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This is an expository paper ; it is concerned with establishing Langlands' conjecture for an interesting family of irreductible representations of a split reductive p-adic group : the representations which admit non-zero vectors invariant under an Iwahori subgroup. This represents only the tip of the iceberg ; the rest of the iceberg remains to be explored. It is remarkable that equivariant K-theory plays such a central role in this problem. These ideas were developed in [11], [4], [8], [9]. We shall also explain a second approach to the same problem following [12].

1. Affine Hecke algebras.

We recall (cf. e.g. [14, 9.1.6]) that a root datum is a quadruple $(X,Y,R,\check{R})$ where X,Y are free (additive) abelian groups of finite rank with a given perfect pairing $<\ ,\ > : X\times Y\to \mathbb{Z}$ and $R,\check{R}$ are finite subsets of X,Y with a given bijection $\alpha\leftrightarrow\check{\alpha}$, $R\leftrightarrow\check{R}$.

For $\alpha\in R$ we define endomorphisms

$$s_\alpha : X\to X,\ s_\alpha x = x - <x,\check{\alpha}>\alpha$$

and

$$s_\alpha : Y\to Y,\ s_\alpha y = y - <\alpha,y>\check{\alpha}.$$

It is required that for all $\alpha\in R$:

$$<\alpha,\check{\alpha}> = 2$$

$$s_\alpha R = R\ ,\ s_\alpha\check{R} = \check{R}$$

$$2\alpha\notin R.$$

We assume given a basis Π of R : thus any $\beta\in R$ can be written uniquely as

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$\sum_{\alpha\in\Pi} n_\alpha \alpha$ where n_α are integers which are all ≥ 0 or all ≤ 0. It determines a partition $R = R^+ \cup R^-$.

The Weyl group W_0 is defined as the subgroup of GL(X) generated by the $s_\alpha : X \to X$ $(\alpha \in R)$. It is a Coxeter group with set of generators $S = \{s_\alpha | \alpha \in \Pi\}$.

Using the natural action of W_0 on X, we form the semidirect product $W_0.X$ with X normal : the product of $w.x$ and $w'.x'$ is $ww'.(w'^{-1}(x) + x')$.

$W_0.X$ is called the affine Weyl group. This is slightly different from the usual definition : usually one calls affine Weyl group the normal subgroup $W_0.Q$ of $W_0.X$ where Q is the subgroup of X generated by R.

Define a function $\ell : W_0.X \to \mathbb{N}$ by

$$\ell(w.x) = \sum_{\substack{\alpha\in R^+ \\ w(\alpha)\in R^+}} |\langle x,\check{\alpha}\rangle| + \sum_{\substack{\alpha\in R^+ \\ w\alpha\in R^-}} |1+\langle x,\check{\alpha}\rangle| .$$

(See [5].)

Let $\Omega = \{w.x \in W_0.X | \ell(w.x) = 0\}$

$\tilde{S} = \{w.x \in W_0.Q | \ell(w.x) = 1\}$.

Then $(W_0.Q,S)$ is a Coxeter group with (W_0,S) a parabolic subgroup, and Ω is an (abelian) subgroup of $W_0.X$ complementary to $W_0.Q$ and normalizing $\tilde{S}$.

Let A be the algebra $\mathbb{C}[\underline{\underline{q}}^{1/2}, \underline{\underline{q}}^{-1/2}]$ where $\underline{\underline{q}}^{1/2}$ is an indeterminate. Let H be the free A-module with basis $T_{w.x}$, $(w,x \in W_0.X)$. According to Iwahori-Matsumoto [5], there is a unique structure of associative A-algebra on H such that :

$T_{w.x} T_{w'.x'} = T_{(w.x)(w'.x)}$ if $\ell(w.x)(w'.x')) = \ell(w.x) + \ell(w'.x')$

$(T_{w.x}+1)(T_{w.x}-\underline{\underline{q}}) = 0$ if $w.x \in \tilde{S}$. The unit element is $T_{w.x}$ where $w.x$ is the neutral element of $W_0.X$.

