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On the representations of reductive groups with disconnected centre

G. Lusztig^{*}

1. We consider a connected reductive algebraic group defined over a finite field $\mathbb{F}_q$ with Frobenius map $F : G \rightarrow G$.

Let $\tilde{G}^F$ denote the set of irreducible representations up to isomorphism of the finite group G^F over $\overline{\mathbb{Q}}_\ell$ (ℓ is a prime not dividing q). In the case where the centre Z_G of G is connected, a parametrization for $\tilde{G}^F$ was given in [4] ; this is extended here to the general case (i.e. we allow Z_G to be disconnected). The proof will be by a reduction to the case where Z_G is connected using a method in [4, 14.1]. The results of this paper were obtained during the summer of 1983 and were announced in [5].

2. We denote by $G^\star$ a connected reductive group defined over $\mathbb{F}_q$, dual to G , as in [2]. We again denote by F the corresponding Frobenius map. (The same notation will be used for the Frobenius map of any algebraic variety defined over $\mathbb{F}_q$). As in [2] we have a natural bijection

$$\{(T', \theta)\} \bmod G^F\text{-conjugacy} \leftrightarrow \{(T, s)\} \bmod G^{\star F}\text{-conjugacy}$$

where T' (resp. T) runs over the F -stable maximal torus of G (resp. $G^\star$), $\theta : T'^F \rightarrow \overline{\mathbb{Q}}_\ell^\star$ is a character and s is an element of T^F . If (T', θ) , (T, s) correspond in this way we consider the virtual representation $R_{T'}^G(\theta)$ defined in [2] ; we shall also write $R_T^G(s)$ instead of $R_{T'}^G(\theta)$.

For a semisimple element $s \in G^{\star F}$, let $(\hat{G}^F)_s$ be the set of all $\rho \in \hat{G}^F$ appearing with non-zero multiplicity in $R_T^G(s)$ for some F -stable maximal torus $T \subset G$.

The subset $(\hat{G}^F)_s$ of $\hat{G}^F$ depends only on the $G^{\star F}$ -conjugacy class of s . We

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have a partition $\hat{G}^F = \bigsqcup_s (\hat{G}^F)_s$ where s runs over the semisimple elements of $G^{\star F}$ up to $G^{\star F}$ -conjugacy. (See [2], [3]).

3. Let $\pi : G \rightarrow G_{\text{ad}}$ be the adjoint quotient of G . We have a natural isomorphism

$$G_{\text{ad}}^F / \pi(G^F) \cong (Z_G / Z_G^O)_F$$

(the subscript F denotes F -coinvariants, i.e. largest quotient on which F acts trivially) ; it is defined by the correspondence $G_{\text{ad}}^F \ni g \rightarrow \dot{g}^{-1}F(\dot{g}) \in Z_G$, where $\dot{g} \in G$ satisfies $\pi(\dot{g}) = g$.

The group G_{ad}^F acts naturally on G^F by automorphisms $g : g_1 \rightarrow \dot{g}g_1\dot{g}^{-1}$ ($g, \dot{g}$ as above). Hence G_{ad}^F acts naturally on $\hat{G}^F$. Clearly this action is trivial on the subgroup $\pi(G^F)$ hence it induces an action of $G_{\text{ad}}^F / \pi(G^F)$ on $\hat{G}^F$. It is easy to check that this action (extended by linearity to virtual representations) leaves fixed each $R_T^G(s)$; hence it leaves stable each subset $(\hat{G}^F)_s$ of $\hat{G}^F$. We have thus defined an action of $(Z_G / Z_G^O)_F$ on $(\hat{G}^F)_s$.

