Astérisque

CHRISTINE BESSENRODT

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Astérisque, tome 181-182 (1990), p. 11-29

<http://www.numdam.org/item?id=AST_1990__181-182__11_0>

SOME NEW BLOCK INVARIANTS COMING FROM COHOMOLOGY

BY

Christine BESSENRODT

1 Introduction

In the usual setup of the representation theory of finite groups we are given a finite group G and a ring A of coefficients, and we want to study the modules over the ring $\mathcal{A} = AG$. Typical coefficient rings are the ring $\mathbb{Z}$, the p-adic numbers $\mathbb{Z}_p$, or fields. For many properties of these modules, we can 'forget' the group G and just need to know the algebra $\mathcal{A}$. Now suppose that p is a prime dividing the order of G, and let A be a complete discrete valuation ring with residue field of characteristic p or a field of characteristic p. There are some very fruitful invariants in integral and modular representation theory which are defined with explicit reference to the given group G. The most prominent among these are the vertex of an indecomposable AG-module and the defect group of a p-block, or the kernels of modules and blocks.

Now it is natural to ask:

(1) What informations on G can we read off from $\mathcal{A}$?

(2) What happens to the invariants mentioned above, if we choose another group basis in $\mathcal{A}$, i.e. a subgroup $H \leq U(\mathcal{A})$ such that $\mathcal{A} = AH$ and $|H| = |G|$?

In particular, question (1) includes the classical isomorphism problem which was formulated by G. Higman in 1940 and later also posed by Brauer [7]:

$$\text{Does } \mathbb{Z}G \simeq \mathbb{Z}H \text{ imply } G \simeq H \text{ ?}$$

This problem has stimulated a lot of research, and the last few years have seen quite some progress, in particular in the work of Roggenkamp and Scott [22]. They have obtained positive answers for some classes of groups also to the much stronger Zassenhaus conjecture, which asks whether another (normalised) group basis for $\mathbb{Z}G$ must even be conjugate to G by a unit in $\mathbb{Q}G$. For more details and the history of the isomorphism problem the reader is referred to the articles by Roggenkamp and Scott, the books by Passman [20] and Sehgal [25], and the survey article by Sandling [23].

Roggenkamp and Scott have also dealt with other integral coefficient rings, such as the p-adic numbers $\mathbb{Z}_p$. For these, too, they could prove the Zassenhaus conjecture for nilpotent groups. For $A = \mathbb{Z}_p$ and G a p-group, Weiss [26] succeeded in proving the strong theorem that any finite subgroup of $V(\mathbb{Z}_pG)$, the augmentation 1 units in $\mathbb{Z}_pG$, is conjugate in $V(\mathbb{Z}_pG)$ to a subgroup of G. For $A = F$ a field of characteristic p, it is still an open question whether the group algebra of a p-group determines the group G. The earliest result to this question goes back to Deskins [13], who proved that an abelian p-group is determined by its modular group algebra. It is also known that the answer is positive for small p-groups and for various special classes of p-groups. The proofs are usually rather computational, and it seems hard to transfer them from the case of p-groups to general groups.

So for these coefficient rings there are rather few results to question (1) for general finite groups. On the other hand, by using the classification of the finite simple groups, Kimmerle-Lyons-Sandling [17] showed that $\mathbb{Z}G$ determines the composition factors of G. They also proved that $\mathbb{Z}G$ determines whether the Sylow subgroups of G are abelian, hamiltonian or of certain other types, and in these cases they can obtain the structure of these groups [16].

For a coefficient ring like $\mathbb{Z}_p$ or a field of characteristic p, there is at least some hope that the group ring AG determines the structure of a Sylow p-subgroup. Motivated by the recent successes, Scott asked the following more general question, which is of type (2) (see [24]):

> Given a p-block B of $\mathbb{Z}_pG$, are its defect groups determined up to conjugation and 'suitable' normalisation, independently of the group G ?

Also, Alperin pointed out that it is even open whether the isomorphism type of the defect groups is determined by B.

In our investigation we will focus mainly on the modular group algebra FG; of course, this also implies results for the integral situation.

In the following sections we present a contribution to the question posed by Scott and Alperin. Our leading idea will be that the problem of determining the isomorphism type of a defect group of a block falls into two parts: first one would like to obtain the defect group algebra from the block algebra, and then one needs a positive answer to the isomorphism problem for p-groups (as mentioned above, this is true for $\mathbb{Z}_p$, but open for fields of characteristic p). In fact, we will be more modest, and we will just try to compute certain new invariants of the defect group algebra from the block algebra. It turns out that for many types of p-groups these invariants are the same for the defect group algebra and the block algebra, and in the abelian case they even suffice to determine the isomorphism type of the defect group.

