu+m}} + \ldots + \partial_{x_{\nu+(k-1)m}})^2 \varphi^{(km)} \Big)(x, x, \ldots, x) = \\
&\sum_{1 \leq \nu \leq m} \ \sum_{0 \leq \alpha \leq k-1} \ \sum_{0 \leq \beta \leq k-1} \Big(\partial_{x_{\nu+\alpha m}} \partial_{x_{\nu+\beta m}} \varphi^{(km)} \Big)(x, x, \ldots, x) = \\
&\sum_{1 \leq \nu \leq m} \ \sum_{0 \leq \alpha \leq k-1} \ \sum_{0 \leq \gamma \leq k-1} \Big(\partial_{x_{\nu+\alpha m}} \partial_{x_{\nu+(\alpha+\gamma)m}} \varphi^{(km)} \Big)(x, x, \ldots, x) = \\
&\qquad \sum_{1 \leq \mu \leq km} \ \sum_{0 \leq \gamma \leq k-1} \Big(\partial_{x_\mu} \partial_{x_{\mu+\gamma m}} \varphi^{(km)} \Big)(x, \ldots, x).
\end{aligned}$$

(Here we use the cyclic convention : $x_{j+km} = x_j$.) Hence :

$$\begin{aligned}
\Delta f(x) = &(\Delta \varphi^{(km)})(x, x, \ldots, x) + \\
&\sum_{1 \leq \mu \leq km} \ \sum_{1 \leq \gamma \leq k-1} \Big(\partial_{x_\mu} \partial_{x_{\mu+\gamma m}} \varphi^{(km)} \Big)(x, \ldots, x).
\end{aligned} \tag{4.20}$$

Similarly, since $(\partial_{x_{\nu+m}} \varphi^{(km)})(x, \ldots, x) = (\partial_{x_\nu} \varphi^{(km)})(x, \ldots, x)$ (by (4.19) with m replaced by km) :

$$\begin{aligned}
(\nabla f)^2 = &\sum_{1 \leq \nu \leq m} \Big((\partial_{x_\nu} \varphi)(x, \ldots, x) + \\
&(\partial_{x_{\nu+m}} \varphi)(x, \ldots, x) + \ldots + (\partial_{x_{\nu+(k-1)m}} \varphi)(x, \ldots, x) \Big)^2 = \\
&k^2 \sum_{1 \leq \nu \leq m} \Big((\partial_{x_\nu} \varphi)(x, \ldots, x) \Big)^2 = k \sum_{1 \leq \nu \leq km} \Big((\partial_{x_\nu} \varphi^{(km)})(x, \ldots, x) \Big)^2.
\end{aligned} \tag{4.21}$$

Still with k fixed, we put $\widetilde{\varphi}^{(m)} = k^{-1}\varphi^{(km)}(x,\dots,x)$, $\widetilde{E}^{(m)} = k^{-1}E^{(km)}$. Then :

$$V^{(m)}(x) - \frac{1}{2}(\nabla\widetilde{\varphi}^{(m)})^2 + \frac{1}{2}h\Delta\widetilde{\varphi}^{(m)} - h\widetilde{E}^{(m)} = -\frac{1}{2}hk^{-1}\sum_{1\le\mu\le km}\ \sum_{1\le\gamma\le k-1}\left(\partial_{x_\mu}\partial_{x_{\mu+\gamma m}}\varphi^{(km)}\right)(x,\dots,x). \tag{4.22}$$

We now add one more assumption. We assume that for sufficiently large m :

(4.23) $V = V^{(m)}$ satisfies the assumption (2.9) with $B = \ell^\infty_\rho$, for some family $\rho^{(m)}$ with the properties (4.24), and that with the same ρ :

$$\sup_x\left\|\nabla^3 V^{(m)}\right\|_{(\ell^\infty_\rho\otimes\ell^1_{1/\rho}\otimes\ell^\infty)^*},$$

$$\sup_x\left\|\nabla^3 V^{(m)}\right\|_{(\ell^\infty\otimes\ell^1\otimes\ell^\infty)^*}$$

$$\sup_x\left\|\nabla^4 V^{(m)}\right\|_{(\ell^\infty_\rho\otimes\ell^1_{1/\rho}\otimes\ell^\infty\otimes\ell^\infty)^*},$$

$$\sup_x\left\|\nabla^4 V^{(m)}\right\|_{(\ell^\infty\otimes\ell^1\otimes\ell^\infty\otimes\ell^\infty)^*}$$

are all finite and bounded by some constant which is independent of m.

