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ON THE SPACE OF MAPS BETWEEN R-LOCAL CW COMPLEXES

by

D.J. Anick¹ and E. Dror Farjoun

1. Summary of Results and Notations

The papers [A1,A2] introduced and studied a differential graded Lie algebra (dgl) associated as a model to certain spaces. Building on that work, we construct in this note a simplicial skeleton for the space of pointed maps between two R-local simply-connected CW complexes ($R \subset \mathbb{Q}$). The construction entails two steps. First is the construction, in the category of dgl's, of a cosimplicial resolution and an associated "function complex" valid in a range of dimensions; and second is the connection with the topological mapping space via the above-mentioned models.

1.1. A function complex for dgl's. Let $R = \mathbb{Z}[(p-1)!]^{-1} \subset \mathbb{Q}$ for a prime p , and let L, M be free r -reduced dgl's over R having all generators in dimensions below rp ($r \geq 1$). We will construct a simplicial set, to be denoted $\underline{\text{hom}}(L, M)$, which serves in a range of dimensions as a function complex in the sense of Dwyer and Kan [DK]. Our construction is explicit, in terms of generators and differentials; it is something which could be implemented on a computer. When L and M arise as models for finite spaces X and Y , this means that a simplicial model for the pointed mapping space Y^X is computable in a range of dimensions.

1.2. The range of dimensions. When X and Y are R-local r -connected CW complexes ($r \geq 1$), whose dimensions m_X and m_Y are bounded above by m and by rp respectively ($m < rp$), we may associate to them the dgl models L_X and L_Y . Then Y^X has the d -type of $\underline{\text{hom}}(L_X, L_Y)$, where

$$d = \min(rp - 1, r + 2p - 3) - m .$$

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Beyond dimension d , $\underline{\text{hom}}(L_X, L_Y)$ is still defined, but its connection with the geometry becomes much hazier.

1.3. Relation to tame homotopy. In view of [D] and [DK], one may associate to a pair of tame spaces (S, T) a function complex in the category of simplicial Lazard algebras. This function complex is homotopy equivalent (as a simplicial set) with the pointed mapping space T^S . When T is not tame, however, it is not obvious how one would obtain information about T^S through this technique. The desire to handle the non-tame case motivated this paper. Instead of requiring spaces to be tame, we require them to be R -local, and we restrict the dimensions where their cells may occur.

(The referee has proposed that Dwyer's functor may be able to be specialized suitably to the category of r -connected simplicial sets generated in dimension $\leq m$. This specialization, call it S , might yield information about T^S when S belongs to CW_r^m . To accomplish this, one would attempt to use S in largely the same way that we have used L in this paper.)

1.4. Notations. We work over a fixed subring R of the rationals, and we denote by p the least non-inverted prime, i.e.,

$$p = \inf\{n \in \mathbb{Z}_+ \mid n^{-1} \notin R\}. \text{ In general, then, } \mathbb{Z}[(p-1)!]^{-1} \subseteq R \subseteq \mathbb{Q}.$$

As in tame homotopy, the relevant dimension ranges vary with a connectivity parameter r , where $r \geq 1$. Following [A1, A2] we introduce several categories.

- SS denotes the category of simplicial sets.
- TOP is the category of pointed topological spaces and pointed continuous maps.
- $CW_r^n(R)$ denotes the full subcategory of TOP , consisting of r -connected R -local CW complexes of dimension $\leq n$. "Dimension" means as an R -local cell complex, e.g., the local n -sphere belongs to $ObCW_r^n(R)$ even though it has topological dimension $n + 1$.
- $HoCW_r^n(R)$ is the category obtained from $CW_r^n(R)$ by collapsing (pointed) homotopy classes of maps.
- $DGL(R)$ is the category of connected dgL 's over R . A dgL is free if it is free as a Lie algebra (ignoring the differential); in this case we write it as $(L(V), \delta)$, where the R -module of

generators $V = \bigoplus_{i=1}^{\infty} V_i$ is free and positively graded, and the differential δ has degree -1 .

