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J. J. GÜEMES

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ON THE HOMOLOGY CLASSES FOR THE COMPONENTS OF SOME FIBRES
OF SPRINGER'S RESOLUTION

J.J. Güemes

ABSTRACT: We compute the homology classes of the components of the fibres of Springer's resolution in terms of Schubert classes when the unipotent element is of "one hook" type.

0. Introduction

Let G be a connected reductive group over \mathbb{C} . Denote by β the variety of all Borel subgroups of G . If u is a unipotent element of G , the fibre of Springer's resolution β_u is the variety of Borel subgroups containing u . The inclusion $\beta_u \hookrightarrow \beta$ induces a homomorphism of homology groups $H_*(\beta_u; \mathbb{Z}) \rightarrow H_*(\beta; \mathbb{Z})$, which is injective if $G = GL_n(\mathbb{C})$ [8]. When w runs over the elements in the Weyl group W of G , the Schubert classes $[\bar{X}_w]$ form a basis of $H_*(\beta; \mathbb{Z})$ [2]. If C is a component of β_u , it defines a homology class in $H_*(\beta_u; \mathbb{Z})$, whose image in $H_*(\beta; \mathbb{Z})$ is denoted by $[C]$. We can then write

$$[C] = \sum_{w \in W} n_C(w) [\bar{X}_w] \quad \text{with } n_C(w) \in \mathbb{Z}$$

In this paper we shall consider the case with $G = GL_n(\mathbb{C})$ and with u a unipotent element whose Jordan decomposition is of "one hook" type, i.e. such that there is at most one Jordan block of size greater than one. The result in that case is that $n_C(w)$ is the cardinal of a set of reduced expressions of w , depending on C . We believe that a similar result could be true in general, at least for $GL_n(\mathbb{C})$. For example, we have obtained such a result in the case that the Jordan decomposition of u has only two blocks.

We want to express our deep gratitude to Professor Springer. He proposed the problem [13], and inspired all our work. He also read the paper and implemented it considerably. The clarity the reader can find comes from him.

1. Combinatorial results about tableaux and permutations

1.1. A good reference for some terminology about tableaux is Macdonald's book [10], for links between tableaux and reduced decompositions the reader is referred to [5], [9], [14].

Consider "strict standard staircase tableaux" with entries in the set $\{1, \dots, n-1\}$, i.e. tableaux T for a partition $(m, m-1, \dots, 1)$ such that the integers $a_{p,q}$ in the place (p, q) satisfy $a_{p,q} < a_{p+1,q}$, $a_{p+1,q} \leq a_{p,q+1}$ (the columns are strictly increasing and the diagonals are increasing, it follows that the rows are strictly increasing).

In the symmetric group S_n , let $s_{p,q}$ denote the transposition (p, q) , let s_i be the fundamental transposition $s_{i,i+1}$, $1 \leq i \leq n-1$. Let $l(w)$ denote the length of an element $w \in S_n$ and $w < w'$ the Bruhat order relative to the set of generators $(s_i) 1 \leq i \leq n-1$ [3].

Denote by $|X|$ the cardinality of a set X .

Associate to such a tableau T a permutation $w = w_T \in S_n$, namely $w = c_1 \dots c_m$ where $c_p = s_{a_{m-p+1,p}} \dots s_{a_{1,p}}$.

We say T is reduced if $l(w_T) = \frac{1}{2}m(m+1)$.

1.2. We list a number of properties.

1.2.1. Write $r_p = s_{a_{p,1}} \dots s_{a_{p,m-p+1}}$. Then $w = r_m \dots r_1$.

1.2.2. If T is reduced then $a_{p+1,q} = a_{p,q+1}$ implies $a_{p,q+1} = a_{p,q} + 1$.

1.2.3. Let i defined by $w_i = 1$. Then $a_{p,q} = p+q-1$ for $p+q \leq i$ and $a_{1,i} > i$.

Define the number $\tau = \tau_p = \tau(T, p)$ as follows: in the p^{th} column of T we have

$$a_{1,p} = a_{2,p-1} = \dots = a_{\tau,p} - \tau + 1 < a_{\tau+1,p} - \tau.$$

1.2.4. For $p < q \leq i-1$ we have $\tau_p > \tau_q$. If $p < i$ then $w_p = \tau_p + 1$. Hence if $p < q \leq i$ then $w_p > w_q$.

1.2.5. If T is reduced and $a_{1,i} = i+1$ then $\tau_i < \tau_{i-1}$. Moreover $\tau_{i-1} = \tau_i + 1$ if and only if $\tau_{i+2} = w(i-1) < w(i+1)$ and $\tau_{i-1} > \tau_i + 1$ if and only if $w(i-1) > w(i+1) = \tau_i + 2$.

