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VLADIMIR DOBRIC
MICHAEL B. MARCUS
MICHEL WEBER

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The Distribution of Large Values of the Supremum of a Gaussian Process

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I. Introduction. Let $\{X(t), t \in T\}$, T some index set, be a real separable centered Gaussian process with $\sup_{t\in T} X(t) < \infty$ a.s. In this paper we will be concerned with estimates of $P(\sup_{t\in T} X(t) > u)$ for large values of u in the case when $EX^2(t)$ achieves its maximum value at a finite number of points. Our results are extensions of a recent theorem of Talagrand (Theorem 1.2 below) which is based on Borell's inequality. As an interesting application of Talagrand's result and ours we get sharp estimates of the tail probability of the ℓ^p norms of sequences of independent normal random variables that we do not think are obtainable by more direct or elementary methods. Before presenting our results we will place them in the context of some recent work on this subject.

When $\sup_{t\in T} X(t) < \infty$ a.s., $P(\sup_{t\in T} X(t) > u)$ is "almost" the same as the probability that $\sigma g > u$ where

$$\sigma^2 = \sup_{t\in T} EX^2(t) \tag{1.1}$$

and g is a normal random variable with mean zero and variance 1. A great deal of work has gone into making the word "almost" precise. In 1970 Landau, Shepp and Marcus [**9**], [**11**] and independently Fernique [**4**] (see also [**8**, Chapter II, Theorem 4.8]) showed that for all $\epsilon > 0$ and $u > u(\epsilon)$ sufficiently large

$$P\left(\sup_{t\in T} X(t) > u\right) \le exp\left(-\frac{u^2}{2\sigma^2 + \epsilon}\right) \tag{1.2}$$

Note that (1.2) is already sharp enough to give the standard type of large deviation result for the supremum of Gaussian processes, i.e. that $\sup_{t\in T} X(t) < \infty$ a.s. implies

$$\lim_{u\to\infty} \frac{\log P\left(\sup_{t\in T} X(t) > u\right)}{u^2} = -\frac{1}{2\sigma^2} \tag{1.3}$$

The best possible general result on the distribution of $\sup_{t\in T} X(t)$ is given by the well known Theorem of C. Borell [**1**].

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THEOREM 1.1 (BORELL). *Let $\{X(t), t \in T\}$, T some index set, be a real separable centered Gaussian process and let σ be as defined in (1.1). Set*

$$\psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-v^2/2}\, dv \tag{1.4}$$

and assume that $P(\sup_{t\in T} X(t) \geq \omega) \leq 1/2$. Then for all $u \geq \omega$

$$P\left(\sup_{t\in T} X(t) > u\right) \leq \psi\left(\frac{u-\omega}{\sigma}\right) \tag{1.5}$$

and for all u

$$P\left(\sup_{t\in T} X(t) > u\right) \leq 2\psi\left(\frac{u-\omega}{\sigma}\right) \tag{1.6}$$

From now on we shall assume that $\sigma = 1$. We note that (1.2) implies that there exists a constant $C(\epsilon)$, depending on ϵ , such that

$$P\left(\sup_{t\in T} X(t) > u\right) \leq C(\epsilon) e^{\epsilon u^2} \psi(u) \qquad \forall \epsilon > 0 \tag{1.7}$$

whereas Theorem 1.1 shows that

$$P\left(\sup_{t\in T} X(t) > u\right) \leq C(\omega) e^{\omega u} \psi(u) \tag{1.8}$$

for some constant $C(\omega)$, depending on ω. The bound in (1.8) can not be improved, however, it is too strong in general. Let $\{B(t), t \in [0,1]\}$ be Brownian motion. Then, as is well known, for $\lambda \geq 0$

$$P(\sup_{t\in[0,1]} B(t) > \lambda) = 2P(B(1) > \lambda) = 2\psi(\lambda)$$

There are many specific results for stationary Gaussian processes that improve upon (1.8). For example in 1969, Pickands [12] obtained sharp estimates for the distribution of the large values of the supremum of $\{Y(t), t \in [0,1]\}$, a centered stationary Gaussian process with $EY^2(t) = 1$ and

$$(E|Y(s) - Y(t)|^2)^{1/2} = \sqrt{2}\,|s-t|^\alpha \qquad 0 < \alpha \leq 1 \tag{1.9}$$

which showed, in particular, that

$$P(\sup_{t\in[0,1]} Y(t) > \lambda) \sim \lambda^{1/\alpha} \psi(\lambda) \qquad \text{as } \lambda \to \infty$$

(We write $a \sim b$ to indicate that there exist constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$). Results of this sort have also been obtained by Weber [14], [15]. One characterization of these early results is that they apply to specific processes with smooth increments variance as in (1.9) and do not give very much insight into the distribution of the maximum of Gaussian processes in general.

Recently Berman [**2**], [**3**] introduced the following problem. Describe those real separable centered Gaussian processes $\{X(t), t \in T\}$, T a compact metric space, with $\sup_{t\in T} EX^2(t) = 1$, such that

$$\lim_{u\to\infty} \frac{P(sup_{t\in T} X(t) > u)}{\psi(u)} = 1 \tag{1.10}$$

This problem has been given a complete solution by Talagrand [**13**] who showed that (1.10) is equivalent to a condition on the local modulus of continuity of $\{X(t), t \in T\}$ in the neighborhood of the element $\tau \in T$ for which $EX^2(\tau) = 1$. (Note that, necessarily, the maximum of $EX^2(t)$ can occur at, at most, one value of $t \in T$. Since, if $EX^2(t_1) = EX^2(t_2) = 1$ and $E|X^2(t_1) - X^2(t_2)| \neq 0$ then

$$\lim_{\lambda\to\infty} \frac{P(X(t_1) \vee X(t_2))}{\psi(\lambda)} = 2 \tag{1.11}$$

This is easy to check. See also Lemma 4.2).

THEOREM 1.2 (TALAGRAND). *Let $\{X(t), t \in T\}$, T a compact metric space, be a real separable centered Gaussian process with continuous covariance. Assume that $\{X(t), t \in T\}$ has almost surely bounded sample paths so that $\sup_{t\in T} X(t) < \infty$ a.s. Then (1.10) is equivalent to the following conditions:*

(1) *There exists a unique $\tau \in T$ such that $\sup_{t\in T} EX^2(t) = EX^2(\tau) = 1$, and*
(2) $E\sup_{a(t)\geq 1-h^2}(X(t) - a(t)X(\tau)) = o(h)$

where $a(t) = EX(t)X(\tau)$.

We were able to find the following Corollary of Theorem 1.2 which heightened our interest in Talagrand's result.

COROLLARY 1.3. *Let $2 < p < \infty$ and $\{g_k\}_{k=1}^{\infty}$ be independent normal random variables with mean zero and variance σ_k^2 where $1 = \sigma_1 > \sigma_2 \geq \sigma_3 \geq \dots$ and $\sum_{k=1}^{\infty} \sigma_k^p \leq \infty$, so that $\sum_{k=1}^{\infty} |g_k|^p < \infty$ a.s. Then*

$$\lim_{u\to\infty} \frac{P\left(\left(\sum_{k=1}^{\infty} |g_k|^p\right)^{1/p} > u\right)}{\psi(u)} = 2 \tag{1.12}$$

If $1 = \sigma_1 = \sigma_2 = \dots = \sigma_n > \sigma_{n+1} \geq \sigma_{n+2} \geq \dots$, then

$$\lim_{u\to\infty} \frac{P\left(\left(\sum_{k=1}^{\infty} |g_k|^p\right)^{1/p} > u\right)}{\psi(u)} = 2n \tag{1.13}$$

We have not been able to find these results in the literature and think that they are new and not obtainable by classical real variable methods. It seems useful to give a proof of (1.12) in order to motivate the work in this paper. The proof of (1.13) will be given in Section IV.

PROOF OF COROLLARY 1.3, (1.12): That 2 is a lower bound in (1.12) is obvious since it is achieved by $|g_1|$ alone. Let q satisfy $1/p + 1/q = 1$ and define

$$T = \left\{ \{b_k\} : \sum_{k=1}^{\infty} \left| \frac{b_k}{\sigma_k} \right|^q \leq 1 \right\}$$

Since $\sum_{k=1}^{\infty} \sigma_k^p < \infty$, T is a compact subset of ℓ^q. Let $\{\eta_k\}_{k=1}^{\infty}$ be i.i.d. normal random variables with mean zero and variance 1 and $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers. Observe that

$$\sup_{\{b_k\} \in T} \sum_{k=1}^{\infty} b_k \eta_k \stackrel{\mathcal{D}}{=} \left(\sum_{k=1}^{\infty} |g_k|^p \right)^{1/p} \tag{1.14}$$

where " $\stackrel{\mathcal{D}}{=}$ " denotes equality in distribution. Define

$$X(\{b_k\}) = \sum_{k=1}^{\infty} b_k \eta_k \tag{1.14a}$$

and consider $\{ X(\{b_k\}), \{b_k\} \in T \}$. Note that

$$\sup_{\{b_k\} \in T} EX^2(\{b_k\}) = \sup_{\{b_k\} \in T} \sum_{k=1}^{\infty} b_k^2 = \sigma_1^2 = 1 \tag{1.14b}$$

Furthermore we see that $X(\{b_k\})$ has variance 1 at

$$(1, 0, 0, \ldots) = \tau^+$$

and

$$(-1, 0, 0, \ldots) = \tau^-$$

Let $T^+ = (b_1 \geq 0) \cap T$ and $T^- = (b_1 \leq 0) \cap T$. In order to complete the proof of (1.12) it is enough to show that

$$\lim_{u \to \infty} \frac{P\left(\sup_{\{b_k\} \in T^+} \sum_{k=1}^{\infty} b_k \eta_k > u\right)}{\psi(u)} = 1 \tag{1.15}$$

This we do by applying Theorem 1.2. Note that $X(\tau^+) = \eta_1$. Therefore for $t = \{b_k\}_{k=1}^{\infty} \in T^+$

$$a(t) = EX(t)X(\tau^+) = b_1$$

and

$$X(t) - a(t)X(\tau^+) = \sum_{k=2}^{\infty} b_k \eta_k \tag{1.16}$$

We now evaluate the term in Theorem 1.2, (2).

$$E \sup_{a(t)\geq 1-h^2,\{b_k\}\in T^+} \sum_{k=2}^{\infty} b_k\eta_k = E \sup_{b_1\geq 1-h^2,\{b_k\}\in T^+} \sum_{k=2}^{\infty} b_k\eta_k \tag{1.17}$$

$$= E \sup_{\sum_{k=2}^{\infty}\left|\frac{b_k}{\sigma_k}\right|^q \leq qh^2+o(h^2)} \left(\sum_{k=2}^{\infty}\left|\frac{b_k}{\sigma_k}\right|^q\right)^{1/q}\left(\sum_{k=2}^{\infty}|g_k|^p\right)^{1/p} = C(p,h)\left(\sum_{k=2}^{\infty}\sigma_k^p\right)^{1/p} h^{2/q}$$

where $\lim_{h\to\infty} C(p,h) = C'(p)$ is a constant depending only on p. Combining (1.16) and (1.17) we see that for $p > 2$ (i.e. $q < 2$)

$$E \sup_{a(t)\geq 1-h^2,\{b_k\}\in T^+} (X(t) - a(t)X(\tau^+)) \sim h^{2/q} \qquad \text{as } h \to 0 \tag{1.18}$$

Therefore, (1.15) follows from Theorem 1.2. This completes the proof of (1.12) of Corollary 1.3.

We see from (1.18) that condition (2) of Theorem 1.2 is not satisfied when $p = q = 2$. This raises the question, can Theorem 1.2 be extended to include this case or, more generally, in the notation of Theorem 1.2, what is the relationship between

$$L(h) = E \sup_{a(t)\geq 1-h^2} (X(t) - a(t)X(\tau)) \tag{1.19}$$

and functions $\ell(u)$ such that

$$\lim_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\ell(u)\psi(u)} \leq 1, \quad (\geq 1) \tag{1.20}$$

These questions are taken up in this paper.