- $DGL_r^m(R)$ denotes the full subcategory of $DGL(R)$ whose objects have the form $(L(V), \delta)$ where $V = \bigoplus_{i=r}^m V_i$, i.e., they are free with all generators occurring in dimensions r through m , inclusive.
- L denotes the model, introduced in [A1], which carries $CW_r^{m+1}(R)$ to $DGL_r^m(R)$ when $m < rp$.

1.5. Distinguished morphisms in $DGL_r^m(R)$. The category $DGL_r^m(R)$ cannot be made into a closed model category, but we will find it convenient to distinguish three classes of morphisms anyway. Call $f \in \text{Mor } DGL_r^m(R)$ a weak equivalence if it induces an isomorphism on homology of universal enveloping algebras. It is a cofibration if it splits as an inclusion of free Lie algebras (ignoring the differential), and it is a fibration if it is surjective in dimensions above r . Trivial fibrations are simultaneously fibrations and weak equivalences.

2. Function Complexes in $DGL_r^m(R)$

We will now investigate the possibility of doing homotopy theory in $DGL_r^m(R)$. The dimension limitation, viz., the "m" in $DGL_r^m(R)$, spoils our hope of doing so in the sense of Quillen [Q] or even Baues [B]. We cannot dispense entirely with the bound m , because dgL 's exhibit a variety of undesirable behaviors when generator dimensions are permitted to exceed rp . On the other hand, the canonical constructions of turning a map into a fibration or cofibration tend to increase the dimensions of generators, and thus they eventually bump us out of any fixed $DGL_r^m(R)$.

An alternate approach is suggested in [T] and [A1]. We may define for $m < rp$ a homotopy relation on morphisms by utilizing a certain cylinder construction, which raises by one the maximum generator dimension. The gap between m and rp then offers us a "breathing space" in which we can perform the standard constructions approximately $(rp - m)$ times, and thus higher homotopy information is obtainable up to dimension (approximately) $rp - m$. This cylinder construction, known as the Tanré cylinder, is recalled next.

2.1. The Tanré cylinder. This is developed in [T] and [A1] so we provide here only a brief overview. Given a dgL $L = (L(V), \mathcal{S})$ in $DGL_r^m(R)$, where $m < rp$, Tanré associates to it another dgL in $DGL_r^{m+1}(R)$, denoted $IL = (IL(V), I\mathcal{S})$. Taking the set of weak equivalences to be as in 1.5, the dgL IL is a valid cylinder object on L in the sense of [Q] or [B]. In particular, I comes with natural weak equivalences $j_0, j_1: id \rightarrow I$, and if $L \xrightleftharpoons[f]{g} M$ are two morphisms in $DGL_r^m(R)$, then f and g are homotopic if and only if $f \circ g$ factors through IL . Collapsing homotopy classes gives us a category which we denote by $HoDGL_r^m(R)$.

We remark that I is not a functor, although $I f: IL \rightarrow IM$ exists non-canonically for each $f: L \rightarrow M$ in $MorDGL_r^m(R)$. However, I does satisfy the weak naturality condition $I f \circ j_0(L) = j_0(M) \circ f$, $I f \circ j_1(L) = j_1(M) \circ f$.

2.2. Constructing the cosimplicial resolution. We construct next an initial segment of a cosimplicial resolution for objects in $DGL_r^m(R)$. We shall use it to define a function complex between two such dgL's. We follow as closely as possible the standard procedure, due to Dwyer and Kan [DK], for constructing cosimplicial resolutions in any closed model category. By a cosimplicial resolution for an object A we mean a (not necessarily functorial) diagram

$$(1) \quad A \rightleftarrows \underset{\sim}{\Delta^1} A \rightleftarrows \underset{\sim}{\Delta^2} A \dots \underset{\sim}{\Delta^n} A \dots$$

satisfying the usual cosimplicial identities. In (1), each arrow is a weak equivalence; the coface maps are cofibrations, while the codegeneracies are fibrations. (See [DK, Section 4.3] for a precise definition.)

