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Serre's conjecture on Galois representations attached to Weil curves with additive reduction

Joan-C. LARIO

1. Introduction: terminology and facts.— Let E be an elliptic curve defined over $\mathbb{Q}$ which is supposed to be modular, i.e. E is a Weil curve, and denote by $F(z) = \sum A_n e^{2\pi i n z}$ the weight 2 newform attached to E by the Eichler-Shimura congruences.

Fix a prime $p > 7$. We shall be interested in which cases E has additive reduction at p , excluding the Kodaira reduction types I_ν^* ($\nu \geq 0$) which are related to the potentially semi-stable case. Thus p divides exactly twice the geometric conductor N_E of the elliptic curve E .

After [Ed 89] we say that E is p -vertical if E has bad but potentially good ordinary reduction at p , and that E is p -horizontal if E has bad but potentially good supersingular reduction at p . Recall that these conditions can be given in terms of the Hasse invariant (cf. [Hu 87], pag.248) of E and, moreover, one gets:

$$E \text{ is } p\text{-vertical} \iff p \equiv 1 \pmod{e},$$

where e is the least common multiple of the multiplicities of the irreducible components of the special fibre of the stable model; that is

$$e = \begin{cases} 6 & \text{if } p\text{-type}(E) = II, II^*; \\ 4 & \text{if } p\text{-type}(E) = III, III^*; \\ 3 & \text{if } p\text{-type}(E) = IV, IV^*. \end{cases}$$

The Galois module E_p of the p -torsion points of E gives rise to a continuous and odd representation

$$\rho : G_{\mathbb{Q}} \rightarrow \text{Aut}(E_p) \simeq \text{GL}_2(\mathbb{F}_p),$$

which is almost always absolutely irreducible (cf. [Ma 78]).

Serre's conjecture (3.2.4?) in [Se 87] predicts, in this case, the existence of a Hecke cusp form (mod p)

$$f(q) = \sum a_n q^n$$

of type $(N_\rho, k_\rho, \varepsilon_\rho)$ satisfying

$$a_n \equiv A_n \pmod{p}, \quad \text{for all } n \text{ prime to } N_E.$$

The level N_ρ , the weight k_ρ and the character ε_ρ are given by a precise recipe in [Se 87] and, depending on the Néron model of E , they have been computed, for instance in [Ba-La 91].

In [Ba-La 91], we verify (3.2.4?) for the Galois representations defined by the p -torsion points of the p -vertical Weil curves. In this paper our purpose is to emphasize the difference between the p -vertical and the p -horizontal cases in order to check Serre's conjecture. Several numerical examples, collected by computer calculations, lead us to give a conjecture which implies (3.2.4?) for the horizontal case.

2. Lowering the level (ordinary case).— First, we shall consider a general situation. Let

$$F(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$$

be a newform of type (N, k, ε) , defined over $\overline{\mathbb{Q}}$. If $\alpha : (\mathbb{Z}/M\mathbb{Z})^* \rightarrow \mathbb{C}^*$ is a Dirichlet character modulo M , then the twisted form

$$F \otimes \alpha(z) = \sum_{n=1}^{\infty} A_n \alpha(n) e^{2\pi i n z} \quad (\alpha(n) = 0 \text{ if } \text{g.c.d.}(n, M) \neq 1)$$

is a Hecke cusp form of type $(N', k, \varepsilon \alpha^2)$, where

$$N' = \text{l.c.m.}(N, M \cdot \text{conductor}(\varepsilon), M^2).$$

If M is prime to the level N , then $F \otimes \alpha$ is a newform of type $(NM^2, k, \varepsilon \alpha^2)$; otherwise, the form $F \otimes \alpha$ can either be or not be a newform! The question is to decide when it is.

After Li's work [Li 75], one has a nice criterion to decide whether a Hecke cusp form is new or not. More precisely, consider the operators K and H_N defined by

$$G|H_N = G \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \quad \text{and} \quad G|K(z) = \overline{G(-\bar{z})},$$

for each cusp form $G(z) = \sum_{n=1}^{\infty} B_n e^{2\pi i n z}$ of type (N, k, ε) . We have

PROPOSITION. (cf. [Li 75]). *Let $G \in S_k(N, \varepsilon)$ be a Hecke cusp form. Then G is a newform of type (N, k, ε) if and only if the functional equation $G|K|H_N = \gamma G$ holds for a certain complex constant γ of absolute value 1.*

Let us go back to the case of elliptic curves. Let F be the newform attached to the Weil curve E as above and let $N_E = Np^2$ be the conductor of E . Choose an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$ and let $\psi : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow \overline{\mathbb{Q}}$ be the Dirichlet character which satisfies

$$\psi(n)n \equiv 1 \pmod{\mathfrak{P}},$$

where $\mathfrak{P}$ is the place of $\overline{\mathbb{Q}}$ dividing p fixed by our embedding.