4. We fix a semisimple element $s \in G^{\star F}$ and we denote $H = Z_{G^{\star}}(s)$. If $x \in H^F$, then conjugation by x is an automorphism of H^O (over $\mathbb{F}_q$) ; hence it defines an automorphism of $\hat{H}^{OF}$ which leaves stable the set $(\hat{H}^{OF})_1$ of unipotent representations. If $x \in H^{OF}$, the corresponding automorphism of $(\hat{H}^{OF})_1$ is trivial, so we have a natural action of H^F / H^{OF} on $(\hat{H}^{OF})_1$.

5. With these notations, we can now state our main result.

Proposition 5.1. There exists a surjective map

$\psi : (\hat{G}^F)_s \rightarrow (\hat{H}^{OF})_1$ mod action of H^F / H^{OF} with the following properties.

The fibres of ψ are precisely the orbits of the action of $(Z_G / Z_G^O)_F$ on $(\hat{G}^F)_s$, (see Sec.3). If $\emptyset$ is a H^F / H^{OF} -orbit on $(\hat{H}^{OF})_1$ and Γ is the stabilizer in H^F / H^{OF} of an element in $\emptyset$, then the fibre $\psi^{-1}(\emptyset)$ has precisely $|\Gamma|$ elements. If $\rho \in \psi^{-1}(\emptyset)$ and T is an F -stable maximal torus of $G^{\star}$ containing s , then

$$(\rho : R_T^G(s))_{G^F} = \varepsilon_G \varepsilon_H \sum_{\bar{\rho} \in \bar{\Theta}} (\bar{\rho} : R_T^{H^O}(1))_{H^{OF}}.$$

Here $(:)$ denotes the standard inner product of virtual representations and $\varepsilon_G = (-1)^{\sigma(G)}$, $\sigma(G) = \mathbb{F}_q$ -rank of G ; in $R_T^{H^O}(1)$, 1 stands for the trivial character of T^F .

Remarks. The sets $(\hat{H}^{OF})_1$ are described explicitly in [4]; they are insensitive to the centre of H^O . The action of outer automorphisms on the set of unipotent representations of a connected reductive group is easy to describe; for example when that group is simple modulo its centre, of type $\neq D_{2n}$, this action is trivial. The multiplicities $(\bar{\rho} : R_T^{H^O}(1))_{H^F}$ are also described explicitly in [4].

Hence the proposition gives an explicit parametrization of $(\hat{G}^F)_s$ and explicit formulas for the multiplicities $(\rho : R_T^G(s))_{G^F}$.

6. Now let G' be a connected reductive group over $\mathbb{F}_q$ with connected centre. Let $s' \in G'^{\star F}$ be semisimple and let $H' = Z_{G', \star}(s')$. Then both groups $(Z_{G'}/Z_{G'}^O)_F$ and H'^F/H'^{OF} are trivial and from 5.1 we obtain the following known result.

Corollary 6.1. [4]. If $Z_{G'}$ is connected, then there exists a bijection
 $\psi : (\hat{G}'^F)_{s'} \longrightarrow (\hat{H}'^{OF})_1$ such that $(\rho' : R_{T'}^{G'}(s'))_{G'^F} = \varepsilon_{G'} \varepsilon_{H'} (\bar{\rho}' : R_{T'}^{H'^O}(1))_{H'^{OF}}$
for any $\rho' \in (\hat{G}'^F)_{s'}$, and any F -stable maximal torus T' of G' containing s' ;
here $\bar{\rho}' = \psi(\rho')$.

7. If G is as in Sec. 1, we say that $i : G \rightarrow G'$ is a regular imbedding if G' is a connected reductive group over $\mathbb{F}_q$ with connected centre, i is an isomorphism of G with a closed subgroup of G' and $i(G)$, G' have the same derived subgroup.

We shall need the following simple result.

Lemma 7.1. [1, 2.3.2]. If G is semisimple and $i : G \rightarrow G'$, $\bar{i} : G \rightarrow \bar{G}$ are regular imbeddings, then there exists a connected reductive group G'' over $\mathbb{F}_q$ and regular imbeddings $j : G' \rightarrow G''$, $\bar{j} : \bar{G} \rightarrow G''$ such that $j \circ i = \bar{j} \circ \bar{i}$.