The A-algebra H is called the affine Hecke algebra.

An alternative description for H has been given by Bernstein and Zelevinskii (unpublished, but see [10]). Let H_0 be the subalgebra of H spanned by the elements $T_{w.0}$ $(w \in W)$. Let Γ be the group algebra of X over A. Thus Γ

has an A-basis $\{\theta_x | x \in X\}$ and $\theta_x\theta_{x'} = \theta_{x+x'}$ $(x,x' \in X)$. Let $H' = H_0 \otimes \Gamma$; it is a free A-module with basis $T_{w.0} \otimes \theta_x$ $(w \in W_0$, $x \in X)$. There is a unique structure of associative A-algebra on H' such that properties (a)-(b) below hold :

(a) $h \to h \otimes 1$ and $\gamma \mapsto 1 \otimes \gamma$ are A-algebra homomorphisms $H_0 \to H'$, $\Gamma \to H'$.

(b) Let us write $T_{w.0}$ instead of $T_{w.0} \otimes 1$ and θ_x instead of $1 \otimes \theta_x$. Then $T_{w.0} \cdot \theta_x = T_{w.0} \otimes \theta_x$ and

$$\theta_x \cdot T_{s.0} = T_{s.0} \cdot \theta_{s(x)} + (\underline{\underline{q}}-1) \frac{\theta_x - \theta_{s(x)}}{1 - \theta_{-\alpha}}$$

for all $x \in X$, $\alpha \in \Pi$, where $s = s_\alpha$. (The fraction above is an element of Γ).

The relationship between H, H' is a follows. There is a unique A-algebra isomorphism $H \xrightarrow{\sim} H'$ which is the identity on H_0 and which maps $T_{1.x}$ to $\underline{\underline{q}}^{\ell(1.x)/2}\theta_x$ for any $x \in X$ such that $\langle x,\check{\alpha}\rangle \geq 0$ for all $a \in \Pi$.

We shall identify H and H' via this isomorphism. For any complex number $q \in \mathbb{C}^\star$ we shall denote $H_q = H \underset{A}{\ } \mathbb{C}$ where $\mathbb{C}$ is regarded as an A-module with $\underline{\underline{q}}^{1/2}$ acting as multiplication by $q^{1/2}$, a fixed square root of q.

2. Relation with representations of p-adic groups.

Let F be a p-adic field whose residue field has a finite number q of elements. Let $\bar{F}$ be an algebraic closure of F.

Let G be a connected reductive group defined over F such that G has a maximal torus T defined and split over F. To G one can associate in the usual way a root datum

$(X(T), Y(T), R',\check{R}')$: we define $X(T) = \mathrm{Hom}(T, F^\star)$, $Y(T) = \mathrm{Hom}(F^\star, T)$; R' is the set of roots, $\check{R}'$ is the set of coroots. We assume that $X(T)=Y, Y(T)=X$, $R'=\check{R}$, $\check{R}'=R$, where $(X,Y,R,\check{R})$ is the root datum in Sec.1.

Let I be an Iwahori subgroup of $G(F)$; this is a certain compact open subgroup of $G(F)$. Let $\mathcal{H}$ be the $\mathbb{C}$-algebra of all I-biinvariant functions with compact support $f : G(F) \to \mathbb{C}$ with respect to convolution product.

A $\mathbb{C}$-basis is given by the characteristic functions of the I-I double cosets, which are parametrized by the elements of $W_0.X$; these functions multiply in the same way as the basis elements $T_{w.x}$ of H_q (see [5]). It follows that the algebra $\mathcal{H}$ is naturally isomorphic to H_q.