We ask for which values of $j \in \{0, \dots, p-2\}$ the twisted form $F \otimes \psi^j$ fails to be new. Prof. D. B. Zagier suggested to us to apply the following test which is an immediate consequence of Li's result.

COROLLARY. *Keep the above notations. If there exist a complex number $z \in \mathbb{H}$ such that*

$$\left| \frac{\sum_{n=1}^{\infty} \overline{A_n} \overline{\psi^j}(n) e^{-2\pi i n / N_E z}}{N_E z^2 \sum_{n=1}^{\infty} A_n \psi^j(n) e^{2\pi i n z}} \right| - 1 \neq 0,$$

then $F \otimes \psi^j$ is not new.

On a VAX 8600 at the Facultat d'Informàtica de Barcelona we have obtained the following numerical examples, by taking $z = 2i/\sqrt{N_E} \in \mathbb{H}$ and a few number (around 500) of Fourier coefficients for $F \otimes \psi^j$.

For the elliptic curve 338 A1 (cf. [Cre 91]),

$$E : y^2 + xy = x^3 - x^2 + x + 1$$

of conductor $N_E = 2 \cdot 13^2$, we get the following data:

j	TEST
0	0.0000000000
1	0.0000000000
2	2.6699959395
3	0.0000000000
4	0.0000000000
5	0.0000000000
6	0.0000000000
7	0.0000000000
8	0.0000000000
9	0.0000000000
10	2.6699959395
11	0.0000000000

Observe that the Hecke cusp forms $F \otimes \psi^2$ and $F \otimes \psi^{10}$ don't seem to be newforms. Indeed, as we shall see later, they are not newforms.

Another example is provided by the elliptic curve

$$E : y^2 + xy + y = x^3 - 39x - 27$$

of conductor $N_E = 43^2$ (cf. [Ed-Gr-To 90]). In this case we get:

$\begin{array}{c} j \\ 14 \\ 28 \end{array}$	$\begin{array}{c} \text{TEST} \\ 5.5513673695 \\ 5.5513673695 \end{array} ,$
--	--

and zero for all the others values of j . Now, the Hecke cusp forms $F \otimes \psi^{14}$ and $F \otimes \psi^{28}$ are not newforms.

In the previous examples E is vertical at $p = 13, 43$, respectively; indeed, in both cases we have $p \equiv 1 \pmod{3}$ and, following Tate's algorithm [Ta 75], we find that p -type(E) is equal to II and IV , respectively.

Actually, we are able to say what happens in the general p -vertical case. If ℓ denotes a prime which does not divide the conductor of E , then one can prove that the restriction to an inertia group I_p at p of the ℓ -adic representation ρ_ℓ attached to $F \otimes \psi^{\frac{p-1}{e}}$ is given by

$$\rho_\ell(I_p) = \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}.$$

Therefore, we obtain

PROPOSITION. (cf. [Ba-La 91]). *If E is a p -vertical Weil curve as above, there exists a newform*

$$G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p-1}{e}}),$$

having the same eigenvalue system as the twisted form $F \otimes \psi^{\frac{p-1}{e}}$.

Similar arguments do not work when E is p -horizontal. Consider the following example: the elliptic curve 605 A1, in [Cre 91],

$$E : y^2 + xy = x^3 - x^2 - 1414x - 44027$$

of conductor $N_E = 5 \cdot 11^2$ has 11-type IV^* . Since $11 \equiv 2 \pmod{3}$, E is 11-horizontal. Running our program we get:

j	TEST
0	0.0000000000
1	0.0000000000
2	0.0000000000
3	0.0000000000
4	0.0000000000
5	0.0000000000
6	0.0000000000
7	0.0000000000
8	0.0000000000
9	0.0000000000

Actually, all the twisted forms $F \otimes \psi^j$ are newforms.

3. Lowering the level (supersingular case).— Keep the notations as above and let $v^{ss} = \frac{p+1}{e}$. Since we are interested in Serre's conjecture (3.2.4?), we shall assume without loss of generality that p -type $(E) = II, III, IV$. Indeed, the exclusion of the cases with asterisk remain justified by considering the minimal or companion representations as in [La 91].