Let us review the Dwyer-Kan construction for a closed model category C . Given an object A , a cylinder on A is an object IA which provides the first stage of a cosimplicial resolution for A . That is, IA fits into a diagram

$$(2) \quad A \begin{array}{c} \xrightarrow{i_0} \\ \xrightarrow{i_1} \end{array} A \times A \xrightarrow{c} IA \xrightarrow{q} A$$

such that c is a cofibration, q is a trivial fibration, and both composites are the identity on A . This $I(\)$ need not be a functor, but we do assume the compatibility of $j_0 = ci_0$ and $j_1 = ci_1$ with any $I f$. Typically I arises by factoring the

folding morphism $A \times A \xrightarrow{\nabla} A$ into a cofibration followed by a trivial fibration.

Assuming one has such an I , let $\underset{\sim}{\Delta}^0$ be the identity functor and let $\underset{\sim}{\Delta}^1$ be the functor $\underset{\sim}{\Delta}^1 A = A \times A$. Then let $\underset{\sim}{\Delta}^1$ be the push-out of $A \xleftarrow{\nabla} \underset{\sim}{\Delta}^1 A \xrightarrow{j_0} I \underset{\sim}{\Delta}^1 A$. It is obvious how $\underset{\sim}{\Delta}^1 A$ serves as the first stage in the cosimplicial resolution (1).

Inductively, suppose the first $(n - 1)$ stages of (1) have been constructed. Let F_A be the functor from the category of faces of the simplicial complex $\underset{\sim}{\Delta}^n$ and inclusions among them (see 3.2) to C , which takes a k -simplex to $\underset{\sim}{\Delta}^k A$, and an inclusion to the appropriate arrow of (1). Let $\underset{\sim}{\Delta}^n A$ be $\text{colim}(F_A)$ and let $\underset{\sim}{\Delta}^n A$ be the push-out of

$$(3) \quad A \xleftarrow{\nabla} \underset{\sim}{\Delta}^n A \xrightarrow{j_0} I \underset{\sim}{\Delta}^n A .$$

We wish to perform the Dwyer-Kan construction in the category $\text{DGL}_r^m(R)$, which is not a closed model category. Let us check precisely which axioms are used. Assuming the existence of I , we need: closure under finite colimits for diagrams of cofibrations; that the push-out of a (resp. trivial) cofibration exists and is a (resp. trivial) cofibration; that two out of three of f and g and gf being weak equivalences makes the third a weak equivalence; and the left lifting property for cofibrations with respect to trivial fibrations. When we take I to be I , the category $\text{DGL}_r^m(R)$ satisfies these four axioms, for $m \leq rp$.

However, as we have noted, the Tanré cylinder construction I applied to a $\text{dgl } L$ having some m -dimensional generators will have some $(m+1)$ -dimensional generators. Inductively, $\underset{\sim}{\Delta}^n L$ lies in $\text{DGL}_r^{m+n}(R)$. This dimension shift, along with the constraint $m + n \leq rp$, is what confines us to an initial segment of a cosimplicial resolution (1).

We have actually verified

LEMMA 2.3. When $m + n \leq rp$, there are constructions

$$\underset{\sim}{\Delta}^n, \underset{\sim}{\Delta}^{n+1}: \text{DGL}_r^m(R) \rightarrow \text{DGL}_r^{m+n}(R).$$

Applied to a dgL $L \in \text{ObDGL}_r^m(R)$, they come with homomorphisms that provide the first $rp - m$ stages of a cosimplicial resolution (1) for L .