In our extension of Theorem 1.2 we are forced to introduce another condition. As in Theorem 1.2 assume that $1 = EX^2(\tau) > EX^2(t)$ for $t \neq \tau$. Recall that $a(t) = EX(t)X(\tau)$ and note that

$$\sup_{a(t)\geq 1-h^2} E(X(t) - a(t)X(\tau))^2 = \sup_{a(t)\geq 1-h^2} EX^2(t) - a^2(t) \leq 2h^2 \tag{1.21}$$

We will require that there exists an $\epsilon > 0$ such that

$$\sup_{a(t)\geq 1-h^2} E(X(t) - a(t)X(\tau))^2 \leq (2-\epsilon)h^2 \qquad \forall h \in [0,\bar{h}] \tag{1.22}$$

for some constant $\bar{h} > 0$. Condition (1.22) is satisfied under the hypotheses of Theorem 1.2 since

$$\begin{aligned} E \sup_{a(t)\geq 1-h^2} (X(t) - a(t)X(\tau)) &\geq \frac{1}{2} \sup_{a(t)\geq 1-h^2} E|X(t) - a(t)X(\tau)| \\ &\geq \frac{1}{\sqrt{2\pi}} \sup_{a(t)\geq 1-h^2} \left(E|X(t) - a(t)X(\tau)|^2\right)^{1/2} \end{aligned} \tag{1.23}$$

Therefore whenever the left side of (1.23) is $o(h)$, (1.22) is satisfied.

We can now give a sample of the results obtained in Sections II and III.

THEOREM 1.4. *Let $\{X(t), t \in T\}$, T a compact metric space, be a real separable centered Gaussian process with continuous covariance and almost surely bounded sample paths. Assume that there exists a unique $\tau \in T$ such that $\sup_{t\in T} EX^2(t) = EX^2(\tau) = 1$. Let $\omega_i(h)$, $i = 1,2$, $\omega_i(0) = 0$, be concave for $h \in [0, \bar{h}]$ for some $\bar{h} > 0$. Define*

$$h_i(u) = \sup\left\{ h : \frac{\omega_i(h)}{h^2} = u \right\} \qquad i = 1,2 \tag{1.24}$$

Then if, for $h \in [0, \bar{h}]$,

$$\omega_1(h) \leq E \sup_{a(t)\geq 1-h^2} (X(t) - a(t)X(\tau)) \leq \omega_2(h) \tag{1.25}$$

where $a(t) = EX(t)X(\tau)$, and if (1.22) is satisfied, there exist constants k_1, k_2 such that for all $u > u_0$ sufficiently large

$$e^{k_1 u\omega_1(h_1(u))} \leq \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\psi(u)} \leq e^{k_2 u\omega_2(h_2(u))} \tag{1.26}$$

If

$$\limsup_{h\to 0} \frac{1}{\omega_1(h)} E \sup_{a(t)\geq 1-h^2} (X(t) - a(t)X(\tau)) \geq 1 \tag{1.27}$$

then, $\forall\epsilon > 0$

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\psi(u) exp(\, k_1(1-\epsilon)u\omega_1(h_1(u))\,)} \geq 1 \tag{1.28}$$

More information on the constants k_1 and k_2 are given in Sections II and III. In these Sections results are also obtained under less restrictive conditions on $\omega_1(h)$ than concavity. Moreover the conditions that (1.22) holds or that $\sup_{t\in T} E^2X(t) = 1$ for only one element $\tau \in T$ are only used for the upper bound in (1.26). These conditions are not used at all in Section II in which we consider the lower bounds in (1.26) and (1.28).

As an example of Theorem 1.4 suppose that

$$\omega_2(h) = Ch^\alpha \qquad 0 < \alpha \leq 1 \tag{1.29}$$

in (1.25), for some constant C. Then

$$h_2(u) = \left(\frac{C}{u}\right)^{1/(2-\alpha)} \qquad \text{and} \qquad u\,\omega_2(h_2(u)) = C^{2/(2-\alpha)}u^{(2-2\alpha)/(2-\alpha)} \tag{1.30}$$

If $\alpha = 1$, $u\,\omega_2(h_2(u)) = C^2$. Therefore, by these remarks and (1.26), we see that $\omega_2(h) \leq Ch$, for $h \in [0, \bar{h}]$ for some $\bar{h} > 0$, implies

$$1 \leq \limsup_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\psi(u)} \leq e^{kC^2} \tag{1.31}$$

for some constant k independent of C. If $L(h) = o(h)$, C can be taken as close to zero as we wish. Thus (1.26) shows that (1) and (2) of Theorem 1.2 imply (1.10).

In Section II we consider the relation between $\ell(u)$ and $L(h)$ so that upper bounds in (1.20) imply upper bounds for $L(h)$. These are used in a contrapositive argument to show that lower bounds for $L(h)$ imply lower bounds for the limit in (1.20). The lower bound in (1.26) is proved in this Section. In Section III we consider the situation in which upper bounds for $L(h)$ imply upper bounds for the limit in (1.20). The upper bound in (1.26) is obtained in this Section. In both of these Sections results are also obtained under less restrictive hypotheses than the ones used in Theorem 1.4. Our proofs closely follow Talagrand's proof of Theorem 1.2 but are more precise because we are considering a more general situation. Our main innovation is to recognize the significance of (1.22) for this method of proof.

Section IV is devoted to examples. We prove Corollary 1.3 and consider the tail of the probability distribution of $(\sum_{k=1}^n |g_k|^p)^{1/p}$ for $1 \le p \le 2$, where $\{g_k\}_{k=1}^n$ are independent normal random variables with mean zero and variance σ_k^2 . Our estimates in the case $p \le 2$ do not follow from Theorem 1.2 but require an extension of Theorem 1.4 to cover the case in which $L(h) \sim h$ and the variance of the process has a finite number of maxima.

More conventional Gaussian processes for which the variance has a unique maximum are cosine transforms of time changed Brownian motion. Let $\{W(t), t \in [0,1]\}$ be a stationary Gaussian process with mean zero and $EW^2(t) = 1$. It is well known that a version of such a process is given by

$$W(t) = \int_0^\infty \cos \lambda t \, dB(F(\lambda)) + \int_0^\infty \sin \lambda t \, dB'(F(\lambda)) \tag{1.32}$$

where B and B' are independent Brownian motions and F is a distribution function on $[0,\infty)$ which uniquely determines the process. Let

$$X(t) = \int_0^\infty \cos \lambda t \, dB(F(\lambda)) \qquad t \in [0,1] \tag{1.33}$$

Clearly, $EX^2(0) = 1$ and $EX^2(t) < 1$, for $t \in (0,a]$ for some $a > 0$, as long as F does not have a jump of size 1 at $\{0\}$. Moreover, we will show in Section IV that under mild conditions on $E|X(t+u) - X(t)|^2$, $\{X(t), t \in [0,a]\}$ satisfies (1.22). For different distribution functions F we obtain processes of the type $\{X(t), t \in [0,a]\}$ with a wide range of associated functions $L(h)$ as in (1.19), (with $\tau = 0$). For these processes Theorem 1.4 gives examples of a wide range of functions $k(u)$ such that

$$0 < \liminf_{u\to\infty} \frac{K(u)}{k(u)} \le \limsup_{u\to\infty} \frac{K(u)}{k(u)} < \infty \tag{1.34}$$

where

$$K(u) = \log\left(\frac{P\left(\sup_{t\in T} X(t) > u\right)}{\psi(u)}\right)$$

This is also done in Section IV.

V. Dobric and M. Weber held visiting positions at the City College of CUNY and The Courant Institute of Mathematical Sciences, respectively, while this research was carried out. They are grateful for the hospitality they received during their visits.

II. Lower bounds. Let $\{X(t),\ t \in T\}$, T some index set, be a real bounded separable centered Gaussian process normalized so that

$$\sup_{t \in T} EX^2(t) = 1 \tag{2.1}$$

Furthermore assume that $T' = \{t : EX^2(t) = 1\}$ is not empty and chose some $\tau \in T'$. Following Talagrand [**13**] we define

$$a(t) = EX(t)X(\tau) \tag{2.2}$$

$$Z(t) = X(t) - a(t)X(\tau) \tag{2.3}$$

and, for $0 \leq h < 1$,

$$T_h = \{t : a(t) \geq 1 - h^2\} \tag{2.4}$$

We will consider

$$L(h) = E \sup_{t \in T_h} Z(t) \tag{2.5}$$

and

$$S(h) = \sup_{t \in T_h} \left(EZ^2(t)\right)^{1/2} \tag{2.6}$$

Let us note that since $Z(t) = X(t) - X(\tau) + (1 - a(t))X(\tau)$, it follows, since $\tau \in T_h$, that

$$\frac{1}{2} E \sup_{t \in T_h} |X(t) - X(\tau)| - h^2 \leq L(h) \leq E \sup_{t \in T_h} |X(t) - X(\tau)| + h^2 \tag{2.6a}$$

i.e. that in most interesting cases $L(h)$ is equivalent to the expected value of the local modulus of continuity of $X(t)$ in the neighborhood of $X(\tau)$. We also define

$$\varphi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) \tag{2.7}$$

The following lemma, along with Theorem 1.1, states several inequalities that are critical in this paper.

LEMMA 2.1. *Let $\{Y(t),\ t \in T\}$, T some index set, be a real separable centered Gaussian process. Assume that $P(\sup_{t\in T} Y(t) \geq \omega) \leq \frac{1}{2}$ and let $\sigma = \sup_{t\in T} E(Y^2(t))^{\frac{1}{2}}$. Then*

$$E\sup_{t\in T} Y(t) \leq \omega + \frac{\sigma}{\sqrt{2\pi}} \tag{2.8}$$

If $P(\sup_{t\in T}|Y(t)| \leq s) > 0$ then

$$E\sup_{t\in T}|Y(t)| \leq \frac{4s}{P(\sup_{t\in T}|Y(t)| \leq s)} \tag{2.9}$$

Also, for $\psi(u)$ as defined in (1.4)

$$\frac{1}{\sqrt{2\pi}}\frac{1}{u+1}exp\left(-\frac{u^2}{2}\right) \leq \psi(u) \leq \frac{4}{3}\frac{1}{\sqrt{2\pi}}\frac{1}{u+1}exp\left(-\frac{u^2}{2}\right) \qquad \forall u \geq 0 \tag{2.10}$$

and

$$\psi((a^2+b^2)^{\frac{1}{2}}) \leq \psi(a)exp\left(-\frac{b^2}{2}\right) \qquad \forall a, b > 0 \tag{2.11}$$

The first inequality is a simple consequence of (1.5) and is given in [**5**, Proposition 3.2.1]. The second inequality follows from an inequality of Fernique on the norm of a Gaussian process [**5a**]. The inequality in (2.10) is proved in [**6**] and the last inequality follows immediately from the change of variables $t = (b^2+y^2)^{1/2}$ in

$$\psi\left((a^2+b^2)^{1/2}\right) = \frac{1}{\sqrt{2\pi}}\int_{(a^2+b^2)^{1/2}}^{\infty} e^{-t^2/2}\,dt$$

The next lemma is a careful rendering of the first part of the proof of the Theorem in [**13**].

