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The boundary values of generalized Dirichlet series and a problem of Chebyshev

J. KACZOROWSKI*

1. Introduction and statement of results

In 1853 Chebyshev asserted in a letter to M. Fuss that there are more primes $p \equiv 3 \pmod{4}$ than $p \equiv 1 \pmod{4}$. S. Knapowski and P. Turán in their well-known series of papers on comparative prime number theory [5] write, after quoting Littlewood's result that $\pi(x, 4, 1) - \pi(x, 4, 3)$ changes sign infinitely many times as $x \rightarrow \infty$, the following lines: *one feels that Chebyshev's vague formulation could also be interpreted so as*

$$(1.1) \quad \lim_{Y \rightarrow \infty} N(Y)/Y = 0,$$

where $N(Y)$ denotes the number of integers $m \leq Y$ with the property

$$(1.2) \quad \pi(m, 4, 1) \geq \pi(m, 4, 3)$$

(cf. also [6], page 26). They support this conjecture by referring to Shanks [7], who found that (1.2) is not fulfilled for $m \leq 26860$, is then fulfilled for $m = 26861$ and $m = 26862$, and is again false for $26863 \leq m \leq 616768$. They also ask the following general question ([5], Problem 7).

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For fixed positive integers a, b and q such that $(a, q) = (b, q) = 1$, $a \not\equiv b \pmod{q}$, what is the asymptotical behaviour of $N_{a,b}(Y)$ for $Y \rightarrow \infty$, where $N_{a,b}(Y)$ denotes the number of integers $m \leq Y$ with

$$\pi(m, q, a) \geq \pi(m, q, b) \quad ?$$

Our aim is to prove a general result concerning boundary values of Dirichlet series and to show its relevance to Chebyshev's problem. As a corollary we obtain the following theorem.

THEOREM 1. Suppose a and q are positive integers satisfying $(a, q) = 1$, $a \not\equiv 1 \pmod{q}$ and let the Generalized Riemann Hypothesis (G.R.H.) be true for Dirichlet L -series $\pmod{q}$. Then there exist two constants $0 < c_1 < c_2 < 1$ such that the inequalities

$$c_1 Y \leq N_{a,1}(Y) \leq c_2 Y$$

hold for all sufficiently large Y .

This shows that the Knapowski-Turán conjecture (1.1) is false at least when we accept the G.R.H.

The basic tool used in the proof of Theorem 1 is a result concerning generalized Dirichlet series which seems to be of an independent interest. For the sake of brevity, let $\mathcal{A}$ denote the set of all functions

$$(1.3) \quad F(z) = \sum_{n=1}^{\infty} a_n e^{i w_n z}, \quad z = x + iy, \quad y > 0$$

satisfying the following conditions:

1. $0 \leq w_1 < w_2 < \dots$ are real numbers.
2. $a_n \in \mathbb{C}$, $n = 1, 2, 3, \dots$
3. There exists a non-negative integer B such that

$$(1.4) \quad \sum_{n=2}^{\infty} |a_n| w_n^{-B} < \infty.$$

4. There exists a non-negative number L_0 such that for every x , $|x| \geq L_0$, the limit

$$P(x) = \lim_{y \rightarrow 0+} \operatorname{Re} F(x + iy)$$

exists and represents a locally bounded function of $x \in \mathbb{R} \setminus [-L_0, L_0]$.

Moreover, let

$$\alpha(F) = \inf_{\substack{y>0 \\ x \in \mathbb{R}}} \operatorname{Re} F(x + iy), \quad \beta(F) = \sup_{\substack{y>0 \\ x \in \mathbb{R}}} \operatorname{Re} F(x + iy).$$

It was proved in [4] that if $F \in \mathcal{A}$ and $\alpha(F) < u < \beta(F)$ then there exists a positive number $l = l(u, F)$ such that

$$(1.5) \quad \inf_{x \in I} P(x) < u < \sup_{x \in I} P(x)$$

for every interval $I \subset \mathbb{R} \setminus [-L_0, L_0]$ of length $\geq l$.

This result is of importance to the prime number theory being a substitute for Ingham's method [1], [2]. Now we impose somewhat stronger conditions on F and we estimate the measure of the set of x satisfying (1.5).

THEOREM 2. *Let $F \in \mathcal{A}$ and suppose that*

$$(1.6) \quad \|P\|^2 = \sup_{|t| > L_0 + 1} \int_0^1 |P(x + t)|^2 dx < \infty.$$

Then for every real number u satisfying $\alpha(F) < u < \beta(F)$ there exist positive constants $l = l(u, F)$ and $d_1 = d_1(u, F)$ such that

$$(1.7) \quad |\{x \in I : P(x) > u\}| \geq d_1$$

and

$$(1.8) \quad |\{x \in I : P(x) < u\}| \geq d_1$$

for every interval $I \subset \mathbb{R} \setminus [-L_0, L_0]$ of length $\geq l$ (where $|A|$ denotes the Lebesgue measure of a set $A \subset \mathbb{R}$).