Definition 2.4. For $L \in \text{ObDGL}_r^m(R)$, $M \in \text{ObDGL}(R)$, let $\underset{\sim}{\Delta}^n$ be as in Lemma 2.3 for $n \leq rp - m$. Define the function complex between L and M , denoted $\underline{\text{hom}}(L, M)$, to be the simplicial set consisting of $\text{Hom}_{\text{DGL}(R)}(\underset{\sim}{\Delta}^n L, M)$ in dimension n when $n \leq rp - m$, and consisting of degeneracies only, above dimension $rp - m$.

Remark 2.5. Definition 2.4 may depend upon choices made during the construction of $\underset{\sim}{\Delta}^n L$. The results that we are interested in will hold regardless of which choices were made. More importantly, the definition depends upon m and r , in the sense that the relevant dimension range will vary according to which $\text{DGL}_r^m(R)$ we view a given L as lying in. In practice, of course, we will want to use the largest possible r and the smallest possible m . In this paper, the intended r and m will always be apparent from the context.

3. Constructing the Simplicial Map

Having constructed $\underline{\text{hom}}(L, M)$ for dgL's, we turn our attention to its connection with the pointed mapping space Y^X . We have mentioned the dgL model L for pointed R -local CW complexes. We will define a simplicial map \hat{L} from a skeleton of Y^X to $\underline{\text{hom}}(L(X), L(Y))$.

3.1. The model L . In [A1] the first author showed that for any $X \in \text{ObCW}_r^{m+1}(R)$ with $m < rp$ there exists $L \in \text{ObDGL}_r^m(R)$ such that UL is an Adams-Hilton model for X . We write $L(X)$ for this L . One has a similar assertion and notation for maps. The passage from X to L is not functorial, since X does not canonically determine

L ; nor does a map $f: X \rightarrow Y$ uniquely determine $L(f)$, even after $L(X)$ and $L(Y)$ have been fixed. However, $L(f)$ is determined up to homotopy, and hence $L(X)$ is determined up to homotopy type. In spite of this indeterminacy, the function complex between such models always does the right thing up to a certain dimension.

The main advantage of L as a model for X is that it is built directly from a cellular decomposition of X , so it is fairly small and accessible to computations.

3.2. Review of Y^X . The pointed mapping space Y^X may be viewed as the simplicial set

$$(4) \quad Y^X = \{ \text{Hom}_{\text{TOP}}(|\Delta^n| \times X, Y) \}_{n \geq 0}.$$

Here Δ^n is the standard simplicial complex whose geometric realization is the standard n -simplex, and \times denotes the half-smash. The subcomplex of Δ^n obtained by removing the n -simplex is denoted, as usual, by $\dot{\Delta}^n$.

Denote by $\text{sd}(\Delta^n)$ (resp. $\text{sd}(\dot{\Delta}^n)$) the first barycentric subdivision of Δ^n (resp. $\dot{\Delta}^n$). Whenever $X \in \text{ObCW}_r^m(R)$, then an easy Kunneth formula argument shows that $|\text{sd}(\Delta^n)| \times X$ and $|\text{sd}(\dot{\Delta}^n)| \times X$ belong to $\text{ObCW}_r^{m+n}(R)$ (cf. 4.4 for a discussion of CW structures). As long as $m + n \leq rp$, a model $L(|\text{sd}(\Delta^n)| \times X)$ exists for $|\Delta^n| \times X$.

LEMMA 3.3. For $X \in \text{ObCW}_r^m(R)$, $m + n \leq rp$, one can choose models such that there are isomorphisms

$$(5) \quad \begin{aligned} L(|\text{sd}(\Delta^n)| \times X) &\approx \dot{\Delta}^n L(X), \quad \text{and} \\ L(|\text{sd}(\dot{\Delta}^{n+1})| \times X) &\approx \dot{\Delta}^{n+1} L(X). \end{aligned}$$

Furthermore, the model L applied to the coface and codegeneracy maps

$$|\text{sd}(\Delta^n)| \times X \rightleftarrows |\text{sd}(\Delta^{n+1})| \times X$$

may be taken to be the coface and codegeneracy homomorphisms mentioned in Lemma 2.3, for $L = L(X)$.

