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Some remarks on equivariant bundles and classifying spaces

by J. P. May

Let Π be a normal subgroup of a topological group Γ with quotient group G; subgroups are understood to be closed. A principal $(\Pi;\Gamma)$-bundle is the projection to orbits $E \to E/\Pi$ of a Π-free Γ-space E. (Function spaces excepted, our Γ-spaces are to be of the homotopy type of Γ-CW complexes, and similarly for other groups.) For a G-space X, let $\mathcal{B}_G(\Pi;\Gamma)(X)$ denote the set of equivalence classes of principal $(\Pi;\Gamma)$-bundles over X. For a space X, let $\mathcal{B}(\Pi)(X)$ denote the set of equivalence classes of principal Π-bundles over X. Let X_G denote the Borel construction $EG \times_G X$ associated to a G-space X. We write

$$\mathcal{B}_G(\Pi;\Gamma)(EG \times X) = \mathcal{B}(\Pi;\Gamma)(X_G)$$

to emphasize that this set depends only on X_G as a space over BG. Equivalently, $\mathcal{B}(\Pi;\Gamma)(X_G)$ is the set of equivalence classes of free Γ-spaces P with a given equivalence $P/\Pi \cong EG \times X$ of G-bundles over $P/\Gamma \cong X_G$. We shall see that the calculation of this set reduces to a nonequivariant lifting problem, and we think of it as essentially a problem in ordinary nonequivariant bundle theory. In fact, in the classical case $\Gamma = G \times \Pi$, passage from P to P/G specifies a natural bijection

$$\Theta\colon \mathcal{B}(\Pi;G \times \Pi)(X_G) \to \mathcal{B}(\Pi)(X_G).$$

The projection $EG \times X \to X$ induces a natural map

$$\Psi\colon \mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G).$$

In the classical case, $\Phi = \Theta\Psi$ is just the Borel construction on bundles. One of our goals is to determine how near the passage Ψ from equivariant bundle theory to ordinary bundle theory is to being an isomorphism. For example, we shall obtain the following result, which is essentially just an exercise in covering space theory.

THEOREM 1. *If Γ is discrete, then Ψ: $\mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G)$ is a bijection for any G-space X. If Π (but not necessarily G) is discrete, then Φ: $\mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ is a bijection for any G-space X.*

We shall see that the following deeper result is a consequence of the Sullivan conjecture. The phrase "(strong) mod p equivalence" will be explained in due course.

THEOREM 2. *Let G be an extension of a torus by a finite p-group. If Γ is a compact Lie group, then the natural transformation Ψ: $\mathcal{B}_G(\Pi;\Gamma)(X) \to \mathcal{B}(\Pi;\Gamma)(X_G)$ is represented by a mod p equivalence of classifying G-spaces. Therefore, if Π is a compact Lie group, then Φ: $\mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ is represented by a mod p equivalence of classifying G-spaces. If G is a finite p-group, then the transformations Ψ and Φ are represented by strong mod p equivalences of classifying G-spaces.*

Restricting Π instead of G, we obtain the following theorem, which is the main result of [7].

THEOREM 3. *If G and Π are compact Lie groups with Π Abelian, then Φ: $\mathcal{B}_G(\Pi;G \times \Pi)(X) \to \mathcal{B}(\Pi)(X_G)$ is a bijection for any G-space X.*

In a preprint version of this paper, the following assertion was claimed as a theorem.

ASSERTION 4. *Under the hypotheses of theorem 3, there is also a natural bijection*

$$\mathcal{B}G(\Pi;G \times \Pi)(X) \cong \mathcal{B}(\Pi)(X/G) \times Nat(\pi_0(X), R_\pi).$$

The fact that this assertion is false was discovered by John Wicks, a student at Chicago, who showed that, with $\Pi = S^1$ and $G = Z_2$, it implies an incorrect calculation of characteristic classes. Since the nature of the assertion and the mistake in its proof may be of interest, we shall discuss these matters in an Appendix.