LEMMA 2.2. *Let $\{X(t), t \in T\}$, T some index set, be a real centered bounded separable Gaussian process with $EX^2(t) \leq 1$ and such that there exists a $\tau \in T$ for which $EX^2(\tau) = 1$. Define*

$$\theta = \sup_{t\in T}(1 - a(t)) \tag{2.12}$$

where $a(t)$ is given in (2.2) and let $m \geq$ median of $\sup_{t\in T} X(t)$. Then for all $u \geq 8(1 \vee m)$

$$E\sup_{t\in T} X(t) \leq \frac{2}{u}\log\left(2\frac{P(\sup_{t\in T} X(t) > u)}{\psi(u)} - 1\right) + \theta(u + \sqrt{2/\pi}) + \sqrt{\theta/\pi} \tag{2.13}$$

and

$$E\sup_{t\in T} X(t) \leq C\left(\frac{2}{u}\log\left(2\frac{P(\sup_{t\in T} X(t) > u)}{\psi(u)} - 1\right) + u\theta\right) + \sqrt{2/\pi}\,\theta \tag{2.14}$$

where $C = 20$.

PROOF: Define

$$\epsilon(u) = \frac{P(\sup_{t\in T} X(t) > u)}{\psi(u)} - 1 \qquad \forall u \in R \tag{2.15}$$

Note that if $\sup_{t\in T} E|X(t)-a(t)X(\tau)|^2 = 0$ then (2.13) and (2.14) are trivially true. If not then $\epsilon(u) > 0$, $\forall u \in R$. We shall assume that the latter is the case in the rest of this proof.

$$P\left(\sup_{t\in T} X(t) > u\right) = \int_{-\infty}^{\infty} P\left(\sup_{t\in T} X(t) > u \mid X(\tau) = y\right)\varphi(y)dy = \int_{-\infty}^{\infty} \eta(y)\varphi(y)dy \tag{2.16}$$

where

$$\eta(y) = P\left(\sup_{t\in T}(Z(t) + a(t)y) > u\right) \tag{2.17}$$

Since $\eta(y) \geq P(Z(\tau) + y > u)$ and $Z(\tau) = 0$ a.s., it follows that $\eta(y) = 1$ for $y \in (u,\infty)$. Therefore

$$\int_{-\infty}^{u} \eta(y)\varphi(y)dy = \epsilon(u)\psi(u) \tag{2.18}$$

and, for any $0 < v < u$

$$\delta(u,v) = \inf_{v\leq y\leq u} \eta(y) \leq \frac{\epsilon(u)\psi(u)}{\psi(v) - \psi(u)} \tag{2.19}$$

By (2.11)

$$\psi(u) \leq \psi(v)\exp\left(-\frac{u^2 - v^2}{2}\right)$$

so that

$$\delta(u,v) \leq \epsilon(u)\left(\exp\left(\frac{u^2 - v^2}{2}\right) - 1\right)^{-1} \tag{2.20}$$

We will show that for $u \geq 8(1 \vee m)$

$$u > \frac{\log(1 + 2\epsilon(u))}{4(\log 2)(1 \vee m)} \tag{2.21}$$

By (1.5) of Theorem 1.1 and (2.10)

$$\begin{aligned} P\left(\sup_{t\in T} X(t) > u\right) \leq \psi(u-m) &\leq \frac{4}{3\sqrt{2\pi}}\frac{1}{1+u-m}\exp\left(\frac{(u-m)^2}{2}\right) \\ &\leq \frac{4(1+u)}{3(1+u-m)}e^{um}\psi(u) \leq 2e^{um}\psi(u) \end{aligned}$$

and by (2.15) we see that for $u \geq 8(1 \vee m)$

$$\log(1 + 2\epsilon(u)) < 4(\log 2)(1 \vee m)u$$

Thus we have established (2.21) and can choose

$$v = u - \frac{2}{u}\log(1 + 2\epsilon(u)) \tag{2.22}$$

Notice that (2.21) and $u \geq 8(1 \vee m)$ gives

$$u^2 - v^2 = 4\log(1+2\epsilon(u))\left(1 - \frac{\log(1+2\epsilon(u))}{u^2}\right) \geq 4\log(1+2\epsilon(u))\left(1-\frac{\log 2}{2}\right) \tag{2.23}$$

and so we see from (2.20) that

$$\delta(u,v) \leq \frac{1}{4-2\log 2} < 0.4 = q \tag{2.24}$$

Therefore, there exists a $c \in [v,u]$ such that $\eta(c) \leq q$ or, equivalently, such that

$$\begin{aligned} 1-q \leq P\left(\sup_{t\in T}(Z(t)+a(t)c) \leq u\right) &\leq P\left(\sup_{t\in T} Z(t) \leq u - c + c\theta\right) \\ &\leq P\left(\sup_{t\in T} Z(t) < \frac{2\log(1+2\epsilon(u))}{u} + u\theta\right) \end{aligned} \tag{2.25}$$

where, we use (2.22) at the last step. By (2.8)

$$E\sup_{t\in T} Z(t) \leq \frac{2\log(1+2\epsilon(u))}{u} + u\theta + \sqrt{\theta/\pi} \tag{2.26}$$

where we also use

$$EZ^2(t) = 1 - a^2(t) \leq 2(1-a(t)) \leq 2\theta \tag{2.26a}$$

This gives us (2.13) since

$$\left|E\sup_{t\in T} X(t) - E\sup_{t\in T} Z(t)\right| \leq \sqrt{2/\pi}\,\theta \tag{2.27}$$

To obtain (2.14) let us notice that by symmetry and (2.25) it follows that

$$1-2q \leq P\left(\sup_{t\in T}|Z(t)| \leq \frac{2\log(1+2\epsilon(u))}{u} + u\theta\right)$$

Therefore, by (2.9) we have that

$$E\sup_{t\in T}|Z(t)| \leq 20\left(\frac{2\log(1+2\epsilon(u))}{u} + u\theta\right) \tag{2.27a}$$

which together with (2.27) gives (2.14).

COROLLARY 2.3. *Under the same hypotheses as in Lemma 2.2, suppose furthermore that for* $B \geq 1$

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\psi(u)} = B < \infty \tag{2.28}$$

then

$$\limsup_{h\to 0} \frac{1}{h} E \sup_{a(t)\geq 1-h^2} (X(t) - a(t)X(\tau)) \leq 2C\left(2\log(2B-1)\right)^{1/2} \tag{2.29}$$

where C is given in (2.14). In particular, when $B = 1$, this shows that (1.10) implies (2) of Theorem 1.2. (We already mentioned in Section I that (1.10) trivially implies (1) of Theorem 1.2.)

PROOF: Let $T_h = \{t \in T : a(t) \geq 1 - h^2\}$. We use (2.14) of Lemma 2.2 with $T = T_h$ and $u = \delta/h$ for some $\delta > 0$. Since $\theta = h^2$ for $T = T_h$ we have that for all $\epsilon > 0$ and $h \in [0, h(\epsilon)]$ for $h(\epsilon)$ sufficiently small

$$E \sup_{t \in T_h} X(t) \leq C\left(\frac{2h}{\delta}\log(2(B+\epsilon)-1) + h\delta\right) + \sqrt{2/\pi}\, h^2$$

Therefore, with $\delta = (2\log(2(B+\epsilon)-1))^{1/2}$ we get

$$\limsup_{h \to 0} \frac{1}{h} E \sup_{t \in T_h} X(t) \leq 2C\left(2\log(2(B+\epsilon)-1)\right)^{1/2}$$

and since this is true for all $\epsilon > 0$ we have

$$\limsup_{h \to 0} \frac{1}{h} E \sup_{t \in T_h} X(t) \leq 2C\left(2\log(2B-1)\right)^{1/2} \tag{2.30}$$

The statement in (2.29) follows from (2.30) by (2.27).

Let $\omega(h), h \in [0, \bar{h}]$ be a non-negative increasing real valued function that satisfies

$$\lim_{h \to 0} \frac{\omega(h)}{h^2} = \infty \qquad \text{and} \qquad \omega(\bar{h}) < \infty \tag{2.31}$$

We define

$$h_0(u) = \sup\left\{h : \frac{\omega(h)}{h^2} \geq u\right\} \tag{2.32}$$

Note that since $\omega(h)$ is increasing it follows that for those values $h_0(u)$ defined in (2.32)

$$\frac{\omega(\,h_0(u)\,)}{h_0^2(u)} = u \tag{2.33}$$

and, of course, $\lim_{u \to \infty} h_0(u) = 0$.

The next two Theorems are the main results of this section.

THEOREM 2.4. *Let $\{X(t), t \in T\}$, T some index set, be a bounded real valued separable centered Gaussian process with $EX^2(t) \leq 1$ and such that there exists a $\tau \in T$ for which $EX^2(\tau) = 1$. Suppose that*

$$L(h) \geq \omega(h), \qquad \forall h \in [0, \bar{h}] \qquad \text{for some } \bar{h} > 0 \tag{2.34}$$

where $L(h)$ is defined in (2.5). Then for all u sufficiently large

$$P\left(\sup_{t \in T} X(t) > u\right) \geq \frac{1}{2}\psi(u)\left(\exp\left(\frac{u}{4C}\omega(h_0(2Cu))\right) + 1\right) \tag{2.35}$$

where C is the constant in (2.14).

Moreover, if in addition to (2.34)

$$\lim_{h\to 0}\frac{\omega(h)}{h}=\infty \tag{2.36}$$

then, $\forall\epsilon>0$ there exists $u_0(\epsilon)$ such that for $u\geq u_0(\epsilon)$

$$P\left(\sup_{t\in T}X(t)>u\right)\geq\frac{1}{2}\psi(u)\left(\exp\left(\frac{(1-\epsilon)u}{4}\omega(h_0(2u))\right)+1\right) \tag{2.37}$$

PROOF: Let T_h be as given in Corollary 2.3 and note that in this case $\theta=h^2$. We use (2.27a) with $T=T_h$ to get, for $u>8(1\vee m)$, that

$$L(h)=E\sup_{t\in T_h}Z(t)\leq C\left(\frac{2}{u}\log\left(2\frac{P\left(\sup_{t\in T}X(t)>u\right)}{\psi(u)}-1\right)+uh^2\right) \tag{2.38}$$

or, equivalently, that

$$P\left(\sup_{t\in T_h}X(t)>u\right)\geq\frac{1}{2}\left(\exp\left(\frac{u}{2}\left(\frac{L(h)}{C}-h^2u\right)\right)+1\right)\psi(u) \tag{2.39}$$

(As above, to avoid trivialities, we assume $\epsilon(u)>0$, $\forall u\in R$). Now, by (2.34) and the fact that $T_h\subset T$, we have for all $h\in[0,\bar{h}]$ that

$$P\left(\sup_{t\in T}X(t)>u\right)\geq\frac{1}{2}\left(\exp\left(\frac{u}{2}\left(\frac{\omega(h)}{C}-h^2u\right)\right)+1\right)\psi(u) \tag{2.40}$$

For u sufficiently large we take $h=h_0(2Cu)$. Then by (2.33) we get (2.35).

Let us now assume that (2.36) holds in addition to (2.34). We see from (2.26) and (2.15) that

$$L(h)\leq\frac{2}{u}\log\left(2\frac{P\left(\sup_{t\in T}X(t)>u\right)}{\psi(u)}-1\right)+uh^2+\frac{h}{\sqrt{\pi}}$$

As above, this implies that

$$P\left(\sup_{t\in T}X(t)>u\right)\geq\frac{1}{2}\left(\exp\left(\frac{u}{2}\left(\omega(h)-h^2u-\frac{h}{\sqrt{\pi}}\right)\right)+1\right)\psi(u) \tag{2.41}$$

Setting $h=h_0(2u)$ in (2.41) and recalling (2.33) we have that (2.41)

$$\geq\frac{1}{2}\left(\exp\left(\frac{u}{2}\left(\frac{\omega(h_0(2u))}{2}-\frac{h_0(u)}{\sqrt{\pi}}\right)\right)+1\right)\psi(u) \tag{2.42}$$

Therefore by (2.36), for all $\epsilon>0$, there exists a $u_0(\epsilon)$ sufficiently large so that (2.42)

$$\geq\frac{1}{2}\left(\exp\left(\frac{u(1-\epsilon)}{4}\omega(h_0(2u))\right)+1\right)\psi(u)$$

which is (2.37).