Here are a few more details on the course of our investigations. As computations inside the group algebra can usually not easily be translated to the block situation, we introduce a new tool coming from cohomology theory in the second section. For this, we use the complexity of a module, which is a measure for the growth of the dimensions of the projective modules in a minimal projective resolution for the module. This invariant was introduced by Alperin in 1977, and it has attracted much attention since Alperin and Evens [1] have proved their celebrated theorem that the complexity of a module can be determined on the elementary abelian p-subgroups. If $A = F$ is an algebraically closed field, it can also be described as the dimension of a certain variety associated with the module, which was defined by Carlson [11], who also proved many important properties of this variety.

For our purposes, we define a sequence of invariants for a p-block B (or more generally for a union of p-blocks) by looking at the dimensions of modules with a certain complexity belonging to the block B. A few properties of the defect group can easily be read off this sequence, like its order and its rank. The invariants for the whole group algebra are the same as those for the group algebra over a Sylow p-subgroup. We then show that for a defect group for which the invariants already come from trivial source modules, the invariants of the block are the same as those of the defect group algebra. Based on some results of Carlson, one can prove that for the group algebra of an abelian p-group our invariants determine the isomorphism type of the p-group, and we see that they come from trivial source modules. Thus, in particular, the structure of an abelian defect group can be deduced from the invariants for the block (see [5]), but also some other types of p-groups can be handled with this method. Unfortunately, our invariants can not decide whether the defect group is abelian, we have to assume this in advance. In fact, note that so far it is not even known if the whole modular group algebra determines whether the Sylow p-subgroups are abelian. In the last section we calculate the sequence of invariants for various p-groups.

I would like to thank the Deutsche Forschungsgemeinschaft for financial support during the time while this research was done at the University of Essen. It forms part of a manuscript which was accepted as a Habilitationsschrift by the Department of Mathematics at the University of Duisburg.

Let us fix some notation for the following. By G we will always denote a finite group, and by F a field of characteristic $p > 0$. Furthermore, R will always be a complete discrete valuation ring of characteristic 0 with residue field of characteristic $p > 0$, which we will then also denote by F. We assume that the quotient field of R is sufficiently large relative to G, so that it is a splitting field for G and its subgroups. The ring A will be one of the rings R or F, and an AG-module is always supposed to be finitely generated and free over A. For an AG-module M we denote by $c_G(M)$ the complexity of M (see e.g. [3]). For $n \in \mathbb{N}$ we write n_p for the highest p-power dividing n. Other standard notations and terminology may be found in the books by Benson [3] and Feit [14].

2 Some new invariants for group algebras and blocks

In this section we want to introduce some new invariants for blocks and group algebras, which are derived from looking at modules of a certain complexity; we refer the reader to Benson [3] and the papers by Alperin-Evens [1], Avrunin-Scott [2] and Carlson [11] for the properties of the complexity and the variety of a module. For the isomorphism problem we want to exploit the relationship between the complexity and the rank of an AG-module.

Now let us come to the precise definition of our invariants.

Let $A \in \{R, F\}$ as before and let $\mathcal{A}$ be a union of p-blocks of AG. For $i \in \mathbb{N}_0$ we set

$$M_i(\mathcal{A}) = \{V \mid V \textit{ indecomposable } AG\textit{-module in } \mathcal{A} \textit{ and } c_G(V) \leq i\}.$$

Let $|G|_p = p^a$. Via the following recursive procedure we define a sequence of invariants associated with $\mathcal{A}$:

$$\begin{aligned} n_1 &= n_1(\mathcal{A}) &= min\{n \in \mathbb{N} \mid p^{a-n} \mid rank_A V \textit{ for all } V \in M_1(\mathcal{A})\}, \\ n_i &= n_i(\mathcal{A}) &= min\{n \in \mathbb{N} \mid p^{(a-\sum_{j=1}^{i-1} n_j)-n} \mid rank_A V \textit{ for all } V \in M_i(\mathcal{A})\}, \end{aligned}$$

for all $i \geq 2$. For abbreviation we will write $\mathbf{n}(\mathcal{A}) = (n_i(\mathcal{A}))_{i \in \mathbf{N}}$.

PROPOSITION 2.1 *Let B be a p-block of AG with D as a defect group, $r = r(D)$ the rank of D, and $|D| = p^d$. Then the sequence $\mathbf{n}(B) = (n_i)_{i \in \mathbf{N}}$ has the following properties:*

(i) $n_i = 0$ *for all* $i > r$, *and* $n_r > 0$.

(ii) $\sum_{i=1}^{r} n_i = d$.

