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Astérisque, tome 191 (1990), p. 109-209

<http://www.numdam.org/item?id=AST_1990__191__109_0>

André-Quillen Cohomology and the Bousfield-Kan Spectral Sequence

by Paul G. Goerss*

In this paper we investigate the Bousfield-Kan spectral sequence [6], [8]

$$(1) \qquad Ext^s_{\mathcal{UA}}(H^*X, H^*S^t) \Rightarrow \pi_{t-s}X_p$$

where $H^*X = H^*(X, \mathbb{F}_p)$ is cohomology with coefficients in the prime field $\mathbb{F}_p$, $\mathcal{UA}$ is the category of augmented unstable algebras over the Steenrod algebra, S^t is the t-sphere and, X_p is the p-completion of the pointed space X. This spectral sequence, an unstable Adams spectral sequence, is a major tool in attempts to compute or understand the homotopy groups of spaces.

Because $\mathcal{UA}$ is not an abelian category, $Ext^*_{\mathcal{UA}}(H^*X, H^*S^t)$ must be defined using a cotriple and, hence, a simplicial resolution of the algebra H^*X. Our first point is to notice that if $s\mathcal{UA}$ is the category of simplicial objects in $\mathcal{UA}$, then there is a contravariant functor

$$(2) \qquad H^*_{\mathcal{QA}} : s\mathcal{UA} \longrightarrow nn\mathbb{F}_p$$

to the category of bigraded $\mathbb{F}_p$ vector spaces that generalizes $Ext_{\mathcal{UA}}$ in the following sense: if Λ is an object in $\mathcal{UA}$, then we may regard Λ as the constant simplicial object that is Λ in every simplicial degree and every face and degeneracy operator is the identity. Then we will have the equation

$$(3) \qquad [H^s_{\mathcal{QA}}\Lambda]_t \cong Ext^s_{\mathcal{UA}}(\Lambda, H^*S^t)$$

where $[H^s_{\mathcal{QA}}\Lambda]_t$ denotes the elements of degree t in the graded vector space $H^s_{\mathcal{QA}}\Lambda$.

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* The author was partially supported by the National Science Foundation

For $A \in s\mathcal{UA}$, $H^*_{Q\mathcal{A}}A$ is a cohomology of A in the sense of Quillen [17]; in fact, this is the sort of cohomology of algebras that has been studied extensively by André and Quillen [1], [18]. Hence the title of the paper.

We will take the observation of the existence of $H^*_{Q\mathcal{A}}$ in two directions. The first is this: the category of simplicial objects $s\mathcal{UA}$ has a structure of a closed model category and, as such, we can do the usual homotopy theoretic constructions. In particular, if $f : A \to B$ is a morphism in $s\mathcal{UA}$, then f has a homotopy cofiber $M(f)$ and there is a long exact sequence in cohomology similar to the one that Quillen called a transitivity sequence [18]:

$$(4) \qquad \cdots \to H^{n-1}_{Q\mathcal{A}}A \to H^n_{Q\mathcal{A}}M(f) \to H^n_{Q\mathcal{A}}B \xrightarrow{H^*_{Q\mathcal{A}}f} H^n_{Q\mathcal{A}}A \to \cdots.$$

Here we need the full generality of $s\mathcal{UA}$. For, even if $f : \Lambda \to \Gamma$ is a morphism of constant simplicial algebras of the type considered in equation (3), $M(f)$ is not necessarily such an object. In fact, if we define the homotopy of an object $A \in s\mathcal{UA}$ by

$$\pi_* A = H_*(A, \partial)$$

where ∂ is the alternating sum of the face operators in A, then for a morphism $f : \Lambda \to \Gamma$ of constant simplicial algebras

$$\pi_* M(f) \cong Tor^{\Lambda}_*(\mathbf{F}_p, \Gamma).$$

We extend the long exact sequence in cohomology to the homotopy spectral sequence. The Bousfield-Kan spectral sequence is an example of the homotopy spectral sequence of a cosimplicial space. Given a fibrant cosimplicial space Z, there is a spectral sequence [8]

$$\pi^s \pi_t Z \Rightarrow \pi_{t-s} Tot(Z)$$

where $Tot(Z)$ is a kind of "codiagonal" of Z given by the mapping space of cosimplicial spaces

$$Tot(Z) = map(\Delta, Z)$$

where Δ is the cosimplicial space that in cosimplicial degree s is the standard s-simplex $\Delta[s]$. Now if Z is a cosimplicial space, then H^*Z is a simplicial object in $\mathcal{UA}$ and we will notice that we can extend the Bousfield-Kan spectral sequence to a spectral sequence

$$[H^s_{Q\mathcal{A}} H^* Z]_t \Rightarrow \pi_{t-s} Tot(Z)_p. \tag{5}$$

If X is a space and $Z = X$ is the constant simplicial space, then $H^*Z = H^*X$ is a constant simplicial algebra we can combine equation (3) with equation (5) to obtain the spectral sequence (1).

In particular, if $f : Z \to Y$ is a morphism of cosimplicial spaces, we will define a new cosimplicial space F so that $H^*F \cong M(f^*)$ — the homotopy cofiber of f^* in $s\mathcal{UA}$ — and so that there is a homotopy fibration sequence of spaces

$$Tot(F) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p.$$

Further, there will be a diagram of spectral sequences

$$\begin{array}{ccccccccc}
\to & [H^s_{Q\mathcal{A}} H^* Z]_t & \to & [H^s_{Q\mathcal{A}} H^* Y]_t & \xrightarrow{\partial} & [H^{s+1}_{Q\mathcal{A}} H^* F]_t & \to \\
 & \Downarrow & & \Downarrow & & \Downarrow & \\
\to & \pi_{t-s} Tot(Z)_p & \to & Tot_{t-s} Tot(Y)_p & \xrightarrow{\delta} & \pi_{t-s-1} Tot(F) & \to
\end{array}$$

where the top row comes from the long exact sequence (4) and the bottom row is the long exact sequence of the fibration sequence. The hard work here is to produce the commutative diagram of spectral sequences

$$\begin{array}{ccc}
[H^s_{Q\mathcal{A}} H^* Y]_t & \xrightarrow{\partial} & [H^{s+1}_{Q\mathcal{A}} H^* F]_t \\
\Downarrow & & \Downarrow \\
\pi_{t-s} Tot(Y)_p & \xrightarrow{\delta} & \pi_{t-s-1} Tot(F).
\end{array}$$

This is done in section 5.

There are other ramifications to the idea that $s\mathcal{UA}$ is a closed model category. Among them are the notions of universal infinite cycles and universal differentials for the spectral sequence (5). Although these were noted by Bousfield and Kan [6] and have been extended by the work of Bousfield [3], and are related to Barratt's desuspension spectral sequence, as rediscovered by Hopkins, they have not been systematically studied from our point of view. We undertake this study, beginning in section 3, but extending our computations into further sections.

The universal cycle, for example, is a cosimplicial space $\bar{\mathsf{F}}^{\cdot}_{p}S(s,t)$ whose cohomology is relevant to the Quillen cohomology of equation (2) in the following way. There is an object $K(s,t)_{+} \in s\mathcal{UA}$ that corepresents cohomology in the usual way: there is a universal class $\iota \in [H^{s}_{Q\mathcal{A}}K(s,t)_{+}]_{t}$ and an isomorphism

$$[A, K(s,t)_{+}]_{s\mathcal{UA}} \xrightarrow{\cong} [H^{s}_{Q\mathcal{A}}A]_{t}$$

given by

$$f \longmapsto f^{*}(\iota).$$

$[\ ,\ ]_{s\mathcal{UA}}$ denotes the morphisms in the homotopy category associated to the closed model category on $s\mathcal{UA}$. Then, we have for the universal cycle $\bar{\mathsf{F}}^{\cdot}_{p}S(s,t)$, a weak equivalence in $s\mathcal{UA}$

$$H^{*}\bar{\mathsf{F}}^{\cdot}_{p}S(s,t) \xrightarrow{\simeq} K(s,t)_{+}$$

and, hence, an equation

$$H^{*}_{Q\mathcal{A}}H^{*}\bar{\mathsf{F}}^{\cdot}_{p}S(s,t) \cong H^{*}_{Q\mathcal{A}}K(s,t)_{+}.$$

Furthermore $Tot(\bar{\mathsf{F}}^{\cdot}_{p}S(s,t)) \simeq S^{t-s}_{p}$. Hence we get a spectral sequence

$$H^{*}_{Q\mathcal{A}}K(s,t)_{+} \Rightarrow \pi_{*}S^{t-s}_{p}$$

that is universal in the following sense. Suppose that, for a fibrant cosimplicial space Z,

$$\alpha \in [H^{s}_{Q\mathcal{A}}H^{*}Z]_{t}$$

survives to E_{∞} in the spectral sequence (5), and detects

$$x \in \pi_{t-s}Tot(Z)_{p}.$$

Then there is a morphism in $s\mathcal{UA}$

$$f : H^{*}Z \to K(p,q)_{+}$$

corepresenting α and the resulting map

$$H^{*}_{Q\mathcal{A}}f : H^{*}_{Q\mathcal{A}}K(s,t)_{+} \to H^{*}_{Q\mathcal{A}}H^{*}Z$$

will fit into a diagram of spectral sequences, at least when $t-s>1$:

$$\begin{array}{ccc} H^*K(s,t)_+ & \Rightarrow & \pi_* S_p^{t-s} \\ \downarrow H^*_{\mathcal{QA}} f & & \downarrow \\ H^*_{\mathcal{QA}} H^* Z & \Rightarrow & \pi_* Tot(Z)_p \end{array}$$

where, under the map $\pi_* S^{t-s} \longrightarrow \pi_* Tot(Z)_p$ the identity map passes to x. This is discussed in section 3 and the computation of $H^*_{\mathcal{QA}} K(s,t)_+$ is considered in section 9.

The second direction we take the existence of the Quillen cohomology functor $H^*_{\mathcal{QA}}$ is this: if $H^*_{\mathcal{QA}}$ is truly a cohomology theory, it should support products and operations. This is, in fact, the case. The work of Bousfield and Kan [7] can be interpreted to prove the existence of a commutative bilinear product

$$[\ ,\ ] : H^s_{\mathcal{QA}} A \otimes H^{s'}_{\mathcal{QA}} A \rightarrow H^{s+s'+1}_{\mathcal{QA}} A$$

satisfying the Jacobi identity and adding internal degree. In the spectral sequence this product will converge to the Whitehead product; that is, there is a commutative diagram of spectral sequences

$$\begin{array}{ccc} [H^s_{\mathcal{QA}} H^* Z]_t \otimes [H^{s'}_{\mathcal{QA}} H^* Z]_{t'} & \Rightarrow & \pi_{t-s} Tot(Z) \otimes \pi_{t'-s'} Tot(Z) \\ \downarrow [\ ,\ ] & & \downarrow [\ ,\ ] \\ [H^{s+s'+1}_{\mathcal{QA}} H^* Z]_{t+t'} & \Rightarrow & \pi_{t+t'-(s+s')-1} Tot(Z) \end{array}$$

where the right vertical map is the Whitehead product in homotopy.

If we specialize to the prime 2, there are also operations. These are homomorphisms

$$P^i : H^s_{\mathcal{QA}} A \rightarrow H^{s+i+1}_{\mathcal{QA}} A$$

doubling internal degree, and satisfying an unstable condition, a formula relating the Whitehead product to the operations, and a set of relations among themselves. In particular, we might call these operations "divided Whitehead products" because $P^i = 0$ if $i > s$ and

$$P^s(x) = [x, x].$$

The exact statement of the various relations is given at the beginning of section 7.

An interesting fact about these operations is that they do not, in general, commute with the differentials in the Bousfield-Kan spectral sequence. We prove this by examining the universal example mentioned above; in fact, portions of sections 7, 8 and 9 are devoted to investigating the operations in the spectral sequence of the universal infinite cycle:

$$H^*_{Q\mathcal{A}}K(s,t)_+ \Rightarrow \pi_* S^{t-s}_p.$$

At this point the reader might think that the work to be done here is highly theoretical, and this is largely the case. However, the last two sections of this paper are devoted to tools for computation, and there we make a serious attempt to compute and understand in detail the functor $H^*_{Q\mathcal{A}}$. We will use a composite functor spectral sequence due, in principal, to Haynes Miller [19] to begin computing the cohomology of the universal examples mentioned above and to undertake other projects, including trying to understand to what extent the Bousfield-Kan spectral sequence (1) satisfies the Hilton-Milnor Theorem. It turns out that we need the full generality of $H^*_{Q\mathcal{A}}$ to address this question, even if we are only trying to understand $Ext_{\mathcal{U}\mathcal{A}}$. There are other tools available for computation, among them the work of André [1], and [12], which owes a debt to the work of Miller [19, Section 4].

There are also numerous examples scattered throughout, and clearly marked as such. These are intended to provide some concreteness to our work and to explain the relevance on the project to the work of others. In particular, see section 4.

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Notation and conventions: Because cohomology algebras are more intuitive than homology coalgebras, we work with the former. However, we sacrifice generality, especially in convergence statements about homotopy spectral sequences. Therefore, we often make *finite type* hypotheses. A graded $\mathbb{F}_p$ vector space is of finite type if, for every n, the elements of degree n form a finite vector space.

A *space* is a simplicial set, usually pointed; that is, the space comes equipped with a chosen basepoint. If we are in a situation where the spaces are not pointed, we say so.

$\mathbf{F}_p$ is the field with p elements for some prime p, $\mathcal{A}$ is the mod p Steenrod algebra, and all homology and cohomology of spaces is with $\mathbf{F}_p$ coefficients.

If $\mathbf{C}$ is a category, then $s\mathbf{C}$ will denote the category of simplicial objects in $\mathbf{C}$ and $n\mathbf{C}$ will denote the category of graded objects in $\mathbf{C}$. In particular, $n\mathbf{F}_p$ will denote the category of graded $\mathbf{F}_p$ vector spaces and $nn\mathbf{F}_p$ the category of bigraded vector spaces.

If V is a simplicial vector space, we define

$$\pi_* V = H_*(V, \partial)$$

where

$$\partial = \sum_{i=0}^{s} (-1)^i d_i : V_s \to V_{s-1}$$

is the alternating sum of the face operators. If V is a cosimplicial vector space, we set

$$\pi^* V = H^*(V, \partial^*)$$

where ∂^* is the alternating sum of the coface operators. These definitions can be extended to any category with a forgetful functor to the category of vector spaces, graded vector spaces, or abelian groups. If V is a simplicial graded vector space, the $\pi_* V$ is a bigraded vector space. We refer to the elements of $[\pi_s V]^t$ as being of external degree s and internal degree t.

Part I: Quillen Cohomology and the Bousfield-Kan Spectral Sequence

1. The Bousfield-Kan Spectral Sequence I

This preliminary section is devoted to the definition of the basic object of study and to establishing notation. The Bousfield-Kan spectral sequence [6, 8] is an Adams-type spectral sequence passing from the homology of a space X to the homotopy of its p-completion X_p. A good introduction to this spectral sequence is given in Section 1 of Miller's paper [19].

We begin by defining some categories. Fix a prime p and let $\mathbb{F}_p$ be the field with p elements. We let $\mathcal{UA}$ be the category of unstable algebras over the Steenrod algebra. Thus $H \in \mathcal{UA}$ is a graded, commutative, supplemented $\mathbb{F}_p$ algebra that supports an action by the Steenrod algebra and so that the two structures are related by the Cartan formula and by the *unstable* condition: if $p > 2$, then

$$\mathrm{P}^n(x) = \begin{cases} 0, & \text{if } deg(x) < 2n; \\ x^p, & \text{if } deg(x) = 2n. \end{cases}$$

and if p = 2, then

$$\mathrm{Sq}^n(x) = \begin{cases} 0, & \text{if } deg(x) < n; \\ x^2, & \text{if } deg(x) = n. \end{cases}$$

The symbol $deg(x)$ means the degree of x as an element of the graded algebra H; the vector space of elements of degree n will be denoted by H^n.

If X is a pointed (based) space, then $H^*\mathrm{X} = H^*(X, \mathbb{F}_p)$ is an object of $\mathcal{UA}$.

There is a simpler, associated category $\mathcal{U}$ – the category of unstable modules over the Steenrod algebra $\mathcal{A}$. This is the full sub-category of the category of modules over $\mathcal{A}$ specified by the conditions that $M \in \mathcal{U}$ if

$$\beta^\epsilon \mathrm{P}^n(x) = 0 \qquad \text{if } deg(x) < 2n + \epsilon$$

$$\mathrm{Sq}^n(x) = 0 \qquad \text{if } deg(x) < n.$$

$\mathcal{U}$ is an abelian category.

The augmentation ideal functor $I : \mathcal{UA} \to \mathcal{U}$ has a left adjoint U; for example,

$$H^*S^{2k+1} \cong U(\Sigma^{2k+1}\mathbb{F}_p)$$

where $\Sigma^{2k+1}\mathbb{F}_p$ is the trivial $\mathcal{A}$ module of dimension 1 over $\mathbb{F}_p$ concentrated in degree $2k+1$.

Next consider the forgetful functor $J : \mathcal{U} \to n\mathbb{F}_p$ where $n\mathbb{F}_p$ is the category of graded $\mathbb{F}_p$ vector spaces. This, too has a left adjoint $P : n\mathbb{F}_p \to \mathcal{U}$; indeed, if V is of finite type, then

$$P(V) = PH^*K(V^*)$$

where the right hand side is the primitives in the indicated Hopf algebra, V^* is the graded vector space dual, and $K(V^*)$ is a generalized Eilenberg-MacLane space with $\pi_*K(V^*) \cong V^*$

As a consequence of the existence of P, the augmentation ideal functor $I : \mathcal{UA} \to n\mathbb{F}_p$ has a left adjoint G; namely

$$G = U \circ P \text{ or } G(V) = U(P(V)).$$

If V is of finite type, then $G(V) \cong H^*K(V^*)$. The composite functors

$$\overline{G} = G \circ I : \mathcal{UA} \to \mathcal{UA}$$

$$\overline{P} = P \circ J : \mathcal{U} \to \mathcal{U}$$

both have the structure of a cotriple on the respective category; that is, there are natural transformations

$$\epsilon_H : \overline{G}(H) \to H \text{ and } \epsilon_M : \overline{P}(M) \to M$$

$$\eta_H : \overline{G}(H) \to \overline{G}^2(H) \text{ and } \eta_M : \overline{P}(M) \to \overline{P}^2(M)$$

and these are related in such a manner that we may form the simplicial objects $\overline{G}.(H) \in s\mathcal{UA}$ and $\overline{P}.(M) \in s\mathrm{U}$. For example,

$$\overline{G}_n(H) = \overline{G}^{n+1}(H)$$

and

$$d_i : \overline{G}_n \to \overline{G}_{n-1}$$

is defined by

$$d_i = \overline{G}^i \epsilon \overline{G}^{n-i}, \qquad 0 \le i \le n$$

and

$$s_i : \overline{G}_n(H) \to \overline{G}_{n+1}(H)$$

is given by

$$s_i = \overline{G}^i \eta \overline{G}^{n-i}, \qquad 0 \le i \le n.$$

Both $\overline{G}.(H)$ and $\overline{P}.(M)$ are *augmented* simplicial objects in the sense that ϵ induces maps

$$\epsilon_H : \overline{G}_0(H) \to H \text{ and } \epsilon_M : \overline{P}_0(M) \to M$$

such that $\epsilon d_0 = \epsilon d_1$. More than this ϵ induces isomorphisms

$$\pi_* \overline{G}.(H) \cong H \text{ and } \pi_* \overline{P}.(M) \cong M$$

concentrated in external degree 0. The retraction that guarantees these isomorphisms is given by the inclusions in $n\mathbb{F}_p$

$$IH \to I\overline{G}(H) \qquad \text{and} \qquad M \to \overline{P}(M)$$

adjoint to the identity.

Thus $\epsilon : \overline{G}.(H) \to H$ and $\epsilon : \overline{P}.(M) \to M$ may be regarded as acyclic resolutions in the relevant category and we may define *Ext* – the right derived functors of *Hom* in the category – by

$$Ext^s_{\mathcal{UA}}(H, K) = \pi^s Hom_{\mathcal{UA}}(\overline{G}.(H), K)$$

and

$$Ext^s_{\mathcal{U}}(M, N) = \pi^s Hom_{\mathcal{U}}(\overline{P}.(M), N).$$

We need π^* because these Hom functors are contravariant.

To obtain a spectral sequence with E_2-term of the form $Ext_{\mathcal{UA}}$, Bousfield and Kan proceed as follows. If X is a space (that is, a pointed simplicial set), let $\mathbf{F}_p(X)$ denote the simplicial vector space on the simplicial set X and let $\bar{\mathbf{F}}_p X = \mathbf{F}_p(\mathrm{X})/\mathbf{F}_p(*)$ where $* \in X$ is the basepoint. Then $\bar{\mathbf{F}}_p(\ )$ has the structure of a triple on the category of spaces and one obtains (in a manner dual to the process above) an augmented cosimplicial space

$$X \to \bar{\mathbf{F}}_p^{\cdot} X$$

where $\bar{\mathbf{F}}_p^{(s)} X = \bar{\mathbf{F}}_p \circ \ldots \circ \bar{\mathbf{F}}_p X$ with the composition taken s+1 times. Since

$$\pi_* \bar{\mathbf{F}}_p X = \bar{H}_* X$$

and $\bar{\mathbf{F}}_p X$ is a simplicial vector space, we have that

$$H^* \bar{\mathbf{F}}_p X \cong \bar{G}(H^* X)$$

as an unstable algebra. Thus for any space Y and H^*X of finite type, we have isomorphisms

$$\begin{aligned} \pi^s \pi_t map_*(Y, \bar{\mathbf{F}}_p^{\cdot} X) &\cong \pi^s [\Sigma^t Y, \bar{\mathbf{F}}_p^{\cdot} X] \\ &= \pi^s Hom_{\mathcal{UA}}(H^* \bar{\mathbf{F}}_p^{\cdot} X, H^* \Sigma^t Y) \end{aligned}$$

since simplicial vector spaces are Eilenberg-MacLane spaces. So

$$\pi^s \pi_t map_*(Y, \bar{\mathbf{F}}_p^{\cdot} X) \cong Ext^s_{\mathcal{UA}}(H^* X, H^* \Sigma^t Y) \tag{1.1}$$

wherever this makes sense; that is, for t > 0 if s $\geq$ 1.

Now, Bousfield and Kan noticed that given a fibrant cosimplicial space $Z^{\cdot}$, there is a spectral sequence

$$\pi^s \pi_t Z^{\cdot} \Rightarrow \pi_{t-s} Tot(Z^{\cdot}) \tag{1.2}$$

where Tot($Z^{\cdot}$) is the simplicial set of cosimplicial maps

$$\mathrm{Tot}(Z^{\cdot}) = \mathrm{map}(\Delta, Z^{\cdot})$$

where Δ is the cosimplicial space with $\Delta^s = \Delta[\mathrm{s}]$, the standard s-simplex. The definition of this spectral sequence will be spelled out in section 5. If we define the p-completion of a space X by the equation

$$X_p = Tot(\bar{\mathbf{F}}_p^{\cdot} X)$$

and set $Z^{\cdot} = map_*(Y, \bar{\mathbb{F}}^{\cdot}_p Z)$, we have that

$$(1.3) \qquad Tot\ map_*(Y, \bar{\mathbb{F}}^{\cdot}_p X) = map_*(Y, Tot(\bar{\mathbb{F}}^{\cdot}_p X)) = map_*(Y, X_p).$$

Combining (1.1), (1.2), and (1.3) we obtain the Bousfield-Kan spectral sequence:

$$(1.4) \qquad Ext^s_{\mathcal{UA}}(H^*X, H^*\Sigma^t Y) \Rightarrow \pi_{t-s} map_*(Y, X_p)$$

We insist, to make the conclusions above, that H^*X and H^*Y be of finite type. Convergence of the spectral sequence of (1.4) is not automatic, but follows when H^*Y is finite. See [8].

The relationship between X and X_p is not evident either. There is a natural map

$$\eta : X \to X_p$$

and under various hypotheses on the fundamental group of X, η is an isomorphism in homology with $\mathbb{F}_p$ coefficients and the induced map

$$\pi_n \eta : \pi_n X \to \pi_n X_p$$

is a suitably defined $\mathbb{F}_p$ completion. This will be true if, for example, X is simply connected or nilpotent. See [8] for details.

One of the purposes of this paper is to explore $Ext_{\mathcal{UA}}$. An initial step is the following result, deceptive in its simplicity:

Proposition 1.5: Let $M \in \mathcal{U}$ and $K \in \mathcal{UA}$. Then there is a natural isomorphism

$$Ext^s_{\mathcal{UA}}(U(M), K) \cong Ext^s_{\mathcal{U}}(M, IK)$$

where $IK \in \mathcal{U}$ is the augmentation ideal of K.

This is proved in [5], among other places.

2. The Quillen cohomology of unstable algebras

The purpose of this section is to extend the definition of $Ext_{\mathcal{UA}}$ to a larger category and, therfore, obtain greater flexibility for calculation.