THEOREM 2.5. *Let $\{X(t), t \in T\}$, T some index set, be a bounded real valued separable centered Gaussian process with $EX^2(t) \le 1$ and such that there exists a $\tau \in T$ for which $EX^2(\tau) = 1$. Assume that there exists a decreasing sequence $\{h_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} h_k = 0$, such that*

$$L(h_k) \ge \omega(h_k) \qquad \forall\, k \ge 1 \tag{2.43}$$

where ω also satisfies

$$\frac{\omega(h)}{h^2} \quad \text{is strictly decreasing} \qquad \forall\, h \in [0, \bar{h}] \tag{2.44}$$

Then, for C the constant in (2.14)

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\frac{1}{2}\psi(u)\left(\exp\left(\frac{u}{4C}\omega(h_0(2Cu))\right) + 1\right)} \ge 1 \tag{2.45}$$

Moreover, if in addition to (2.43), (2.36) holds, then for all $\epsilon > 0$

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\frac{1}{2}\psi(u)\left(\exp\left(\frac{(1-\epsilon)u}{4}\omega(h_0(2u))\right) + 1\right)} \ge 1 \tag{2.46}$$

PROOF: Condition (2.44) implies that $\omega(h)/h^2$ is invertible for $h \in [0, \bar{h}]$ and that its inverse is continuous on $[0, \bar{h}]$. Thus, in particular, $h_0(2Cu)$ takes all values in $[0, h']$ for some $h' > 0$. Therefore, for all k sufficiently large we can choose u_k such that

$$h_0(2Cu_k) = h_k \tag{2.47}$$

It follows from (2.40) with $u = u_k$ and $h = h_k$ and (2.47) that

$$P\left(\sup_{t\in T} X(t) > u_k\right) \ge \frac{1}{2}\left(\exp\left(\frac{u}{4}\omega(h_0(2Cu_k))\right) + 1\right)\psi(u) \tag{2.48}$$

which gives (2.45). A similar argument applied to the second part of the proof of Theorem 2.4 gives (2.46).

REMARK 2.6: In the previous results we have the expressions $\omega(h_0(2Cu))$ and $\omega(h_0(2u))$. If ω is concave we can remove the constants $2C$ and 2 from the arguments of $h_0(\cdot)$, since, for all $k \ge 1$

$$ku\omega(h_0(ku)) = \frac{\omega^2(h_0(ku))}{h_0^2(ku)} \ge \frac{\omega^2(h_0(u))}{h_0^2(u)} = u\omega(h_0(u))$$

where we use the obvious fact that $h(ku) \le h(u)$. Therefore

$$\omega(h_0(ku)) \ge \frac{\omega(h_0(u))}{k} \tag{2.49}$$

We see that (2.49) along with Theorem 2.4 gives the left side of (1.26). Finally let us note that when ω is concave (2.44) is satisfied. Therefore (1.28) follows from Theorem 2.5.

III Upper Bounds. We continue with the notation defined in the beginning of Section II. The main result of this Section is the following:

THEOREM 3.1. *Let $\{X(t), t \in T\}$, T a compact metric space, be a real separable centered Gaussian process with continuous covariance. Assume that there exists a unique $\tau \in T$ such that $\sup_{t\in T} EX^2(t) = EX^2(\tau) = 1$. Let $\omega(h)$, $h \in [0,\bar{h}]$ be a non-decreasing function with $\omega(0) = 0$ and $\omega(\bar{h}) < \infty$. Assume that there exists an $\bar{h} > 0$ such that for all $h \in [0,\bar{h}]$*

$$L(h) \leq \beta\omega(h) \tag{3.1}$$

for some constant β, and a number $\eta < \sqrt{2}$ such that

$$S(h) \leq \eta h \tag{3.2}$$

Then there exist constants d_1 and $d_2 \geq 0$ such that

$$\limsup_{u\to\infty} \frac{P\left(\sup_{t\in T} X(t) > u\right)}{\psi(u)\left(C_\eta + \exp\left(d_1 u\, \omega(h_0(d_2 u))\right)\right)} \leq 1 \tag{3.3}$$

where $C_\eta > 0$ satisfies $\lim_{\eta\to 0} C_\eta = 0$, and h_0 is as defined in (2.32).

PROOF: Since the covariance $a(t) = EX(t)X(\tau)$ is continuous, for all $0 < \alpha < 1$ there exists a $0 < \delta < 1$ such that

$$\sup_{\{t:a(t)\leq 1-\alpha^2\}} EX^2(t) \leq 1-\delta \tag{3.4}$$

Therefore, by (1.6) of Theorem 1.1

$$P\left(\sup_{\{t:a(t)\leq 1-\alpha^2\}} X(t) > u\right) \leq 2\psi\left(\frac{u-\omega}{1-\delta}\right)$$

for some finite number ω, because $\{X(t), t\in T\}$ is almost surely bounded. Thus

$$\limsup_{u\to\infty} \frac{P\left(\sup_{\{t:a(t)\leq 1-\alpha^2\}} X(t) > u\right)}{\psi(u)} = 0 \tag{3.5}$$

Therefore, in order to obtain (3.3) we need only consider $P\left(\sup_{t\in T_\alpha} X(t) > u\right)$, where T_α is defined in (2.4). Following Talagrand [**13**], let $0 < \epsilon < 1$ and consider for $n = 0, 1, \ldots$

$$A_n = \left\{t \in T : a(t) \geq 1 - \epsilon^{2n}\alpha^2\right\} \tag{3.6}$$

$$B_n = A_n \setminus A_{n+1} \tag{3.7}$$

$$\omega_n = 2E \sup_{t \in A_n} Z(t) \tag{3.8}$$

$$E_u = \left\{ \omega : X(t) \le \inf_n \left(\frac{u - \omega_n}{1 - \epsilon^{2n+2}\alpha^2} \wedge u \right) \right\} \tag{3.9}$$

Note that

$$P\left(\sup_{t \in A_n} Z(t) \le \omega_n \right) \ge \frac{1}{2} \tag{3.9a}$$

We have

$$P\left(\sup_{t \in T_\alpha} X(t) > u \right) \le P\left(E_u^c\right) + P\left(\left(\sup_{t \in T_\alpha} X(t) > u \right) \cap E_u \right) \tag{3.10}$$

The first term on the right in (3.10)

$$P\left(E_u^c\right) = \sup_n \left\{ \psi\left(\frac{u - \omega_n}{1 - \epsilon^{2n+2}\alpha^2} \right) \vee \psi(u) \right\} \tag{3.11}$$

Note that if

$$u \le \frac{\omega_n}{\epsilon^{2n+2}\alpha^2} \qquad \text{then} \qquad \frac{u - \omega_n}{1 - \epsilon^{2n+2}\alpha^2} \le u$$

Therefore, in order to evaluate (3.11) we must consider

$$\sup_n \left\{ \psi\left(\frac{u - \omega_n}{1 - \epsilon^{2n+2}\alpha^2} \right) : u \le \frac{\omega_n}{\epsilon^{2n+2}\alpha^2} \right\} \tag{3.12}$$

Set

$$F_n(u) = u - \frac{\omega_n - \epsilon^{2n+2}\alpha^2 u}{1 - \epsilon^{2n+2}\alpha^2}$$

When $u \le \omega_n/(\epsilon^{2n+2}\alpha^2)$ we have, for $u \ge 1$

$$\begin{aligned} \psi\left(\frac{u - \omega_n}{1 - \epsilon^{2n+2}\alpha^2} \right) &= \psi(u) + \frac{1}{\sqrt{2\pi}} \int_{F(u)}^{u} e^{-s^2/2}\, ds \\ &\le \psi(u) + \frac{1}{\sqrt{2\pi}} \frac{\omega_n}{1-\alpha^2} \exp(-\frac{1}{2}F^2(u)) \le \psi(u)\left(1 + 2\frac{u\omega_n}{1-\alpha^2} \exp\left(\frac{u\omega_n}{1-\alpha^2}\right)\right) \end{aligned} \tag{3.13}$$

where we use (2.10) at the last step. We now observe that by (3.1) and (3.8)

$$\omega_n \le 2\beta\omega(\alpha\epsilon^n) \tag{3.14}$$

and so $u \le \omega_n/(\epsilon^{2n+2}\alpha^2)$ implies

$$u \le \frac{2\,\beta\,\omega(\alpha\epsilon^n)}{\epsilon^2(\alpha\epsilon^n)^2}$$

Therefore by (2.32)

$$\alpha\epsilon^n \le h_0\left(\frac{\epsilon^2 u}{2\beta}\right) \tag{3.15}$$

Putting (3.11), (3.13), (3.14) and (3.15) together we have

$$P(E_u^c) \le \psi(u)\left(1 + \frac{4\beta}{1-\alpha^2} u\,\omega(h_0(\frac{\epsilon^2 u}{2\beta}))\, exp(\frac{2\beta}{1-\alpha^2})\, u\,\omega(h_0(\frac{\epsilon^2 u}{2\beta}))\right) \tag{3.16}$$

Let

$$H(u) = \inf_n \left\{ \frac{u - \omega_n}{1 - \epsilon^{2n+2}\alpha^2} \wedge u \right\}$$

The second term on the right in (3.10)

$$\begin{aligned} P\left(\left(\sup_{t\in T_\alpha} X(t) > u\right) \cap E_u\right) &= \int_{-\infty}^{\infty} P\left(\sup_{t\in T_\alpha} X(t) > u \mid X(\tau) = y\right) \varphi(y) I_{[y\le H(u)]}(y)\,dy \\ &= \int_{-\infty}^{\infty} P\left(\sup_{t\in T_\alpha} (Z(t) + a(t)y) > u\right) \varphi(y) I_{[y\le H(u)]}(y)\,dy \end{aligned} \tag{3.17}$$

since $Z(t)$ and $X(\tau)$ are mutually independent. We will designate the last integral in (3.17) as $I_1 + I_2$ where

$$I_1 = \int_{-\infty}^{0} \qquad \text{and} \qquad I_2 = \int_0^u$$

Obviously

$$I_1 \le \frac{1}{2} P\left(\sup_{t\in T_\alpha} Z(t) > u\right)$$

Note that, by definition, $T_\alpha = A_0$, $\omega_0 = 2E\left(\sup_{t\in A_0} Z(t)\right)$ and by (3.2)

$$\sup_{t\in T_\alpha} \left(EZ^2(t)\right)^{1/2} = S(\alpha) \le \eta\alpha < 1$$

for α sufficiently small. Therefore, by Theorem 1.1, (1.5)

$$I_1 \le \psi\left(\frac{u - \omega_0}{\eta\alpha}\right) \qquad \forall u \ge \omega_0$$

so that

$$I_1 = o(\psi(u)) \qquad \text{as } u \to \infty \tag{3.18}$$

The key point in the proof is the estimation of I_2. We have

$$I_2 \le \sum_{n=0}^{\infty} \int_0^u I_{[y\le H(u)]}(y) P\left(\sup_{t\in B_n} Z(t) > u - y\left(1 - \epsilon^{2n+2}\alpha^2\right)\right) \varphi(y)\,dy$$

We set

$$\sigma_n = \sup_{t\in B_n} \left(EZ^2(t)\right)^{1/2} \le S(\epsilon^n\alpha) \qquad \forall n \ge 0 \tag{3.19}$$

and apply Theorem 1.1, (1.5) to get

$$I_2 \le \sum_{n=0}^{\infty} \int_0^u I_{[y\le H(u)]}(y)\,\psi\left(\frac{u - y\left(1 - \epsilon^{2n+2}\alpha^2\right) - \omega_n}{\sigma_n}\right) \varphi(y)\,dy \tag{3.20}$$

which, by the change of variables $y = u - z$

$$\le \sum_{n=0}^{\infty} \int_0^u I_{[g(u,z,n)\ge\omega_n]}(z)\, \psi\left(\frac{g(u,z,n)-\omega_n}{\sigma_n}\right) \varphi(u-z)\, dz$$

where we set

$$g(u,z,n) = u\epsilon^{2n+2}\alpha^2 + z\left(1-\epsilon^{2n+2}\alpha^2\right)$$