Proof. The first part of (i) holds because $M_r(B)$ is the set of all indecomposable modules in B, and (ii) follows from the well-known fact that there is an indecomposable module of height 0 in B. The second assertion in (i) is a consequence of the fact that modules of complexity $< r$ have always dimension divisible by p^{a-d+1} (this will follow from a later result; or see [6]). □

Remark. We see from the above that our invariants determine the rank of the defect group of a block, which is no surprise since it is known that the rank equals the maximal complexity of a module in the block. For the whole group G, not only the order of its maximal elementary abelian p-subgroups is given by cohomology theory, but Quillen [21] has even proved that the minimal prime ideals of the mod p cohomology ring of G are in one-to-one correspondence with the conjugacy classes of maximal elementary abelian p-subgroups. So the group algebra FG determines the number of these conjugacy classes.

Knowing the rank of the defect groups already suffices to handle the smallest cases. Let us here give an immediate application for the smallest non-abelian situation. Remember that the order of the defect groups is always easily obtainable from the dimensions of the modules in the block.

PROPOSITION 2.2 *Let B be a p-block of FG with defect groups of order 8. Then the isomorphism type of the defect groups is determined by B.*

Proof. Since we know the rank of the defect groups from the dimensions of the modules in B, we only have to be able to distinguish between the groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ and D_8, respectively $\mathbb{Z}_8$ and Q_8. Now if the defect groups are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, then B is of wild representation type, if they are dihedral or quaternion, then B is of tame type, and if they are cyclic, B is of finite type. Thus we are done. □

Sometimes it will also be useful to have a short notion for the following invariants, which are just as good as the n_i's defined above:

$$l_i = l_i(B) = max\{p^l \mid p^l \mid rank_A V \text{ for all } V \in M_i(B)\},$$

for all $i \in \mathbb{N}$.
Note that we always have $l_0(B) = p^a$ by Brauer [8], so usually we will just consider the sequence $\mathrm{l}(B) = (l_i(B))_{i \in \mathbb{N}}$, starting from $i = 1$.

The relationship between these sequences is given by:

LEMMA 2.3 *Let B be a p-block of AG, $\mathrm{n}(B) = (n_i)_{i\in\mathbb{N}}$, $\mathrm{l}(B) = (l_i)_{i\in\mathbb{N}}$. Then we have for all $i \in \mathbb{N}$:*

$$\begin{aligned} l_i &= p^{a-\sum_{j=1}^{i} n_j}, \\ p^{n_i} &= \frac{l_{i-1}}{l_i}. \end{aligned}$$

Now a word on the dependency on the ring A is in order. Note that for a ring extension $A \subseteq A'$ we just have $l_i(AG) \geq l_i(A'G)$, and if we go from R to its residue field F we have $l_i(RG) \geq l_i(FG)$. This follows from the fact that the complexity of a module is well-behaved in these situations.

Another easy property of these invariants is contained in the following lemma:

LEMMA 2.4 *If $X \leq G$, then for all $i \in \mathbb{N}$ we have:*

(i) $l_i(AX) \leq l_i(AG)$.
(ii) $\sum_{j=1}^{i} n_j(AX) \leq \sum_{j=1}^{i} n_j(AG)$,
or equivalently: $l_i(AG) \leq l_i(AX)|G : X|_p$.

In general, it is not at all clear whether the isomorphism type of the group algebra AP of a Sylow p-subgroup P of G is determined by the group algebra AG. Even for many invariants of AP, it is not known how to compute them given just the group algebra AG. So it is nice to see that the invariants given above are well-behaved in this respect:

COROLLARY 2.5 *If P is a Sylow p-subgroup of G, then* $\mathrm{l}(AP) = \mathrm{l}(AG)$ *and* $\mathrm{n}(AP) = \mathrm{n}(AG)$.

Of course, now the next problem is: does this even hold blockwise, i.e. are the invariants for a block B the same as the invariants for AD, where D is a defect group of B ? We will later give a positive answer at least for certain types of defect groups.

The easiest situation for which we know the invariants defined above is, of course, the case of a group G with cyclic or generalised quaternion Sylow p-subgroup of order p^a where we just have $n_1(AG) = a$. Another easy case is treated in the following result.

LEMMA 2.6 *Let P be a Sylow p-subgroup of G, $|P| = p^a$ as before. If P has a cyclic or generalised quaternion subgroup of index p, then the invariants of AG are:*

$$n_1(AG) = a - 1 \ , \ n_2(AG) = 1 \ .$$

Proof. By the previous corollary, $n_i(AG) = n_i(AP)$ for all $i \in \mathbb{N}$. Of course, $r(P) = 2$, so $n_i(AP) = 0$ for $i > 2$. Inducing the trivial module from a cyclic or generalised quaternion subgroup of index p to P, gives an indecomposable periodic AP-module of rank p, hence $n_1(AP) \geq a - 1$. Now Proposition 2.1 proves the assertion. □

REMARK 2.7 *At least for $A = F$, the invariants do not characterise groups with a Sylow p-subgroup of the type above. Take for example $G = P$ to be extraspecial of order p^3 and exponent p. If $F \neq GF(p)$, then by a result of Carlson [9] FP has an indecomposable FP-module of dimension p, so also in this case we have $n_1(FP) = 2$ and $n_2(FP) = 1$.*

3 A new lower bound for the p-part in the rank of a lattice

We will now improve Green's lower bound for the p-part in the rank of a lattice, in which we get an extra factor which comes from the invariants defined above. A first step in this direction was the bound given in [6] which will be obtained as a special case here. Whereas there only the rank of the vertex, i.e. its elementary abelian subgroups were considered, we will now take the invariants for the vertex group algebra into account. Also compare the results in [5], where we had looked at abelian subgroups of the vertex.