PROOFS

1.2.1.* If $p > u, q > v$ then $a_{p,q} - a_{u,v} \geq 2$. It follows that $s_{a_{p,q}}$ and $s_{a_{u,v}}$ commute. The proof follows from this observation.

1.2.2.* Let T be not necessarily reduced. We show by induction on $m-i$ that $l(r_m \dots r_i) > l(r_m \dots r_i s_{a_{i,j}})$ ($1 \leq i \leq m, 1 \leq j \leq m-i+1$). This is clear if $j = m-i+1$ or $a_{i,j+1} > a_{i,j} + 1$, otherwise $s_{a_{i,j}} s_{a_{i,j+1}} s_{a_{i,j}} = s_{a_{i,j+1}} s_{a_{i,j}} s_{a_{i,j+1}}$, $a_{i,j+1} = a_{i+1,j}$ (diagonals increase) so that $r_i s_{a_{i,j}} = s_{a_{i+1,j}} r_i$. By the induction hypothesis we have $l(r_m \dots r_{i+1} s_{a_{i+1,j}}) < l(r_m \dots r_{i+1})$, whence the asserted inequality.

It follows that if $a_{p+1,q} = a_{p,q+1}$ and $a_{p,q} + 1 < a_{p,q+1}$, we have $l(r_m \dots r_{p+1} s_{a_{p+1,q}}) < l(r_m \dots r_{p+1})$, showing that T is not reduced.

1.2.3.* We have $w^{-1} = r_1^{-1} \dots r_m^{-1}$ and r_p fixes 1 if $p \geq 2$, so $i = w^{-1}(1) = r_1^{-1}(1) = s_{a_{1,m}} s_{a_{1,m-1}} \dots s_{a_{1,1}}(1)$. Since $a_{1,p} \geq p$ it follows that $a_{1,p} = p$ for $p \leq i-1, a_{1,p} > p$ for $p \geq i$.

1.2.4.* That $\tau_p > \tau_q$ if $p < q < i$ follows from the definitions (T is a strict tableau). Now c_j fixes p if $j > p, c_p = p + \tau_p$ for $p < i$ and $c_j t = t-1$ if $j < t \leq j + \tau_j$; thus because $p < i$ we have $wp = c_1 \dots c_p(p) = c_1 \dots c_{p-1}(p + \tau_p) = \tau_p + 1$.

1.2.5.* That $\tau_i < \tau_{i-1}$ follows from (1.2.2). From (1.2.4) we have $w(i-1) = \tau_{i-1} + 1$. Also $w(i+1) = r_m \dots r_1(i+1) = r_m \dots r_2(i+2) = \dots = r_m \dots r_{\tau_i+1}(i + \tau_i + 1)$; now r_{τ_i+1} fixes $i + \tau_i + 1, r_{\tau_i+2}(i + \tau_i + 1) = \tau_i + 2$ if $\tau_{i-1} > \tau_i + 1$ and $r_{\tau_i+2}(i + \tau_i + 1) > \tau_i + 2$ if $\tau_{i-1} = \tau_i + 1$. The result now follows from the observation that r_j fixes $\{1, \dots, \tau_i + 2\}$ if $j > \tau_i + 2$.

1.3. The following known result (see [6, pg. 156] as reference) is useful.

LEMMA 1.- Let $w \in S_n$, assume $1 \leq p \leq q \leq n$. We have $l(ws_{p,q}) < l(w)$ if and only if $wp > wq$, moreover in this case $l(w) - l(ws_{p,q}) = 1 + 2|\{k \text{ s.t. } p < k < q \text{ and } wp > wk > wq\}|$. As a consequence if $ws > w$ for some fundamental transposition s and $l(ws_{p,q}) = l(w)$ then $ws(p) > ws(q)$ and there is no k with $p < k < q$ and $ws(p) > ws(k) > ws(q)$.

LEMMA 2.- Let $w = w_T, w' = w_{T'}$, be permutations corresponding to tableaux $T = (a_{p,q}), T' = (a'_{p,q})$ as in the beginning of the section. Suppose there are positive integers $t, j, k, j < k$, with $a_{p,q} = a'_{p,q}$ if $q \neq t$ or $p > k, a'_{p,t} = a_{p,t} = t + p - 1$ if $1 \leq p < j$ and $a'_{p,t} = a_{p,t} + 1 = t + p$ if $j \leq p \leq k$. Then $w^{-1}w'$ is the cyclic permutation (t, b, c) , where b is defined by $wb = j$ and c by $w'c = k + 1$.

PROOF.- Write $w = c_1 \dots c_m$, $w' = c'_1 \dots c'_m$ then $c_p = c'_p$ if $p \neq t$ and $c_t(h) \neq c'_t(h)$ exactly for three values of h , namely $h=t$, $t+j$, $t+k$. Therefore $w^{-1}w'$ is a cyclic permutation (a,b,c) . Moreover $wt = k+1$ and $w't = c_1 \dots c_{t-1}(t+j-1) = j$, so we can take $a = t$ and b,c defined by $wb = w't = j$, $w'c = wt = k+1$.