By (2.10), (3.19) and (3.2) the last line in (3.20)

$$\le \psi(u) \sum_{n=0}^{\infty} \int_0^u I_{[g(u,z,n)\ge\omega_n]}(z)\,(u+1)\exp\left(uz - \frac{(g(u,z,n)-\omega_n)^2}{2(\eta\epsilon^n\alpha)^2}\right) dz \tag{3.21}$$

$$\le 2\psi(u) \sum_{n=0}^{\infty} u \int_0^u \exp\left(uz - \frac{1}{2}(u\gamma_n + z\delta_n - C_n)^2\right) dz$$

where

$$\gamma_n = \frac{\epsilon^{n+2}\alpha}{\eta} \qquad \delta_n = \frac{1-\epsilon^{2n+2}\alpha^2}{\eta\epsilon^n\alpha} \qquad C_n = \frac{\omega_n}{\eta\epsilon^n\alpha} \tag{3.22}$$

Let us note that

$$uz - \frac{1}{2}(u\gamma_n + z\delta_n - C_n)^2$$
$$= -\frac{1}{2}(\gamma_n u - C_n)^2 + \frac{1}{2}\left(\frac{(1-\delta_n\gamma_n)u}{\delta_n} + C_n\right)^2 - \frac{1}{2}\left(\delta_n z - \left(\frac{(1-\delta_n\gamma_n)u}{\delta_n} + C_n\right)\right)^2$$

Substituting this into (3.21) and integrating on z from $-\infty$ to ∞ we get

$$I_2 \le 2\sqrt{2\pi}\,\psi(u) \sum_{n=0}^{\infty} \frac{u}{\delta_n} \exp\left(-\frac{1}{2}(\gamma_n u - C_n)^2 + \frac{1}{2}\left(C_n + \left(\frac{1-\delta_n\gamma_n}{\delta_n}\right)u\right)^2\right) \tag{3.23}$$

$$= 2\sqrt{2\pi}\psi(u) \sum_{n=0}^{\infty} \frac{u}{\delta_n} \exp\left(\frac{uC_n}{\delta_n} - \frac{1}{2}\left(\frac{2\delta_n\gamma_n - 1}{\delta_n^2}\right)u^2\right)$$

By (3.22)

$$\eta^2(2\gamma_n\delta_n - 1) \ge (2(1-\epsilon^2\alpha^2)\epsilon^2 - \eta^2) \equiv a_\eta > 0 \tag{3.24}$$

That is, we choose $0 < \epsilon < 1$ and $0 < \alpha < 1$ so that $a_\eta > 0$. This means that if η is close to $\sqrt{2}$, ϵ must be close to 1 and α must be close to zero.

We now see that

$$\frac{(2\delta_n\gamma_n - 1)}{\delta_n^2} \ge a_\eta \epsilon_n^{2n}\alpha^2 \tag{3.25}$$

By (3.14) and (3.22)–(3.25) we have

$$I_2 \le k(\alpha,n)\psi(u) \sum_{n=0}^{\infty} u\epsilon^n \exp\left(-\frac{1}{2}a_\eta\alpha^2u^2\epsilon^{2n} + \frac{2\beta}{1-\alpha^2}\,u\,\omega(\alpha\epsilon^n)\right) \tag{3.26}$$

where

$$k(\alpha, n) = 2\sqrt{2\pi}\,\frac{\alpha\eta}{1-\alpha^2}$$

We write

$$\begin{aligned}(3.27)\quad I_2 \le k(\alpha,n)\psi(u)\Bigg(&\sum_{\{n:u\ge C\frac{\omega(\alpha\epsilon^n)}{\epsilon^{2n}}\}} u\epsilon^n \exp\left(-\frac{1}{4}a_\eta \alpha^2 u^2 \epsilon^{2n}\right)\\ &+\sum_{\{n:u\le C\frac{\omega(\alpha\epsilon^n)}{\epsilon^{2n}}\}} u\epsilon^n \exp\left(\frac{2\beta}{1-\alpha^2}\,u\,\omega(\alpha\epsilon^n)\right)\Bigg) = k(\alpha,n)\psi(u)(I_3+I_4)\end{aligned}$$

where

$$C = \frac{8\beta}{\alpha^2(1-\alpha^2)a_\eta} \tag{3.28}$$

In order to estimate I_3 let us note that for $0<\epsilon<1$ and $\delta = (a_\eta^{1/2}\alpha)/2$ we have

$$\sum_{n=0}^{\infty} u\epsilon^n \exp\left(-(\delta u\epsilon^n)^2\right) \le \sum_{\{n:u\epsilon^n\delta\le 1\}} u\epsilon^n + \frac{1}{\delta}\sum_{\{n:u\epsilon^n\delta>1\}} (u\epsilon^n\delta)^{-1} \le \frac{2}{\delta}\sum_{n=0}^{\infty}\epsilon^n = \frac{2}{\delta(1-\epsilon)} \tag{3.29}$$

where we use the inequality $x^2e^{-x^2}\le 1$. Thus we have

$$I_3 \le \frac{4}{(\alpha^2 a_\eta)^{1/2}(1-\epsilon)} \tag{3.29a}$$

In order to estimate I_4 we define

$$m = \min\left\{n : \frac{C\omega(\alpha\epsilon^n)}{\epsilon^{2n}} \ge u\right\}$$

Then, obviously

$$\frac{\omega(\alpha\epsilon^{m-1})}{\epsilon^{2m-2}} < \frac{u}{C} \le \frac{\omega(\alpha\epsilon^m)}{\epsilon^{2m}} \tag{3.30}$$

(Note that if m is uniformly bounded for all u then there is nothing to prove).

By the definition of $h_0(u)$

$$h_0\left(\frac{u}{C\alpha^2}\right) \ge \alpha\epsilon^m \tag{3.31}$$

We now see, by (3.30) and the fact that $\omega(h)$ is non-decreasing, that

$$I_4 \le C\sum_{n\ge m}\frac{\omega(\alpha\epsilon^m)}{\epsilon^{2m}}\epsilon^n \exp\left(\frac{2\beta C\,\omega^2(\alpha\epsilon^m)}{(1-\alpha^2)\epsilon^{2m}}\right)$$

Therefore

$$I_4 \le C\left(\frac{\omega(\alpha\epsilon^m)}{\epsilon^m}\exp\left(\frac{2C\beta\omega^2(\alpha\epsilon^m)}{(1-\alpha^2)\epsilon^{2m}}\right)\right)\sum_{j=0}^{\infty}\epsilon^j = \frac{C\omega(\alpha\epsilon^m)}{(1-\epsilon)\epsilon^m}\exp\left(\frac{2C\beta\omega^2(\alpha\epsilon^m)}{(1-\alpha^2)\epsilon^{2m}}\right) \tag{3.32}$$

Furthermore, by (3.30) and (3.31)

$$\frac{u}{C} \ge \frac{\omega(\alpha\epsilon^{m-1})}{\epsilon^{2m-2}} \ge \epsilon^2\,\frac{\omega(\alpha\epsilon^m)}{\epsilon^{2m}} \tag{3.33}$$

Thus

$$\frac{\omega^2(\alpha\epsilon^m)}{\epsilon^{2m}} \le \frac{u\,\omega(\alpha\epsilon^m)}{\epsilon^2 C} \le \frac{u\,\omega\left(h_0\left(\frac{u}{C\alpha^2}\right)\right)}{\epsilon^2 C}$$

Substituting this in (3.32) we get

$$I_4 \le \frac{C^{1/2}}{(1-\epsilon)\epsilon}\left(u\,\omega(h_0(\frac{u}{C\alpha^2}))\right)^{1/2}\exp\frac{2\beta}{(1-\alpha^2)\epsilon^2}u\,\omega(h_0(\frac{u}{C\alpha^2})) \tag{3.34}$$

We can put this all together to get a bound for $P\left(\sup_{t\in T_\alpha}X(t)>u\right)$. Let

$$d_1' = \frac{\epsilon^2}{2\beta}\wedge 1 \tag{3.35}$$

and

$$d_2' = \frac{(1-\alpha^2)a_\eta}{8\beta}\wedge 1 \tag{3.36}$$

and notice that since $h_0(u)$ decreases as u increases. We have

$$h_0\left(\frac{\epsilon^2 u}{2\beta}\right) \le h_0(d_1' u) \tag{3.37}$$

and

$$h_0\left(\frac{u}{C\alpha^2}\right) \le h_0(d_2' u) \tag{3.38}$$

By (2.6a), (3.5), (3.10), (3.16), (3.18), (3.27), (3.28), (3.29a), (3.34), (3.37) and (3.38) we have

(3.39)

$$\begin{aligned}P\left(\sup_{t\in T}X(t)>u\right) \le \psi(u)\Bigg(1+\frac{4\beta}{1-\alpha^2}u\,\omega(h_0(d_1'u))\exp\frac{2\beta}{1-\alpha^2}\,u\,\omega(h_0(d_1'u))+\frac{8\sqrt{2\pi}\,\eta}{(1-\alpha^2)a_\eta^{1/2}}\\ +\frac{8\sqrt{\beta\pi}\,\eta}{(1-\alpha^2)^{3/2}a_\eta^{1/2}\epsilon(1-\epsilon)}(u\,\omega(h_0(d_2'u)))^{1/2}\exp\frac{2\beta}{\epsilon^2(1-\alpha^2)}\,u\,\omega(h_0(d_2'u))+o(\psi(u))\Bigg)\end{aligned}$$

for $u\ge u_0$ sufficiently large and $0<\alpha\le\alpha_0$ sufficiently small and $0<\epsilon<1$ large enough so that $a_\eta = \; 2(1-\epsilon^2\alpha^2)\epsilon^2-\eta^2 \;>0$.

If we take $d_2 = d_1' \vee d_2'$ and d_1 large enough we get (3.3). It seems that the constant

$$C_\eta = \frac{8\sqrt{2\pi}\,\eta}{(1-\alpha^2)a_\eta^{1/2}(1-\epsilon)} \tag{3.40}$$

is unavoidable. It is needed to cover the case when $\liminf_{h\to 0} \frac{\omega(h)}{h} = 0$ but $\limsup_{h\to 0} \frac{\omega(h)}{h} > 0$. If $\liminf_{h\to 0} \frac{\omega(h)}{h} > 0$, C_η can be absorbed into the exponential term by altering d_1.

REMARK 3.2: The constants d_1 and d_2 in Theorem 3.1 approach infinity as η approaches $\sqrt{2}$. If η is small these constants need not be too big. In order to obtain the upper bound in (1.26) from (3.3) we note that when ω is concave (and not identically 0) then it exceeds bh for some $b > 0$ for $h \in [0, \bar{h}]$ for some $\bar{h} > 0$. Therefore we can absorb the term C_η into the exponential term by making d_1 sufficiently large. Also note that by (2.49) if $d_2 \le 1$

$$\omega\left(h_0(d_2 u)\right) \le \frac{\omega\left(h_0(u)\right)}{d_2} \tag{3.41}$$

Of course, if $d_2 \ge 1$ then $\omega(h_0(d_2 u)) \le \omega(h_0(u))$. We have already pointed out in Section I that Theorem 1.4 shows that (1) and (2) of Theorem 1.2 imply (1.10).

IV. Examples. We have several interesting applications of Theorems 1.2 and 1.4. We begin with a very elementary lemma which we prove for the convenience of the reader.