The three theorems above are direct interpretations of results about equivariant classifying spaces, namely Theorems 5, 9, and 10 below. There is a universal example $E(\Pi;\Gamma) \to B(\Pi;\Gamma)$ of a principal $(\Pi;\Gamma)$-bundle. Up to Γ-homotopy type, the Γ-space $E(\Pi;\Gamma)$ is characterized by the requirement that, for $\Omega \subset \Gamma$, the fixed point space $E(\Pi;\Gamma)^\Omega$ be contractible if $\Omega \cap \Pi = e$ and empty otherwise.

By universality, we have a natural bijection

$$(*)\qquad \mathcal{B}G(\Pi;\Gamma)(X) \cong [X, B(\Pi;\Gamma)]_G,$$

where homotopy classes of unbased G-maps are understood. In particular, we have natural bijections

$$\mathcal{B}G(\Pi;\Gamma)(EG \times X) \cong [EG \times X, B(\Pi;\Gamma)]_G \cong [X, Map(EG, B(\Pi;\Gamma))]_G.$$

Let $p: X_G \to BG$ be the evident bundle and let $q: \Gamma \to G$ be the quotient homomorphism. Let $[X_G, B\Gamma]/BG$ be the set of homotopy classes of maps $f: X_G \to B\Gamma$ such that $Bq \circ f = p$ and define

$\mathrm{Sec}(EG, B\Gamma)$ to be the G-space of maps $\varphi: EG \to B\Gamma$ such that $Bq\circ\varphi = p: EG \to BG$. A central idea in this paper is the modelling of classifying spaces by such spaces of sections. We introduce this idea by observing that the previous bijections are equivalent to

$$(\#) \qquad \mathcal{B}(\Pi;\Gamma)(X_G) \cong [X_G, B\Gamma]/BG \cong [X, \mathrm{Sec}(EG, B\Gamma)]_G.$$

This should be clear from the equivalent bundle theoretic descriptions of the left sides already given, but we want to see it directly on the classifying space level. Since $E\Gamma$ is Π-free, the universal property of $E(\Pi;\Gamma)$ gives a Γ-map $\nu: E\Gamma \to E(\Pi;\Gamma)$, unique up to Γ-homotopy. The Γ-map $(Eq,\nu): E\Gamma \to EG \times E(\Pi;\Gamma)$ is clearly a Γ-homotopy equivalence, where Γ acts through q on EG, and it is a fiber Γ-homotopy equivalence provided we choose a model for $E\Gamma$ such that $Eq: E\Gamma \to EG$ is a Γ-fibration. Passing to orbits over Γ by first passing to orbits over Π and then over G, we obtain a homotopy equivalence

$$B\Gamma \to EG \times_G B(\Pi;\Gamma) = B(\Pi;\Gamma)_G$$

over BG. (Lemma 11 at the end will generalize this equivalence.) We have an evident G-homeomorphism $\mathrm{Sec}(EG, X_G) \cong \mathrm{Map}(EG, X)$ for any G-space X, and there results a G-homotopy equivalence

$$\xi: \mathrm{Sec}(EG, B\Gamma) \to \mathrm{Sec}(EG, B(\Pi;\Gamma)_G) \cong \mathrm{Map}(EG, B(\Pi;\Gamma)).$$

Via the projection $EG \to pt$ and use of a chosen homotopy inverse to ξ, we obtain a G-map

$$\alpha: B(\Pi;\Gamma) \to \mathrm{Sec}(EG, B\Gamma)$$

which induces the transformation Ψ under the isomorphisms $(*)$ and $(\#)$. In order to prove Theorem 1, we model $E(\Pi;\Gamma)$ as a space of sections and use this model to obtain an explicit description of α. In the classical case $\Gamma = G \times \Pi$, we agree to abbreviate

$$E_G(\Pi) = E(\Pi;G \times \Pi) \quad \text{and} \quad B_G(\Pi) = B(\Pi;G \times \Pi);$$

here $B\Gamma = BG \times B\Pi$ and therefore $Sec(EG, B\Gamma) \cong Map(EG, B\Pi)$.

