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NEAR—COHOMOLOGY OF HILBERT COMPLEXES AND TOPOLOGY OF NON—SIMPLY CONNECTED MANIFOLDS.

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Introduction

In an earlier paper [5] we introduced some new homotopy invariants of compact non—simply connected manifolds (possibly with boundary) or finite CW —complexes. In terms of these invariants the heat kernel invariants of closed non—simply connected manifolds [9] (see also [4]) can be expressed and thus their homotopy invariance can be proved.

Note that both invariants in [5] and [9] are expressed in terms of L^2 —de Rham complex on the universal covering, using the deck transformation action of the fundamental group in differential forms. The use of the combinatorial Laplacians leads to the same invariants as was proved by A. Efremov [3].

In this paper we follow the abstract setting from [5] and give a refined formulation of the abstract result there. This leads to a new notion of near—cohomology for Hilbert complexes. We take a special family of quadric cones depending on a small positive parameter and consisting of cochains which have coboundaries which are small with respect to the distance of the cochains to the space of all cocycles. Heuristically this means that we take cochains with small coboundaries modulo cochains close to cocycles. (Instead of cochains close to cocycles we could also take cochains close to coboundaries which would remind cohomology more but it just adds cohomology as a direct summand.) Near—cohomology are germs of such families of quadric cones modulo

an equivalence relation which naturally arises if we consider homotopy equivalence of Hilbert complexes with morphisms given by bounded linear operators. Then near-cohomology becomes a homotopy invariant.

Adding a von Neumann algebra structure to the Hilbert complex we can transform near-cohomology to a set of positive-valued functions of the small parameter up to an equivalence. These functions are defined as maximal von Neumann dimensions of linear spaces which belong to the cones. The equivalence is given by estimates of these functions with dilatated arguments.

Applying these constructions to the de Rham L^2 -complex on the universal covering of a compact manifolds (with the von Neumann algebras consisting of operators commuting with deck transformations on differential forms) we obtain invariants which were introduced and studied in [5].

Note that the idea that there may be topology invariants lying near cohomology was first formulated in [8].

1. Hilbert complexes and their near-cohomology.

A. Let us consider a sequence

$$E : 0 \rightarrow E_0 \xrightarrow{d_0} E_1 \rightarrow \dots \rightarrow E_k \xrightarrow{d_k} E_{k+1} \rightarrow \dots \xrightarrow{d_{N-1}} E_N \rightarrow 0,$$

where E_k is a Hilbert space and the differential $d_k : E_k \rightarrow E_{k+1}$ is a closed densely defined linear operator (with the domain $D(d_k)$). This sequence is called a *Hilbert complex* if $d_{k+1} \circ d_k = 0$ on $D(d_k)$ or, equivalently, $\text{Im } d_k \subset \text{Ker } d_{k+1}$. Note that $\text{Ker } d_k$ is always a closed linear subspace in E_k .

Let E' be another Hilbert complex of the same length N (if the lengths differ then we can always formally extend the shorter complex by adding zero spaces in the end; so for the sake of simplicity we shall always suppose that all complexes have the same length N). The corresponding spaces and differentials will be denoted E'_k and d'_k .

Definition 1.1. A *morphism* $f : E \rightarrow E'$ of the Hilbert complexes is a collection of *bounded* linear operators $f_k : E_k \rightarrow E'_k$ such that

$$f_{k+1}d_k \subset d'_k f_k,$$

which means that $f_{k+1}d_k = d'_k f_k$ on $D(d_k)$. In particular we require that $f_k(D(d_k)) \subset D(d'_k)$.

If $f : E \rightarrow E'$ and $g : E' \rightarrow E''$ are two morphisms of Hilbert complexes then their composition $g \circ f : E \rightarrow E''$ is a morphism defined as the collection of compositions $g_k \circ f_k$, $k = 0, 1, \dots, N$.

Definition 1.2. Let $f, g : E \rightarrow E'$ be two morphisms of the same Hilbert complexes. A *homotopy* (between f and g) is a collection T of *bounded* linear operators $T_k : E_k \rightarrow E'_{k-1}$ such that

$$f_k - g_k - T_{k+1}d_k \subset d'_{k-1}T_k, \quad k = 0, 1, \dots, N,$$

or equivalently, $f_k - g_k = T_{k+1}d_k + d'_{k-1}T_k$ on $D(d_k)$ (in particular this means that $T_k(D(d_k)) \subset D(d'_{k-1})$). If there exists a homotopy between morphisms f and g then f and g are called *homotopic* and we denote it as $f \sim g$. (It is easy to check that being homotopic is really an equivalence relation between morphisms.)