8. With G as in Sec. 1, we fix a regular imbedding $G \rightarrow G'$. To this corresponds by duality a surjective homomorphism $\delta : G'^{\star} \rightarrow G^{\star}$ (over $\mathbb{F}_q$) whose kernel K is a central torus in $G'^{\star}$. We have a natural isomorphism $K^F \xrightarrow{\sim} \text{Hom}(G'^F/G^F, \Phi_{\ell}^{\star})$, $k \mapsto \theta_k$. We consider the action of K^F on $\hat{G}'^F$ given by $k : \rho' \rightarrow \rho' \otimes \theta_k$. The action of $k \in K^F$ on $\hat{G}'^F$ defines a bijection $(\hat{G}'^F)_{s_1} \xrightarrow{\sim} (\hat{G}'^F)_{ks_1}$ for any semisimple $s_1 \in G'^{\star F}$.

Now let $s' \in G'^{\star F}$ be semisimple, $H' = Z_{G'^{\star}}(s')$. Let $K_{S'}^F$ be the set of all $k \in K^F$ which map $(\hat{G}'^F)_{s'}$ into itself or, equivalently,

$$K_{S'}^F = \{k \in K^F \mid ks' \text{ is conjugate to } s' \text{ under } G'^{\star F}\}.$$

If $s = \delta(s') \in G^{\star F}$, and $H = Z_{G^{\star}}(s)$ we have a natural isomorphism $H^F/H^{OF} \simeq K_{S'}^F$, defined by the correspondence $H^F \ni x \rightarrow s'^{-1}\dot{x}s'\dot{x}^{-1} \in K^F$ where $\dot{x} \in G'^{\star F}$ satisfies $\delta(\dot{x})=x$. (Note that $\delta : G'^{\star F} \rightarrow G^{\star F}$ is surjective). Using this isomorphism the action of $K_{S'}^F$ on $(\hat{G}'^F)_{s'}$ becomes an action of H^F/H^{OF} on $(\hat{G}'^F)_{s'}$. Now δ defines a surjective homomorphism $H'^O \rightarrow H^O$ with kernel K , hence a bijection $(\hat{H}'^{OF})_1 \xrightarrow{\sim} (\hat{H}^{OF})_1$. Using this, the action of H^F/H^{OF} on $(\hat{H}'^{OF})_1$ in Sec 4 becomes an action on $(\hat{H}^{OF})_1$. We shall need the following strengthening of 6.1.

Proposition 8.1. The isomorphism ψ in 6.1 can be chosen to be compatible with the action of H^F/H^{OF} on $(\hat{G}'^F)_{s'}$ and $(\hat{H}'^{OF})_1$ defined above.

Proof. (a) Assume first that G is almost simple, simply connected. If G is a classical group, then ψ in 6.1 is uniquely determined ; in the remaining cases (with one exception) either H is connected (and there is nothing to prove) or ψ is uniquely determined. In these cases the result follows easily. The exception is : G of type E_7 , $s \in G^{\star F}$ is such that $H = Z_{G^{\star}}(s)$ has two components and H^O modulo its centre is of type E_6 . There are two representations in $(\hat{G}'^F)_{s'}$ which are not distinguished by their multiplicities in the $R_T^{G'}(s')$. We must show that they are in the same orbit of $H^F/H^{OF} \cong \mathbb{Z}/2$. If they are not, they would remain irreducible on restriction to G'^F . But their restrictions to G'^F are reducible by an argument in [4, p.353].