THEOREM 5. *Let Γ act through $q: \Gamma \to G$ on EG and by conjugation on the space $Sec(EG, E\Gamma)$ of maps $\varphi: EG \to E\Gamma$ such that $Eq \circ \varphi = id$. Then $Sec(EG, E\Gamma)$ satisfies the fixed point criteria characterizing $E(\Pi;\Gamma)$, hence the orbit space $Sec(EG, E\Gamma)/\Pi$ is a model for $B(\Pi;\Gamma)$. With this model, $\alpha: B(\Pi;\Gamma) \to Sec(EG, B\Gamma)$ is the G-map induced by $Sec(id, p)$, where $p: E\Gamma \to B\Gamma$ is the universal Γ-bundle. If Γ is discrete, then α is a homeomorphism. If $\Gamma = G \times \Pi$, then $Map(EG, E\Pi)$ is a model for $E_G(\Pi)$, $Map(EG, E\Pi)/\Pi$ is a model for $B_G\Pi$, $\alpha: B(\Pi;\Gamma) \to Map(EG, B\Pi)$ is induced by $p: E\Pi \to B\Pi$, and α is a homeomorphism if Π is discrete.*

When Γ is discrete, elementary covering space theory shows that any map $\varphi: EG \to B\Gamma$ such that $Bq \circ \varphi = p$ lifts to a section $\tilde{\varphi}: EG \to E\Gamma$ of Eq and that any two such lifts are in the same Π-orbit. The last homeomorphism is seen similarly, and Theorem 1 is an immediate consequence of these homeomorphisms.

To prove Theorem 5, we need a kind of topological analog of the standard comparison of projective and acyclic resolutions.

LEMMA 6. *Let G be a topological group, let X be a free G-CW complex, and let Y be a nonequivariantly contractible G-space. Then the space $Map_G(X,Y)$ of G-maps $X \to Y$ is contractible.*

PROOF. If $X = G \times K$ for a space K, then $\mathrm{Map}_G(X, Y) \cong \mathrm{Map}(K, Y)$ and the conclusion is clear. Since $\mathrm{Map}_G(?, Y)$ converts pushouts to pullbacks, G-cofibrations to fibrations, and colimits to limits, the conclusion follows in general by use of the cell structure on X.

PROOF OF THEOREM 5. Recall that we have a fiber Γ-homotopy equivalence $(Eq,\nu)\colon E\Gamma \to EG \times E(\Pi;\Gamma)$ over EG. Applying the functor $\mathrm{Map}(EG, ?)$ and restricting to the fiber over $\mathrm{id} \in \mathrm{Map}(EG, EG)$, we obtain a Γ-homotopy equivalence

$$\mathrm{Sec}(EG, E\Gamma) \to \mathrm{Map}(EG, E(\Pi;\Gamma)).$$

Let $\Omega \subset \Gamma$. Since EG is Π-trivial and $E(\Pi;\Gamma)$ is Π-free, there are no Ω-maps $EG \to E(\Pi;\Gamma)$ if $\Omega \cap \Pi \neq e$. If $\Omega \cap \Pi = e$, then Ω acts freely via q on EG while $E(\Pi;\Gamma)$ is Ω-contractible since $E(\Omega;\Gamma)^\Lambda$ is contractible for all $\Lambda \subset \Omega$. Therefore $\mathrm{Map}_\Omega(EG, E(\Pi;\Gamma))$ is contractible. The compatibility of $\mathrm{Sec}(\mathrm{id}, p)$ with the earlier map α is checked by an easy diagram chase.

To prove Theorem 2, we must first obtain a nonequivariant description of the fixed point maps α^H. At least if Γ is a Lie group, the fixed point structure of the G-space $B(\Pi;\Gamma)$ is given as follows [6, Thm 10]. Let $N_\Gamma\Omega$ and $Z_\Gamma\Omega$ be the normalizer and centralizer of Ω in Γ. If $\Omega \cap \Pi = e$, then an easy check shows that $\Pi \cap N_\Gamma\Omega = \Pi \cap Z_\Gamma\Omega$; we agree to write Π^Ω for this intersection.