Hilbert complexes E, E' are called *homotopy equivalent* if there exists morphisms $f : E \rightarrow E'$ and $g : E' \rightarrow E$ such that $g \circ f \sim \text{Id}_E$, $f \circ g \sim \text{Id}_{E'}$ where Id_E and $\text{Id}_{E'}$ are identity morphisms of the corresponding Hilbert complexes. We shall denote the homotopy equivalence between E and E' as $E \sim E'$.

Definition 1.3. E is called a *retract* of E' if there exist morphisms $f : E \rightarrow E'$ and $g : E' \rightarrow E$ such that $g \circ f \sim \text{Id}_E$. In this case f (resp. g) is called a *homotopy inclusion* (resp. *homotopy retraction*) map.

Remark. Cohomology spaces $H^k(E) = \text{Ker } d_k / \text{Im } d_{k-1}$ and reduced cohomology spaces $\overline{H}^k(E) = \text{Ker } d_k / \overline{\text{Im } d_{k-1}}$ are homotopy functors in the category of Hilbert complexes with morphisms and homotopy as before.

B. Now let us introduce the following quadric cones, depending on the degree k and on a positive parameter λ :

$$B_{\lambda}^{(k)} = \{\omega | \omega \in E_k / \text{Ker } d_k, \| d_k \omega \| \leq \lambda \| \omega \|_{\text{mod Ker } d}\},$$

where $\| \omega \|_{\text{mod Ker } d}$ is the norm in the quotient space $E_k / \text{Ker } d_k$, $\| d_k \omega \|$ means the norm of $d_k \omega$ in E_{k+1} . It is understood that in this definition we should only take co-sets in $E_k / \text{Ker } d_k$ defined by elements $\omega \in D(d_k)$ to make $d_k \omega$ well defined. So $B_{\lambda}^{(k)}$ becomes a conic set in the Hilbert space $E_k / \text{Ker } d_k$ which can also be identified with $(\text{Ker } d_k)^{\perp}$ (the orthogonal complement of $\text{Ker } d_k$ in E_k).

Lemma 1.4. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a closed linear operator with the domain $D(A)$. Then for every $\lambda > 0$ the set

$$C_{\lambda, A} = \{x | x \in D(A), \| Ax \| \leq \lambda \| x \|\}$$

is closed in $\mathcal{H}_1$.

Proof. Suppose that x is in the closure of $C_{\lambda, A}$. Without loss of generality we may assume that $\| x \| = 1$. Then we easily obtain that there exist $x_{\gamma} \in C_{\lambda, A}$ such that

$$\lim_{\Gamma} \| x_{\gamma} - x \| = 0, \| Ax_{\gamma} \| \leq \lambda \| x_{\gamma} \|, \gamma \in \Gamma,$$

where Γ is a directed set. Taking a cofinal subset of Γ we may further suppose that $\| x_{\gamma} \| \leq 1 + \varepsilon$ whatever fixed $\varepsilon > 0$. Changing Γ again we may suppose that there exists $w - \lim_{\Gamma} Ax_{\gamma} = y$ (weak limit is taken in $\mathcal{H}_2$). Then we have

$$\| y \| \leq \liminf_{\Gamma} \| Ax_{\gamma} \| \leq \lambda \liminf_{\Gamma} \| x_{\gamma} \| \leq \lambda \| x \|\}$$

Now the pair $\{x, y\}$ is in the weak closure of the graph of the operator A in $\mathcal{H}_1 \times \mathcal{H}_2$. The graph is a closed linear subspace, hence it is weakly closed. Therefore $x \in D(A)$ and $y = Ax$. Hence $x \in C_{\lambda, A}$ as required. $\square$

Applying Lemma 1.4 to $\mathcal{H}_1 = E_k / \text{Ker } d_k$, $\mathcal{H}_2 = E_{k+1}$ and $A = d_k$ we see that $B_{\lambda}^{(k)}$ is a closed cone in $E_k / \text{Ker } d_k$ for every

$\lambda > 0$. Now let us look what happens to these cones when we apply morphisms of Hilbert complexes.