CONJECTURE (ss.?). *If E is a p -horizontal Weil curve as above, there exists a newform*

$$G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p+1}{e}}),$$

having the same eigenvalue system (mod $\mathfrak{P}$) as the twisted form $F \otimes \psi^{\frac{p+1}{e}}$; i.e., such that for all n prime to N_E we have

$$B_n \equiv A_n \psi^{v^{ss}}(n) \pmod{\mathfrak{P}}.$$

Moreover, G is $\mathfrak{P}$ -ordinary if and only if $\rho|_{D_p}$ is not irreducible. Here D_p denotes a decomposition group for p .

First of all, we are going to show a numerical example of the lowering of the level predicted by the conjecture.

Consider the elliptic curve E given by the Weierstraß model

$$y^2 + xy = x^3 + 3x + 1;$$

it is the curve 242 A1 in [Cre 91] and has conductor $N_E = 2 \cdot 11^2$. The special fibre of the Néron model over the local ring $\mathbb{Z}_{11}$ has reduction type II; therefore, E is 11-horizontal. Moreover, E has no $\mathbb{Q}$ -rational isogenies of degree 11, and then

$$\rho : G_{\mathbb{Q}} \rightarrow \text{Aut}(E_{11}) \simeq \text{GL}_2(\mathbb{F}_{11})$$

is irreducible; since $v_{11}(c_4) = 1$, we find that $\rho|_{D_{11}}$ is reducible.

Choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_{11}$ such that the character

$$\psi : (\mathbb{Z}/11\mathbb{Z})^* \rightarrow \overline{\mathbb{Q}}^*, \quad \psi(2) = e^{2\pi i/10},$$

satisfies

$$\psi(2) \equiv 2^{-1} \pmod{\mathfrak{P}_{11}},$$

where $\mathfrak{P}_{11}$ is the place of $\overline{\mathbb{Q}}$, over 11, determined by the embedding.

Our conjecture (ss.?) predicts the existence of a cusp form

$$G(q) = \sum B_n q^n \in S_2(22, \psi^4)$$

satisfying the congruences

$$B_n \equiv A_n \psi^2(n) \pmod{\mathfrak{P}_{11}}$$

for all odd integers n prime to 11.

In this case, we find

$$\dim S_2(22, \psi^4) = 1.$$

The Eichler-Selberg trace formula (cf. [Hij-Pi-She 90]) allows us to obtain the Fourier coefficients of the unique normalized newform of this type.

An efficient implementation of this formula is due to J. Quer; its program, written in UBASIC, find the first coefficients of the newform $G(q) = \sum_{n \geq 1} B_n q^n$ in $S_2(22, \psi^4)$:

$$\begin{aligned} B_3 &= -\zeta^4 - \zeta^2 - 2\zeta - 2 \\ B_5 &= -2\zeta^3 - 2 \\ B_7 &= -2\zeta^4 - 2\zeta^3 - 4\zeta^2 - 2\zeta \\ B_{13} &= -2\zeta^2 - 2\zeta - 2 \\ B_{17} &= -4\zeta^4 - 4\zeta^3 - 5\zeta^2 - 5\zeta - 4 \\ B_{19} &= -2\zeta^4 - 6\zeta^3 - 2\zeta^2 - 5\zeta - 5 \\ &\dots \end{aligned}$$

where $\zeta = e^{2\pi i/5}$. The coefficients B_n can be rewritten taking into account that $\cos \pi/5$ is the unique positive (double) root of the polynomial $16X^5 - 20X^3 + 5X + 1$ and that

$$i \cos \frac{\pi}{10} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} - \frac{\sqrt{5}-1}{4}.$$

On the other hand, computing the first coefficients of the L -series of the elliptic curve E , we find the following:

ℓ	2	3	5	7	11	13	17	19	23	29	31	37	41	43
A_ℓ	—	—2	—3	—2	—	—5	—3	—2	6	3	2	—7	—3	—8

Finally, since $\sqrt{5} \equiv 4 \pmod{11}$ and

$$\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} = e^{2\pi i/10} \equiv 2^{-1} \equiv 6 \pmod{\mathfrak{p}_{11}},$$

we get

ℓ	$\psi^2(\ell)$	B_ℓ	$A_\ell \psi^2(\ell) - B_\ell$ (mod $\mathfrak{p}_{11}$)
2	$\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$	—	—
3	$\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$	$-\frac{\sqrt{5}+1}{2}(\psi^4(3) + 1)$	0
5	$\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$	$(\sqrt{5} - 1)(\psi^8(5) + 1) - 2$	0
7	$\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$	$-(\sqrt{5} - 1)\psi^6(7) + 2$	0
11	—	—	—
13	$\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$	$-(\sqrt{5} - 1)\psi^2(13)$	0
17	$\cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$	$-\frac{\sqrt{5}+1}{2}\psi^8(17) + 1$	0
19	$\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$	$(2\sqrt{5} - 5)(\psi^4(19) + 1)$	0
23	1	$-(\sqrt{5} + 1)$	0
29	$\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$	$(\sqrt{5} - 5)\psi^6(29) + 2\sqrt{5}$	0
31	$\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$	$2\psi^2(31)$	0
37	$\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$	$6\psi^6(37) - 3(\sqrt{5} + 1)$	0
41	$\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$	$\frac{7-5\sqrt{5}}{2}(\psi^4(41) + 1)$	0
43	1	$\frac{3}{2}(3\sqrt{5} + 1)$	0
47	$\cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$	$(2\sqrt{5} - 6)(\psi^4(47) + 1)$	0

Since a modular form of given weight and level (mod p) cannot start with a very high power of q (cf. [Gr 90], pag.499), note that in this case we have truly proved the congruences $B_n \equiv A_n \psi^2(n) \pmod{\mathfrak{p}_{11}}$ for all n .

4. Galois representations attached to Weil curves. —

We deduce Serre's conjecture (3.2.4?) for the remaining case of p -horizontal Weil curves from our conjecture (ss.?).

THEOREM. *Assume conjecture (ss.?) is true. Then Serre's conjecture (3.2.4?) is true for all irreducible representations*

$$\rho : G_{\mathbb{Q}} \rightarrow \text{Aut}(E_p)$$

provided that E is a p -horizontal Weil curve as above.

PROOF. Consider the twisted representation

$$\rho\left(\frac{p+1}{e}\right) := \rho \otimes \chi^{-\frac{p+1}{e}},$$

where χ is the p th cyclotomic character. It is easy to see that if $\rho\left(\frac{p+1}{e}\right)$ satisfies Serre's conjecture so does ρ .

The reason to consider $\rho\left(\frac{p+1}{e}\right)$ instead of ρ is that the invariant weight for $\rho\left(\frac{p+1}{e}\right)$ is less than $p+1$. Namely (cf. [Ba-La 91]),

$$k_{\rho\left(\frac{p+1}{e}\right)} = \begin{cases} \frac{p+1}{e}(e-2) & \text{if } \rho|_{D_p} \text{ reduces;} \\ p+1-2\frac{p+1}{e} & \text{if } \rho|_{D_p} \text{ is irreducible.} \end{cases}$$

Let $G(z) = \sum B_n e^{2\pi i n z} \in S_2(Np, \psi^{2\frac{p+1}{e}})$ be the newform attached to E as in conjecture (ss.?). We have (cf. Lemma 4, [Ba-La 91]) that

$$\text{Tr}(G E_{1,\psi}^{-2\frac{p+1}{e}+(p-1)}) \in S_{k_{\rho\left(\frac{p+1}{e}\right)}}(N, 1)$$

with $\text{Tr}(G E_{1,\psi}^{-2\frac{p+1}{e}+(p-1)}) \equiv G \pmod{\mathfrak{P}}$ if $\rho|_{D_p}$ is reducible, and

$$\text{Tr}(G E_{1,\psi}^{-2\frac{p+1}{e}+2(p-1)}) \in S_{k_{\rho\left(\frac{p+1}{e}\right)}}(N, 1)$$

with $\text{Tr}(G E_{1,\psi}^{-2\frac{p+1}{e}+2(p-1)}) \equiv G \pmod{\mathfrak{P}}$ if $\rho|_{D_p}$ is irreducible, where Tr denotes the trace operator and $E_{1,\psi}$ denotes the Eisenstein series of weight one attached to ψ .

Now, we see that the twisted representation $\rho\left(\frac{p+1}{e}\right)$ arises from a Hecke cusp form $(\text{mod } p)$ of type $(N, k_{\rho\left(\frac{p+1}{e}\right)}, 1)$. If necessary, since N is prime to p , the results in [Jo-Li 89] bring the level N to N_p .

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