Here the property of ρ should be :

(4.24) For $j \in \mathbb{Z}/m\mathbb{Z}$ we have : $e^{-\kappa} \le \rho(j+1)/\rho(j) \le e^{\kappa}$. Moreover $\rho(0) = 1$ and we have $\rho(j+1)/\rho(j) = e^{\kappa}$ for $C \le j \le \frac{1}{2}m - C$, $\rho(j+1)/\rho(j) = e^{-\kappa}$ for $-(\frac{1}{2}m - C) \le j \le -C$, with $\kappa > 0$ and C independent of m.

It follows from our earlier estimates that

(4.25) $M^j_{\ell^\infty_\rho}(\varphi)$, $M^j_{\ell^\infty}(\varphi)$ for $j = 3,4$ are bounded by a constant independent of m (when m is sufficiently large),

are bounded by a constant independent of m (when m is sufficiently large), and using this fact for $\varphi^{(km)}$ (with k fixed) we shall estimate the right hand side, F of (4.22). To shorten the formulas, we take $k=2$, but everything works the same way for any fixed $k \geq 2$. Then we have :

$$F = -\frac{1}{2}h \sum_{1\leq\mu\leq m} \Big(\partial_{x_\mu}\partial_{x_{\mu+m}}\varphi^{(2m)}\Big)(x,x) \ . \tag{4.26}$$

From (2.2), it follows that

$$(\partial_{x_1}\partial_{x_{1+m}}\varphi^{(2m)})(x,x) = \mathcal{O}(1)e^{-\kappa m} \ , \tag{4.27}$$

uniformly in x and in m. Using (4.25), we also get :

$$\Big|\nabla\Big((\partial_{x_1}\partial_{x_{1+m}}\varphi^{(2m)})(x,x)\Big)\Big|_1 = \mathcal{O}(1)e^{-\kappa m} \ , \tag{4.28}$$

$$\Big\|\nabla^2\Big((\partial_{x_1}\partial_{x_{1+m}}\varphi^{(2m)})(x,x)\Big)\Big\|_{(\ell^\infty\otimes\ell^\infty)^*} = \mathcal{O}(1)e^{-\kappa m} \ . \tag{4.29}$$

For instance, the last estimate follows from :

$$\Big\langle\nabla^2\Big((\partial_{x_1}\partial_{x_{1+m}}\varphi^{(2m)})(x,x)\Big),\nu\otimes\mu\Big\rangle =$$
$$\Big\langle(\nabla^4\varphi^{(2m)})(x,x), e_1\otimes e_{1+m}\otimes(\nu_1\otimes\mu_1+\nu_1\otimes\mu_2+\nu_2\otimes\mu_1+\nu_2\otimes\mu_2)\Big\rangle \ ,$$

where $\nu_1 = (\nu,0)$, $\nu_2 = (0,\nu)$ etc., and the fact that $\|e_1\|_{\infty,\rho} = \mathcal{O}(1)$, $\|e_{1+m}\|_{1,1/\rho} = \mathcal{O}(e^{-\kappa m})$. Since $\varphi^{(2m)}$ is invariant under cyclic permutation of the coordinates (cf. (4.19)), we have (4.27)-(4.29) also in the case when $\partial_{x_1}\partial_{x_{1+m}}$ is replaced by $\partial_{x_\mu}\partial_{x_{\mu+m}}$, so by (4.26) :

$$F(x),\ |\nabla F(x)|_1\ ,\ \|\nabla^2 F(x)\|_{(\ell^\infty\otimes\ell^\infty)^*} = \mathcal{O}(1)mhe^{-\kappa m}. \tag{4.30}$$