According to [2], [3] the irreducible admissible representations of $G(F)$ wich have non-zero I-invariant vectors are in natural bijection with the simple $\mathcal{H}$-modules. (The bijection associates to the representation V of $G(F)$ its space V^I of I-invariant vectors, regarded as an $\mathcal{H}$-module in a natural way). Thus an interesting part of the representation theory of $G(F)$ is captured by the algebra $\mathcal{H} = H_q$.

This justifies the study of simple H_q-modules.

3. The Langlands dual.

We consider a complex connected reductive group G, with Lie algebra $\mathfrak{g}$. We can associate a root datum to G just as for G, in terms of a maximal torus of G.

It will be more convenient to define it in an intrinsic way. Let $\mathcal{B}$ be the variety of Borel subalgebras of $\mathfrak{g}$. Let X be the set of isomorphism classes of algebraic G-equivariant line bundles on $\mathcal{B}$. (This is an abelian group under $\otimes$). Let P be a conjugacy class of parabolic subalgebras of $\mathfrak{g}$ of semisimple rank 1 and let $\pi : \mathcal{B} \to P$ be the natural $\mathbb{P}^1$-bundle. Let $L_P \in X$ be the tangent bundle along the fibres of π. Let $h_P : X \to \mathbb{Z}$ be defined by $h_P(L) = m$, where $m+1$ = Euler characteristic of $L \in X$ restricted to any fibre of π (regarded as a coherent sheaf). Then h_P is a homomorphism so it is an element of $Y = \mathrm{Hom}(X,\mathbb{Z})$. Let $s_P : X \to X$ be defined by $s_P(L) = L \otimes L_P^{-h_P(L)}$. The s_P for varying P generate the Weyl group $W \subset GL(X)$. We set $\Pi = \{L_P | P \text{ as above}\} \subset X$, $\check{\Pi} = \{h_P | P \text{ as above}\} \subset Y$, $R = W\Pi \subset X$, $\check{R} = W\check{\Pi} \subset Y$. Then $R, \check{R}$ are naturally in bijection and $(X,Y,R,\check{R})$ is a root datum. We assume that it is the same as the one in Sec.1.

This means that G is the Langlands dual of G.

4. The Deligne-Langlands conjecture.

According to the general Langlands philosophy, the irreductible admissible representations of $G(F)$ should correspond to certain objects related to the geometry of G. For those representations of $G(F)$ which have non-zero vectors invariant by the Iwahori subgroup, this philosophy predicts (using the reformulation in Sec.2) that the simple H_q-modules should correspond to G-conjugacy classes of pairs (s,N), where $s \in G$ is semisimple, $N \in \mathfrak{g}$ is nilpotent and $\mathrm{Ad}(s)N = qN$. This statement, known as the Deligne-Langlands conjecture, has been verified for GL_n by Bernstein and Zelevinskii [1], [15]. In that case the correspondence is a bijection. In general it is not a bijection. In [10] it was suggested that in order to make it a bijection, to (s,N) one should add a third ingredient ρ, an irreducible representation of the finite group $\frac{Z(s,N)}{Z^0(s,N)}$ appearing in the homology $H_\star(\mathcal{B}_N^s,\mathbb{Q})$ where $\mathcal{B}_N^s = \{b \in \mathcal{B} | N \in b,\ Ad(s)b = b\}$. (Here $Z(s,N) = \{g \in G | gs = sg,$ $Ad(g)N = N\}$; it acts naturally on $\mathcal{B}_N^s$). This was suggested by an analogy with Springer's work on W-modules and by working out examples corresponding to subregular N.

In the rest of this paper we shall assume that G has simply connected derived group. We now state :

Theorem 4.1. [9] Let $q \in \mathbb{C}^\star$ be a complex number which is not a root of 1. Then the simple H_q-modules (up to isomorphism) are in the natural bijection with the G-conjugacy classes of triples (s,N,ρ) as above.

The bijection in the theorem will be constructed in Sec.5 using in essential way methods of equivariant K-theory. The approach to the Deligne-Langlands conjecture using equivariant K-theory has been developed in [11], [8] ; the conjecture itself is proved in [9].

