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ON THE MAGNITUDE OF THE LEAST PRIMITIVE ROOT

by

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1. Let p be an odd prime number. We define

$g(p)$ = the least positive integer which is a primitive root mod p,
$G(p)$ = the least prime which is a primitive root mod p.

In most cases, $g(p)$ are very small. For example, among the 19862 odd primes ≤ 223051, $g(p) = 2$ happens for 7429 primes (37.4 %), $g(p) = 3$ happens for 4518 primes (22.8 %), and $g(p) \leq 6$ holds for about 80 % of these primes. And we can support this fact by a probabilistic argument. In fact, for a given prime p, there are $p-1$ invertible residue classes, among which $\varphi(p-1)$ residue classes are primitive modulo p, where φ denotes Euler's totient function. Therefore, on the assumption of good distribution of the primitive residue classes mod p, we can surmise that,

(1) for almost all prime $p, g(p)$ is not very far from $\dfrac{p-1}{\varphi(p-1)} + 1$.

The function $(p-1)/\varphi(p-1)$ fluctuates irregularly, but we can prove the asymptotic formula :

$$\pi(x)^{-1} \sum_{\substack{p \leq x \\ p:\text{prime}}} \frac{p-1}{\varphi(p-1)} = C + O\left(\frac{\log\log x}{\log x}\right), C = \prod_{p:\text{prime}} \left(1 + \frac{1}{(p-1)^2}\right) \doteq 2.827.$$

So, we can guess that

(2) for almost all prime p, $\dfrac{p-1}{\varphi(p-1)}$ is not very far from the constant C ,

and, combining (1) and (2), we can expect that,

(3) for almost all p, $g(p)$ is not very far from the constant $C+1$.

So, it seems very natural to conjecture that, for any monotone increasing positive function $\psi(x)$ tending to $+\infty$, we have an estimate

$$|\{p \leq x \ ; \ g(p) > \psi(p)\}| \ = \ o(\pi(x)). \tag{4}$$

In this direction, we have already a lot of results :

- BURGESS [1] : $g(p) \ll p^{(1/4)+\varepsilon}$, for any $\varepsilon > 0$,
- WANG [12] : under the assumption of the Generalized Riemann Hypothesis (G.R.H.),

$$g(p) \ll (\log p)^2 \omega(p-1)^6 \ ,$$

where $\omega(n)$ denotes the number of distinct prime divisors of n.

- If we take $\psi(x) = C$, the constant function, then we can prove, from MATTHEWS' result about ARTIN's conjecture [10] that, under G.R.H. ,

$$|\{p \leq x \ ; \ g(p) > C\}| \ = \ A_c \pi(x) + o(\pi(x)) \ ,$$

where A_c is a positive constant depending on C, with $0 < A_c \leq 1$.

The last result shows that our conjecture (4) does not hold for the constant function. So, we are interested in the problem, when $\psi(x)$ is a function tends to $+\infty$ rather slowly, is our conjecture (4) true or not ?

Our first result shows that our conjecture is true, under the assumption of G.R.H..

THEOREM 1. ([11]). *We assume G.R.H.. Let $\psi(x)$ be a monotone increasing positive function with the properties*

$$\lim_{x\to\infty} \psi(x) = +\infty, \psi(x) \ll (\log x)^A \text{ for some } A > 0, \psi(x) \ll \psi(x(\log x)^{-1}).$$

Then we have

$$|\{p \leq x \ ; \ G(p) > \psi(p)\}| \ \ll \ \pi(x)(\log \psi(x))^{-1}.$$

This is a result about $G(p)$, but the trivial inequality $g(p) \leq G(p)$ implies that the same estimate still holds for $g(p)$, which verifies (4).

To clarify the contents of our theorem, we take, for example, $\psi(x) = \log\log x$. Then we have $g(p) \leq G(p) \leq \log\log p$, except for $O\left(\frac{\pi(x)}{\log\log\log x}\right)$ primes, whose density is zero.