THEOREM 7. *For $H \subset G$, $B(\Pi;\Gamma)^H = \coprod B\Pi^\Omega$, where the union runs over the Π-conjugacy classes of subgroups $\Omega \subset \Gamma$ such that $\Omega \cap \Pi = e$ and $q(\Omega) = H$; $B(\Pi;\Gamma)^H$ is empty if there are no such subgroups Ω.*

LEMMA 8. *For* $\Omega \subset \Gamma$ *such that* $\Omega \cap \Pi = e$ *and* $q(\Omega) = H$, *define* $\mu: H \times \Pi^{\Omega} \to \Gamma$ *by* $\mu(q(\lambda), \pi) = \lambda\pi$ *and note that* $q\circ\mu = i\circ\pi_1$. *The restriction of* α^H *to* $B\Pi^{\Omega}$ *is the adjoint of the classifying map*

$$B\mu: BH \times B\Pi^{\Omega} = B(H \times \Pi^{\Omega}) \to B\Gamma.$$

PROOF. Let $\tilde{\alpha}: EG \times E(\Pi;\Gamma) \to E\Gamma$ be a Γ-homotopy equivalence over EG inverse to (Eq,ν). Since the adjoint of α is obtained from $\tilde{\alpha}$ by passage to orbits and since $B\Pi^{\Omega} = E(\Pi;\Gamma)^{\Omega}/\Pi^{\Omega}$ as a subspace of $B(\Pi;\Gamma)$, it suffices to observe that the restriction of $\tilde{\alpha}$ to the free contractible $(H \times \Pi^{\Omega})$-space $EG \times E(\Pi;\Gamma)^{\Omega}$ is μ-equivariant:

$$\tilde{\alpha}(yq(\lambda), x\pi) = \tilde{\alpha}(yq(\pi\lambda), x\lambda\pi) = \tilde{\alpha}((y,x)\lambda\pi) = (\tilde{\alpha}(y,x))\lambda\pi$$

for $y \in EG$, $x \in E(\Pi;\Gamma)^{\Omega}$, $\lambda \in \Omega$, and $\pi \in \Pi^{\Omega}$.

Given this interpretation of α^H, Theorem 2 follows directly from the application of the Sullivan conjecture to the study of maps between classifying spaces given by Dwyer and Zabrodsky [3] and Notbohm [10]. We say that a map $f: X \to Y$ is a mod p equivalence if f induces an isomorphism on mod p homology. We say that f is a strong mod p equivalence if the following conditions hold.

(i) f induces an isomorphism $\pi_0(X) \to \pi_0(Y)$;

(ii) f induces an isomorphism $\pi_1(X,x) \to \pi_1(Y,f(x))$ for any $x \in X$;

(iii) f induces an isomorphism $H_*(\tilde{X}_x,Z_p) \to H_*(\tilde{Y}_{f(x)},Z_p)$ for any $x \in X$, where $\tilde{X}_x$ and $\tilde{Y}_{f(x)}$ are the universal covers of the components of X and Y containing x and $f(x)$.

We say that a G-map $f: X \to Y$ is a (strong) mod p equivalence if $f^H: X^H \to Y^H$ is a (strong) mod p equivalence for each $H \subset G$. The results of Dwyer and Zabrodsky and of Notbohm admit the following interpretation (their G and Π playing opposite roles from ours).

THEOREM 9. *If Γ is a compact Lie group and G is an extension of a torus by a finite p-group, then the G-map $\alpha\colon B(\Pi;\Gamma) \to Sec(EG, B\Gamma)$ is a mod p equivalence. If G is a finite p-group, then α is a strong mod p equivalence.*

When $\Gamma = G \times \Pi$, $Sec(BH, B\Gamma) = Map(BH, B\Pi)$ and the second statement is Dwyer and Zabrodsky's [3, 1.1] while the first result is Notbohm's [10,1.1]. When $G = Z_p$, the result is [3, 4.5]. The result for general extensions follows from the result for trivial extensions exactly as in the deduction of [3, 4.5] from [3, 4.4]. Incidentally, as observed by Notbohm [private communication], the components of α^H induce injections but not surjections on the fundamental groups of corresponding components when G is an extension of a non-trivial torus by a finite p-group.