LEMMA 4.1. *Let ξ and η be normal random variables with mean zero and variance 1 satisfying $E\xi\eta = a < 1$. Then*

$$P(\xi > u, \eta > u) = o(\psi(u)) \qquad \text{as } u \to \infty \tag{4.1}$$

where $\psi(u)$ is given in (1.4).

PROOF: Let $u > 0$. We have

$$P(\psi > u, \eta > u) = \int_u^\infty \psi\left(\frac{u - ay}{1-a^2}\right)\varphi(y)\,dy \tag{4.2}$$

where φ is defined in (2.7). If $a \le 0$ (4.2) is less than or equal to $\psi(u/(1-a^2))\psi(u)$. If $a > 0$ (4.2)

$$= \int_u^{\frac{u(1+a)}{2a}} \psi\left(\frac{u-ay}{1-a^2}\right)\varphi(y)\,dy + \int_{\frac{u(1+a)}{2a}}^\infty \psi\left(\frac{u-ay}{1-a^2}\right)\varphi(y)\,dy$$

$$\le \psi\left(\frac{u}{2(1+a)}\right)\psi(u) + \psi\left(\frac{u(1+a)}{2a}\right) = o(\psi(u))$$

as $u \to \infty$ since $a < 1$ and $\frac{1+a}{2a} > 1$.

The next lemma enables us to complete the proof of Corollary 1.3.

LEMMA 4.2. *Let $\xi_1, \cdots, \xi_n$ be normal random variables with mean zero and variance 1 such that $E\xi_i\xi_j < 1$, $i \neq j$. Then*

$$\lim_{u\to\infty} \frac{P(\sup_{1\le i\le n} \xi_i > u)}{\psi(u)} = n \tag{4.3}$$

PROOF: The upper bound is obvious since

$$P\left(\sup_{1\le i\le n} \xi_i > u\right) \le \sum_{i=1}^{n} P(\xi_i > u) = n\psi(u)$$

Let $A_i = \{\xi_i > u\}$. Then, as is well known,

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \ge \sum_{i=1}^{n} P(A_i) - \sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n} P(A_i \cap A_j) \tag{4.4}$$

The lower bound in (4.3) follows from (4.4) since for $i \neq j$

$$P(A_i \cap A_j) = o(\psi(u)) \qquad \text{as } u \to \infty$$

as we showed in Lemma 4.1.

PROOF OF COROLLARY 1.3, (1.13): We consider $X(\{b_k\})$ as defined in (1.14a), however, we now see that

$$\sup_{\{b_k\}\in T} EX^2(\{b_k\}) = 1$$

for the $2n$ sequences $\pm e_j$, $j = 1, \cdots, n$, where e_j is an element of the cannonical basis of R^n, (i.e. $e_j = (0, \cdots, 1, 0, \cdots)$ where 1 appears in the j–th coordinate and all the other coordinates are zero). Let $\tilde{e}_{2j-1} = e_j$ and $\tilde{e}_{2j} = -e_j$, $j = 1, \cdots, n$. Define

$$\|\{b_k\}\| = \left(\sum_{k=1}^{\infty} \left|\frac{b_k}{\sigma_k}\right|^q\right)^{1/q}$$

and let

$$T_m = \left\{ \{b_k\} \in T : \|\{b_k\} - \tilde{e}_m\| < \frac{1}{2} \right\} \qquad \forall m = 1, \cdots, 2n \tag{4.5}$$

Note that by continuity

$$\sup_{\{b_k\}\in T - \bigcup_{m=1}^{2n} T_m} \sum_{k=1}^{\infty} b_k^2 \le 1 - \epsilon \tag{4.6}$$

for some $\epsilon > 0$. Therefore, by Theorem 1.1

$$P\left(\sup_{\{b_k\}\in T - \bigcup_{m=1}^{2n} T_m} X(\{b_k\}) > u\right) \le \psi\left(\frac{u-\omega}{1-\epsilon}\right) = o(\psi(u)) \tag{4.7}$$

as $u \to \infty$, where ω is defined in Theorem 1.1 and is finite since $\{X(\{b_k\}), \{b_k\} \in T\}$ is finite a.s.

We now show that $\forall m = 1, \cdots, 2n$

$$\lim_{u\to\infty} \frac{P\left(\sup_{\{b_k\}\in T_m} X(\{b_k\}) > u\right)}{\psi(u)} = 1 \tag{4.8}$$

Without loss of generality we can take $m = 1$, in this case, we see from (4.5) that $b_1 > 1/2$ and therefore $T_1 \subset T^+$ for T^+ as defined in Corollary 1.3. It is also clear from (4.5) that T_1 contains only one point of maximum variance of $X(\{b_k\})$ namely $\tilde{e}_1$. The rest of the proof of (4.8) follows the proof of (1.12) exactly since the only place that we used $\sigma_2 < 1, \cdots, \sigma_n < 1$ was in (1.14b). It is now easy to obtain (1.13). We get the lower bound from Lemma 4.2 by considering $\{X(\tilde{e}_m)\}_{m=1}^{2n}$. For the upper bound we use (4.7), (4.8) and the obvious inequality

$$P\left(\sup_{\{b_k\}\in T} X(\{b_k\}) > u\right) \le \sum_{m=1}^{2n} P\left(\sup_{\{b_k\}\in T_m} X(\{b_k\}) > u\right) + P\left(\sup_{\{b_k\}\in T-\bigcup_{m=1}^{2n} T_m} X(\{b_k\}) > u\right) \tag{4.9}$$

This completes the proof of Corollary 1.3, (1.13).

When $p = 2$ we get the following Corollary of Theorem 1.4.

COROLLARY 4.3. *Let $\{g_k\}_{k=1}^\infty$ be independent normal random variables with mean zero and variance σ_k^2 where $1 = \sigma_1 > \sigma_2 \ge \sigma_3 \ge \cdots$ and $\sum_{k=1}^\infty \sigma_k^2 < \infty$. Then*

$$\limsup_{u\to\infty} \frac{P\left(\left(\sum_{k=1}^\infty |g_k|^2\right)^{1/2} > u\right)}{\psi(u)} \le e^{C^2} \tag{4.10}$$

where $C = C(\sum_{k=1}^\infty \sigma_k^2, (1-\sigma_2)^{-1})$ is a real valued function of $\{\sigma_k\}_{k=1}^\infty$ which is large when $\sum_{k=1}^\infty \sigma_k^2$ and, or $(1-\sigma_2)^{-1}$ is large.

PROOF: Consider the proof of Corllary 1.3, (1.12) given in Section I but with $p = q = 2$. In particular, consider

$$P\left(\sup_{\{b_k\}\in T^+} X(\{b_k\}) > u\right) \tag{4.11}$$

By (1.17) we have

$$E \sup_{a(t)\ge 1-h^2, \{b_k\}\in T^+} (X(t) - a(t)X(\tau^+)) = C_2 \left(\sum_{k=2}^\infty \sigma_k^2\right)^{1/2} h \tag{4.12}$$

Furthermore

(4.13)

$$\sup_{a(t)\geq 1-h^2,\{b_k\}\in T^+} \left(E|X(t)-a(t)X(\tau)|^2\right)^{1/2} = \sup_{\sum_{k=2}^{\infty}|\frac{b_k}{\sigma_k}|^2\geq 2h^2+o(h^2)} \sum_{k=2}^{\infty} b_k^2 \leq 2\sigma_2 h^2 + o(h^2)$$

and, of course, $EX^2(\tau^+)=1$. We can now use Theorem 1.4 applied to $\{X(\{b_k\}),\{b_k\}\in T^+\}$ since (4.13) shows that (1.22) is satisfied. We use (4.12) to define $\omega_2(h)$ in (1.25). Thus we get, from (1.26), that

$$\frac{P\left(\sup_{t\in T^+} X(\{b_k\})>u\right)}{\psi(u)} \leq e^{k_2 u\omega(h_2(u))} \tag{4.14}$$

We use (4.14) and (1.30) to get (4.10). The remark on the size of C as a function of $\sum_{k=2}^{\infty}\sigma_k^2$ comes from (1.30). That C goes to infinity as $(1-\sigma_2)^{-1}$ does comes from an examination of the constants in the proof of Theorem 3.1, (i.e. $\sigma_2\to 1$ implies $\eta\to\sqrt{2}$). We must also consider the process for $t\in T^-$ but this just adds a factor of 2 on the right in (4.14) which we absorb in to the exponent in (4.10).

A sharper result than (4.10) exists in the literature [7]. When $p=2$, sharp estimates for $P(\sum_{k=1}^{\infty}|g_k|^2\geq u^2)$ can be obtained by using Laplace transforms and Tauberian Theorems. However, even in this case, no simple methods that we can find, such as exponential Chebysev inequalities or truncations, imply that the right side of (4.10) is finite. Since transform methods are not available when $p\neq 2$ we believe the Corollary 1.3 and Corollary 1.4 above are new results.

When $p=2$ our methods do not apply if $1=\sigma_1=\sigma_2>\sigma_3\cdots$ (as they did in Corollary 1.3, (1.13)) since in this case

$$EX^2(\{b_k\})=1 \quad \forall b_1,b_2 \text{ such that } \quad b_1^2+b_2^2=1$$

i.e., the maximum points of the variance of $X(\{b_k\})$ are not isolated. As everyone knows, in this case

$$P\left(\left(\sum_{k=1}^{\infty}|g_k|^2\right)^{1/2}\geq u\right)\geq P\left(\left(g_1^2+g_2^2\right)^{1/2}\geq u\right)\sim u\psi(u)$$

as $u\to\infty$ which is not the same as (4.10). Note, however, that in this case (1.22) is not satisfied.

We can not apply our methods to obtain upper bounds for the probability distribution of $\sum_{k=1}^{\infty}|g_k|^p$ when $1<p<2$, where, as above, $\{g_k\}_{k=1}^{\infty}$ are independent normal random variables with mean zero and variance σ_k^2. To see why we write, as in (1.14)

$$\left(\sum_{k=1}^{\infty}|g_k|^p\right)^{1/p} \stackrel{\mathcal{D}}{=} \sup_{\sum_{k=1}^{\infty}|\frac{b_k}{\sigma_k}|^q\leq 1} \sum_{k=1}^{\infty} b_k\eta_k \tag{4.15}$$

where $\{\eta_k\}_{k=1}^{\infty}$ are i.i.d. normal random variables with mean zero and variance 1 and $1/p+1/q = 1$. However, now $q > 2$ and so

$$E\left(\sum_{k=1}^{\infty} b_k\eta_k\right)^2 = \sum_{k=1}^{\infty} b_k^2 \leq \left(\sum_{k=1}^{\infty}\left|\frac{b_k}{\sigma_k}\right|^q\right)^{2/q}\left(\sum_{k=1}^{\infty}\sigma_k^{2q/(q-2)}\right)^{(q-2)/q} \tag{4.16}$$

with equality whenever

$$|b_k| = C_q\sigma_k^{q/(q-2)}$$

for some constant C_q depending only on q. Therefore

$$\sup_{\sum_{k=1}^{\infty}|\frac{b_k}{\sigma_k}|^q\leq 1} E\left(\sum_{k=1}^{\infty} b_k\eta_k\right)^2 = \left(\sum_{k=1}^{\infty}\sigma_k^{2q/(q-2)}\right)^{(q-2)/q}$$

whenever $b_k = \pm C_q\sigma_k^{q/(q-2)}$. Thus the process $\sum_{k=1}^{\infty} b_k\eta_k$ has an infinite number of points of maximum variance and hence these points cannot be isolated. However we can obtain estimates of the probability distribution of $\sum_{k=1}^{n}|g_k|^p$.