Let $s\mathcal{UA}$ be the category of simplicial unstable algebras over the Steenrod algebra. We already have an example of an object of this category: $\bar{G}.(H)$ with $H \in \mathcal{UA}$. Another example — admittedly a trivial one — is a constant simplicial object: if $H \in \mathcal{UA}$, then we may regard H as an object of $s\mathcal{UA}$ by letting $H_n = H$ for all n and setting all face a degeneracy operators to be the identity.

The initial observation is that $s\mathcal{UA}$ has a structure of a closed model category in the sense of Quillen. There are weak equivalences, fibrations and cofibrations satisfying the axioms CM1–CM5 of [17]. We now supply the definitions. Notice that for $A \in s\mathcal{UA}$, we have that $\pi_* A$ is a bigraded, supplemented, commutative $\mathbf{F}_p$-algebra, that $\pi_0 A \in s\mathcal{UA}$, and that for each $n > 0, \pi_n A \in \mathcal{U}$. Furthermore, $\pi_0 A$ is a quotient of A_0 and the quotient map

$$A_0 \to \pi_0 A$$

defines a map of simplicial algebras

$$\epsilon : A \to \pi_0 A$$

where $\pi_0 A$ is regraded as a constant simplicial algebra. If $A = \bar{G}.H$ as in the previous section, then this augmentation is the one given there:

$$\bar{G}.H \to \pi_0 \bar{G}.H \cong H.$$

If $f : A \to B$ is a morphism in $s\mathcal{UA}$, we obtain a diagram

$$\begin{array}{ccc} A & \xrightarrow{\epsilon} & \pi_0 A \\ \downarrow f & & \downarrow \pi_0 f \\ B & \xrightarrow{\epsilon} & \pi_0 B \end{array}$$

and hence a canonical map in $s\mathcal{UA}$

$$(f, \epsilon) : A \to B \times_{\pi_0 B} \pi_0 A$$

where the target is the evident pullback. The morphism f will be called *surjective on components* if this map is a surjection.

Definition 2.1:1.) A morphism $f : A \to B$ in $s\mathcal{UA}$ is a *weak equivalence* if

$$\pi_* f : \pi_* A \to \pi_* B$$

is an isomorphism.

2.) $f : A \to B$ is a *fibration* if it is a surjection on components; f is an *acyclic fibration* if it is a fibration and a weak equivalence.

3.) $f : A \to B$ is a *cofibration* if for every acyclic fibration $p : X \to Y$ in $s\mathcal{UA}$, there is a morphism $B \to X$ so that is the following diagram both triangles commute:

$$\begin{array}{ccc} A & \to & X \\ \downarrow f & \nearrow & \downarrow p \\ B & \to & Y. \end{array}$$

As specializations of these ideas we have *fibrant* and *cofibrant* objects. We write $\mathbb{F}_p$ for the terminal and the initial object of $s\mathcal{UA}$. Then we say that $A \in s\mathcal{UA}$ is cofibrant if the unit map $\eta : \mathbb{F}_p \to A$ is a cofibration. Similarly, we say that A is fibrant if the augmentation $\epsilon : A \to \mathbb{F}_p$ is a fibration. Every object in $s\mathcal{UA}$ is fibrant, so we say no more about this concept.

The following now follows from Theorem 4, pII.4.1 of [17].

Proposition 2.2: With the notions of weak equivalence, fibration, and cofibration defined above, $s\mathcal{UA}$ is a closed model category.

Of course, cofibrations are somewhat mysterious objects and difficult to recognize at this point. We will now be more concrete.

Let $G : n\mathbb{F}_p \to \mathcal{UA}$ be the left adjoint to augmentation ideal functor I. This functor was discussed in the previous section. We will call a morphism $f : A \to B$ in $s\mathcal{UA}$ *almost-free* if, for every n ≥ 0, there is a sub-vector space $V_n \subseteq IB_n$ and maps of vector spaces

$$\delta_i : V_n \to V_{n-1}, \qquad 1 \leq i \leq n$$

$$\sigma_i : V_n \to V_{n+1}, \qquad 0 \le i \le n$$

so that the evident extension

$$A_n \otimes G(V_n) \to B_n$$

is an isomorphism for each n and there are commutive diagrams, with the horizontal maps isomorphisms:

$$\begin{array}{ccc} A_n \otimes G(V_n) & \xrightarrow{\cong} & B_n \\ \downarrow {\scriptstyle d_i \otimes G\delta_i} & & \downarrow {\scriptstyle d_i} \\ A_{n-1} \otimes G(V_{n-1}) & \xrightarrow{\cong} & B_{n-1} \end{array}$$

for i ≥ 1 and

$$\begin{array}{ccc} A_n \otimes G(V_n) & \xrightarrow{\cong} & B_n \\ \downarrow {\scriptstyle s_i \otimes G\sigma_i} & & \downarrow {\scriptstyle s_i} \\ A_{n+1} \otimes G(V_{n+1}) & \xrightarrow{\cong} & B_{n+1} \end{array}$$

for i ≥ 0. Only d_0 is *not* induced up from n$\mathbf{F}_p$. The following result (which is implicit in Quillen, section II.4) can be proved exactly as the corresponding result in section 3 of [19,20].

Theorem 2.3: Almost-free morphisms are cofibrations.

Proposition 2.4: Any morphism $f : A \to B$ in $s\mathcal{UA}$ may be factored canonically as

$$A \xrightarrow{i} X \xrightarrow{p} B$$

with i almost-free and p an acyclic fibration.

We will prove Proposition 2.4, as the construction will prove useful in the later discussion. To begin, let $H \in \mathcal{UA}$. Then we may define the category $H/\mathcal{UA}$ to be the category of objects under H; that is, objects $K \in \mathcal{UA}$ equipped with a morphism $H \to K$ in $\mathcal{UA}$ making K into an H-algebra. The augmentation ideal functor $I : H/\mathcal{UA} \to n\mathbf{F}_p$ has a left adjoint

$$G^H(V) = H \otimes G(V).$$

This pair of adjoint functors yields a cotriple $\bar{G}^H : H/\mathcal{UA} \to H/\mathcal{UA}$ and, as in the previous section, this yields an augmented simplicial object

$$\bar{G}^H_{\cdot} K \to K$$

for any object $K \in H/\mathcal{UA}$. If $H = \mathbb{F}_p$, this is exactly the situation of the previous section.

Now, let $f : A \to B$ be a morphism in $s\mathcal{UA}$. Then the last paragraph yields an augmented bisimplicial algebra

$$\bar{G}^A_{\cdot,\cdot} B \to B \tag{2.5}$$

with

$$\bar{G}^A_{p,q} B = (\bar{G}^{A_q})^{p+1} B_q.$$

Let

$$\bar{G}^A B = diag(\bar{G}^A_{\cdot,\cdot} B)$$

be the resulting diagonal simplicial algebra. Thus, we have factored $f : A \to B$ as

$$A \to \bar{G}^A B \to B. \tag{2.6}$$

The first map is almost-free, the second map is a fibration, and the construction is canonical and functorial in f. We need only show that $\bar{G}^A B \to$ B is an acyclic fibration. But, since $\bar{G}^A B$ is the diagonal simplicial algebra of $\bar{G}^A_{\cdot,\cdot} B$, we may filter $\bar{G}^A_{\cdot,\cdot} B$ by degree in q to obtain a spectral sequence converging to $\pi_* \bar{G}^A B$. But since $\pi_* \bar{G}^{A_q}_{\cdot} B_q \cong B_q$, and the isomorphism is induced by the augmentation, the result follows.

The great strength of the construction of (2.6) is precisely that $\bar{G}^A B$ is the diagonal of a bisimplicial algebra. This allows the construction of many spectral sequences.

As a bit of notation, if $f = \eta : \mathbb{F}_p \to B$ we abbreviate $\bar{G}^{\mathbb{F}_p} B$ as $\bar{G}_{\cdot} B$ in keeping with the conventions of the previous section.

Indeed, consider the case where $H \in \mathcal{UA}$ is regarded as a constant simplcial algebra and we take the morphism f to be the be the unit map $\mathbb{F}_p \to H$. Then the construction of (2.6) yields an acyclic fibration $X \to H$

with X cofibrant. The reader should note that $X = \bar{G}.(H)$, as in the previous section and that the acyclic fibration is the augmentation

$$\bar{G}.(H) \to H.$$

The next obvious subject to bring up is the definition of the homology of an object in the model category $s\mathcal{UA}$ — after all, if $s\mathcal{UA}$ is supposed to be a good place to do homotopy theory, it must have a good notion of homology. However, in order to make sure that our constructions are well-defined, we need technical lemma on homotopies. For this, of course, we need the definition of homotopy. Notice that in $s\mathcal{UA}$, tensor product is the coproduct and if $A \in s\mathcal{UA}$, then the algebra multiplication

$$\mu : A \otimes A \to A$$

is the "fold" map; that is, multiplication supplies the canonical map from the coproduct from A to itself. Factor μ as a cofibration followed by an acyclic fibration

$$A \otimes A \xrightarrow{i} Cy(A) \xrightarrow{p} A.$$

By Proposition 2.4 this may be done functorially in A. Cy(A) is a cylinder object on A. Then two morphisms $f, g : A \to B$ in $s\mathcal{UA}$ are homotopic if there is a morphism H making the following diagram commute

$$\begin{array}{ccc} A \otimes A & \xrightarrow{i} & Cy(A) \\ \downarrow{\scriptstyle f \vee g} & & \downarrow{\scriptstyle H} \\ B & \xrightarrow{=} & B \end{array}$$

where $f \vee g = \mu(f \otimes g)$. If $f = g$ and we let H be the composite

$$Cy(A) \xrightarrow{p} A \xrightarrow{f} B$$

we obtain the constant homotopy from f to itself. The reader is invited to prove that homotopy defines an equivalence relation on the set of maps from an object A to an object B.

We can specialize these notions somewhat. If $h : C \to A$ is another morphism in $s\mathcal{UA}$ and $f, g : A \to B$ are two maps, then we say that f and g are *homotopic under* C if $fh = gh$ and there is some homotopy from f to

g which restricts to the constant homotopy on fh. If $q : B \to D$ is a map, then there is a corresponding notion of a *homotopy over* D.

The following, then, is the lemma that we need to show that our definitions of homology and cohomology will be well-defined.The proof is in [18] as Proposition 1.3.

Lemma 2.7: Let $f : A \to B$ be a cofibration and $p : X \to Y$ be an acyclic fibration. Then any two solutions $B \to X$ in the diagram

$$\begin{array}{ccc} A & \to & X \\ \downarrow f & \nearrow & \downarrow p \\ B & \to & Y \end{array}$$

are homotopic under A and over Y.

In the following $\mathcal{A}$ denotes the Steenrod algebra.

Definition 2.8: Let $A \in s\mathcal{UA}$. Define $H_*^{Q\mathcal{A}}A$ as follows. Choose an acyclic fibration

$$p : X \to A$$

with X cofibrant in $s\mathcal{UA}$ and set

$$H_*^{Q\mathcal{A}}A = \pi_*(\mathbb{F}_p \otimes_{\mathcal{A}} QX).$$

Define $H^*_{Q\mathcal{A}}A$ by

$$H^*_{Q\mathcal{A}}A = (H_*^{Q\mathcal{A}}A)^* = Hom_{\mathbf{F}_p}(H_*^{Q\mathcal{A}}A, \mathbb{F}_p).$$

Remark 2.9: It is a consequence of Lemma 2.7 that $H_*^{Q\mathcal{A}}A$ is well-defined and functorial in A. It is also a consequence of Lemma 2.7 that if

$$f : A \to B$$

is a weak equivalence in $s\mathcal{UA}$ then

$$H_*^{Q\mathcal{A}}f{:}H_*^{Q\mathcal{A}}A \to H_*^{Q\mathcal{A}}B$$

is an isomorphism.

Example 2.10.1.) Let X be space. Then we may regard H^*X as a constant simplicial algebra in $s\mathcal{UA}$. Then, as mentioned above, the augmented simplicial algebra

$$\bar{G}.H^*X \to H^*X$$

is an acylic fibration in $s\mathcal{UA}$ and $\bar{G}.H^*X$ is an almost-free and, hence cofibrant object in $s\mathcal{UA}$. Then we have

$$\begin{aligned} H^*_{Q\mathcal{A}}H^*X &\cong Hom_{\mathbf{F}_p}(H_*^{Q\mathcal{A}}H^*X, \mathbf{F}_p) \\ &\cong \pi^* Hom_{\mathbf{F}_p}(\mathbf{F}_p \otimes_{\mathcal{A}} \bar{G}.H^*X, \mathbf{F}_p) \end{aligned}$$

by the universal coefficient theorem for fields. Therefore, we have, in internal degree t,

$$\begin{aligned} [H^*_{Q\mathcal{A}}H^*X]_t &\cong \pi^* Hom_{\mathbf{F}_p}(\mathbf{F}_p \otimes_{\mathcal{A}} Q\bar{G}.H^*X, \Sigma^t\mathbf{F}_p) \\ &\cong \pi^* Hom_{\mathcal{UA}}(\bar{G}.H^*X, H^*S^t) \\ &\cong Ext^*_{\mathcal{UA}}(H^*X, H^*S^t) \end{aligned}$$

Thus $H^*_{Q\mathcal{A}}$ is one way to generalize $Ext_{\mathcal{UA}}$.

Example 2.10.2.) As an example of a simplicial algebra with interesting higher homotopy, we offer the bar construction. Let $H \in \mathcal{UA}$ and let $\bar{B}(H)$ be the bar construction on H. Then $\bar{B}(H) \in s\mathcal{UA}$ and

$$\pi_*\bar{B}(H) \cong Tor_*^H(\mathbf{F}_p, \mathbf{F}_p).$$

This bigraded algebra is a Hopf algebra, a divided power algebra, and more. We will see below in (2.15) that

$$H^s_{Q\mathcal{A}}\bar{B}(H)_t \cong Ext^{s-1}_{\mathcal{UA}}(H, H^*S^t).$$

This offers a new perspective for computing $Ext_{\mathcal{UA}}$.

We end this section with an example of the flexibility that general objects in $s\mathcal{UA}$ supplies. This is the long exact sequence of a cofibration in

$s\mathcal{UA}$ – a long exact sequence related to Quillen's transitivity sequence [18]. Let $f : A \to B$ be a morphism in $s\mathcal{UA}$. Using the construction of (2.6), form the commutative square

$$\begin{array}{ccc} \bar{G}.A & \xrightarrow{\bar{G}.f} & \bar{G}.B \\ \downarrow p_A & & \downarrow p_B \\ A & \xrightarrow{f} & B \end{array}$$

and factor $\bar{G}.f$ as an almost-free map followed by an acyclic fibration

$$\bar{G}.A \xrightarrow{i} X \xrightarrow{p} \bar{G}.B.$$

Then define the *mapping cone* of the morphism f by the equation

$$M(f) = \mathbb{F}_p \otimes_{\bar{G}.A} X.$$

$M(f)$ is almost-free and, hence, cofibrant. Lemma 2.7 implies that $M(f)$ is unique up to homotopy equivalence and functorial in f up to homotopy. (A homotopy equivalence is a weak equivalence with a homotopy inverse.) We could use the construction of (2.6) to make $M(f)$ strictly functorial.

Proposition 2.11: There is a long exact sequence in homology

$$\cdots \to H_n^{Q\mathcal{A}} A \xrightarrow{H_*^{Q\mathcal{A}} f} H_n^{Q\mathcal{A}} B \to H_n^{Q\mathcal{A}} M(f) \to H_{n-1}^{Q\mathcal{A}} A \to \cdots$$

and a long exact sequence in cohomology

$$\cdots \to H^{n-1}_{Q\mathcal{A}} A \to H^n_{Q\mathcal{A}} M(f) \to H^n_{Q\mathcal{A}} B \xrightarrow{H^*_{Q\mathcal{A}} f} H^n_{Q\mathcal{A}} A \to \cdots.$$

Proof: The cohomology result is obtained from the homology result by dualizing. To prove the homology result, notice that since $\bar{G}.A$ is almost-free and i is an almost-free morphsim, the sequence of simplicial algebras

$$\bar{G}.A \xrightarrow{i} X \to \mathbb{F}_p \otimes_{\bar{G}.A} X$$

yields a short exact sequence of simplicial vector spaces

$$0 \to \mathbb{F}_p \otimes_{\mathcal{A}} Q\bar{G}.a \to \mathbb{F}_p \otimes_{\mathcal{A}} QX \to \mathbb{F}_p \otimes_{\mathcal{A}} Q(\mathbb{F}_p \otimes_{\bar{G}.A} X) \to 0$$

Since $p : X \to \bar{G}_{.}B$ is an acyclic fibration and the composition of cofibrations is a cofibration, we have that

$$\pi_*\mathbb{F}_p \otimes_{\mathcal{A}} X \cong H_*^{Q\mathcal{A}} B$$

and the result follows.

The higher homotopy of $M(f)$ is often non-trivial, even if $\pi_* A$ and $\pi_* B$ are concentrated in degree 0. For computational purposes, we have the following result, from [17, Theorem II.6.b)]. Let $f : A \to B$ be a morphism in $s\mathcal{UA}$.

Proposition 2.12: There is a first quadrant spectral sequence of algebras

$$Tor_p^{\pi_* A}(\mathbb{F}_p, \pi_* B)_q \Rightarrow \pi_{p+q} M(f).$$

Notice that if $f : H \to K$ is a map of constant simplicial objects in $s\mathcal{UA}$, then this result implies that

$$\pi_* M(f) \cong Tor_*^H(\mathbb{F}_p, K).$$

Of particular interest is the case where B $= \mathbb{F}_p$ is the terminal object in $s\mathcal{UA}$ and $f = \epsilon : A \to \mathbb{F}_p$ is the augmentation. Because the cofiber of a the map to the terminal object deserves to be called a suspension, we define the *suspension* of A by the equation

$$\Sigma A = M(\epsilon).$$

Since $H_*^{Q\mathcal{A}} \mathbb{F}_p = 0$, 2.11 says that there are isomorphisms

$$H_n^{Q\mathcal{A}} \Sigma A \cong H_{n-1}^{Q\mathcal{A}} A \qquad n \geq 1$$

(2.13) $$H^n_{Q\mathcal{A}} \Sigma A \cong H^{n-1}_{Q\mathcal{A}} A \qquad n \geq 1$$

and

$$H_0^{Q\mathcal{A}} \Sigma A = 0 = H^0_{Q\mathcal{A}} \Sigma A.$$

The suspension has other properties that are worth recording here. For example, from [12] we have that there is a homotopy associative coproduct

$$\psi : \Sigma A \longrightarrow \Sigma A \otimes \Sigma A$$

that gives $\pi_*\Sigma A$ the structure of a Hopf algebra that is connected in the sense that $\pi_0\Sigma A = \mathbb{F}_p$. This coproduct can be used to turn the spectral sequence, obtained as a corollary to Proposition 2.12

$$Tor_*^{\pi_* A}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*\Sigma A \tag{2.14}$$

into a spectral sequence of Hopf algebras.

To specialize even further, if we regard $H \in \mathcal{UA}$ as a constant simplicial algebra, then the spectral sequence of (2.14) collapses and we obtain an isomorphism of Hopf algebras

$$\pi_*\Sigma H \cong Tor_*^H(\mathbb{F}_p, \mathbb{F}_p).$$

Finally, the work of Miller [19,Section 5;20] implies that if $\bar{B}(H)$ is the bar construction, then there is a weak equivalence is $s\mathcal{UA}$

$$\Sigma H \longrightarrow \bar{B}(H). \tag{2.15}$$

Thus (2.13) and (2.10.1) sustain the claims of Example 2.10.2.

2.16: The homotopy category. Associated to $s\mathcal{UA}$ and the closed model category structure we have on $s\mathcal{UA}$ there is an associated homotopy category. This category has the same objects as $s\mathcal{UA}$ and morphisms

$$[A, B]_{s\mathcal{UA}} = Hom_{s\mathcal{UA}}(X, B)/\sim$$

where $\sim$ denotes the equivalence relation generated by homotopy and $p : X \longrightarrow A$ is an acyclic fibration with X cofibrant. Lemma 2.7 implies that $[A, B]_{s\mathcal{UA}}$ is well-defined. A morphism in the homotopy category may be represented by a diagram

$$A \xleftarrow{p} X \xrightarrow{f} B$$

and an *isomorphism* in the homotopy category is such a diagram where f is a weak equivalence. This homotopy category is relatively simple because every object in $s\mathcal{UA}$ is fibrant.

Notice that for $f : A \rightarrow B$, the mapping cone $M(f)$ is well-defined in the homotopy category and that ΣA is co-group object in the homotopy category.

3. The Bousfield-Kan Spectral Sequence II

In this section we show that the Quillen cohomology of the previous section is the E_2 term of a more general spectral sequence than that described in section 1. This spectral sequence will converge to the homotopy groups of the total space of a cosimplicial space that is often interesting in applications. We end the section with some examples: a universal infinite cycle and a universal r-cycle.

If X is a (pointed) space, let $X \to \bar{\mathbb{F}}_p^{\cdot} X$ be the augmented cosimplicial space of the first section. Then if we let $Z = Z^{\cdot}$ be a fibrant cosimplicial space. Then we may use the functor $\bar{\mathbb{F}}_p^{\cdot}(\ )$ to define an augmented bi-cosimplicial space

$$Z \to \bar{\mathbb{F}}_p^{\cdot} Z^{\cdot} \tag{3.1}$$

by setting

$$(\bar{\mathbb{F}}_p^{\cdot} Z^{\cdot})^{(s,t)} = \bar{\mathbb{F}}_p^{s+1} Z^t$$

and letting the augmentation $Z^t \to \bar{\mathbb{F}}_p^{\cdot} Z^t$ define the augmentation for (3.1). Define $\bar{\mathbb{F}}_p^{\cdot} Z$ by the equation

$$\bar{\mathbb{F}}_p^{\cdot} Z = diag(\bar{\mathbb{F}}_p^{\cdot} Z^{\cdot}).$$

Thus

$$(\bar{\mathbb{F}}_p^{\cdot} Z)^s = \bar{\mathbb{F}}_p^{s+1} Z^s.$$

The augmentation of (3.1) induces a canonical map of cosimplicial spaces

$$\eta : Z \to \bar{\mathbb{F}}_p^{\cdot} Z.$$

Now let us consider the induced map of simplicial algebras

$$H^*\eta : H^*\bar{\mathbb{F}}_p^{\cdot} Z \to H^* Z.$$

An examination of the definitions of (2.5) and (2.6) demonstrate that we have a natural commutative square with the vertical maps isomorphisms:

$$\begin{array}{ccc} H^*\bar{\mathbb{F}}_p^{\cdot} Z & \xrightarrow{H^*\eta} & H^* Z \\ \downarrow \cong & & \downarrow = \\ \bar{G}_{\cdot} H^* Z & \xrightarrow{p} & H^* Z \end{array} \tag{3.3}$$

In particular, we have proven the following result.

Lemma 3.4: In the category $s\mathcal{UA}$

$$H^*\eta : H^*\bar{\mathbb{F}}_p^{\cdot} Z \to H^* Z$$

is an acylic fibration with $H^*\bar{\mathbb{F}}_p^{\cdot} Z$ almost-free.

The following result now delineates the affect of the construction (3.2) in homotopy.

Lemma 3.5: Let Z be a fibrant cosimplicial space. Suppose that

$$\pi^s H_t Z = 0, \qquad t - s \leq 1$$

and, for all n and sufficiently large s,

$$\pi^s H_{s+n} Z = 0.$$

Then

$$Tot(\eta) : Tot(Z) \to Tot(\bar{\mathbb{F}}_p^{\cdot} Z)$$

is the Bousfield-Kan $\mathbb{F}_p$-completion of $Tot(Z)$.

We postpone the proof to record a corollary of the previous two lemmas.

Corollary 3.6: Let Z be a fibrant cosimplicial space so that $\pi_s H^t Z$ is finite for all s and t, $\pi_s H^t Z = 0$ or all $t - s \leq 1$ and $\pi_s H^{s+n} Z = 0$ for all n and sufficiently large s. Then there is a convergent spectral sequence

$$[H^s_{Q\mathcal{A}}(H^* Z)]_t \Rightarrow \pi_{t-s} Tot(Z)_p.$$

Proof: This is the homotopy spectral sequence of the cosimplicial space $\bar{\mathbb{F}}_p^{\cdot} Z$:

$$\pi^s \pi_t \bar{\mathbb{F}}_p^{\cdot} Z \Rightarrow \pi_{t-s} Tot(Z)_p.$$

We must notice that under the finiteness hypotheses listed, we have

$$\begin{aligned}\pi^s \pi_t \bar{\mathbb{F}}_p^{\cdot} Z &\cong \pi^s Hom_{\mathbb{F}_p}(\mathbb{F}_p \otimes_{\mathcal{A}} H^* \bar{\mathbb{F}}_p^{\cdot} Z, H^* S^t) \\ &\cong [H^s_{Q\mathcal{A}}(H^* Z)]_t.\end{aligned}$$

The result now follows from Lemma 3.5.