Proof. This is easily deduced by induction on n . At each stage, L can be chosen to commute with colimits of inclusions of CW complexes [A1, Theorem 8.5i], with cylinders [A2, Lemma 5], and with push-outs in which one map is CW and the other is an inclusion into a cylinder [A2, Lemma 6].

PROPOSITION 3.4. Let $X \in \text{ObCW}_r^m(\mathbb{R})$ where $m \leq rp$, and let $Y \in \text{ObCW}_r^{rp}(\mathbb{R})$. There is a homomorphism of simplicial sets

$$(6) \quad \hat{L}: (Y^X)^{rp-m} \rightarrow \underline{\text{hom}}(L(X), L(Y)).$$

The source of (6) is the $(rp-m)$ -skeleton of the simplicial set (4). For each $f \in (Y^X)^{rp-m}$, $\hat{L}(f)$ may be interpreted as a valid L -model for f .

Proof. We build \hat{L} dimension by dimension. Assume we have the simplicial map

$$\hat{L}^{n-1}: (Y^X)^{n-1} \rightarrow \underline{\text{hom}}(L(X), L(Y)).$$

For each element $f: |\Delta^n| \times X \rightarrow Y$, view f as a map from the CW complex $|\text{sd}(\Delta^n)| \times X$ to Y . Consider

$$(7) \quad \underset{\sim}{\Delta^n} L(X) \xrightarrow[\text{by (5)}]{\approx} L(|\text{sd}(\Delta^n)| \times X) \xrightarrow{L(f)} L(Y).$$

This composite belongs to the dimension n part of $\underline{\text{hom}}(L(X), L(Y))$

if $n \leq rp - m$. Thus we may extend \hat{L}^{n-1} to $\hat{L}^n: (Y^X)^n \rightarrow$

$\underline{\text{hom}}(L(X), L(Y))$ by defining $\hat{L}^n(f)$ to be the composite (7). The

only subtlety is the requirement that \hat{L}^n is to be a simplicial map, i.e., compatible with faces and degeneracies. This in turn requires that we utilize the flexibility inherent in our choices for $L(f)$.

We are supposing that \hat{L}^{n-1} is simplicial, i.e., these choices have been made compatibly below dimension n . Given $f: |\Delta^n| \times X \rightarrow Y$, let \dot{f} denote the restriction $\dot{f}: |\text{sd}(\dot{\Delta}^n)| \times X \rightarrow Y$, and for $0 \leq i \leq n$ let $f_i: |\text{sd}(\Delta^{n-1})| \times X \rightarrow Y$ denote the further restriction to the i^{th} face of $|\Delta^n|$ half-smashed with X . By our inductive assumption, the $L(f_i)$ are compatible with faces; by [A1, theorem 8.5j] their colimit serves as a valid choice for $L(\dot{f})$. Lastly, use [A1, theorem 8.5h] to extend this choice for $L(\dot{f})$ to some valid model $L(f)$. By Lemma 3.3, the resulting choice for $\hat{L}(f)$ remains compatible with faces and degeneracies.

PROPOSITION 3.5. Let $X \in \text{ObCW}_r^t(\mathbb{R})$, $Y \in \text{ObCW}_r^{\text{FP}}(\mathbb{R})$, where $t = \min(rp - 1, r + 2p - 3)$. Then \hat{L} induces a bijection

$$\pi_0(\hat{L}): \pi_0(Y^X) \rightarrow \pi_0(\underline{\text{hom}}(L(X), L(Y))).$$

If instead $X \in \text{ObCW}_r^{t+1}(\mathbb{R})$, then $\pi_0(\hat{L})$ is a surjection.