5. Equivariant K-theory.

Let M be a linear algebraic group over $\mathbb{C}$. An M-variety is an algebraic variety over $\mathbb{C}$ with an algebraic action of M. If Z is an M-variety, let $K^M(Z)$ be the Grothendieck group of the category of M-equivariant coherent sheaves on Z. Then $R_M = K^M$ (point) is the Grothendieck group of finite dimensional algebraic representations of M. Note that R_M is a commutative ring and $K^M(Z)$ is an R_M-module in a natural way using tensor product.

Let Z' be another M-variety and let $f : Z \to Z'$ be an M-equivariant morphism. If f is smooth, then the inverse image $f^\star : K^M(Z') \to K^M(Z)$ is well defined ; if f is proper, then the direct image $f_\star : K^M(Z) \to K^M(Z')$ is well defined : it is defined using an alternating sum of higher direct images.

Now let $\phi : \mathcal{E}_0 \to \mathcal{E}_1$ be an M-equivariant map of M-equivariant vector bundles on Z, and let Z' be a closed M-subvariety of Z such that ϕ is an isomorphism on all fibres over Z-Z'. Let $\mathcal{F}$ be an M-equivariant coherent sheaf on X. Let $\mathcal{K}_0$ (resp. $\mathcal{K}_1$) be the kernel (resp. cokernel) of $\mathcal{F} \otimes \mathcal{E}_0 \xrightarrow{1\otimes\phi} \mathcal{F} \otimes \mathcal{E}_1$. Then $\mathcal{K}_0$, $\mathcal{K}_1$ are M-equivariant coherent sheaves on Z such that $\mathcal{K}_0 | Z\text{-}Z' = 0$, $\mathcal{K}_1 | Z\text{-}Z' = 0$. Let $\mathcal{I}$ be the coherent sheaf of functions on Z which vanish on Z'. For any $i \geq 0$, there is a · ll defined M-equivariant coherent sheaf $\overline{K}_0^i$ (resp. $\overline{K}_1^i$) on Z' whose extension to Z by 0 outside Z' is $I^i K_0 / I^{i+1} K_0$ (resp. $I^i K_1 / I^{i+1} K_1$). For large i we have $I^i K_0 = I^i K_1 = 0$ hence $\overline{K}_0^i = \overline{K}_1^i = 0$; now $F \to \sum_i (-1)^i \overline{K}_0^i - \sum_i (-1)^i \overline{K}_1^i$ defines a homomorphism $\gamma_\phi : K^M(Z) \to K^M(Z')$.

6. Construction of H-modules.

Fix a nilpotent element $N \in \mathfrak{g}$. Let $M(N) = \{(g,\lambda) \in G \times \mathbb{C}^\star \mid \mathrm{Ad}(g)N = \lambda N\}$. If $(s,q) \in M(N)$ is a semisimple element we denote by $M(s,q)$ the smallest algebraic diagonalizable) subgroup of $M(N)$ containing (s,q). Let $\mathcal{B}_N = \{b \in \mathcal{B} \mid N \in b\}$. Note that $M(N)$ acts on $\mathcal{B}_N$ by $(g,\lambda) : b \mapsto \mathrm{Ad}(g)b$. In particular, $M(s,q)$ acts on $\mathcal{B}_N$ and therefore $K^{M(s,q)}(\mathcal{B}_N)$ is an $R_{M(s,q)}$-module. Now $(s,q) \in M(s,q)$ defines a ring homomorphism $h : R_{M(s,q)} \to \mathbb{C}$ (it attaches to an $M(s,q)$-module the trace of (s,q) on that module). This makes $\mathbb{C}$ into an $R_{M(s,q)}$-module, hence we can form $E = K^{M(s,q)}(\mathcal{B}_N) \underset{R_{M(s,q)}}{\otimes} \mathbb{C}$. On this complex vector space we want to define endomorphisms corresponding to the generators of the algebra H_q.