THEOREM 3.1 *Let V be an AG-module. Suppose V is D-projective for some p-group $D \leq G$. Let $X \leq D$ and $c = c_X(V_X)$, then*

$$|G:D|_p\, l_c(AX) \;\Big|\; rank_A V.$$

Proof. Let P be a Sylow p-subgroup of G with $D \leq P$. Then $V_P = \oplus_i V_i$, where the V_i are indecomposable AP-modules. As V is D-projective, each V_i is D_i-projective for $D_i = D^{x_i} \cap P$, with a suitable $x_i \in G$. Thus $V_i \simeq U_i{}^P$ for an indecomposable FD_i-module U_i.

Set $X_i = X^{x_i} \cap D_i = X^{x_i} \cap P$. As $U_{i|X_i} \mid V_{X_i}$, we have

$$c_{X_i}(U_{i|X_i}) \leq c_{X_i}(V_{X_i}) \leq c_X(V_X) = c,$$

and hence $l_c(AX_i) \mid rank_A U_i$. Since $X_i \leq_G X$, we know that

$$l_c(AX) \leq l_c(AX_i)\,|X:X_i|\,.$$

Furthermore, we have

$$|X| = |X^{x_i}| = |X_i||X^{x_i} : X^{x_i} \cap D_i| \leq |X_i||D^{x_i} : D_i| = \frac{|X_i||D|}{|D_i|}\,,$$

and these inequalities together yield:

$$|P:D|\, l_c(AX) \;\Big|\; |P:D_i|\, l_c(AX_i) \;\Big|\; |P:D_i|\, rank_A U_i = rank_A V_i,$$

for all i . Hence $|P:D|\, l_c(AX)$ divides $rank_A V$. □

The theorem immediately implies the following result for the invariants of blocks:

COROLLARY 3.2 *Let B be a block of AG with defect group D. Then for all $i \in \{1,\ldots,r(D)\}$ we have:*

$$l_i(AD)\,|G:D|_p \leq l_i(B)\,,$$

or equivalently:

$$\sum_{j=1}^{i} n_j(B) \leq \sum_{j=1}^{i} n_j(AD).$$

Of course, in the above we would like to have equality, i.e. $\mathbf{n}(B) = \mathbf{n}(AD)$. This would be the generalisation of Corollary 2.5 to blocks that we are looking for. We will later obtain this at least for a certain class of p-groups.

4 Trivial source modules

In this section, we define a set of invariants for a group G, which is related to the (n_i)-sequence for AG.

The sequence $\mathbf{t}(G) = (t_i(G))_{i \in \mathbb{N}}$ is defined recursively by

$$p^{\sum_{j=1}^{i} t_j(G)} = max\{\,|P| \;\big|\; P \leq G \text{ a } p\text{-group},\ r(P) \leq i\}.$$

Already from the definition it is clear that G has the same invariants as its Sylow p-subgroups. Corresponding to the invariants $l_i(AG)$, we also set

$$s_i(G) = min\{|G : P|_p \;\big|\; P \leq G \text{ a } p\text{-group}, r(P) \leq i\}$$

and $\mathbf{s}(G) = (s_i(G))_{i \in \mathbb{N}}$. These invariants can also be defined via special modules. We consider the set

$$T_i(G) = \{V \mid V \in M_i(AG),\ V \text{ is a trivial source module}\}.$$

In this context, 'trivial source AG-module' is always supposed to mean a module which has trivial source with respect to G. Note that here it doesn't matter whether $A = R$ or $A = F$. Let again $|G|_p = p^a$; since the trivial module over a p-group P has complexity $r(P)$, we have:

$$t_i = t_i(G) = min\{n \in \mathbb{N} \;\big|\; p^{(a - \sum_{j=1}^{i-1} t_j) - n} \mid rank_A V, \text{ for all } V \in T_i(G)\}.$$

Furthermore, the invariants $s_i(G)$ satisfy

$$s_i = s_i(G) = max\{p^s \;\big|\; p^s \mid rank_A V \text{ for all } V \in T_i(G)\}\ ,$$

and the relationship between the invariants (s_i) and (t_i) is given by

$$s_i = p^{a - \sum_{j=1}^{i} t_j} \quad \text{and} \quad t_i = \frac{s_{i-1}}{s_i}\ .$$