2. Combinatorial correspondences

2.1. Let \mathcal{L} be the set of tableaux $T = (a_{p,q})$ as in section 1, with $w(i+1) = 1$.

Let \mathcal{M} be the set of tableaux with $w_i = 1$ and $a_{1,i} = i+1$.

Let \mathcal{N} be the set of tableaux with $w(i-1) = 1$, $a_{1,i-1} = i$ and $a_{1,i} = i+1$.

2.2. Define a map $\psi: \mathcal{L} \rightarrow \mathcal{M}$ as follows: ψT is obtained by replacing the numbers $i, i+1, \dots, i+\tau_i-1$ in the i^{th} column of T by $i+1, i+2, \dots, i+\tau_i$. Define $\bar{\psi}: \mathcal{M} \rightarrow \mathcal{N}$ similarly (change i for $i-1$). Define $e = e_T$ by $w_{\psi T} e = \tau_i + 1$ ($\tau_i = \tau(T, i)$) if $T \in \mathcal{L}$ (similarly if $T \in \mathcal{N}$).

If $T \in \mathcal{N}$ then $\tau_{i-1} > \tau_i$ by definitions. Define a map $\chi: \mathcal{N} \rightarrow \mathcal{M}$ as follows: χT is obtained by replacing the numbers $i, i+1, \dots, i+\tau_i$ in the $(i-1)^{\text{th}}$ column of T by $i-1, i, \dots, i+\tau_i-1$.

We list a number of results.

2.2.1. We have $\bar{\psi} \circ \chi = \text{identity}$, in particular χ is injective.

2.2.2. Suppose $T \in \mathcal{L}$, then $e > i+1$, $w_T = w_{\psi T} s_i s_{i,1}$ and ψT is reduced if T is reduced.

2.2.3. Suppose $T \in \mathcal{N}$ is reduced, then $w_T = w_{\chi T} s_i s_{i-1, i+1}$ and χT is reduced.

2.2.4. We have that $\psi T = \psi T'$ and $e_T = e_{T'}$, implies $T = T'$.

2.2.5. If $T \in \mathcal{M}$ is reduced and $l(ws_{i,t} s_{t, i+1}) = l(w)$ for some $t < i$, then $t = i-1$ and $\tau_{i-1} = \tau_i + 1$, in particular $T = \chi \bar{\psi} T$, $\bar{\psi} T \in \mathcal{N}$.

2.2.6. Given $T' \in \mathcal{M}$ reduced, $e > i+1$, $l(w_T s_i s_{i,1}) = l(w_{T'})$ then there exists a reduced $T \in \mathcal{L}$ with $\psi T = T'$ and $e = e_T$, if and only if $w_{T'} e \leq \tau(T', i) + 1$.

PROOFS

2.2.1.* This follows from the constructions of the maps.

2.2.2.* Write $w_{\psi T} = c_1 \dots c_m$ as in section 1. We have:

$$e = w_{\psi T}^{-1}(\tau_i + 1) = c_m^{-1} \dots c_1^{-1}(\tau_i + 1) = c_m^{-1} \dots c_2^{-1}(\tau_i + 2) = \dots = c_m^{-1} \dots c_{i+1}^{-1}(\tau_i + i + 1).$$

Now c_{i+1}, \dots, c_m fix $\{1, \dots, i+1\}$ therefore $e > i+1$.

Applying lemma 2 to $T, \psi T$ with $t=i, j=1$ and $k=\tau_i$ we obtain $w_T = w_{\psi T}(i, e, i+1)$ because $b = i+1$ and $c=e$, thus $w_T = w_{\psi T} s_i s_{i,e}$.

We shall prove $w_{\psi T}(i+1) > \tau_{i+1} = w_{\psi T} e_T$. Then we shall have $l(w_T) < l(w_{\psi T} s_{i,e}) = l(w_{\psi T} s_i)$ (c.f. lemma 1), thus ψT is reduced if T is. Write $w_{\psi T} = r_m \dots r_i$ then $w_{\psi T}(i+1) = r_m \dots r_1(i+1) = r_m \dots r_{\tau_i+1}(i+\tau_i+1)$, now r_m, \dots, r_{τ_i+1} fix $\{1, \dots, \tau_i\}$ and $w_{\psi T}(i+1) \neq \tau_{i+1}$ because $e > i+1$.