We define for $x \in X$, $\theta_x : K^{M(s,q)}(\mathcal{B}_N) \to K^{M(s,q)}(\mathcal{B}_N)$ by $\theta_x(F) = F \otimes L_x$ where L_x is the G-equivariant line bundle on $\mathcal{B}$ indexed by x. (We regard L_x as a $G \times \mathbb{C}^\star$-equivariant line bundle on $\mathcal{B}$ with $\mathbb{C}^\star$ acting trivially, and we retrict it to $\mathcal{B}_N$; the restriction is an $M(s,q)$-equivariant line bundle on $\mathcal{B}_N$). This is $R_{M(s,q)}$-linear, hence it induces a $\mathbb{C}$-linear map $\theta_x : E \to E$.

Now let $\mathcal{P}$ be a conjugacy class of parabolic subalgebras of $\mathfrak{g}$ of semi-simple rank 1. Let $\mathcal{P}_N$ be the set of all $p \in \mathcal{P}$ such that $N \in p$. Consider its inverse image $\pi^{-1}(\mathcal{P}_N)$ under the natural map $\pi : \mathcal{B} \to \mathcal{P}$. Then π restricts to

$$\pi' : \mathcal{B}_N \to \mathcal{P}_N \qquad \text{(a proper map)}$$

and to

$$\pi'' : \pi^{-1}(\mathcal{P}_N) \to \mathcal{P}_N \qquad \text{(a } \mathbb{P}^1\text{-bundle)}.$$

Let $\mathcal{L}$ be the line bundle on $\pi^{-1}(\mathcal{P}_N)$ whose fibre at b is p/b where p is the unique subalgebra in $\mathcal{P}$ containing b. It is the restriction of a $G \times \mathbb{C}^\star$-equivariant line bundle on $\mathcal{B}$, hence it is $M(s,q)$-equivariant. It has a canonical section defined by the image of $N \in \mathfrak{p}$ in $\mathfrak{p}/\mathfrak{b}$. This section is not $M(s,q)$-equivariant, but it becomes so if $\mathcal{L}$ is replaced by $\lambda^{-1} \otimes \mathcal{L}$. (Here λ is the trivial line bundle on which $M(s,q)$ acts in the fibre direction by multiplication with the character $pr_2 : M(s,q) \to \mathbb{C}^\star$; λ^{-1} denotes the dual of that line bundle. The sections of $\mathcal{L}$ are the same as the sections of $\lambda^{-1} \otimes \mathcal{L}$). Our section of $\lambda^{-1} \otimes \mathcal{L}$ vanishes exactly over $\mathcal{B}_N \subset \pi^{-1}(P_N)$. It defines a map of line bundles $\mathbb{C} \to \lambda^{-1} \otimes \mathcal{L}$; taking duals we find a map of line bundles $\Phi : \lambda \otimes \mathcal{L}^{-1} \to \mathbb{C}$ which is an isomorphism outside $\mathcal{B}_N$. It gives rise by the construction in Sec.5 to a map

$$\gamma_\Phi : K^{M(s,q)}(\pi^{-1}(P_N)) \to K^{M(s,q)}(\mathcal{B}_N).$$

We define an operator $\underline{\underline{q}} - T_{s_P} : K^{M(s,q)}(\mathcal{B}_N) \to K^{M(s,q)}(\mathcal{B}_N)$ as the composition $\gamma_\Phi . (\pi'')^\star . (\pi')_\star$. This operator is $R_{M(s,q)}$-linear hence it defines by extension of scalars a $\mathbb{C}$-linear map $\underline{\underline{q}} - T_{s_P} : E \to E$.