Moreover, it is clear that for all i

$$\sum_{j=1}^{i} t_j(G) \leq \sum_{j=1}^{i} n_j(AG)\,.$$

If B is a block of AG, we define similarly

$$T_i(B) = \{V \in B \mid V \in T_i(G)\}.$$

Corresponding to the invariants $t_i(G)$ and $s_i(G)$, we define the invariants $t_i(B)$ and $s_i(B)$ by replacing the set $T_i(G)$ by $T_i(B)$. Here, one should always keep in mind that these invariants depend on the chosen group basis G, so we should write more precisely $t_i(B) = t_i(B; G)$. Clearly, $l_i(B) \leq s_i(B)$ since $T_i(B) \subseteq M_i(B)$.

These invariants have similar properties as the ones defined before:

LEMMA 4.1 *If $X \leq G$, then for all $i \in \mathbb{N}$ we have:*

(i) $s_i(AX) \leq s_i(AG)$.

(ii) $\sum_{j=1}^{i} t_j(AX) \leq \sum_{j=1}^{i} t_j(AG)$,

or equivalently: $s_i(AG) \leq s_i(AX)\,|G:X|_p$.

This observation also allows to write down a version of Theorem 3.1 for trivial source modules:

THEOREM 4.2 *Let $D \leq G$ be a p-group, V an indecomposable D-projective AG-module with trivial source. Let $X \leq D$ and $c = c_X(V_X)$. Then*

$$|G:D|_p\, s_c(D) \,\Big|\, rank_A V.$$

Before we can prove the next theorem, we need a result on the existence of certain trivial source modules in a given block B (see [5]).

PROPOSITION 4.3 *Let $Q \leq G$ be a p-group, B a p-block of AG with a defect group $D \geq Q$. Then B has an indecomposable AG-module U with source A_Q such that*

$$|G:Q|_p = (rank_A U)_p$$

(i.e.: U is of vertex-height 0).

We can now show that for a block these new invariants do only depend on its defect group.

THEOREM 4.4 *Let B be a block of AG with defect group D. Then we have*

$$s_i(B) = s_i(D)\,|G:D|_p \text{ for all } i,$$

or equivalently, $\mathrm{t}(B) = \mathrm{t}(D)$.

Proof. By Theorem 4.2 we have: $s_i(B) \geq s_i(D)\,|G:D|_p$.
Now let P be a subgroup of D with $r(P) \leq i$ and $|D:P| = s_i(D)$. By the preceding proposition, A_P is a source for a vertex-height 0 module V in B, so $c_G(V) = r(P) \leq i$ and hence by definition of the $s_i(B)$ we have

$$s_i(B) \,\Big|\, (rank_A V)_p = s_i(D)\,|G:D|_p.$$

Thus $s_i(B) \leq s_i(D)\,|G:D|_p$, which proves the assertion. □

As we will see later, for some types of p-groups D the group theoretic invariants $(t_i(D))$ are the same as the algebra invariants $(n_i(AD))$. The invariants of blocks with such defect groups are also under control:

COROLLARY 4.5 *Let D be a p-group for which $\mathbf{n}(AD) = \mathbf{t}(D)$ (or equivalently, $\mathbf{l}(AD) = \mathbf{s}(D)$). Then any block B of AG with defect group D satisfies:*

$$\mathbf{n}(B) = \mathbf{t}(B) = \mathbf{n}(AD)\ ,$$

or equivalently, $\mathbf{l}(B) = \mathbf{s}(B) = \mathbf{l}(AD)$.

Proof. By Corollary 3.2 and the theorem above we obtain:

$$l_i(B) \leq s_i(B) = s_i(D)\,|G:D|_p = l_i(AD)\,|G:D|_p \leq l_i(B)\,. \ \square$$

5 Some remarks and questions

Now suppose that we already know that our given block B has a defect group D satisfying the condition in Corollary 4.5, but without knowing the invariants of D. This is for example the case if we know for some reason that the defect groups are abelian. Then we want to compute the invariants $\mathbf{n}(B) = \mathbf{n}(AD)$ to get some information on D.

Of course, the sets $M_i(B)$ are usually too large to use them in practice to determine the $n_i(B)$'s. But Corollary 4.5 implies that the $n_i(B)$'s can be computed with the sets $M_i(B)$ replaced by the much smaller sets

$$\widetilde{M}_i(B) = \{V \in M_i(B) \,\big|\, rank_A V < |G|\},$$

which still do not depend on a special group basis, and these are finite if $A = F$ is finite!