(b) Assume next that $G = G_1 \times G_2 \times \dots \times G_n$ with almost simple,

simply connected factors G_i permuted by F and that $G' = G'_1 \times G'_2 \times \dots \times G'_n$ where $G_i \rightarrow G'_i$ are regular imbeddings over an extension of $\mathbb{F}_q$ and the G_i are again permuted by F . In this case we group together the factors in the various orbits of F and we are reduced to the case where F permutes cyclically the indices. In that case we have $G^F = G_1^{F^n}$, $G'^F = G'^{F^n}$ and the result follows by applying (a) to G_1 and G'_1 instead of G and G' .

(c) Assume that G is simply connected. We decompose G in a product $G_1 \times G_2 \times \dots \times G_n$ as in (b); we imbed it in $\bar{G} = \bar{G}_1 \times \dots \times \bar{G}_n$ where $G \rightarrow \bar{G}$ is like $G \rightarrow G'$ in (b). Let $G' \rightarrow G''$, $\bar{G} \rightarrow G''$ be as in 7.1. We can find $s'' \in G''^{\star F}$ which maps to $s' \in G'^{\star F}$ and to some element $\bar{s} \in \bar{G}^{\star F}$ under $G'^{\star} \leftarrow G''^{\star} \rightarrow \bar{G}^{\star}$. Since G' , G'' , $\bar{G}$ have connected centre, we get by restriction bijections $(\hat{G}^F)_s \xleftarrow{\sim} (\hat{G}''^F)_{s''} \xrightarrow{\sim} (\hat{\bar{G}}^F)_{\bar{s}}$. There are compatible with the actions of H^F/H^{OF} . Since the case $(G \rightarrow \bar{G}, \bar{s})$ is handled by (b), the cases $(G \rightarrow G'', s'')$ and $(G \rightarrow G', s')$ follow.

(d) Assume that G is the derived group of G' . We can find a connected reductive group $\tilde{G}'$ over $\mathbb{F}_q$ with simply connected derived group $\tilde{G}$ and a surjective homomorphism $\tilde{G}' \rightarrow G'$ (over $\mathbb{F}_q$) whose kernel is a central torus in $\tilde{G}'$. Then $\tilde{G}'$ has connected centre and $\tilde{G}' \rightarrow G'$ restricts to a finite covering $\tilde{G} \rightarrow G$. We have $G'^{\star} \subset \tilde{G}'^{\star}$ hence s' can be considered as an element of $\tilde{G}'^{\star F}$. Let $\tilde{s}$ be the image of $s \in G^{\star F}$ under the finite covering $G^{\star} \rightarrow \tilde{G}^{\star}$. Let $\tilde{H} = Z_{\tilde{G}^{\star}}(\tilde{s})$. We have a natural imbedding $H^F/H^{OF} \rightarrow \tilde{H}^F/\tilde{H}^{OF}$, induced by $G^{\star} \rightarrow \tilde{G}^{\star}$. Composition with $\tilde{G}'^F \rightarrow G'^F$ defines a bijection $(\hat{G}^F)_s \xrightarrow{\sim} (\hat{\tilde{G}}^F)_{\tilde{s}}$. Applying (c) to $(\tilde{G} \rightarrow \tilde{G}', s')$ we deduce the desired result for $(G \rightarrow G', s')$.

(e) We now consider the general case. Let G'' be the derived group of G . Let s'' be the image of $s \in G^{\star F}$ under $G^{\star} \rightarrow G''^{\star}$ and let $H'' = Z_{G''^{\star}}(s'')$. We have a natural imbedding $H^F/H^{OF} \rightarrow H''^F/H''^{OF}$. Applying (d) to $(G'' \rightarrow G', s')$ we deduce the desired result for $(G \rightarrow G', s')$. This completes the proof.

9. Let $A \subset B$ be finite groups such that A is normal in B and B/A is abelian. Then the abelian group B/A acts naturally on $\hat{A}$ (this is induced by the action of B on A by conjugation) and the abelian group $\hat{B}/\hat{A}$ acts naturally

on $\hat{B}$ by tensor product. The proofs of the results in this section are standard, and will be omitted.