Let us consider a morphism of Hilbert complexes $f : E \rightarrow E'$ defined by a collection of bounded linear operators $f_k : E_k \rightarrow E'_k$, $k = 0, \dots, N$. Then $f_k(\text{Ker } d_k) \subset \text{Ker } d'_k$ so f_k naturally defines a bounded linear operator

$$\hat{f}_k : E_k / \text{Ker } d_k \rightarrow E'_k / \text{Ker } d'_k.$$

Theorem 1.5. Let a Hilbert complex E be a retract of E' and $f : E \rightarrow E'$, $g : E' \rightarrow E$ be corresponding homotopy inclusion and retraction maps. Let $B_\lambda^{(k)}$, $'B_\lambda^{(k)}$ be families of cones defined as before in E, E' respectively. Then there exist $C > 0$ and $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$

$$(i) \quad \hat{f}_k(B_\lambda^{(k)}) \subset 'B_{C\lambda}^{(k)};$$

$$(ii) \quad \|\omega\| \leq C \|\hat{f}_k \omega\| \text{ if } \omega \in B_\lambda^{(k)}.$$

Proof. Let us consider $\omega \in B_\lambda^{(k)}$ and let $\omega_1 \in (\text{Ker } d_k)^\perp$ represent ω i.e. $\omega_1 \bmod \text{Ker } d_k = \omega$. Then $\|d_k \omega_1\| \leq \lambda \|\omega_1\|$. It follows that $f_k \omega_1 \in D(d'_k)$, $d'_k f_k \omega_1 = f_{k+1} d_k \omega_1$ and

$$\|d'_k(f_k \omega_1)\| = \|f_{k+1}(d_k \omega_1)\| \leq \|f_{k+1}\| \|d_k \omega_1\| \leq \lambda \|f_{k+1}\| \|\omega_1\|$$

Now we should estimate $\|\omega_1\|$ by $C_1 \|f_k \omega_1\|_{\text{mod Ker } d'}$ provided $\lambda \in (0, \lambda_0)$ with a small $\lambda_0 > 0$ with a constant C_1 which does not depend on ω_1 or λ . Then (i) and (ii) will follow. Let us split $f_k \omega_1$ into the sum

$$f_k \omega_1 = \omega'_1 + \omega'_2$$

with $\omega'_2 \in \text{Ker } d'_k$, $\omega'_1 \perp \text{Ker } d'_k$. Hence

$$\| f_k \omega_1 \|_{\text{mod Ker } d'} = \| \omega'_1 \|.$$

Now let us use a homotopy T between $g \circ f$ and Id_E . In particular we have

$$\text{Id}_E - g_k \circ f_k = T_{k+1} d_k + d_{k-1} T_k \text{ on } D(d_k).$$

hence

$$\begin{aligned} \omega_1 &= g_k(f_k \omega_1) + T_{k+1} d_k \omega_1 + d_{k-1} T_k \omega_1 = \\ &= g_k(\omega'_1 + \omega'_2) + T_{k+1} d_k \omega_1 + d_{k-1} T_k \omega_1. \end{aligned}$$

Clearly $g_k \omega'_2 \in \text{Ker } d_k$, hence

$$\omega_1 \equiv g_k \omega'_1 + T_{k+1} d_k \omega_1 \text{ mod Ker } d_k.$$

It follows that

$$\begin{aligned} \| \omega_1 \| &\leq \| g_k \omega'_1 + T_{k+1} d_k \omega_1 \| \leq \| g_k \| \| \omega'_1 \| + \| T_{k+1} \| \| d_k \omega_1 \| \leq \\ &\leq \| g_k \| \| \omega'_1 \| + \lambda \| T_{k+1} \| \| \omega_1 \|, \end{aligned}$$

hence

$$\| \omega_1 \| \leq \frac{\| g_k \| \| \omega'_1 \|}{1 - \lambda \| T_{k+1} \|} \leq 2 \| g_k \| \| \omega'_1 \|,$$

if $\lambda \in (0, \lambda_0)$ where $\lambda_0 = (2 \| T_{k+1} \|)^{-1}$. This gives the required estimate that proves the Theorem. $\square$