THEOREM 4.4. *Let $\{g_k\}_{k=1}^{n}$ be independent normal random variables with mean zero and variance σ_k^2 and $1 < p < 2$. Define*

$$\sigma = \left(\sum_{k=1}^{n}\sigma_k^{2q/(q-2)}\right)^{(q-2)/(2q)}$$

where $1/p + 1/q = 1$. Then there exists a real valued function $K = K(\sigma, n, p)$ such that for $u \geq u_0(\sigma, n, p)$ sufficiently large

$$2^n \leq \frac{P\left(\left(\sum_{k=1}^{n}|g_k|^p\right)^{1/p} > u\right)}{\psi(\frac{u}{\sigma})} \leq K^n \tag{4.17}$$

where K goes to infinity as p approaches 2.

PROOF: Let $T = \{\{b_k\} : \sum_{k=1}^{n}|\frac{b_k}{\sigma_k}|^q \leq 1\}$. As above we consider $\{X(\{b_k\}), \{b_k\} \in T\}$ where $X(\{b_k\}) = \sum_{k=1}^{n} b_k\eta_k$ for $\{\eta_k\}_{k=1}^{n}$ i.i.d. normal random variables with mean zero and variance 1. For simplicity we will sometimes denote the elements of T by the letter t. As in (4.15) we have

$$\left(\sum_{k=1}^{n}|g_k|^p\right)^{1/p} \stackrel{D}{=} \sup_{t\in T} X(t) \tag{4.18}$$

and, following (4.16)

$$EX^2(t) = \sum_{k=1}^{n} b_k^2 \leq \left(\sum_{k=1}^{n}\left|\frac{b_k}{\sigma_k}\right|^q\right)^{2/q}\left(\sum_{k=1}^{n}\sigma_k^{2q/(q-2)}\right)^{(q-2)/q} \tag{4.19}$$

Without loss of generality we can normalize $EX^2(t)$ by taking

$$\sum_{k=1}^{n} \sigma_k^{2q/(q-2)} = 1 \tag{4.20}$$

By (4.19) and the discussion prior to the statement of this Theorem we have

$$\sup_{t\in T} EX^2(t) \leq 1$$

with equality whenever $b_k = \pm\sigma_k^{q/(q-2)}$, $\forall k = 1,\cdots,n$. We see that $\{X(t), t \in T\}$ has 2^n points of maximum variance. Let

$$\delta_k = \sigma_k^{q/(q-2)} \qquad k = 1,\cdots,n \tag{4.21}$$

Let $\tau \equiv \{\delta_k\}_{k=1}^n$ and note that $\tau \in T$ and $EX^2(\tau) = 1$ implies that

$$\sum_{k=1}^{n} \left|\frac{\delta_k}{\sigma_k}\right|^q = 1 \qquad \text{and} \qquad \sum_{k=1}^{n} \delta_k^2 = 1 \tag{4.22}$$

We write

$$b_k = \delta_k - u_k \qquad \forall k = 1,\cdots,n. \tag{4.23}$$

Let T_ϵ be a neighborhood of τ (say in the ℓ^2 metric) chosen so that

$$|u_k| \leq \epsilon\delta_k \qquad \forall k = 1,\cdots,n \tag{4.24}$$

where $\epsilon > 0$ is small enough so that $EX^2(t) < 1$, $\forall t \in T_\epsilon$ such that $t \neq \tau$. We will show that for all $0 < \epsilon < \frac{2}{q-1}$ and $u \geq u_0(n,\epsilon)$ for u_0 sufficiently large

$$\frac{P\left(\sup_{t\in T_\epsilon\cap T} X(t) > u\right)}{\psi(u)} \leq \exp\left(k_2 \frac{2}{q-1}(1+\epsilon)n\right) \tag{4.25}$$

One can check that (4.25) implies the upper bound in (4.17) by precisely the same argument used in the proof of Corollary 1.3, (1.13), which was given earlier in this Section. (We incorporate the factor 2^n into K^n).

We will now obtain the upper bound in (4.25). We have

$$a(t) = EX(t)X(\tau) = \sum_{k=1}^{n} \delta_k(\delta_k - u_k) = 1 - \sum_{k=1}^{n} \delta_k u_k \tag{4.26}$$

Therefore, by (4.22) and (4.26)

$$E(X(t) - a(t)X(\tau))^2 = EX^2(t) - a^2(t) \leq \sum_{k=1}^{n} u_k^2 \tag{4.27}$$

and by (4.26)

$$1 - h^2 \le a(t) \le 1 \qquad \text{if and only if} \qquad 0 \le \sum_{k=1}^{n} \delta_k u_k \le h^2 \tag{4.28}$$

The key point in this proof is to show that $\forall \epsilon > 0$, there exists an $\bar{h}(\epsilon)$ such that for $h \in [0, \bar{h}(\epsilon)]$

$$\sup_{t \in T_\epsilon \cap T,\, a(t) \ge 1-h^2} E(X(t) - a(t)X(\tau))^2 \le \frac{2(1-\epsilon)^{2-q}}{q-1} h^2 \tag{4.29}$$

The point of this is that since $q > 2$ we can choose ϵ such that $2(1-\epsilon)^{2-q}/(q-1) < 2$. Therefore when we later apply Theorem 1.4 to $\{ X(t), t \in T_\epsilon \cap T \}$ we will have that (1.22) is satisfied.

By Taylor's Theorem applied to $|\delta_k - u_k|^q$ where u_k satisfies (4.24) $\forall\, k = 1, \cdots, n$ we see that

$$\sum_{k=1}^{n} \left| \frac{\delta_k - u_k}{\sigma_k} \right|^q = \sum_{k=1}^{n} \left| \frac{\delta_k}{\sigma_k} \right|^q - q \sum_{k=1}^{n} \frac{\delta_k^{q-1} u_k}{\sigma_k^q} + \frac{q(q-1)}{2} \sum_{k=1}^{n} \frac{|\delta_k - c_k|^{q-2} u_k^2}{\sigma_k^q} \tag{4.30}$$

where $|c_k| \le \epsilon \delta_k$, $\forall\, k = 1, \cdots, n$. Recalling (4.22) we see that $\{\delta_k - u_k\}_{k=1}^n \in T \cap T_\epsilon$ implies that

$$\sum_{k=1}^{n} \frac{|\delta_k - c_k|^{q-2} u_k^2}{\sigma_k^q} \le \frac{2}{q-1} \sum_{k=1}^{n} \frac{\delta_k^{q-1} u_k}{\sigma_k^q} \tag{4.31}$$

which implies by (4.21) and the bound on $|c_k|$ that

$$\sum_{k=1}^{n} u_k^2 \le \left(\frac{2}{q-1}\right)(1+\epsilon') \sum_{k=1}^{n} \delta_k u_k \tag{4.32}$$

where $\epsilon' = |1-\epsilon|^{2-q} - 1$ can be made arbitrarily small depending on ϵ.

Combining (4.32) and (4.28) we see that

$$t \in T \cap T_\epsilon \cap \{ t : a(t) \ge 1 - h^2 \} \equiv T^* \subset \{ \{u_k\} : \sum_{k=1}^{n} u_k^2 \le \left(\frac{2}{q-1}\right)(1+\epsilon')h^2 \} \tag{4.33}$$

(Recall that t identifies the points $\{\delta_k - u_k\}_{k=1}^n$). Combining (4.27) and (4.33) we get (4.29).

Following (2.6a) we have

$$L(h) = E \sup_{t \in T^*} (X(t) - a(t)X(\tau)) \le E \sup_{t \in T^*} (X(t) - X(\tau)) + h^2 \tag{4.34}$$

Note that

(4.35)

$$E \sup_{t \in T^*} (X(t) - X(\tau)) = E \sup_{t \in T^*} \sum_{k=1}^{n} u_k \eta_k \le \sup_{t \in T^*} \left(\sum_{k=1}^{n} u_k^2 \right)^{1/2} n^{1/2} \le \left(\frac{2}{q-2}(1+\epsilon')n \right)^{1/2} h$$

by (4.33). By Theorem 1.4 and the example immediately following it we get (4.25). An estimate for the constant k_2 can be obtained from Theorem 3.1. It goes to infinity as $2/(q-1)$ gets closer to 2. The lower bound in (4.25) follows from Lemma 4.2.

To round out the picture we consider $p = 1$ and obtain the following simple consequence of Lemma 4.2.

THEOREM 4.5. *Let $\{g_k\}_{k=1}^n$ be independent normal random variables with mean zero and variance σ_k^2. Let*

$$\sigma = \left(\sum_{k=1}^n \sigma_k^2\right)^{1/2} \tag{4.36}$$

Then

$$\lim_{u\to\infty} \frac{P\left(\sum_{k=1}^n |g_k| > u\right)}{\psi(\frac{u}{\sigma})} = 2^n \tag{4.37}$$

PROOF: Note that

$$\sum_{k=1}^n |g_k| = \sup_{b_k=\pm 1} \sum_{k=1}^n b_k g_k \tag{4.38}$$

As usual we define the process $X(\{b_k\}) = \sum_{k=1}^n b_k g_k$ and consider $\{X(\{b_k\}) : \{b_k\} \in T\}$ where $T = \{\{b_k\} : b_k = \pm 1,\ k = 1, \cdots, n\}$ contains 2^n points. Since

$$EX^2(\{b_k\}) = \sum_{k=1}^n \sigma_k^2 \qquad \forall \{b_k\} \in T \tag{4.39}$$

this process has 2^n points of maximum variance. Therefore the lower bound in (4.37) follows from Lemma 4.2. The upper bound in (4.37) is trivial since T only contains 2^n points.

In all the examples we have given so far we have either had $L(h) \sim h$ or $L(h) = o(h)$ as $h \to 0$. In these cáses the function $k(u)$ in (1.34) can always be taken to be 1. However, the cosine transform portion of a stationary Gaussian process leads to examples of other types of behavior in (1.34). Let $\{W(t), t \in [0,1]\}$ and $\{X(t) \in [0,1]\}$ be defined in (1.32) and (1.33) and let

$$Y(t) = \int_0^\infty \sin \lambda t \, dB'(F(\lambda)) \qquad t \in [0,1] \tag{4.40}$$

Suppose that

$$\int_0^\infty \sin^2 \lambda t \, dF(\lambda) > 0 \qquad t \in (0,a] \tag{4.41}$$

for some $0 < a < 1$. Then $\{X(t), t \in [0,a]\}$ has a unique point of maximum variance at $t = 0$, where, by definition, $EX^2(0) = 1$. Let

$$\sigma(u) = (E|W(t+u) - W(t)|^2)^{1/2} \tag{4.42}$$

Continuing with our usual notation we see that

$$a(t) = EX(t)X(0) = EW(t)W(0) = 1 - \frac{\sigma^2(t)}{2} \tag{4.43}$$

and so

$$a(t) \geq 1 - h^2 \qquad \text{if and only if} \qquad \sigma^2(t) \leq 2h^2 \tag{4.44}$$

In order to apply Theorem 1.4 to $\{X(t), t \in [0,a]\}$ we must verify (1.22). The next Lemma gives conditions for this.

LEMMA 4.6. *For σ and $X(t)$ as defined above*

$$\liminf_{u\to\infty} \frac{\sigma^2(2u)}{\sigma^2(u)} > \delta > 0 \tag{4.45}$$

implies, $\forall\, h \in [0, \bar{h}]$ for $\bar{h}$ sufficiently small, that

$$\sup_{a(t)\geq 1-h^2} E|X(t) - a(t)X(0)|^2 \leq (2 - \frac{\delta}{2})\, h^2 \tag{4.46}$$

PROOF: We have

$$E|X(t) - a(t)X(0)|^2 = EX^2(t) - a(t) + a(t)(1 - a(t))$$
$$\leq EX^2(t) - a(t) + (1 - a(t))$$

so that

$$\sup_{a(t)\geq 1-h^2} E|X(t) - a(t)X(0)|^2 \leq \sup_{a(t)\geq 1-h^2} EX^2(t) - a(t) + h^2 \tag{4.47}$$

By definition

$$EX^2(t) = \int_0^\infty \cos^2 \lambda t\, dF(\lambda) = \frac{1}{2}(1 + a(2t))$$

and so by (4.43)

$$EX^2(t) - a(t) = \frac{1}{2}\left(\sigma^2(t) - \frac{\sigma^2(2t)}{2}\right) \tag{4.48}$$

Thus we see by (4.48) and (4.44) that the left side of (4.47)

$$\leq \sup_{\sigma^2(t)\leq 2h^2} \frac{1}{2}\left(\sigma^2(t) - \frac{\sigma^2(2t)}{2}\right) + h^2$$

which, by (4.45) gives (4.46).