In the very special case where G is a p-group one might even take:

$$\hat{M}_i(FG) = \{V \in M_i(FG) \mid V \text{ principal right ideal in } FG \text{ with symmetric Loewy series, dim } V \text{ a p-power}\}$$

Now it is natural to ask:

(1) How can we compute the n_i's ?

(2) For which p-groups D do we have $\mathbf{n}(AD) = \mathbf{t}(D)$?

(3) Which properties of D can we deduce from the $n_i(AD)$'s, or even from the $n_i(B)$'s, when B is a block with defect group D ?

First a remark concerning question (3). Unfortunately, for a block B of FG, the invariants $n_i(B)$ cannot even determine whether the defect groups of B are abelian, for a non-abelian p-group D the $n_i(AD)$'s cannot even determine the exponent. The easiest example for this is provided by $\mathbb{Z}_8$ and the quaternion group of order 8, which both just have $n_1 = 3$. For $p \neq 2$, consider the following examples. By Lemma 2.6 the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ and the extraspecial group of order p^3 and exponent p^2 both have invariants $n_1 = 2$ and $n_2 = 1$. By Remark 2.7, the extraspecial group of order p^3 and exponent p has the same invariants for $A = F \neq GF(p)$. So the groups of order p^3 and rank 2 cannot be distinguished by their (n_i)-sequence.

But at least for the case $A = \mathbb{Z}$, Kimmerle and Sandling proved that the group ring $\mathbb{Z}G$ determines whether the Sylow subgroups are abelian [16].

On the other hand, even the exponent of the defect groups of a p-block B of FG can be obtained from the algebra B by a result of Külshammer [18].

6 Invariants for abelian p-groups

The answers to questions (1) and (3) of the previous section for abelian p-groups are very satisfying.

Before we state the main result of this section, we introduce the following definition: an abelian p-group X is said to be of type $(n_1 \geq \ldots \geq n_s)$ if $X \simeq \mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_s}}$. For the details of the proofs in this section we refer to [5].

THEOREM 6.1 *Let G be an abelian p-group of type $(m_1 \geq \ldots \geq m_r)$, then*

$$m_i = n_i(AG) = t_i(G) \text{ for all } i.$$

The crucial result for the proof of the Theorem is the following proposition which generalises a result of Carlson on periodic modules [10; 5.1]. One can prove it by a modification of the proof given there.

PROPOSITION 6.2 *Let G be an abelian p-group, F an algebraically closed field of characteristic $p > 0$ and M an FG-module with $c = c_G(M)$. Then there exists a group $G' \subseteq U(FG)$ such that:*

(i) $G \simeq G'$ and $G' \hookrightarrow FG$ induces an isomorphism $FG' \simeq FG$.

(ii) There is a subgroup H in G' with $r(G'/H) = c$ and M_H projective.

From the theorem above and Theorem 3.1 we can now deduce the following generalisation of the bounds given in [4] and [6] (see [5]):

THEOREM 6.3 *Let V be an AG-module, $A \in \{R, F\}$. Suppose V is D-projective for some p-group $D \leq G$, $X \leq D$ is an abelian p-group of type $(m_1 \geq \ldots \geq m_s)$ and $c = c_X(V_X)$. Then*

$$|G : D|_p \frac{|X|}{p^{m_1} \cdots p^{m_c}} \mid rank_A V .$$

Furthermore, if we assume that we have a block with abelian defect groups, then we can determine the isomorphism type of the defect groups. This answers, in the abelian case, the question raised by Scott and Alperin even for modular group algebras. Note that it was conjectured by Brauer that the degrees of the ordinary irreducible characters in B indicate whether the defect groups of B are abelian.

THEOREM 6.4 *Let B be a p-block of AG with abelian defect group D of rank r. Then D is of type $(n_1(B) \geq \ldots \geq n_r(B))$.*

Proof. Apply Corollary 4.5 and Theorem 6.1. □

In connection with the classical isomorphism problem and the result of Kimmerle and Sandling [16] we also state:

COROLLARY 6.5 *If G has abelian Sylow p-subgroups, then the isomorphism type of the Sylow p-subgroups is determined by AG (in particular, it is determined by $\mathbb{Z}G$).*

7 Invariants for various p-groups

Suppose we already know the invariants of the p-group P. We want to determine the invariants of a group which is a direct product of P with a 'small' abelian group. We will see that p-groups with the property defined below are well-behaved in this situation.

DEFINITION 7.1 *Let P be a p-group with invariants $n_j = n_j(AP)$, for $1 \leq j \leq r = r(P)$. We say that the invariants $n_1, \dots, n_i$ come from abelian subgroups of P, if there exists an abelian subgroup D of P of type $(m_1 \geq \dots \geq m_s)$, where $s \geq i$, such that*

$$\frac{|D|}{p^{m_1} \cdots p^{m_i}} = \frac{|P|}{p^{n_1} \cdots p^{n_i}}.$$

In other words:

$$l_i(AD) = s_i(D) = l_i(AP).$$

We say that the invariants of P come from abelian subgroups if the above is satisfied for all $i \leq r$.