2.2.3.* We have $w_{\chi T} = w_{\psi \chi T} s_{i-1} s_{i-1,e} = w_T s_{i-1} s_{i-1,e}$, $e = e_{\chi T}$ (c.f. 2.2.2., 2.2.1.). Now $w_T e = \tau(\chi T, i-1)+1$ (definition) and $\tau(\chi T, i-1) = \tau(T, i)+1$ by construction. We shall prove $w_T(i+1) = \tau(T, i)+2$ then $e = i+1$ and $w_T = w_{\chi T} s_i s_{i-1, i+1}$. We have:

$w_T(i+1) = c_1 \dots c_m(i+1) = c_1 \dots c_i(i+1) = c_1 \dots c_{i-1}(i+\tau_i+1) = c_1 \dots c_{i-2}(i+\tau_i)$, T is reduced and $\tau_{i-2} > \tau_{i-1}$ (c.f. 1.2.5.), hence

$$c_1 \dots c_{i-2}(i+\tau_i) = c_1 \dots c_{i-3}(i+\tau_i-1) = \dots = c_1(\tau_i+3) = \tau_i+2$$

Also $l(w_T) < l(w_{\chi T} s_{i-1, i+1}) = l(w_{\chi T} s_i)$ (use that $w_T(i-1) = 1$ c.f. lemma 1), thus χT is reduced.

2.2.4.* If $T, T' \in \mathcal{L}$ and $\psi T = \psi T'$, $e_T = e_{T'}$, implies $\tau(T, i) = \tau(T', i)$ (definition of e). Then the construction of $\psi T = \psi T'$ shows $T = T'$.

2.2.5.* We have $ws_i(i+1) = wi = 1$. We deduce $t = i-1$ (c.f. 1.2.4., and lemma 1). We must have also (c.f. lemma 1) $ws_i(i) = w(i+1) > ws_i(i-1) = w(i-1)$, then (1.2.5) implies $\tau_{i-1} = \tau_i+1$.

2.2.6.* If $T' = \psi T$, $e = e_T$ then by definition $w_{T'} e = \tau_i+1 \leq \tau(T', i)+1$ (construction). Conversely if $w_{T'} e \leq \tau(T', i)+1$ put $k+1 = w_{T'} e$ and obtain T from T' by replacing the numbers $i+1, \dots, i+k$ in the i^{th} column of T' by the numbers $i, \dots, i+k-1$ (T is a strict tableau because $\tau(T', i-1) > \tau(T', i)$ (c.f. 1.2.5.)). Then $\psi T = T'$, $w_{\psi T} e = w_{T'} e = k+1 = \tau(T, i)+1$, i.e. $e = e_T$, and T is reduced because $w_T = w_{T'} s_i s_{i,e}$ (c.f. 2.2.2).

2.3. Let $T \in \mathcal{M}$ be reduced and let $t < i$ be such that $l(ws_i s_{t,i}) = l(w)$, then t is uniquely determined by T (c.f. 1.2.4. and lemma 1). We have also (lemma 1 and 1.2.4) $w(i+1) < wt = \tau_t+1$.

In this situation construct $\lambda T \in \mathcal{M}$ in the following way:

Replace the numbers $t+w(i+1)-1, \dots, t+\tau_t-1$ in the t^{th} column of T by the numbers $t+w(i+1), \dots, t+\tau_t$. (We obtain a strict tableau because $w(t+1) < w(i+1)$ (c.f. lemma 1).

Define $k = k_T$ by $w_{\lambda_T} k = \tau(T, t) + 1$.

We list a number of results.

2.3.1. We have $k > i + 1$ and $w_T s_i s_{t,i} = w_{\lambda_T} s_i s_{i,k}$.

2.3.2. λT is reduced and there is no reduced $T' \in \mathcal{L}$ with $\psi T' = \lambda T$ and $k = e_{T'}$.

2.3.3. Suppose we have $\lambda T = \lambda T'$, $k_T = k_{T'}$, for two reduced $T, T' \in \mathcal{M}$ with $l(w_T s_i s_{t,i}) = l(w_{T'})$, $l(w_T s_i s_{t',i}) = l(w_{T'})$, then $t = t'$ and $T = T'$.

2.3.4. Let $T \in \mathcal{M}$ be reduced, let $e > i + 1$ be such that $l(w s_i s_{i,e}) = l(w)$, and suppose that there is no reduced $T' \in \mathcal{L}$ with $T = \psi T'$ and $e = e_{T'}$. Then there is a reduced $T' \in \mathcal{M}$ and $t < i$ with $l(w_T s_i s_{t,i}) = l(w_{T'})$, $T = \lambda T'$ and $e = k_{T'}$.

PROOFS

2.3.1.* Write $w_{\lambda T} = c_1 \dots c_m$ then

$$k = w_{\lambda T}^{-1}(\tau_t + 1) = c_m^{-1} \dots c_1^{-1}(\tau_t + 1) = c_m^{-1} \dots c_t^{-1}(\tau_t + t) = c_m^{-1} \dots c_{t+1}^{-1}(\tau_t + t + 1).$$

Observe that $c_j^{-1}(s) > j + \tau_j$ if $j < i$ and $s > j + \tau_j$. Also c_m, \dots, c_i fix $\{1, \dots, i\}$, we conclude $k > i$.