The following result is an application of Theorem 1.4.

THEOREM 4.7. *Let $\{g_k\}_{k=0}^\infty$ be i.i.d. normal random variables with mean zero and variance 1. Consider, for $\delta > 1$,*

$$X(t) = C_\delta \sum_{k=0}^\infty k^{-\delta} g_k \cos 2^k t \qquad t \in [0, \frac{\pi}{2}] \tag{4.49}$$

where we set $0^{-\delta} = 1$ and $C_\delta = (\sum_{k=0}^\infty k^{-2\delta})^{-1/2}$. Then there exist constants $0 < k_1 < k_2 < \infty$ such that for all $u \geq u_0(\delta)$, for $u_0(\delta)$ sufficiently large,

$$\exp\left(k_1 u^{1/\delta}\right) \leq \frac{P\left(\sup_{t\in[0,\pi/2]} X(t) > u\right)}{\psi(u)} \leq \exp\left(k_2 u^{1/\delta}\right) \tag{4.50}$$

PROOF: For simplicity let us set $a_k = C_\delta k^{-\delta}$, $\forall k \geq 0$. As above we define

$$Y(t) = C_\delta \sum_{k=0}^{\infty} k^{-\delta} g'_k \sin 2^k t \qquad t \in [0, \frac{\pi}{2}]$$

where $\{g'_k\}_{k=0}^{\infty}$ is an independent copy of $\{g_k\}_{k=0}^{\infty}$, and $W(t) = X(t) + Y(t)$, $t \in [0, \pi/2]$. It is easy to see $EY^2(t) > 0$, $\forall t \in [0, \pi/2]$ so $\{X(t), t \in [0, \pi/2]\}$ has a unique point of maximum variance 1 at $t = 0$.

Let σ be defined as in (4.42). It is easy to see that, at least,

$$\sigma^2(2t) > \frac{1}{2}\sigma^2(t) \qquad \forall t \in [0, \bar{t}] \tag{4.51}$$

for some $\bar{t} \geq 0$. Thus $\{X(t), t \in [0, \pi/2]\}$ satisfies (1.22). (To see (4.51) note that for all t sufficiently small

$$\sigma^2(2t) - \frac{1}{2}\sigma^2(t) = 4\sum_{k=1}^{\infty}\left(a_{k-1}^2 - \frac{1}{2}a_k^2\right) \sin^2\left(\frac{2^k t}{2}\right) - 2a_0^2 \sin^2\left(\frac{t}{2}\right)$$

$$\geq 2\left(a_0^2 - \frac{1}{2}a_1^2\right)t^2 - \frac{1}{2}a_0^2 t^2 \geq \frac{1}{2}t^2 C_\delta^2$$

since $a_0 = a_1$).

Let

$$h_N = \left(\sum_{k=N}^{\infty} a_k^2\right)^{1/2} \tag{4.52}$$

we will show that (see (2.5))

$$L(h_N) \sim \sum_{k=N}^{\infty} a_k \qquad \text{as } N \to \infty \tag{4.53}$$

By (4.44) and (2.6a)

$$L(h_N) \leq E \sup_{\sigma(t) \leq \sqrt{2}\, h_N} |X(t) - X(0)| + h_N^2$$

Note that

(4.54)

$$\begin{aligned} E \sup_{\sigma(t) \leq \sqrt{2}h,\, t \in [0,\pi/2]} |X(t) - X(0)| &\leq E \sup_{\sigma(t) \leq \sqrt{2}h,\, t \in [0,\pi/2]} |W(t) - W(0)| \\ &\leq E \sup_{\sigma(s-t) \leq \sqrt{2}h;\, s,t \in [0,\pi/2]} |W(t) - W(s)| \\ &\leq k\left(\int_0^{\sqrt{2}h} \left(\log N([0, \frac{\pi}{2}], \epsilon)\right)^{1/2} d\epsilon + h\left(\log\log\frac{k}{h}\right)^{1/2}\right) \end{aligned}$$

for some constant k independent of h, where, at the last step, we use a version of Dudley's Theorem (see [**10**, Chapter II, Theorem 3.1], and note that $N([0,\pi/2],\epsilon)$ is the minimum number of open balls, in the metric $d(s,t)=\sigma(|s-t|)$, that covers $[0,\pi/2]$).

Define $m(\epsilon)=\lambda\{t\in[0,\pi/2]:\sigma(t)<\epsilon\}$ where λ is Lebesque measure and set

$$\epsilon_m = 5\left(\sum_{k=m}^{\infty} a_k^2\right)^{1/2}$$

Then, since

$$\sigma^2(t)=4\sum_{k=0}^{\infty} a_k^2\sin^2\left(\frac{2^k t}{2}\right)\le\sum_{k=0}^{m-1}a_k^2 4^k t^2+4\sum_{k=m}^{\infty}a_k^2 \tag{4.55}$$

we see that for $t\in[0,2^{-m}]$, $\sigma(t)<\epsilon_m$ and so $m(\epsilon_m/2)>2^{-m}$.

Therefore, as is well known, (see e.g. [**10**, Chapter II, Lemma 1.1])

$$N(\epsilon_m)\equiv N([0,\pi/2],\epsilon_m)\le K'2^m \qquad \forall m\ge 0$$

for some numerical constant K'. It follows that

$$\begin{aligned}\int_0^{h_N}\left(\log N([0,\tfrac{\pi}{2}],\epsilon)\right)^{1/2}d\epsilon &\le \sum_{m=N}^{\infty}\int_{\epsilon_{m+1}}^{\epsilon_m}\left(\log N([0,\tfrac{\pi}{2}],\epsilon)\right)^{1/2}d\epsilon \\ &\le C'\sum_{m=N}^{\infty}m^{1/2}(\epsilon_m-\epsilon_{m+1})\le 5C'\sum_{m=N}^{\infty}\frac{m^{1/2}a_m^2}{(\sum_{k=m}^{\infty}a_m^2)^{1/2}}\le C\sum_{k=N}^{\infty}a_k\end{aligned} \tag{4.56}$$

for some numerical constants C and C', where we use the facts that $\epsilon_m-\epsilon_{m+1}\le(\epsilon_m^2-\epsilon_{m+1}^2)/\epsilon_m$ and a_k is regularly varying as $k\to\infty$. Combining (4.54) and (4.56) we get the upper bound in (4.53). For the lower bound we note that for any index set T'

$$\begin{aligned}E\sup_{t\in T'}|X(t)-X(0)| &\ge E\sup_{t\in T'}\left|\sum_{k=2N}^{\infty}a_k g_k(1-\cos 2^k t)\right| \\ &\ge E\sup_{t\in T'}\left|\sum_{k=2N}^{\infty}a_k g_k\cos 2^k t\right|-E\left|\sum_{k=2N}^{\infty}a_k g_k\right|\end{aligned} \tag{4.57}$$

Also, it follows as in (4.55) that

$$\sigma(t)\le\sqrt{2}h_N \qquad \forall t\in[0,4^{-N}] \tag{4.58}$$

and so by (4.57) and (4.58) we have

$$\begin{aligned}L(h_N) &= \frac{1}{2}E\sup_{\sigma(t)\le\sqrt{2}h_N,t\in[0,\pi/2]}|X(t)-X(0)|-h_N^2 \\ &\ge\frac{1}{2}E\sup_{t\in[0,4^{-N}]}\left|\sum_{k=2N}^{\infty}a_k g_k\cos 2^k t\right|-\sqrt{2/\pi}\left(\sum_{k=2N}^{\infty}a_k^2\right)^{1/2}-h_N^2 \\ &\ge c'\sum_{k=2N+1}^{\infty}a_k-\sqrt{2/\pi}\left(\sum_{k=2N}^{\infty}a_k^2\right)^{1/2}\ge c\left(\sum_{k=2N+1}^{\infty}a_k\right)-h_N^2\end{aligned} \tag{4.59}$$

for constants c' and c independent of N, where we use the fact that the cosine series is a lacunary series and, of course, the nature of the $\{a_k\}_{k=0}^{\infty}$. Thus, taking into account the actual values of the $\{a_k\}_{k=0}^{\infty}$, we get (4.53).

Let

$$\omega(h_N) = \sum_{k=N}^{\infty} a_k \qquad \forall N > 0 \tag{4.60}$$

and following (1.24) set

$$u_N = \frac{\omega(h_N)}{h_N^2} = \frac{\sum_{k=N}^{\infty} a_k}{\sum_{k=N}^{\infty} a_k^2} \sim \frac{1}{a_N} = N^{\delta}$$

as $N \to \infty$, by regular variation. Therefore

$$u_N \omega(h_N) \sim N = u_N^{1/\delta} \qquad \text{as } N \to \infty$$

This shows that

$$u\omega(h(u)) \sim u^{1/\delta} \qquad \text{as } u \to \infty \tag{4.61}$$

along the sequences u_N and $h(u_N) = h_N$. The proof of Theorem 4.7 follows from (4.53), (4.60), (4.61) and Theorem 1.4 by extrapolation. (Note that the concavity assumption is alright since $\omega(h_N)/h_N \sim N^{1/2}$. In any case one could also use Theorems 2.5 and 3.3).

REMARK 4.8: In finding the upper bound for $L(h)$ in (4.54) we passed to the stationary process, which, obviously, does not have a unique point of maximum variance. This doesn't seem to matter as long as the process isn't too regular. On the other hand it is clear from Lemma 4.2 that the stationary process can not satisfy (1.10). However, the cosine part alone can if the $\{a_k\}_{k=0}^{\infty}$ go to zero fast enough. For example let $a_k = C_\beta 2^{-k\beta}$, $\forall\, k \geq 0$ where $\beta > 2$ and C_β is chosen such that $\sum_{k=0}^{\infty} a_k^2 = 1$ and consider the process

$$X(t) = \sum_{k=0}^{\infty} a_k g_k \cos 2^k t \qquad t \in [0, \frac{\pi}{2}] \tag{4.62}$$

for σ as defined in (4.42). It is easy to check that

$$\left\{ t : \sigma(t) \leq \sqrt{2}h \right\} \subset [0, \frac{3h}{a_0}]$$

Therefore by (2.6a) and (4.44)

$$\begin{aligned} L(h) - h^2 &= E \sup_{\sigma(t) \leq \sqrt{2}h} |X(t) - a(t)X(0)| \\ &\leq E \sup_{t \in [0, 3h/a_0]} \sum_{k=0}^{\infty} a_k g_k \sin^2 \frac{2^k t}{2} \leq \left(\frac{3h}{2a_0}\right)^2 \sum_{k=0}^{\infty} a_k 2^{2k} \leq C'_\beta h^2 \end{aligned}$$

for some constant C'_β depending on β. It follows, by Theorem 1.2, that the process in (4.62) satisfies (1.10).

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Vladimir Dobric
Department of Mathematics
Lehigh University
Bethleham, PA 18015
U.S.A.

Michael B. Marcus
Department of Mathematics
The City College of CUNY
New York, NY 10031
U.S.A.

Michel Weber
Université Louis Pasteur
7 rue René Descartes
67084 Strasbourg Cedex
FRANCE