Remark: If Z is *not* a fibrant cosimplicial space, we still get a spectral sequence

$$H^*_{Q\mathcal{A}} H^* Z \Rightarrow \pi_* Tot(\bar{\mathbb{F}}^{\cdot}_p Z)$$

because $\bar{\mathbb{F}}^{\cdot}_p Z$ is fibrant, being group-like in the sense of Bousfield and Kan [8, X.4.9]. But we are not able to identify the the abuttment with the $\mathbb{F}_p$-completion of $Tot(Z)$. Indeed, $Tot(Z)$ may be uninitersting, but $Tot(\bar{\mathbb{F}}^{\cdot}_p Z)$ might be of great interest. We will give some examples below where this generality is of importance.

To prove Lemma 3.5, we need the following result of Bousfield [2, Theorem 3.5].

Theorem 3.7: Let Z be a fibrant cosimplicial space. Then there is a natural second quadrant spectral sequence

$$\pi^s H_t Z \Rightarrow H_{t-s} Tot(Z)$$

If $\pi^s H_t Z = 0$ for $t - s \leq 1$ and $\pi^s H_{s+n} Z = 0$ for all n and sufficiently large s, the spectral sequence converges and $Tot(Z)$ is simply connected.

Proof of Lemma 3.5: By the tower lemmas of Bousfield and Kan [8, III.6.2] $Tot(\bar{\mathbb{F}}^{\cdot}_p Z)$ is $\mathbb{F}_p$-complete. Theorem 3.7 and Lemma 3.4 imply that

$$H_* Tot(Z) \to H_* Tot(\bar{\mathbb{F}}^{\cdot}_p Z)$$

is an isomorphism. The result now follows from the universal property of $\mathbb{F}_p$-completion.

We complete this section with a sequence of examples to justify the generality.

Example 3.8: Let X be a pointed, fibrant space and let $Z^{\cdot} = X$ be the constant cosimplicial space on X; that is, $Z^s = X$ for all s and every coface and codegeneracy map is the identity. Then the construction of (3.2) and the spectral sequence of Corollary 3.6 yield the Bousfield-Kan spectral sequence of the first section. To identify the E_2 terms of these two spectral sequences we use Example 2.10.1.

Example 3.9: This example contructs a universal infinite cycle for the spectral sequence of Corollary 3.6.

We begin with some remarks on simplicial unstable algebras. If V is a simplicial graded $\mathbf{F}_p$-vector space, then we may define a trivial simplicial algbra V_+ as follows. For each n, give V_n the structure of a trivial A-module and let

$$[V_+]_n = V_n \oplus \mathbf{F}_p$$

be the trivial algbera; that is, the augmentation ideal of $[V_+]_n$ is V_n and $(V_n)^2 = 0$. The face and degeneracy maps of V_+ are the obvious ones and a moment's thought will demonstrate that

$$\pi_*(V_+) \cong (\pi_* V)_+$$

where $(\pi_* V)_+$ is the evident bigraded trivial algebra.

In particular, we let $K(p,q)$ be the simplicial graded vector space with

$$\pi_* K(p,q) \cong \Sigma^q \mathbf{F}_p$$

concentrated in π_p — we will say that the non-zero bidegree is in *external* degree p and *internal* degree q. For any object $A \in s\mathcal{UA}$, choose an acyclic fibration $p : X \to A$ with X cofibrant. Then, in the language of 2.15, we have

$$\begin{aligned}[A, K(p,q)_+]_{s\mathcal{UA}} &\cong [\mathbf{F}_p \otimes_{\mathcal{A}} QX, K(p,q)]_{sn\mathbf{F}_p} \\ &\cong Hom_{nn\mathbf{F}_p}(\pi_*(\mathbf{F}_p \otimes_{\mathcal{A}} QX), \pi_* K(p,q)) \\ &\cong (H^p_{Q\mathcal{A}} A)_q\end{aligned} \tag{3.10}$$

where $[A, B]_C$ means the homotopy classes of maps in the relevant category. The second isomorphism in (3.10) follows from the fact that a homotopy class of maps in the category of simplicial vector spaces is completely determined by the map on homotopy.

The conclusion to be drawn from (3.10) is that the functor $H^p_{Q\mathcal{A}}(\ )_q$ is a corepresentable functor — as any functor we label cohomology should be — and that $K(p,q)_+ \in s\mathcal{UA}$ acts as an Eilenberg-MacLane space in this category. Therefore, $H^*_{Q\mathcal{A}}K(p,q)_+$ is a good thing to compute. If we can do the computation for all p and q we will have computed the "algebra" of cohomology operations.

The next point of this example is that if $p \leq q$ (or $q-p \geq 0$), then there is a cosimplicial space $S(p,q)$ so that $H^*S(p,q) \cong K(p,q)_+$ in $s\mathcal{UA}$. Let Δ be the cosimplicial space with $\Delta[s]$ the standard s-simplex and let $sk_n(\ )$ be the n-skeleton functor. Then, for $q-p \geq 0$, let

$$S(p,p) = \Delta/sk_{p-1}\Delta$$

and

$$S(p,q) = \Sigma^{q-p}S(p,p).$$

In [6] it was shown that $H^*S(p,q) \cong K(p,q)_+$ and that S(p,q) has the following universal property. There is a class $\iota \in \pi^p\pi_q S(p,q)$ that is the *universal* infinite cycle in the sense that if Z is a fibrant cosimplicial space and $z \in \pi^p\pi_q Z$ survives to E_∞ in the homotopy spectral sequence

$$\pi^s\pi_t Z \Rightarrow \pi_{t-s}Tot(Z)$$

then there is a morphism of cosimplicial spaces

$$f : S(p,q) \to Z$$

so that

$$\pi^*\pi_* f(\iota) = z.$$

We will see this in section 5. Now $S(p,q)$ is not evidently fibrant; however, we can preform the construction (3.2) nonetheless and obtain

$$\eta : S(p,q) \to \bar{\mathsf{F}}^{\cdot}_p S(p,q).$$

Lemma 3.5 will no longer be valid, however. But, $\bar{\mathsf{F}}^{\cdot}_p S(p,q)$ is a fibrant cosimplicial space — being group-like — and Theorem 3.7 and the fact that $Tot(\bar{\mathbf{F}}^{\cdot}_p S(p,q))$ is p-complete imply that if $q-p>1$, then there is a homology equivalence

$$S^{q-p} \rightarrow Tot(\bar{\mathbf{F}}^{\cdot}_p S(p,q));$$

that is, $Tot(\bar{\mathbf{F}}^{\cdot}_p S(p,q))$ is the $\mathbf{F}_p$ completion of the sphere S^{q-p}. Therefore, we obtain a spectral sequence

$$(3.11) \qquad [H^s_{\mathcal{Q}\mathcal{A}} K(p,q)_+]_t \Rightarrow \pi_{t-s} S^{q-p}$$

where we, as is customary, confuse the sphere with its $\mathbf{F}_p$-completion. This spectral sequence is related to Barratt's desuspension spectral sequence [see 2, Section 4] and also [15, Section 3]. This spectral sequence is also universal in the following sense. Let Z be a fibrant cosimplicial space and, with $q-p>1$,

$$z \in \pi^p \pi_q \bar{\mathbf{F}}^{\cdot}_p Z \cong [H^p_{\mathcal{Q}\mathcal{A}} H^* Z]_q$$

a permanent cycle in the Bousfield-Kan spectral sequence. Then, by the remarks made on $S(p,q)$ above, there is a morphism of cosimplicial spaces

$$f : S(p,q) \rightarrow \bar{\mathbf{F}}^{\cdot}_p Z$$

so that $\pi^* \pi_* f(\iota) = z$. Then there is a commutative diagram

$$\begin{array}{ccc} S(p,q) & \xrightarrow{f} & \bar{\mathbf{F}}^{\cdot}_p Z \\ \downarrow \eta & & \downarrow \eta \\ \bar{\mathbf{F}}^{\cdot}_p S(p,q) & \xrightarrow{\bar{\mathbf{F}}^{\cdot}_p f} & \bar{\mathbf{F}}^{\cdot}_p \bar{\mathbf{F}}^{\cdot}_p Z. \end{array}$$

Since we have that

$$H^*\eta : H^* \bar{\mathbf{F}}^{\cdot}_p \bar{\mathbf{F}}^{\cdot}_p Z \rightarrow H^* \bar{\mathbf{F}}^{\cdot}_p Z$$

is a weak equivalence in $s\mathcal{U}\mathcal{A}$

$$\eta : \bar{\mathbf{F}}^{\cdot}_p Z \rightarrow \bar{\mathbf{F}}^{\cdot}_p \bar{\mathbf{F}}^{\cdot}_p Z$$

induces an isomorphism of spectral sequences. Thus, if we confuse

$$\iota \in \pi^p \pi_q S(p,q)$$

with its image under $\pi^*\pi_*\eta$ in

$$\pi^p\pi_q\bar{\mathsf{F}}^{\cdot}_p S(p,q) \cong [H^p_{\mathcal{Q}\mathcal{A}}K(p,q)_+]_q$$

we obtain a diagram of spectral sequences

$$(3.12)\qquad \begin{array}{ccc} [H^s_{\mathcal{Q}\mathcal{A}}K(p,q)_+]_t & \Rightarrow & \pi_{t-s}S^{q-p} \\ \downarrow {\scriptstyle H^*_{\mathcal{Q}\mathcal{A}}f} & & \downarrow {\scriptstyle \pi_* Tot(\bar{\mathsf{F}}^{\cdot}_p f)} \\ [H^s_{\mathcal{Q}\mathcal{A}}H^*Z]_t & \Rightarrow & \pi_{t-s}Tot(Z)_p \end{array}$$

so that $H^*_{\mathcal{Q}\mathcal{A}}f(\iota) = z$.

Thus we conclude that not only does $K(p,q)_+$ corepresent cohomology, but that this phenomenon extends in a precise way to the Bousfield-Kan spectral sequence as well.

Example 3.13: There is also a universal r-cycle. Let Δ and $sk_n(\ )$ be as in the previous example and set, for $r \geq 2$,

$$D(r,p,p) = sk_{p+r-1}\Delta / sk_{p-1}\Delta$$

and for $q - p \geq 0$

$$D(r,p,q) = \Sigma^{q-p}D(r,p,p).$$

These cosimplicial spaces have the following universal property: there are classes

$$\iota \in \pi^p\pi_q D(r,p,q)$$

and

$$\vartheta \in \pi^{p+r}\pi_{q+r-1}D(r,p,q)$$

so that ι survives to E_r in the homotopy spectral sequence and

$$d_r(\iota) = \vartheta.$$

This differential is universal in this sense: if Z is a fibrant cosimplicial space and $x \in \pi^p\pi_q Z$ survives to E_r in the homotopy spectral sequence for Z, and if

$$d_r(x) = y,$$

then there exists a morphism of cosimplicial spaces $f : D(r,p,q) \to Z$ so that

$$\pi^*\pi_* f(\iota) = x \qquad and \qquad \pi^*\pi_* f(\vartheta) = y.$$

$D(r,p,q)$ may not be fibrant, so the homotopy spectral sequence for this cosimplicial space must be adjusted as follows: there is a fibrant cosimplicial space $\hat{D}(r,p,q)$ and a homotopy spectral sequence

$$\pi^s\pi_t D(r,p,q) \Rightarrow \pi_{t-s}Tot(\hat{D}(r,p,q)).$$

However, this technicality will be avoided completely below.

Bousfield and Kan [6] have computed the homology spectral sequence

$$\pi^s H_t D(r,p,q) \Rightarrow H_{t-s}\hat{D}(r,p,q).$$

Let

$$h : \pi^*\pi_* D(r,p,q) \to \pi^* H_* D(r,p,q)$$

be the map induced by the Hurewicz homomorphism. Then $\pi^* H_* D(r,p,q)$ is of dimension 3 over $\mathbb{F}_p$ with basis

$$1 \in \pi^0 H_0 D(r,p,q)$$

$$h(\iota), h(\vartheta) \in \pi^* H_* D(r,p,q).$$

Since h induces a map of spectral sequences

$$d_r h(\iota) = h(\vartheta).$$

Thus the homology spectral sequence

$$\pi^* H_* D(r,p,q) \Rightarrow H_* Tot(\bar{\mathbb{F}}_p^{\cdot} D(r,p,q))$$

implies that $Tot(\bar{\mathbb{F}}_p^{\cdot} D(r,p,q))$ is contractible — if $q-p>1$.

Therefore, in the homotopy spectral sequence, $q-p>1$,

$$[H^s_{\mathcal{QA}} H^* D(r,p,q)]_t \Rightarrow \pi_{t-s} Tot(\bar{\mathbb{F}}_p^{\cdot} D(r,p,q))$$

we have $E_\infty = 0$. Incidentally,

$$H^*D(r,p,q) \cong (\bar{H}^*D(r,p,q))_+.$$

Finally, arguing as for 3.12, we see that if Z is a fibrant cosimplicial space and $x, y \in H^*_{Q\mathcal{A}}H^*Z$ are so that $d_r x = y$, then there is a map of spectral sequences

$$\begin{array}{ccc} H^*_{Q\mathcal{A}}H^*D(r,p,q) & \Rightarrow & \pi_*Tot(\bar{\mathbb{F}}^\cdot_p D(r,p,q)) = 0 \\ \downarrow {\scriptstyle H^*_{Q\mathcal{A}}f} & & \downarrow \\ H^*_{Q\mathcal{A}}H^*Z & \Rightarrow & \pi_*Tot(Z)_p \end{array}$$

This should imply the existence of many formal differentials.

4. Fibrations and the Bousfield-Kan Spectral Sequence

In this section we discuss certain fibration sequences of cosimplicial spaces, demonstrate the relationship between these and fibrations of spaces, and show how these behave with respect to the spectral sequence of the previous section. We close with some examples from the work of Mahowald. This section is one of the major justifications for the generality of the previous two sections.

The first remark to make is that there is another way to generalize the construction, for a space X

$$X \to \bar{\mathbb{F}}^\cdot_p X$$

of the first section. This will produce a relative version of this cosimplicial space. We do this by producing a triple on the category of spaces over a fixed space Y. This construction is the object used in [11] to define a fibre-wise completion of X.

Fix a pointed space Y and let $f : X \to Y$ be a map of pointed spaces. Define

$$(\bar{\mathbb{F}}_p)_Y X = Y \times \bar{\mathbb{F}}_p X \tag{4.1}$$

and give $(\bar{\mathbb{F}}_p)_Y(\ )$ the structure of a triple with the following structure maps. Define

$$\eta : X \to (\bar{\mathbb{F}}_p)_Y X$$

by

$$\eta = f \times \eta : X \to Y \times \bar{\mathsf{F}}_p X$$

where the second η is $\eta : X \to \bar{\mathsf{F}}_p X$ — the unit of the triple $\bar{\mathsf{F}}_p(\ )$. Define

$$\epsilon : (\bar{\mathsf{F}}_p)^2_Y X \to (\bar{\mathsf{F}}_p)_Y X$$

to be the composite

$$Y \times \bar{\mathsf{F}}_p(Y \times \bar{\mathsf{F}}_p X) \xrightarrow{1 \times \bar{\mathsf{F}}_p \pi_2} Y \times \bar{\mathsf{F}}^2_p X \xrightarrow{1 \times \epsilon} Y \times \bar{\mathsf{F}}_p X$$

where π_2 is the projection onto the second factor and $\epsilon : \bar{\mathsf{F}}^2_p X \to \bar{\mathsf{F}}_p X$ is the structure map for the triple $\bar{\mathsf{F}}_p(\ )$.

One easily checks that there are commutative diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\eta} & Y \times \bar{\mathsf{F}}_p X & = & (\bar{\mathsf{F}}_p)_Y X \\ \downarrow f & & \downarrow \pi_1 & & \\ Y & \xrightarrow{=} & Y & & \end{array}$$

where π_1 is projection onto the first factor and

$$\begin{array}{ccccccc} (\bar{\mathsf{F}}_p)^2_Y X & = & Y \times \bar{\mathsf{F}}_p(Y \times \bar{\mathsf{F}}_p X) & \xrightarrow{\epsilon} & Y \times \bar{\mathsf{F}}_p X & = & (\bar{\mathsf{F}}_p)_Y X \\ & & \downarrow \pi_1 & & \downarrow \pi_1 & & \\ & & Y & \xrightarrow{=} & Y & & \end{array}$$

and, thus, $((\bar{\mathsf{F}}_p)_Y, \eta, \epsilon)$ is a triple on the category of spaces over Y. This is the category whose objects are maps $f : X \to Y$ and whose morphisms are commutative diagrams. Let

(4.2) $$X \to (\bar{\mathsf{F}}_p)^{\cdot}_Y X$$

be the resulting augmented cosimplicial space over Y. Notice that if we prefer, we could say that there is a map of cosimplicial spaces

(4.3) $$(\bar{\mathsf{F}}_p)^{\cdot}_Y X \to Y$$

where Y is regarded as a constant cosimplicial space. Notice that the construction (4.2) is natural in the map $f : X \to Y$.

Therefore, we can generalize this relative construction to cosimplicial spaces. Suppose that

$$f : Z \to Y$$

is a map of cosimplicial spaces, with Y not necessarily a constant cosimplicial space. Then we can form the bi-cosimplicial space $(\bar{\mathsf{F}}_p)^{\cdot\cdot}_Y Z$ with

$$(\bar{\mathsf{F}}_p)^{(s,t)}_Y Z = (\bar{\mathsf{F}}_p)^{s+1}_{Y^t} Z^t$$

with the obvious vertical and horizontal coface and codegeneracy maps. Define

$$(\bar{\mathsf{F}}_p)^{\cdot}_Y Z = diag(\bar{\mathsf{F}}_p)^{\cdot\cdot}_Y Z. \tag{4.4}$$

The augmentation of (4.2) yields an augmentation

$$i : Z \to (\bar{\mathsf{F}}_p)^{\cdot}_Y Z$$

and the projection of (4.3) yields a natural projection

$$p : (\bar{\mathsf{F}}_p)^{\cdot}_Y Z \to Y$$

so that the composite

$$Z \xrightarrow{i} (\bar{\mathsf{F}}_p)^{\cdot}_Y Z \xrightarrow{p} Y$$

is the original map $f : Z \to Y$.

Lemma 4.5: There is an isomorphism

$$i_* : \pi^* H_* Z \to \pi^* H_* (\bar{\mathsf{F}}_p)^{\cdot}_Y Z.$$

Proof: By (4.4), there is a spectral sequence

$$\pi^t \pi^s H_* (\bar{\mathsf{F}}_p)^{\cdot\cdot}_Y Z \Rightarrow \pi^{s+t} H_* (\bar{\mathsf{F}}_p)^{\cdot}_Y Z.$$

But

$$\pi^s H_* (\bar{\mathsf{F}}_p)^{\cdot,t}_Y Z = \pi^s H_* (\bar{\mathsf{F}}_p)^{\cdot}_{Y^t} Z^t = \begin{cases} H_* Z^t, & \text{if } s = 0; \\ 0, & \text{if } s > 0. \end{cases}$$

and the result follows.

Remark 4.6: In actual fact, much more is true. There is a commutative diagram

$$\begin{array}{ccccc} H^*Y & \xrightarrow{p^*} & H^*(\bar{\mathbb{F}}_p)^{\cdot}_Y Z & \xrightarrow{i^*} & H^*Z \\ \downarrow = & & \downarrow & & \downarrow = \\ H^*Y & \xrightarrow{i} & \bar{G}^{H^*Y}_{\cdot} H^*Z & \xrightarrow{p} & H^*Z \end{array}$$

where the bottom row is the factoring of the morphism $f^* : H^*Y \to H^*Z$ as an almost-free map followed by an acyclic fibration constructed in Section 2. Thus p^* is almost free in $s\mathcal{UA}$.

Now suppose that Y is a group-like cosimplicial space. Then for any cosimplicial space Z and any map $f : Z \to Y$, one easily checks that $(\bar{\mathbb{F}}_p)^{\cdot}_Y Z$ is group-like and that

$$p : (\bar{\mathbb{F}}_p)^{\cdot}_Y Z \to Y$$

is a (level-wise) surjection of group-like objects in the category of cosimplicial spaces. Any such is a fibration in the category of cosimplicial spaces. If we let $*$ denote the initial object in the category of cosimplicial spaces, then we may define the fiber $F(p)$ of p by the pull-back diagram

$$\begin{array}{ccc} F(p) & \to & (\bar{\mathbb{F}}_p)^{\cdot}_Y Z \\ \downarrow & & \downarrow p \\ * & \to & Y. \end{array}$$

Lemma 4.7: If $F(p)$ is the fiber of $p : (\bar{\mathbb{F}}_p)^{\cdot}_Y Z \to Y$ with Y group-like, then $F(p)$ is group-like and there is a natural isomorphism

$$H^*F(p) \cong \mathbb{F}_p \otimes_{H^*Y} H^*(\bar{\mathbb{F}}_p)^{\cdot}_Y Z.$$

Proof: For each s,

$$(\bar{\mathbb{F}}_p)^s_Y Z = (\bar{\mathbb{F}}_p)^{s+1}_{Y^s} Z^s = Y^s \times \bar{\mathbb{F}}_p((\bar{\mathbb{F}}_p)^s_{Y^s} Z^s).$$

Thus, for each s, there is a fibration sequence induced by p:

$$F(p)^s \to Y^s \times \bar{\mathbb{F}}_p((\bar{\mathbb{F}}_p)^s_{A^s} S^s) \xrightarrow{\pi_1} Y^s.$$

In particular,

$$F(p)^s = \bar{\mathbb{F}}_p((\bar{\mathbb{F}}_p)^s_{Y^s} Z^s)$$

and the result follows.

Because $F(p)$ is fibrant, $Tot(F(p))$ is a meaningful object from the point of view of homotopy theory. In particular, there is a fibration sequence in homotopy

$$Tot(F(p)) \to Tot((\bar{\mathbb{F}}_p)^{\cdot}_Y Z) \xrightarrow{Tot(p)} Tot(Y). \tag{4.8}$$

This follows from [8,p.277].

Now consider the case of an arbitrary map of fibrant cosimplicial spaces $f : Z \to Y$. Applying the functor $\bar{\mathbb{F}}^{\cdot}_p(\ )$ to this map, we obtain a map of group like cosimplicial spaces

$$\bar{\mathbb{F}}^{\cdot}_p f : \bar{\mathbb{F}}^{\cdot}_p Z \to \bar{\mathbb{F}}^{\cdot}_p Y.$$

If we apply the construction of (4.4) we obtain a factoring of $\bar{\mathbb{F}}^{\cdot}_p f$:

$$\bar{\mathbb{F}}^{\cdot}_p Z \xrightarrow{i} X = (\bar{\mathbb{F}}_p)^{\cdot}_{\bar{\mathbb{F}}^{\cdot}_p Y} \bar{\mathbb{F}}^{\cdot}_p Z \xrightarrow{p} \bar{\mathbb{F}}^{\cdot}_p Y$$

where p is a fibration and

$$i_* : \pi^* H_* \bar{\mathbb{F}}^{\cdot}_p Z \to \pi^* H_* X$$

is an isomorphism. This last implies that there is a homotopy equivalence

$$Tot(\bar{\mathbb{F}}^{\cdot}_p Z) \simeq Tot(X).$$

Let $F(p)$ be the fiber of $p : X \to \bar{\mathbb{F}}^{\cdot}_p Y$. Then, in light of Remark 4.6, Lemma 4.7, and the material before Proposition 2.12, we have that

$$\pi^* \pi_* F(p) \cong H^*_{Q\mathcal{A}} M(f^*)$$

where $M(f^*)$ is the mapping cone of $f^* : H^*Y \to H^*Z$ in $s\mathcal{UA}$. Therefore there is a spectral sequence

$$[H^s_{Q\mathcal{A}} M(f^*)]_t \Rightarrow \pi_{t-s} F(p).$$

Thus we obtain a fiber sequence up to homotopy

$$Tot(F(p)) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p$$

and a long exact sequence of E_2-terms:

$$\to [H^s_{Q\mathcal{A}} M(f^*)]_t \to [H^s_{Q\mathcal{A}} H^* Z]_t \xrightarrow{f^*} [H^s_{Q\mathcal{A}} H^* Y]_t \xrightarrow{\partial} [H^{s+1}_{Q\mathcal{A}} M(f^*)]_t \to$$

We would like the morphism ∂ to be induced by a morphism of spectral sequences. In the next section we will prove the following result.

Theorem 4.9: There is a diagram of spectral sequences

$$\begin{array}{ccc} [H^s_{Q\mathcal{A}} H^* Y]_t & \Rightarrow & \pi_{t-s} Tot(X)_p \\ \downarrow \partial & & \downarrow \delta \\ [H^{s+1}_{Q\mathcal{A}} M(f^*)]_t & \Rightarrow & \pi_{t-s-1} Tot(F(p)). \end{array}$$

where $\delta : \pi_{t-s} Tot(X)_p \to \pi_{t-s-1} Tot(F(p))$ is the boundary map induced from the homotopy fibration sequence

$$Tot(F(p)) \to Tot(Z)_p \xrightarrow{Tot(f)} Tot(Y)_p.$$

We close the section with a sequence of examples applying this technology.