Of course, Theorem 2 is a restatement of Theorem 9. Some discussion of the significance of the represented form of the result is in order. For G-spaces Y, [9] constructs a functorial "fundamental groupoid G-space πY" and a natural G-map $\chi\colon Y \to \pi Y$. For $H \subset G$, $\chi^H\colon Y^H \to (\pi Y)^H$ induces a bijection on components and an isomorphism between the fundamental groups of corresponding components, while each component of $(\pi Y)^H$ has trivial higher homotopy groups. For $y \in Y^G$, let $\tilde{Y}_y$ be the homotopy fibre of χ regarded as a based map with respect to the basepoints y and $\chi(y)$. Then $(\tilde{Y}_y)^H$ is the homotopy fibre of the restriction of χ^H to the component of Y^H containing y. Clearly $\tilde{Y}_y$ is G-simply connected, in the sense that all of its fixed point spaces are simply connected. We can p-adically complete G-simply connected (or G-nilpotent) G-spaces and characterize the completion in terms of the usual homological characterization of completion on H-fixed point spaces for all H [8]. If $f: Y \to Z$ is a strong mod p equivalence, then the map $\pi f: \pi Y \to \pi Z$ and the p-adic completions $\hat{f}_p: (\tilde{Y}_y)\hat{}_p \to (\tilde{Z}_{f(y)})\hat{}_p$ for $y \in Y^G$ are all G-homotopy equivalences and so induce bijections on application of the functor $[X,?]_G$.

The following result is the represented equivalent of Theorem 3 and was proven in [7]. (The maps studied in [7] were defined a bit differently, but an easy diagram chase gives the conclusion in the form stated.) Recall that a G-map $f: Y \to Z$ is said to be a weak G-equivalence if each $f^H: Y^H \to Z^H$ is a weak equivalence and that $f_*: [X, Y]_G \to [X, Z]_G$ is then a bijection for any G-CW complex X.

THEOREM 10. *If* Π *and* Γ *are compact Lie groups with* Π *Abelian, then* $\alpha: B_G(\Pi) \to Map(EG, B\Pi)$ *is a weak* G-*equivalence.*

As a final remark, we give an equivariant generalization of the usual Borel construction model for the classifying space of an extension.

LEMMA 11. *Let* $\Lambda \subset \Pi \subset \Gamma$, *where* Λ *and* Π *are normal subgroups of the topological group* Γ. *Then, as* (Γ/Π)-*spaces,*

$$B(\Pi;\Gamma) \simeq E(\Pi/\Lambda;\ \Gamma/\Lambda) \times_{\Pi/\Lambda} B(\Lambda;\Gamma).$$

PROOF. For $\Omega \subset \Gamma$, $\Omega \cap \Pi = e$ if and only if both $\Omega \cap \Lambda = e$ and $\Theta \cap (\Pi/\Lambda) = e$, where Θ is the image of Ω in Π/Λ. Therefore, as Γ-spaces,

$$E(\Pi;\Gamma) \simeq E(\Pi/\Lambda;\ \Gamma/\Lambda) \times E(\Lambda;\Gamma)$$

by the characteristic behavior on fixed point sets. Now pass to Π-orbits by first passing to Λ-orbits and then to (Π/Λ)-orbits.

APPENDIX

Let G and Π be compact Lie groups. A Π-bundle over X/G may be regarded as a G-trivial $(\Pi;G \times \Pi)$-bundle, and it determines a $(\Pi;G \times \Pi)$-bundle over X by pullback. This gives a natural map

$$\zeta\colon \mathcal{B}(\Pi)(X/G) \to \mathcal{B}_G(\Pi;G \times \Pi)(X).$$

When Π is Abelian, the false proof of Assertion 4 to be described here would show that ζ is a naturally split injection.

The complementary factor would be $\mathrm{Nat}(\pi_0(X), R_\pi)$, which we proceed to define. Let $\mathcal{O}$ be the topological category of orbit G-spaces G/H and G-maps between them. Let $h\mathcal{O}$ be its homotopy category. For any n and any G-space X, there is an evident contravariant functor $\pi_n(X)\colon h\mathcal{O} \to \mathrm{Sets}$ which sends G/H to $\pi_n(X^H)$. There is also a contravariant functor $R_\pi\colon \mathcal{O} \to \mathrm{Sets}$ which sends G/H to the set of Π-conjugacy classes of Lie group homomorphisms $H \to \Pi$; R_π factors through $h\mathcal{O}$ since homotopic homomorphisms lie in the same Π-conjugacy class by the Montgomery-Zippin theorem [2, 38.1]. Let $\mathrm{Nat}(\pi_0(X), R_\pi)$ be the set of natural transformations $\pi_0(X) \to R_\pi$.