PROPOSITION 7.2 *Let X be an abelian p-group. Suppose the invariants $n_1, \dots, n_i$ of the p-group P come from an abelian subgroup which is of type $(m_1 \geq \cdots \geq m_{s(i)})$ with $p^{m_i} \geq exp X$, for all $i \leq k$. Set $G = P \times X$. Then*

$$n_i(AG) = n_i(AP)\,, \textit{for all } i \leq k,$$

or equivalently: $l_i(AG) = l_i(AP)\,|X|$.
Furthermore, if $r = r(P)$ then $l_{i+r}(AG) = l_i(AX) = s_i(X)$ for $i \geq 1$.

Proof. By Lemma 2.4 we have $l_i(AG) \leq |X|\, l_i(AP)$ for all $i \leq r(G)$. Now let D be an abelian subgroup of P of type $(m_1 \geq \dots \geq m_s)$ with $m_i \geq exp X$ such that

$$\frac{|D|}{p^{m_1} \cdots p^{m_i}} = \frac{|P|}{p^{n_1} \cdots p^{n_i}}.$$

Suppose V is an AG-module with $c_G(V) \leq i$, then we apply Theorem 6.3 with the abelian subgroup $D \times X$ of G to obtain:

$$\frac{|G|}{p^{n_1} \cdots p^{n_i}} = \frac{|P||X|}{p^{n_1} \cdots p^{n_i}} = \frac{|D \times X|}{p^{m_1} \cdots p^{m_i}} \,\Big|\, rank_A V\,,$$

hence

$$\frac{|G|}{p^{n_1} \cdots p^{n_i}} \leq l_i(AG)$$

and thus: $l_i(AG) = l_i(AP)\,|X|$ for all i such that the first i invariants of P come from abelian subgroups with large enough i-th invariant. □

COROLLARY 7.3 *Let X be an elementary abelian p-group and suppose that all invariants of the p-group P come from abelian subgroups. Then the invariants of $G = P \times X$ are:*

$$\begin{aligned}(n_1(AG), \ldots, n_r(AG)) &= (n_1(AP), \ldots, n_{r_1}(AP), n_1(AX), \ldots, n_{r_2}(AX)) \\ &= (n_1(AP), \ldots, n_{r_1}(AP), 1, \ldots, 1),\end{aligned}$$

where $r = r(G)$, $r_1 = r(P)$ and $r_2 = r(X)$.

EXAMPLES.

(a) Let Q_{2^n} be the generalised quaternion group of order 2^n. Then the proposition above is applicable with any abelian 2-group X of exponent $\leq 2^{n-1}$, say X is of type $(m_1 \geq \ldots \geq m_s)$. So $G = Q_{2^n} \times X$ has invariants:

$$\mathbf{t}(G) = \mathbf{n}(AG) = (n_1, \ldots, n_{s+1}) = (n, m_1, \ldots, m_s).$$

In particular, the non-abelian hamiltonian 2-groups are among the groups handled with this example.

(b) Let P be dihedral or semidihedral of order 2^n. In both cases, the invariants of P are $(n-1, 1)$ and they come from abelian subgroups. So we can apply our result with an elementary abelian 2-group X.

(c) Let P be extraspecial of order p^3 and exponent p^2, then P has invariants $(2, 1)$. If P is extraspecial of order p^3 and exponent p, then assuming that $F \neq GF(p)$ we also have $\mathbf{n}(FP) = (2, 1)$. In both cases the invariants come from abelian subgroups.

(d) Let $P = \mathbb{Z}_p \wr \mathbb{Z}_p$. Then P has invariants $\mathbf{t}(P) = (2, \underbrace{1, \ldots, 1}_{p-1}) = \mathbf{n}(AP)$, and these come from abelian subgroups. Again, we can apply the corollary.

Note that the groups in example (a) and the extraspecial groups of exponent p can easily be distinguished from the abelian groups with the same invariants. One just has to use the fact that the exponent is also determined by the modular group algebra, and for these two types the exponent is not equal to its first invariant as is the case with the abelian groups.

Since even the exponent of a defect group is determined by the block [18], Corollary 4.5 now leads to the following improved version of Theorem 6.4:

THEOREM 7.4 *Let B be a p-block of AG, and assume that a defect group D of B is abelian or of type (a) above. Let* $\mathbf{n}(B) = (n_1 \geq \ldots \geq n_r)$.