Example 4.10: Let $Z = *$ be the initial object in the category of cosimplicial spaces. (Remember that all our spaces and morphisms are pointed.) Let Y be fibrant cosimplicial space so that $\pi^s H_t Y = 0$ for $t-s \leq 1$ and $\pi^s H_{s+n} Y = 0$ for all s and sufficiently large n. Then there is a natural weak equivalence

$$Tot(F(p)) \simeq \Omega Tot(Y)_p$$

because $Tot(Z)$ is contractible. On the other hand,

$$\epsilon = f^* : H^* Y \to H^* Z = \mathbf{F}_p$$

so $M(f^*) = \Sigma H^* Y$ in the terminology of 2.13. Therefore,

$$\partial : [H^s_{Q\mathcal{A}} H^* Y]_t \to [H^{s+1}_{Q\mathcal{A}} M(f^*)]_t \cong [H^{s+1}_{Q\mathcal{A}} \Sigma H^* Y]_t$$

is an isomorphism for all s and t, and we get a commutative diagram of spectral sequences, where the vertical maps are isomorphisms:

$$\begin{array}{ccc} [H^s_{Q\mathcal{A}} H^* Y]_t & \Rightarrow & \pi_{t-s} Tot(Y)_p \\ \downarrow \partial & & \downarrow \partial \\ [H^{s+1}_{Q\mathcal{A}} \Sigma H^* Y]_t & \Rightarrow & \pi_{t-s-1} \Omega Tot(X)_p \end{array}$$

Also, the Hurewicz map induces a map of spectral sequences

$$\begin{array}{ccc} [H^s_{Q\mathcal{A}} \Sigma H^* Y]_t & \Rightarrow & \pi_{t-s} \Omega Tot(Y)_p \\ \downarrow & & \downarrow \\ [\pi_s \Sigma H^* Y]^*_t & \Rightarrow & H_{t-s} \Omega Tot(Y)_p. \end{array}$$

This of particular importance if $Y = \bar{\mathbb{F}}^{\cdot}_p X$ for some pointed space X. Then (2.12) implies that

$$[\pi_s \Sigma H^* Y]_t \cong Tor^{H^* X}_s(\mathbb{F}_p, \mathbb{F}_p)_t$$

and, of course,

$$[H^s_{Q\mathcal{A}} \Sigma H^* Y]_t \cong Ext^{s-1}_{\mathcal{UA}}(H^* X, H^* S^t).$$

The upshot, then, is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^{s-1}_{\mathcal{UA}}(H^* X, H^* S^t) & \Rightarrow & \pi_{t-s} \Omega X_p \\ \downarrow & & \downarrow \\ Cotor^{H_* X}_s(\mathbb{F}_p, \mathbb{F}_p)_t & \Rightarrow & H_{t-s} \Omega X_p. \end{array} \tag{4.11}$$

The homology spectral sequence is easily seen to be isomorphic to the Eilenberg-Moore spectral sequence. A similar construction has been used by Bousfield and Curtis [5] and Bousfield and Kan [6].

It is worth pointing out that the spectral sequence of (2.12)

$$Tor^{\pi_* H^* Y}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_* \Sigma H^* Y$$

collapses for other examples than the example of $Y = \bar{\mathbb{F}}^{\cdot}_p X$; for example, it will collapse for either of the examples 3.8 or 3.13.

Example 4.12: In this example, we investigate the suspension homomorphism. Let $\bar{\mathsf{F}}_p(\ )$ be the underlying functor of the triple described in section 1. Then there is an evident natural map

$$\Sigma^k \bar{\mathsf{F}}_p X \to \bar{\mathsf{F}}_p \Sigma^k X$$

and this, in turn, induces a map of cosimplicial spaces

$$e_k : \bar{\mathsf{F}}^{\cdot}_p X \to \Omega^k \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X.$$

Since $Tot(\Omega^k Z) \cong \Omega^k Tot(Z)$ for any fibrant cosimplicial space Z, and since

$$\pi^s \pi_t \Omega^k \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X \cong \pi^s \pi_{t+k} \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X$$

we obtain a diagram of spectral sequences

$$\begin{array}{ccc} Ext^s_{\mathcal{UA}}(H^*X, H^*S^t) & \Rightarrow & \pi_{t-s}X_p \\ \downarrow \pi^*\pi_* e_k & & \downarrow E_k \\ Ext^s_{\mathcal{UA}}(H^*\Sigma^k X, H^*S^{t+k}) & \Rightarrow & \pi_{t-s+k}\Sigma^k X_p. \end{array}$$

where E_k is the suspension homomorphism.

Now, from the work of Mark Mahowald, it is known that, for the case $X = S^n$, the algebraic suspension homomorphism $\pi^*\pi_* e_k$ fits into a long exact sequence. The work we have done here allows us to give name — from the point of view of homological algebra — to the third term in this long exact sequence and, perhaps, more flexibility for computation. The following lemma will help us to identify the E_2 term of various spectral sequences.

Lemma 4.13: Let Z be a cosimplicial space so that, for every s, Z^s is homotopy equivalent to an Eilenberg-MacLane space and $\pi_* Z^s$ is a graded F_p vector space. Then, for all s and t, we have that the homomorphism induced by the augmentation

$$\pi^s \pi_t Z \to \pi^s \pi_t \bar{\mathsf{F}}^{\cdot}_p Z$$

is an isomorphism.

Proof: $\bar{\mathsf{F}}^{\cdot}_p Z = diag \bar{\mathsf{F}}^{\cdot,\cdot}_p Z$ where $\bar{\mathsf{F}}^{p,q}_p Z = \bar{\mathsf{F}}^{p+1}_p Z^q$. If we filter $\pi_* \bar{\mathsf{F}}^{\cdot,\cdot}_p Z$ be degree in q, we obtain a spectral sequence

$$\pi^q \pi^p \pi_* \bar{\mathsf{F}}^{\cdot,\cdot}_p Z \Rightarrow \pi^{p+q} \pi_* \bar{\mathsf{F}}^{\cdot}_p Z.$$

Because of the hypotheses on Z^q, we have

$$\pi^p \pi_* \bar{\mathsf{F}}^{\cdot}_p Z^q \cong \begin{cases} \pi_* Z^q, & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

The result follows.

The hypothesis of Lemma 4.13 applies to both $Z = \bar{\mathsf{F}}^{\cdot}_p X$ and, especially, $Z = \Omega^k \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X$. Therefore,

$$[H^s_{Q\mathcal{A}} H^* \bar{\mathsf{F}}^{\cdot}_p X]_t \cong Ext^s_{\mathcal{U}\mathcal{A}}(H^* X, H^* S^t)$$

and

$$[H^s_{Q\mathcal{A}} H^* \Omega^k \bar{\mathsf{F}}^{\cdot}_p \Sigma^k X]_t \cong Ext^s_{\mathcal{U}\mathcal{A}}(H^* \Sigma^k X, H^* S^{t+k})$$

and we obtain a long exact sequence

$$\begin{aligned} \to [H^s_{Q\mathcal{A}} M(e^*_k)]_t \to Ext^s_{\mathcal{U}\mathcal{A}}(H^* X, H^* S^t) & \\ \xrightarrow{\pi^* \pi_* e_k} Ext^s_{\mathcal{U}\mathcal{A}}(H^* \Sigma^k X, H^* S^{t+k}) & \to [H^{s+1}_{Q\mathcal{A}} M(e^*_k)]_t \to \end{aligned}$$

And if $C(E_k)$ is the homotopy fiber in the homotopy fibration sequence

$$C(E_k) \to X \xrightarrow{E_k} \Omega^k \Sigma^k X$$

then Theorem 4.9 implies that there is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^s_{\mathcal{U}\mathcal{A}}(H^* \Sigma^k X, H^* S^{t+k}) & \Rightarrow & \pi_{t+k-s} \Sigma^k X_p \\ \downarrow & & \downarrow \\ [H^{s+1}_{Q\mathcal{A}} M(e^*_k)]_t & \Rightarrow & \pi_{t-s-1} C(E_k)_p. \end{array}$$

Since there are techniques for computing $H^*_{Q\mathcal{A}} A$ from knowledge of $\pi_* A$ (see [13]), it would be nice to know $\pi_* M(e^*_k)$. In principal, this can be done as follows. First of all,

$$\pi_* H^* \bar{\mathsf{F}}^{\cdot}_p X \cong H^* X$$

concentrated in external degree 0. On the other hand

$$\pi_* H^* \Omega^k \bar{\mathbf{F}}_p^{\cdot} \Sigma^k X$$

can (and this is the part that is only in principal) be computed using the derived functors of Lannes's mapping object functors [16]. For example — and here we offer only the prime 2, $k = 2$, and $X = S^n$:

$$\pi_* H^* \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2} \cong \Lambda(i_n) \otimes \Gamma[x_{2n+1}, x_{2n+2}, y_j]$$

where

$$i_n \in \pi_0 H^n \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2}$$

$$x_j \in \pi_1 H^j \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2}$$

and

$$y_j \in \pi_{2^j+1} H^{2^j(4n+5)} \Omega^2 \bar{\mathbf{F}}_2^{\cdot} S^{n+2}, \qquad j \geq 0.$$

Λ and Γ denote the exterior and divided power algebras respectively.

Once $\pi_* H^* \Omega^k \bar{\mathbf{F}}_p^{\cdot} \Sigma^k X$ is computed, one can appeal to the spectral sequence of (2.12) to compute $\pi_* M(e_k^*)$. In the case of

$$e_2^* : H^* \Omega^2 \bar{\mathbf{F}}_p^{\cdot} S^{n+2} \longrightarrow H^* \bar{\mathbf{F}}_p^{\cdot} S^n$$

this spectral sequence will collapse.

5. The homotopy spectral sequence and twisted products

The purpose of this section is two-fold. First, we explain in detail how the homotopy spectral sequence of a fibrant cosimplicial space is constructed and, second, we use this explanation to prove Theorem 4.9. This theorem defines a boundary map in a "long-exact sequence" of spectral sequences. We begin with the first project.

Let Z be a fibrant cosimplicial space. If Y is a cosimplicial space, let $map(Y, Z)$ be the space of maps between Y and Z. The n-simplices of $map(Y, Z)$ are maps of cosimplicial spaces

$$\Delta[n] \times Y \to Z$$

where $\Delta[n]$ is the standard n-simplex. If Y, Z are pointed, let $map_*(Y, Z)$ denote the space of pointed maps between Y and Z. The n-simplices of this space are pointed maps of cosimplicial spaces

$$\Delta[n]_+ \wedge Y \to Z$$

where $+$ denotes a disjoint basepoint.

If Z is pointed and fibrant, there is a homotopy spectral sequence

$$\pi^s \pi_t Z \Rightarrow \pi_{t-s} Tot(Z)$$

where $Tot(Z) = map(\Delta, Z)$ and Δ is the cosimplicial space that is $\Delta[n]$ in cosimplicial degree n. This spectral sequence is a tower of fibrations

$$(5.1) \qquad \begin{array}{ccccccccc} \cdots & \to & Tot_3(Z) & \to & Tot_2(Z) & \to & Tot_1(Z) & \to & Tot_0(Z) \\ & & \uparrow & & \uparrow & & \uparrow & & \| \\ & & F_3 Z & & F_2 Z & & F_1 Z & & F_0 Z \end{array}$$

Here

$$Tot_n(Z) = map(sk_n \Delta, Z)$$

where $sk_n(\ )$ is the n-skeleton functor and the fibrations

$$Tot_n(Z) \to Tot_{n-1}(Z)$$

are determined by the inclusion $sk_{n-1}\Delta \to sk_n\Delta$. Thus the fiber is the mapping space

$$F_nZ = map_*(sk_n\Delta/sk_{n-1}\Delta, Z).$$

Here and elsewhere we make the convention that

$$X/sk_{-1}Y = X_+.$$

Bousfield and Kan have given a description of F_nZ. Let

$$M^nZ \subseteq Z^n \times \cdots \times Z^n$$

be the *matching space* given by

$$M^nZ = \{(z^0, z^1, \ldots, z^n) \mid s^iz^j = s^{j-1}z^i,\ 0 \le i < j \le n\}$$

where the s^i are the codegeneracies in Z. There is a natural map

$$s : Z^n \to M^{n-1}Z$$

given by

$$s(z) = (s^0z, s^1z, \ldots, s^{n-1}z).$$

The condition that Z be fibrant is equivalent to the condition that s be a fibration for all n. Let N^nZ be defined by the fibration sequence

(5.2) $$N^nZ \to Z \to M^{n-1}Z.$$

Bousfield and Kan now prove [8,X.6], using the fact that

$$[sk_n\Delta/sk_{n-1}\Delta]^n = S^n$$

that we have natural isomorphims

$$F_nZ \cong \Omega^nN^nZ$$

and

(5.3) $$\pi_t\Omega^nN^nZ \cong \pi_{t+n}N^nZ \cong N^n\pi_{t+n}Z$$

where $N^n\pi_{t+n}Z$ is the n^{th} group in the normalized cochain complex of the cosimplicial group $\pi_{t+n}Z$. Furthermore Bousfield [10.4 of 3] shows that the composite

$$\pi_t F_n Z \longrightarrow \pi_t Tot_n(Z) \longrightarrow \pi_{t-1}F_{n+1}Z$$

induced by the fibrations of (5.1) is equivalent, under the isomorphisms of (5.3) to

$$(5.4) \qquad (-1)^t\partial : N^n\pi_{t+n}Z \longrightarrow N^{n+1}\pi_{t+n}Z$$

where ∂ is the boundary operator of $N\pi_{t+n}Z$. Thus if we use the tower (5.1) to define a spectral sequence with

$$E_1^{s,t} = \pi_{t-s}F_sZ \cong N^s\pi_tZ$$

then the spectral sequence reads, because of (5.4),

$$E_2^{s,t} \cong \pi^s\pi_tZ \Rightarrow \pi_{t-s}Tot(Z)$$

where we have used the identification

$$Tot(Z) = \varprojlim Tot_n(Z).$$

This is the spectral sequence *under* $Tot(Z)$. We can build the same spectral sequence from a tower *over* $Tot(Z)$. This is often more convenient, especially as it allows one to use pointed mapping spaces at all times. First notice that

$$Tot(Z) = map_*(\Delta_+, Z).$$

Call this Tot^0Z. If $n \geq 1$, define

$$Tot^n(Z) = map_*(\Delta/sk_{n-1}\Delta, Z).$$

The fibration sequences

$$sk_{n-1}\Delta \longrightarrow \Delta \longrightarrow \Delta/sk_{n-1}\Delta$$

give rise to a diagram of fibration sequences

$$\begin{array}{ccccc} & & & & F_nZ \\ & & & & \downarrow \\ Tot^{n+1}(Z) & \rightarrow & Tot(Z) & \rightarrow & Tot_n(Z) \\ \downarrow & & \downarrow = & & \downarrow \\ Tot^n(Z) & \rightarrow & Tot(Z) & \rightarrow & Tot_{n-1}(Z) \\ \downarrow & & & & \\ F_nZ & & & & \end{array}$$

and, hence, to a tower of fibrations

$$(5.5)\qquad \begin{array}{cccccccc} \cdots & \rightarrow & Tot^2(Z) & \xrightarrow{p_1} & Tot^1(Z) & \xrightarrow{p_0} & Tot^0(Z) & = Tot(Z). \\ & & \downarrow k_2 & & \downarrow k_1 & & \downarrow k_0 & \\ & & F_2Z & & F_1Z & & F_0Z & \end{array}$$

If we apply homotopy to this tower of fibrations, we obtain a spectral sequence with

$$E_1^{s,t} = \pi_{t-s}F_sZ$$

and standard arguments show that we have produced a spectral sequence isomorphic to the usual one.

The universal examples of section 3 are easily explained using the tower (5.5). Notice that, if $x \in \pi^s\pi_tZ$ is an infinite cycle detecting $\alpha \in \pi_{t-s}Tot(Z)$, then there is a diagram

$$\begin{array}{ccccc} S^{t-s} & \xrightarrow{f_s} & Tot^sZ & \xrightarrow{k_s} & F_sZ \\ \| & & \downarrow & & \\ S^{t-s} & \xrightarrow{f} & Tot(Z) & & \end{array}$$

so that the homotopy class of f is α and so that k_sf_s represents x. The adjoint of the map

$$f_s : S^{t-s} \rightarrow map_*(\Delta/sk_{s-1}\Delta, Z)$$

yields a map

$$S(s,t) = S^{t-s} \wedge \Delta/sk_{s-1}\Delta \rightarrow Z$$

demonstrating the claim that $S(s,t)$ forms some sort of universal infinite cycle. The universal differential can be discussed in the same way.

We now turn to the discussion of the boundary maps between homotopy spectral sequences. To isolate the key point in the argument, we make the following definition.

Definition 5.6: A fibration sequence of pointed cosimplicial spaces

$$F \xrightarrow{i} Z \xrightarrow{p} Y$$

is called a *twisted product* if there are isomorphisms of pointed simplicial sets, $n > 0$,

$$\Theta^n : Z^n \to Y^n \times X^n$$

and commutative diagrams

$$\begin{array}{ccc} Z^n & \xrightarrow{\Theta^n} & Y^n \times F^n \\ \downarrow p & & \downarrow p_1 \\ Y^n & \xrightarrow{=} & Y^n \end{array}$$

where p_1 is the projection, and so that

$$(d^i \times d^i)\Theta^n = \Theta^{n+1} d^i, \qquad i > 0$$

and

$$(s^i \times s^i)\Theta^n = \Theta^{n-1} s^i, \qquad i \geq 0$$

where d^i and s^i are the appropriate coface and codegeneracy operators.
Only d^0 does not commute with the Θ^n and provides the twisting.

One easily checks that a twisted product is a fibrations sequence of cosimplicial spaces.

Lemma 5.7: Let $f : Z \to Y$ be any morphism of cosimplicial spaces. Then the fibration sequence

$$F(p) \to (\bar{\mathsf{F}}_p)_{\bar{\mathsf{F}}^{\cdot}_p Y} \bar{\mathsf{F}}^{\cdot}_p Z \to \bar{\mathsf{F}}^{\cdot}_p Y$$

of Section 4 is a twisted product.

Proof: This is a matter of examining the definitions. Indeed, if Y is a space, then $(\bar{\mathsf{F}}_p)_Y(\ ) = Y \times \bar{\mathsf{F}}_p(\ )$ and this splitting extends to the fibration sequence.

Remark: The class of twisted products contains more than the examples provided by this lemma. Indeed, if $F \to Z \to Y$ is a twisted product and X is a pointed space, than

$$map_*(X, F) \to map_*(X, Z) \to map_*(X, Y)$$

is also a twisted product.

Proposition 5.8: If $F \to Z \xrightarrow{p} Y$ is a twisted product, then there are isomorphisms of spaces

$$N^n Z \xrightarrow{\cong} N^n Y \times N^n F$$

for all $n \geq 0$ and a diagram

$$\begin{array}{ccc} N^n Z & \xrightarrow{\cong} & N^n Y \times N^n F \\ \downarrow Np & & \downarrow p_1 \\ N^n Y & \xrightarrow{=} & N^n Y. \end{array}$$

Proof: The matching spaces $M^n Z$ and the fibration sequences

$$N^n Z \to Z^n \xrightarrow{s} M^{n-1} Z$$

that define the spaces $N^n Z$ depend only on the codegeneracies of Z. Since $\Theta^n : Z^n \xrightarrow{\cong} Y^n \times F^n$ commutes with codegeneracies, the result follows.

Corollary 5.9: In the normalized cochain complex $N\pi_* Z$ there is an isomorphism

$$N^n \pi_* Z \cong N^n \pi_* Y \times N^n \pi_* F$$

and the isomorphism commute with the projection to $N\pi_* Y$. There is a long exact sequence

$$\cdots \to \pi^s \pi_t F \to \pi^s \pi_t Z \to \pi^s \pi_t Y \xrightarrow{\partial} \pi^{s+1} \pi_t F \to \cdots$$

Proof: The isomorphisms follow from Proposition 5.8 and the isomorphism of (5.3). The long exact sequence is now induced by the short exact sequence of cochain complexes

$$0 \to N\pi_*F \to N\pi_*Z \to N\pi_*Y \to 0.$$

Remark 5.10.1.) One easily checks that the long exact sequence obtained by combining Lemma 5.7 with Corollary 5.9 is the same as that obtained in Proposition 2.11 and used in Theorem 4.9.

2.) The map $\partial : \pi^s\pi_tY \to \pi^{s+1}\pi_tF$ has a canonical description on the cochain level given as follows. If $\alpha \in \pi^s\pi_tY$ is the residue class of the cocycle $y \in N^s\pi_tY$, then 5.9 identifies an element $z \in N^s\pi_tZ$ that passes, under the isomorphism, to $(y, 0)$. The coboundary ∂z passes to $(0, w)$ for some $w \in N^{s+1}\pi^tF$ and $\partial\alpha$ is the residue class of w.

In light of Lemma 5.7 and Remark 5.10.1, Theorem 4.9 is subsumed in the following result.

Theorem 5.11: Let $F \to Z \to Y$ be a twisted product of fibrant pointed cosimplicial spaces. Then there is a diagram of spectral sequences

$$\begin{array}{ccc} \pi^s\pi_tY & \Rightarrow & \pi_{t-s}Tot(Y) \\ \downarrow \partial & & \downarrow \delta \\ \pi^{s+1}\pi_tF & \Rightarrow & \pi_{t-s-1}Tot(F) \end{array}$$

where δ is induced by the fibration sequence of spaces

$$Tot(F) \to Tot(Z) \to Tot(Y).$$

This follows from the following omnibus lemma. We use the notation of (5.5).

Lemma 5.12.1.) There are maps

$$f^n : \pi_*\Omega Tot^n(Y) \to \pi_*Tot^{n+1}(F)$$

and

$$\partial^n : \pi_*\Omega^2 F_n Y \to \Omega F_{n+1} F$$

and a commutative diagram

$$\begin{array}{ccccccc}
\pi_*\Omega^2 F_n Y & \to & \pi_*\Omega Tot^n Y & \to & \pi_*\Omega Tot^{n-1} Y & \xrightarrow{\delta} & \pi_{*-1}\Omega^2 F_n Y \\
\downarrow \partial^n & & \downarrow f^n & & \downarrow f^{n-1} & & \downarrow \partial^n \\
\pi_*\Omega F_{n+1} F & \to & \pi_* Tot^{n+1}(F) & \to & \pi_* Tot^n(F) & \xrightarrow{\delta} & \pi_{*-1}\Omega F_{n+1} F
\end{array}$$

where the rows arise from the fibration sequences induced from (5.5).

2.) Under the isomorphisms

$$\pi_i\Omega^2 F_n Y \cong N^n \pi_{i+n+2} Y \qquad \text{and} \qquad \pi_i \Omega F_{n+1} F \cong N^{n+1}\pi_{i+n+2} F$$

the map ∂^n induces the map

$$\partial : \pi^n \pi_t Y \to \pi^{n+1}\pi_t F.$$

The proof will occupy the rest of the section. The delicate point is to produce f^n and ∂^n in a natural enough way to demonstrate the commutivity of the diagram of 5.12.1. We give the technique we will use, which exploits Proposition 5.8. Let $\mathcal{H}\wr(\nabla\mathcal{S}_*)$ be the homotopy category of pointed cosimplicial spaces [8,X].

Definition 5.13: An object $D \in \mathcal{H}\wr(\nabla\mathcal{S}_*)$ will be called a $d_1^{s,t}$ *model* if there is an isomorphism in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$,

$$D(s,t,1) = \Sigma^{t-s} sk_s\Delta / sk_{s-1}\Delta \to D.$$

We can conclude that there is prefered generator

$$\iota_s \in N^s \pi_t D$$

and

$$0 \neq \partial\iota_s \in N^{s+1}\pi_t D.$$

We call D a d_1 *model* if there is an isomorphism in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$

$$\vee_k \ D(s_k, t_k, 1) \to D$$

for some finite indexing set $\{k\}$. A map $f : D \to D'$ between d_1 models will be called a *projection/injection* if under the prefered isomorphisms in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$ f corresponds to a projection onto one wedge summand followed by inclusion to another.

The key fact we will use use is this.

Lemma 5.14: Let $F \to Z \to Y$ be twisted product of pointed fibrant cosimplicial spaces and let D be a cofibrant d_1 model. Then the map

$$\pi_* map_*(D, Z) \to \pi_* map_*(D, Y)$$

is split surjective. Furthermore, this splitting is natural with respect to projection/inclusions $D \to D'$ of d_1 models.

Proof: Let

$$\vee_k\ D(s_k, t_k, 1) \to D$$

be the given isomorphism in $\mathcal{H}\wr(\nabla\mathcal{S}_*)$. Using this and [8,p.277] we obtain isomorphisms

$$\begin{aligned} \pi_* map_*(D, Z) &\cong \times_k\ \pi_* map_*(D(s_k, t_k, 1), Z) \\ &\cong \times_k\ \pi_* \Omega^{t_k} N^{s_k} Z. \end{aligned}$$

The result now follows from Proposition 5.8.