Applying lemma 2 to T , $\lambda T, t$, $w(i+1)$, τ_t we obtain $w_T = w_{\lambda T}(t, c, b)$ where $b = i + 1$ and $w_{\lambda T} c = \tau_t + 1$. Therefore $k \neq i + 1$ and $w_T s_i s_{t,i} = w_{\lambda T} s_i s_{i,k}$.

2.3.2.* We have $l(w_{\lambda T} s_i s_{i,k}) = l(w_T s_i s_{t,i}) = l(w_T)$ (c.f. 2.3.1). We shall prove $w_{\lambda T}(i+1) > \tau_t + 1 = w_{\lambda T} k$. Then $l(w_{\lambda T} s_i) > l(w_{\lambda T} s_i s_{i,k}) = l(w_T)$ (lemma 1) and λT is reduced.

Write $w_T = c_1 \dots c_m$, $w_{\lambda T} = c'_1 \dots c'_m$. Also $w(i+1) < \tau_t + 1$ by definition of t . Then $w(i+1) = c_1 \dots c_t(t + w(i+1))$ and $w_{\lambda T}(i+1) = c_1 \dots c_{t-1} c'_t(t + w(i+1)) = c_1 \dots c_{t-1}(u)$ where $u \geq t + \tau_t + 1$, hence $w_{\lambda T}(i+1) \geq \tau_t + 2$.

We have $w_{\lambda T} k = \tau_t + 1 > \tau_i + 1$ (c.f. 1.2.4, 1.2.5). Hence the second part of the statement follows from (2.2.6).

2.3.3.* In this situation $w_T s_i s_{t,i} = w_{T'} s_i s_{t',i}$ (c.f. 2.3.1). If $t \neq t'$ suppose for instance $t' < t$, then $w_{T'} t' > w_T t$ (c.f. 1.2.4) and $w_{T'} t' = w_{T'}(i+1) > w_{T'} t' = w_T t$ against the definition of t' .

Therefore $t = t'$ and $w_T = w_{T'}$. We conclude $T = T'$ because they can only differ in one column.

2.3.4.* In this situation $w_e > \tau_{i+1}$ (c.f. 2.2.6). We deduce $\tau_{i-1} = \tau_i + 1$ because if $\tau_{i-1} > \tau_i + 1$ then $w(i+1) = \tau_i + 2$ (c.f. 1.2.5) against $w_e < w(i+1)$ (c.f. lemma 1). We have in fact $w_e > \tau_{i-1} + 1$ ($w_e \neq \tau_{i-1} + 1 = w(i-1)$) (c.f. 1.2.4) because $e \neq i-1$.

Define t to be the smallest number with $w_e > \tau_t + 1$.

We claim: the t^{th} column of T is of the form, $t, t+1, \dots, t+\tau_t-1, t+\tau_t+1, t+\tau_t+2, \dots, t+w_e-1, \dots$. If $w = c_1 \dots c_m$ this is equivalent to $c_t(t+\tau_t+1) \geq t+w_e$. Suppose $c_t(t+\tau_t+1) < t+w_e$, because $w_e \leq \tau_{t-1} + 1$ (by definition and 1.2.4), we deduce $c_1 \dots c_t(t+\tau_t+1) \leq w_e$. If we show $c_{t+1} \dots c_m(i+1) = t+\tau_t+1$, we arrive to the contradiction $w(i+1) \leq w_e$ and the claim follows.

For $j = i-1$ we have $c_{j+1} \dots c_m(i+1) = c_i(i+1) = i+\tau_i+1 = i+\tau_{i-1} = j+\tau_j+1$.

Now we show $c_j(j+\tau_j+1) = j-1+\tau_{j-1}+1$ if $t < j < i$ by decreasing induction.

If the j^{th} column of T has the form $j, j+1, \dots, j+\tau_j-1, j+\tau_j+1, j+\tau_j+2, \dots, \dots, j+\tau_j+s, \dots$ then $\tau_{j-1} \geq \tau_j+s+1$ (c.f. 1.2.2). If $\tau_{j-1} \geq \tau_j+s+1$ then $w(i+1) < w(j-1) = \tau_{j-1}+1$, this is against $w(i+1) > \tau_t+1$ (c.f. 1.2.4). We have $\tau_{j-1} = \tau_j+s+1$ and the result follows.

Now construct T' by replacing the numbers $t+\tau_t+1, \dots, t+w_e-1$ in the t^{th} column of T by the numbers $t+\tau_t, \dots, t+e-2$.