We now compare (4.22) :

$$V^{(m)}(x) - \frac{1}{2}(\nabla\widetilde{\varphi}^{(m)})^2 + \frac{1}{2}h\Delta\widetilde{\varphi}^{(m)} - h\widetilde{E}^{(m)} = F$$

and

$$V^{(m)}(x) - \frac{1}{2}(\nabla\varphi^{(m)})^2 + \frac{1}{2}h\Delta\varphi^{(m)} - hE^{(m)} = 0 \tag{4.31}$$

as in section 2. Taking the gradient of the difference gives :

$$\begin{aligned}\frac{1}{2}(\nabla\widetilde{\varphi}^{(m)}+\nabla\varphi^{(m)})\cdot\partial_x(\nabla\widetilde{\varphi}^{(m)}-\nabla\varphi^{(m)})+&\\ \frac{1}{2}(\nabla^2\widetilde{\varphi}^{(m)}+\nabla^2\varphi^{(m)})(\nabla\widetilde{\varphi}^{(m)}-\nabla\varphi^{(m)}) =&\\ -\nabla F + \frac{1}{2}h\Delta(\nabla\widetilde{\varphi}^{(m)}-\nabla\varphi^{(m)}).&\end{aligned} \tag{4.32}$$

Here we recall that $\nabla^2\varphi^{(m)} = 1 + \nabla^2\psi^{(m)}$ with $\left\|\nabla^2\psi^{(m)}\right\|_{(\ell^\infty\otimes\ell^1)^*} \leq \widetilde{\theta}$. Using this with m replaced by $2m$, we get :

$$\left\langle \nabla^2\widetilde{\varphi}^{(m)}(x), \nu\otimes\mu \right\rangle = \frac{1}{2}\left\langle (\nabla^2\varphi^{(2m)})(x,x), (\nu,\nu)\otimes(\mu,\mu) \right\rangle =$$

$$\langle \nu,\mu\rangle + \frac{1}{2}\left\langle (\nabla^2\psi^{(2m)})(x,x), (\nu,\nu)\otimes(\mu,\mu) \right\rangle .$$

The absolute value of the last term is $\leq \frac{1}{2}\widetilde{\theta}\,|\nu|_\infty\, 2\,|\mu|_1 = \widetilde{\theta}\,|\nu|_\infty\,|\mu|_1$, so $\nabla^2\widetilde{\varphi}^{(m)} = 1 + \nabla^2\widetilde{\psi}^{(m)}$ with

$$\left\|\nabla^2\widetilde{\psi}^{(m)}\right\|_{(\ell^\infty\otimes\ell^1)^*} \leq \widetilde{\theta}\ . \tag{4.33}$$

The same argument as in section 2 then gives :

$$\left|\nabla(\widetilde{\varphi}^{(m)} - \varphi^{(m)})\right|_1 \leq (1-\widetilde{\theta})^{-1}\sup_x |\nabla F(x)|_1 = \mathcal{O}(1)mhe^{-\kappa m}. \tag{4.34}$$

Taking the scalar product of (4.32) with the constant vector t and differentiating in the constant direction s, we get as in section 2 :

$$\begin{aligned} &\frac{1}{2}(\nabla\widetilde{\varphi}^{(m)} + \nabla\varphi^{(m)})\cdot\partial_x\left\langle \nabla^2(\widetilde{\varphi}^{(m)} - \varphi^{(m)}), s\otimes t\right\rangle + \\ &\quad 2\left\langle \nabla^2(\widetilde{\varphi}^{(m)} - \varphi^{(m)}), s\otimes t\right\rangle + \Big\langle \nabla^2(\widetilde{\varphi}^{(m)} - \nabla\varphi^{(m)}), \\ &\quad \frac{1}{2}(\nabla^2\widetilde{\psi}^{(m)} + \nabla^2\psi^{(m)})(s)\otimes t + s\otimes\frac{1}{2}(\nabla^2\widetilde{\psi}^{(m)} + \nabla^2\psi^{(m)})(t)\Big\rangle + \\ &\quad \left\langle \frac{1}{2}\nabla^3(\widetilde{\varphi}^{(m)} + \varphi^{(m)}), (\nabla\widetilde{\varphi}^{(m)} - \nabla\varphi^{(m)})\otimes s\otimes t\right\rangle = \\ &\quad -\langle \nabla^2 F, s\otimes t\rangle + \frac{1}{2}h\Delta\left\langle \nabla^2(\widetilde{\varphi}^{(m)} - \varphi^{(m)}), s\otimes t\right\rangle . \end{aligned} \tag{4.35}$$