Proof. For $L, M \in \text{ObDGL}_r^{\text{FP}-1}(\mathbb{R})$, $f, g: L \rightarrow M$ extends over IL if and only if it extends over $\underline{\Delta}^1 L$. Thus $\pi_0(\underline{\text{hom}}(L, M))$ coincides with the (Tanré-induced) set of homotopy classes $[L; M]$. Also, this diagram commutes:

$$(8) \quad \begin{array}{ccccc} \pi_0(Y^X) & \xleftarrow{\approx} & \pi_0((Y^X)^1) & \xrightarrow{\pi_0(\hat{L})} & \pi_0(\underline{\text{hom}}(L(X), L(Y))) \\ \downarrow \approx & & \downarrow & & \downarrow \approx \\ [X; Y] & \xleftarrow{=} & [X; Y] & \xrightarrow{(L)_\#} & [L(X); L(Y)] \end{array} ,$$

where we have put $m = rp - 1$. By [A2, Theorem 3] the arrow $(L)_\#$ of (8) is a bijection. When $\dim(X) = t + 1$, use $(Y^X)^0$ in place of $(Y^X)^1$ in (8); then the upper left arrow and $(L)_\#$ are surjections, hence so is $\pi_0(\hat{L})$.

4. The d-type of Y^X

We conclude by showing that the simplicial map \hat{L} of (6) is a homotopy equivalence in a range of dimensions. We fix the notation

$$(9) \quad t = \min(rp - 1, r + 2p - 3).$$

4.1. Simplicial d-type. Let A and B denote simplicial sets, and let $d \geq 0$. A d-equivalence is a simplicial map $g: A \rightarrow B$ such that, for every choice of base point $a_0 \in (A)_0$, g induces a bijection on π_n for $n < d$ and a surjection on π_d . We say that B and B' have the same (d-1)-type if and only if there is a simplicial set A which comes with d-equivalences $B \xleftarrow{g} A \xrightarrow{g'} B'$. "Same (d-1)-type" is an equivalence relation because, if $B'' \xleftarrow{g''} A'' \xrightarrow{g'''} B' \xrightarrow{g'} A' \rightarrow B'$ are d-equivalences, letting A'' be the fiber-homotopy pull-back of g

and g' leads to d -equivalences $A \leftarrow A'' \rightarrow A'$. (For an alternate approach to $(d-1)$ -type, see [B, p. 364].) Note that the skeleton inclusion $A^d \rightarrow A$ is always a d -equivalence. Lastly, the condition on π_0 amounts to the requirement that g induce a bijection on path-components (resp., a surjection, if $d = 0$).

Two spaces having the same d -type tells us that their homotopy groups $\pi_n(\)$ are isomorphic for $n \leq d$, but it tells us much more than this. For instance, the spaces S^2 and $CP^\infty \times S^3$ have isomorphic π_n for all n ; they have the same 2-type ($S^2 \leftarrow S^2 \vee S^3 \rightarrow CP^\infty \times S^3$) but not the same 3-type.

We assert (see 4.7) that Y^X and $\underline{\text{hom}}(L(X), L(Y))$ have the same d -type, for a certain d .

4.2. Relative homotopy in $DGL_r^m(R)$. We need the concept of a relative homotopy, for dgL 's. First let us review the concept for spaces. Let W be a pointed space and let X be a subspace; we fix a pointed map $\phi: X \rightarrow Y$. Denote by $\text{Hom}_{\text{TOP}}(W, Y)_\phi$ the set of all extensions of ϕ over W . Two maps in $\text{Hom}_{\text{TOP}}(W, Y)_\phi$ are homotopic rel X , denoted $f \simeq_X g$, if and only if there is a homotopy $F: W \times [0, 1] \rightarrow Y$ such that $F|_{W \times 0} = f$, $F|_{W \times 1} = g$, and $F(w, s) = \phi(w)$ for $w \in X$. Denote by $[W; Y]_\phi$ the set of \simeq_X -equivalence classes. We will be especially interested in the case where W is a CW complex and X is a subcomplex.