A principal $(\Pi;G \times \Pi)$-bundle over G/H determines and is determined by an element of $R_\pi(G/H)$. A principal $(\Pi;G \times \Pi)$-bundle over X determines a natural transformation $\pi_0(X) \to R_\pi$ by pulling the bundle back along G-maps $G/H \to X$ which represent elements of $\pi_0(X^H)$. This gives a natural map

$$\rho\colon \mathcal{B}(\Pi;G \times \Pi)(X) \to \mathrm{Nat}(\pi_0(X), R_\pi).$$

When Π is Abelian, the false proof of Assertion 4 would show that ρ is a naturally split surjection. A left inverse λ would construct a global bundle over X from compatible bundles over the domains of the representative G-maps $G/H \to X$. Given λ, a natural bijection

$$\mathcal{B}(\Pi)(X/G) \times \mathrm{Nat}(\pi_0(X), R_\pi) \to \mathcal{B}G(\Pi;G \times \Pi)(X)$$

would be obtained by using the Abelian structure of Π to add bundles in the images of the transformations ζ and λ.

The following is the represented equivalent of Assertion 4.

ASSERTION 12. *There is a weak G-equivalence*

$$B\Pi \times K(R_\pi,0) \to Map(EG, B\Pi),$$

where G acts trivially on $B\Pi$.

To explain this assertion, we must say a bit about diagrams of G-spaces and about Eilenberg-MacLane G-spaces $K(\pi,0)$. Define an $\mathcal{O}$-space to be a continuous contravariant functor from $\mathcal{O}$ to the category of spaces; a map of $\mathcal{O}$-spaces is a natural transformation. A G-space X determines the $\mathcal{O}$-space ΦX specified by $(\Phi X)(G/H) = X^H$. Conversely, by Elmendorf [4, Thm1], an $\mathcal{O}$-space T determines a G-space ΨT and an $\mathcal{O}$-map $\varepsilon\colon \Phi\Psi T \to T$ such that each component map $\varepsilon\colon (\Psi T)^H \to T(G/H)$ is a homotopy equivalence. In particular, with $H = e$ and $T = \Phi X$, the G-map $\varepsilon\colon \Psi\Phi X \to X$ is a weak G-equivalence. With the evident notion of homotopy in the category of $\mathcal{O}$-spaces, a slight refinement of [3, Thm 2] gives an adjunction on the level of homotopy classes of maps

$$(\perp) \qquad [X, \Psi T]_G \cong [\Phi X, T]_{\mathcal{O}}$$

when X has the homotopy type of a G-CW complex.

A space Y is homotopically discrete if each of its components is contractible, that is, if the discretization map $\delta: Y \to \pi_0 Y$ is a homotopy equivalence. A G-space Y is homotopically discrete if each Y^H is homotopically discrete. These are the $K(\pi,0)$'s referred to above, where π is a continuous functor from $\mathcal{O}$ to discrete spaces or, equivalently, a functor from the homotopy category $h\mathcal{O}$ to sets. Given such a functor π, we can construct $K(\pi,0)$ by setting $K(\pi,0) = \Psi\pi$; $(\perp)$ and the discreteness of π then give

$$[X, K(\pi,0)]_G \cong [\Phi X, \pi]_{\mathcal{O}} \cong \mathrm{Nat}(\pi_0(X), \pi).$$

Since we obviously have $[X, B\Pi]_G \cong [X/G, B\Pi]$, it is now clear that Assertion 12 implies Assertion 4.