As in section 2 we conclude that :

$$\left\|\nabla^2(\widetilde{\varphi}^{(m)} - \varphi^{(m)})\right\|_{(\ell^\infty\otimes\ell^\infty)^*} = \mathcal{O}(1)mhe^{-\kappa m}. \tag{4.36}$$

Combining (4.22), (4.31), we get :

$$\begin{aligned} &m^{-1}(h\widetilde{E}^{(m)} - hE^{(m)}) = \\ &\quad -m^{-1}F - (2m)^{-1}(\nabla\widetilde{\varphi}^{(m)} + \nabla\varphi^{(m)})\cdot(\nabla\widetilde{\varphi}^{(m)} - \nabla\varphi^{(m)}) + \\ &\quad (h/2m)\Delta(\widetilde{\varphi}^{(m)} - \varphi^{(m)})\ , \end{aligned} \tag{4.37}$$

and choosing x such that $\nabla\widetilde{\varphi}^{(m)} + \nabla\varphi^{(m)} = 0$ at x, we get from (4.30), (4.36), (4.37) and Lemma 1.2 of **[S1]** :

$$m^{-1}(h\widetilde{E}^{(m)} - hE^{(m)}) = \mathcal{O}(h+h^2)e^{-\kappa m}\ . \tag{4.38}$$

Summing up, we have proved :

Theorem 4.1. *Let* $V^{(m)} = V^{(m)}(x_1, x_2, \ldots, x_m)$, $m = 1, 2, \ldots$ *be a sequence of potentials with* $V^{(m)}(0) = 0$, $\nabla V^{(m)}(0) = 0$, *which for* m *large enough satisfy the assumptions* (4.18) *(with a fixed* $k \geq 2$ *),* (4.19), (4.23). *Let* $\mu(m)$ *be the smallest eigenvalue of* $-\frac{1}{2}h^2\Delta + V^{(m)}$. *Then for sufficiently large* m *(uniformly in* h *):*

$$(km)^{-1}\mu(km) - m^{-1}\mu(m) = \mathcal{O}(h + h^2)e^{-\kappa m} \ . \tag{4.39}$$

If $\lim_{m\to\infty} \mu(m)/m \underset{\text{def}}{=} \mu(\infty)$ *exists (as we know under certain assumptions, cf. section 3 and* **[HS]***), then* (4.39) *gives:*

$$m^{-1}\mu(m) - \mu(\infty) = \mathcal{O}(h + h^2)e^{-\kappa m} \ . \tag{4.40}$$

5. Application to a model related to statistical mechanics

In **[HS]** we studied the following model operator (inspired by **[K]**):

$$P_m = -h^2\Delta + V^{(m)}(x) \tag{5.1}$$

on $\mathbb{R}^m$, where:

$$V^{(m)}(x) = \frac{1}{4}\Sigma x_j^2 - \Sigma \log ch\Big(\sqrt{\frac{\nu}{2}}(x_j + x_{j+1})\Big) \quad \text{with } j \in \mathbb{Z}/m\mathbb{Z} \tag{5.2}$$

and assumed that ν is fixed in $]0, \frac{1}{4}[$. We keep the same assumption on ν and we then know (**[HS]**) that $V^{(m)}$ is strictly convex and vanishes to the second order at 0. If $f(t) = \log ch t$, then $f'(t) = sh t/ch t$, $f''(t) = (ch t)^{-2}$, and hence (as we saw in **[HS]**):

$$\partial_{x_j}^2 V^{(m)}(x) = \frac{1}{2} - \frac{1}{2}\nu\Big((ch\sqrt{\frac{\nu}{2}}(x_{j-1}+x_j))^{-2} + (ch\sqrt{\frac{\nu}{2}}(x_j+x_{j+1}))^{-2}\Big), \tag{5.3}$$

$$\partial_{x_j}\partial_{x_{j+1}} V^{(m)}(x) = -\frac{1}{2}\nu(ch\sqrt{\frac{\nu}{2}}(x_j + x_{j+1}))^{-2}, \tag{5.4}$$

$$\partial_{x_j}\partial_{x_k} V^{(m)}(x) = 0 \text{ if } j - k \not\equiv -1, 0, 1 \mod (m). \tag{5.5}$$