The lemma we use to construct the maps of 5.12.1 is the following.

Proposition 5.15: Let $A \to D \to C$ be a cofiber sequence of cofibrant pointed cosimplicial spaces and let D be a d_1 model. Let $F \to Z \to Y$ be a twisted product of pointed fibrant cosimplicial spaces. Then there is a map

$$\pi_* map_*(C, Y) \to \pi_* map_*(A, F).$$

This map is natural with respect to diagrams

$$\begin{array}{ccccc} A & \to & D & \to & C \\ \downarrow & & \downarrow g & & \downarrow \\ A' & \to & D' & \to & C' \end{array}$$

where g is a projection/inclusion of d_1 models.

Proof: Let $i : D \to C$ and $p : Z \to Y$ be the given maps. Then there is a pull-back diagram of fibrations

$$\begin{array}{ccc} map_*(i,p) & \to & map_*(D,Z) \\ \downarrow & & \downarrow \\ map_*(C,Y) & \to & map_*(D,Y) \end{array}$$

where $map_*(i,p)$ is the mapping space with n-simplices pointed commutative diagrams

$$\begin{array}{ccc} \Delta[n]_+ \wedge D & \to & Z \\ \downarrow {\scriptstyle 1\wedge i} & & \downarrow {\scriptstyle p} \\ \Delta[n]_+ \wedge C & \to & Y \end{array}$$

The splitting of the previous result yields a splitting

$$\pi_* map_*(C,Y) \to \pi_* map_*(i,p).$$

The result follows by composing with the natural map

$$\pi_* map_*(i,p) \to \pi_* map_*(A,F).$$

The naturality clause follows from the naturality clause of 5.14.

The proof of 5.12.1 now depends on making a good choice of cofibration sequences. Because the category of pointed cosimplicial spaces is a closed model category, we can make the following assertions. Let

$$\begin{array}{ccc} A_1 & \to & B_1 \\ \downarrow & & \downarrow \\ A_2 & \to & B_2 \end{array}$$

be a homotopy commutative diagram of cosimplicial spaces. Then there is a commutative diagram

$$(5.16) \qquad \begin{array}{ccccc} A_1 & \to & X_1 & \to & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ Z_2 & \to & X_2 & \to & Y_2 \\ \downarrow & & \downarrow & & \downarrow \\ Z_3 & \to & X_3 & \to & Y_3 \end{array}$$

where the rows and columns are cofibration sequences and the square

$$\begin{array}{ccc} A_1 & \to & X_1 \\ \downarrow & & \downarrow \\ Z_2 & \to & X_2 \end{array}$$

is equivalent to the original square in the sense that there are weak equivalences

$$X_1 \to B_1 \qquad X_2 \to B_2 \qquad Z_2 \to A_2$$

and the diagrams

$$\begin{array}{ccc} A_1 & \xrightarrow{=} & A_1 \\ \downarrow & & \downarrow \\ Z_2 & \to & A_2 \end{array} \qquad \begin{array}{ccc} Z_2 & \to & X_2 \\ \downarrow & & \downarrow \\ A_2 & \to & B_2 \end{array}$$

$$\begin{array}{ccc} A_1 & \to & X_1 \\ \downarrow = & & \downarrow \\ A_1 & \to & B_1 \end{array}$$

commute and the following diagram commutes up to homotopy

$$\begin{array}{ccc} X_1 & \to & X_2 \\ \downarrow & & \downarrow \\ B_1 & \to & B_2. \end{array}$$

The diagram produced depends on the choice of homotopy. If the original square commutes exactly, then there is a canonical choice of homotopy: the constant homotopy.

We can now turn to the proof of Lemma 5.12.

Proof of Lemma 5.12.1: For every $n > 0$, there is a cofibration sequence

$$sk_n\Delta/sk_{n-1}\Delta \to \Delta/sk_{n-1}\Delta \to \Delta/sk_n\Delta.$$

By using a functorial mapping cylinder construction, this cofibration sequence yields a sequence

$$\Delta/sk_{n-1}\Delta \to \Delta/sk_n\Delta \xrightarrow{\lambda_n} B(n, n+1)$$

where $B(n, n+1) \simeq \Sigma sk_n\Delta/sk_{n-1}\Delta$ and is, hence, a $d_1^{n,n+1}$ model. In fact, if $\iota_{n+1} \in N^{n+1}\pi_{n+1}\Delta/sk_n\Delta$ is the generator, then under the induced map

$$N\lambda_n : N^{n+1}\pi_{n+1}\Delta/sk_n\Delta \to N^{n+1}\pi_{n+1}B(n, n+1)$$

we have that

$$N\lambda_n(\iota_{n+1}) = \partial\iota_n$$

where $\iota_n \in N^n\pi_{n+1}B(n, n+1)$ is the generator.

Now, inclusion of skeleta induces a diagram

$$\begin{array}{ccccc} sk_n\Delta/sk_{n-1}\Delta & \to & \Delta/sk_{n-1}\Delta & \to & \Delta/sk_n\Delta \\ \downarrow \gamma & & \downarrow & & \downarrow \\ sk_{n+1}\Delta/sk_n\Delta & \to & \Delta/sk_n\Delta & \to & \Delta/sk_{n+1}\Delta \end{array}$$

where γ is the constant map. Then, because the mapping cylinder construction is functorial, we get a commutative diagram with γ' constractable:

$$\begin{array}{ccc} \Delta/sk_n\Delta & \xrightarrow{\lambda_n} & B(n, n+1) \\ \downarrow & & \downarrow \gamma' \\ \Delta/sk_{n+1}\Delta & \xrightarrow{\lambda_{n+1}} & B(n+1, n+2) \end{array}.$$

Then, by applying the construction of (5.16) to this square, we obtain a diagram

$$(5.17) \quad \begin{array}{ccccc} \Delta/sk_n\Delta & \to & D(n, n+1) & \to & S(n, n+1) \\ \downarrow & & \downarrow \gamma'' & & \downarrow i \\ S(n+2, n+2) & \to & D(n+1, n+2) & \to & S(n+1, n+2) \\ \downarrow & & \downarrow j & & \downarrow \\ D'(n+1, n+2) & \xrightarrow{f} & Z & \xrightarrow{g} & D''(n, n+2) \end{array}$$

where

5.17.1) in $\mathcal{H}o(\nabla\mathcal{S}_*)$ we have isomorphisms $S(s, t) \cong \Sigma^{t-s}\Delta/sk_{s-1}\Delta$ in $\mathcal{H}o(\nabla\mathcal{S}_*)$ and, under these isomorphisms, the map i is isomorphic to the projection

$$\Sigma\Delta/sk_{n-1}\Delta \to \Sigma\Delta/sk_n\Delta;$$

5.17.2) $D(s, t)$ and $D'(s, t)$ are $d_1^{s,t}$ models; and

5.17.3) there is an isomorphism in $\mathcal{H}o(\nabla\mathcal{S}_*)$

$$Z \cong D(n+1,n+2) \vee \Sigma D(n,n+1)$$

and j is isomorphic to the inclusion.

Then, applying Proposition 5.15, we obtain, for any twisted product $F \to Z \to Y$ of fibrant cosimplicial spaces, a diagram

$$\begin{array}{lccc} & \pi_* map_*(D'(n,n+2),Y) & \to & \pi_* map_*(D'(n+1,n+2),F) \\ & \downarrow & & \downarrow \\ (5.18) & \pi_* map_*(S(n+1,n+2),Y) & \to & \pi_* map_*(S(n+2,n+2),F) \\ & \downarrow & & \downarrow \\ & \pi_* map*(S(n,n+1),Y) & \to & \pi_* map_*(\Delta/sk_n\Delta,F). \end{array}$$

Now we use 5.17.1-3) and the fact that for cosimplicial spaces A and B, we have

$$\pi_* map_*(\Sigma A,B) \cong \pi_* \Omega map_*(A,B)$$

to define the maps f_n and ∂_n. Indeed,

$$\begin{aligned} \pi_* map_*(D'(n,n+2),Y) &\cong \pi_* map_*(\Sigma^2 sk_n\Delta/sk_{n-1}\Delta,Y) \\ &\cong \pi_*\Omega^2 F_n Y \end{aligned}$$

and

$$\begin{aligned} \pi_* map_*(D'(n+1,n+2),F) &\cong \pi_* map_*(\Sigma sk_{n+1}\Delta/sk_n\Delta,F) \\ &\cong \pi_*\Omega F_{n+1} F. \end{aligned}$$

So the first row of (5.18) and these isomorphisms defines ∂_n. For f_n, use the fact that for a fibrant cosimplicial space W

$$\begin{aligned} \pi_* map_*(S(s,t),W) &\cong \pi_* map_*(\Sigma^{t-s}\Delta/sk_{s-1}\Delta,W) \\ &\cong \pi_*\Omega^{t-s} Tot^s W. \end{aligned}$$

and the bottom row of (5.18).

The diagram (5.18) now demonstrates the commutivity of the three of the four squares of the diagram in 5.12.1. To get the final commutative square, recapitulate this argument, beginning with the square

$$\begin{array}{ccc} S(n+2,n+2) & \to & D(n+1,n+2) \\ \downarrow & & \downarrow j \\ D'(n+1,n+2) & \xrightarrow{f} & Z. \end{array}$$

This completes the proof of 5.12.1 and leaves only the following.

Proof of 5.12.2: Let

$$D'(n+1,n+2) \xrightarrow{f} Z \xrightarrow{g} D'(n,n+2) \tag{5.19}$$

be the cofibration sequence of 5.17. We examine f and g in homotopy. Because of the commutative square

$$\begin{array}{ccc} D(n+1,n+2) & \to & S(n+1,n+2) \\ \downarrow j & & \downarrow \\ Z & \xrightarrow{g} & D'(n,n+2) \end{array}$$

we understand the composition gj: if $\iota_{n+1} \in N^{n+1}\pi_{n+2}D(n+1,n+2)$ is the generator, then, by (5.4)

$$N(gj)\iota_{n+1} = -\partial\iota_n$$

where $\iota_n \in N^n\pi_{n+2}D'(n,n+2)$ is the generator. Thus, if

$$j_n \in N^n\pi_{n+2}Z$$

and

$$j_{n+1} \in N^{n+1}\pi_{n+2}Z$$

are the generators obtained from the isomorphisms in $\mathcal{H}o(\nabla\mathcal{S}_*)$

$$Z \cong D(n+1,n+2) \vee \Sigma D(n,n+1) \cong D(n+1,n+2) \vee D'(n,n+2)$$

and if $\iota_{n+1} \in N^{n+1}\pi_{n+2}D'(n+1,n+2)$ is the generator, then we have

$$Ng(j_n) = \iota_n$$

and

$$Ng(j_{n+1}) = -\partial\iota_n.$$

And, because the composition gf is constant, we conclude that

$$Nf(\iota_{n+1}) = j_{n+1} + \partial j_n.$$

The result now follows because ∂_n is obtained from 5.19, using 5.15, and because of the canonical description of ∂ given in Remark 5.10.2

Part II: Products and Operations in Quillen Cohomology

In the last four sections of this paper, we will define and explore the Whitehead product and the operations that appear in the E_2 term of the Bousfield-Kan spectral sequence. Then we will discuss to what extent these products and operations commute with the differentials and, hence, are reflected in the homotopy groups of spaces. Sections 8 and 9 are devoted to methods of computation and calculations in the universal examples of section 3. For these final sections we will restrict attention to the prime 2, although many of the results immediately generalize to other primes. What has not been generalized are the operations of section 7.

6. Products in Quillen cohomology

In this section, we expand on some work of Bousfield and Kan and show that there is a product in the spectral sequence

$$H^*_{Q\mathcal{A}} H^* Z \Rightarrow \pi_* Tot(Z)_2.$$

This product will satisfy the Jacobi identity and abut to the Whitehead product in homotopy. In the next section we will show that there are Steenrod operations related to this product.

To begin with, it is useful to make the following definition:

Definition 6.1: A cosimplicial space Z is a $\mathbf{F}_2$-*like* if each Z^s is a simplicial $\mathbf{F}_2$ vector space for each s, and the coface and codegeneracy maps

$$d^i : Z^{s-1} \rightarrow Z^s, \qquad 1 \leq i \leq s$$

and

$$s^i : Z^{s+1} \rightarrow Z^s, \qquad 0 \leq i \leq s$$

are all maps of simplicial vector spaces. Only d^0 is not necessarily a map of simplicial vector spaces. In addition, a morphism of $\mathbf{F}_2$-like cosimplicial spaces is a map $f : Z \rightarrow Y$ of $\mathbf{F}_2$-like cosimplicial spaces so that each

$$f^s : Z^s \rightarrow Y^s$$

is a morphism of simplicial vector spaces.

Remark 6.2: In light of the constructions of Section 3, given any cosimplicial space Z, we can form the augmented cosimplicial space $Z \to \bar{\mathbf{F}}_2^{\cdot} Z$ and $\bar{\mathbf{F}}_2^{\cdot} Z$ is $\mathbf{F}_2$-like. Since

$$H^*_{Q\mathcal{A}} H^* Z \Rightarrow \pi_* Tot(Z)_2$$

is the homotopy spectral sequence of $\bar{\mathbf{F}}_2^{\cdot} Z$ all the subsequent results apply to the this generalization of the Bousfield-Kan spectral sequence. For example, see (6.6) below.

Also, if Z is $\mathbf{F}_2$-like and $H_* Z^s$ is of finite type for each s, then

$$\pi_* Z \cong (\mathbf{F}_2 \otimes_{\mathcal{A}} QH^* Z)^*$$

so

$$\pi^* \pi_* Z \cong H^*_{Q\mathcal{A}} H^* Z.$$

In [7], Bousfield and Kan demonstrate how to put a Whitehead product into the homotopy spectral sequence of an $\mathbf{F}_2$-like cosimplicial space. Define a map, for each s,

$$\zeta : Z^s \wedge Z^s \to Z^{s+1}$$

by

$$\zeta(u \wedge v) = d^0(u+v) - (d^0(u) + d^0(v)). \tag{6.2}$$

Here we use the fact that Z^{s+1} is a vector space. Notice that ζ measures the deviation of d^0 from being a vector space homomorphism.This allows one to define a pairing

$$\omega_* : \pi_t Z^s \otimes \pi_{t'} Z^s \to \pi_{t+t'} Z^{s+1}$$

as follows. Define a map

$$\wedge : \pi_t Z^s \otimes \pi_{t'} Z^s \to \pi_{t+t'} Z^s \wedge Z^s$$

by the smash product pairing and let ω_* be the composition

$$\pi_t Z^s \otimes \pi_{t'} Z^s \xrightarrow{\wedge} \pi_{t+t'} Z^s \wedge Z^s \xrightarrow{\pi_* \zeta} \pi_{t+t'} Z^{s+1}.$$

The following can now be proved exactly as in Chapter III of [7].

Theorem 6.3: Let Z be an $\mathbf{F}_2$-like cosimplicial space. Then the pairing ω_* induces a product in the homotopy spectral sequence

$$\pi^s \pi_t Z \cong [H^s_{Q\mathcal{A}} H^* Z]_t \Rightarrow \pi_{t-s} Tot(Z)$$

abutting to the Whitehead product

$$[\ ,\ ] : \pi_* Tot(Z) \otimes \pi_* Tot(Z) \to \pi_* Tot(Z).$$

That is, there is a product

$$[\ ,\ ] : [H^s_{Q\mathcal{A}} H^* Z]_t \otimes [H^{s'}_{Q\mathcal{A}} H^* Z]_{t'} \to [H^{s+s'+1}_{Q\mathcal{A}} H^* Z]_{t+t'}$$

and a diagram of spectral sequences

$$\begin{array}{ccc} [H^s_{Q\mathcal{A}} H^* Z]_t \otimes [H^{s'}_{Q\mathcal{A}} H^* Z]_{t'} & \Rightarrow & \pi_{t-s} Tot(Z) \otimes \pi_{t'-s'} Tot(Z) \\ \downarrow [\ ,\ ] & & \downarrow [\ ,\ ] \\ [H^{s+s'+1}_{Q\mathcal{A}} H^* Z]_{t+t'} & \Rightarrow & \pi_{t+t'-(s+s')-1} Tot(Z) \end{array}$$

Remark 6.4: More technically, this should be phrased as follows. Let $\{E_r^{s,t} Z\}$ denote the homotopy spectral sequence. Then there exist natural products

$$[\ ,\ ] : E_r^{s,t} Z \otimes E_r^{s',t'} Z \to E_r^{s+s'+1,t+t'} Z$$

so that

1.) the product on $E_1 Z$ is induced by ω_*;

2.) $d_r[u,v] = [d_r u, v] + [u, d_r v]$;

3.) the product on $E_{r+1} Z$ is induced by the product on $E_r Z$ and the product on $E_\infty Z$ is induced by the product on $E_r Z$, $r < \infty$;

4.) the product on E_∞ is also induced by the Whitehead product on $\pi_* Tot(Z)$.

Bousfield and Kan also prove the following result.

Proposition 6.5: The product

$$[\ ,\ ] : [H^s_{Q\mathcal{A}}H^*Z]_t \otimes [H^{s'}_{Q\mathcal{A}}H^*Z]_{t'} \to [H^{s+s'+1}_{Q\mathcal{A}}H^*Z]_{t+t'}$$

is commutative, bilinear and satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

It follows that the product on E_rZ also satisfies the conclusion on Proposition 6.5.

In particular, let us consider the case where we are analyzing the cosimplicial space $\bar{\mathsf{F}}_2^{\cdot}Z$ where Z is a fibrant cosimplicial space. Then the homotopy spectral sequence reads

$$H^*_{Q\mathcal{A}}H^*Z \Rightarrow \pi_*Tot(Z)_2.$$

Therefore we obtain from the above results a commutative bilinear product

$$(6.6) \qquad [\ ,\ ] : H^s_{Q\mathcal{A}}H^*Z \otimes H^{s'}_{Q\mathcal{A}}H^*Z \to H^{s+s'+1}_{Q\mathcal{A}}H^*Z.$$

This product adds the internal degree and abutts to the Whitehead product in the homotopy groups $\pi_*Tot(Z)_2$.

Notice that the product on $H^*_{Q\mathcal{A}}H^*Z$ is defined for *any* cosimplicial space Z because $\bar{\mathsf{F}}_2^{\cdot}Z$ is always F_2-like. We just have to be careful what the spectral sequence abuts to. See the examples at the end of section 3.

The rest of this section is devoted to studying the product (6.6).

The first thing to notice is that this product is actually intrinsic to $H^*_{Q\mathcal{A}}(\ )$ and does not depend on the existence of a cosimplicial space. To see this, let $A \in s\mathcal{U}\mathcal{A}$ be an almost-free unstable simplicial algebra. Then for all $s \geq 0$

$$A_s = G(V_s)$$

for some graded vector space V_s. Of course, $G : n\mathsf{F}_2 \to \mathcal{U}\mathcal{A}$ is the left adjoint to the augmentation ideal functor. The vector space diagonal

$$\Delta : V_s \to V_s \oplus V_s$$

yields, after applying G, a coproduct

$$\psi_s = G\Delta : A_s = G(V_s) \to G(V_s) \otimes G(V_s) = A_s \otimes A_s$$

that gives A_s the structure of a commutative, cocommutative Hopf algebra with conjugation in $\mathcal{UA}$. In particular, for any $\Lambda \in \mathcal{UA}$

$$Hom_{\mathcal{UA}}(A_s, \Lambda)$$

is a group; indeed

$$\begin{aligned} Hom_{\mathcal{UA}}(A_s, \Lambda) &\cong Hom_{\mathcal{UA}}(G(V_s), \Lambda) \\ &\cong Hom_{n\mathbf{F}_2}(V_s, I\Lambda) \end{aligned}$$

and all isomorphisms are group isomorphisms. Hence $Hom_{\mathcal{UA}}(A_s, \Lambda)$ is an $\mathbf{F}_2$ vector space. Now, because A is almost-free,

$$d_i : A_s \to A_{s-1}, \qquad 1 \leq i \leq s$$

and

$$s_i : A_s \to A_{s+1}, \qquad 0 \leq i \leq s$$

are maps of Hopf algebras. Only d_0 is not necessarily a map of Hopf alegbras; hence, it makes sense to measure the deviation of d_0 from being a Hopf algebra map. Define

$$\bar{\xi} : A_s \to A_{s-1} \otimes A_{s-1}$$

to be the product, in the group $Hom_{\mathcal{UA}}(A_s, A_{s-1} \otimes A_{s-1})$, of

$$(d_0 \otimes d_0)\psi_s : A_s \to A_{s-1} \otimes A_{s-1}$$

and

$$\psi_{s-1} d_0 : A_s \to A_{s-1} \otimes A_{s-1}.$$

The morphism $\bar{\xi}$ actually factors through a subalgebra of $A_{s-1} \otimes A_{s-1}$. For $\Lambda, \Gamma \in \mathcal{UA}$, define the product $\Lambda \times_{\mathbf{F}_2} \Gamma \in \mathcal{UA}$ by the pull-back diagram (of simplicial graded vector spaces)

$$\begin{array}{ccc} \Lambda \times_{\mathbf{F}_2} \Gamma & \to & \Gamma \\ \downarrow & & \downarrow \epsilon \\ \Lambda & \xrightarrow{\epsilon} & \mathbf{F}_2 \end{array}$$

If X and Y are pointed spaces, then

$$H^*(X \vee Y) \cong H^*X \times_{\mathbf{F}_2} H^*Y.$$

There is a natural map $\Lambda \otimes \Gamma \to \Lambda \times_{\mathbf{F}_2} \Gamma$ given by

$$u \otimes v \longmapsto (u\eta\epsilon(v), \eta\epsilon(u)v)$$

and we may define $\Lambda \wedge \Gamma$ by the pull-back diagram

$$\begin{array}{ccc} \Lambda \wedge \Gamma & \rightarrow & \Lambda \otimes \Gamma \\ \downarrow \epsilon & & \downarrow \\ \mathbf{F}_2 & \xrightarrow{\eta} & \Lambda \times_{\mathbf{F}_2} \Gamma. \end{array}$$

If X and Y are pointed spaces, then

$$H^*(X \wedge Y) \cong H^*X \wedge H^*Y.$$

Finally, notice for $A \in s\mathcal{UA}$ almost-free, there is a factoring

$$\begin{array}{ccc} A_s & \xrightarrow{\xi} & A_{s-1} \wedge A_{s-1} \\ \downarrow = & & \downarrow \\ A_s & \xrightarrow{\bar{\xi}} & A_{s-1} \otimes A_{s-1} \end{array} \tag{6.7}$$

To see this, one need only check that the two composites

$$A_s \xrightarrow{\bar{\xi}} A_{s-1} \otimes A_{s-1} \xrightarrow{\epsilon\otimes 1} A_{s-1}$$

and

$$A_s \xrightarrow{\bar{\xi}} A_{s-1} \otimes A_{s-1} \xrightarrow{1\otimes\epsilon} A_{s-1}$$

are the trivial map

$$\eta\epsilon : A_s \to A_{s-1}.$$

For the morphism $\epsilon \otimes 1$, say, this is equivalent to showing that

$$(\epsilon \otimes 1)(d_0 \otimes d_0)\psi_s = (\epsilon \otimes 1)\psi_{s-1}d_0 : A_s \to A_{s-1}.$$

But this is obvious. A similiar argument can be given in the other case and that completes the definition of the map ξ of (6.7).

Notice that is $A = H^*Z$ where Z is an $\mathbf{F}_2$-like cosimplicial space, then

$$\xi = \zeta^* : H^*Z^s \to H^*Z^{s-1} \wedge H^*Z^{s-1}. \tag{6.8}$$

Thus we have algebraically copied the topological construction.

To define the product on cohomology of the simplicial unstable algebra $A \in s\mathcal{UA}$, we need the following lemmas. Let $Q(\ )$ denote the indecomposables functor.

Lemma 6.9: For $\Lambda, \Gamma \in \mathcal{UA}$, there are natural maps

$$Q(\Lambda \wedge \Gamma) \to Q\Lambda \otimes Q\Gamma$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q(\Lambda \wedge \Gamma) \to (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\Lambda) \otimes (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\Gamma).$$

Proof: The map $\Lambda \wedge \Gamma \to \Lambda \otimes \Gamma$ induces a map

$$I(\Lambda \wedge \Gamma) \to I\Lambda \otimes I\Gamma$$

where $I(\ )$ is the augmentation ideal functor. The result follows by investigating this map.