Next we note that $M(N,s) = \{(g,\lambda) \in M(N) | gs = sg\}$ acts on $\mathcal{B}_N$ (restriction of $M(N)$-action) and it commutes with the action of $M(s,q)$. For any $m \in M(N,s)$ and any $M(s,q)$-equivariant coherent sheaf F on $\mathcal{B}_N$, we can consider the inverse image $m^\star F$; it is again an $M(s,q)$-equivariant coherent sheaf on $\mathcal{B}_N$. This defines an action of $M(N,s)$ on $K^{M(s,q)}(\mathcal{B}_N)$, which is $R_{M(s,q)}$-linear, hence it defines an action of $M(N,s)$ on E. For any irreducible $\mathbb{C}$-representation ρ of $M(N,s)$, trivial on $M^0(N,s)$, we consider $E_\rho = \mathrm{Hom}_{M(N,s)}(\rho,E)$. The operators θ_x, $\underline{\underline{q}} - T_{s_P}$ on E commute with the action of $M(N,s)$ hence they define analogous operators on E_ρ.

We can now indicate the construction of the bijection in Theorem 4.1. Assume that $q \in \mathbb{C}^\star$ is not a root of 1. One shows that the operators θ_x, $\underline{\underline{q}} - T_{s_P}$ define an H_q-module structure on E_ρ ($\underline{\underline{q}}$ acts as multiplication by q). One shows that $E_\rho \neq 0$ if and only if ρ appears in $H_\star(\mathcal{B}_N^s, Q)$ regarded as a $M(N,s)$-module in a natural way. If $E_\rho \neq 0$ then E_ρ has a unique simple quotient H_q-module $\overline{E}_\rho$. Then $(s,q,\rho) \to \overline{E}_\rho$ is the required bijection. (Note that $\dfrac{Z(s,N)}{Z^0(s,N)} = \dfrac{M(s,N)}{M^0(s,N)}$).

The proof of 4.1 given in [9] (and the statements given there) involve equivariant topological K-homology $K_{top}(\ )$ instead of Grothendieck's K-theory of coherent sheaves, which was used only as a heuristic guide. Subsequently, as a consequence of [3] it became known that the natural map

$$K^{M(s,q)}(\mathcal{B}_N) \underset{R_{M(s,q)}}{\otimes} \mathbb{C} \longrightarrow K^{M(s,q)}_{top}(\mathcal{B}_N) \underset{R_{M(s,q)}}{\otimes} \mathbb{C}$$

is an isomorphism. Indeed, using the localization theorem (Atiyah, Segal, Thomason) in the two kinds of K-theory we see that it is enough to show

$$K(\mathcal{B}_N^s) \otimes \mathbb{C} \xrightarrow{\sim} K_{top}(\mathcal{B}_N^s)$$

with non equivariant K-groups). This follows from the main result of [3] which asserts that for $\mathcal{B}_N^s$, the integral homology in even degrees is isomorphic to the Chow group, while in odd degrees it is zero.

This allows us to define the bijection 4.1 in terms K-theory of coherent sheaves ; we note however that topological K-homology seems to be still needed in the proofs.

7. Roots of unity.

The statement of Theorem 4.1 is not true in general when q is a root of 1 (for example for $G = SL_2$, $q = -1$). However, it is true for $q = 1$ when it can be deduced from Springer's results on W-modules (an observation of S.Kato [6]). It is likely that the statement of theorem 4.1 remains true for any $q \in \mathbb{C}^*$ such that

(a) $$\sum_{y \in W_0} q^{\ell(y)} \neq 0 ;$$

thus it can only fail for finitely many roots of unity.

We will show that for $q \in \mathbb{C}^*$, $q \neq 1$, the inequality (a) is equivalent with each of the following two statements (b), (c) below.

(b) $\det(q-w) \neq 0$ for all $w \in W_0$ (in the standard reflection representation of W_0).

(c) For any semisimple element $s \in G$, the eigenspace $\mathfrak{g}_q = \{\xi \in \mid \mathrm{Ad}(s)\xi = q\xi\}$ consists entirely of nilpotent elements.