(a) Assume that any $\rho \in \hat{B}$ restricts to a multiplicity free representation of A . Then there is a unique bijection

$$\hat{A} \bmod \text{action of } B/A \leftrightarrow \hat{B} \bmod \text{action of } B/\hat{A}$$

with the following properties. Let θ be a B/A -orbit on $\hat{A}$ and let θ' be the corresponding $B/\hat{A}$ -orbit on $\hat{B}$. Then if $\rho_0 \in \theta'$, we have $\rho_0|_A = \sum_{\tau \in \theta} \tau$; if $\tau_0 \in \theta$, we have $\text{ind}_A^B \tau_0 = \sum_{\rho \in \theta'} \rho$. Moreover, the stabilizer of ρ_0 in $B/\hat{A}$ and the stabilizer of τ_0 in B/A are orthogonal to each other under the natural duality $B/A \times B/\hat{A} \rightarrow \overline{\mathbb{Q}}_\ell^\star$.

We now want to find conditions which should imply that the assumptions of (a) holds.

(b) If B/A is cyclic then the assumption of (a) is automatically satisfied.

(c) Assume now that any $\rho \in \hat{B}$ has stabilizer I_ρ in $B/\hat{A}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ or $\{e\}$. Let $\hat{B}' = \{\rho \in \hat{B} | \rho|_A \text{ is multiplicity free}\}$, $\hat{B}'' = \hat{B} - \hat{B}'$, $\hat{A}' = \{\tau \in \hat{A} | \tau \text{ appears in } \rho|_A \text{ some } \rho \in \hat{B}'\}$, $\hat{A}'' = \hat{A} - \hat{A}'$. Then the conclusions of (a) hold if $\hat{A}, \hat{B}$ are replaced by $\hat{A}', \hat{B}'$. If $\rho \in \hat{B}''$ then $I_\rho = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\rho|_A = 2\tau$, $\tau \in \hat{A}''$; moreover $\text{Ind}_A^B \tau = 2\rho$ and $\rho \mapsto \tau$ is a bijection $\hat{B}'' \xrightarrow{\sim} \hat{A}''$.

Let $x_i = \#\{\rho \in \hat{B} | |I_\rho| = i\}$, ($i = 1, 2, 4$) and $y = \#\hat{B}''$. Then $|\hat{B}| = x_1 + x_2 + x_4$, $|\hat{A}| = \frac{x_1}{p} + 4 \frac{x_2 + y}{p} + 16 \frac{x_4 - y}{p}$, ($p = |B/A|$). Hence if we assume also that $|\hat{A}| = \frac{x_1}{p} + 4 \frac{x_2}{p} + 16 \frac{x_4}{p}$, then $y = 0$, so that the assumption of (a) is again satisfied.

10. Proposition. Let $G \subset G'$ be a regular imbedding (Sec. 7). For any $\rho' \in \hat{G}'^F$, the restriction $\rho'|_{G^F}$ is multiplicity free.

Proof. a) Assume first that G is almost simple, simply connected and that $\dim Z_{G'} \leq 1$ except that $\dim Z_{G'} = 2$ when $G = \text{Spin}_{4n}$ and $\text{char } \mathbb{F}_q \neq 2$. If $\dim Z_{G'} \leq 1$, or if $G = \text{Spin}_{4n}$ is non-split over $\mathbb{F}_q$ (of odd characteristic) then G'^F/G^F is cyclic

and we may use 9(b). If $G = \text{Spin}_{4n}$ is split over F_q of odd characteristic we use 9(c) as follows. First note that in this case the set $\hat{G}^F$ and the action of $(G^F/G^F)^\wedge$ on it are determined explicitly by Proposition 8.1. From this we can compute explicitly the numbers x_1, x_2, x_4 in 9(c) for $A = G^F$, $B = G^F$. On the other hand we can count directly the number of conjugacy classes in the split $\text{Spin}_{4n}(F_q)$. This is the same as $|\hat{A}|$. We then compare $|\hat{A}|$ and $\frac{x_1}{p} + 4 \frac{x_2}{p} + 16 \frac{x_4}{p}$ ($p = |B/A|$) and find that they are equal. We can apply 9(c) and we see that the proposition holds.