We apply this theorem to the function

$$(1.9) \quad \begin{aligned} F_{a,b}(z) = & -2e^{-z/2} \frac{1}{\phi(q)} \sum_{x \pmod{q}} (\overline{\chi(a)} - \overline{\chi(b)}) K(z, \chi') \\ & - \frac{2}{\phi(q)} \sum_{x \pmod{q}} (\overline{\chi(a)} - \overline{\chi(b)}) m(\tfrac{1}{2}, \chi), \end{aligned}$$

where $q \geq 2$, $0 < a, b < q$, $(a, q) = (b, q) = 1$, $a \not\equiv b \pmod{q}$ are integers, K denotes the K -function as introduced in [3]:

$$K(z, \chi') = \sum_{\gamma > 0} e^{\rho z}, \quad z = x + iy, \quad y > 0$$

(the summation being taken over all non-trivial $L(s, \chi')$ zeros ρ with positive imaginary parts γ); χ' is the primitive Dirichlet character induced by χ , and $m(\frac{1}{2}, \chi)$ is the multiplicity of a zero of $L(s, \chi)$ at $s = \frac{1}{2}$ (we put $m(\frac{1}{2}, \chi) = 0$ when $L(s, \chi) \neq 0$). We obtain the following corollaries.

COROLLARY 1. *Suppose the G.R.H. is true for Dirichlet L -functions (mod q). Then for every real number u satisfying $\alpha(F_{a,b}) < u < \beta(F_{a,b})$ there exist positive constants $c_0 = c_0(u, q)$ and $d_0 = d_0(u, q)$ such that*

$$(1.10) \quad \left| \left\{ T \leq t \leq c_0 T : \psi(t, q, a) - \psi(t, q, b) > u\sqrt{t} \right\} \right| \geq d_0 T$$

and

$$(1.11) \quad \left| \left\{ T \leq t \leq c_0 T : \psi(t, q, a) - \psi(t, q, b) < u\sqrt{t} \right\} \right| \geq d_0 T$$

for sufficiently large T .

COROLLARY 2. *Suppose the G.R.H. is true for Dirichlet L -functions (mod q) and let $(a, q) = 1$, $a \not\equiv 1 \pmod{q}$. Then for every positive u there exist $c_1 = c_1(u, q) > 0$ and $d_1 = d_1(u, q) > 0$ such that*

$$(1.12) \quad \# \{ Y \leq m \leq c_1 Y : \psi(m, q, a) - \psi(m, q, 1) > u\sqrt{m} \} \geq d_1 Y,$$

$$(1.13) \quad \# \{ Y \leq m \leq c_1 Y : \psi(m, q, a) - \psi(m, q, 1) < -u\sqrt{m} \} \geq d_1 Y,$$

$$(1.14) \quad \# \{ Y \leq m \leq c_1 Y : \pi(m, q, a) - \pi(m, q, 1) > u\sqrt{m}/(\log m) \} \geq d_1 Y,$$

and

$$(1.15) \quad \# \{ Y \leq m \leq c_1 Y : \pi(m, q, a) - \pi(m, q, 1) < -u\sqrt{m}/(\log m) \} \geq d_1 Y,$$

for all sufficiently large Y .

Let us remark that our Theorem 1 follows at once from Corollary 2; it is sufficient therefore to prove this corollary only.

Applying Theorem 2 to the function

$$\begin{aligned} F_a(z) = & -2e^{-z/2} \frac{1}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)} K(z, \chi') \\ & - \frac{2}{\phi(q)} \sum_{\chi \pmod{q}} \overline{\chi(a)} m(\tfrac{1}{2}, \chi), \\ & (z = x + iy, \quad y > 0, \quad (a, q) = 1) \end{aligned}$$

in place of $F_{a,b}$, we can prove results analogous to Corollaries 1 and 2 for the remainders $\psi(t, q, a) - t/\phi(q)$, $\psi(m, q, 1) - m/\phi(q)$, and $\pi(m, q, 1) - \text{li } x/\phi(q)$.