Corollary 1.6. Suppose that Hilbert complexes E, E' are homotopy equivalent and this equivalence is given by the morphisms $f : E \rightarrow E'$ and $g : E' \rightarrow E$. Then there exist constants $C > 0$, $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ and for every $k = 0, \dots, N$

$$(i) \quad \hat{f}(B_\lambda^{(k)}) \subset {}'B_{C\lambda}^{(k)}; \quad \hat{g}({}'B_\lambda^{(k)}) \in B_{C\lambda}^{(k)};$$

$$(ii) \quad \| \omega \| \leq C \| \hat{f} \omega \|, \quad \omega \in B_\lambda^{(k)}; \quad \| \omega' \| \leq C \| \hat{g} \omega' \|, \quad \omega' \in {}'B_\lambda^{(k)}.$$

Now we can introduce an appropriate notion of near-cohomology of a Hilbert complex. This will be done along the lines that can be traced in Corollary 1.6.

Definition 1.7. *Special family of quadric cones* in a Hilbert space E is a family of closed subsets $B_\lambda \subset E$ defined for all $\lambda > 0$ as follows:

$$B_\lambda = \{x \mid x \in D(A), \|Ax\| \leq \lambda \|x\|\},$$

where $A : E \rightarrow E_1$ is a closed densely defined linear operator (E_1 is another Hilbert space) with the domain $D(A)$, the norms $\|Ax\|$ and $\|x\|$ are taken in E_1 and E respectively.

Two such families $B_\lambda \subset E$ and $'B_\lambda \subset E'$ are called *equivalent* if there exist two bounded linear operators: $f : E \rightarrow E'$, $g : E' \rightarrow E$ and positive constants $C > 0$, $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$

$$(i) \quad f(B_\lambda) \subset 'B_{C\lambda}, \quad g('B_\lambda) \subset B_{C\lambda};$$

$$(ii) \quad \|x\| \leq C \|fx\|, \quad x \in B_\lambda; \quad \|x'\| \leq C \|gx'\|, \quad x' \in 'B_\lambda.$$

So in fact up to the equivalence only the germ of the family B_λ near 0 is important.

Definition 1.8. Let E be a Hilbert complex. Its *near cohomology* $NH^k(E)$ of degree k is the equivalence class of the special family of quadric cones $B_\lambda^{(k)}$.

Corollary 1.6 means then that the near-cohomology is a homotopy invariant of the Hilbert complex if the homotopy equivalence is defined as a chain homotopy equivalence with bounded morphisms and homotopy operators as in Definitions 1.1 and 1.2.

Remark. All results of this Section can be easily extended to complexes of reflexive Banach spaces.

2. Von Neumann structure

Von Neumann structure on a Hilbert complex allows to transform near-cohomology to some simpler invariants: to make the same kind of transfer from homotopy to the Betti numbers.

First we shall recall some necessary definitions (see e.g. [2]). Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators in $\mathcal{H}$. A *von Neumann algebra* of operators in $\mathcal{H}$ is a subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ satisfying the following conditions:

- (i) $A \ni \text{Id}_{\mathcal{H}}$, $\mathcal{A}$ is a $*$ -algebra (i.e. $A \in \mathcal{A} \Rightarrow A^* \in \mathcal{A}$),
- (ii) $\mathcal{A}$ is closed in the weak operator topology.

Let $\mathcal{A}^+ = \{A \mid A \in \mathcal{A}, A \geq 0\}$. A *trace* $\text{Tr}_{\mathcal{A}}$ on $\mathcal{A}$ is a map $\text{Tr}_{\mathcal{A}} : \mathcal{A}^+ \rightarrow [0, +\infty]$ satisfying the following conditions:

- (i) $\text{Tr}_{\mathcal{A}}(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \text{Tr}_{\mathcal{A}} A_1 + \lambda_2 \text{Tr}_{\mathcal{A}} A_2$ if $\lambda_i \in [0, +\infty]$, $A_i \in \mathcal{A}^+$, $i = 1, 2$,
- (ii) $\text{Tr}_{\mathcal{A}}(AA^*) = \text{Tr}_{\mathcal{A}}(A^*A)$ for every $A \in \mathcal{A}$,
- (iii) If $A_{\gamma} \in \mathcal{A}^+$ and $A_{\gamma} \nearrow A$ then $\text{Tr}_{\mathcal{A}} A_{\gamma} \rightarrow \text{Tr}_{\mathcal{A}} A$ (normality);
- (iv) $\text{Tr}_{\mathcal{A}} A = \sup\{\text{Tr}_{\mathcal{A}} B \mid 0 \leq B \leq A, B \in \mathcal{A}, \text{Tr}_{\mathcal{A}} B < \infty\}$ for every $A \in \mathcal{A}^+$ (semi-finiteness);
- (v) $\text{Tr}_{\mathcal{A}} A = 0$, $A \in \mathcal{A}^+ \Rightarrow A = 0$ (faithfulness).

If a trace $\text{Tr}_{\mathcal{A}}$ is given on $\mathcal{A}$ then we can define *von Neumann dimension* $\dim_{\mathcal{A}}$. It is defined on all closed subspaces $L \subset \mathcal{H}$ which are *affiliated* with $\mathcal{A}$ i.e. such that $P_L \in \mathcal{A}$ where P_L is the orthogonal projection in $\mathcal{H}$ with the image L . Then we write $L \eta \mathcal{A}$ and $\dim_{\mathcal{A}} L = \text{Tr}_{\mathcal{A}} P_L$.

Definition 2.1. Let E be a Hilbert complex. A *von Neumann structure on E* is a collection of von Neumann algebras $\mathcal{A}_k \subset \mathcal{B}(E_k)$ for all $k = 0, \dots, N$, and a trace $\text{Tr}_{\mathcal{A}}$ on every algebra $\mathcal{A}_k$ (we denote all the traces $\text{Tr}_{\mathcal{A}}$ for all k for simplicity of notations because it does not lead to a confusion), provided $\text{Ker } d_k$ is affiliated with $\mathcal{A}_k$ for every k .

Now modelling the well known variational principle (Glazman's Lemma) for the operators $d_k^* d_k$ we can introduce the following functions which will imitate the eigenvalue distribution function of the discrete spectrum.

Definition 2.2.

$$F_k(\lambda) = \sup_{L \subset B_\lambda^{(k)}} \dim_{\mathcal{A}} L$$

Here it is convenient to identify $E_k / \text{Ker } d_k$ with $(\text{Ker } d_k)^\perp$ so L can be considered as a closed linear subspace in E_k (such that in fact $L \subset (\text{Ker } d_k)^\perp$), hence $\dim_{\mathcal{A}} L$ makes sense.

Since the cones $B_\lambda^{(k)}$ increase with λ the function $F_k(\lambda)$ is an increasing function on $(0, \infty)$. If $F_k(\lambda_0) < \infty$ for some $\lambda_0 > 0$ then $F_k(+0) = 0$.

Now let us introduce morphisms and homotopy equivalence for Hilbert complexes with von Neumann structure.

Definition 2.3. Let E, E' be Hilbert complexes with von Neumann structures,

$f : E \rightarrow E'$ a morphism of Hilbert complexes. Then f is called *compatible* with von Neumann structures if the following condition is satisfied:

(C) Suppose that $L \subset E_k, L \eta \mathcal{A}_k$ and there exists $C > 0$ such that

$$\|x\| \leq C \|f_k x\|, \quad x \in L.$$

Then $f_k(L) \eta \mathcal{A}'_k$ and $\dim_{\mathcal{A}'} f_k(L) = \dim_{\mathcal{A}} L$.

Roughly speaking this means that the morphism f conserves the von Neumann dimension of a subspace provided this subspace is mapped by f isomorphically (in topological sense).

Definition 2.4. Let E, E' be Hilbert complexes with von Neumann structures. They are called *homotopy equivalent* if there exist morphisms of Hilbert complexes compatible with von Neumann structure $f : E \rightarrow E', g : E' \rightarrow E$, such that $f \circ g \sim \text{Id}_{E'}$, $g \circ f \sim \text{Id}_E$. (Here homotopy between morphisms is understood as in Sect. 1 without any additional compatibility conditions). E is called a *retract* of E' if there exist morphisms (again compatible with von Neumann structure) $f : E \rightarrow E', g : E' \rightarrow E$ such that $g \circ f \sim \text{Id}_E$. The following theorem is an immediate corollary of Theorem 1.5.