(b) Assume next that G is almost simple, simply connected but there is now no restriction on $\dim Z_G$. We can find a regular imbedding $G \rightarrow \bar{G}$ which is like $G \rightarrow G'$ in (a). Let $G' \rightarrow G''$, $\bar{G} \rightarrow G''$ be as in 7.1. We have natural surjective maps $\hat{G}^F \leftarrow \hat{G}''^F \rightarrow \hat{\bar{G}}^F$ defined by restriction. Let $\rho' \in \hat{G}^F$ and let $\rho'' \in \hat{G}''^F$ be such that $\rho''|_{G^F} = \rho'$. Let $\bar{\rho} = \rho''|_{\bar{G}^F}$. By (a), $\bar{\rho}|_{G^F}$ is multiplicity free; the restrictions $\rho'|_{G^F}$, $\bar{\rho}|_{G^F}$ coincide, hence $\rho'|_{G^F}$ is multiplicity free.

(c) Assume that G is simply connected. Let $G \rightarrow \bar{G} \rightarrow G''$, $G' \rightarrow G''$ be as in 8(c). Arguing as in (b) we see that we can replace G' by $\bar{G}$ in which case we can use the case (b).

(d) Assume that G is semisimple. Let $\tilde{G} \rightarrow \tilde{G}'$ be as in 8(d). Then

$$\begin{array}{c} \downarrow \quad \downarrow \\ G \rightarrow G' \end{array}$$

$\tilde{\rho}' = \rho'|_{\tilde{G}^F}$ is irreducible since $\tilde{G}^F \rightarrow G^F$ is surjective. By (c), $\tilde{\rho}'|_{\tilde{G}^F}$ is multiplicity free. But $(\rho'|_{G^F})|_{\tilde{G}^F} = \tilde{\rho}'|_{\tilde{G}^F}$. Hence $\rho'|_{G^F}$ is multiplicity free.

(e) We now consider the general case. Let G'' be the derived group of G . By (d), $\rho'|_{G''^F}$ is multiplicity free. But $(\rho'|_{G^F})|_{G''^F} = \rho'|_{G''^F}$ hence $\rho'|_{G^F}$ is multiplicity free. This completes the proof.

11. Proof of Proposition 5.1. Let $G \rightarrow G'$ be a regular imbedding (Sec.7), and let $s' \in G'^{\star F}$ be an element which maps to $s \in G^{\star F}$ under the corresponding homomorphism $G'^{\star} \rightarrow G^{\star}$. Let $H' = Z_{G', \star}^O(s')$.

The action of G^F/G^F on $\hat{G}^F$ (see Sec.9) factors through the action of

$G_{\text{ad}}^F/\pi(G^F)$ in Sec.3. (We have a natural surjective homomorphism $G^F/G^F \rightarrow G_{\text{ad}}^F/\pi(G^F)$).

Hence in the statement of 5.1. we can replace "orbits of $(\mathbb{Z}_G/\mathbb{Z}_G^O)_F$ " by "orbits of G^F/G^F ". The map ψ in the proposition is defined as the composition

$$\begin{aligned}
 & (\hat{G}^F)_S \\
 & \downarrow \\
 & (\hat{G}^F)_S \text{ mod action of } G^F/G^F \\
 & \downarrow \quad \quad \quad , \text{ see Sec. 10 and 9(a)} \\
 & \bigcup_{k \in K^F} (\hat{G}^F)_{S'k} \text{ mod action of } K^F \cong (G^F/G^F)^\sim \\
 & \downarrow \quad \quad \quad , \text{ see Sec. 8} \\
 & (\hat{G}^F)_S, \text{ mod action of } K_S^F = H^F/H^{OF} \\
 & \downarrow \quad \quad \quad , \text{ see Sec. 8.1} \\
 & (\hat{H}^{OF})_1 \text{ mod action of } H^F/H^{OF} \\
 & \downarrow \quad \quad \quad , \text{ see Sec. 8} \\
 & (\hat{H}^{OF})_1 \text{ mod action of } H^F/H^{OF}.
 \end{aligned}$$