2. Proof of Theorem 2

For a real $\delta > 0$ we consider the subsidiary function

$$F_\delta(z) = \sum_{n=1}^{\infty} a_n S^N(\delta w_n) e^{i w_n z}, \quad z = x + iy, \quad y > 0,$$

where

$$S(\nu) = \begin{cases} (\sin \nu)/\nu, & \nu \neq 0, \\ 1, & \nu = 0 \end{cases}$$

and $N = B + 2$. Since $S(\nu) \leq \min(1, 1/|\nu|)$, the sum F_δ is absolutely convergent for $y \geq 0$. Moreover, $F_\delta \rightarrow F$ as $\delta \rightarrow 0$ almost uniformly on the upper half-plane and thus $\alpha(F_\delta) \rightarrow \alpha(F)$ and $\beta(F_\delta) \rightarrow \beta(F)$ as $\delta \rightarrow 0$. Let us fix a δ_0 , $0 < \delta_0 < \frac{1}{2}$, so small that $\alpha(F_{\delta_0}) < u < \beta(F_{\delta_0})$.

From (1.4) it follows that the sum in (1.3) absolutely and uniformly converges in every closed half-plane $y \geq y_0$ with $y_0 > 0$. Hence we can integrate $F(z)$ term by term. Thus for $y > 0$

$$F_{\delta_0}(z) = \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \cdots \int_{-\delta_0}^{\delta_0} F(z + t_1 + \dots + t_N) dt_1 \dots dt_N.$$

We take real parts and make $y \rightarrow 0+$. Using the Lebesgue bounded integration theorem we obtain

$$(2.1) \quad \text{Re } F_{\delta_0}(x) = \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \cdots \int_{-\delta_0}^{\delta_0} P(x + t_1 + \dots + t_N) dt_1 \dots dt_N$$

for $|x| > L_0 + N\delta_0$.

Let us consider now the following two cases.

Case 1. $\alpha(F)\beta(F) < 0$. Then of course $\alpha(F) < 0$ and $\beta(F) > 0$. Obviously it suffices to prove (1.7) for positive u only. Moreover, (1.8) follows from (1.7) by considering $-F$ instead of F . Let hence u be positive and let us fix u_1 satisfying

$$(2.2) \quad u < u_1 < \alpha(F_{\delta_0}).$$

$\text{Re } F_{\delta_0}(x)$ is almost periodic in the sense of Bohr. Hence there exists a positive constant $l_1 = l_1(u_1, F, \delta_0)$ such that every interval of length $\geq l_1$ contains a real number x such that

$$(2.3) \quad \operatorname{Re} F_{\delta_0}(x) \geq u_1.$$

Let now $I \subset \mathbb{R} \setminus [-L_0, L_0]$ be an interval of length $\geq l_1 + N\delta_0$. Let x_0 be its middle point and let x satisfying (2.3) be such that $|x - x_0| \leq l_1/2$. Let

$$\mathbf{A} = \{t \in I : P(t) > u\}$$

$$\mathbf{B} = \{(t_1, \dots, t_N) : |t_j| < \delta_0 \ (j = 1, 2, \dots, N), \ x + t_1 + \dots + t_N \in \mathbf{A}\}$$

$$\mathbf{C} = [-\delta_0, \delta_0]^N \setminus \mathbf{B}.$$

Using (2.1) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} u_1 &\leq \operatorname{Re} F_{\delta_0}(x) \\ &= \frac{1}{(2\delta_0)^N} \left(\int_{\mathbf{B}} \dots \int + \int_{\mathbf{C}} \dots \int \right) P(x + t_1 + \dots + t_N) dt_1 \dots dt_N \\ &\leq u + \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \dots \int_{-\delta_0}^{\delta_0} \int_{\substack{\mathbf{A} - (x+t_1+\dots+t_{N-1}) \\ |t_N| < \delta_0}} P(x + t_1 + \dots + t_N) dt_N \dots dt_1 \\ &\leq u + \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \dots \int_{-\delta_0}^{\delta_0} \int_{\substack{\mathbf{A} - (x+t_1+\dots+t_{N-1}) \\ |t_N| < \delta_0}} |\mathbf{A}|^{\frac{1}{2}} \|P\| dt_1 \dots dt_N \\ &= u + |\mathbf{A}|^{\frac{1}{2}} \|P\| / (2\delta_0). \end{aligned}$$