We can then write:

$$\nabla^2 V^{(m)}(x) = \frac{1}{2}(I + A(x)) \ , \tag{5.6}$$

$$
(5.7)\qquad A(x)=\begin{pmatrix} d_1(x) & c_1(x) & 0\dots & 0 & c_m(x)\\ c_1(x) & d_2(x) & c_2(x) & & 0\\ 0 & & & & 0\\ \cdot & & & \cdot & \\ 0 & & c_{m-2}(x) & d_{m-1}(x) & c_{m-1}(x)\\ c_m(x) & 0\dots & \dots & c_{m-1}(x) & d_m(x)\end{pmatrix}
$$

where

$$
(5.8)\qquad |d_j(x)|\le 2\nu\ ,\quad |c_j(x)|\le\nu\ .
$$

We may also notice that $d_j(0)=-2\nu$, $c_j(0)=-\nu$.

Let $\rho:\mathbb{Z}/m\mathbb{Z}\to\,]0,\infty[$ satisfy :

$$
(5.9)\qquad e^{-\delta}\le \mu(j+1)/\mu(j)\le e^{\delta}\ ,
$$

where $\mu(j)=\rho(j+1)/\rho(j)$. Then the argument after (4.8) shows that

$$
(5.10)\qquad \|A(x)\|_{\mathcal{L}(\ell^p_\rho,\ell^p_\rho)}\le 2\nu(1+e^{\delta}\sup_{1\le k\le m}\frac{1}{2}(\mu(k)+\mu(k)^{-1})).
$$

Let $\kappa>0$ satisfy

$$
(5.11)\qquad 2\nu(1+ch\,\kappa)<1\ ,\quad \text{i.e. } \kappa<ch^{-1}((1-2\nu)/2\nu)\ .
$$

Then, if we choose $\delta>0$ sufficiently small, it follows that :

$$
(5.12)\qquad \|A(x)\|_{\mathcal{L}(\ell^p_\rho,\ell^p_\rho)}\le\theta<1\ ,
$$

for some fixed θ, provided that ρ satisfies (5.9) and :

$$
(5.13)\qquad e^{-\kappa}\le\rho(j+1)/\rho(j)\le e^{\kappa}\ .
$$

We can clearly find such a ρ which also satisfies (4.24).

A part from the factor $\frac{1}{2}$ in (5.6) and the fact that there is no "$\frac{1}{2}$" in (5.1) (which is not essential, as can be seen by a scaling in h), we have then verified the part of (4.23) which concerns the Hessian of $V^{(m)}$. The remaining parts of (4.23) (concerning the higher order Hessians of $V^{(m)}$ are easy to check, and it is also clear that we have (4.18), (4.19), so we can apply Theorem 4.1 and get :

Theorem 5.1. *Let* $\mu(m;h)$ *be the lowest eigenvalue of the operator* (5.1), (5.2), *and assume that* $0 \leq \nu < \frac{1}{4}$. *If* $\kappa > 0$ *satisfies* (5.11), *then for* ν , κ *fixed we have uniformly with respect to* h *:*

$$\mu(\infty, h) - \mu(m;h)/m = \mathcal{O}(h + h^2)e^{-\kappa m} \ , \quad m \to \infty \ . \tag{5.14}$$

Here $\mu(\infty;h)$ *denotes the limit of* $\mu(m;h)/m$ *as* m *tends to infinity. (The existence of the limit was established in* **[HS]** *and also follows from Theorem 3.1.)*

REMARK 5.2. In analogy with (4.1) we can write

$$\nabla^2 V^{(m)}(0) = \Big(\frac{1}{2} - \nu\Big)(I - (2\nu/(1-2\nu))\frac{1}{2}(\tau_{-1} + \tau_1)) \ , \tag{5.15}$$

so if we compare (4.8) and (5.11), we see that Theorem 5.1 produces a decay rate which is equal to the (probably optimal) one that we get for the quadratic approximations of $V^{(m)}$, by applying (4.8). We have therefore every reason to believe that the set of exponents in Theorem (5.14) is optimal, and by applying the WKB results of [**HS, S1S2,**], it seems quite possible to prove that so is the case, if we require uniformity in h, as in (5.14).

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