The properties of ψ follow easily from 9(a) ; for the multiplicity formula we use that :

$$R_T^G(s) = R_{T'}^{G'}(s')|_{G^F} \quad (T' = \text{inverse image of } T \subset G'^\star \rightarrow G^\star)$$

$$\text{ind}_{G^F}^{G'^F}(\rho) = \sum_{\rho' \in \theta'} \rho' + \text{representation of } G'^F \text{ outside } (\hat{G}^F)_S, \quad ,$$

where $\theta' \subset (\hat{G}^F)_S$, is the H^F/H^{OF} -orbit determined by ρ . θ' corresponds to a H^F/H^{OF} orbit θ'_1 on $(\hat{H}^{OF})_1$ and to a H^F/H^{OF} orbit θ on $(\hat{H}^{OF})_1$. We have

$$\begin{aligned}
 (\rho : R_T^G(s))_{G^F} &= (\rho : R_{T'}^{G'}(s')|_{G^F})_{G^F} \\
 &= (\text{ind}_{G^F}^{G'^F}(\rho) : R_{T'}^{G'}(s'))_{G^F} \\
 &= \left(\sum_{\rho' \in \theta'} \rho' : R_{T'}^{G'}(s') \right)_{G^F}
 \end{aligned}$$

$$\begin{aligned}
 &= \varepsilon_{G'} \varepsilon_{H'} \sum_{\rho_1' \in \Theta_1'} (\rho_1' : R_{T'}^{H'^O}(1))_{H', OF} \\
 &\quad \text{by 6.1, 8.1,} \\
 &= \varepsilon_G \varepsilon_H \sum_{\bar{\rho} \in \bar{\Theta}} (\bar{\rho} : R_T^{H^O}(1))_{H^{OF}} .
 \end{aligned}$$

This completes the proof.

12. In the setup of Sec.4, we say that an irreducible representation of H^F is unipotent if its restriction to H^{OF} is a sum of unipotent representations of H^{OF} . Let $(\hat{H}^F)_1$ be the set of unipotent representations of H^F . It is easy to see that 9(a) (for $H^{OF} \subset H^F$) provides a surjective map

$$\psi' : (\hat{H}^F)_1 \rightarrow (\hat{H}^{OF})_1 \text{ mod action of } H^F/H^{OF}$$

with the following property : the fibres of ψ' and ψ over the same point have the same cardinal. Hence there exists a bijection

$$\psi'' : (\hat{G}^F)_S \xrightarrow{\sim} (\hat{H}^F)_1$$

such that $\psi = \psi' \circ \psi''$.

13. The parametrizations of $\hat{G}^F$ considered here and in [4] are to some extent non-canonical. It is likely that these will be canonical when they will be related to character sheaves. Note also that the crucial part of our proof (the multiplicity 1 statement in Sec. 10 for $\text{Spin}_{4n}(\mathbb{F}_q)$) involves some very long and unpleasant computations of the number of conjugacy classes and unpleasant computations of the number of conjugacy classes and irreducible representations of $\text{Spin}_{4n}(\mathbb{F}_q)$. Although these computations give the desired results, they don't show why the result holds. One can give a somewhat more satisfactory proof, using character sheaves on Spin_{4n} .

14. The parametrization of $\hat{G}^F$ given in [5] is in terms of a dual group over $\mathbb{C}$ rather than over $\mathbb{F}_q$; however that parametrization is equivalent to the one given here.

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