For a G-space X, the discretization maps of fixed point spaces specify an $\mathcal{O}$-map $\delta: \Phi X \to \pi_0(X)$, and application of Ψ therefore gives a natural G-map $X \simeq \Psi\Phi X \to K(\pi_0(X),0)$. It seems reasonable to expect this map to admit a section, but it usually doesn't. To obtain a section, it would suffice to obtain a right inverse $\pi_0(X) \to \Phi X$ to δ, but there is usually no such natural choice of basepoints of components of fixed point spaces. This train of thought leads to a

"PROOF OF ASSERTION 12". The intuition is that there should be such a section of δ when $X = \mathrm{Map}(EG, B\Pi)$. With the standard functorial construction of EG, we have the two continuous covariant functors B and B' from $\mathcal{O}$ to spaces specified on objects by $B(G/H) = EH/H$ and $B'(G/H) = EG/H$. We may identify ΦX with the contravariant functor $\mathrm{Map}(B', B\Pi)$. Therefore

(A) $$\mathrm{Map}(EG, B\Pi) \simeq \Psi\Phi\mathrm{Map}(EG, B\Pi) = \Psi\mathrm{Map}(B', B\Pi).$$

On the other hand, passage to classifying maps defines an $\mathcal{O}$-map $\beta\colon R_\pi \to \mathrm{Map}(B, B\Pi)$. By [7, Prop. 4],

$$B\colon \mathrm{Hom}(G, \Pi) \to [BG, B\Pi]$$

is a bijection. Therefore $\pi_0\mathrm{Map}(B, B\Pi) = R_\pi$ and we have a map

(B) $$\Psi\beta\colon K(R_\pi,0) \to \Psi\mathrm{Map}(B, B\Pi).$$

It seems reasonable to expect there to be a weak G-equivalence

(C) $$\Psi\mathrm{Map}(B, B\Pi) \simeq \Psi\mathrm{Map}(B', B\Pi).$$

Given this, $\Psi\beta$ would transport under the equivalences (A) and (C) to give the desired section

$$\lambda\colon K(R_\pi,0) \to \mathrm{Map}(EG, B\Pi).$$

Letting $\varsigma\colon B\Pi \to \mathrm{Map}(EG, B\Pi)$ be induced by the projection $EG \to pt$ and φ be the product on $\mathrm{Map}(EG, B\Pi)$ induced by the product on the topological Abelian group $B\Pi$, the composite

$$\varphi\circ(\varsigma,\lambda)\colon B\Pi \times K(R_\pi,0) \to \mathrm{Map}(EG, B\Pi)$$

would then be a weak G-equivalence (compare [7, p.173]).

In fact, (C) fails. The obvious way to try to prove (C) would be to exploit the equivalences $B(G/H) = EH/H \to EG/H = B'(G/H)$ induced by the inclusions $EH \to EG$. However, these equivalences fail to define a map $B \to B'$ of $\mathcal{O}$-spaces. The requisite naturality fails, as we see by taking $H = e$ and observing that the map from the point Ee into EG cannot be a G-map.

Assertion 12 would imply an incorrect calculation of the characteristic classes of principal $(\Pi, G \times \Pi)$-bundles in Bredon cohomology. For a commutative ring k, a k-module valued coefficient system is a contravariant functor from $h\mathcal{O}$ to the category of k-modules. Write $\mathrm{Ext}^{*}_{h\mathcal{O}}$ for the Ext functor in the resulting Abelian category of $h\mathcal{O}$-k-modules. For a contravariant functor π: $h\mathcal{O} \to$ Sets, let $k\pi$ denote the $h\mathcal{O}$-k-module obtained by letting $k\pi(G/H)$ be the free k-module generated by $\pi(G/H)$. Let G and Π be compact Lie groups with Π Abelian and let M be an $h\mathcal{O}$-k-module, where k is a commutative ring such that $H^{*}(B\Pi; k)$ is k-free. Then there is a universal coefficients spectral sequence converging from $H^{*}(B\Pi; k) \otimes_k \mathrm{Ext}^{*}_{h\mathcal{O}}(kR_{\pi}, M)$ to $H^{*}_{G}(B_G(\Pi); M)$ [11]. Assertion 12 would imply that $E_2 = E_{\infty}$ in this spectral sequence, and this conclusion is usually false.

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