We may assume that G is semisimple. It is well known that $\det(\underline{\underline{q}}-w)$ divides $(\sum_{y \in M_0} \underline{\underline{q}}^{\ell(y)}).(\underline{\underline{q}}-1)^r$, ($r$ = rank of W_0) as polynomials in $\mathbb{Z}[\underline{\underline{q}}]$. Hence (a) => (b).

It is also well known that

$$|W_0| = \sum_{w\in W_0} (-1)^{\ell(w)} \left(\sum_{y\in W_0} \underline{\underline{q}}^{\ell(y)} \right) (\underline{\underline{q}}-1)^r \cdot \det(\underline{\underline{q}}-w)^{-1} .$$

Hence (b) => (a).

Assume that (b) doesn't hold. Then we can find a maximal torus T of G with Lie algebra $\underline{t}$ and an element $\dot{w} \in N(T)$ such that $(q-\mathrm{Ad}(\dot{w}))\xi = 0$ for some $\xi \in \underline{t}-0$. We may assume that $\dot{w}$ is of finite order hence semisimple ; we see that (c) doesn't hold. Thus we have (c) => (b).

Assume now that (c) doesn't hold. Let $s \in G$ be a semisimple element and ξ be a non-nilpotent element such that $\mathrm{Ad}(s)\xi = q\xi$. The same identity is then satisfied by the semisimple part of ξ so that we can assume that ξ is semisimple, non-zero. Let $G' = \{g \in G | \mathrm{Ad}(g)\xi \in \mathbb{C}^\star.\xi\}$ and let $\psi : G' \to \mathbb{C}^\star$ be the homomorphism defined by $\psi(g) = \lambda$ where $\mathrm{Ad}(g)\xi = \lambda\xi$. If $\mathrm{Ad}(g)\xi = \lambda\xi$ with λ not root of 1 then ξ is clearly nilpotent, a contradiction. Hence the image of ψ contains only roots of 1. Being a closed subgroup of $\mathbb{C}^\star$, the image of ψ must be finite. Since the centralizer $Z_G(\xi)$ is connected we have $\ker \psi = Z_G(\xi) = (G')^0$. Hence $\psi^{-1}(q)$ is a connected component of G', so it contains some element of finite order. Hence we can assume that s has finite order. Let γ be the space of all maximal tori of $Z_G(\xi)$. It is well known that γ has the same rational cohomology with compact support as an affine space. Now s acts on γ by conjugation. By the fixed point theorem it follows that $\gamma^s \neq 0$ so that there exists a maximal torus T of $Z_G(\xi)$ normalized by s. Let $\underline{t}$ be the Lie algebra of T. Then $\xi \in \underline{t}$ and $\mathrm{Ad}(s) : \underline{t} \to \underline{t}$ has ξ as a q-eigenvector. Hence $\det(q-\mathrm{Ad}(s),\underline{t}) = 0$. But $\mathrm{Ad}(s)$ acts on $\underline{t}$ as an element of the Weyl group of T and we see that (b) doesn't hold. Thus (b) => (c). The equivalence of (a), (b), (c) is proved.

8. Simple $\mathbb{C}[W_0X]$-modules and simple H_q-modules.

We shall indicate a procedure which establishes a bijection

$$\text{(a)} \qquad \left\{\begin{matrix}\text{simple } H_q\text{-modules} \\ \text{up to isomorphism}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{simple } \mathbb{C}[W_0X]\text{-modules} \\ \text{up to isomorphism}\end{matrix}\right\}$$

when q is not a root of 1.

The proofs can be found in [12].