For the next lemma, we need some notation. If $f, g : A_s \to \Lambda$ with $A \in s\mathcal{UA}$ almost-free, let $f * g$ denote the product of f and g in the group $Hom_{\mathcal{UA}}(A_s, \Lambda)$; that is, $f * g$ is the composite

$$A_s \xrightarrow{\psi_s} A_s \otimes A_s \xrightarrow{f \otimes g} \Lambda \otimes \Lambda \to \Lambda$$

where the last map is multiplication. Notice that

$$Q(f * g) = Qf + Qg; QA_s \to Q\Lambda \tag{6.10.1}$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q(f * g) = \mathbf{F}_2 \otimes_{\mathcal{A}} Qf + \mathbf{F}_2 \otimes_{\mathcal{A}} Qg : \mathbf{F}_2 \otimes_{\mathcal{A}} QA_s \to \mathbf{F}_2 \otimes_{\mathcal{A}} Q\Lambda. \tag{6.10.2}$$

Thus, the next result will allow us to compute boundary homomorphisms in various chain complexes.

Lemma 6.11: Let $A \in s\mathcal{UA}$ be almost-free. Then if

$$\xi; A_s \longrightarrow A_{s-1} \wedge A_{s-1}$$

is the map of (6.7), we have

1.) $(d_i \wedge d_i)\xi = \xi d_{i+1}, \qquad i \geq 1$; and

2.) $(d_0 \wedge d_0)\xi = [\xi d_0] * [\xi d_1]$.

Proof: These are simple consequences of the simplicial identities; we will do 2.)

It is sufficient to show that for

$$\bar{\xi} : A_s \longrightarrow A_{s-1} \otimes A_{s-1}$$

we have the equation

$$(d_0 \otimes d_0)\bar{\xi} = [\bar{\xi} d_0] * [\bar{\xi} d_1].$$

This is because the map

$$Hom_{\mathcal{UA}}(A_s, A_{s-2} \wedge A_{s-2}) \longrightarrow Hom_{\mathcal{UA}}(A_s, A_{s-2} \otimes A_{s-2})$$

is an injection. However,

$$\bar{\xi} = [(d_0 \otimes d_0)\psi] * [\psi d_0]$$

where the coproducts ψ_s and ψ_{s-1} are abbreviated to ψ. Now, since A is almost-free, the coproduct ψ commutes with d_i for $i \geq 1$:

$$(d_i \otimes d_i)\psi = \psi d_i, \qquad i \geq 1.$$

Thus we may compute, using the facts that $Hom_{\mathcal{UA}}(A_s, \Lambda)$ is an $\mathbf{F}_2$-vector space and that $d_0 d_1 = d_0 d_0$:

$$\begin{aligned}[(d_0 \otimes d_0)\bar{\xi}] * [\bar{\xi} d_1] &= [(d_0 \otimes d_0)^2\psi] * [(d_0 \otimes d_0 \psi d_0] * [(d_0 \otimes d_0)^2\psi] * [\psi d_0 d_0] \\ &= [(d_0 \otimes d_0)\psi d_0] * [\psi d_0 d_0] \\ &= \bar{\xi} d_0.\end{aligned}$$

The result follows.

We now use ξ to define a product in the cohomology of a simplicial unstable algebra. Let $A \in s\mathcal{UA}$. Since A is weakly equivalent to an almost-free object, we may assume that A is almost-free. Applying Lemma 6.9, we know that ξ induces maps of degree -1:

(6.12.1) $$Q\xi : QA \to QA \otimes QA$$

and

(6.12.2) $$\mathbf{F}_2 \otimes_{\mathcal{A}} \xi : \mathbf{F}_2 \otimes_{\mathcal{A}} QA \to \mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA.$$

By (6.10) and Lemma 6.11, we know that these are maps of chain complexes. We can use the second to define a product

$$[\ ,\ ] : H^p_{Q\mathcal{A}} A \otimes H^q_{Q\mathcal{A}} A \to H^{p+q+1}_{Q\mathcal{A}} A$$

as the map induced by the map of cochain complexes

$$(\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \otimes (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \to (\mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \xrightarrow{\mathbf{F}_2 \otimes_{\mathcal{A}} \xi} (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*.$$

The first map is the canonical homomorphism from $V^* \otimes W^* \to (V \otimes W)^*$ and we use the Eilenberg-Zilber Theorem to give a natural isomorphism

$$H^*[(\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \otimes (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*] \cong H^*_{Q\mathcal{A}} A \otimes H^*_{Q\mathcal{A}} A.$$

Notice that if $A = H^* Z$ for some $\mathbf{F}_2$-like cosimplicial space, then

$$\pi_* Z = (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*$$

and, thus, in light of 6.8, this product agrees with the one given by Theorem 6.3.

The following is obvious.

Proposition 6.13: The product

$$[\ ,\ ] : H^p_{Q\mathcal{A}} A \otimes H^q_{Q\mathcal{A}} A \to H^{p+q+1}_{Q\mathcal{A}} A$$

is bilinear, commutative, and adds internal degree.

More importantly, perhaps, the product is commutative on the chain level. We record this fact in the following result. If V is any vector space, let $T : V \otimes V \to V \otimes V$ be the switch map $T(u \otimes v) = v \otimes u$. The next result follows from the definitions.

Lemma 6.14: We have equality between the following morphisms:

$$Q\xi = TQ\xi : QA \to QA \otimes QA$$

and

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi = T\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi : \mathbf{F}_2 \otimes_{\mathcal{A}} QA \to \mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA.$$

7. Operations in Quillen Cohomology

Whenever one has a cohomology theory with a product that is commutative on the cochain level, then one has naturally defined Steenrod or divided product operations. Hence the results of the last section will yield "divided Whitehead squares." The purpose of this section is to define and explore the properties of these operations. In particular, we will note at the end of the section that these operations do not, in general, commute with the differentials in the Bousfield-Kan spectral sequence.

We will prove the following result.

Theorem 7.1: Let $A \in s\mathcal{U}\mathcal{A}$ be a simplicial unstable algebra. Then there are natural homomorphisms

$$P^i : H^q_{Q\mathcal{A}} A \to H^{q+i+1}_{Q\mathcal{A}} A$$

so that

1.) there is an *unstable* condition:

$$P^i(x) = 0 \qquad if \quad i < 2 \quad \text{or} \quad i > q$$

and

$$P^q(x) = [x, x]$$

where [,] is the product of the previous section;

2.) for all $x, y \in H^*_{Q\mathcal{A}}A$ and all i, there is a *Cartan Formula*:

$$[x, P^i(y)] = 0;$$

3.) there are *Adem Relations* for $j \geq 2i$:

$$P^j P^i = \sum_{s=j-i+1}^{j+i-2} \binom{2s-j-1}{s-i} P^{i+j-s} P^s.$$

Remark: If an element $x \in H^s_{Q\mathcal{A}} H^* Z$ with $s = 0$ or 1 survives to E_∞ in the homotopy spectral sequence and detects an element $\alpha \in \pi_* Tot(Z)_2$ it is not immediately apparent what detects the Whitehead product $[\alpha, \alpha]$, since $[x, x] = P^s(x) = 0$. This will be the case, for example, if $Z = S^n$ regarded as a constant cosimplicial space and

$$\iota \in [H^0_{Q\mathcal{A}} H^* S^n]_n \cong Ext^0_{\mathcal{UA}}(H^* S^n, H^* S^n)$$

detects the identity map. Since $[\iota, \iota] = 0$ in the E_2 term, the Whitehead product of the identity $1 \in \pi_n S^n$ with itself must be detected by an element in $Ext^s_{\mathcal{UA}}(H^* S^n, H^* S^t)$ with $t - s = 2n - 1$ and $s \geq 2$. If $n \neq 2^k - 1$ for some k, then it is known that $s = 2$.

There are several ways to define the operations P^i. The classical way is to appeal to the following lemma. If V is a simplicial $\mathbf{F}_2$-vector space, let $C(V)$ be the chain complex obtained by setting $C(V)_n = V_n$ and $\partial = \sum_{i=0}^n d_i$. Let T denote any switch map interchanging factors.

Lemma 7.2: Let V and W be simplicial $\mathbf{F}_2$-vector spaces. Then there are *higher Eilenberg-Zilber maps*:

$$D_i : C(V \otimes W) \to [C(V) \otimes C(W)]_{n+i}$$

so that

1.) D_0 is a chain map and a chain equivalence; and

2.) for $i \geq 1$

$$\partial D_i + D_i \partial = D_{i-1} + T D_{i-1} T.$$

These are standard and essentially unique. See [10].

We use these maps to define the operations. First assume $A \in s\mathcal{UA}$ almost-free,

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi : \mathbf{F}_2 \otimes_{\mathcal{A}} QA \to \mathbf{F}_2 \otimes_{\mathcal{A}} QA \otimes \mathbf{F}_2 \otimes_{\mathcal{A}} QA$$

be the chain coproduct of degree -1 defined in (6.12.1). Define a function

$$S^i : (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^* \to (\mathbf{F}_2 \otimes_{\mathcal{A}} QA)^*$$

of degree $i+1$ by setting, for α of degree q

(7.3) $S^i(\alpha) = (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi)^* D^*_{q-i}(\alpha \otimes \alpha) + (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi)^* D^*_{q-i+1}(\alpha \otimes \partial\alpha).$

Here we let $D_i = 0$ if $i < 0$. The one easily checks, using Lemma 6.14, that

(7.3.1) $$\partial S^i(\alpha) = (\mathbf{F}_2 \otimes_{\mathcal{A}} Q\xi)^* D^*_{q+1-i}(\partial\alpha \otimes \partial\alpha) = S^i(\partial\alpha).$$

Let

$$P^i = \pi^* S^i : H^*_{Q\mathcal{A}} A \to H^*_{Q\mathcal{A}} A.$$

If $A \in s\mathcal{UA}$ is not almost-free, choose an acyclic fibration $X \to A$ and define the operations in $H^*_{Q\mathcal{A}} X \cong H^*_{Q\mathcal{A}} A$.

Since D_0 is the Eilenberg-Zilber chain equivalence, the following is clear

$$P^q(x) = [x, x].$$

This is part of Theorem 7.1.1.

Now, in order to establish the properties of the operations and to prove certain other facts about the structure of the functor $H^*_{Q\mathcal{A}}(\ )$, we establish the connection between this cohomology of simplicial unstable algebras and the ordinary André-Quillen cohomology of simplicial commutative algebras over a field. We will use only the field $\mathbf{F}_2$, but much of what say here will hold at other primes as well.

Let $\mathbf{A}$ be the category of graded, commutative, supplemented $\mathbf{F}_2$ algebras and let $s\mathbf{A}$ be the associated simplicial category. There is a forgetful functor $\mathcal{UA} \to \mathbf{A}$. As with $s\mathcal{UA}$, $s\mathbf{A}$ is a closed model category with a distinguished sub-catgeory of abelian objects and, hence, there is a notion of homology and cohomology. This goes back to André [1] and Quillen [18].

To be specific, we first say that weak equivalences, fibrations, and cofibrations are defined exactly as they were for $s\mathcal{UA}$ in Section 2. In particular, we have a notion of almost-free objects defined using the symmetric algebra functor

$$S : n\mathbf{F}_2 \to \mathbf{A}$$

left adjoint to the augmentation ideal functor I. Then for $\Lambda \in \mathbf{A}$, we obtain an augmented simplicial object

$$\bar{S}_.\Lambda \to \Lambda$$

from the cotriple $\bar{S} = S \circ I$. Then, as in (2.5), we obtain, for $A \in s\mathbf{A}$, an augmented bisimplicial algebra

$$\bar{S}_{.,.}A \to A$$

and, if we set $\bar{S}_.A = diag\bar{S}_{.,.}A$, then we have an acyclic fibration

$$\bar{S}_.A \to A$$

in $s\mathbf{A}$ with $\bar{S}_.A$ almost-free and, hence, cofibrant in $s\mathbf{A}$. This is all gone into in detail in [19] and [20]. Therefore, we define

$$H_*^Q A = \pi_* Q\bar{S}_.A \tag{7.4}$$

and

$$H_Q^* A = \pi^*(Q\bar{S}_.A)^*.$$

The appropriate analog of 2.7 implies that these are well-defined functors of A, independent of the choice of $\bar{S}_.A$. Indeed, we may replace the acyclic fibration $\bar{S}_.A \to A$ by any acyclic fibration $X \to A$ with X cofibrant in $s\mathbf{A}$. $H_Q^*(\ )$ supports a great deal of structure; indeed, H_Q^* is a functor from $s\mathbf{A}$ to the category $\mathcal{W}$ defined in the following definition.

Definition 7.5: Let $\mathcal{W}$ be the subcategory of bigraded $\mathbf{F}_2$-vector spaces defined as follows: $W = \{W_n^p\}$ is an object in $\mathcal{W}$ if

1.) there is a commutative bilinear product

$$[\ ,\ ] : W^p \otimes W^q \to W^{p+q+1}$$

that adds internal degree and satisfies the Jacobi identity;

2.) there are homomorphisms

$$P^i : W^q \to W^{q+i+1}$$

doubling internal degree, such that if $i < 2$ or $i > q$

$$P^i = 0$$

and if $i = q$

$$P^i(x) = [x, x]$$

and if $j \geq 2i$, then

$$P^j P^i = \sum_{s=j-i+1}^{j+i-2} \binom{2s-j-1}{s-i} P^{i+j-s} P^s$$

and for all x, y and i

$$[x, P^i(y)] = 0;$$

3.) there is a quadratic operation

$$\beta : W^0 \to W^1$$

doubling internal degree and so that for all

$$x, y \in W^0$$

$$\beta(x+y) = \beta(x) + \beta(y) + [x, y]$$

and for all $x \in W$ and $y \in W^0$

$$[\beta(y), x] = [y, [y, x]].$$

A morphism in $\mathcal{W}$ preserves this structure.

Theorem 7.6:[14] André-Quillen cohomology defines a functor

$$H_Q^* : s\mathbf{A} \to \mathcal{W}.$$

The product and operations in $H_Q^*(\ )$ are defined in exactly the same fashion as the product and operations in $H_{Q\mathcal{A}}^*(\ )$. In fact, we note the following fact: if $A \in s\mathcal{UA}$ is almost-free, then under the forgetful functor $s\mathcal{UA} \to s\mathbf{A}$, A passes to an almost-free object in $s\mathbf{A}$. Thus we have, by the remarks after the definition of H_*^Q, that

$$\pi_* QA \cong H_*^Q A$$

and the quotient map

$$QA \to \mathbb{F}_2 \otimes_{\mathcal{A}} QA$$

induces a map

$$H_*^Q A \to H_*^{Q\mathcal{A}} A. \tag{7.7}$$

This fact will be exploited in the following sections.

Now, however, we wish to exploit Theorem 7.6. To do this, we define a functor $\mathcal{UA} \to \mathbf{A}$ that kills all Steenrod operations except possibly the squaring (or top) operation. Let $\Lambda \in \mathcal{UA}$. Define

$$J(\Lambda) \subseteq \Lambda$$

to be the ideal generated by all elements of the form

$$Sq^I(x) = Sq^{i_1} \cdots Sq^{i_s}(x)$$

so that

$$e(I) = i_1 - i_2 \cdots - i_s < deg(x)$$

and Sq^I in the augmentation ideal of $\mathcal{A}$. $J(\Lambda)$ is a functor of Λ and we may set

$$\Theta\Lambda = \Lambda / J(\Lambda).$$

Notice that $J(\Lambda)$ is not necessarily invariant under the action of the Steenrod algebra; hence Θ defines a functor $\Theta : \mathcal{UA} \to \mathbf{A}$, but not a functor to $\mathcal{UA}$.

Let $G : n\mathbf{F}_2 \to \mathcal{UA}$ and $S : n\mathbf{F}_2 \to \mathbf{A}$ be the left adjoints to the augmentation ideal functors. The next result describes a few properties of the functor Θ.

Proposition 7.8:1.) For $V \in n\mathbf{F}_2$ there is a natural isomorphism

$$\Theta G(V) \cong S(V);$$

2.) for $V \in n\mathbf{F}_2$, there is a natural isomorphism

$$\mathbb{F}_2 \otimes_{\mathcal{A}} QG(V) \cong Q\Theta G(V);$$

3.) for $A \in s\mathcal{UA}$ almost-free

$$H^*_Q\Theta A \cong H^*_{Q\mathcal{A}}A;$$

4.) if $\wedge : \mathcal{UA} \times \mathcal{UA} \to \mathcal{UA}$ is the smash product defined in the previous section, then for all $V, W \in n\mathbf{F}_2$ there is a ntural isomorphism

$$\Theta(G(V) \wedge G(W)) \cong \Theta G(V) \wedge \Theta G(W).$$

Proof: Parts 1.) and 2.) are obvious. For part 3.), if $A \in s\mathcal{UA}$ is almost-free, then part 1.) implies that ΘA is almost-free in $s\mathbf{A}$. Hence

$$\begin{aligned} H^*_{Q\mathcal{A}}A &\cong \pi^*(\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* \\ &\cong \pi^*(Q\Theta A)^* \qquad \text{by part 2.)} \\ &\cong H^*_Q\Theta A \end{aligned}$$

For part 4.), the definition of $\wedge$ implies that the following isomorphisms are sufficient to imply the result:

$$\begin{aligned} \Theta(G(V) \otimes G(W)) &\cong \Theta G(V \oplus W) \\ &\cong S(V \oplus W) \qquad \text{by part 1.)} \\ &\cong S(V) \otimes S(W) \cong \Theta G(V) \otimes \Theta G(W) \end{aligned}$$

and

$$\Theta(G(V) \times_{\mathbf{F}_2} G(W)) \cong \Theta G(V) \times_{\mathbf{F}_2} \Theta G(W)$$

by direct calculation. The result now follows.

We can now prove the result stated at the beginning of the section.

Proof of Theorem 7.1: We may assume that $A \in s\mathcal{UA}$ is almost-free. Let

$$\xi : A \to A \wedge A$$

be the comultiplication map used to define the product and operations in $H^*_{Q\mathcal{A}}A$. Then

$$\Theta\xi : \Theta A \to \Theta(A \wedge A) \cong \Theta A \wedge \Theta A$$

is used to define the product and coproduct in $H^*_Q\Theta A$. Here we use Propositon 7.8.4. The result now follows from Theorem 7.6 and Proposition 7.8.3.

A similar argument now proves the following result, independent of the work of Bousfield and Kan.

Proposition 7.9: The product

$$[\ ,\ ] : H^*_{Q\mathcal{A}}A \otimes H^*_{Q\mathcal{A}}A \to H^*_{Q\mathcal{A}}A$$

satisfies the Jacobi identity: for all $x, y, z \in H^*_{Q\mathcal{A}}A$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Now let $f : A \to B$ be morphism in $s\mathcal{UA}$ and

$$\partial : H^s_{Q\mathcal{A}}A \to H^{s+1}_{Q\mathcal{A}}M(f)$$

the boundary map in the long exact sequence of the resulting cofibration sequence, as in Proposition 2.11. Since one the f the focuses of this paper has been on how this map behaves with respect to the homotopy spectral sequence, we would like to know how it behaves with respect to the product and operations.

Lemma 7.10:1.) Let $x \in H^*_{Q\mathcal{A}}A$. Then, for all i

$$\partial P^i(x) = P^i(\partial x).$$

2.) For all $x \in H^*_{Q\mathcal{A}}A$ and $y \in H^*_{Q\mathcal{A}}M(f)$

$$[\partial x, y] = 0.$$

Proof: The map ∂ is the connecting homomorphism obtained from a short exact sequence of cochain complexes. See 2.11. Part 1.) follows from investigating formula (7.3.1) and part 2.) follows from the naturality of the homomorphism D_0 of Lemma 7.2.

Corollary 7.11: Let $A \in s\mathcal{UA}$ and let ΣA be the suspension of A. Then for all $x, y \in H^*_{Q\mathcal{A}}\Sigma A$

$$P^i(x) = 0 \qquad \text{for } i \geq deg(x)$$

and

$$[x, y] = 0.$$

Proof: This follows from the fact that

$$\partial : H^s_{Q\mathcal{A}}A \to H^{s+1}_{Q\mathcal{A}}\Sigma A$$

is an isomorphism and the previous lemma.

To obtain some initial understanding of how the operations behave in the homotopy spectral sequence, we consider the universal examples of section 3

$$H^*_{Q\mathcal{A}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

where the sphere is completed at 2 and we assume that $q - p > 1$. Let $\iota = \iota_{p,q} \in H^p_{Q\mathcal{A}}K(p,q)_+$ be the universal class. If $p = 1$, the Theorem 7.1.1 implies that

$$P^i(\iota) = 0$$

for all i. Thus, if

$$j \in H^p_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+$$

is the suspension of this class, Lemma 7.10 implies that

$$P^i(j) = 0 \tag{7.12}$$

for all i. By considering the results of section 4, we see that there is spectral sequence

$$H^*_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+ \Rightarrow \pi_*\Omega^{p-1}S^{q-1}.$$

Now, in the homotopy category associated to $s\mathcal{UA}$, let

$$e : \Sigma^{p-1}K(1,q)_+ \to K(p,q)_+$$

corepresent $j \in H^p_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+$. The results of sections 4 and 5 yield a diagram of spectral sequences which is long exact on the E_2 terms

(7.13)

$$\begin{array}{ccccccccc}
\to & H^*_{Q\mathcal{A}}M(e) & \xrightarrow{f} & H^*_{Q\mathcal{A}}K(p,q)_+ & \xrightarrow{e^*} & H^*_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+ & \to \\
& \Downarrow & & \Downarrow & & \Downarrow & \\
\to & \pi_*C(p-1) & \to & \pi_*S^{q-p} & \xrightarrow{E_{p-1}} & \pi_*\Omega^{p-1}S^{q-1} & \to
\end{array}$$

where E_{p-1} is the suspension homomorphism and $C(p-1)$ is the homotopy fiber. Now (7.12) implies that

$$e^*P^i(\iota_{p,q}) = 0$$

for all i; hence, for each i, $2 \le i \le p$ there must exist a non-zero class

$$y_i \in [H^{p+i+1}_{Q\mathcal{A}}M(e)]_{2q}$$

so that

$$f(y_i) = P^i(\iota_{p,q})$$

where f is the map in (7.13). We now investigate the behavior of the class y_i in the diagram of spectral sequences induced by the Hurewicz homomorphism

$$\begin{array}{ccc}
H^*_{Q\mathcal{A}}M(e) & \Rightarrow & \pi_*C(p-1) \\
\downarrow h^* & & \downarrow h \\
\pi^*M(e)^* & \Rightarrow & H_*C(p-1)
\end{array}$$

where the lower spectral sequence is the homology spectral sequence of section 3. In fact, we will prove the following result. It is known from the calculations of [14, Appendix B] that $h^* y_i \neq 0$.

Proposition 7.14: The class $h^* y_i$ survives to E_∞ in the homology spectral sequence

$$\pi^* M(e)^* \Rightarrow H_* C(p-1)$$

and detects the unique non-zero class in

$$H_{2q-(p+i+1)} C(p-1).$$

What is more, we will identify this non-zero class and show that it often cannot be in the image of the Hurewicz homomorphism

$$h : \pi_* C(p-1) \to H_* C(p-1).$$

We will thus conclude the following corollary:

Corollary 7.15: If $p \geq 3$ and $q - p$ is an odd number, then y_{p-1} does not survive to E_∞ in the homotopy spectral sequence

$$H^*_{Q\mathcal{A}} M(e) \Rightarrow \pi_* C(p-1).$$

Then we will use the calculations of the next two sections to prove that a similar statement can actually be made about $P^i(\iota_{p,q})$; in other words, since $\iota_{p,q}$ survives to E_∞ in the spectral sequence

$$H^*_{Q\mathcal{A}} K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

and detects the identity, the operations P^i don't necessarily commute with the differentials.

The computation necessary for proving Proposition 7.14, begins with a familiar, but disguised calculation: most homotopy theorists have used the Eilenberg-Moore spectral sequence to compute $H_* W(p-1)$ and the same computation — bigraded, if you will — is used to compute $\pi^* M(e)^*$.

First we compute $\pi_*\Sigma^{p-1}K(1,q)_+$. This is done by double induction on the succesive spectral sequences, $0 \le m \le p-1$,

$$Tor^{\pi_*\Sigma^m K(1,q)_+}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow \pi_*\Sigma^{m+1}K(1,q)_+ \Rightarrow H^*\Omega^{m+1}S^{q-1}.$$

Both these spectral sequences must collpase for every m for vector space dimension reasons. Here we use the fact that we know $H_*\Omega^{m+1}S^{q-1}$. Next we investigate the succesive spectral sequences

$$Tor^{\pi_* K(p,q)_+}(\mathbf{F}_2, \mathbf{F}_2) \Rightarrow \pi_* M(e) \Rightarrow H^*C(p-1)$$

and conclude from our knowledge of $H_*W(p-1)$ that both these spectral sequences must collapse.

In particular, we can make the following conclusions. If

$$\mathbf{R}\mathrm{P}^m_n = \mathbf{R}\mathrm{P}^m/\mathbf{R}\mathrm{P}^{n-1}$$

is the stunted real projective space with cells in dimensions n through m, then we know that there is map

$$\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-2}_{q-p} \to W(p-1) \tag{7.16}$$

and for $2q-2p-1 \le k \le 2q-p-3$

$$H_k C(p-1) \cong \mathbf{F}_2$$

generated by the image of the non-zero class in $H_k\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-1}_{q-p}$. Thus, the map of (7.14) induces an isomorphism on homotopy groups in a range of degrees.