Let $L \rightarrow K$ be a cofibration in $DGL_r^m(R)$, $m < rp$; we identify L with a sub- dgL of K . Let $M \in \text{Ob} DGL(R)$, and fix a dgL homomorphism $\lambda: L \rightarrow M$. Denote by $\text{Hom}_{DGL(R)}(K, M)_\lambda$ the set of all extensions of λ over K .

Although we have stressed that the Tanré cylinder I is not natural, there is a cofibration $IL \rightarrow IK$ which extends the given cofibration $L \rightarrow K$. Let $q_L: IL \rightarrow L$ denote the trivial fibration which extends the fold map $L \rightarrow L$. Two dgL homomorphisms in $\text{Hom}_{DGL(R)}(K, M)_\lambda$ are homotopic rel L , denoted $f \simeq g$, if and only if there exists $F: IK \rightarrow M$ whose restriction to IL is λq_L . Denote the set of \simeq -equivalence classes by $[K; M]_\lambda$.

PROPOSITION 4.3. Let $W \in \text{ObCW}_r^t(R)$, let X be a subcomplex, and let $Y \in \text{ObCW}_r^{\text{rp}}(R)$. Fix a map $\phi: X \rightarrow Y$ and fix a model

$\lambda = L(\phi): L(X) \rightarrow L(Y)$. Then L induces a bijection

$$(10) \quad \text{Ho}(L): [W; Y]_{\phi} \rightarrow [L(W); L(Y)]_{\lambda},$$

in which a \simeq -class $[f]_{\phi}$ is sent to the \simeq -class $[L(f)]_{\lambda}$. If instead

$W \in \text{ObCW}_r^{t+1}(R)$, then (10) is a surjection.

Proof. One may easily adapt the proof of [A2, Theorem 3] to cover this situation as well. One needs only to be careful always to choose $L(f)$ for $f: W \rightarrow Y$ so as to extend the model λ for $f|_X$.

4.4. Homomorphisms induced by \hat{L} . We intend to study the

homomorphisms induced by the \hat{L} of (6) on homotopy groups. Let $X \in \text{ObCW}_r^m(R)$, $m \leq \text{rp}$, and $Y \in \text{ObCW}_r^{\text{rp}}(R)$. Fix a map $\phi: X \rightarrow Y$ and view Y^X as the simplicial set (4); thus $\phi \in (Y^X)_0$. Fix $n \geq 0$ and take as base point the 0^{th} vertex $v_0 \in |\text{sd}(\dot{\Delta}^{n+1})|$. Henceforth, when we write S^n , we will intend S^n to be viewed as the CW realization $|\text{sd}(\dot{\Delta}^{n+1})|$ with base point v_0 (i.e., as a CW complex, S^n has one cell for each non-degenerate simplex of $\text{sd}(\dot{\Delta}^{n+1})$). Let $W = S^n \times X$. The CW structures on S^n and on X give us a CW structure on W ; note that $W \in \text{ObCW}_r^{m+n}(R)$. We identify X with the subcomplex $v_0 \times X$ of W . Clearly, $[W; Y]_{\phi}$ makes sense.

We consider the same setup in $\text{DGL}_r^{\text{rp}}(R)$. Let $L \in \text{ObDGL}_r^m(R)$, $m < \text{rp}$, $M \in \text{ObDGL}(R)$. When $m + n \leq \text{rp}$, $\dot{\Delta}^n L$ is defined, and we may include L into $\dot{\Delta}^n L$ "at the 0^{th} vertex" (see (1)). Thus L is viewed as a sub-dgL of $K = \dot{\Delta}^{n+1} L$, and $[K; M]_{\lambda}$ makes sense for any given $\lambda: L \rightarrow M$. When $L = L(X)$, we may by Lemma 3.3 identify K with $L(W)$. Then the inclusion of the sub-dgL L into K is a valid L -model for the subcomplex inclusion $X \rightarrow W$ described above.