If $\exp D = p^{n_1}$*, then the defect group D is abelian of type* $(n_1 \geq \ldots \geq n_r)$*, otherwise we have* $D \simeq Q_{2^{n_1}} \times \mathbb{Z}_{2^{n_2}} \times \cdots \times \mathbb{Z}_{2^{n_r}}$*, where* $(n_1 > n_2 \geq \ldots \geq n_r)$.

In particular, if the defect groups of B are known to be hamiltonian, then their isomorphism type is determined by the block algebra.

Let us make a few observations on the examples given above. In all cases, the sequence (n_i) is decreasing. Furthermore, for all $i \leq r_p(G)$ the invariants $n_i(AG)$ are non-zero. Is this always the case ? Another obvious question is whether there is a good characterisation of p-groups whose invariants come from abelian subgroups.

At the end of this section let us look at the p-groups of order p^4, where $p \geq 3$. A list of these groups can be found in Huppert [15, p.346]. The modular isomorphism problem for these groups was solved by Passman [19]. Using the commutator quotient, the centre, the Brauer-Jennings-Zassenhaus M-series, and in some critical cases the kernel size of certain canonical maps, he showed that the isomorphism type of all of these groups is determined by their group algebra over $GF(p)$.

Remember that p-groups P with $\mathbf{t}(P) = \mathbf{n}(AP)$ are especially good, because then we have $\mathbf{n}(B) = \mathbf{n}(AP)$ for a block B with defect group P. We will see below that several groups of order p^4 satisfy this condition. But even together with the knowledge of the exponent our invariants do not suffice to distinguish between all the non-abelian groups of order p^4.

Non-abelian groups of order p^4, for $p \geq 3$

(1) $G = < x, y \mid x^{p^3} = y^p = 1, x^y = x^{1+p^2} >$

This group is metacyclic of exponent p^3. By Lemma 2.6 we have

$$\mathbf{n}(AG) = (3, 1) = \mathbf{t}(G) ,$$

and the invariants clearly come from its abelian subgroups.

(2) $G = < x, y \mid x^{p^2} = y^{p^2} = 1, x^y = x^{1+p} >$

The group G is metacyclic of exponent p^2. We claim:

$$\mathbf{n}(AG) = (2, 2) = \mathbf{t}(G) .$$

So the invariants do not come from the abelian subgroups.

Proof. Let U be an indecomposable periodic FG-module. As $V_G(U)$ is a line [12], U is projective on $<x^p>$ or on $<y^p>$, hence it is projective on $<x>$ or on $<y>$. Thus p^2 divides $dim_F U$. This also implies the assertion for $A = R$. □

(3) Let G be the central product of an extraspecial group P of order p^3 and exponent p with $C = \mathbb{Z}_{p^2}$. Here we have:

$$\mathbf{t}(G) = (2,2) \text{ and } \mathbf{n}(FG) = (3,1) \text{ , for } F \neq GF(p).$$

In particular, the n-invariants come from abelian subgroups.

Proof. We already know that P has a periodic module of dimension p on which $Z(P)$ acts trivially, so we can consider this as an FG-module on which C acts trivially. As it is periodic on the maximal elementary abelian p-subgroups of G, it is a periodic FG-module. This proves $\mathbf{n}(FG) = (3,1)$. □

(4) If G is the direct product of an extraspecial group of order p^3 and exponent p with a $\mathbb{Z}_p$ or the semidirect product of a ${\mathbb{Z}_p}^3$ with a $\mathbb{Z}_p$, then we have:

$$\mathbf{t}(G) = (1,2,1) \text{ and } \mathbf{n}(FG) = (2,1,1) \text{ , for } F \neq GF(p),$$

and the $\mathbf{n}$-sequence does come from abelian subgroups.

(5) If G is one of the three non-split extensions of ${\mathbb{Z}_p}^3$ with $\mathbb{Z}_p$, then the invariants are:

$$\mathbf{t}(G) = (2,1,1) = \mathbf{n}(AG) \text{ ,}$$

so the $\mathbf{n}$-sequence comes from abelian subgroups.

(6) For the groups of the form

$$G = < x, y, z \mid x^p = y^p = z^{p^2} = [y,z] = 1, y^x = yz^{sp}, z^x = zy > \text{ ,}$$

where $s = 1$ resp. a quadratic non-residue modulo p, we note that $\mathbf{t}(G) = (2,2)$. But it is not clear whether there is a periodic module with rank only divisible by p but not by p^2.

(7) For $p = 3$ we have to replace one of the groups under (5) by the group:

$$G = < x, y, z \mid x^9 = y^3 = [x,y] = 1, x^3 = z^3, x^z = xy, y^z = x^3 y > \text{ ,}$$

for which we also have $\mathbf{t}(G) = (2,2)$. Again, there could be a periodic module whose rank is only divisible by 3.

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C.BESSENRODT
Fachbereich Mathematik
Univ. Duisburg
Lotharst. 65
4100 Duisburg 1, R.F.A.