We have $w_{T'} = \tau_t + 1$ (c.f. 1.2.4), $w_{T'}, t = w_e$ (by construction of T') and $w_{T'}(i+1) = \tau_t + 1$ (we had $c_{t+1} \dots c_m(i+1) = t+\tau_t+1$). Then (by lemma 2) $w_{T'} = w(i+1, t, e)$ and $w_{s_i} s_{i,e} = w_{T', s_i} s_{t,i}$.

We have $l(w_{T', s_i} s_{t,i}) = l(w_{s_i} s_{i,e}) = l(w)$ and $w_e = w_{T'}, t > w_{T'}(i+1) = \tau_t + 1$, then $l(w_{T'}, s_i) > l(w)$ (c.f. lemma 1) and T' is reduced.

Also $l(w_{T', s_i} s_{t,i}) = l(w_{T'})$. One see easily $\lambda T' = T$ (we have $w_{T'}(i+1) = \tau_t + 1$) and $e = k_{T'}$.

3. The main result

3.1. We start with a unipotent u in the general linear group $Gl(n, \mathbb{C})$, which in Jordan normal form has a block of size $n-m$ and m blocks of size one.

Let us recall that the variety β_u can be identified with the variety of flags fixed by u .

It is known [11], that the standard tableaux of shape $(n-m, 1, \dots, 1)$ parametrize the components of β_u . Here we shall follow the convention that a standard tableau has strictly decreasing rows and columns.

3.1.1. THEOREM.- The expression of the homology class of a component C of β_u corresponding to the tableau T_C in terms of Schubert classes is $[C] = \sum [\bar{X}_w^T]$, where T runs over the set of reduced tableaux $T = (a_{p,q})$ with $a_{1,q} = a_q$ $1 \leq q \leq m$.

n	...	
a _m		
⋮	T _C	
a ₁		

3.1.2 Example.- The component C corresponding to the tableau Tb is non singular and is not a Schubert cycle. There is only one reduced tableau corresponding to

6	4	2
5		
3	Tb	
1		

the component, namely

1	3	5
2	4	
3		

Thus $[C] = [\bar{X}_w]$ with $w = s_3 s_2 s_1 s_4 s_3 s_5$ ($w = 415263$) and $[C]$ is a single Schubert class. The corresponding Schubert cycle is singular and has different Poincaré polynomial and intersection homology Poincaré polynomial ($P_C = (q^2+q+1)(q+1)^4$, $P_w = (q^3+3q^2+2q+1)(q+1)^3$, $IHP_w = (q+1)^6$).

3.2. Let us recall some results on the components and on the action of the Weyl group.

Let Δ be the root system of a reductive group; Π is a system of simple roots; Δ_+ is the set of positive roots; \langle, \rangle is the duality pairing between roots and coroots; β' is the coroot associated to the root β ; $s_\beta \in W$ is the reflection defined by β [3].

3.2.1. According to Bernstein-Gelfand-Gelfand [1 th. 3.12] and Demazure [4 pg.80] the action of a simple reflection $s = s_\alpha, \alpha \in \Pi$, on Schubert classes is given by:

$$(1) \quad s[\bar{X}_w] = \begin{cases} -[\bar{X}_w] & \text{if } ws < w \\ [\bar{X}_w] + \sum_{\beta \in \Delta_+ - \{\alpha\}} \langle \alpha, \beta' \rangle [\bar{X}_{w s s_\beta}] & \text{if } ws > w \end{cases}$$

$l(w s s_\beta) = l(w)$

Here $[\bar{X}_w]$ is the Schubert class corresponding to $w \in W$.

3.2.2 Let \mathcal{P}_s be the variety of parabolic lines of type s [15], and $\Pi: \beta \rightarrow \mathcal{P}_s$ the natural projection. Following Hotta [7] we say that a pair of components (C, C') form an s-pair if $\Pi(C') \subset \Pi(C)$ but $\Pi(C') \neq \Pi(C)$; in particular C and C' intersect in codimension one.

The action of s on the homology classes of the components is given by:

$$s[C] = \begin{cases} -[C] & \text{if } \dim \Pi(C) < \dim C \\ [C] + \sum n_{C,C'} [C'] & \text{if } \dim \Pi(C) = \dim C \end{cases}$$

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Here the summation is over all the components C' with (C, C') an s -pair and the numbers $n_{C, C'}$ are strictly positive integers [8].

REMARK.- In the formulas of Demazure and Hotta we observe

- i) $ws < w$ if and only if the Schubert cycle corresponding to w contains lines of type s .
- ii) $\dim \Pi(C) < \dim C$ if and only if the component C contains lines of type s .

3.3. In our case the Weyl group W is S_n and the set of positive roots Δ_+ is in one-to-one correspondence with set of transpositions in S_n .