Now, if $2 \le i \le p$, then $2q-2p-1 \le 2q-(p+i+1) \le 2q-p-3$ and there is a unique class, $2 \le i \le p$,

$$x_i \in \pi^{p+i+1}M(e)^*_{2q}$$

that detects the non-zero class in

$$H_{2q-(p+i+1)}W(p-1)$$

in the homology spectral sequence. Finally, in the diagram

$$\begin{array}{ccc} H^*_{Q\mathcal{A}}M(e) & \xrightarrow{f} & H^*_{Q\mathcal{A}}K(p,q)_+ \\ \downarrow h^* & & \\ \pi^*M(e)^* & & \end{array}$$

there is a class $y_i \in H^*_{Q\mathcal{A}}M(e)$ so that

$$f(y_i) = P^i(\iota_{p,q}) \in [H^{p+i+1}_{Q\mathcal{A}}K(p,q)_+]_{2q}$$

and

$$h^* y_i = x_i.$$

The first statement was asserted and proved above, and the second statement follows from the fact (proved in [14]) that $h^* y_i \neq 0$. This completes the proof of Proposition 7.14.

Proof of Corollary 7.15: Notice that the map

$$\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-2}_{q-p} \to W(p-1)$$

which is an equivalence in a range of degrees $k \leq 2q-p-3$, demonstrates that very often the non-zero class in $H_k W(p-1)$, $2p-2p-1 \leq k \leq 2q-p-3$ is not in the image of the Hurewicz map. In particular, if $p \geq 3$ and $q-p$ is odd, and if $z \in H^{2(q-p)-1}C(p-1)$ is the non-zero class, then $\mathrm{Sq}^1 z \neq 0$. Thus, the non-zero class in $H_{2(q-p)}C(p-1)$ is not in the image of the Hurewicz homomorphism. Hence there exists an r so that $d_r y_{p-1} \neq 0$.

In fact, we make the following remark, which is perhaps more philosophy than mathematics. If we regard

$$H^*_{Q\mathcal{A}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

as a desuspension spectral sequence — and we will explore this point more in the next few sections — we could say that the elements $P^i(\iota_{p,q})$ are the "cells" of $\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-2}_{q-p}$ in Toda's desuspension long exact sequence

$$\cdots \to \pi_n(\Sigma^{q-p-1}\mathbf{R}\mathrm{P}^{q-1}_{q-p}) \to \pi_n S^{q-p} \xrightarrow{E_{p-1}} \pi_{n+p-1}S^{q-1} \to \cdots$$

valid for $n \leq 2p-p-3$. This sequence is derived from the fiber sequence of (7.13) using the inclusion of (7.16).

8. Miller's Composite Functor Spectral Sequence

Because the functor $\mathbf{F}_2 \otimes_{\mathcal{A}} Q(\ )$ — the main object of study in this paper — can be written as the composite functor

$$\mathbf{F}_2 \otimes_{\mathcal{A}} Q(\ ) = \mathbf{F}_2 \otimes_{\mathcal{A}} (\ ) \circ Q(\ )$$

it is no surpise that there is a composite functor spectral sequence converging to $H^*_{Q\mathcal{A}}A$ for $A \in s\mathcal{UA}$. The purpose of this section is to explore this spectral sequence — due, in principal, to Haynes Miller — and to prepare the way for the computations of the next section.

If $A \in s\mathcal{UA}$, we may define H^Q_*A and H^*_QA — the homology and cohomology based on the indecomposables functor — in a manner analogous to $H^{Q\mathcal{A}}_*A$ and $H^*_{Q\mathcal{A}}A$. Let

$$p : X \to A$$

be an acyclic fibration in $s\mathcal{UA}$ with X almost-free. Then we let

$$H^Q_*A = \pi_*QX$$

and

$$H^*_QA = \pi^*(QX)^*. \tag{8.1}$$

Actually, we defined H^Q_*A and H^*_QA in the previous section, and we must show that this new defintion agrees with this one. This is proved by Miller in [19, Section 2]. In fact, the forgetful functor $s\mathcal{UA} \to s\mathbf{A}$ carries almost-free objects to almost-free objects, and the appropriate analog of Lemma 2.7 for the category $s\mathbf{A}$ implies that the two definitions agree with each other.

So let $X \to A$ be an acyclic fibration with X almost-free in $s\mathcal{UA}$. Since X_m is an unstable algebra for each m, QX_m is an unstable $\mathcal{A}$ module and, hence, H^Q_mA is an unstable $\mathcal{A}$ module for each m. More is true, however. Since for $x \in X^n_m$, $\mathrm{Sq}^n(x) = x^2$, we actually have that H^m_QA is a suspension in the category of unstable $\mathcal{A}$ modules. We now give this category a name. Let $\mathcal{U}_0 \subseteq \mathcal{U}$ be the full sub-catgeory specified by the condition that $M \in \mathcal{U}_0$ if an only if for all $x \in M^n$,

$$\mathrm{Sq}^i x = 0 \qquad \text{for } i \geq n.$$

The suspension functor defines an isomorphism of categories

$$(8.2) \qquad \Sigma : \mathcal{U} \xrightarrow{\cong} \mathcal{U}_0.$$

The above remarks now imply that we have a functor for each m

$$(8.3) \qquad H^Q_m(\ ) : s\mathcal{U}\mathcal{A} \to \mathcal{U}_0.$$

Because of the isomorphism of categories given in 8.2, homological algebra in $\mathcal{U}_0$ is basically the same as homological algebra in $\mathcal{U}$. Indeed, if $M \in \mathcal{U}_0$, then there is a natural isomorphism

$$(8.4) \qquad Ext^s_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2) \cong Ext^s_{\mathcal{U}}(\Sigma^{-1} M, \Sigma^{t-1} \mathbb{F}_2)$$

for all s and t. To see this, note that the if $P : n\mathbb{F}_2 \to \mathcal{U}$ is left adjoint to the forgetful functor F, the composite

$$\bar{P} = P \circ F : \mathcal{U} \to \mathcal{U}$$

forms a cotriple and

$$Ext^s_{\mathcal{U}}(N, \Sigma^t \mathbb{F}_2) \cong \pi^s Hom_{\mathcal{U}}(\bar{P}_\cdot(N), \Sigma^t \mathbb{F}_2).$$

Furthermore,

$$\bar{P}' = \Sigma \circ \bar{P} \circ \Sigma^{-1} : \mathcal{U}_0 \to \mathcal{U}_0$$

forms a cotriple on $\mathcal{U}_0$ and

$$Ext^s_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2) \cong \pi^s Hom_{\mathcal{U}_0}(\bar{P}'_\cdot(M), \Sigma^t \mathbb{F}_2).$$

A simple comparison of the two defintions, using the fact that Σ is exact, now yields the equation (8.4) above.

Let us write $Hom_{\mathcal{U}_0}(M, \mathbb{F}_2)$ and $Ext^s_{\mathcal{U}_0}(M, \mathbb{F}_2)$ for the graded vector spaces with, in degree t,

$$Hom_{\mathcal{U}_0}(M, \mathbb{F}_2)_t = Hom_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2)$$

and

$$Ext^s_{\mathcal{U}_0}(M, \mathbb{F}_2)_t = Ext^s_{\mathcal{U}_0}(M, \Sigma^t \mathbb{F}_2).$$

The following is basically the spectral sequence of Miller's, found in section 2 of [19].

Theorem 8.5: For $A \in s\mathcal{UA}$, there is a spectral sequence of graded vector spaces

$$Ext^p_{\mathcal{U}_0}(H^Q_q A, \mathbb{F}_2) \Rightarrow H^{p+q}_{Q\mathcal{A}} A.$$

Proof: Our proof is no different than Miller's, or the proof given for any composite functor spectral sequence. We may assume that A is amost free in $s\mathcal{UA}$. Form the augmented bi-cosimplicial vector space

$$\lambda : Hom_{\mathcal{U}_0}(QA_q, \mathbb{F}_2) \to C^{p,q} = Hom_{\mathcal{U}_0}(\bar{P}'_p QA_q, \mathbb{F}_2) \tag{8.6}$$

and note that

$$Hom_{\mathcal{U}_0}(QA_q, \mathbb{F}_2) \cong Hom_{n\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\mathcal{A}} QA_q, \mathbb{F}_2) \cong (\mathbb{F}_2 \otimes_{\mathcal{A}} QA_q)^*.$$

Filtering $C^{\cdot,\cdot}$ by degree in q, we obtain a spectral sequence with

$$E_1^{p,q} \cong \begin{cases} Hom_{\mathcal{U}_0}(QA_q, \mathbb{F}_2), & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

This is because QA_m is projective in $\mathcal{U}_0$. Here we use the fact that, since A is almost-free, $A_m \cong G(V)$ for some graded vector space V and $QG(V) = P'(V)$. Therefore, the spectral sequence has the form

$$E_2^{p,q} \cong \begin{cases} H^q_{Q\mathcal{A}} A, & \text{if } p = 0; \\ 0, & \text{if } p > 0. \end{cases}$$

Thus λ, as in (8.6), induces an isomorphism

$$\lambda^* : H^*_{Q\mathcal{A}} A \to H^* C^{\cdot,\cdot}.$$

Therefore, filtering $C^{\cdot,\cdot}$ by degree in p, we obtain a spectral sequence abutting to $H^*_{Q\mathcal{A}} A$ with

$$E_1^{p,q} \cong Hom_{\mathcal{U}_0}(P'_p H^Q_q A, \mathbb{F}_2).$$

This is because

$$Hom_{\mathcal{U}_0}(P'(\ \ ), \mathbb{F}_2)$$

is an exact functor. Hence

$$E_2^{p,q} \cong Ext^p_{\mathcal{U}_0}(H_q^Q A, \mathbb{F}_2).$$

This finishes the proof.

Notice that the differentials in this spectral sequence are of the form

$$d_r : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}.$$

Before proceeding, let us do an example.

Example 8.7: Let us use this spectral sequence to begin to compute

$$Ext^*_{\mathcal{UA}}(H^*S^n, H^*S^t).$$

The spectral sequence in this case reads

$$Ext^p_{\mathcal{U}_0}(H_q^Q H^*S^n, \Sigma^t\mathbb{F}_2) \Rightarrow [H^{p+q}_{\mathcal{QA}} H^*S^n]_t \cong Ext^{p+q}_{\mathcal{UA}}(H^*S^n, H^*S^t)$$

where H^*S^n is regarded as a constant simplicial algebra in $s\mathcal{UA}$. $H_*^Q H^*S^n$ has been known for years (see [4]):

$$H_0^Q H^*S^n \cong \mathbb{F}_2$$

concentrated in degree n,

$$H_0^Q H^*S^n \cong \mathbb{F}_2$$

concentrated in degree $2n$, and

$$H_q^Q H^*S^n = 0 \qquad \text{if } q \geq 2.$$

Thus the spectral sequence becomes a long exact sequence

$$\cdots \longrightarrow E_2^{p-1,1} \xrightarrow{d_2} E_2^{p+2,0} \longrightarrow Ext^{p+2}_{\mathcal{UA}}(H^*S^n, H^*S^n) \longrightarrow E_2^{p+1,1} \longrightarrow \cdots$$

which, using the relationship between $Ext_{\mathcal{U}_0}(\ ,\ )$ and $Ext_{\mathcal{U}}(\ ,\ )$ described above, yields a long exact sequence

$$\begin{aligned}\to Ext_{\mathcal{U}}^{p-1}(\Sigma^{2n-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2) \xrightarrow{d_2} Ext_{\mathcal{U}}^{p+2}(\Sigma^{n-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2)\\ \to Ext_{\mathcal{UA}}^{p+2}(H^*S^n, H^*S^t)\\ \to Ext_{\mathcal{U}}^{p+1}(\Sigma^{2n-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2) \xrightarrow{d_2}\end{aligned}$$

This is easily seen to be the algebraic EHP sequence, much studied by Mahowald and others, perhaps most exhaustively in [9].

Next notice that there is an edge homomorphism

$$e : H^q_{Q\mathcal{A}}A \to E^{0,q} \subseteq Hom_{\mathcal{U}_0}(H_q^{Q\mathcal{A}}A, \mathbb{F}_2) \subseteq H^q_Q A. \tag{8.8}$$

Since

$$Hom_{\mathcal{U}_0}(QA, \mathbb{F}_2) \cong Hom_{n\mathbb{F}_2}(\mathbb{F}_2 \otimes_{\mathcal{A}} QA, \mathbb{F}_2)$$

one easily sees that this edge homomorphism is given by the dual of the map

$$H_q^Q A \to H_q^{Q\mathcal{A}} A$$

induced by the quotient map

$$QA \to \mathbb{F}_2 \otimes_{\mathcal{A}} QA.$$

Now we know that $H^*_Q A$ and $H^*_{Q\mathcal{A}}A$ support a great deal of structure, including the product and operations as defined in the previous two sections. Furthermore, we will know that the edge homomorphism of (8.8) preserves the product and operations — see 8.14 below. Thus, it makes sense that the product and operations in $H^*_Q A$ should induce a product and operations in the composite functor spectral sequence and these should abut to the product and operations in $H^*_{Q\mathcal{A}}A$. In preparation for the computations of the next section, we show that this is in fact the case.

We will use the techniques of Singer [21], modified slightly. The modification is necessary, as our product is non-associative and induced by a chain map of degree -1, instead of degree 0. Let

$$C = C^{\cdot,\cdot} = Hom_{\mathcal{U}_0}(\bar{P}'_{\cdot} QA_{\cdot}, \mathbb{F}_2)$$

be the bi-cosimplicial vector space used to define the spectral sequence in (8.5). In order to apply Singer's line of argument, we must define a product

$$\xi'' : C^{\cdot,\cdot} \otimes C^{\cdot,\cdot} \to C^{\cdot,\cdot}$$

so that, if λ is the augmentation of (8.5), then there is a commutative diagram

$$(8.9)\qquad \begin{array}{ccc} (\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* \otimes (\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* & \xrightarrow{\lambda\otimes\lambda} & C \otimes C \\ \downarrow \xi' & & \downarrow \xi'' \\ (\mathbb{F}_2 \otimes_{\mathcal{A}} QA)^* & \xrightarrow{\lambda} & C \end{array}$$

Here $\xi' = (\mathbb{F}_2 \otimes Q\xi)^*$ where $\mathbb{F}_2 \otimes Q\xi$ is as in Lemma 6.14. But this is simpy done. Let $\bar{P} : \mathcal{U} \to \mathcal{U}$ be the cotriple used above to define $Ext_{\mathcal{U}}$. Then, for $M, N \in \mathcal{U}$, there is a canonical map

$$\bar{P}(M \otimes N) \to \bar{P}(M) \otimes \bar{P}(N).$$

Since the cotriple $\bar{P}' : \mathcal{U}_0 \to \mathcal{U}_0$ is defined by the equations $\bar{P}' = \Sigma \circ \bar{P} \circ \Sigma^{-1}$, we can then define, for $A \in s\mathcal{U}\mathcal{A}$ almost-free, a map

$$\zeta : \bar{P}'_{\cdot} QA \to \bar{P}' QA \otimes \bar{P}' QA$$

by the composition

$$\begin{aligned} \bar{P}'_p Q A_q &\xrightarrow{\bar{P}'_q Q\xi} \bar{P}'_q(Q(A_{q-1} \wedge A_{q-1})) \\ &\to \bar{P}'_p(QA_{q-1} \otimes QA_{q-1}) \\ &\to \bar{P}'_p QA_{q-1} \otimes \bar{P}'_p QA_{q-1} \end{aligned}$$

where ξ is the as in (6.7) and we use (6.7) for the second map. Then we apply the functor $Hom_{\mathcal{U}_0}(\ ,\mathbb{F}_2)$ to obtain ξ'' satisfying the requirements of (8.9).

The following lemma is needed to use Singer's results. Let V be a bisimplicial vector space — for example, we could let $V = \bar{P}'_{\cdot} QA$ or $\bar{P}'_{\cdot} QA \otimes \bar{P}'_{\cdot} QA$, where we take the degree-wise tensor product. Let

$$\partial^h : V_{p,q} \to V_{p-1,q}$$

and

$$\partial^v : V_{p,q} \to V_{p,q-1}$$

be the horizontal and vertical boundary operators obtained by taking an appropriate sum of face maps. Let ζ be as above. Then

$$\zeta : (\bar{P}'_{\cdot} QA)_{p,q} \to (\bar{P}'_{\cdot} QA \otimes \bar{P}'_{\cdot} QA)_{p,q-1}.$$

Lemma 8.11: ζ commutes with the horizontal and vertical boundary operators:

$$\partial^h \zeta = \zeta \partial^h \qquad \text{and} \qquad \partial^v \zeta = \zeta \partial^v.$$

Proof: ζ actually commutes with the horizontal face maps, by the naturality of the construction of (8.10). That ζ commutes with vertical boundary operator is a consequence of 6.11.

The following results are now a direct consequence of Singer's work. Let $\{E_r A\}$ denote Miller's composite functor spectral sequence.

Proposition 8.11: For $2 \leq r \leq \infty$, there is a commutative, bilinear product

$$[\ ,\ ] : E_r^{p,q} A \otimes E_r^{p',q'} A \to E_r^{p+p',q+q'+1} A$$

so that 1.) $d_r[x, y] = [d_r x, y] + [x, d_r y]$:

2.) the product of $E_{r+1} A$ is induced from the product on $E_r A$:

3.) the product on $E_\infty A$ is induced by the product on $E_r A$, with $r < \infty$ and is induced by the product on $H^*_{Q\mathcal{A}} A$.

More informally, we can say that there is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^p_{\mathcal{U}_0}(H^Q_q A, \mathbb{F}_2) \otimes Ext^{p'}_{\mathcal{U}_0}(H^Q_{q'} A, \mathbb{F}_2) & \Rightarrow & H^{p+q}_{Q\mathcal{A}} A \otimes H^{p'+q'}_{Q\mathcal{A}} A \\ \downarrow [\ ,\ ] & & \downarrow [\ ,\ ] \\ Ext^{p+p'}_{\mathcal{U}_0}(H^Q_{q+q'+1} A, \mathbb{F}_2) & \Rightarrow & H^{p+p'+q+q'+1}_{Q\mathcal{A}} A \end{array}$$

There are also operations. Some of these are only defined up to indeterminacy, which we now define. Let $\{E_rA\}$ be the composite functor spectral sequence. Define

$$B_s^{p,q}A \subseteq E_r^{p,q}A, \qquad s \geq r$$

to be the the vector space of elements that survive to $E_s^{p,q}A$ and have zero residue class in $E_s^{p,q}A$. An element $y \in E_r^{p,q}A$ is defined up to indeterminacy s if y is a coset representative for a particular element in $E_r^{p,q}A/B_s^{p,q}A$.

Proposition 8.13: For $2 \leq r \leq \infty$, there are operations

$$P^i : E_t^{p,q}A \to E_r^{p,q+i+1}A, \qquad 0 \leq i \leq q$$

and operations of indeterminacy $2r-2$

$$P^i : E_r^{p,q}A \to E_r^{p+i-q,2q+1}A, \qquad q \leq i \leq p+q$$

so that

1.) $P^{p+q}(x) = [x,x]$ modulo indeterminacy;
2.) if $d_r(x) = y$ then

$$\begin{aligned} d_rP^i(x) &= P^i(y), && 0 \leq i \leq q-r+1; \\ d_{i-q+2r-1}P^i(x) &= P^i(y), && q-r+1 \leq i \leq q; \\ d_{2r-1}P^i(y) &= P^i(y), && q \leq i \leq p+q \end{aligned}$$

modulo appropriate indeterminacy;

3.) the operations on E_rA are determined by the operations on $E_{r'}A$ for $r' < r \leq \infty$, up to indeterminacy; and
4.) the operations on $E_\infty A$ are determined by the operations

$$P^i : H_{QA}^q A \to H_{QA}^q A.$$

In other words, for $0 \leq i \leq q$ there is a diagram of spectral sequences

$$\begin{array}{ccc} Ext^p_{\mathcal{U}_0}(H_q^Q A, \mathbb{F}_2) & \Rightarrow & H_{QA}^{p+q}A \\ \downarrow P^i & & \downarrow P^i \\ Ext^p_{\mathcal{U}_0}(H_{q+i+1}^Q A, \mathbb{F}_2) & \Rightarrow & H_{QA}^{p+q+i+1}A \end{array}$$

and a similar diagram for $q \leq i \leq p+q$.

Remarks: 1.) Notice that there is never any indeterminacy at E_2A. Also notice that $P^i(x) \in E_rA$ determines a well-defined element in $E_{2r-1}A$. Hence $P^i : E_\infty A \to E_\infty A$ is well-defined.

2.) In 8.13.2 it is assumed that $P^i(x)$, $i > q-r+1$ survives to an appropriate E_sA so that the statements make sense.

Singer's work implies the following result about the edge homomorphism

$$e : H^*_{Q\mathcal{A}}A \to H^*_Q A$$

of (8.8).

Proposition 8.14: The edge homomorphism commutes with products and operations:

$$e[x,y] = [e(x), e(y)]$$

and

$$eP^i(x) = P^i(e(x)).$$

We now turn to understanding the operations at E_2A:

$$P^i : Ext^p_{\mathcal{U}_0}(H^Q_q A, \Sigma^t \mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(H^Q_{q+i+1}A, \mathbf{F}_2), \qquad 0 \leq i \leq q.$$

For this we need to understand how the operations commute with the action of the Steenrod algebra. Notice that for every q, $H^q_Q A$ is a right module over the Steenrod algebra; that is, the is an action of the Steenrod operations

$$(\cdot)\mathrm{Sq}^j : [H^q_Q A]_n \to [H^q_Q A]_{n-j}.$$

The following lemma relates the P^i to the Sq^j.

Proposition 8.15: For $A \in s\mathcal{U}\mathcal{A}$ and $x, y \in H^q_Q A$

$$\begin{aligned}
[x,y]\mathrm{Sq}^j &= \sum_{a+b=j} [x\mathrm{Sq}^a, y\mathrm{Sq}^b] \\
P^i(x)\mathrm{Sq}^{2j} &= P^i(x\mathrm{Sq}^j) \\
P^i(x)\mathrm{Sq}^{2j+1} &= 0.
\end{aligned}$$

Proof: We return to the definition of the operations given in section 7. Let $x \in H^q_Q A$ be the residue class of the the cycle $\alpha \in QA^*_q$. Then $P^i(x)$ is the residue class of $\xi' D^*_{q-i}(\alpha \otimes \alpha)$, where we write ξ' for $(\mathbf{F}_2 \otimes Q\xi)^*$. The naturality of ξ and the higher Eilenberg-Zilber maps, and the Cartan formula for Steenrod operations now imply that

$$\begin{aligned}
[\xi^* D^*_{q-i}(\alpha \otimes \alpha)]\mathrm{Sq}^j &= \xi^* D^*_{q-i}(\sum_{0 \leq a \leq j} \alpha \mathrm{Sq}^a \otimes \alpha \mathrm{Sq}^{j-a}) \\
&= \xi^* D^*_{q-i}(\alpha \mathrm{Sq}^{j/2} \otimes \alpha \mathrm{Sq}^{j/2}) \\
&\quad + \partial \xi^* D^*_{q-i+1}(\sum_{0 \leq a < j/2} \alpha \mathrm{Sq}^a \otimes \alpha \mathrm{Sq}^{j-a})
\end{aligned}$$

where $\mathrm{Sq}^{j/2} = 0$ is j if odd. The formula involving the product is proved the same way, using D_0.

To apply this, we dualize the operations P^i and obtain operations acting on the right

$$(\cdot)P^i : H^Q_{q+i+1}A \to H^Q_q A$$

that halve the internal degree in the sense that they are identically zero on elements of odd internal degree. Interpreting Proposition 8.15 in this context, we have

$$(8.16) \qquad (x\mathrm{Sq}^{2j})P^i = \mathrm{Sq}^j(xP^i).$$

Thus $(\cdot)P^i$ is not quite a morphism in $\mathcal{U}_0$. This can be rectified as follows. Let

$$\Phi : \mathcal{U} \to \mathcal{U}$$

be the doubling functor. That is, for $M \in \mathcal{U}$

$$(\Phi M)^n = \begin{cases} M^m, & \text{if } n = 2m; \\ 0, & \text{if } n = 2m+1 \end{cases}$$

with the Steenrod algebra action given by

$$\begin{aligned}
\mathrm{Sq}^{2j}\phi(x) &= \phi(\mathrm{Sq}^j(x)) \\
\mathrm{Sq}^{2j+1}\phi(x) &= 0
\end{aligned}$$

where ϕ is the isomorphism between $(\Phi M)^{2m}$ and M^m. Notice that Φ restricts to a functor $\Phi : \mathcal{U}_0 \to \mathcal{U}_0$.