Hence

$$|\mathbf{A}| \geq \left(2\delta_0(u_1 - u) \|P\|^{-1} \right)^2.$$

Hence it is enough to take

$$d(u, F) = \left(2\delta_0(u_1 - u) \|P\|^{-1} \right)^2$$

and

$$(2.4) \quad l(u, F) = l_1(u_1, F, \delta_0) + N\delta_0.$$

Case 2. $\alpha(F)\beta(F) \geq 0$. Replacing if necessary F by $-F$ we can assume that $\alpha(F) \geq 0$. Then (1.7) can be proved exactly in the same way as in Case 1. To prove (1.8) let us fix u_1 satisfying $\max(\alpha(F), \alpha(F_{\delta_0})) < u_1 < u$. Let l_1 , I and x have the same meaning as previously with (2.3) replaced by

$$(2.3)' \quad \operatorname{Re} F_{\delta_0}(x) \leq u_1.$$

Let moreover

$$\mathbf{A}_1 = \{t \in I : P(t) < u\}$$

$$\mathbf{B}_1 = \{(t_1, \dots, t_N) : |t_j| < \delta_0 \ (j = 1, 2, \dots, N), \ x + t_1 + \dots + t_N \in \mathbf{A}_1\}$$

$$\mathbf{C}_1 = [-\delta_0, \delta_0]^N \setminus \mathbf{B}_1.$$

Then

$$\begin{aligned} u_1 &\geq \operatorname{Re} F_{\delta_0}(x) \geq \frac{1}{(2\delta_0)^N} \int_{-\delta_0}^{\delta_0} \dots \int_{-\delta_0}^{\delta_0} P(x + t_1 + \dots + t_N) dt_1 \dots dt_N \\ &\geq u(2\delta_0)^{-N} \mu(\mathbf{C}_1) \\ &= u(2\delta_0)^{-N} ((2\delta_0)^N - \mu(\mathbf{B}_1)), \end{aligned}$$

μ being the N -dimensional Lebesgue measure. Hence

$$\mu(\mathbf{B}_1) \geq (2\delta_0)^N (1 - u_1/u).$$

But

$$\begin{aligned} \mu(\mathbf{B}_1) &= \int_{-\delta_0}^{\delta_0} \dots \int_{-\delta_0}^{\delta_0} \int_{\substack{\mathbf{A}_1 - (x + t_1 + \dots + t_{N-1}) \\ |t_N| < \delta_0}} dt_N \dots dt_1 \\ &\leq |\mathbf{A}_1| (2\delta_0)^{N-1} \end{aligned}$$

and consequently

$$|\mathbf{A}_1| \geq 2\delta_0(1 - u_1/u).$$

We obtain (1.8) with the same $l(u, F)$ as in (2.4) and

$$d_1(u, F) = 2\delta_0(1 - u_1/u).$$

The proof is complete.

3. Proof of the corollaries

We apply Theorem 2 to the function $F_{a,b}$ defined by (1.9). It belongs to the class $\mathcal{A}$; condition 4. is satisfied with $L_0 = 0$ (cf. [3]). Condition (1.6) can be proved as follows. For positive y we have by term-by-term integration

$$\begin{aligned} & \int_0^1 |\operatorname{Re} F_{a,b}(x+t+iy)|^2 dx \\ & \ll 1 + \sum_{\gamma>0} \sum_{\gamma'>0} \frac{1}{\gamma} \frac{1}{\gamma'} e^{-(\gamma+\gamma')y} \left| \int_0^1 e^{i(\gamma-\gamma')x} dx \right| \\ & \ll 1 + \sum_{\gamma>0} \sum_{\gamma'>0} \frac{1}{\gamma} \frac{1}{\gamma'} \min(1, |\gamma - \gamma'|^{-1}) \ll 1 \end{aligned}$$

uniformly in $t \in \mathbb{R}$ (γ and γ' denote imaginary parts of non-trivial zeros of all Dirichlet L -functions (mod q)); (1.6) therefore follows making $y \rightarrow 0+$ and using the Lebesgue bounded integration theorem. Finally it is proved in [4], page 242, that

$$P(x) = e^{-x/2}(\psi(e^x, q, a) - \psi(e^x, q, b)) + O(xe^{-x/2}).$$

Hence (1.10) and (1.11) follow from (1.7) and (1.8) by the change of variable $t = e^x$; this proves Corollary 1.

To prove Corollary 2 observe that by (3.3), (4.3) and (8.11) of [3] we have

$$\operatorname{Re} F_{1,a}(re^{i\phi}) = \frac{1}{\pi}(\phi - \pi/2) \log r + O(1) \quad \text{for } 0 < r < 1, 0 < \phi < \pi,$$

and hence $\alpha = -\infty$ and $\beta = +\infty$. Using this, Corollary 1 and the obvious remark that $\psi(t, q, a) - \psi(t, q, 1) = \psi([t], q, a) - \psi([t], q, 1)$ we obtain (1.12) and (1.13). Inequalities for $\pi(x, q, a) - \pi(x, q, 1)$ follow from what we have just proved and the partial summation.

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