Denote by $\beta_{i,j}$ the positive root corresponding to the transposition $s_{i,j} = (i,j)$. The roots $e_i = \beta_{i,i+1}$ $1 \leq i \leq n-1$ corresponding to the fundamental transpositions $s_i = (i,i+1)$ form a system of simple roots. Say that a component contains lines of type i if it contains lines of type s_i . Say that two components form an i -pair if they form an s_i -pair.

3.3.1. PROPOSITION [12 pg. 87].- The component corresponding to the tableau T_C contains exactly lines of type $\{a_1, a_2, \dots, a_m\}$.

n	
a_m	
⋮	
⋮	
a_1	T_C

The intersection pattern of these components is known [16], in particular we have:

3.3.2. PROPOSITION.- Two components intersect in codimension one if and only if the corresponding tableaux differ by a transposition of consecutive integers not lying in the same row or column.

As a consequence there are for given i and C at most two components C' with (C, C') an i -pair.

We see that the homology classes of the components are a very special basis for the action of the Weyl group (Springer representations), i.e. the matrices of the fundamental reflections have 1's and -1's in the diagonal and 0's or positive integers outside. For "hook" components we have:

3.3.3. PROPOSITION.- All the integers $n_{C, C'}$ in Hotta's formula (2) are 1.

PROOF.- Take a component A whose tableau has i in the first column and $i+1 \leq n-1$ in the first row. Let B be the component obtained by interchanging i and $i+1$. Assume:

- a) $i \geq 2$ and $i-1$ is in the first column. Let C be the component whose tableau is obtained by interchanging i and $i-1$ in B .

b) $i+1 < n-1$ and $i+2$ is in the first column. Let D be the component whose tableau is obtained by interchanging $i+1$ and $i+2$ in A .

Then

$$s_i [B] = [B] + n_{BA}^i [A] + n_{BC}^i [C],$$

$$s_{i+1} s_i [B] = -[B] + n_{BA}^i ([A] + n_{AB}^{i+1} [B] + n_{AD}^{i+1} [D]) - n_{BC}^i [C],$$

$$s_i s_{i+1} s_i [B] = -[B] - n_{BA}^i [A] - n_{BC}^i [C] - n_{BA}^i [A] + n_{BA}^i n_{AB}^{i+1} ([B] + n_{BA}^i [A] + n_{BC}^i [C]) + n_{BC}^i [C] - n_{BA}^i n_{AD}^{i+1} [D],$$

$$s_{i+1} [B] = -[B],$$

$$s_i s_{i+1} [B] = -[B] - n_{BA}^i [A] - n_{BC}^i [C],$$

$$s_{i+1} s_i s_{i+1} [B] = [B] - n_{BA}^i ([A] + n_{AB}^{i+1} [B] + n_{AD}^{i+1} [D]) + n_{BC}^i [C]$$

But $s_i s_{i+1} s_i [B] = s_{i+1} s_i s_{i+1} [B]$ and the homology classes of the components form a basis, so comparing coefficients:

$$-1 + n_{BA}^i n_{AB}^{i+1} = 1 - n_{BA}^i n_{AB}^{i+1} \quad \text{and} \quad n_{BA}^i n_{AB}^{i+1} = 1 \quad \text{therefore} \quad n_{BA}^i = n_{AB}^{i+1} = 1$$

If the assumptions a) and b) are not satisfied the components C or D do not appear but the proof is the same.

3.4. The proof of theorem 3.1.1. is by double induction on the length of the first row and on the integer in the upper right-hand corner of the tableau.

Read the tableau
$$\begin{array}{c} n \dots b_1 \\ a_m \\ \vdots \\ a_1 \end{array}$$
 as the "word" $a_1 \dots n \dots b_1$.

i) = case $12 \dots n$. If the length of the first row of the tableau is one, the unipotent is the identity and $\beta_u = \beta$. On the other hand $s_{n-1} \dots s_1 s_{n-1} \dots s_2 \dots s_{n-1}$ is w_0 the longest element in S_n which corresponds to $[\beta]$.

ii) case $a_1 \dots a_m n \dots b_3 b_2 1$. If the number in the upper right-hand corner of the tableau is 1, all the flags in the component have for one-dimensional subspace a fixed line $[12]$.

The component is isomorphic to the component given by the tableau $a_{1-1}, a_{2-1}, \dots, n-1, \dots, b_{3-1}, b_{2-1}$; the isomorphism is given by the natural isomorphism between Flags $(n-1)$ and the Schubert variety in Flags (n) of flags

that contain the fixed line, and through this isomorphism the Schubert cycle corresponding to the permutation $w' \in S_{n-1}$ goes to the one corresponding to $w \in S_n$ ($w(1)=1$, $w(i) = w'(i-1)+1$ if $i > 1$). (See for instance [6, Chapter III §4]). If one writes $w' = s_{\alpha_1} \dots s_{\alpha_k}$ as product of fundamental transpositions then $w = s_{\alpha_1+1} \dots s_{\alpha_k+1}$. The result now follows immediately.

iii) Case $1 \ 2 \dots i \ \hat{i+1} \dots n \dots b_2 \ i+1$. By induction we have now a fixed component A corresponding to a tableau of this form (if the tableau is $1 \ 2 \dots m \ n \ n-1 \dots \ m+1$ the proof is the same as in the first step of the induction).