Then, equation (8.16) implies that $(\cdot)P^i$ induces a homomorphism in $\mathcal{U}_0$

$$\rho_i : H^Q_{q+i+1}A \to \Phi H^Q_q A$$

and, hence, a natural map

$$\rho_i^* : Ext^p_{\mathcal{U}_0}(\Phi H^Q_q A, \Sigma^{2t}\mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(H^Q_{q+i+1}A, \Sigma^{2t}\mathbf{F}_2).$$

We will define a canonical map, for $M \in \mathcal{U}_0$

$$\mathrm{Sq}_0 : Ext^p_{\mathcal{U}_0}(M, \Sigma^t\mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(\Phi M, \Sigma^{2t}\mathbf{F}_2)$$

and then appeal to Singer's work to claim that

$$P^i = \rho_i^* \circ \mathrm{Sq}_0$$

where P^i is as in (8.15).

The map Sq_0 is a familiar one to those working with stable Ext — there it is also known as Sq_0. Let $M \in \mathcal{U}_0$. The natural map of graded vaector spaces

$$M \to \bar{P}'M$$

adjoint to the identity $\bar{P}'M \to \bar{P}'M$ determines a map of vector spaces

$$\Phi M \to \Phi\bar{P}'M$$

and, hence, a natural morphism in $\mathcal{U}_0$

$$\bar{P}'\Phi M \to \Phi\bar{P}'M.$$

This, in turn, determines a map of simplicial objects in $\mathcal{U}_0$

$$\bar{P}'_{\cdot}\Phi M \to \Phi\bar{P}'_{\cdot}M.$$

Since Φ is an exact functor, we get a composite

$$\begin{aligned} Ext^p_{\mathcal{U}_0}(M, \Sigma^t\mathbf{F}_2) &\cong \pi^p Hom_{\mathcal{U}_0}(\Phi\bar{P}'_{\cdot}M, \Sigma^{2t}\mathbf{F}_2) \\ &\to \pi^p Hom_{\mathcal{U}_0}(\bar{P}'_{\cdot}\Phi M, \Sigma^{2t}\mathbf{F}_2) \\ &\cong Ext^p_{\mathcal{U}_0}(\Phi M, \Sigma^{2t}\mathbf{F}_2). \end{aligned} \tag{8.17}$$

This composite is Sq_0. The following is now immediately obvious from [Proposition 5.1 of 21].

Proposition 8.18: There is an equality of homomorphisms

$$P^i = \rho_i^* \circ \mathrm{Sq}_0 : Ext^p_{\mathcal{U}_0}(H^Q_q A, \Sigma^t \mathbf{F}_2) \to Ext^p_{\mathcal{U}_0}(H^Q_{q+i+1}A, \Sigma^{2t}\mathbf{F}_2)$$

for $0 \leq i \leq q$.

9. The cohomology of abelian objects

In this section we make some calculations with Miller's spectral sequence, including a computation — in terms of unstable Ext groups — of the E_2 term of the homotopy spectral sequence associated to the universal examples of section 3.

We begin by defining abelian obejects and recalling the Hilton-Milnor Theorem in the category $s\mathcal{UA}$. Let $A, B \in s\mathcal{UA}$. Then their tensor product $A \otimes B$ is the categorical coproduct of A and B; their product $A \times_{\mathbf{F}_2} B$ was defined in section 6. $A \in s\mathcal{UA}$ is an abelian group object if

$$Hom_{s\mathcal{UA}}(B, A)$$

is an abelian group for all $B \in s\mathcal{UA}$. This turns out to be equivalent to the following: A is an abelian group object if there is a morphism

$$\mu : A \times_{\mathbf{F}_2} A \to A$$

in $s\mathcal{UA}$ and a commutative diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f} & A \times_{\mathbf{F}_2} A \\ \downarrow m & & \downarrow \mu \\ A & \xrightarrow{=} & A \end{array}$$

where $f(a \otimes b) = (a\eta\epsilon(b), \eta\epsilon(a)b)$ and m is the algebra multiplication. One easily sees that this implies that

$$A \cong M_+$$

where $M \in s\mathcal{U}_0$ is a simplicial object in the category $\mathcal{U}_0$ and

$$(\ )_+ : s\mathcal{U}_0 \longrightarrow s\mathcal{U}\mathcal{A}$$

is the functor that sets, for $N \in s\mathcal{U}_0$

$$N_+ = N \oplus \mathbb{F}_2$$

with $\mathbb{F}_2$ the unit, N the augmentation ideal and $N^2 = 0$. Thus, in particular, we have for an abelian object $A \cong M_+$

$$Hom_{s\mathcal{U}\mathcal{A}}(B, A) \cong Hom_{s\mathcal{U}_0}(QB, M)$$

where B is the indecomposables functor. Hence the functors $(\ )_+ : s\mathcal{U}_0 \to s\mathcal{U}\mathcal{A}$ and $Q : s\mathcal{U}\mathcal{A} \to s\mathcal{U}_0$ form an adjoint pair. Finally, notice that for $M_1, M_2 \in s\mathcal{U}_0$

$$(M_1)_+ \times_{\mathbb{F}_2} (M_2)_+ \cong (M_1 \times M_2)_+.$$

The Hilton-Milnor Theorem discusses the homotopy type of

$$\Sigma[(M_1)_+ \times_{\mathbb{F}_2} (M_2)_+] \cong \Sigma[(M_1 \times M_2)_+]$$

in $s\mathcal{U}\mathcal{A}$, where $\Sigma : s\mathcal{U}\mathcal{A} \to s\mathcal{U}\mathcal{A}$ is the suspension functor of section 3.

We need some further notation. The category $s\mathcal{U}_0$ is a category of modules and, as such, is equivalent to a category of chain complexes. Therefore, it is easy to construct a suspension functor

$$\sigma : s\mathcal{U}_0 \longrightarrow s\mathcal{U}_0$$

so that there is a natural isomorphism

$$\pi_n \sigma M = \begin{cases} \pi_{n-1} M; & \text{if } n \geq 1; \\ 0; & \text{if } n = 0. \end{cases}$$

Now let L be the free algebra on two elements x_1, x_2. Let B be the Hall basis for L [22, p. 512]. Then $b \in B$ is an iterated Lie product in the elements x_1 and x_2. Let

$$\begin{aligned} i(b) &= \text{the number of appearances of } x_1 \text{ in } b \\ j(b) &= \text{the number of appearances of } x_2 \text{ in } b \\ \ell(b) &= i(b) + j(b) \end{aligned}$$

and if $M_1, M_2 \in s\mathcal{UA}$, define $M(b) \in s\mathcal{UA}$ by

$$M(b) = \sigma^{\ell(b)-1} M_1^{\otimes i(b)} \otimes M_2^{\otimes j(b)} \tag{9.1}$$

where $N^{\otimes k}$ means the tensor product of N with itself k times.

Theorem 9.2 (Hilton-Milnor)[12]: Let M_1 and M_2 be objects in $s\mathcal{U}_0$. Then there is a weak equivalence in $s\mathcal{UA}$

$$\Sigma[(M_1)_+ \times_{\mathbb{F}_2} (M_2)_+] \to \otimes_{b \in L} \Sigma[M(b)_+].$$

Remarks 9.3.1) The relationship to the usual Hilton-Milnor Theorem for spaces is given by the following: if X and Y are spaces, then $H^*\Sigma X \cong (\bar{H}^*\Sigma X)_+$ and

$$H^*\Sigma X \vee \Sigma Y \cong (\bar{H}^*\Sigma X)_+ \times_{\mathbb{F}_2} (\bar{H}^*\Sigma Y)_+.$$

Then, regarding $H^*\Sigma X \vee \Sigma Y$ as a constant simplicial algebra, we obtain a simplicial unstable algebra

$$\Sigma(H^*\Sigma X \vee \Sigma Y)$$

and, by example 4.8, a spectral sequence

$$H^*_{Q\mathcal{A}}\Sigma(H^*\Sigma X \vee \Sigma Y) \Rightarrow \pi_*\Omega(\Sigma X \vee \Sigma Y).$$

On the one hand

$$\pi_*\Omega(\Sigma X \vee \Sigma Y).$$

is computed using the classical Hilton-Milnor Theorem, and on the other hand

$$H^*_{Q\mathcal{A}}\Sigma(H^*\Sigma X \vee \Sigma Y)$$

is computed using Theorem 9.2 and the next remark.

9.3.2) Notice that if $A, B \in s\mathcal{UA}$, then there is a natural isomorphism

$$\mathbb{F}_2 \otimes_{\mathcal{A}} Q(A \otimes B) \cong \mathbb{F}_2 \otimes_{\mathcal{A}} QA \oplus \mathbb{F}_2 \otimes_{\mathcal{A}} QB$$

and, hence, a natural isomorphism

$$H^*_{Q\mathcal{A}} A \otimes B \cong H^*_{Q\mathcal{A}} A \times H^*_{Q\mathcal{A}} B.$$

Thus there is a sequence of isomorphisms

$$H^*_{Q\mathcal{A}}[(M_1)_+ \times_{\mathbf{F}_2} (M_2)_+] \cong H^{*+1}_{Q\mathcal{A}} \Sigma[(M_1)_+ \times_{\mathbf{F}_2} (M_2)_+]$$

and

$$H^*_{Q\mathcal{A}} \Sigma[(M_1)_+ \times_{\mathbf{F}_2} (M_2)_+] \cong \times_{b \in L} H^*_{Q\mathcal{A}} \Sigma M(b)_+$$

and

$$H^{*+1}_{Q\mathcal{A}} \Sigma M(b)_+ \cong H^*_{Q\mathcal{A}} M(b)_+.$$

In particular, this serves to compute

$$Ext_{\mathcal{U}\mathcal{A}}(H^* \Sigma X \vee \Sigma Y, \mathbf{F}_2)$$

as a product of $H^*_{Q\mathcal{A}} M(b)_+$ where, in $s\mathcal{U}_0$

$$M(b) = \sigma^{\ell(b)-1} (\bar{H}^* \Sigma X)^{\otimes i(b)} \otimes (\bar{H}^* \Sigma Y)^{\otimes j(b)}$$

is $s\mathcal{U}_0$. *A priori*, one might have expected this *Ext* group to split as a product of *Ext* groups. This turns out not to be the case — $H^*_{Q\mathcal{A}}$ turns out to be the necessary generalization in this case. The reader is encouraged, as an example to consider the case where $X = Y = S^n$ for some n. Compare 9.3.4 below.

9.3.3) To compute $H^*_{Q\mathcal{A}} M_+$ for $M \in s\mathcal{U}_0$ — that is, to compute the cohomology of abelian objects in $s\mathcal{U}\mathcal{A}$ — it is sufficient to compute $H^*_{Q\mathcal{A}} N_+$ for all $N \in s\mathcal{U}_0$ indecomposable in the sense that it has no non-trivial direct summands. Notice that one needs the full generality of the Hilton-Milnor Theorem and the preceeding remark even if one is only interested in $Ext_{\mathcal{U}\mathcal{A}}(M_+, \mathbf{F}_2)$ with $M \in \mathcal{U}_0$.

9.3.4) In particular, if $D(r,p,q)$ is as is example 3.13, we have a weak equivalence in $s\mathcal{U}\mathcal{A}$

$$H^* D(r,p,q) \simeq K(p,q)_+ \times_{\mathbf{F}_2} K(p+r, q+r-1)_+.$$

Thus, to compute $H^*_{Q\mathcal{A}}H^*D(r,p,q)$ — of interest because it is the E_2 term of the universal r-differential in the Bousfield-Kan spectral sequence — it is sufficient to compute

$$H^*_{Q\mathcal{A}}K(s,t)_+$$

for all s and t. This follows from Theorem 9.2 and the fact that there is a weak equivalence in $s\mathcal{U}_0$

$$K(s,t)\otimes K(s',t') \simeq K(s+s',t+t')$$

and

$$\sigma K(s,t) \simeq K(s+1,t).$$

This project will occupy the rest of the section.

So saying, let $M_+ \in s\mathcal{U}\mathcal{A}$ be an abelian object. We look to Miller' spectral sequence

$$Ext^s_{\mathcal{U}_0}(H^Q_r M_+, \mathbb{F}_2) \Rightarrow H^{s+r}_Q M_+$$

for our computations. This is plausible because $H^Q_* M_+$ is known. In fact, we will record $H^*_Q M_+$ and dualize. Let $\mathcal{W}$ be the category of Theorem 7.5. If $A \in s\mathcal{U}\mathcal{A}$, then $H^*_Q A \in \mathcal{W}$.

The forgetful functor $\mathcal{W} \to nn\mathbb{F}_2$ from $\mathcal{W}$ to the category of bigraded $\mathbb{F}_2$ vector spaces has a left adjoint

$$U : nn\mathbb{F}_2 \to \mathcal{W}$$

and one of the main theorem of [14] implies the following.

Theorem 9.4: If $M \in s\mathcal{U}_0$ with $\pi_* M$ of finite type. Then

$$H^*_Q M_+ \cong U(\pi_* M^*).$$

Now, $H^n_Q M_+$ is a right module over the Steenrod algbera and this structure is a consequence of Theorem 9.4 and the formulas

$$\begin{aligned}
[x,y]\mathrm{Sq}^k &= \sum_{i+j=k} [x\mathrm{Sq}^i, y\mathrm{Sq}^j] \\
P^i(x)\mathrm{Sq}^{2k} &= P^i(x\mathrm{Sq}^k) \\
P^i(x)\mathrm{Sq}^{2k+1} &= 0 \\
\beta(x)\mathrm{Sq}^k &= \beta(x\mathrm{Sq}^{k/2}) + \sum_{i<k/2} [x\mathrm{Sq}^i, x\mathrm{Sq}^{k-i}]
\end{aligned} \tag{9.5}$$

where $\mathrm{Sq}^{k/2} = 0$ if k is odd. The first three formulas are in Proposition 8.15; the fourth is in [14, Section 4].

Example 9.6: Consider the example of $M = K(p,q)$. For $p = 0$ we have $K(0,q)_+ \cong H^*S^q$ regarded as a constant simplicial algebras and

$$H^0_Q K(0,q)_+ \cong \mathbf{F}_2$$

concentrated in degree q and generated by a class ι and

$$H^1_Q K(0,q)_+ \cong \mathbf{F}_2$$

concentrated in degree $2q$ and generated by $\beta(\iota)$. Miller's spectral sequence for this example was calculated in 8.7.

If $p > 0$ and $\iota \in H^p_Q K(p,q)_+$ is the non-zero class of degree q, then a basis for $H^*_Q K(p,q)_+$ as a bigraded $\mathbf{F}_2$ vector space is given by the elements

$$P^{i_1} P^{i_2} \dots P^{i_s}(\iota) \tag{9.6.1}$$

where $2 \leq i_t < 2i_{t+1}$ for all t and $2 \leq i_s \leq p$. Furthermore, the equations of (9.5) imply that for each s $H^s_Q K(p,q)_+$ has the structure of a trivial module over the Steenrod algebra.

In particular

$$H^*_Q K(1,q)_+ \cong \mathbf{F}_2 \tag{9.6.2}$$

concentrated in bidegree $(1,q)$. In this case, then, Miller's spectral sequence must collapse and we have

$$\begin{aligned}[H^s_{Q\mathcal{A}}K(1,q)_+]_t &\cong Ext^{s-1}_{\mathcal{U}_0}(\Sigma^q\mathbb{F}_2,\Sigma^t\mathbb{F}_2)\\ &\cong Ext^{s-1}_{\mathcal{U}}(\Sigma^{q-1}\mathbb{F}_2,\Sigma^{t-1}\mathbb{F}_2)\end{aligned}.$$

Thus the spectral sequence

$$H^*_{Q\mathcal{A}}K(1,q)_+ \Rightarrow \pi_*S^{q-1}$$

quaranteed by 3.11 is, in fact, the same spectral sequence one obtains from Proposition 1.5 and the Bousfield-Kan spectral sequence.

The final result of this section expands on these computations.

Theorem 9.7: For all $p \geq 1$, the composite functor spectral sequence

$$Ext^s_{\mathcal{U}_0}(H^Q_tK(p,q)_+,\mathbb{F}_2) \Rightarrow H^{s+t}_QK(p,q)_+$$

collapses.

Before proving this, we establish some notation and state a lemma. We have a prefered basis for $H^*_QK(p,q)_+$; namely, all, elements of the form

$$P^I(\iota) = P^{i_1}\cdots P^{i_k}(\iota)$$

with $2 \leq i_t < 2i_{t+1}$ for all t and $2 \leq i_k \leq p$. Call such I *allowable*, and let $\ell(I) = s$ and $e(I) = i_k$.

If M is a trivial module over the Steenrod algebra, then

$$Ext^s_{\mathcal{U}_0}(M,\Sigma^t\mathbb{F}_2) \cong \times_m\ Ext^s_{\mathcal{U}_0}(\Sigma^m\mathbb{F}_2,\Sigma^t\mathbb{F}_2)\otimes M^*_m$$

Where $M^m \subseteq M$ are the elements of degree m. Hence, since $(H^Q_*A)^* \cong H^*_QA$,

$$Ext^s_{\mathcal{U}_0}(H^Q_tK(p,q)_+,\mathbb{F}_2) \cong \times_m\ Ext^s_{\mathcal{U}_0}(\Sigma^m\mathbb{F}_2,\mathbb{F}_2)\otimes H^t_QK(p,q)_+.$$

If $\iota \in [H^p_Q K(,p,q)_+]_q$ is the fundamental class, then the properties of the operations P^i imply that

$$P^I(\iota) = P^{i_1} \cdots P^{i_k}(\iota) \in [H_Q^{p+i_1+\cdots+i_k+k} K(p,q)_+]_{2^{\ell(I)}q}.$$

Thus, if $\langle P^I(\iota)\rangle \subseteq H^*_Q K(p,q)_+$ is the sub-vector space generated by $P^I(\iota)$, we have

$$\text{(9.8)} \qquad Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathsf{F}_2) \cong \times_I \, Ext^s_{\mathcal{U}_0}(\Sigma^{2^{\ell(I)}q}\mathsf{F}_2, \mathsf{F}_2) \otimes \langle P^I(\iota)\rangle$$

where the product is taken over all allowable I so that

$$\text{(9.8.1)} \qquad e(I) \leq p$$

and

$$\text{(9.8.2)} \qquad p + i_1 + \cdots + i_k + k = t.$$

I can be empty, in which case $P^I(\iota) = \iota$ and $e(I) = 0$.

Proof of 9.7: First, since

$$\text{(9.9)} \qquad E_2^{s,t} \cong Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathsf{F}_2) = 0$$

for $t < p$ and the differentials have the form

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}$$

all differentials vanish on $E_r^{s,p}$ for all $r \geq 2$ and all $p \geq 1$.

Second, we have

$$E_2^{0,a} \cong H^a_Q K(p,q)_+$$

and this vector space is spanned by elements of the form $P^I(\iota)$. Thus, from 8.13.2 and 8.14, we have that all differentials vanish on $E_r^{0,a}$ for $r \geq 2$. Also, if $0 \neq P^I(\iota) \in E_2^{0,a}$, then $P^I(\iota) \neq 0$ in $H^t_{Q\mathcal{A}} K(p,q)_+$. Let

$$f : K(p,q)_+ \longrightarrow K(a,b)_+$$

be the morphism in the homotopy category of $s\mathcal{UA}$ (see 2.16) corepresenting

$$P^I(\iota) \in H^a_Q K(p,q)_+.$$

Of course

$$a = p + i_1 + i_1 + \cdots + i_k + k$$

and

$$b = 2^{\ell(I)}q.$$

The morphism f, then induces a diagram of spectral sequences

$$\begin{array}{ccc} Ext^s_{\mathcal{U}_0}(H^Q_t K(a,b)_+, \mathbb{F}_2) & \Rightarrow & H^{s+t}_{Q\mathcal{A}} K(a,b)_+ \\ \downarrow E_2 f & & \downarrow H^*_{Q\mathcal{A}} f \\ Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathbb{F}_2) & \Rightarrow & H^{s+t}_{Q\mathcal{A}} K(p,q)_+. \end{array}$$

Then (9.9) for (a,b) implies that all differentials vanish on

$$Ext^s_{\mathcal{U}_0}(\Sigma^{2^{\ell(I)}q}\mathbb{F}_2, \mathbb{F}_2) \otimes \langle P^I(\iota)\rangle \subseteq Ext^s_{\mathcal{U}_0}(H^Q_t K(p,q)_+, \mathbb{F}_2).$$

The result now follows from (9.8).

The following is a consequence of Theorem 9.7 and the equation (9.8).

Corollary 9.10: There is an isomorphism of vector spaces for $p \geq 1$

$$H^n_{Q\mathcal{A}} K(p,q)_+ \cong \times_I \; Ext^s_{\mathcal{U}_0}(\Sigma^{2^{\ell(I)}q}\mathbb{F}_2, \mathbb{F}_2) \otimes \langle P^I(\iota)\rangle$$

where the product is over all allowable I so that $e(I) \leq p$ and

$$s + p + i_1 + \cdots i_k + k = n.$$

Of course, by 8.4, we have

$$Ext^s_{\mathcal{U}_0}(\Sigma^m\mathbb{F}_2, \Sigma^t\mathbb{F}_2) \cong Ext^s_{\mathcal{U}}(\Sigma^{m-1}\mathbb{F}_2, \Sigma^{t-1}\mathbb{F}_2)$$

and the latter is a familiar, if somewhat intractable, object. Finally the action of the operations

$$P^i : H^n_{Q\mathcal{A}} K(p,q)_+ \to H^{n+i+1}_{Q\mathcal{A}} K(p,q)_+$$

can be computed up to filtration using 8.8 and 9.10.

We now say in what sense the spectral sequence

$$H^*_{Q\mathcal{A}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

is a desuspension spectral sequence. We assume that $p \geq 1$. Then , as in (7.13), we obtain a diagram of spectral sequences

$$\begin{array}{ccc} H^*_{Q\mathcal{A}}K(p,q)_+ & \Rightarrow & \pi_* S^{q-p} \\ \downarrow e^* & & \downarrow E_{p-1} \\ H^*_{Q\mathcal{A}}\Sigma^{p-1}K(1,q)_+ & \Rightarrow & \pi_*\Omega^{p-1}S^{q-1} \end{array}$$

where E_{p-1} is the suspension homomorphism. Using Corollary 9.10 and the fact that

$$e^* P^I(\iota_{p,q}) = 0$$

— where $\iota_{p,q} \in [H^p_{Q\mathcal{A}}K(p,q)_+]_q$ is the generator — we see that e^* is surjective; indeed it is isomorphic (under the isomorphisms of 9.10) to projection onto the factor

$$Ext^*_{\mathcal{U}_0}(\Sigma^q \mathbf{F}_2, \mathbf{F}_2).$$

Since

$$Ext^*_{\mathcal{U}_0}(\Sigma^q \mathbf{F}_2, \mathbf{F}_2) \cong Ext^*_{\mathcal{U}}(\Sigma^{q-1}\mathbf{F}_2, \mathbf{F}_2)$$

and the latter is the E_2 term of a spectral sequence for computing $\pi_* S^{q-1}$, the other factors in $H^*_{Q\mathcal{A}}K(p,q)_+$ are present to correct the computation to a calculation of $\pi_* S^{q-p}$.

We end this paper with a calculation that demonstrates that not all the operations P^i commute with differentials in the Bousfield-Kan spectral sequence. Let $\alpha : S^{n-1} \to S^{n-1}$ be the identity map and let

$$h_0 \in Ext^1_{\mathcal{U}_0}(\Sigma^n \mathbf{F}_2, \Sigma^{n+1}\mathbf{F}_2)$$

be the element detecting $2\alpha \in \pi_{n-1}S^{n-1}$. Then we can let

$$P^p(\iota)h_0 \in Ext^1_{\mathcal{U}_0}(\Sigma 2q\mathbf{F}_2, \Sigma^{2q+1}\mathbf{F}_2) \otimes \langle P^p(\iota)\rangle$$

stand for the non-zero class.

Proposition 9.11: Let $p \geq 3$ and $q-p$ be an odd number. Then in the spectral sequence

$$H^*_{\mathcal{QA}}K(p,q)_+ \Rightarrow \pi_* S^{q-p}$$

there is a differential

$$d_2 P^{p-1}(\iota) = P^p(\iota)h_0.$$

Proof: We refer to the calculations of Corollary 7.15 and consider the diagram of spectral sequences of 7.13:

$$\begin{array}{ccc} H^*_{\mathcal{QA}}M(e^*) & \Rightarrow & \pi_* C(p-1) \\ \downarrow f & & \downarrow \\ H^*_{\mathcal{QA}}K(p,q)_+ & \Rightarrow & \pi_* S^{q-p}. \end{array}$$

Corollary 9.10 implies that there is a unique class $y_i \in H^*_{\mathcal{QA}}M(e^*)$ so that

$$f(y_i) = P^i(\iota).$$

Corollary 9.10 also implies that

$$0 \neq y_p h_0 \in [H^{2p+2}_{\mathcal{QA}}M(e^*)]_{2q+1}$$

and that

$$[H^s_{\mathcal{QA}}M(e^*)]_t = 0$$

for $t-s = 2(q-p)$ and $s < 2q$. Hence the calculation given in the proof of 7.15 implies that

$$d_2 y_{p-1} = y_p h_0.$$

The result follows.

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December 1988

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