Now let $X \in \text{ObCW}_r^m(R)$, $m \leq \text{rp}$, $Y \in \text{ObCW}_r^{\text{rp}}(R)$, as above. Choose an \hat{L} as in Proposition 3.4. Let $\lambda = \hat{L}(\phi)$, which is a valid L -model

for ϕ . For $n \leq rp - m$, consider the diagram

$$\begin{array}{ccc}
 \text{Hom}_{\text{SS}}(\Delta^{n+1}, v_0; (Y^X)^{rp-m}, \phi) & \xrightarrow{(\hat{L})_*} & \text{Hom}_{\text{SS}}(\Delta^{n+1}, v_0; \underline{\text{hom}}(L(X), L(Y)), \lambda) \\
 \alpha \downarrow & & \downarrow \approx \\
 \text{Hom}_{\text{SS}}(\Delta^{n+1}, v_0; Y^X, \phi) & & \\
 \approx \downarrow & & \\
 \text{Hom}_{\text{TOP}}(|\Delta^{n+1}| \times X, Y)_{\phi} & & \text{Hom}_{\text{DGL}(\mathbb{R})}(\Delta^{n+1} L(X), L(Y))_{\lambda} \\
 = \downarrow & & \downarrow \approx \\
 \text{Hom}_{\text{TOP}}(W, Y)_{\phi} & \xrightarrow{L'} & \text{Hom}_{\text{DGL}(\mathbb{R})}(L(W), L(Y))_{\lambda} .
 \end{array}$$

(11)

Because all the vertical arrows in (11) are bijections, there is a unique L' which makes the diagram commute. The following lemma follows easily from the construction of \hat{L} .

LEMMA 4.5. For any choice of \hat{L} as in Proposition 3.4, the function L' of (11) satisfies this: for any $f \in \text{Hom}_{\text{TOP}}(W, Y)_{\phi}$, $L'(f)$ is a valid L -model for f .

The reader may now check that the equivalence relations that we have on the various sets in (11) are compatible with the arrows, and lead to the diagram

$$\begin{array}{ccc}
 \pi_n((Y^X)^{rp-m}, \phi) & \xrightarrow{(\hat{L})_*} & \pi_n(\underline{\text{hom}}(L(X), L(Y)), \lambda) \\
 \alpha_* \downarrow & & \downarrow \approx \\
 \pi_n(Y^X, \phi) & & \\
 \approx \downarrow & & \\
 \text{Hom}_{\text{TOP}}(W, Y)_{\phi} / (\approx) & & \text{Hom}_{\text{DGL}(\mathbb{R})}(\Delta^{n+1} L(X), L(Y))_{\lambda} / (\approx) \\
 = \downarrow & & \downarrow \approx \\
 [W; Y]_{\phi} & \xrightarrow{(L')_*} & [L(W); L(Y)]_{\lambda} .
 \end{array}$$

(12)

The following two facts are also clear.

LEMMA 4.6. (a) In (12), $(L')_*$ coincides with $Ho(L)$ of (10). (b) In (12), α_* is bijective if $m + n < rp$ and surjective if $m + n = rp$.

THEOREM 4.7. Let $X \in \text{ObCW}_r^m(\mathbb{R})$, $m \leq t + 1$, $Y \in \text{ObCW}_r^{rp}(\mathbb{R})$. Put $d = t - m$ (cf. (9)). The simplicial map \hat{L} of (6) is a $(d+1)$ -equivalence. Consequently, the simplicial sets Y^X and $\underline{\text{hom}}(L(X), L(Y))$ have the same d -type.

Proof. The condition on π_0 is actually given by Proposition 3.5.
When $t - m \geq n > 0$, $(\hat{L})_{\#}$ of (12) is bijective, by 4.3 and 4.6.
When $n = t - m + 1$, $(\hat{L})_{\#}$ of (12) is surjective, again by 4.3 and 4.6.

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