Let B the component corresponding to the tableau which we obtain by interchanging i and $i+1$ in the tableau of A.

We will assume $i \geq 2$ and we shall not treat the case $i=1$ separately, (because the proof in that situation is a particular case of the proof for $i \geq 2$).

Let C be the component corresponding to the tableau which we obtain by interchanging $i-1$ and i in the tableau of B.

Put $s = s_i$, and let $e = e_i$ be the corresponding fundamental root (i is fixed).

Define Γ_A as the set of reduced tableaux $T=(a_{p,q})$ with $a_{1,q}=a_q$ $1 \leq q \leq m$. Define Γ_B, Γ_C similarly.

We have $\Gamma_A \subset \mathcal{L}$, $\Gamma_B \subset \mathcal{M}$, $\Gamma_C \subset \mathcal{N}$

Hotta formula gives $s[B] = [B] + [A] + [C]$

By the induction hypothesis the main result is true for B and C, so we have:

$$[B] = \sum_{T \in \Gamma_B} [\bar{X}_{w_T}] \quad , \quad [C] = \sum_{T \in \Gamma_C} [\bar{X}_{w_T}]$$

Now $\Gamma_B \subset \mathcal{M}$ then $w_T i=1$ if $T \in \Gamma_B$ (c.f. 1.2.3) and $w_T s > w_T$. Thus by Demazure formula

(1) $[A] = \sum_{T \in \Gamma_A} [\bar{X}_{w_T}]$ is equivalent to:

$$(*) \quad \sum_{T \in \Gamma_A} [\bar{X}_{w_T}] + \sum_{T \in \Gamma_C} [\bar{X}_{w_T}] = \sum_{(T,\beta) \in I} \langle e, \beta' \rangle [\bar{X}_{w_T s s_\beta}]$$

where I is the set of pairs (T,β) with $T \in \Gamma_B$, $\beta \in \Delta_+ - \{e\}$ and $l(w_T s s_\beta) = l(w_T)$.

We prove (*) by "counting" terms in both sides of the equality. Put $I = I_+ \cup I_-$ where $(T,\beta) \in I_+$ if $\langle e, \beta' \rangle = 1$ and $(T,\beta) \in I_-$ if $\langle e, \beta' \rangle = -1$.

We shall construct two maps:

$$\begin{aligned} \phi: \Gamma_A \cup \Gamma_C &\longrightarrow I_+ \text{ injective and satisfying } w_{T'} = w_T s s_\beta \text{ if } (T, v) = \phi(T') \\ \psi: I_- &\longrightarrow I_+ \text{ -im } \phi \text{ bijective and satisfying } w_{T'} s s_{\beta'} = w_T s s_\beta \text{ if } (T, \beta) = \psi(T', \beta') \end{aligned}$$

Define ϕ by:
$$\phi(T) = \begin{cases} (\psi T, \beta_{i,e}) & e = e_T \text{ if } T \in \Gamma_A \\ (\lambda T, \beta_{i-1,i+1}) & \text{if } T \in \Gamma_C \end{cases}$$

ϕ is well defined and satisfies the previous requirement (c.f. 2.2.2, 2.2.3).
 ϕ is injective (c.f. 2.2.4, 2.2.1).

Suppose (T, v) is in I_- . Then β has the form $\beta = \beta_{t,i}$ with $t < i$ (if β has the form $\beta = \beta_{i+1,e}$ with $e > i+1$, then because $w_{T'} = 1(w_T) < 1(w_{T'} s_i) < 1(w_{T'} s_i s_\beta)$ (c.f. lemma 1)).

Define ψ by:

$$\psi(T, \beta) = (\lambda T, k_T)$$

ψ is well defined and satisfies the requirement (c.f. 2.3.2, 2.3.1). ψ is injective (c.f. 2.3.3).

If $(t, \beta) \in I_+$ and β has the form $\beta = \beta_{t,i+1}$ then (T, β) is in $\text{im } \phi$ (c.f. 2.2.5), it follows from (2.3.4) that ψ is surjective.

Example.- We have a non trivial example for the tableau

9	7	6	4	3	.
8					
5					
2					
1					

Here the cardinalities of the sets $\Gamma_A, \Gamma_B, \Gamma_C$ are 10, 4, 3 respectively. Moreover I_- is a non empty set.

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Javier Güemes
Dpto. Algebra y Geometría
Facultad de Ciencias
Universidad de Valladolid
47005 Valladolid (SPAIN)