Theorem 2.5. Let E, E' be Hilbert complexes with von Neumann structures and E a retract of E' . Denote by F_k, F'_k the

functions defined for E, E' according to Definition 2.2. Then there exist $C > 0, \lambda > 0$ such that for every $k = 0, \dots, N$

$$F_k(\lambda) \leq F'_k(C\lambda), \quad \lambda \in (0, \lambda_0).$$

Corollary 2.6. Suppose that Hilbert complexes E, E' with von Neumann structure are homotopy equivalent. Then there exist $C > 0, \lambda_0 > 0$ such that

$$F_k(C^{-1}\lambda) \leq F'_k(\lambda) \leq F_k(C\lambda), \quad \lambda \in (0, \lambda_0).$$

This corollary tells that the asymptotics of F_k and F'_k near zero coincide in a weak sense. In particular let us introduce

$$\beta_k = \liminf_{\lambda \downarrow 0} \frac{\log F_k(\lambda)}{\log \lambda}$$

and let β'_k mean the same number for F'_k .

Corollary 2.7. If E, E' are as in Corollary 2.6 then $\beta_k = \beta'_k$ for all $k = 0, \dots, N$.

Hence β_k is a homotopy invariant of the Hilbert complex E with the von Neumann structure. We can also introduce an equivalence relation between functions F_k, F'_k given by the inequalities in Corollary 2.6. Then the equivalence class of F_k will be a homotopy invariant of the Hilbert complex E with the von Neumann structure.

3. Geometric examples.

Let X be a compact Riemannian manifold (possibly with a piecewise smooth boundary), M its universal covering with the lifted from X Riemannian metric. Then let us take $E_k = L^2 \wedge^k(M)$, the Hilbert space of all square integrable exterior differential forms of degree k on M . Let us define d_k as the de Rham exterior differential on E_k with the maximal domain i.e.

$$D(d_k) = \{\omega | \omega \in L^2 \wedge^k(M), d\omega \in L^2 \wedge^{k+1}(M)\},$$

where $d\omega$ is understood in the sense of distributions. Thus we obtain a Hilbert de Rham complex $L^2 \wedge^\bullet(M)$. Its near-cohomology

are homotopy invariants of X if homotopy invariance is understood already in the usual topology sense (for the proof see reasoning given in [5], Sect. 5). Note that the group $\Gamma = \pi_1(X)$ acts on M by deck transformations and a more general example can be obtained if we consider a more general discrete group Γ acting without fixed points as a discrete group of isometries of a Riemannian manifold M (with boundary) so that the orbit space $X = M/\Gamma$ is compact. Then similarly defined near-cohomology will be homotopy invariants in the homotopy category of Γ -manifolds and Γ -maps.

The action of Γ by isometries on the spaces $L^2 \wedge^k(M)$ (induced by the change-of-variable maps on differential forms) allows to introduce a von Neumann structure on $L^2 \wedge^\bullet(M)$ if we define

$$\mathcal{A}_k = \{A | A \in \mathcal{B}(L^2 \wedge^k(M)), A\gamma^* = \gamma^*A \text{ for every } \gamma \in \Gamma\}$$

(where γ^* is the change-of-variable map on $L^2 \wedge^k(M)$ given by γ) and take $\text{Tr}_{\mathcal{A}} = \text{Tr}_{\Gamma}$, the Γ -trace introduced by M. Atiyah in [1]. It is shown in [5] that the heat-kernel invariants introduced in [9] (see also [4]) for the case of manifolds without boundary can be expressed in terms of the functions F_k and the numbers β_k , and in this way the homotopy invariance of the heat-kernel invariants can be proved. Note that the result of A. Efremov [3] means really the coincidence of the near-cohomology of a closed manifold and its simplicial approximation.

Another geometrical example naturally arises if we consider a foliation with a transverse measure on a compact manifold. The arguments from [3] can be applied here too. This fact was independently noticed by J.L.Heitsh and C. Lazarov.

Note finally that some calculations of heat-kernel invariants made by J. Lott (see [6],[7]) allow to make some conclusions about the numbers β_k (and sometimes even calculate them, e.g. for the case when M is the hyperbolic space).

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