2. Here we consider the average value of $g(p)$.

It is already proved in 1967 by BURGESS-ELLIOTT [2] that

$$\pi(x)^{-1}\sum_{p\leq x} g(p) \ll (\log x)^2(\log\log x)^4.$$

We can improve this estimate, under G.R.H. , as follows :

THEOREM 2. ([8]). *We assume G.R.H.. Then we have, for any* $\varepsilon > 0$,

$$\pi(x)^{-1}\sum_{p\leq x} g(p) \leq \pi(x)^{-1}\sum_{p\leq x} G(p) \ll (\log x)(\log\log x)^{1+\varepsilon}.$$

Making use of the same argument, we have the following corollary. Let $n_2(p)$ be the least quadratic non-residue mod p, MONTGOMERY proved in 1971 that, under G.R.H., $n_2(p) = \Omega((\log p)(\log\log p))$.

(Remark. Very recently, GRAHAM and RINGROSE proved unconditionally that $n_2(p) = \Omega((\log p)(\log\log\log p))$ cf.[9]). Since $g(p) \geq n_2(p)$, under G.R.H. we have

$$g(p) = \Omega((\log p)(\log\log p)) . \tag{5}$$

Now, we can prove that the primes which satisfy the inequality (5) are rather exceptional :

COROLLARY. *We assume G.R.H.. Let B be an arbitrary positive constant, then we have, for any* $\varepsilon > 0$,

$$|\{p \leq x \ ; \ g(p) \geq B(\log p)(\log\log p)\}| \ll \pi(x)(\log x)^{(-1/2+\varepsilon)},$$

where the constant implied by the $\ll$-symbol depends only on B and ε.

3. We want to think about our problem from a little different point of view. We define

$n_k(p)$ = the least positive integer which is not a k-th power residue mod p,

$r_k(p)$ = the least prime which is a k-th power residue mod p,

then, $n_k(p)$ and $r_k(p)$ have the similar property as $g(p)$ and $G(p)$, respectively. In fact, among $p-1$ invertible residue classes mod p, there are $(1-k^{-1})(p-1)$ classes which are not k-th power residue mod p, and, on the assumption of good distribution of these classes, we can expect that $n_k(p)$ is not very far from the constant $k(k-1)^{-1}+1$, etc. Concerning $n_k(p)$ and $r_k(p)$, more than twenty years ago, ELLIOTT obtained the following asymptotic relations (cf.[3], [4] , see also [5], [6], [7]) :

- If $\delta < 4\ \exp(1-k^{-1})$, then

$$\pi(x)^{-1}\sum_{p\leq x} n_k(p)^\delta \ = \ C_{k,\delta}+o(1), \quad \text{as } x\to+\infty,$$

where $C_{k,\delta}$ is a constant depending only on k and δ.

- If $\delta < 4$, then

$$\pi(x)^{-1}\sum_{p\leq x} r_2(p)^\delta \ = \ D_\delta + O\left(\exp\left(-D\frac{\log\log x}{\log\log\log x}\right)\right)\ ,\ D>0\ ,$$

where D_δ is a constant depending on δ.

- If $k\geq 3$, then there exists a constant $\delta(k)<1$, and for any $\delta<\delta(k)$,

$$\pi(x)^{-1}\sum_{p\leq x} r_k(p)^\delta \ = \ D_{k,\delta}+o(1)\ ,\ \text{as } x\to+\infty. \tag{6}$$

where $D_{k,\delta}$ is a constant depending only on k and δ.

Therefore it seems very natural to seek the same asymptotic formula for the averages of $g(p)^\delta$ and $G(p)^\delta$. And actually, we have

THEOREM 3. ([8]). *We assume G.R.H.. If $\delta<\frac{1}{2}$, then we can prove the asymptotic relation :*

$$\begin{cases} \pi(x)^{-1}\sum_{p\leq x} g(p)^\delta = E_\delta+o(1)\ , \\ \pi(x)^{-1}\sum_{p\leq x} G(p)^\delta = E'_\delta+o(1)\ , \end{cases} \tag{7}$$

where E_δ and E'_δ are constants depending only on δ.

So, in some sense, by Theorem 3 we arrived at the same stage with (6) under the assumption of G.R.H..

The asymptotic relations (7) are likely to be true for $\delta = 1$, but it seems very difficult to prove it, if we assume G.R.H. only.

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