Let $\omega.\zeta$, $\omega'.\zeta'$ be two elements of $W_0.X$ $(\omega,\omega' \in \Omega$, $\zeta,\zeta' \in W_0Q)$. Since $(W_0Q,\tilde{S})$ is a Coxeter group, the polynomials $P_{\zeta,\zeta'}$ of [7] are well defined. We define $P_{\omega\zeta,\zeta'\zeta'}$ to be $P_{\zeta,\zeta'}$ when $\omega = \omega'$ and 0 when $\omega \neq \omega'$. As in [7] we consider for each $w.x \in W_0.X$ the element

$$C_{wx} = \sum_{v.y \in W_0.X} (-1)^{\ell(wx)-\ell(vy)} \underline{\underline{q}}^{\frac{\ell(wx)}{2} - \ell(vy)} P_{v.y,wx}(\underline{\underline{q}}^{-1}) T_{v.y} \in H.$$

The element C_{wx} $(wx \in W_0X)$ form an A-basis of H. Hence we have

$$C_{wx}\, C_{w'x'} = \sum_{w''x''} h_{wx,w'x',w''x''}\, C_{w''x''}$$

where $h_{wx,w'x',w''x''} \in A$.

There is a unique function $a : W_0X \to \mathbb{N}$ such that for any $w''x'' \in W_0X$, $q^{a(w''w'')/2} h_{wx,w'x',w''x''}$ is a polynomial in $\underline{\underline{q}}^{1/2}$ for all $wx, w'x' \in W_0X$ and it has non-zero constant term for some $wx, w'x'$.

Let $\underline{J}$ be the $\mathbb{C}$-vector space with basis $\{t_{wx} \mid wx \in W_0X\}$.
There is a unique structure of associative $\mathbb{C}$-algebra on $\underline{J}$ such that

$$t_{wx}.t_{w'x'} = \sum_{w''x'' \in W_0X} (\text{const. term of } (-1)^{a(w''x'')} \underline{\underline{q}}^{\frac{1}{2}a\,(w''w'')} . h_{wx,w'x',w''x''})\, t_{w''x''}.$$

This algebra has a unit element of form $1 = \sum_{d \in \mathcal{D}} t_d$ where $\mathcal{D}$ is a certain set of involutions in W_0X. For any $q \in \mathbb{C}^\star$, the $\mathbb{C}$-linear map $\psi_q : H \to \underline{J}$ defined by

$$\psi_q(C_w) = \sum_{\substack{d \in \mathcal{D} \\ vz \in W_0X \\ a(vz)=a(d)}} h_{wx,d,vz}\,(q^{1/2})\, t_{vz}$$

is a $\mathbb{C}$-algebra homomorphism preserving 1. (Here $h_{wx,d,vz}(q^{1/2})$ is the evaluation of $h_{wx,d,vz} \in A$ at $\underline{\underline{q}}^{1/2} = q^{1/2}$). Moreover, ψ_q is injective. Thus all algebras $H_q (q \in \mathbb{C}^\star)$ appear as subalgebras of a single $\mathbb{C}$-algebra $\underline{J}$.

Let M be a simple H_q-module (resp. J-module). We attach to M an integer $a = a_M$ by the following two requirements :

$C_{wx}M = 0$ (resp. $t_{wx}M = 0$) for all $wx \in W_0X$, $a(wx) > a$.

$C_{wx}M \neq 0$ (resp. $t_{wx}M \neq 0$) for some $wx \in W_0X$, $a(wx) = a$.

Theorem 8.1. Assume that $q \in \mathbb{C}^{*}$ is either 1 or is not a root of 1. There is a unique bijection

$$\text{(b)} \quad \left\{\frac{\text{simple } H_q\text{-modules}}{\text{up to isomorphism}}\right\} \longrightarrow \left\{\frac{\text{simple } J\text{-modules}}{\text{up to isomorphism}}\right\}.$$

$(M \to M')$ with the following properties :

$a_{M'} = a_M$ and the restriction of M' to H_q (via ψ_q) is an H_q-module with exactly one composition factor isomorphic to M and all other composition factors of form $\overline{M}$, $a_{\overline{M}} < a_M$.

The proof of this result given in [12] makes use of the main results of [9] among other things. Applying (b) once for $q = 1$ and once for q not a root of 1 we obtain the bijection (a). (Note that $H_1 = \mathbb{C}[W_0X]$).

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